On permutation characters of finite groups

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Abstract

Let G be a finite group and M be a maximal subgroup of G. We call every irreducible constituent χ of $(1_M)^G$ a $\mathcal{P}$-character of G with respect to M. In this paper, we prove that if all $\mathcal{P}$-characters of G are monomial, then G is solvable, which solves a question posed by Qian and Yang.

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1 Introduction

Throughout this paper, G always denotes a finite group, p is a prime and π is a set of some primes. As usual, let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. It is well-know that the structure of G is heavily determined by $\mathrm{cd}(G)$, and there are a lot of classical theorems on this subject. For example, see [1, 3].

Recall that $\chi \in \mathrm{Irr}(G)$ is monomial if χ is induced from a linear character of a subgroup of G, and that G is an M-group if all its irreducible characters are monomial. We know that M-groups are solvable and that every solvable group is contained in an M-group. More interesting results on M-groups can be found in [2].

Let M be a maximal subgroup of G. Then G acts on the set of right cosets of M and it induces a transitive permutation character $(1_M)^G$. Let χ be an irreducible constituent of $(1_M)^G$. Followed by [4], we call χ a $\mathcal{P}$-character of G with respect to M, and denote by $\mathrm{Irr}_{\mathcal{P}}(G)$ the set of $\mathcal{P}$-characters of G and $\mathrm{cd}_{\mathcal{P}}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_{\mathcal{P}}(G)\}$. If in addition $|G : M|$ is a π-number where π is a set of primes, then χ is called a $\mathcal{P}_\pi$-character. Let $\mathrm{Irr}_{\mathcal{P}_\pi}(G)$ be the set of $\mathcal{P}_\pi$-characters of G. Clearly, if χ is a $\mathcal{P}$-character of a solvable group, then χ is a $\mathcal{P}_p$-character for some prime p. Moreover, a $\mathcal{P}$-character of a solvable group is monomial and monolithic (see [4, Proposition 2.3]).

In [4], the authors posed several questions, one of which is whether G is solvable if all $\mathcal{P}$-characters of G are monomial. In the following theorem, we give an affirmative answer.

Theorem A. *Let G be a group and assume that all $\mathcal{P}$-characters of G are monomial. Then G is solvable.*

In the following theorem, we prove a π-variation of [4, Proposition 2.5].

Theorem B. *Let G be a π-solvable group. Then all $\mathcal{P}_\pi$-character degrees of G are π-number if and only if G has a normal π-complement.*

Theorem C. *Let G be π-separable. If all members of* $\mathrm{Irr}(G)$ *are $\mathcal{P}_\pi$-characters, then G is a π-group.*

Theorem C suggests that perhaps if G is a π-group, then $\mathrm{Irr}_{\mathcal{P}}(G) = \mathrm{Irr}(G)$. However, this is false in general. An easy counterexample in a solvable group is to take G to be $SL(2,3)$ and $\pi = \{2,3\}$. Then the irreducible characters of degree 2 of G are not $\mathcal{P}$-characters.

Note also that if $\mathrm{Irr}_{\mathcal{P}}(G) = \mathrm{Irr}(G)$, then G is not necessarily solvable. As pointed out by G. Navarro, $\mathrm{PSL}(2,27)$ is such an example.

By Corollary 2.6 of [4], a finite group G is nilpotent if and only if $\mathrm{cd}_{\mathcal{P}}(G) = \{1\}$. Let $G = A_5$, the alternating group of degree 5. Then $\mathrm{cd}_{\mathcal{P}}(G) = \{1,4,5\}$. After checking many examples, it seems reasonable to conjecture that if $|\mathrm{cd}_{\mathcal{P}}(G)| \leq 2$, then G is solvable. In this paper, however, we cannot yet prove it.

2 Proofs

Proof of Theorem A. Suppose the theorem does not hold, and let G be a counterexample of minimal order.

We first show that G is not a non-abelian simple group. Assuming that this result false, we derive a contradiction. Let K be a proper subgroup of G with the largest possible order. Then K is a maximal subgroup of G. Let φ be a nonprincipal $\mathcal{P}$-character of G with respect to K. Note that such φ really exists by Frobenius Reciprocity. By assumption, φ is monomial, so we may assume that $\varphi = \lambda^G$, where λ is a linear character of some subgroup K_1 of G. Since

$$(1_K)^G(1) = |G:K| \geq 1 + \varphi(1) = 1 + |G:K_1|,$$

we must have that $|K_1| > |K|$ and so $K_1 = G$. It follows that φ is a nonprincipal linear character of G and thus $|G:G'| \geq 2$, a contradiction.

Let N be a minimal normal subgroup of G. Then $N < G$. Let $\chi \in \mathrm{Irr}(G/N)$ be a $\mathcal{P}$-character of G/N with respect to the maximal subgroup M/N. Then $\chi \in \mathrm{Irr}(G)$ be a $\mathcal{P}$-character of G with respect to the maximal subgroup M. By assumption, $\chi = \eta^G$, where η is a linear character of some subgroup M_1 of G. Then $N \leq \ker \chi = \bigcap_{x \in G} (\ker \eta)^x \leq \ker \eta \leq M_1$. Thus, $\chi \in \mathrm{Irr}(G/N)$ is monomial. By induction hypothesis, G/N is solvable. Then N is not solvable. Clearly, N is the unique minimal subgroup of G and $\Phi(G) = 1$.

Let H be a maximal subgroup of G such that N is not contained in H and H has the largest possible order. Then $\mathrm{Core}_G(H) = 1$. Let ψ be a nonprincipal $\mathcal{P}$-character of G with respect to H. If $\ker \psi \neq 1$, then $N \leq \ker \psi$, and so $\ker \psi \not\leq H$. It follows that $G = H\ker \psi$ and ψ_H is irreducible. Since $[\psi_H, 1_H] = [\psi, (1_H)^G] > 0$, we have that $\psi_H = 1_H$. In particular, $H \leq \ker \psi$, a contradiction. So $\ker \psi \neq 1$.

Now, by assumption, we may assume that $\psi = \mu^G$, where μ is a linear character of some subgroup H_1 of G. If $H_1 = G$, then $\psi = \mu$ is a linear constituent of $(1_H)^G$, so $\psi_H = 1_H$. This implies that $H \leq \ker \psi$, a contradiction. So $H_1 < G$. Since

$$(1_H)^G(1) = |G : H| \geq 1 + \psi(1) = 1 + |G : H_1|,$$

we have that $|H_1| > |H|$. By the choice of H, we get that $N \leq H_1$. It follows that $1 \neq N = N' \leq H_1' \leq \ker \mu$, and $N \leq \bigcap_{x \in G} (\ker \mu)^x = \ker \mu^G = \ker \psi$, a contradiction. The proof is complete. $\square$

For the proof of Theorem B, we need the following Lemma.

Lemma 2.1. [4, Proposition 2.3 (2)] *Let G be a solvable group and $G = N \rtimes H$, where H is a maximal subgroup of G and N is a minimal normal subgroup of G. Then the map $\Theta : \chi \mapsto \mathrm{Irr}(\chi_N)$ is a bijection of $\mathrm{Irr}(1_H)^G$ onto the set of G-orbits of $\mathrm{Irr}(N)$. Furthermore, if $\lambda \in \mathrm{Irr}(\chi_N)$, then λ is extendible to $I_G(\lambda)$, and we have $\chi(1) = |G : I_G(\lambda)|$, where $I_G(\lambda)$ is the inertia group of λ in G.*

Proof of Theorem B. Suppose that H is a normal π-complement of G, and let χ be a $\mathcal{P}_\pi$-character of G with respect to a maximal subgroup M. Then $|G : M|$ is a π-number. Since H is a normal Hall π'-subgroup of G, we have that $H \leq M$ and G/H is a solvable π-group. By definition, since χ is an irreducible constituent of $(1_M)^G$, it follows that χ lies over 1_M and 1_H lies under χ. So $H \leq \ker(\chi)$ and $\chi(1)$ is a π-number, as desired.

Conversely, suppose that all $\mathcal{P}_\pi$-character degrees of G are π-number. Let N be a minimal normal subgroup G. By induction hypothesis, G/N has a normal π-complement NK/N, where K is a Hall π'-subgroup of G. Clearly, N is the unique minimal subgroup of G.

In what following, it is sufficient to show that NK has a normal Hall π'-subgroup.

Since G is π-solvable, N is a π'-group or elementary abelian p-group for some prime $p \in \pi$. If N is a π'-group, then NK is a π'-group and NK is a normal Hall π'-subgroup, as desired.

Now suppose N is an elementary abelian p-group, where $p \in \pi$. If $\Phi(G) \neq 1$, then $N \leq \Phi(G)$. By the Frattini argument, $G = NKN_G(K) = NN_G(K)$. Since $N \leq \Phi(G)$, it follows that $G = N_G(K)$, so $L \trianglelefteq G$, and thus $L \trianglelefteq K$.

Thus, we may assume that $\Phi(G) = 1$, and N is not contained in $\Phi(G)$. Choose a maximal subgroup M of G such that $N \nleq M$. Then $G = N \rtimes M$, and M is not normal in G.

Let $\lambda \in \mathrm{Irr}(N)$. By Lemma 2.1, there exists an irreducible constituent of $(1_M)^G$, say χ, such that λ is an irreducible constituent of χ_N, and $\chi(1) = |G : I_G(\lambda)|$. By the hypothesis, $\chi(1)$ is a π-number. So we may assume that $NK \leq T$, where $T = I_G(\lambda)$. By Lemma 2.1 again, we see that λ extends to $\mathrm{Irr}(T)$. As a result, λ extends to some NK. By [2, Theorem 3.17], NK has a normal Hall π'-subgroup. $\square$

Proof of Theorem C. We suppose that $\mathrm{Irr}_{\mathcal{P}}(G) = \mathrm{Irr}(G)$, and we proceed by induction on $|G|$ to show that G must be a π-group. We can assume that $|G| > 1$, and we let N be minimal normal in G. Since all members of $\mathrm{Irr}(G/N)$ are $\mathcal{P}_\pi$-characters, the inductive hypothesis implies that G/N is a π-group, and thus it suffices to show that N is a π-group.

Recall that a minimal normal subgroup of G must be either a π'-group or a π-group. Let χ be a $\mathcal{P}_\pi$-character of G with respect to a maximal subgroup M of G, and $|G : M|$ be a π-number. Then M contains some Hall π'-group K of G. It follows that $O_{\pi'}(G) \leq K \leq M$, and so $O_{\pi'}(G) \leq \mathrm{Core}_G(M) = \ker \chi$.

Now, if N is a π'-group, then $N \subseteq \ker(\chi)$ for all $\chi \in \mathrm{Irr}_{\mathcal{P}}(G) = \mathrm{Irr}(G)$, and since $N > 1$, this is a contradiction. It follows that N is a π-group, as wanted. $\square$

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