ZETA ZEROS ON THE CRITICAL LINE

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Abstract. Montgomery in 1973 introduced the pair correlation method to study the vertical distribution of Riemann zeta-function zeros. This work assumed the Riemann Hypothesis (RH). One striking application was a short proof that at least 2/3 of zeta-zeros are simple zeros, the first result of its type. Over the last 50 years, most work on pair correlation of zeta-zeros has continued to assume RH. Here we show that if RH could be removed from Montgomery's simple zero proof, then this would also give a proof that 2/3 of the zeros are on the critical line. While at present we cannot unconditionally obtain this result, we have proved in joint work that this is true if we assume all the zeros are in a narrow enough region centered on the critical line. In separate joint work we use the same idea with a related pair correlation method to prove that assuming the Pair Correlation Conjecture, also without the assumption of RH, we obtain 100% of the zeros are simple and on the critical line.

1. Introduction

In 1973 Montgomery [Mon73] proved, assuming the Riemann Hypothesis (RH), that at least 2/3 of the zeros of the Riemann zeta-function are simple. Prior to this, it was not known either unconditionally or on RH that there were even infinitely many simple zeros. Currently, by a refined version of Levinson's method, K. Pratt, N. Robles, A. Zaharescu, and D. Zeindler [PRZZ20] in 2020 proved that at least 41.7% of the zeros of $\zeta(s)$ are on the critical line, and at least 40.7% of the zeros are on the critical line and simple. By large scale computation with careful error analysis, Platt and Trudgian [PT21] in 2021 showed that in the critical strip up to height 3,000,175,332,800, there are exactly 12,363,153,437,138 zeros and all of them are on the critical line and are simple.

Conditionally, under the RH assumption one can obtain that at least 67.92% of the zeros of $\zeta(s)$ are simple using Montgomery's pair correlation method [CGdL20], and with a different method [CGG98, BHB13] one obtains at least 70.37% of the zeros being simple. One may ask why we care about Montgomery's pair correlation method to detect simple zeros, if one could obtain a better proportion of simple zeros using a different method. In fact, Chirre, Gonçalves, and de Laat [CGdL20]'s 67.92% result appeared years after Bui and Heath-Brown [BHB13] proved the 70.37% proportion, which also shows a specific interest in Montgomery's pair correlation method. We will not dive into further details on these papers. What really has became the main motivation is the fact that Montgomery's pair correlation method has wide applications, including direct applications to the distribution of prime numbers. This is demonstrated in several papers [GM78, HBG84, GM87, Gol88] and many more following them. In particular, Montgomery's method has a huge flexibility that leaves a vast room for improvements, one of which we will clarify in this exposé.

In this exposé we introduce a simple method using pair correlation of zeros of $\zeta(s)$ that, on assuming additional information on the zeros, proves conditionally that there are many zeros on the critical line. Three main points to underline in our results are that 1) we can remove RH in the study of pair correlation of zeros of $\zeta(s)$, 2) we can replace RH with weaker assumptions to recover results on simple zeros that have been only obtained under RH, and that 3) for the first time, the pair correlation of zeros implies that most of the zeros lie on the critical line.

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2. Zeros of the Riemann zeta-function

Let $s = \sigma + it$ denote any point in the complex plane. The meromorphic function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, and is analytic elsewhere. For $\sigma > 1$, we have the Euler product formula $\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^s)^{-1}$, where the product runs over the primes p. Thus we see that $\zeta(s) \neq 0$ in the half-plane $\sigma > 1$ since the individual terms in the Euler product are not zero and converge to 1 as $p \to \infty$. By the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where $\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)$, we now obtain the analytic continuation of $\zeta(s)$ for $\sigma < 0$ where we see the only zeros in this half-plane are the *trivial zeros* $s = -2, -4, -6, -8, \ldots$ coming from the factor $\sin(\pi s/2)$ in $\chi(s)$. Thus all the complex zeros of $\zeta(s)$ lie within the strip $0 \leq \sigma \leq 1$. Each of the two proofs of the prime number theorem in 1896 by Hadamard and by de la Vallée Poussin gave different proofs that there are no zeros on the line $\sigma = 1$, and therefore all the zeros must lie in the *critical strip* $0 < \sigma < 1$.

There is one further property of the Riemann zeta-function that is so easy to prove that its importance can often be overlooked; namely that there are no zeros on the real axis $s = \sigma$ for $0 \leq \sigma < 1$. This follows from the formula

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \quad \text{for} \quad 0 < \sigma < 1,$$

since this shows that $(1 - 2^{1-\sigma})\zeta(\sigma) > 0$ so that $\zeta(\sigma) < 0$ on the real line for $0 < \sigma < 1$. We also see immediately from the functional equation that $\zeta(0) = -1/2$. Note that this last formula gives the analytic continuation of $\zeta(s)$ in the critical strip.

If an analytic function $F(s)$ is real on a segment of the real axis it will satisfy the reflection principle that $F(\bar{s}) = \overline{F(s)}$ in the symmetric region on either side of that segment where F is analytical. Denoting the complex zeros of the Riemann zeta-function by $\rho = \beta + i\gamma$, we see $\zeta(\rho) = 0 = \zeta(\bar{\rho})$ and the functional equation gives as well $\zeta(\rho) = 0 = \zeta(1-\rho)$. Thus, all nontrivial zeros come in pairs symmetric with the real axis $\rho = \beta + i\gamma$ and $\bar{\rho} = \beta - i\gamma$, and zeros with $\beta \neq 1/2$ come in symmetric quadruples with respect to both the real axis and the *critical line* (namely, the vertical line $\sigma = 1/2$): $\rho = \beta + i\gamma,\ \bar{\rho} = \beta - i\gamma,\ 1 - \rho = 1 - \beta - i\gamma,\ 1 - \bar{\rho} = 1 - \beta + i\gamma$. This elementary fact turns out to be fundamental later in this paper.

Since the zeros in the critical strip below the real axis are a symmetric mirror of the zeros above the real axis, we only need to study the zeros above the real axis. Thus, we let

$$N(T) := \#'\{\rho = \beta + i\gamma \ : \ 0 < \gamma \leq T\} = \sum_{0 < \gamma \leq T} 1, \tag{2.1}$$

where $\#'$ means the number of elements in the set counted with multiplicity, that is if ρ is a zero with multiplicity m_ρ, then we have m_ρ copies of ρ in the sequence of zeros. Thus, for example, a triple zero gets counted 3 times in $N(T)$.

In 1905 von Mangoldt proved that, for $T \geq 3$,

$$N(T) := \sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{2.2}$$

and we have the useful special cases

$$N(T) \sim \frac{T}{2\pi} \log T \quad \text{as } T \to \infty, \qquad \text{and} \qquad N(T+1) - N(T) = O(\log T). \tag{2.3}$$

We define the *average spacing between consecutive zeros* by

$$\frac{\text{Length of } [0, T]}{N(T)} \sim \frac{T}{\frac{T}{2\pi} \log T} \sim \frac{2\pi}{\log T}, \quad \text{as } T \to \infty. \tag{2.4}$$

3. Montgomery's RH proof of simple zeros

In this section we assume RH so that $\rho = 1/2 + i\gamma$, and prove the following result.

Theorem 1 (Montgomery). *Assuming RH, at least 2/3 of the zeros of $\zeta(s)$ are simple.*

The proof is based on the following inequality

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma=\gamma'}} 1 \le \sum_{0<\gamma,\gamma'\le T} \left(\frac{\sin(\frac{1}{2}(\gamma-\gamma')\log T)}{\frac{1}{2}(\gamma-\gamma')\log T}\right)^2 = \left(\frac{4}{3}+o(1)\right)\frac{T}{2\pi}\log T. \tag{3.1}$$

Here the first inequality is trivially true. The asymptotic formula for the Fejér kernel sum over differences of zeros requires RH; the interested reader may see the proof in [Mon73]. In thinking about these double sums it may be helpful to consider the formula

$$\sum_{0<\gamma,\gamma'\le T} \left(\frac{\sin(\frac{1}{2}(\gamma-\gamma')\log T)}{\frac{1}{2}(\gamma-\gamma')\log T}\right)^2 = 2\int_0^1 \Bigg|\sum_{0<\gamma\le T} T^{i\alpha\gamma}\Bigg|^2 (1-\alpha)\,d\alpha, \tag{3.2}$$

which follows immediately from the elementary formula

$$2\,\mathrm{Re}\int_0^1 e^{iw\alpha}(1-\alpha)\,d\alpha = \left(\frac{\sin\frac{1}{2}w}{\frac{1}{2}w}\right)^2. \tag{3.3}$$

Thus in (3.1) the double sums can be thought of as two copies of the sequence of zeros $0<\gamma\le T$, where this sequence itself already has m_γ copies of each zero of multiplicity m_γ.

Now consider the first sum in (3.1) with the condition that $\gamma = \gamma'$. If we think of the terms as ordered pairs (γ,γ') where $\gamma=\gamma'$, then a simple zero gets counted once, but a double zero with $m_\gamma = 2$ gets counted 4 times, since $\gamma_1=\gamma_2=\gamma_1'=\gamma_2'$ and we get the 4 terms (γ_1,γ_1'), (γ_1,γ_2'), (γ_2,γ_1'), (γ_2,γ_2'). Thus a zero of multiplicity m_γ gets counted m_γ^2 times. The proof of Montgomery's Simple Zero Theorem is now just two lines.

Proof of Theorem 1. We have

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma=\gamma'}} 1 = \sum_{0<\gamma\le T} m_\gamma \le \left(\frac{4}{3}+o(1)\right)\frac{T}{2\pi}\log T, \tag{3.4}$$

and, using (2.1) and (2.3),

$$\begin{aligned}\sum_{\substack{0<\gamma\le T\\ \rho\ \text{simple}}} 1 &\ge \sum_{0<\gamma\le T}(2-m_\gamma) = 2\sum_{0<\gamma\le T}1 - \sum_{0<\gamma\le T} m_\gamma\\ &\ge \left(2-\frac{4}{3}+o(1)\right)\frac{T}{2\pi}\log T = \left(\frac{2}{3}+o(1)\right)N(T) = \left(\frac{2}{3}+o(1)\right)\sum_{0<\gamma\le T}1.\end{aligned}$$

$\square$

4. Montgomery's Simple Zero Method if RH is not assumed

For 50 years it never seemed to occur to anyone (including the first-named author of this paper) to seriously think about this proof when RH is not assumed. As we will see in a moment, the key idea here is that, despite the title of this section, not assuming RH is not enough; what you have to do is take seriously the possibility that RH is false, and consider it without prejudice. We actually spent almost two years proving (3.1) holds with conditions weaker than RH before we realized that these results could be applied to obtain critical zeros. The source of this mental block comes from the equation

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma=\gamma'}} 1 \;=\; \sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ \gamma=\gamma'}} 1. \tag{4.1}$$

The sum on the left looks like we are assuming RH because there is no β dependence evident in the equation. The equation on the right hints at the β dependence, since $\rho=\beta+i\gamma$ and $\rho'=\beta'+i\gamma'$, but it is still easy to think RH does not matter because it is equal to the sum on the left.

We need to go back to the formulas in Section 2. With respect to the sum (4.1), we want to classify pairs of zeros which lie on the same horizontal line. Note that under the assumption of RH, these can only be the same zeros, thus

$$\sum_{\substack{\rho=\frac12+i\gamma\\ 0<\gamma\le T}} \sum_{\substack{\rho'=\frac12+i\gamma'\\ 0<\gamma'\le T\\ \gamma=\gamma'}} 1 \;=\; \sum_{\substack{\rho=\rho'\\ 0<\gamma,\gamma'\le T\\ \beta=\beta'=\frac12}} 1 \;=\; \sum_{\substack{\rho=\frac12+i\gamma\\ 0<\gamma\le T}} m_\rho. \tag{4.2}$$

Without assuming RH, there "may" be zeros $\rho=\beta+i\gamma$, $\beta\neq 1/2$, lying off the critical line, and each one of these is matched with the symmetric zero on the opposite side of the critical line $1-\overline{\rho}=1-\beta+i\gamma$. Hence, if $\beta\neq 1/2$ then the pairs of zeros $(\beta+i\gamma, 1-\beta+i\gamma)$ and $(1-\beta+i\gamma,\beta+i\gamma)$ are solutions of the equation $\gamma=\gamma'$. We call these "symmetric diagonal terms". Thus, in addition to the sum (4.2), the summation (4.1) also includes an additional sum over these symmetric diagonal terms. Note also since we are not assuming RH, the sum over the diagonal terms is

$$\sum_{\substack{\rho=\rho'\\ 0<\gamma,\gamma'\le T}} 1 \;=\; \sum_{\substack{\rho=\beta+i\gamma\\ 0<\gamma\le T}} m_\rho.$$

Is this the end? Apparently not. Since we are working with pairs of zeros, if there are three or more zeros on the same horizontal line, we can also have terms which are not diagonal and not symmetric. To conclude, we have

$$\sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ \gamma=\gamma'}} 1 \;=\; \sum_{\substack{\rho\\ 0<\gamma\le T}} m_\rho + \sum_{\substack{\rho\\ 0<\gamma\le T\\ \beta\neq\frac12}} m_\rho + \sum_{\substack{\rho\neq\rho'\\ 0<\gamma,\gamma'\le T\\ \beta+\beta'\neq 1,\,\gamma=\gamma'}} 1, \tag{4.3}$$

where the first sum on the right-hand side comes from the diagonal terms, the second is given by the symmetric diagonal terms, and all the rest in the third sum are non-symmetric horizontal terms.

We can now immediately prove our RH-free version of Montgomery's simple zero result.

Theorem 2. *Suppose there exists a constant C where $1\le C<2$ and that, as $T\to\infty$,*

$$\sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ \gamma=\gamma'}} 1 \le \Big(C+o(1)\Big)\frac{T}{2\pi}\log T. \tag{4.4}$$

Then asymptotically at least the proportion $2-C$ of the zeros of $\zeta(s)$ are simple; at least the proportion $2-C$ of the zeros of $\zeta(s)$ are on the critical line; and if $1\le C<3/2$ then at least the proportion $3-2C$ of the zeros of $\zeta(s)$ are simple zeros on the critical line.

Proof of Theorem 2. We first give a remark about the restriction on the constant C in our assumption. Note that

$$\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ \gamma=\gamma'}} 1 \ \ge \ \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ \rho=\rho'}} 1 \ = \ \sum_{\substack{\rho \\ 0<\gamma\le T}} m_\rho \ \ge \ \sum_{\substack{\rho \\ 0<\gamma\le T}} 1 = N(T) \ \sim \ \frac{T}{2\pi}\log T \tag{4.5}$$

as $T\to\infty$. Thus C in our assumption has to satisfy $C\ge 1$.

Substituting (4.3) into (4.4), we have

$$\sum_{\substack{\rho \\ 0<\gamma\le T}} m_\rho + \sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta\ne\frac12}} m_\rho + \sum_{\substack{\rho\ne\rho' \\ 0<\gamma,\gamma'\le T \\ \beta+\beta'\ne 1,\, \gamma=\gamma'}} 1 \le \Big(C+o(1)\Big)\frac{T}{2\pi}\log T. \tag{4.6}$$

In this inequality, removing any sum on the left-hand side results in a smaller quantity since each is non-negative. First, we have

$$\sum_{\substack{\rho \\ 0<\gamma\le T}} m_\rho \le \Big(C+o(1)\Big)\frac{T}{2\pi}\log T,$$

which is (3.4) with $C=4/3$, and the identical argument used there gives

$$\sum_{\substack{0<\gamma\le T \\ \rho \text{ simple}}} 1 \ge \Big(2-C+o(1)\Big)\sum_{0<\gamma\le T} 1, \tag{4.7}$$

which proves that the proportion of simple zeros of $\zeta(s)$ is $\ge 2-C+o(1)$.

Next, for the symmetric diagonal terms, we have

$$\sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta\ne 1/2}} m_\rho \le \Big(C+o(1)\Big)\frac{T}{2\pi}\log T - \sum_{\substack{\rho \\ 0<\gamma\le T}} m_\rho.$$

Recalling (4.5) and plugging in the last inequality there into the above inequality, we obtain

$$\sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta\ne 1/2}} m_\rho \le \Big(C-1+o(1)\Big)\frac{T}{2\pi}\log T. \tag{4.8}$$

Thus

$$\sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta=1/2}} 1 = \sum_{\substack{\rho \\ 0<\gamma\le T}} 1 - \sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta\ne 1/2}} 1 \ge N(T) - \sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta\ne 1/2}} m_\rho \ge \Big(2-C+o(1)\Big)\frac{T}{2\pi}\log T,$$

and we have proved that the proportion of zeros of $\zeta(s)$ on the critical line is $\ge 2-C+o(1)$. Finally, using (4.7) and (4.8) we obtain

$$\begin{aligned}\sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta=\frac12 \text{ and } \rho \text{ simple}}} 1 \ &= \ \sum_{\substack{\rho \\ 0<\gamma\le T \\ \rho \text{ simple}}} 1 \ - \ \sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta\ne\frac12 \text{ and } \rho \text{ simple}}} 1 \ge \ \Big(2-C+o(1)\Big)\frac{T}{2\pi}\log T - \sum_{\substack{\rho \\ 0<\gamma\le T \\ \beta\ne\frac12}} m_\rho \\ &\ge \ \Big(3-2C+o(1)\Big)\frac{T}{2\pi}\log T,\end{aligned}$$

which proves, if $1\le C<3/2$, that the proportion of simple zeros of $\zeta(s)$ on the critical line is $\ge 3-2C+o(1)$. $\square$

5. Two Results on Critical Zeros

Using Theorem 2, we proved in [BGSTB25] the following result.

Theorem 3 (Baluyot, Goldston, Suriajaya, Turnage-Butterbaugh)**.** *Assuming all of the zeros of $\zeta(s)$ with $T < \gamma \le 2T$ lie in the region*

$$B_b = \{s = \sigma + it \ : \ \left|\sigma - \frac{1}{2}\right| < \frac{b}{2\log T}, \ T < t \le 2T\}. \tag{5.1}$$

Then if $b = 0.3185$, we have for the zeros of $\zeta(s)$ with $T < \gamma \le 2T$ that at least 2/3 are simple, at least 2/3 are on the critical line, and at least 1/3 are both simple and on the critical line.

Can we get that asymptotically 100% of the zeros are on the critical line by this method? Since the proportion of simple zeros is equal to that of critical zeros in Theorem 2, we need to use a conditional result on pair correlation that does not assume RH and implies that 100% of the zeros are simple.

Pair Correlation Conjecture (PCC). *For fixed $\lambda > 0$ and $T \to \infty$,*

$$\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ 0<\gamma-\gamma'\le \frac{2\pi\lambda}{\log T}}} 1 \sim \left(\int_0^\lambda \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du\right) \frac{T}{2\pi}\log T.$$

Recall that Montgomery's Pair Correlation Conjecture (PCC) was the first suggestion that zeros of L-functions obey a random matrix eigenvalue model; for $\zeta(s)$, it is the Gaussian Unitary Ensemble (GUE) model.

Since **PCC** only concerns the non-zero vertical distance between zeros, it is unaffected by RH. On the other hand, nothing in **PCC** by itself implies directly that any zeros are on the critical line. Montgomery's reasoning to support **PCC** used RH, but Gallagher and Mueller [GM78] proved that **PCC** implies asymptotically 100% of the zeros are simple using a different method based on unconditional results of Selberg and Fujii. However, since RH was assumed and used in many parts of [GM78], we clarified this in [GLSS25a] to show the results on **PCC** hold without RH. As a consequence, by Theorem 2, we also obtain that asymptotically 100% of the zeros are on the critical line.

Theorem 4 (Goldston, Lee, Suriajaya, Schettler)**.** *The Pair Correlation Conjecture implies asymptotically 100% of the zeros of the $\zeta(s)$ are simple and on the critical line.*

The proof of Theorem 4 actually depends on the following property of **PCC**.

Essential Simplicity (ES). *If $\lambda \to 0$ as $T \to \infty$, then we have*

$$\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\gamma-\gamma'|\le \frac{2\pi\lambda}{\log T}}} 1 = \Big(1 + o(1)\Big)\frac{T}{2\pi}\log T. \tag{5.2}$$

Clearly (5.2) implies $C = 1$, thus Theorem 2 follows from **ES**. This reduces the proof of Theorem 4 to proving without RH that **PCC** implies **ES**.

Other pair correlation conjectures besides **PCC** also imply **ES**. In [GLSS25b] we proved that the Alternative Hypothesis (AH) gives a different pair correlation density function from **PCC**, which in one form implies **ES** and thus Theorem 4 with **PCC** replaced by that form of AH.

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