Topological BF Theory construction of twisted dihedral quantum double phases from spontaneous symmetry breaking

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Nonabelian topological orders host exotic anyons central to quantum computing, yet established realizations rely on case-by-case constructions that are often conceptually involved. In this work, we present a systematic construction of nonabelian dihedral quantum double phases based on a continuous $O(2)$ gauge field. We first formulate a topological $S[O(2) \times O(2)]$ BF theory, and by identifying the Wilson loops and twist operators of this theory with anyons, we show that our topological BF theory reproduces the complete anyon data, and can incorporate all Dijkgraaf–Witten twists. Building on this correspondence, we present a microscopic model with $O(2)$ lattice gauge field coupled to Ising and rotor matter whose Higgsing yields the desired dihedral quantum double phase. A perturbative renormalization group analysis further indicates a direct transition from this phase to a $U(1)$ Coulomb or chiral topological phase at a stable multicritical point with emergent $O(3)$ symmetry. Our proposal offers an alternative route to nonabelian topological order with promising prospects in synthetic gauge field platforms.

Introduction.— Topological orders are novel quantum phases of matter whose description lies beyond the traditional Landau paradigm of symmetry-breaking [1]. They are instead characterized by long-range entanglement [2] and emergent gauge fields [3–5], exhibiting unusual properties such as ground state degeneracy tied to the topology of space [1] and fractionalized anyonic quasiparticles with nontrivial statistics [6]. These properties are universal and robust against arbitrary local perturbations [7], making topological orders a natural platform for fault-tolerant quantum computation [8]. Topological orders have been observed in fractional quantum Hall systems [9–11]; certain quantum spin and cold atom systems also offer promising routes to their realization [3, 4, 12].

Among the topological orders, the *nonabelian* ones play an essential role in realizing universal quantum computing [13], making the understanding and engineering of nonabelian topological orders a central task. Great efforts have been put into the design [14, 15] of *dihedral* quantum doubles (QDs), for which explicit lattice constructions exist, notably Kitaev's QD Hamiltonians [8] and Levin–Wen string-net models [5]. Yet they directly work with discrete degrees of freedom and the recipe to create a nonabelian gauge field is quite involved; they do not offer a systematic route to incorporate dihedral QDs into a broader hierarchy of gauge theories.

In this work we close that gap by showing that an $S[O(2)\times O(2)]$ gauge theory—a particular class of gauge theory endowed with a continuous $O(2)$ gauge field—reproduces the full anyon data of the dihedral QD with arbitrary *Dijkgraaf–Witten twists* (DWTs) [16]. At the continuum level, we identify a topological BF-type action

$$\mathcal{L}_{\mathrm{BF}} = \frac{in}{2\pi} a\mathrm{d}b, \quad (a,b) \sim (-a,-b), \tag{1}$$

whose gauge-invariant observables map one-to-one onto the (worldlines of) anyons of the dihedral QD. We verify that the fractional statistics are identical to those of the dihedral QD without DWTs, and that all DWTs can be incorporated as attaching self Chern–Simons (CS) terms and introducing twisted boundary conditions to Eq. (1). This construction reveals how a continuous gauge theory belongs to the same universality class as the discrete dihedral QD. Such a correspondence inspires us to construct lattice models for dihedral QD phases whose local degrees of freedom are compact $O(2)$ rotors. We employ field theory analysis to map out the phase diagram, where the dihedral QD phases can naturally arise by condensing the $O(2)$ rotor. In addition, we find that this phase can undergo a direct transition to a $U(1)$ phase through a stable multicritical point with emergent $O(3)$ symmetry.

Quantum double and Dijkgraaf–Witten twist.— The twisted quantum double $D^\omega(G)$ is a systematic construction of certain nonchiral topological orders [8, 17–19]. The physical data of topological order is completely given by a finite group G and a cohomology class $\omega \in H^3(G, U(1))$; the latter encodes the DWT. The physical data refers to the low energy gapped excitations (the anyons) and their statistics: an anyon $a \in \mathcal{A} = \{a, b, c, ...\}$ is a shorthand for the corresponding excited state $|a\rangle$. The statistics consists of the anyon quantum dimensions d_a, the fusion coefficients N^c_{ab}, the F-symbol F^{abc}_d and the R-symbol R^{ab}_c: the quantum dimensions $\{d_a\}_{a\in\mathcal{A}}$ collectively determine the ground state degeneracy (GSD) of topological order on a genus g surface: $\mathrm{GSD} = \sum_{a\in\mathcal{A}} (d_a)^g$, and any symbol $\mathcal{O}^{cd...}_{ab...}$ governs the transition amplitide for incoming anyons $(a, b, ...)$ and outcoming anyons $(c, d, ...)$. For $D^\omega(G)$, the anyons are labeled by $a = \{(C\ell, \rho)\}$ where $C\ell$ runs over the conjugacy classes of G and ρ the irreducible representations (irreps) of the centralizer of any $g \in C\ell$, see Table I for odd n (even n case is left in [19]). The symbols N^c_{ab}, F^{abc}_d, and R^{ab}_c can all be derived from the algebras of conjugacy classes and irreps of G [20]. The physical data for topological order can be equivalently determined from the modular S and T matrices [21].

Anyon Label	Quantum Dim.	Number of Anyons	Conjugacy Class	Centralizer	Irrep.
1	1	1	$\{1\}$	D_n	$\mathbf{1}_+$
η	1	1	$\{1\}$	D_n	$\mathbf{1}_-$
ψ_l	2	$(n-1)/2$	$\{1\}$	D_n	$\mathbf{2}_l$
$\psi_{l,r}$	2	$n(n-1)/2$	$\{C_n^l, C_n^{n-l}\}$	$\mathbb{Z}_n$	$\mathbf{1}_r$
$\sigma_\pm$	n	2	$\{M_1, ..., M_n\}$	$\mathbb{Z}_2$	$\mathbf{1}_\pm$

TABLE I. Anyon data of $D^\omega(D_n)$ for odd n. There are $(n^2+7)/2$ distinct anyons, and the total quantum dimension is $4n^2$. The anyons with quantum dimension equal to 2 are labeled by ψ's, with $l = 1, ..., (n-1)/2$ and $r = -(n-1)/2, ..., (n-1)/2$.

$S[O(2)\times O(2)]$ BF theory for dihedral quantum double.— It is known that the $\mathbb{Z}_n$ QD can be described by a $U(1)\times U(1)$ BF theory [22–24]. This construction is based on the fact that $\mathbb{Z}_n$ is a subgroup of $U(1)$. Moving to the non-abelian case of dihedral group, since $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ and $O(2) = U(1) \rtimes \mathbb{Z}_2$, it is natural to also expect a D_n QD phase to be described by $O(2)$ gauge fields. Concretely, we propose the following gauge theory:

$$\mathcal{L}_{D_n} = \mathcal{L}_{\mathrm{BF}} + \frac{ik}{2\pi} a\mathrm{d}a, \quad k = 0, 1, ..., n-1. \tag{2}$$

This theory has the same appearance as the $U(1)$ BF gauge theory; however a is an $O(2)$ gauge field arising from gauging the charge conjugation symmetry $\mathbb{Z}_2^C : a \mapsto -a$ [25, 26]. The level of the self CS term, $2k$, guarantees that the theory is bosonic.

The fate of field b in $\mathcal{L}_{\mathrm{BF}}$ under $\mathbb{Z}_2^C$ needs special attention. Crucially, when a is promoted to $O(2)$, b cannot remain invariant under $\mathbb{Z}_2^C$, since $\mathbb{Z}_2^a : a \mapsto -a, b \mapsto b$ does not preserve Eq. (2). Instead, the appropriate symmetry is

$$\mathbb{Z}_2^C : \ a \mapsto -a, \ b \mapsto -b. \tag{3}$$

This defines the theory (2) for the dihedral QD to be $S[O(2)\times O(2)]$ BF theory, with a gauge structure $S[O(2)\times O(2)] := [O(2)\times O(2)]/\mathbb{Z}_2$ [27]. For conciseness, in the rest of this letter we only present the case of odd n; the even n case is treated in detail in [19].

A subtle issue exists within this construction when considering DWTs: the self CS term, judging by its appearance in Eq. (2), can only capture the *even* half of the DWTs of $D^\omega(D_n)$, namely, those that have a trivial value in the $\mathbb{Z}_2$ factor of $H^3(D_n, U(1)) = \mathbb{Z}_{2n} = \mathbb{Z}_n \times \mathbb{Z}_2$. The rest with a nontrivial $\mathbb{Z}_2$ value are *odd* DWTs originated from the nontrivial DWT in the charge conjugation $\mathbb{Z}_2^C$. As we will show below, these *odd* DWTs are correctly reflected in the *twisted* sectors of the Hilbert space of the theory (2).

Correspondence between $D^\omega(D_n)$ and $S[O(2)\times O(2)]$ BF theory.— In order to show that $S[O(2)\times O(2)]$ BF theory is a valid description for the dihedral QD, we analyze the states and sectors of the $S[O(2)\times O(2)]$ BF theory on Riemann surfaces: the states are obtained from those of the $U(1)\times U(1)$ BF theory by gauging the symmetry $\mathbb{Z}_2^C$ defined in (3). Consider a genus g surface Σ_g with noncontractible loops α_i, β_i with $i = 1, 2, ..., g$, where α_i and β_j intersect if and only if $i = j$. On Σ_g, the ground states of a $U(1)\times U(1)$ BF theory are [28–31]

$$\bigotimes_{i=1}^{g} |p_i, q_i\rangle, \quad (p_i, q_i) \sim (p_i+n, q_i) \sim (p_i+2k, q_i+n), \tag{4}$$

where $p_i, q_i \in \mathbb{Z}$ label zero modes (flat configurations) of a and b, respectively. To promote the $U(1)\times U(1)$ gauge structure to $S[O(2)\times O(2)]$, the $\mathbb{Z}_2^C$ symmetry needs to be further gauged, identifying

$$(p_i, q_i) \sim (-p_i, -q_i). \tag{5}$$

This gauging procedure introduces *twisted* sectors in the Hilbert space: gauge fields a and b in a twisted sector change sign (i.e. antiperiodic) across at least one of the noncontractible loops. There are a total of $(2^{2g}-1)$ *twisted* sectors [32]. A detailed counting (see [19]) shows that the number of flat configurations of the $S[O(2)\times O(2)]$ theory matches the GSD of $D^\omega(D_n)$.

A one-to-one correspondence between anyons of $D^\omega(D_n)$ and the operators of the $S[O(2)\times O(2)]$ BF theory can also be drawn. On a torus with $g = 1$ [33], depending on whether there are twists across of tha gauge fields on the loops α and β, the total Hilbert space is resolved into four sectors $\mathcal{H}_{\bullet\bullet}$ with $\bullet\bullet = ++, +-, -+,$ and $--$. The untwisted sector $\mathcal{H}_{++}$ has its states labeled by $|p, q\rangle_{++}$, whereas each of the three twisted sectors contains only one state for odd n.

In the untwisted sector of $S[O(2)\times O(2)]$ BF theory, the operators can be straightforwardly identified with certain anyons of $D^\omega(D_n)$. Starting from the Wilson loop of the $U(1)\times U(1)$ BF theory

$$w_{p,q} = \mathcal{P}\exp\left(i\oint_\gamma pa + qb\right), \tag{6}$$

where $\mathcal{P}$ is path ordering and γ is some closed loop on the torus. The "electric charge" p and "magnetic charge" q are subject to the identification (4). These Wilson loop operators satisfy Abelian fusion rules $w_{p,q} \times w_{p',q'} = w_{p+p',q+q'}$. The $\mathbb{Z}_2^C$ gauge invariant operators of the $S[O(2)\times O(2)]$ BF theory are

$$\begin{aligned} &W_{p,q} = w_{p,q} + w_{p,q}^\dagger, \ (p,q) \neq (0,0), \\ &W_{0,0} = w_{0,0}, \end{aligned} \tag{7}$$

subject to the additional equivalence relation Eq. (5), and we have $|p, q\rangle_{++} = W_{p,q}|0,0\rangle$. The topological spin of $w_{p,q}$, $w_{p,q}^\dagger = w_{n+2k-p,n-q}$, and $W_{p,q}$ is identical. The number of inequivalent nontrivial operators in the BF theory, $(n^2-1)/2$, coincides with the number of anyons with quantum dimension 2 in $D^\omega(D_n)$, evidencing a one-to-one correspondence. The correspondence is further revealed by the fusion rule of $W_{p,q}$, given by

$$W_{p,q} \times W_{p',q'} = W_{p+p',q+q'} + W_{p-p',q-q'}, \tag{8}$$

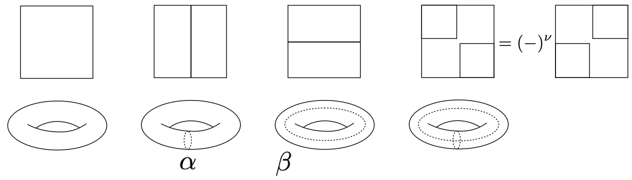

FIG. 1. $\mathbb{Z}_2$ sign of the S and T transformations.

for $(p,q) \neq (\pm p', \pm q')$. Therefore, these operators in $S[O(2)\times O(2)]$ BF theory all have a quantum dimension $d=2$, consistent with that of the ψ anyons in $D^{\omega}(D_n)$. $W_{p,p}$ are identified with $\psi_{l=p}$ in Table I, while the operators $\{W_{p,q}\}_{p\neq q}$ are identified with $\{\psi_{l,r}\}$.

The Wilson loop operators discussed above create states within the untwisted sectors. For the $S[O(2)\times O(2)]$ BF theory, there also exist twist operators creating states that live in the twisted sectors. Below we show that these twist operators can be identified with the remaining anyons of $D^{\omega}(D_n)$. This is achieved by matching the topological statistics of twist operators and anyons.

To determine the topological spin and fusion statistics we calculate the S and T matrices for the twist operators, each of which creates the only state $|0,0\rangle_{+-/-+/--}$ in the corresponding twisted sector. Below we illustrate the procedure by calculating the topological spins, and the full analysis can be found in [19] (see also [34–38] for S and T transformations in $U(1)$ gauge theories). These transformations are illustrated in Fig. 1 and read:

$$\begin{aligned}\hat{S}\,|0,0\rangle_{\pm\mp} &= |0,0\rangle_{\mp\pm}\,,\ \hat{S}\,|0,0\rangle_{--} = (-)^{\nu}\,|0,0\rangle_{--}\,,\\ \hat{T}\,|0,0\rangle_{\pm\mp} &= |0,0\rangle_{\pm-}\,,\ \hat{T}\,|0,0\rangle_{--} = (-)^{\nu}\,|0,0\rangle_{-+}\,.\end{aligned} \tag{9}$$

Importantly, they contain a $\mathbb{Z}_2$ sign [39] labeled by $\nu = 0,1$, originating from whether the $\mathbb{Z}_2^C$ part of the gauge structure has a nontrivial DWT ($\nu=1$) or not ($\nu=0$) [39, 40]. Here ν precisely captures the $\mathbb{Z}_2$ factor of $H^3(D_n, U(1)) = \mathbb{Z}_n\times\mathbb{Z}_2$. To see its effect, define

$$\begin{aligned}|1\rangle &= \frac{1}{\sqrt{2}}\left(|0,0\rangle_{++} + |0,0\rangle_{+-}\right),\\ |\eta\rangle &= \frac{1}{\sqrt{2}}\left(|0,0\rangle_{++} - |0,0\rangle_{+-}\right),\\ |\sigma_{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|0,0\rangle_{-+} \pm (-i)^{\nu}\,|0,0\rangle_{--}\right),\end{aligned} \tag{10}$$

where a T matrix calculation gives their topological spins $\theta_1 = \theta_\eta = 0$, $\theta_{\sigma_+} = \nu/4$, $\theta_{\sigma_-} = 1/2 + \nu/4$, identical to those of the η and $\sigma_{\pm}$ anyons in Table. I. We see that an effect of the $\mathbb{Z}_2$ sign ν is to change the topological spin of $\sigma_{\pm}$ by 1/4 while preserving that of the other anyons. Physically, the nontrivial $\mathbb{Z}_2^C$ DWT corresponds to the gluing of a $\mathbb{Z}_2^C$ SPT state [40–43], and is reflected in the additional signs of F-symbols [5].

Higgsing $O(2)$ gauge field on lattice.— The $S[O(2)\times O(2)]$ gauge theory described above suggests a natural route to realizing dihedral QDs from Higgsing an $O(2)$ gauge theory. Below, we introduce a lattice construction of the D_n QD phase based on this Higgsing scenario. Unlike previous constructions based on Kitaev's proposal [8, 14, 15], our construction is built in a physically transparent way, featuring a lattice $O(2)$ gauge field [44] that can arise in nematic liquid crystals [45, 46], cold atoms [47, 48], spin liquids [49], and certain quantum Hall systems [50].

Consider the following classical model defined on 3D cubic lattice

$$H = H_{\text{Maxwell}} + \sum_{\langle\mathbf{ij}\rangle} -J_s s_{\mathbf{i}}|\mathbf{u_{ij}}|s_{\mathbf{j}} - J_v \mathbf{v_i}\cdot\mathbf{u_{ij}}\cdot\mathbf{v_j}. \tag{11}$$

The matter fields defined on lattice site $\mathbf{i}$ are the two-component unit-norm real vector field $\mathbf{v_i}$ and the Ising field $s_{\mathbf{i}} = \pm 1$. The $O(2,\mathbb{R})$ gauge field $\mathbf{u_{ij}}$ is a 2-by-2 real orthogonal matrix with determinant $|\mathbf{u_{ij}}| = \pm 1$ defined on the link $\langle\mathbf{ij}\rangle$. The Maxwell term of the $O(2)$ gauge field in Eq. (11) is standard. The $O(2)$ gauge transformations are $s_{\mathbf{i}} \mapsto s'_{\mathbf{i}} = |\mathbf{R_i}|s_{\mathbf{i}}$, $\mathbf{v_i}\mapsto \mathbf{v'_i} = \mathbf{R_i}\cdot\mathbf{v_i}$, $\mathbf{u_{ij}} \mapsto \mathbf{u'_{ij}} = \mathbf{R_i}\cdot\mathbf{u_{ij}}\cdot\mathbf{R_j^T}$, where $\mathbf{R_i}$ is the $O(2)$ gauge transformation matrix transforming under the two-dimensional irrep $\mathbf{2}_n$ of $O(2)$. The Ising field $s_{\mathbf{i}}$ and the vector field $\mathbf{v_i}$ transform under the non-trivial one-dimensional irrep and the two-dimensional irrep $\mathbf{2}_n$ of $O(2)$, respectively. DWTs can also be realized in the model Eq. (11) by self CS terms. In the following we illustrate the phases using the model without DWT, and assume the continuous gauge fields are deconfined due to monopole suppression.

The schematic phase diagram for theory (11) is shown in Fig. 2. For sufficiently small Maxwell coupling constant κ [Fig. 2 (a)], all gauge fields are deconfined, and the condensation of $s_{\mathbf{i}}$ and $\mathbf{v_i}$ can be separately tuned by J_s/T and J_v/T respectively, where T is the temperature. This subdivides the phase region with deconfined gauge fields into four phases. Increasing κ will confine the discrete gauge fields arising from properly Higgsing the $O(2)$ gauge field, as we analyze below.

When $J_v = 0$, the J_s-term describes a $\mathbb{Z}_2$ gauge field interacting with an Ising matter [51]. Increasing J_s/T condenses the Ising field $s_{\mathbf{i}}$, thereby Higgsing the $\mathbb{Z}_2$ part of the $O(2)$ gauge field; the $U(1)$ part stays intact. This marks a 3D Ising transition [51, 52] from an $O(2)$ Coulomb phase to a $U(1)$ Coulomb phase.

The same phase transition happens at $J_v = +\infty$ upon varying J_s/T. Sufficiently large J_v condenses the vector field $\mathbf{v_i}$, thereby Higgsing the $U(1)$ part of the $O(2)$ gauge field to $\mathbb{Z}_n$. Importantly, a residual $\mathbb{Z}_2 \equiv O(2)/U(1)$ gauge redundancy still remains, which together with the unbroken $\mathbb{Z}_n$ gauge structure results in a D_n QD phase. Here the Ising field $s_{\mathbf{i}}$ plays the role of η anyon in $D(D_n)$ for odd n, and its condensation Higgses the $\mathbb{Z}_2$ part of the D_n gauge field through a 3D Ising transition, leaving a $D(\mathbb{Z}_n)$ phase.

When $J_s = +\infty$, the $\mathbb{Z}_2$ part of the $O(2)$ gauge field is Higgsed through an Ising transition, and the theory Eq. (11) becomes a lattice Abelian Higgs model with charge-n matter [53–56]. Increasing J_v/T condenses $\mathbf{v_i}$, thereby Higgsing the $U(1)$ gauge structure to $\mathbb{Z}_n$; this

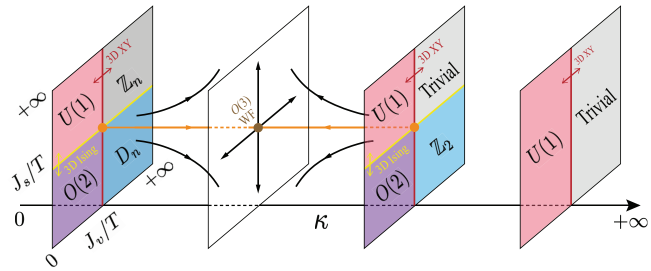

FIG. 2. Schematic phase diagram of the lattice model Eq. (11) for $n > 2$, with schematic RG flows derived from the effective field theory Eq. (12). The relevant RG flow towards the $O(3)$ Wilson-Fisher fixed point is highlighted in orange. The gauge structure of each phase is shown explicitly. The 3D XY and the 3D Ising transitions are marked in red and yellow, respectively. From left to right the three colored planes are phase diagrams for sufficiently small, mediate, and sufficiently large κ, respectively. The relative position of the $O(3)$ Wilson-Fisher fixed point and the phase diagram for intermediate κ is for illustrative purposes only and may differ from the actual situation.

marks a 3D XY transition from the $U(1)$ Coulomb phase to the $D(\mathbb{Z}_n)$ phase [54–56]. Similarly, when $s_{\mathbf{i}}$ is gapped at $J_s = 0$, tuning J_v/T also effects a 3D XY transition between the $O(2)$ Coulomb phase and the $D(D_n)$ phase. A direct transition between $O(2)$ and $\mathbb{Z}_n$ phases can occur when the Ising and the XY transitions meet with each other. The phase diagram is shown in the leftmost panel of Fig. 2.

Stable multicritical point.— It is unnatural to expect a direct transition between $U(1)$ and D_n phases, as neither one is the subgroup of the other. However, as we show below through a perturbative renormalization group (RG) analysis, a direct transition can happen at a stable multicritical point where the four stable phases meet, described by the $O(3)$ Wilson–Fisher (WF) universality class with emergent $O(3)$ symmetry.

In the model (11), the $O(2)$ gauge field can be formally integrated out, generating an effective coupling between the vector and the Ising fields. This is legitimate in the deconfined phases, where the Maxiwell coupling κ remains weak enough to not invalidate the perturbative expansion in κ. The effective field theory reads

$$\begin{aligned} \mathcal{L} \;=\; & |\partial_\mu \mathbf{\Phi}_v|^2 + t_v|\mathbf{\Phi}_v|^2 + r_v|\mathbf{\Phi}_v|^4 + \tilde{\kappa}|\mathbf{\Phi}_v|^2\phi_s^2 \\ & +(\partial_\mu\phi_s)^2 + t_s\phi_s^2 + r_s\phi_s^4, \end{aligned} \tag{12}$$

where $\mathbf{\Phi}_v$ and ϕ_s are the coarse-grained vector and Ising fields, respectively, $t_{v,s} \sim (J^c_{v,s} - J_{v,s})/T$ measures the deviation of the coupling constant $J_{v/s}$ from its critical value $J^c_{v,s}$, and $r_{v,s} > 0$ denotes the self-interaction. The interaction between $\mathbf{\Phi}_v$ and ϕ_s generated by the $O(2)$ gauge field has coupling constant $\tilde{\kappa} \sim \kappa$.

We first note that, in the absence of the $O(2)$ gauge field, the two matter fields are entirely decoupled in model (11), resulting in a multicritical point where the XY and the Ising transition lines intersect. Since the two transitions are described by $O(2)$ and $O(1) = \mathbb{Z}_2$ WF universality classes, the starting point of the perturbative RG analysis for Eq. (12) is the $O(2) \times O(1)$ WF fixed point with $\tilde{\kappa} = 0$ and $r_{v,s} > 0$. At this fixed point, parameters t_v, t_s, and $\tilde{\kappa}$ are relevant, while r_v and r_s are irrelevant. Among the relevant ones, t_v and t_s control the XY and the Ising transitions, respectively, whereas $\tilde{\kappa}$ drives the RG trajectory towards the $O(3)$ WF fixed point of Eq. (12), at which $\mathbf{\Phi}_v$ and ϕ_s combine into an $O(3)$ vector and the system exhibits an emergent $O(3)$ symmetry. Since t_v and t_s perturbations remain relevant in the parameter space of the lattice model, any RG trajectory—as long as it is not the direct trajectory from $O(2) \times O(1)$ to the $O(3)$ WF fixed point—will be driven away from the latter, as illustrated in the left two panels in Fig. 2. This suggests the stability of the multicritical point in the phase diagram. However, if the lattice model allows more relevant perturbations, the multicritical point will be no longer stable, as exemplified in another lattice construction using complex $O(2)$ gauge field given in [19]. We also reiterate that our RG analysis is based on the perturbative expansion in κ, and might be invalidated with a large enough κ. Nevertheless, a Monte Carlo simulation (see [19] for details and a detailed RG argument) suggests that for a wide range of κ the multicritical point remains stable. Such a direct transition between the D_n phase and the $U(1)$ phase at the $O(3)$ WF critical point is similar to the deconfined quantum critical point and SPT transitions [57–64].

Confinement effects.— Increasing of κ also induces a two-step confinement of the gauge fields: the $\mathbb{Z}_n$ part of the gauge fields gets confined before the $\mathbb{Z}_2$ part for $n > 2$ [65, 66]. Thus, there exists an intermediate window of κ where the phase diagram contains four phases, with $O(2)$, $U(1)$, $\mathbb{Z}_2$ and trivial gauge structures, respectively. The universality classes of the transitions among these phases remain unchanged as those in the small κ case, and we conjecture that the multicritical point will still be stable. In addition, the transition between the $D(D_n)$ and the $D(\mathbb{Z}_2)$ phases and that between the $D(\mathbb{Z}_2)$ and the trivial phases are both described by the deconfinement to confinement transition of the $\mathbb{Z}_n$ gauge theory [51, 52]. A direct transition between the $D(D_n)$ phase and the trivial phase is in principle allowed. When κ is sufficiently large, the $\mathbb{Z}_2$ part of the gauge fields is further confined through a 3D Ising transition, leaving the $U(1)$ Coulomb and the trivial the only phases in the phase diagram. The multicritical point also disappears due to confinement of the Ising field $s_{\mathbf{i}}$. The phase diagram slices with intermediate and sufficiently large κ are shown in the right two pannels in Fig. 2.

Conclusion and outlook.— In summary, we provide a new route to realizing nonabelian twisted dihedral quantum double phases $D^\omega(D_n)$ through continuous gauge fields: an $S[O(2) \times O(2)]$ BF theory that exactly reproduces the anyon content and modular data of $D^\omega(D_n)$, and a concrete Higgsing construction in an $O(2)$ lattice gauge model with matter fields that realizes these phases microscopically. By tracking Wilson loops and twist operators to anyons and matching their statistics, we establish a one-to-one correspondence that clarifies how discrete dihedral quantum double phases emerge from continuous gauge structure. In the lattice model, we find the dihedral quantum double phase can meet with a $U(1)$ Coulomb phase at a stable multicritical point governed by the $O(3)$ Wilson–Fisher fixed point, enabling a direct continuous transition in analog to deconfined quantum criticality.

Our proposal for realizing nonabelian topological orders is based on continuous gauge theory which distinguishes ours from most existing proposals [8, 14, 15]. Our lattice construction allows us to access a large set of phases and several phase transitions (3D XY, 3D Ising, and $O(3)$ WF) by tuning a small set of parameters. Our construction can be systematized in the realization of more exotic nonabelian topological orders: for example, the point group quantum doubles via Higgsing $O(3)$ or $SU(2)$ gauge fields. Given the success in the experimental creation of synthetic nonabelian continuous gauge fields [47, 67–71], it is worthwhile to explore similar experiments realizing our proposed field theory (2) and lattice model (11).

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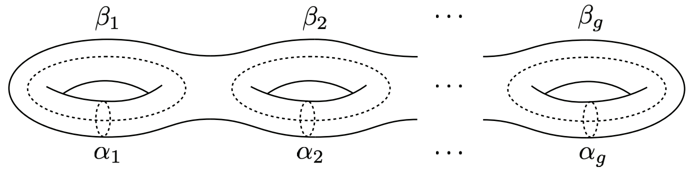

FIG. S1. Genus-g Riemann surface. Homology cycles α_i, β_i with $i = 1, 2, ..., g$ are marked by dashed loops, where α_i and β_j intersect if and only if $i = j$.

Supplemental Material for "Topological BF Theory construction of twisted dihedral quantum double phases from spontaneous symmetry breaking"

SI. DATA OF THE DIHEDRAL QUANTUM DOUBLE

The fusion constants N_{ab}^c for various anyon types can be deduced from Tables SI and SII.

The non-twisted quantum double has trivial fusion and braiding symbols due to the bosonic nature of the anyon statistics. The twisted quantum double models have F-symbols of $D^{[2k+\nu]}(D_n)$ that can be identified with the cocycle $\omega = [2k+\nu] \in H^3(D_n, U(1))$:

$$f_\omega^{g_1,g_2,g_3} = \omega(g_1, g_2, g_3) = e^{2\pi i(2k+\nu)\left(\frac{1}{n^2}(-1)^{s_2+s_3} r_1 \mathrm{Cell}_n[(-1)^{s_3} r_2 + r_3] + \frac{1}{2} s_1 s_2 s_3\right)}, \tag{S1}$$

where the function $\mathrm{Cell}_n[x]$ subtracts (or adds) integer multiples of n to take x back to the interval $(-(n-1)/2, (n-1)/2]$, and we denoted a generic element of D_n by $g = C_n^r M^s$. The R-symbol is

$$r_\omega^{g_1 g_2} = e^{\frac{2\pi i(2k+\nu)}{n^2}(a_2 \mathrm{Cell}_n[(-1)^{s_2} r_2 + 2 s_1 r_2] - s_1 r_2^2) + \frac{i\pi(2k+\nu)}{2} s_1 s_2}, \tag{S2}$$

For $n = 2m$ even, the DWTs are captured by $H^3(D_n, U(1)) = \mathbb{Z}_n \times \mathbb{Z}_2^2$. Here we label the DWTs as $\omega = [k, \nu, \upsilon]$, with indices $\nu, \upsilon \in \{0, 1\}$. We term the ones with $\nu = \upsilon = 0$ as *regular*, and the rest *anomalous*. F-symbols of $D^{[k,\nu,\upsilon]}(D_{2m})$ can be identified with the cocycle $\omega = [k, \nu, \upsilon] \in H^3(D_{2m}, U(1))$:

$$f^{g_1,g_2,g_3} = \omega(g_1, g_2, g_3) = e^{2\pi i\left(\frac{k}{n^2}(-1)^{s_2+s_3} r_1 \mathrm{Cell}[(-1)^{s_3} r_2 + r_3] + \frac{\nu}{2} s_1 s_2 s_3 + \frac{\nu+\upsilon}{2} r_1 s_2 s_3\right)}, \tag{S3}$$

And the R-symbol is

$$r_\omega^{g_1 g_2} = \begin{cases} e^{-\frac{i\pi k}{n} r_2 + \frac{i\pi}{2}(\nu+\upsilon)\mathrm{Cell}_2[m] s_2}, & \text{when } (r_1, s_1) = (m, 0), \\ e^{\frac{2\pi i k}{n^2}(a_2 \mathrm{Cell}_n[(-1)^{s_2} r_2 + 2 s_1 r_2] - s_1 r_2^2) + \frac{i\pi}{2}(\nu s_1 s_2 + (\nu+\upsilon)\mathrm{Cell}_2[r_1] s_2)}, & \text{otherwise.} \end{cases} \tag{S4}$$

SII. EQUIVALENCE BETWEEN $S[O(2) \times O(2)]$ BF THEORY AND D_n QUANTUM DOUBLE

A. Ground State Degeneracy and Quantum Dimension

This section provides a general counting of the ground state degeneracy on genus-g Riemann surfaces, which dictates the anyon quantum dimensions of the theory. However, to extract fusion and braiding, one needs to explicitly calculate the modular S and T matrices, which will be left for the next section.

Gauging $\mathbb{Z}_2^C$ symmetry to promote the $U(1) \times U(1)$ BF theory to $S[O(2) \times O(2)]$ introduces *twisted* sectors in the Hilbert space. Consider a genus-g Riemann surface Σ_g with inequivalent homology cycles (non-contractible loops) α_i, β_i with $i = 1, 2, ..., g$, where α_i and β_j intersect if and only if $i = j$ (see FIG. S1 for illustration). A *twisted* sector of the Hilbert space is defined as the states on which both the gauge fields a and b change sign (i.e. antiperiodic) on at least one of the cycles. All the sectors (including the *untwisted* one) are in one-to-one correspondence with the

Conj. Cls. of D_n (n odd)	1	C^i	M
1	1	C^i	M
C^j	C^j	$C^{i+j}\oplus C^{i-j}$	$2M$
M	M	$2M$	$1\oplus 2\left(\bigoplus_{k=1}^{(n-1)/2} C^k\right)$

TABLE SI. Fusion rule for the conjugacy classes of D_n for odd n. Here $i-j$ and $i+j$ are understood in the mod-n sense. $i,j=1,2,...,(n-1)/2$.

Irreps of D_n (n odd)	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}_i$
$\mathbf{1}_+$	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}_i$
$\mathbf{1}_-$	$\mathbf{1}_-$	$\mathbf{1}_+$	$\mathbf{2}_i$
$\mathbf{2}_j$	$\mathbf{2}_j$	$\mathbf{2}_j$	$\begin{cases}\mathbf{2}_{i-j}\oplus\mathbf{2}_{i+j}, & i\neq j\\ \mathbf{1}_+\oplus\mathbf{1}_-\oplus\mathbf{2}_{2j}, & i=j\end{cases}$

TABLE SII. Fusion rule for the irreps of D_n for odd n. Here $i-j$ and $i+j$ are understood in the mod-n sense. $i,j=1,2,...,(n-1)/2$.

distinct flat connections on Σ_g up to conjugation: $\mathrm{Hom}(\pi_1(\Sigma_g),G)/G$. Therefore, there are a total of $(2^{2g}-1)$ *twisted* sectors.

On Σ_g, the ground states of a $U(1)\times U(1)$ BF theory are

$$\bigotimes_{i=1}^{g}|p_i,q_i\rangle\,, \tag{S5}$$

where $p_i,q_i\in\mathbb{Z}$ label zero modes of a and b, respectively, subject to the identification

$$(p,q)\sim(p+n,q)\sim(p+2k,q+n). \tag{S6}$$

The nontrivial element of the $\mathbb{Z}_2^C$ symmetry acts on these states as

$$\bigotimes_{i=1}^{g}|p_i,q_i\rangle\mapsto\bigotimes_{i=1}^{g}|-p_i,-q_i\rangle\,. \tag{S7}$$

Below we discuss the cases of odd n and even n separately.

Conj. Cls. of D_{2m}	1	C^i	C^m	M	M'
1	1	C^i	C^m	M	M'
C^j	C^j	$\begin{cases}C^{i-j}\oplus C^{i+j}, & i+j\neq m\\ C^{i-j}\oplus 2C^m, & i+j=m\end{cases}$	$C^{m-i}\oplus C^{m+i}$	$\begin{cases}2M, & j \text{ odd}\\ 2M', & j \text{ even}\end{cases}$	$\begin{cases}2M', & j \text{ odd}\\ 2M, & j \text{ even}\end{cases}$
C^m	C^m	$C^{m-j}\oplus C^{m+j}$	$1\oplus 1$	$\begin{cases}2M, & m \text{ odd}\\ 2M', & m \text{ even}\end{cases}$	$\begin{cases}2M', & m \text{ odd}\\ 2M, & m \text{ even}\end{cases}$
M	M	$\begin{cases}2M, & i \text{ odd}\\ 2M', & i \text{ even}\end{cases}$	$\begin{cases}2M, & m \text{ odd}\\ 2M', & m \text{ even}\end{cases}$	$1\oplus C^m\oplus 2\left(\bigoplus_{k=1,k \text{ even}}^{m-1} C^k\right)$	$2\left(\bigoplus_{k=1,k \text{ odd}}^{m-1} C^k\right)$
M'	M'	$\begin{cases}2M', & i \text{ odd}\\ 2M, & i \text{ even}\end{cases}$	$\begin{cases}2M, & m \text{ odd}\\ 2M', & m \text{ even}\end{cases}$	$2\left(\bigoplus_{k=1,k \text{ odd}}^{m-1} C^k\right)$	$1\oplus C^m\oplus 2\left(\bigoplus_{k=1,k \text{ even}}^{m-1} C^k\right)$

TABLE SIII. Fusion rule for the conjugacy classes of D_n for even $n=2m$. Here $i-j$ and $i+j$ are understood in the mod-n sense. $i,j=1,2,...,m-1$.

Irreps of D_{2m}	$\mathbf{1}_1$	$\mathbf{1}_e$	$\mathbf{1}_m$	$\mathbf{1}_f$	$\mathbf{2}_i$
$\mathbf{1}_1$	$\mathbf{1}_1$	$\mathbf{1}_e$	$\mathbf{1}_m$	$\mathbf{1}_f$	$\mathbf{2}_i$
$\mathbf{1}_e$	$\mathbf{1}_e$	$\mathbf{1}_1$	$\mathbf{1}_f$	$\mathbf{1}_m$	$\mathbf{2}_{m-i}$
$\mathbf{1}_m$	$\mathbf{1}_m$	$\mathbf{1}_f$	$\mathbf{1}_1$	$\mathbf{1}_e$	$\mathbf{2}_i$
$\mathbf{1}_f$	$\mathbf{1}_f$	$\mathbf{1}_m$	$\mathbf{1}_e$	$\mathbf{1}_1$	$\mathbf{2}_{m-i}$
$\mathbf{2}_j$	$\mathbf{2}_j$	$\mathbf{2}_{m-j}$	$\mathbf{2}_j$	$\mathbf{2}_{m-j}$	$\begin{cases} \mathbf{2}_{i-j} \oplus \mathbf{2}_{i+j}, & i \neq j \text{ and } i+j \neq m, \\ \mathbf{1}_e \oplus \mathbf{1}_f, & i+j = m, \\ \mathbf{1}_1 \oplus \mathbf{1}_m, & i = j \text{ and } 2j \neq m, \\ \mathbf{1}_1 \oplus \mathbf{1}_e \oplus \mathbf{1}_m \oplus \mathbf{1}_f, & i = j \text{ and } 2j = m. \end{cases}$

TABLE SIV. Fusion rule for the irreps of D_n for even $n = 2m$. Here $i - j$ and $i + j$ are understood in the mod-n sense. $i, j = 1, 2, ..., m-1$.

1. Odd n

For odd n, the state $\otimes_i |p_i = 0, q_i = 0\rangle$ is automatically invariant under the $\mathbb{Z}_2^C$ action. To find the other invariant states, consider the states symmetrized under P^C:

$$\frac{\otimes_i |p_i, q_i\rangle + \otimes_i |-p_i, -q_i\rangle}{\sqrt{2}}, \quad \exists\ (p_i, q_i) \neq (0,0). \tag{S8}$$

These together with $\otimes_i |p_i = 0, q_i = 0\rangle$ form the $(n^{2g}+1)/2$ states in the untwisted sector that are invariant under $\mathbb{Z}_2^C$.

Next we consider the twisted sectors. Since each of these sectors contains the same number of degenerate ground states, without loss of generality we can assume that a and b change sign along only one cycle α_g. Continuity of the gauge potential requires that the states in this twisted sector come in form of

$$\left(\bigotimes_{i'=1}^{g-1} |p_{i'}, q_{i'}\rangle \right) \otimes |p_g = 0, q_g = 0\rangle\,, \tag{S9}$$

which can be deduced from a standard method of considering the double cover of Σ_g to make a and b single-valued. Symmetrizing $\otimes_i |p_{i'}, q_{i'}\rangle$ under $\mathbb{Z}_2^C$ gives rise to $(n^{2g-2}+1)/2$ inequivalent states, which is the same as the number of states in the untwisted sector on Σ_{g-1}.

In total, the ground state degeneracy of the $S[O(2) \times O(2)]$ BF theory with odd n on genus g Riemann surfaces is

$$N_g^{\text{odd}} = \left(2^{2g} - 1\right) \frac{n^{2g-2}+1}{2} + \frac{n^{2g}+1}{2}, \tag{S10}$$

which is the same as the ground state degeneracy of $D^\omega(D_n)$ with odd n. The quantum dimension of anyons is uniquely determined by N_g. Therefore, we have shown that the anyons in the $S[O(2) \times O(2)]$ BF theory with odd n have the same quantum dimension as those in $D^\omega(D_n)$ with odd n.

2. Even n

The ground state degeneracy of the $S[O(2) \times O(2)]$ BF theory on a genus g Riemann surface Σ_g for even n can be calculated in parallel to the odd n case. Note that for even n, besides $|0,0\rangle$, the states

$$\left|\frac{n}{2}, 0\right\rangle, \ \left|k, \frac{n}{2}\right\rangle \ \left|\frac{n}{2}+k, \frac{n}{2}\right\rangle$$

are also invariant under $\mathbb{Z}_2^C$. Therefore, upon gauging, the ground states in the untwisted sectors are symmetrized as

$$\begin{aligned} \frac{\otimes_j |p_j, q_j\rangle + \otimes_j |-p_j, -q_j\rangle}{\sqrt{2}}, &\quad \text{if } \exists\ (p_j, q_j) \notin I, \\ \text{and } \ \otimes_i |p_j, q_j\rangle\,, &\quad \text{if } \forall j,\ (p_j, q_j) \in I, \end{aligned} \tag{S11}$$

where we defined the set

$$I \equiv \{(0,0), (0, \frac{n}{2}), (\frac{n}{2}+k, 0), (\frac{n}{2}+k, n)\}. \tag{S12}$$

The total number of such ground states is $(n^{2g}+2^{2g})/2$. In the twisted sector with a twist along the cycle α_g, the states subject to the continuity of gauge fields a and b read

$$|p_g,q_g\rangle\otimes|p,q\rangle_{g-1}\,,\ \ (p_g,q_g)\in I, \tag{S13}$$

where $|p,q\rangle_{g-1}$ is a shorthand for the symmetrized states in the untwisted sector on a genus $(g-1)$ Riemann surface Σ_{g-1}, given in Eq. (S11). Consequently, the total number of the ground states in this twisted sector is

$$4\times\frac{1}{2}\left(n^{2g-2}+2^{2g-2}\right).$$

Since each twisted sector has the same ground state degeneracy, the total ground state degeneracy of the $S[O(2)\times O(2)]$ BF theory with even n on Σ_g is

$$N_g^{\text{even}}=2\left(2^{2g}-1\right)\left(n^{2g-2}+2^{2g-2}\right)+\frac{n^{2g}+2^{2g}}{2}, \tag{S14}$$

which is consistent with the ground state degeneracy of $D^{\omega}(D_n)$ with even n. As a result, the anyon quantum dimensions in the two theories also match.

B. Topological Spins and Fusion Rules

In this section we present the study of topological spins and fusion rules of anyons via a generalization of the results for the $U(1)\times U(1)$ BF theory and an explicit calculation of modular S and T matrices on a torus with $g=1$. The S and T matrices can fully establish the correspondence between twisted dihedral quantum double and $S[O(2)\times O(2)]$ BF theory.

1. *Odd* n

In the main text we have shown the topological statistics of Wilson loops, and topological spin of twist operators. Here we present a full analysis of the S and T transformations. These transformations map the untwisted sector to itself, but generally map one twisted sector to another twisted sector. More concretely, it yields

$$\begin{aligned}\hat{S}:\ &\mathcal{H}_{\pm\pm}\to\mathcal{H}_{\pm\pm},\ \mathcal{H}_{\pm\mp}\to\mathcal{H}_{\mp\pm},\\ \hat{T}:\ &\mathcal{H}_{+\pm}\to\mathcal{H}_{+\pm},\ \mathcal{H}_{-\pm}\to\mathcal{H}_{-\mp},\end{aligned} \tag{S15}$$

where $\mathcal{H}_{\bullet\bullet}$ are the Hilbert subspaces of the sectors, whose first and second subscripts denote the twist across the holonomy α and β, respectively, with $-(+)$ signals a twist (untwist).

For a $U(1)\times U(1)$ BF theory, which is Abelian and does not possess twisted sectors, the S and T transformations act on the states as

$$\hat{S}\,|p,q\rangle=\frac{1}{n}\sum_{p',q'}\exp\left(2\pi i\frac{n(pq'+qp')-2kqq'}{n^2}\right)|p',q'\rangle\,, \tag{S16}$$

$$\hat{T}\,|p,q\rangle=\exp\left(2\pi i\frac{q(np-kq)}{n^2}\right)|p,q\rangle\,. \tag{S17}$$

These relations revealing the mutual and self statistics of the Abelian anyons and can be obtained from the Verlinde formula and topological spins of the Abelian anyons. For states in the untwisted sector $\mathcal{H}_{++}$ of the $S[O(2)\times O(2)]$ BF theory created by Wilson loops $W_{p,q}$, the S and T transformations read

$$\hat{S}\,|p,q\rangle_{++}=\frac{2}{n}\sum_{(p',q')\neq(0,0)}\cos\left(2\pi\frac{n(pq'+qp')-2kqq'}{n^2}\right)|p',q'\rangle_{++}+\frac{\sqrt{2}}{n}\,|0,0\rangle_{++}\,,\quad (p,q)\neq(0,0), \tag{S18}$$

$$\hat{S}\,|0,0\rangle_{++}=\frac{1}{n}\,|0,0\rangle_{++}+\frac{\sqrt{2}}{n}\sum_{(p',q')\neq(0,0)}|p',q'\rangle_{++}\,, \tag{S19}$$

$$\hat{T}\,|p,q\rangle_{++}=\exp\left(2\pi i\frac{q(np-kq)}{n^2}\right)|p,q\rangle_{++}\,, \tag{S20}$$

where the $S[O(2)\times O(2)]$ states (with subscripts) and the $U(1)\times U(1)$ states (without subscripts) are related by

$$|p,q\rangle_{++} = \frac{1}{\sqrt{2}}\left(|p,q\rangle + |-p,-q\rangle\right) \tag{S21}$$

for $(p,q)\neq(0,0)$, and $|0,0\rangle_{++}=|0,0\rangle$. And for states in twisted sectors

$$\begin{aligned}\hat{S}\,|0,0\rangle_{\pm\mp} &= |0,0\rangle_{\mp\pm}\,,\ \hat{S}\,|0,0\rangle_{--} = (-)^{\nu}\,|0,0\rangle_{--}\,,\\ \hat{T}\,|0,0\rangle_{\pm\mp} &= |0,0\rangle_{\pm-}\,,\ \hat{T}\,|0,0\rangle_{--} = (-)^{\nu}\,|0,0\rangle_{-+}\,.\end{aligned} \tag{S22}$$

Define

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle_{++} + |0,0\rangle_{+-}\right), \tag{S23}$$

$$|\eta\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle_{++} - |0,0\rangle_{+-}\right), \tag{S24}$$

$$|\sigma_{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle_{-+} \pm (-i)^{\nu}\,|0,0\rangle_{--}\right). \tag{S25}$$

as those in the main text. Then the S transformation reads

$$\hat{S}\,|1\rangle = \frac{1}{2n}\,|1\rangle + \frac{1}{2n}\,|\eta\rangle + \frac{1}{2}\,|\sigma_+\rangle + \frac{1}{2}\,|\sigma_-\rangle + \frac{1}{n}\sum_{(p',q')\neq(0,0)}|p',q'\rangle_{++}\,, \tag{S26}$$

$$\hat{S}\,|\eta\rangle = \frac{1}{2n}\,|1\rangle + \frac{1}{2n}\,|\eta\rangle - \frac{1}{2}\,|\sigma_+\rangle - \frac{1}{2}\,|\sigma_-\rangle + \frac{1}{n}\sum_{(p',q')\neq(0,0)}|p',q'\rangle_{++}\,, \tag{S27}$$

$$\hat{S}\,|\sigma_{\pm}\rangle = \frac{1}{2}\left(|1\rangle - |\eta\rangle \pm (-)^{\nu}\,|\sigma_+\rangle \mp (-)^{\nu}\,|\sigma_-\rangle\right), \tag{S28}$$

and the T transformation reads

$$\hat{T}\,|1\rangle = |1\rangle\,,\ \hat{T}\,|\eta\rangle = |\eta\rangle\,,\ \hat{T}\,|\sigma_{\pm}\rangle = \pm i^{\nu}\,|\sigma_{\pm}\rangle\,, \tag{S29}$$

which agrees with the direct calculation of S and T matrices of the quantum double $D^{[2k+\nu]}(D_n)$ using finite group data in Sec. SI. In following we present a simple example, $n=1$, to illustrate the effect of $\mathbb{Z}_2$ sign ν.

For $n=1$, there is no even DWT, and the $U(1)\times U(1)$ BF theory describes a phase without topological order. Therefore, gauging the $\mathbb{Z}_2^C$ symmetry results in the $D(\mathbb{Z}_2)$ phase, where the ψ anyons do not exist. Based on examining the fusion rule and topological spin, we identify the $1,\eta,\sigma_+,\sigma_-$ anyons with the $1,e,m,f$ anyons in the toric code model. Although even DWT is absent, the $\mathbb{Z}_2$ toric code phase does admit an odd DWT: this is the double semion model, which can be constructed by gluing a $\mathbb{Z}_2$ SPT phase to the toric code model. In the double semion model, the fusion rule of the anyons remain the same as the toric code model; however, the topological spins of m and f anyons become $1/4$ and $3/4$, respectively. As expected, the odd DWT comes from the self DWT of the $\mathbb{Z}_2^C$ part of the gauge structure, instead of the (trivial) $U(1)\times U(1)$ part. Thus it is reasonable to expect that for general odd n, the topological properties of the anyons that are solely inherited from the $U(1)\times U(1)$ BF theory, specifically the topological spins of the ψ anyons, should not be alternated by the odd DWT in the $\mathbb{Z}_2^C$ part of the gauge structure. On the other hand, the topological properties of the anyons that are arising from the $\mathbb{Z}_2^C$ gauge structure, namely the topological spins and the S transformations of the $\sigma_\pm$ anyons, should only depend on the parity of the DWT. Indeed, the ψ anyons in $D^{[2k]}(D_n)$ have the same topological spins as those in $D^{[2k+1]}(D_n)$. In addition, for all k and odd n, the σ_+ anyon has a topological spin 0 in $D^{[2k]}(D_n)$ and $1/4$ in $D^{[2k+1]}(D_n)$, and the σ_- anyon has topological spin $1/2$ in $D^{[2k]}(D_n)$ and $3/4$ in $D^{[2k+1]}(D_n)$, respectively. The S matrix in the subspace spanned by $\sigma_\pm$ reads

$$S_\sigma = \pm\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}, \tag{S30}$$

with the plus (minus) sign for even (odd) DWTs, which is independent of n and k. Therefore, we conclude that in the $S[O(2)\times O(2)]$ BF theory, while *even* DWTs are explicitly reflected in the self CS term of level-$2k$, *odd* DWTs are given implicitly by the DWT in the $\mathbb{Z}_2^C$ gauge structure. This is consistent with the explicit calculation of S and T matrices using finite group modular data in the Sec. SI.

Anyon Label	Quantum Dim.	Number of Anyons	Conjugacy Class	Centra-lizer	Irrep.
1_μ	1	4	$\{1\}$	D_n	$\mathbf{1}_\mu$
e_μ	1	4	$\{C_n^{n/2}\}$	D_n	$\mathbf{1}_\mu$
ψ_l	2	$(n-2)/2$	$\{1\}$	D_n	$\mathbf{2}_l$
$\bar{\psi}_l$	2	$(n-2)/2$	$\{C_n^{n/2}\}$	D_n	$\mathbf{2}_l$
$\psi_{l,r}$	2	$n(n-2)/2$	$\{C_n^l, C_n^{n-l}\}$	$\mathbb{Z}_n$	$\mathbf{1}_r$
m_μ	$n/2$	4	$\{M_1, ..., M_n\}$	D_2	$\mathbf{1}_\mu$
f_μ	$n/2$	4	$\{C_nM_1, ..., C_nM_n\}$	D_2	$\mathbf{1}_\mu$

TABLE SV. Anyon data of $D^\omega(D_n)$ for even n. In the first and the last two rows $\mu = 1, e, m, f$ label the four different one dimensional irreps of D_n ($D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$). The direct products of these irreps obey the the same fusion rules as the 1, e, m and f anyons in $\mathbb{Z}_2$ toric code model. For simplicity the trivial anyon 1_1 will be also denoted as 1. There are $(n^2+28)/2$ distinct anyons, and the total quantum dimension is $4n^2$. Here $l = 1, ..., (n-2)/2$ and $r = -(n-2)/2, ..., (n-2)/2$.

2. *Even n*

Similar to the odd n case, the ψ anyons with quantum dimension 2 are identified with the Wilson loop operators of the $S[O(2)\times O(2)]$ BF theory

$$W_{p,q} = 2\mathcal{P}\cos\left(\oint_\gamma pa + qb\right),\ (p,q)\notin I. \tag{S31}$$

Their fusion rule, consistent with those of ψ anyons in $D^\omega(D_n)$, is the same as Eq. (8). To be concrete, here $W_{p,p}$ and $W_{p,\frac{n}{2}-p}$ are identified as ψ_l and $\bar{\psi}_l$, and the other Wilson loop operators are identified with $\psi_{l,r}$. Aside from the Wilson loops and the identity operator, there are 15 twist operators, among which 7 have quantum dimension 1 and 8 have quantum dimension $n/2$. Their topological spin and fusion rule can be determined from the S and T matrices, similar to the odd n case. In the untwisted sector (labeled by $++$), the S and T transformations constrained to the basis of states

$$|{+}{+}\rangle = \left(|0,0\rangle_{++}, \left|\frac{n}{2},0\right\rangle_{++}, \left|k,\frac{n}{2}\right\rangle_{++}, \left|\frac{n}{2}+k,\frac{n}{2}\right\rangle_{++}\right)^{\mathbf{T}}$$

are given by $\hat{S}\,|{+}{+}\rangle = \frac{1}{n}S_0\,|{+}{+}\rangle$, $\hat{T}\,|{+}{+}\rangle = T_0\,|{+}{+}\rangle$, where the matrices S_0 and T_0 read

$$\begin{aligned} S_0 &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & (-)^{\frac{n}{2}} & (-)^{\frac{n}{2}} \\ 1 & (-)^{\frac{n}{2}} & (-)^k & (-)^{\frac{n}{2}+k} \\ 1 & (-)^{\frac{n}{2}} & (-)^{\frac{n}{2}+k} & (-)^k \end{pmatrix}, && \text{(S32)} \\ T_0 &= \mathrm{diag}\{1,1,i^k,i^{n+k}\}. && \text{(S33)} \end{aligned}$$

Apparently the S transformation will also generate states $|p,q\rangle_{++}$ with $(p,q)\notin I$, which are the same as Eq. (S20) and omitted for clarity. For twisted sectors, the actions of S and T still have a $\mathbb{Z}_2$ sign arising from whether $\mathbb{Z}_2^C$ has nontrivial DWT or not, similar to the odd DWTs in the odd n case. After considering this $\mathbb{Z}_2$ sign labeled by $\nu = 0, 1$, the S and T transformations are

$$\begin{aligned} &\hat{S}\,|{\pm}{\mp}\rangle = \frac{1}{2}S_0\,|{\mp}{\pm}\rangle\,, \quad \hat{S}\,|{-}{-}\rangle = \frac{1}{2}(-)^\nu S_0\,|{-}{-}\rangle\,, \\ &\hat{T}\,|{\pm}{\mp}\rangle = T_0\,|{\pm}{-}\rangle\,, \quad \hat{T}\,|{-}{-}\rangle = (-)^\nu T_0\,|{-}{+}\rangle\,. \end{aligned} \tag{S34}$$

Here the basis of states $|{\bullet\bullet}\rangle$ is defined similarly to $|{+}{+}\rangle$ as

$$|{\bullet\bullet}\rangle = \left(|0,0\rangle_{\bullet\bullet}, \left|\frac{n}{2},0\right\rangle_{\bullet\bullet}, \left|k,\frac{n}{2}\right\rangle_{\bullet\bullet}, \left|\frac{n}{2}+k,\frac{n}{2}\right\rangle_{\bullet\bullet}\right)^{\mathbf{T}}.$$

The $1/2$ factor in the S transformation comes from the total number of states equal to $4 = 2^2$ in each twisted sector, similar to the $1/n$ factor in the untwisted sector [see Eq. (S20)].

However, for even n, the dihedral group D_n has a non-trivial center $\mathbb{Z}_2^\pi = \{1, C_n^{n/2}\}$ as shown in Table SV, which commutes with all group elements including the $\mathbb{Z}_2^C$ action. Thus, it is possible that this center $\mathbb{Z}_2^\pi$ has a nontrivial mixed DWT with $\mathbb{Z}_2^C$. This mixed DWT is independent of the self DWT of $\mathbb{Z}_2^C$ (labeled by ν) or $\mathbb{Z}_n \supset \mathbb{Z}_2^\pi$ (labeled by k), and it incarnates in the $S[O(2)\times O(2)]$ BF theory as an additional $\mathbb{Z}_2$ sign labeled by $\upsilon = 0,1$ in the S and T transformations in the twisted sectors:

$$\begin{aligned}
&\hat{S}\left|\pm\mp\right\rangle = \frac{1}{2}S_0\left|\mp\pm\right\rangle,\ \hat{S}\left|--\right\rangle = \frac{1}{2}(-)^\nu S_0\Gamma_\upsilon^2\left|--\right\rangle,\\
&\hat{T}\left|\pm\mp\right\rangle = T_0\left|\pm-\right\rangle,\ \hat{T}\left|--\right\rangle = (-)^\nu T_0\Gamma_\upsilon^2\left|-+\right\rangle,
\end{aligned} \tag{S35}$$

where $\Gamma_\upsilon = \mathrm{diag}\{1,1,i^\upsilon,i^\upsilon\}$. Heuristically, while the $\mathbb{Z}_2^C$ self DWT flips the sign of all the four states in the $--$ sector in an S or T transformation, the mixed DWT only flips the sign of the two states $\left|k,\frac{n}{2}\right\rangle_{--}$ and $\left|\frac{n}{2}+k,\frac{n}{2}\right\rangle_{--}$ dependent on the self DWT in $\mathbb{Z}_2^\pi$, reflecting its mixed nature. Note that the self DWT in $\mathbb{Z}_2^\pi$ is exactly encoded in the level-$2k$ self CS term of the $S[O(2)\times O(2)]$ BF theory, and hence relies on k, since $\mathbb{Z}_2^\pi$ is a subgroup of $\mathbb{Z}_n$.

In parallel to the odd n case, we define

$$\left|\mathbf{1}\right\rangle = (\left|1\right\rangle,\left|1_e\right\rangle,\left|1_m\right\rangle,\left|1_f\right\rangle)^{\mathbf{T}} = \frac{1}{\sqrt{2}}\left(\left|++\right\rangle + \Gamma_\upsilon^{-1}\left|+-\right\rangle\right), \tag{S36a}$$

$$\left|\mathbf{e}\right\rangle = (\left|e_1\right\rangle,\left|e_e\right\rangle,\left|e_m\right\rangle,\left|e_f\right\rangle)^{\mathbf{T}} = \frac{1}{\sqrt{2}}\left(\left|++\right\rangle - \Gamma_\upsilon^{-1}\left|+-\right\rangle\right), \tag{S36b}$$

$$\left|\mathbf{m}\right\rangle = (\left|m_1\right\rangle,\left|m_e\right\rangle,\left|m_m\right\rangle,\left|m_f\right\rangle)^{\mathbf{T}} = \frac{1}{\sqrt{2}}\left(\left|-+\right\rangle + (-i)^\nu\Gamma_\upsilon^{-1}\left|--\right\rangle\right), \tag{S36c}$$

$$\left|\mathbf{f}\right\rangle = (\left|f_1\right\rangle,\left|f_e\right\rangle,\left|f_m\right\rangle,\left|f_f\right\rangle)^{\mathbf{T}} = \frac{1}{\sqrt{2}}\left(\left|-+\right\rangle - (-i)^\nu\Gamma_\upsilon^{-1}\left|--\right\rangle\right). \tag{S36d}$$

Then the S and T transformations in the 16-dimensional subspace spanned by basis Eq. (S36) read

$$S_\mu = \frac{1}{4n}\begin{pmatrix} 2S_0 & 2S_0 & nS_\upsilon & nS_\upsilon \\ 2S_0 & 2S_0 & -nS_\upsilon & -nS_\upsilon \\ nS_\upsilon^{\mathbf{T}} & -nS_\upsilon^{\mathbf{T}} & nS_{\nu,\upsilon} & -nS_{\nu,\upsilon} \\ nS_\upsilon^{\mathbf{T}} & -nS_\upsilon^{\mathbf{T}} & -nS_{\nu,\upsilon} & nS_{\nu,\upsilon}\end{pmatrix}, \tag{S37}$$

$$T_\mu = \mathrm{diag}\{T_0, T_0, i^\nu\Gamma_\upsilon T_0, -i^\nu\Gamma_\upsilon T_0\}, \tag{S38}$$

where $S_\upsilon = \Gamma_\upsilon^{-1}S_0$ and $S_{\nu,\upsilon} = (-)^\nu\Gamma_\upsilon^{-1}S_0\Gamma_\upsilon^{-1}$. The quantum dimensions of the twist operators can be read out from the first row of the S matrix, yielding that 1_μ and e_μ anyons have quantum dimension 1, while m_μ and f_μ anyons have quantum dimension $n/2$, for $\mu = 1,e,m,f$. These are consistent with Table SV. In conclusion, all the $4n$ DWTs of $D^\omega(D_n)$ for even n are captured by $\omega = [k,\nu,\upsilon]$. In the following section we will illustrate the physics of anomalous DWT, especially the mixed one, through a simple example of D_2.

Unlike the $n>2$ case where D_n is a semidirect product of $\mathbb{Z}_n$ and $\mathbb{Z}_2$, the $n=2$ case, D_2, has a direct product structure: $D_2 = \mathbb{Z}_2^\pi \times \mathbb{Z}_2^C$. As a result, there are two ways to describe a $D^\omega(D_2)$ phase. One is the $S[O(2)\times O(2)]$ BF theory proposed in this work

$$\mathcal{L}_{D_2} = \frac{ik}{2\pi}a\mathrm{d}a + \frac{i}{\pi}a\mathrm{d}b, \tag{S39}$$

with $k=0,1$ labeling the *regular* DWTs. Physically, this can be viewed as a toric code (for $k=0$) or double semion (for $k=1$) model with a $\mathbb{Z}_2^C$ orbifold. In this representation, the *anomalous* DWTs are implicitly captured by the S and T transformations on states in the twisted sectors originated from the orbifold. Since $\mathbb{Z}_2^\pi$ and $\mathbb{Z}_2^C$ commute with each other, this orbifold procedure can be equivalently described by stacking, which gives the other way to describe $D^{[k,\nu,\upsilon]}(D_2)$. In this representation, the quantum double phase is viewed as a stacking of two layers of $\mathbb{Z}_2$ quantum double phases

$$\mathcal{L}'_{D_2} = \frac{ik}{2\pi}\alpha\mathrm{d}\alpha + \frac{i}{\pi}\alpha\mathrm{d}\beta + \frac{i\nu}{2\pi}\alpha'\mathrm{d}\alpha' + \frac{i}{\pi}\alpha'\mathrm{d}\beta' + \frac{i\upsilon}{2\pi}\alpha\mathrm{d}\alpha', \tag{S40}$$

here α and α' are $\mathbb{Z}_2^\pi$ and $\mathbb{Z}_2^C$ gauge fields, with β and β' being the Lagrangian multipliers, respectively. A Maxwell term also exists for the field α and α'. As in the $S[O(2)\times O(2)]$ BF theory, $k=0,1$ captures the regular DWTs arising from the self DWT of $\mathbb{Z}_2^\pi$; however, the anomalous DWTs from $\mathbb{Z}_2^C$ is also explicitly captured by the level-2ν self CS term in α'. In addition, the mixed DWT between the two $\mathbb{Z}_2$ gauge structures, which corresponds to a mutual CS term between the two $\mathbb{Z}_2$ gauge fields α and α', is explicitly captured by the υ-term in the stacking theory Eq. (S40). Thus, compared with the $S[O(2)\times O(2)]$ BF theory, the stacking theory Eq. (S40) reveals all the $2^3=8$ possible DWTs.

The anyons in the stacking theory are bound states of two anyons from each $\mathbb{Z}_2$ quantum double layer. We use μ'_μ with $\mu, \mu' = 1, e, m, f$ to label these anyons, where e and m anyons carrying "electric" and "magnetic" charges are coupled to α, α' and β, β', respectively. Under this basis, the S and T matrices of the stacking theory Eq. (S40) are equal to those of the $S[O(2)\times O(2)]$ BF theory calculated in the previous subsection, Eq. (S37) and Eq. (S38), for all (k, ν, υ). This strongly corroborates the $S[O(2)\times O(2)]$ BF theory as an effective theory of the $D^{[k,\nu,\upsilon]}(D_n)$ phase for even n.

The effects of DWTs, which are rather abstract in the $S[O(2)\times O(2)]$ BF theory, can be easily read out from the anyon statistics in the stacking theory Eq. (S40). Consider the intra-layer anyons in the $D(\mathbb{Z}_2^\pi)$ and the $D(\mathbb{Z}_2^C)$ layers (anyons bound with the trivial anyon in the other layer), defined as $1, e_1, m_1, f_1$ and $1, 1_e, 1_m, 1_f$, respectively. Here 1 is the bound state of two trivial anyons in each layer, which becomes the trivial anyon in the stacking theory. Without DWT, both of the two layers are in the $\mathbb{Z}_2$ toric code phase. A regular DWT in the $D(\mathbb{Z}_2^\pi)$ layer with $k = 1$ in Eq. (S40) converts the layer to be in a double semion phase, while the $D(\mathbb{Z}_2^C)$ layer remains toric code. However, an anomalous DWT in the $D(\mathbb{Z}_2^C)$ will also be resulted in a stacking of a toric code phase and a double semion phase, which is equivalent to the effect of the regular DWT. This occasional isomorphism is originated from the duality between the $\mathbb{Z}_2^C$ and $\mathbb{Z}_n = \mathbb{Z}_2^\pi$ subgroups in $D_{n=2}$, and not expected to exist for general $n > 2$. On the other hand, the mixed DWT between $\mathbb{Z}_2^C$ and $\mathbb{Z}_2^\pi$ does not alternate the statistics among the intra-layer anyons; however, the statistics among anyons from different layers will be affected. For example, consider $k = \nu = 0$ and $\upsilon = 1$ case. Compared with a trivial stacking of two toric code layers, in this phase the statistics among 1_μ and among μ_1 remain the same as the toric code model; however, the mutual statistics between 1_m and m_1 becomes non-trivial, with a $\pi/2$ mutual statistical angle. Thus, for generic even n, we expect an anomalous DWT labeled by υ will not affect the statistics among the 1_μ anyons and among the μ_1 anyons, but the mutual statistics between these two groups.

SIII. MONTE CARLO SIMULATION WITH MONOPOLE SUPPRESSION

In this section we discuss the procedure of suppressing monopoles in the simulation, and show the obtained phase diagram. The real $O(2)$ gauge field $\mathbf{u_{ij}}$ can be parameterized by its $U(1)$ part $a_{\mathbf{ij}}$ and $\mathbb{Z}_2$ part $\sigma_{\mathbf{ij}}$ as

$$\mathbf{u_{ij}} = \mathbf{u_{ji}^T} = \begin{pmatrix} \cos n a_{\mathbf{ij}} & \sin n a_{\mathbf{ij}} \\ -\sigma_{\mathbf{ij}} \sin n a_{\mathbf{ij}} & \sigma_{\mathbf{ij}} \cos n a_{\mathbf{ij}} \end{pmatrix} = \mathbf{Z}^{\frac{1+\sigma_{\mathbf{ij}}}{2}} \cdot \left(\mathbf{I} \cos n a_{\mathbf{ij}} + i \mathbf{Y} \sin n a_{\mathbf{ij}} \right), \tag{S41}$$

where $a_{\mathbf{ij}} \in [0, 2\pi)$ and $\sigma_{\mathbf{ij}} = \pm 1$. The Maxwell coupling on a plaquette can be rewritten as

$$H_\square = -\frac{1}{2\kappa} \mathrm{tr}\ \mathbf{u}_{12} \cdot \mathbf{u}_{23} \cdot \mathbf{u}_{34} \cdot \mathbf{u}_{41} = -\frac{1 + \sigma_{12}\sigma_{23}\sigma_{34}\sigma_{41}}{\kappa} \cos n\Psi_\square, \tag{S42}$$

where $\Psi_\square = a_{12}\sigma_{23}\sigma_{34}\sigma_{41} + a_{23}\sigma_{34}\sigma_{41} + a_{34}\sigma_{41} + a_{41}$. Denote $K_\square = (1 + \sigma_{12}\sigma_{23}\sigma_{34}\sigma_{41})/(\kappa T)$ where T is the temperature. Then the Boltzmann weight can be approximated by means of the Villain approximation

$$e^{-\beta H_\square} = \sum_{r_{\bar{\mathbf{i}}\bar{\mathbf{j}}}=-\infty}^{+\infty} \exp\left(-\frac{K_\square}{2} \left(n\Psi_\square - 2\pi r_{\bar{\mathbf{i}}\bar{\mathbf{j}}} \right)^2 \right), \tag{S43}$$

where $\langle \bar{\mathbf{i}}\bar{\mathbf{j}} \rangle$ is the dual lattice link piercing the plaquette $\square$. The monopole suppression condition under the Villain approximation is $\nabla \cdot \mathbf{r}_{\bar{\mathbf{i}}} = 0$ for $\mathbf{r}_{\bar{\mathbf{i}}} = (r_{\bar{\mathbf{i}},\bar{\mathbf{i}}+\hat{\mathbf{x}}}, r_{\bar{\mathbf{i}},\bar{\mathbf{i}}+\hat{\mathbf{y}}}, r_{\bar{\mathbf{i}},\bar{\mathbf{i}}+\hat{\mathbf{z}}})$. It can be enforced in the partition function by introducing the lagrange multiplier $\gamma_{\bar{\mathbf{i}}} \in \mathbb{R}$ defined on the dual site $\bar{\mathbf{i}}$. The partition function for the gauge field with monopole suppression then reads

$$Z = \sum_{\{\sigma=\pm1\}} \sum_{\{r\in\mathbb{Z}\}} \int_{-\infty}^{+\infty} D[a] \int_{-2\pi}^{2\pi} D[\gamma] \prod_{\bar{\mathbf{i}}} \exp\left(i\gamma_{\bar{\mathbf{i}}} \nabla \cdot \mathbf{r}_{\bar{\mathbf{i}}} \right) \prod_\square \exp\left(-\frac{K_\square[a,\sigma]}{2} \left(n\Psi_\square[a,\sigma] - 2\pi r_{\bar{\mathbf{i}}\bar{\mathbf{j}}} \right)^2 \right). \tag{S44}$$

Note that $a_{\mathbf{ij}}$ is now defined through $\mathbb{R}$ due to the Villain approximation. Its periodicity is recovered by the integer field $r_{\bar{\mathbf{i}}\bar{\mathbf{j}}}$.

From Fig. S2 it can be seen that the multicritical point is always stable in the range of κ we consider. A more rigorous argument about the stability of the multicritical point is given as follows. First consider a three-dimensional fixed volume $\tilde{\mathcal{V}}$ (preserved under the RG flow) as a subspace of the five-dimensional phase diagram of the effective field theory (12) in the main text. For such a $\tilde{\mathcal{V}}$ with r_v and r_s remaining irrelevant in it and containing the $O(2)\times O(1)$ WF fixed point, any RG trajectory—as long as it is not the direct trajectory from $O(2)\times O(1)$ to the $O(3)$ WF fixed point—will be driven away from the latter, since t_v and t_s are always relevant in the entire parameter space. This

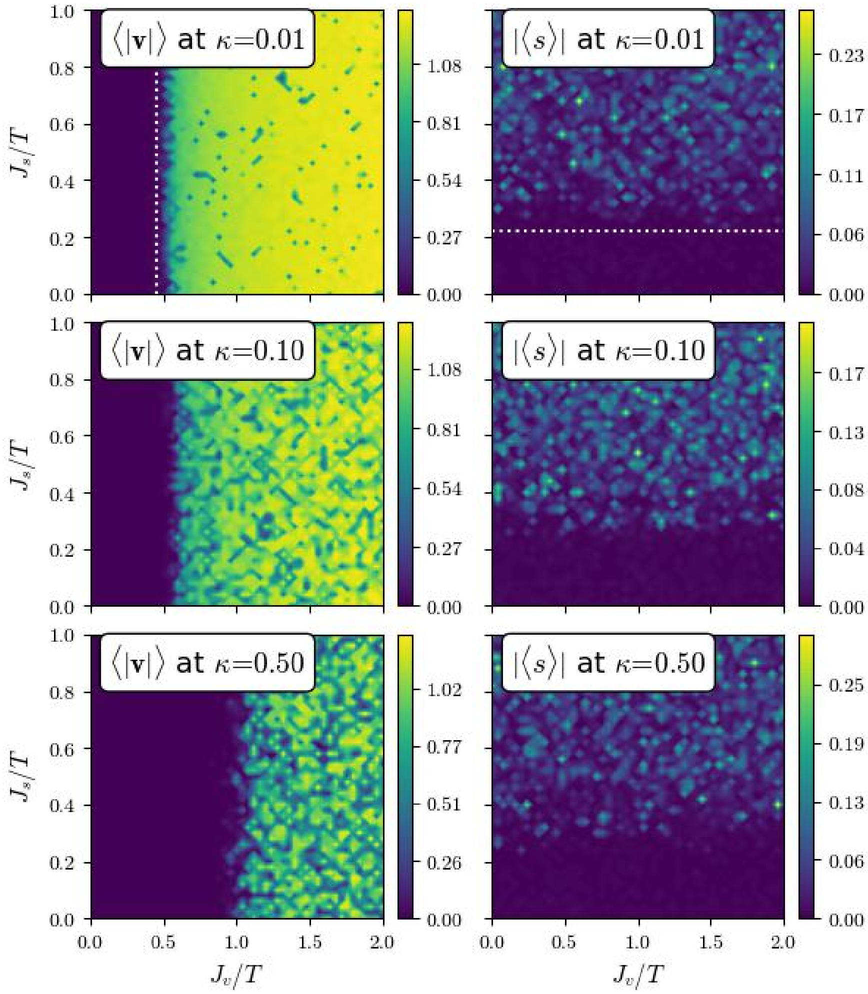

FIG. S2. Numerical phase diagram in the $(J_v/T, J_s/T)$ parameter space for various κ obtained from Monte Carlo simulation for the gauge theory (S44). $|\mathbf{v}|$ and s are the order parameters for the $U(1)$ and $\mathbb{Z}_2$ symmetries, respectively. The vertical dashed line in the upper left panel denotes the critical value for the 3D XY transition, $J_v/T = 0.45422$, and the horizontal dashed line in the upper right panel denotes that for the 3D Ising transition, $J_s/T = 0.22165$.

suggests the stability of the multicritical point within $\tilde{\mathcal{V}}$. On the other hand, the phase diagram $\mathcal{V}$ of the lattice model (11) in the main text spanned by J_v, J_s, and κ is also an embedded three-dimensional subspace. *Assuming* that r_v and r_s are always irrelevant in $\mathcal{V}$, the lattice model phase diagram $\mathcal{V}$ can be deformed to the phase diagram in $\tilde{\mathcal{V}}$ with irrelevant $r_{v,s}$. This assumption is justified by the observation that upon varying J_s and J_v in the lattice model (11), the universality class of transitions between stable phases remains Ising or XY, where the r_v and r_s are indeed irrelevant. Thus, the multicritical point should also be stable in the lattice model.

SIV. THE COMPLEX LATTICE MODEL

In this section we present another lattice model involving complex boson fields, where the $O(2)$ gauge group manifests as $O(2,\mathbb{C})$. Consider the classical theory defined on 3D cubic lattice

$$H = H_{\text{Maxwell}} + \sum_{\langle \mathbf{ij}\rangle} -J_\Phi \mathbf{\Phi}_\mathbf{i}^* \cdot \mathbf{u}_\mathbf{ij} \cdot \mathbf{\Phi}_\mathbf{j} - J_\phi \phi_\mathbf{i} |\mathbf{u}_\mathbf{ij}| \phi_\mathbf{j} + \sum_\mathbf{i} \lambda \phi_\mathbf{i} \left(|\Phi_{1,\mathbf{i}}|^2 - |\Phi_{2,\mathbf{i}}|^2 \right) + \Lambda |\Phi_{1,\mathbf{i}} - \Phi_{2,\mathbf{i}}^*|^2, \tag{S45}$$

where $\mathbf{\Phi_i} = (\Phi_{1,\mathbf{i}}, \Phi_{2,\mathbf{i}})$ is a unit-norm two-component complex boson field, $\phi_\mathbf{i} = \pm 1$ is an Ising field, and $\mathbf{u_{ij}}$ is the $O(2,\mathbb{C})$ gauge field. J_Φ, J_ϕ, λ and Λ are coupling constants. Without losing of generality, we assume $\lambda, \Lambda > 0$. For the matter fields, the $O(2)$ actions read

$$\begin{aligned} &U(1): \ \mathbf{\Phi} \mapsto e^{in\theta\mathbf{Z}} \cdot \mathbf{\Phi}, \ \phi \mapsto \phi, \\ &\mathbb{Z}_2: \ \mathbf{\Phi} \mapsto \mathbf{X} \cdot \mathbf{\Phi}, \ \phi \mapsto -\phi, \end{aligned} \tag{S46}$$

which is different to the real case. Here $\mathbf{X}$ and $\mathbf{Z}$ represent the Pauli matrices. To illustrate the phase diagram of the complex theory, we adopt a Ginzberg-Landau mean-field treatment based on the continuum version of Eq. (S45). The corresponding Ginzberg-Landau potential reads

$$V[\mathbf{\Phi}, \phi] = t_\Phi |\mathbf{\Phi}|^2 + |\mathbf{\Phi}|^4 + t_\phi \phi^2 + \phi^4 + \lambda\phi \left(|\Phi_1|^2 - |\Phi_2|^2\right) + \Lambda |\Phi_1 - \Phi_2^*|^2, \tag{S47}$$

where parameters t_Φ and t_ϕ control the condensation of $\mathbf{\Phi}$ and ϕ, respectively. The coupling constants of the quartic terms are set to be 1 for clarity. Note that the complex theory is closely related to the real theory. Define

$$\Psi_1 = \Phi_1 + \Phi_2^*, \ \Psi_2 = \Phi_1^* - \Phi_2, \tag{S48}$$

The $O(2)$ action on Ψ_1 is the same as that in the real theory, where $U(1)$ and $\mathbb{Z}_2$ act as charge conservation and charge conjugation symmetries, respectively. Assuming Ψ_2 is gapped and decoupled from the low energy sector, the Ginzberg-Landau potential can be rewritten in terms of Ψ_1 as

$$V[\Psi_1, \phi] = t_\Phi |\Psi_1|^2 + |\Psi_1|^4 + t_\phi \phi^2 + \phi^4, \tag{S49}$$

which equals that in the real theory before integrating out the gauge field.

Below we only consider the fully deconfined phases in the complex theory. Similar to the case of the real theory, there are four such phases:

1. The $O(2)$ Coulomb phase: $\langle \Phi_1 \rangle = \langle \Phi_2 \rangle = \langle \phi \rangle = 0$. The matter fields are complex boson $\mathbf{\Phi}$ carrying $\mathbf{2}_n$ irrep and real boson ϕ carrying $\mathbf{1}_-$ irrep.
2. The $U(1)$ Coulomb phase: $\langle \Phi_1 \rangle = \langle \Phi_2 \rangle = 0$, $\langle \phi \rangle \neq 0$. The matter fields are charged-$\pm n$ bosons $\Phi_{1,2}$.
3. The $D(D_n)$ phase: $\langle \Phi_1 \rangle = \langle \Phi_2^* \rangle \neq 0$, $\langle \phi \rangle = 0$. The matter field is the η (1_e) anyon when n is odd (even).
4. The $D(\mathbb{Z}_n)$ phase: $\langle \Phi_1 \rangle \neq 0$, $\langle \Phi_2 \rangle \neq 0$, $\langle \phi \rangle \neq 0$. This is a pure gauge theory phase without matter fields.

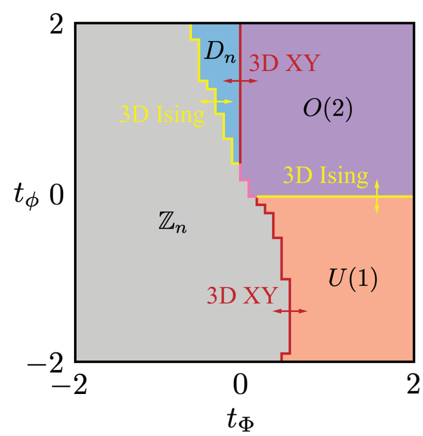

FIG. S3. Mean-field phase diagram of the complex $O(2)$ theory Eq. (S45) with $\lambda = 1.5$ and $\Lambda = 0.2$ when all the possible gauge fields are deconfined. The gauge structure of each phase is shown explicitly. The 3D XY and the 3D Ising transitions are marked in red and yellow, respectively. The direct transition between the $O(2)$ Coulomb phase and the $D(\mathbb{Z}_n)$ phase, marked in pink, could be in a coexistence of the 3D XY and 3D Ising classes.

Assuming that H_{Maxwell} deconfines all the possible gauge fields, we calculate the mean-field phase diagram of the complex $O(2)$ theory Eq. (S45) with $\lambda = 1.5$ and $\Lambda = 0.2$, as shown in FIG. S3. The phase transition patterns are almost the same as the real theory case, while the multicritical point with emergent $O(3)$ symmetry is no longer stable due to the relevant λ-term in Eq. (S45). When Ψ_2 in Eq. (S48) is decoupled from the low energy sector, the complex theory is reduced to the real theory, eliminating the λ-term and recovering the stable multicritical point. The $O(2)$ Coulomb phase and the $D(\mathbb{Z}_n)$ phase can still sandwich a direct transition.