ON EMBEDDING OF PARTIALLY ORDERED SETS IN $(\boldsymbol{\beta}\omega, \leq_{\mathrm{RK}})$

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Abstract. A natural question, which appeared as Problem 61 of [23], asks whether every finite partial order is embeddable in the Rudin–Keisler order on (types of) ultrafilters over a countable set. Although the positive answer, even for all countable partial orders, was proved under CH in [2], the situation in ZFC alone remains widely open. We show that, in ZFC, not only this result of [2] can be reproved, but, moreover, the ordered by inclusion lattices of finite subsets of a set of cardinality $2^{\mathfrak{c}}$, and of countable subsets of a set of cardinality $\aleph_1$, both are embeddable in ultrafilters with any relation lying between the Rudin–Keisler and Comfort orders.

1. Introduction

The question of the structure of the Rudin–Keisler preorder $\leq_{\mathrm{RK}}$ on $\boldsymbol{\beta}\omega$, the set of ultrafilters over ω, and, in particular, of what orders can be isomorphically embedded in it, is nearly as old as its definition itself. Let us recall, without claiming to be exhaustive, several relevant facts. Many of them remain true, sometimes with minor modifications, for ultrafilters over other cardinals; for simplicity, however, we state them here only in the case of ω.

The preorder $\leq_{\mathrm{RK}}$ was defined independently by Keisler [25] and M. E. Rudin [42, 43]; before this, W. Rudin defined the isomorphism of ultrafilters $\cong$ in [41]. The paper [43] contains, among others, the following basic observations on the structure of $\leq_{\mathrm{RK}}$. The equivalence induced by $\leq_{\mathrm{RK}}$ coincides with $\cong$, so $\leq_{\mathrm{RK}}$ induces an order on isomorphism classes (usually called types) of ultrafilters. The preorder $\leq_{\mathrm{RK}}$ includes $\leq_{\mathrm{RF}}$, the Rudin–Frolík preorder. For any nonprincipal $\mathfrak{u} \in \boldsymbol{\beta}\omega$ there are $\mathfrak{c}$ its $\leq_{\mathrm{RK}}$-predecessors and $2^{\mathfrak{c}}$ its $\leq_{\mathrm{RK}}$-successors (where $\mathfrak{c}$ is $2^{\aleph_0}$, the cardinality of continuum). Moreover, $\leq_{\mathrm{RK}}$ is directed, and in fact, $\mathfrak{c}^+$-directed, as follows from more general results by Comfort and Negrepontis [12] (see also [13], Theorem 10.9 and onward). It easily follows that $(\boldsymbol{\beta}\omega, \leq_{\mathrm{RK}})$ contains well-ordered chains of (the largest possible) length $\mathfrak{c}^+$. In [28], Kunen shows that there are $\leq_{\mathrm{RK}}$-incomparable ultrafilters over ω, and in [48], Shelah and M. E. Rudin proved that there are $2^{\mathfrak{c}}$ such ultrafilters. Kunen defined weak P-points and κ-OK-points (a concept weaker than P-point but stronger than weak P-points for $\kappa \geq \aleph_1$) and proves in [29] their existence (without any additional assumptions) in the space $\boldsymbol{\beta}\omega\setminus\omega$ of nonprincipal ultrafilters over ω. Combining these results, Simon showed in [49] that there exist $2^{\mathfrak{c}}$ pairwise $\leq_{\mathrm{RK}}$-incomparable $\aleph_1$-OK ultrafilters (thus weak P-points), the fact we shall use below. In [33], van Mill showed that $\leq_{\mathrm{RK}}$-above each ultrafilter there is a $\mathfrak{c}$-OK ultrafilter (actually, Theorem 4.5.1 of [33] is much stronger: it states this for $\leq_{\mathrm{RB}}$, the Rudin–Blass preorder, and that this can be realized uniformly, by a single finite-to-one map). Product of ultrafilters was defined independently by Kochen in [26] and Frayne, Morel, and Scott in [16] and then studied by Frolík [17], Booth [7], and others. Laflamme in [30] proved that this operation with any nonprincipal fixed argument is an isomorphic embedding of $(\boldsymbol{\beta}\omega, <_{\mathrm{RK}})$ into itself (see also [19], Theorem 2.26); this shows that $<_{\mathrm{RK}}$-above each ultrafilter there is an ordered copy of the whole structure $(\boldsymbol{\beta}\omega, <_{\mathrm{RK}})$. In [7], Booth showed that the ultrapower of $(\omega, <)$ by any nonprincipal ultrafilter embeds in $(\boldsymbol{\beta}\omega, <_{\mathrm{RF}})$; a fortiori, it embeds in $(\boldsymbol{\beta}\omega, <_{\mathrm{RK}})$. It follows by a standard saturation argument therefore that both structures are universal for linearly ordered sets of cardinality $\leq \aleph_1$. The Comfort preorder $\leq_{\mathrm{C}}$ was defined by Comfort (based on the $\mathfrak{u}$-compactness introduced by Bernstein in [1]) and first appeared, probably, in García-Ferreira's papers [19, 20]. Little is known about the structure of $\leq_{\mathrm{C}}$ on $\boldsymbol{\beta}\omega$ except that it includes $\leq_{\mathrm{RK}}$, is $\mathfrak{c}^+$-directed, and there are $2^{\mathfrak{c}}$ pairwise

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$\leq_{\mathrm{C}}$-incomparable ultrafilters; the latter is immediate from Simon's result mentioned above and the fact that $\leq_{\mathrm{C}}$ coincides with $\leq_{\mathrm{RK}}$ on the set of weak P-points ([19], Theorem 2.10 and Corollary 2.11). Using this, we shall embed all finite orders (and more) in any order between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$, including the orders $\leq_\alpha$ defined by the authors in [38].

The facts listed above are established in ZFC alone. More statements can be proved under additional assumptions, primarily using CH (the continuum hypothesis) or some weaker hypotheses. In [41], W. Rudin proved that, under CH, the space $\boldsymbol{\beta}\omega \setminus \omega$ contains $2^{\mathfrak{c}}$ P-points (and each of them can be mapped to each other by some autohomeomorphism of the space). In [43], M. E. Rudin proved, also under CH, that there is no $\leq_{\mathrm{RK}}$-maximal P-points. Pitt in [35] and Mathias in [31] independently proved that, under CH, there is an ultrafilter without P-points $\leq_{\mathrm{RK}}$-below it. The existence of P-point, unlike weak P-points, is unprovable in ZFC; this was shown by Shelah, see [47]. In [9], Choquet showed (in other terms) that CH implies the existence of $\leq_{\mathrm{RK}}$-minimal ultrafilters. Any such ultrafilter is a P-point (but not conversely), so their existence is unprovable in ZFC. In fact, CH implies that the set of $\leq_{\mathrm{RK}}$-minimal ultrafilters has cardinality $2^{\mathfrak{c}}$, and Shelah constructed models in which its cardinality is any prescribed $\kappa \leq \mathfrak{c}$, see [47]. There is a number of natural characterizations of $\leq_{\mathrm{RK}}$-minimal ultrafilters as Ramsey, selective, quasinormal, etc., which were obtained by many authors including Blass, Booth, Kunen, Rowbottom, and others (see [13], Theorem 9.6 and Notes for §9); also, by the mentioned result of [19], $\leq_{\mathrm{RK}}$-minimal ultrafilters are exactly $\leq_{\mathrm{C}}$-minimal weak P-points. In Blass' pioneering work [2], the following results were obtained under CH (or sometimes a weaker assumption). The partially ordered set $(\boldsymbol{\beta}\omega, <_{\mathrm{RK}})$ contains an isomorphic copy of the Boolean lattice $(\mathscr{P}(\omega), \subset)$ as well as of the "long line" (the lexicographic product of ω_1 and the real line). Each ultrafilter has $2^{\mathfrak{c}}$ immediate $<_{\mathrm{RK}}$-successors. For any n, $2 \leq n < \omega$, there are $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$ with exactly n minimal upper bounds, in particular, $(\boldsymbol{\beta}\omega, <_{\mathrm{RK}})$ is not an upper semilattice; also the set of P-points is not $<_{\mathrm{RK}}$-directed (see also [13], Theorem 16.8). Ultrafilters with a prescribed finite number of $<_{\mathrm{RK}}$-predecessors were studied by Blass in [3] and Daguenet in [15]; like $\leq_{\mathrm{RK}}$-minimal ultrafilters, they have natural Ramsey-like characteristics. In contrast to this, under NCF (the near coherence filter principle, introduced by Blass and Shelah in [6]), $(\boldsymbol{\beta}\omega \setminus \omega, <_{\mathrm{RK}})$ is downward directed without a least element (so there is no $\leq_{\mathrm{RK}}$-minimal ultrafilters) and P-points form an its coinitial subset. The consistency of the existence of simple $P_{\aleph_1}$-points and $P_{\aleph_2}$-points simultaneously, claimed in [6], was correctly proved recently by Bräuninger and Mildenberger in [8]. In [4], Blass showed that, under CH, there is an initial segment of $(\boldsymbol{\beta}\omega \setminus \omega, <_{\mathrm{RK}})$ order-isomorphic to the tree $((2^{\mathfrak{c}})^{<\omega_1}, \subset)$ and such that each increasing ω-sequence of its points has a unique supremum. The equality $2^{\mathfrak{c}} = \mathfrak{c}^+$ is equivalent to the existence of a well-ordered cofinal set of $(\boldsymbol{\beta}\omega, <_{\mathrm{RK}})$, as well as of such a set of cardinality $2^{\mathfrak{c}}$, or just a set of cardinality $2^{\mathfrak{c}}$ without incomparable elements; this was stated by Comfort and Negrepontis in [12] and Shelah (see [13], Theorem 10.11 and Corollary 10.15, or [11], Theorem 6.5). Improving on Blass' result mentioned above, Raghavan and Shelah showed in [39] that, under MA (Martin's axiom), $(\boldsymbol{\beta}\omega, <_{\mathrm{RK}})$, and even its subset consisting of P-points, contains an isomorphic copy of the Boolean lattice $(\mathscr{P}(\omega), \subset)/\mathscr{P}_\omega(\omega)$.

Returning to the situation in ZFC alone, it should be noted that little is known about the structure of the Rudin–Keisler order beyond the facts listed above. One natural question that has remained open so far is whether ZFC proves that every finite partially ordered set is isomorphically embeddable in $(\boldsymbol{\beta}\omega, <_{\mathrm{RK}})$; this question has been appeared, although without explicit mention of provability in ZFC, in Hart and van Mill's list of open problems on $\boldsymbol{\beta}\omega$ [23] as Problem 61. As is noted there, it would suffice to embed any finite Boolean algebra, i.e., $(\mathscr{P}(n), \subseteq)$ for any finite n, but even for $n = 3$ the question seems to remain open (for $n = 2$ the solution easily follows from known facts, see [50]). In this paper, we answer this question affirmatively. Actually, we prove in ZFC two stronger statements (Theorems 3 and 4). We show that $(\boldsymbol{\beta}\omega, \leq_{\mathrm{RK}})$ contains isomorphic copies of the lattice $(\mathscr{P}_\omega(2^{\mathfrak{c}}), \subseteq)$ as well as of the lattice $(\mathscr{P}_{\omega_1}(\omega_1), \subseteq)$; moreover, this remains true for $(\boldsymbol{\beta}\omega, R)$ with any relation R lying between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$, in particular, with the relations R_α from [38]. We also prove some facts about weak P-points, which are of an auxiliary importance for the proof of our main results, but are of an independent interest (Theorems 1 and 2).

2. Definitions

In this section, we recall the precise definitions of the concepts mentioned in the introduction. Most of them are given only for the case $X = \omega$, which we consider in this paper, but they can easily be generalized to arbitrary sets X.

Topology. For X a set, $\boldsymbol{\beta}X$ denotes the set of ultrafilters over X. As usual, we suppose $X \subseteq \boldsymbol{\beta}X$ by identifying each $x \in X$ with the principal ultrafilter given by x. A natural topology on $\boldsymbol{\beta}X$ is generated by basic open sets $\widetilde{A} := \{\mathfrak{u} \in \boldsymbol{\beta}X : A \in \mathfrak{u}\}$, for all $A \subseteq X$. The resulting space $\boldsymbol{\beta}X$ is compact, Hausdorff, zero-dimensional (since the sets $\widetilde{A}$ are in fact clopen), moreover, extremally disconnected (the closure of any open set is open), and the *largest* (or *Čech–Stone*) *compactification* of the discrete space X. This latter means that X is dense in $\boldsymbol{\beta}X$ and every (trivially continuous) map h of X into any compact Hausdorff space Y uniquely extends to a continuous map $\widetilde{h}$ of $\boldsymbol{\beta}X$ into Y. (See, e.g., [13, 24].)

Preorders. The *Rudin–Keisler preorder* $\leq_{\mathrm{RK}}$ on $\boldsymbol{\beta}\omega$ is defined by letting, for all $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$,

$$\mathfrak{u} \leq_{\mathrm{RK}} \mathfrak{v} \;\text{ iff }\; \text{there exists } f : \omega \to \omega \text{ such that } \widetilde{f}(\mathfrak{v}) = \mathfrak{u}.$$

Requiring additionally that f be finite-to-one, we obtain the *Rudin–Blass preorder* $\leq_{\mathrm{RB}}$, and that f be, moreover, a permutation, the *isomorphism* relation $\cong$ of ultrafilters. The *Rudin–Frolík preorder* $\leq_{\mathrm{RF}}$ on $\boldsymbol{\beta}\omega$ is defined by letting, for all $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$,

$$\mathfrak{u} \leq_{\mathrm{RF}} \mathfrak{v} \;\text{ iff }\; \text{there exists a one-to-one } f : \omega \to \boldsymbol{\beta}\omega \text{ such that } \operatorname{ran} f \text{ is strongly discrete and } \widetilde{f}(\mathfrak{u}) = \mathfrak{v},$$

where a subspace is *strongly discrete* iff admits a pairwise disjoint family of neighborhoods of its points. (Clearly, in the definition of $\leq_{\mathrm{RK}}$ we could use a continuous $f : \boldsymbol{\beta}\omega \to \boldsymbol{\beta}\omega$ such that $f(\mathfrak{v}) = \mathfrak{u}$, and in the definition of $\leq_{\mathrm{RF}}$, a topological embedding $f : \boldsymbol{\beta}\omega \to \boldsymbol{\beta}\omega$ such that $f(\mathfrak{u}) = \mathfrak{v}$.) We have $\leq_{\mathrm{RB}} \cup \leq_{\mathrm{RF}} \subseteq \leq_{\mathrm{RK}}$, and $\leq_{\mathrm{RF}}$ is *tree-like*, i.e., for any $\mathfrak{u}$, if $\mathfrak{v} \leq_{\mathrm{RF}} \mathfrak{u}$ and $\mathfrak{w} \leq_{\mathrm{RF}} \mathfrak{u}$, then $\mathfrak{v}$ and $\mathfrak{w}$ are $\leq_{\mathrm{RF}}$-comparable. The *Comfort* preorder $\leq_{\mathrm{C}}$ on $\boldsymbol{\beta}\omega$ is defined by letting, for all $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$,

$$\mathfrak{u} \leq_{\mathrm{C}} \mathfrak{v} \;\text{ iff }\; \text{any } \mathfrak{v}\text{-compact space is } \mathfrak{u}\text{-compact},$$

where a space X is $\mathfrak{u}$-*compact* iff $\widetilde{f}(\mathfrak{u}) \in X$ for any $f : \omega \to X$. For all ordinals α, define the relations R_α as follows: $R_0 := \omega \times \boldsymbol{\beta}\omega$, and for all $\alpha > 0$ and $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$,

$$\mathfrak{u} \, R_\alpha \, \mathfrak{v} \;\text{ iff }\; \text{there exists } f : \omega \to \boldsymbol{\beta}\omega \text{ such that } \widetilde{f}(\mathfrak{v}) = \mathfrak{u} \text{ and } f(n) \, R_{<\alpha} \, \mathfrak{v}, \text{ for all } n < \omega,$$

where $R_{<\alpha} := \bigcup_{\beta<\alpha} R_\beta$. The hierarchy of R_α is non-degenerate below ω_1 and lies (for $\alpha > 0$) between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$:

$$R_0 \subset \leq_{\mathrm{RK}} = R_1 \subset R_2 \subset \ldots \subset R_{<\omega} \subset R_\omega \subset \ldots R_{<\alpha} \subset R_\alpha \subset \ldots \subset R_{<\omega_1} = R_{\omega_1} = \leq_{\mathrm{C}}.$$

Moreover, if $2 \leq \alpha \leq \omega_1$, then $R_{<\alpha}$ is a preorder iff α is multiplicatively indecomposable. This allows us to define a hierarchy of preorders lying between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$ by letting, for all $\alpha \leq \omega_1$,

$$\leq_0 := \leq_{\mathrm{RK}} \;\text{ and }\; \leq_{1+\alpha} := R_{<\omega^{\omega^\alpha}}.$$

We speak of the *Rudin–Keisler order* both in the case of the strict order $<_{\mathrm{RK}}$ on $\boldsymbol{\beta}\omega$ and in the case of the (strict or non-strict) order on $\boldsymbol{\beta}\omega/{\cong}$ induced by the preorder $\leq_{\mathrm{RK}}$ on $\boldsymbol{\beta}\omega$, and use the same symbols in both cases. This convention is adopted to other preorders on ultrafilters as well; e.g., the *Comfort order* may relate to $<_{\mathrm{C}}$ on $\boldsymbol{\beta}\omega$ as well as on $\boldsymbol{\beta}\omega/{=_{\mathrm{C}}}$. The question of the existence of an isomorphic embedding of a given order into a given preorder on ultrafilters, which is the topic of this article, can be understood equivalently as the embedding between the corresponding strict orders and as the embedding into its quotient.

Ultrafilter extensions. If $\mathfrak{u}$ is an ultrafilter, define the (self-dual) *ultrafilter quantifier* by letting $\forall^{\mathfrak{u}}x\,\varphi(x)$ iff $\{x:\varphi(x)\}\in\mathfrak{u}$. The (*tensor* or *Fubini*) *product of ultrafilters* $\mathfrak{u},\mathfrak{v}\in\boldsymbol{\beta}\omega$ is an ultrafilter $\mathfrak{u}\otimes\mathfrak{v}\in\boldsymbol{\beta}(\omega\times\omega)$ consisting of $S\subseteq\omega\times\omega$ such that $\forall^{\mathfrak{u}}x\,\forall^{\mathfrak{v}}y\,(x,y)\in S$. More generally, for any n-ary map $f:\omega^n\to Y$, where Y is a discrete space, its *ultrafilter extension* is the n-ary map $\widetilde{f}:(\boldsymbol{\beta}\omega)^n\to\boldsymbol{\beta}Y$ defined by letting, for all $\mathfrak{u}_0,\dots,\mathfrak{u}_{n-1}\in\boldsymbol{\beta}\omega$,

$$\widetilde{f}(\mathfrak{u}_0,\dots,\mathfrak{u}_{n-1}):=\{S\subseteq Y:\forall^{\mathfrak{u}_0}x_0\dots\forall^{\mathfrak{u}_{n-1}}x_{n-1}\,f(x_0,\dots,x_{n-1})\in S\}.$$

(Thus the multiplication of ultrafilters is the ultrafilter extension of the pairing map.) The ultrafilter extension $\widetilde{f}$ of f is its unique extension to $\boldsymbol{\beta}\omega$ that is *right-continuous* w.r.t. ω, i.e., such that, for each $i<n$, the shift $\mathfrak{v}\mapsto\widetilde{f}(k_0,\dots,k_{i-1},\mathfrak{v},\mathfrak{u}_{i+1},\dots,\mathfrak{u}_{n-1})$ is continuous whenever $k_0,\dots,k_{i-1}\in\omega$ and $\mathfrak{u}_{i+1},\dots,\mathfrak{u}_{n-1}\in\boldsymbol{\beta}\omega$. Let us point out that ultrafilter extensions of n-ary relations, together with their similar topological characterization, are also known; moreover, the resulting ultrafilter extensions of models are their largest compactifications, in a certain sense. However, since these concepts will not be used in this article, we omit here their definitions and refer the reader to [44, 45] (see also [37], Introduction). Concerning an interplay between ultrafilter extensions of maps and the defined relations on ultrafilters, we always have $\mathfrak{u}<_{\mathrm{RK}}\mathfrak{u}\otimes\mathfrak{v}$ and $\mathfrak{v}<_{\mathrm{RK}}\mathfrak{u}\otimes\mathfrak{v}$ (both fail for $\le_{\mathrm{RB}}$) as well as $\mathfrak{u}<_{\mathrm{RF}}\mathfrak{u}\otimes\mathfrak{v}$ but generally not $\mathfrak{v}\le_{\mathrm{RF}}\mathfrak{u}\otimes\mathfrak{v}$. Also $\mathfrak{u}\,R_n\,\mathfrak{v}$ iff there exists $f:\omega^n\to\omega$ such that $\widetilde{f}(\mathfrak{v},\dots,\mathfrak{v})=\mathfrak{u}$, and a similar characterization can be provided for R_α with $\omega\le\alpha<\omega_1$ by using certain maps $f:\omega^\omega\to\omega$ (see [38]).

***P*-point variants.** An ultrafilter $\mathfrak{u}\in\boldsymbol{\beta}\omega\setminus\omega$ is a *P-point* iff the intersection of countably many its neighborhoods contains some its neighborhood, and a *weak P-point* iff it does not belong to the closure of any countable set $A\subseteq\boldsymbol{\beta}\omega\setminus(\omega\cup\{\mathfrak{u}\})$. More generally, given a cardinal κ, an ultrafilter $\mathfrak{u}\in\boldsymbol{\beta}\omega\setminus\omega$ is a *P_κ-point* iff the intersection of $<\kappa$ its neighborhoods contains some its neighborhood, and a *weak P_κ-point* iff it does not belongs to the closure of any set $A\subseteq\boldsymbol{\beta}\omega\setminus(\omega\cup\{\mathfrak{u}\})$ of cardinality $|A|<\kappa$. (Thus P-points and weak P-points are obtained when $\kappa=\aleph_1$.) A P_κ-point is *simple* iff it has a basis of cardinality κ. An ultrafilter $\mathfrak{u}$ is *κ-OK* iff for any ω-sequence $(A_n)_{n<\omega}$ of its neighborhoods there exists its *κ-refinement*, i.e., a κ-sequence $(B_\alpha)_{\alpha<\kappa}$ of its neighborhoods such that, for any $n<\omega$ and n-subsequence $(B_{\alpha_i})_{i<n}$, we have $\bigcap_{i<n}B_{\alpha_i}\subseteq A_n$. Obviously, each of the three properties depending on κ (i.e., of being P_κ, weak P_κ, and κ-OK) become stronger as κ increases; any κ-OK point is a weak P_κ-point; and any P-point is κ-OK for all κ. Actually, we shall only need weak P-points for our proofs.

3. Characterizations of weak P-points

In this section, we provide some characterizations of weak P-points, which will be used in the subsequent proof of our main results, and are also of some independent interest.

Clearly, a unary operation on $\boldsymbol{\beta}\omega$ is continuous iff it is an extension of a map on ω into $\boldsymbol{\beta}\omega$. We provide an alternative criterion in terms of binary operations on ω, which is easy but, to our knowledge, not previously published in /the literature.

Lemma 1. *A map $f:\boldsymbol{\beta}\omega\to\boldsymbol{\beta}\omega$ is continuous iff there exist $g:\omega^2\to\omega$ and $\mathfrak{v}\in\boldsymbol{\beta}\omega$ such that*

$$f(\mathfrak{u})=\widetilde{g}(\mathfrak{u},\mathfrak{v})$$

for all $\mathfrak{u}\in\boldsymbol{\beta}\omega$. Moreover, as such a $\mathfrak{v}$ we can take any $\le_{\mathrm{RK}}$-upper bound of $f[\omega]$, the image of ω under f.

Proof. Since the extension $\widetilde{g}:(\boldsymbol{\beta}\omega)^2\to\boldsymbol{\beta}\omega$ is right-continuous, the map f obtained from $\widetilde{g}$ by letting $f(\mathfrak{u}):=\widetilde{g}(\mathfrak{u},\mathfrak{v})$, is automatically continuous. It remains to prove the converse.

Let $f:\boldsymbol{\beta}\omega\to\boldsymbol{\beta}\omega$ be continuos. Using that any countable (and in fact, of cardinality $\mathfrak{c}$) set of ultrafilters there exists an $\le_{\mathrm{RK}}$-upper bound, pick $\mathfrak{v}\in\boldsymbol{\beta}\omega$ such that $f(i)\le_{\mathrm{RK}}\mathfrak{v}$, for all $i\in\omega$. For each $i\in\omega$ we fix some $e_i:\omega\to\omega$ such that $\widetilde{e_i}(\mathfrak{v})=\mathfrak{u}_i$. Consider the map $g:\omega^2\to\omega$ defined by letting $g(i,j):=e_i(j)$, and furthermore, its extension $\widetilde{g}:(\boldsymbol{\beta}\omega)^2\to\boldsymbol{\beta}\omega$. For every fixed $i\in\omega$ we have

$$g_{(i)}(j)=g(i,j)=e_i(j)$$

(where the map $g_{(i)}$ is obtained from g by fixing i as the first argument), hence,

$$\widetilde{g}(i, \mathfrak{v}) = \widetilde{g_{(i)}}(\mathfrak{v}) = \widetilde{e_i}(\mathfrak{v}) = \mathfrak{u}_i = f(i).$$

Now consider $h : \omega \to \boldsymbol{\beta}\omega$ defined by letting $h(i) := \widetilde{g}(i, \mathfrak{v})$. The map h coincides with the restriction of f to ω. Therefore,

$$\widetilde{g}(\mathfrak{u}, \mathfrak{v}) = \widetilde{h}(\mathfrak{u}) = f(\mathfrak{u})$$

for all $\mathfrak{u} \in \boldsymbol{\beta}\omega$, as required. $\square$

The following fact is well known:

Lemma 2. *For every $\mathfrak{u} \in \boldsymbol{\beta}\omega$ and $A \subseteq \boldsymbol{\beta}\omega$, the following are equivalent:*

(i) $\mathfrak{u} \subseteq \bigcup A$;
(ii) $\bigcap A \subseteq \mathfrak{u}$;
(iii) $\mathfrak{u} \in \mathrm{cl}_{\boldsymbol{\beta}\omega} A$.

Proof. This is a routine check.

(i)⇒(ii). Let $\mathfrak{u} \subseteq \bigcup A$. If $S \in \bigcap A$ then $\omega \setminus S$ is not in any $\mathfrak{v} \in A$, hence, $\omega \setminus S \notin \mathfrak{u}$, and so, $S \in \mathfrak{u}$.

(ii)⇒(i). Let $\bigcap A \subseteq \mathfrak{u}$. Let $S \in \mathfrak{u}$. Assume S is not in any $\mathfrak{v} \in A$. Then $\omega \setminus S \in \bigcap A$. Therefore, $\omega \setminus S \in \mathfrak{u}$, a contradiction.

It follows from (i)⇔(ii) that $\mathrm{cl}_{\boldsymbol{\beta}\omega} A = \{\mathfrak{u} \in \boldsymbol{\beta}\omega : \bigcap A \subseteq \mathfrak{u}\}$, whence we get (ii)⇔(iii). $\square$

Lemma 3. *For every $\mathfrak{u} \in \boldsymbol{\beta}\omega$ and countable $A \subseteq \boldsymbol{\beta}\omega$, the following are equivalent:*

(i) $\mathfrak{u} \in \mathrm{cl}_{\boldsymbol{\beta}\omega} A$;
(ii) $\mathfrak{u} = \widetilde{f}(\mathfrak{v})$ *for some* $\mathfrak{v} \in \boldsymbol{\beta}\omega$ *whenever* $f : \omega \to \boldsymbol{\beta}\omega$ *satisfies* $\operatorname{ran} \widetilde{f} \subseteq A$;
(iii) $\mathfrak{u} = f(\mathfrak{v})$ *for some* $\mathfrak{v} \in \boldsymbol{\beta}\omega$ *whenever* $f : \boldsymbol{\beta}\omega \to \boldsymbol{\beta}\omega$ *is continuous and satisfies* $f[\omega] \subseteq A$;
(iv) $\mathfrak{u} = \widetilde{g}(\mathfrak{v}, \mathfrak{w})$ *for some* $\mathfrak{v}, \mathfrak{w} \in \boldsymbol{\beta}\omega$ *whenever* $g : \omega^2 \to \omega$ *satisfies* $\{\widetilde{g}(i, \mathfrak{w}) : i \in \omega\} \subseteq A$.

Proof. (i)⇒(ii). First assume that $A \subseteq \boldsymbol{\beta}\omega$ has an arbitrary cardinality $\kappa \leq 2^{\mathfrak{c}}$, and let $\mathfrak{u}_\alpha$, $\alpha < \kappa$, be an enumeration of A (possibly with repetitions). If $\mathfrak{u} \in \mathrm{cl}_{\boldsymbol{\beta}\omega} A$, let $\mathscr{V} := \{A_S : S \in \mathfrak{u}\}$ where $A_S := \{\alpha < \kappa : S \in \mathfrak{u}_\alpha\}$. As easy to see, for all $S, T \in \mathfrak{u}$ we have

(a) $A_S \neq \emptyset$ (due to $\mathfrak{u} \subseteq \bigcup A$) and
(b) $A_S \cap A_T = A_{S \cap T}$ (since $S \cap T \in \mathfrak{u}$).

Thus $\mathscr{V}$ does not have the empty set and is centered.

Let now $\kappa := \omega$, and suppose that an ultrafilter $\mathfrak{v} \in \boldsymbol{\beta}\omega$ extends $\mathscr{V}$. Let $f : \omega \to \boldsymbol{\beta}\omega$ be such that $\operatorname{ran} f \subseteq A$; w.l.o.g. assume that $\operatorname{ran} f = A$ and f is defined by letting $f(i) := \mathfrak{u}_i$. Then $\mathfrak{u} = \widetilde{f}(\mathfrak{v})$. Indeed, let $S \in \mathfrak{u}$. Then $A_S \in \mathfrak{v}$ and for every $i \in A_S$ we have $S \in f(i)$, whence it follows $\widetilde{f}(\mathfrak{v}) = \mathfrak{u}$.

(ii)⇒(i). If for some $\mathfrak{v} \in \boldsymbol{\beta}\omega$ and $f : \omega \to \boldsymbol{\beta}\omega$ we have $\widetilde{f}(\mathfrak{v}) = \mathfrak{u}$, then $\mathfrak{u} \in \mathrm{cl}_{\boldsymbol{\beta}\omega} \operatorname{ran} f$ (since $\widetilde{f}(\mathfrak{v})$ is the intersection of the sets $\mathrm{cl}_{\boldsymbol{\beta}\omega} f[S]$ for all $S \in \mathfrak{v}$).

The equivalence (ii)⇔(iii) is evident; for (iii)⇔(iv), use Lemma 1. $\square$

The above leads to the following characterizations of weak P-points:

Theorem 1. *For every $\mathfrak{u} \in \boldsymbol{\beta}\omega \setminus \omega$ the following are equivalent:*

(i) $\mathfrak{u}$ *is a weak* P*-point*;
(iia) *for all* $f : \omega \to \boldsymbol{\beta}\omega$ *and* $\mathfrak{v} \in \boldsymbol{\beta}\omega$ *such that* $\mathfrak{u} = \widetilde{f}(\mathfrak{v})$ *there exists* $i \in \omega$ *satisfying* $f(i) = \mathfrak{u}$ *or* $f(i) \in \omega$;
(iib) *for all* $f : \omega \to \boldsymbol{\beta}\omega$ *and* $\mathfrak{v} \in \boldsymbol{\beta}\omega$ *such that* $\mathfrak{u} = \widetilde{f}(\mathfrak{v})$ *one of two sets* $\{i \in \omega : f(i) = \mathfrak{u}\}$ *and* $\{i \in \omega : f(i) \in \omega\}$ *belongs to* $\mathfrak{v}$;
(iiia) *for all* $g : \omega^2 \to \omega$ *and* $\mathfrak{v}, \mathfrak{w} \in \boldsymbol{\beta}\omega$ *such that* $\mathfrak{u} = \widetilde{g}(\mathfrak{v}, \mathfrak{w})$ *there exists* $i \in \omega$ *satisfying* $\widetilde{g}(i, \mathfrak{w}) = \mathfrak{u}$ *or* $\widetilde{g}(i, \mathfrak{w}) \in \omega$;
(iiib) *for all* $g : \omega^2 \to \omega$ *and* $\mathfrak{v}, \mathfrak{w} \in \boldsymbol{\beta}\omega$ *such that* $\mathfrak{u} = \widetilde{g}(\mathfrak{v}, \mathfrak{w})$ *one of two sets* $\{i \in \omega : g(i, \mathfrak{w}) = \mathfrak{u}\}$ *and* $\{i \in \omega : g(i, \mathfrak{w}) \in \omega\}$ *belongs to* $\mathfrak{v}$.

Proof. (i)⇔(iia)⇔(iiia). This follows immediately from Lemma 3.

(iia)⇔(iib). The implication from the right to the left is evident, so let us prove the converse. Assume that neither the set $\{i \in \omega : f(i) = \mathfrak{u}\}$ nor the set $\{i \in \omega : f(i) \in \omega\}$ belong to $\mathfrak{v}$. Then $\mathfrak{v}$ has the set

$$A := \{i \in \omega : f(i) \notin \omega \cup \{\mathfrak{u}\}\}.$$

Pick an arbitrary ultrafilter $\mathfrak{w} \in \boldsymbol{\beta}\omega \setminus (\omega \cup \{\mathfrak{u}\})$ and consider the map $e : \omega \to \boldsymbol{\beta}\omega$ defined by letting

$$e(i) := \begin{cases} f(i) & \text{if } i \in A, \\ \mathfrak{w} & \text{otherwise.} \end{cases}$$

Then $\widetilde{e}(\mathfrak{v}) = \widetilde{f}(\mathfrak{v}) = \mathfrak{u}$, but $e(i) \notin \omega \cup \{\mathfrak{u}\}$ for all $i \in \omega$.

(iiia)⇔(iiib). Likewise. □

Remark 1. Moreover, each of (i)-(iiib) is equivalent to the two following:

- (iva) for all $g : \omega^2 \to \omega$ and $\mathfrak{v} \in \boldsymbol{\beta}\omega$ such that $\mathfrak{u} = \widetilde{g}(\mathfrak{v}, \mathfrak{v})$ there exists $i \in \omega$ satisfying $\widetilde{g}(i, \mathfrak{v}) = \mathfrak{u}$ or $\widetilde{g}(i, \mathfrak{v}) \in \omega$;
- (ivb) for all $g : \omega^2 \to \omega$ and $\mathfrak{v} \in \boldsymbol{\beta}\omega$ such that $\mathfrak{u} = \widetilde{g}(\mathfrak{v}, \mathfrak{v})$ one of two sets $\{i \in \omega : g(i, \mathfrak{v}) = \mathfrak{u}\}$ and $\{i \in \omega : g(i, \mathfrak{v}) \in \omega\}$ belongs to $\mathfrak{v}$.

We prove this fact elsewhere; here it will not be used.

Remark 2. The results of this section can be generalized to larger cardinals (but still in $\boldsymbol{\beta}\omega$).

Lemma 1 is generalized as follows: given $\kappa \leq 2^\omega$, a map $f : \boldsymbol{\beta}\kappa \to \boldsymbol{\beta}\omega$ is continuous iff there exist $g : \kappa \times \omega \to \omega$ and $\mathfrak{v} \in \boldsymbol{\beta}\omega$ such that

$$f(\mathfrak{u}) = \widetilde{g}(\mathfrak{u}, \mathfrak{v})$$

for all $\mathfrak{u} \in \boldsymbol{\beta}\kappa$. Indeed, since $f[\kappa]$, the image of κ under f, is of cardinality $\leq \kappa \leq \mathfrak{c}$, it has an $\leq_{\mathrm{RK}}$-upper bound, so we can apply the same argument as in the proof of Lemma 1.

Lemma 3 is also easily extended to uncountable sets of ultrafilters: if $A \subseteq \boldsymbol{\beta}\omega$ is of cardinality κ, then the following are equivalent:

- (i) $\mathfrak{u} \in \mathrm{cl}_{\boldsymbol{\beta}\omega} A$;
- (ii) $\mathfrak{u} = \widetilde{f}(\mathfrak{v})$ for some $\mathfrak{v} \in \boldsymbol{\beta}\kappa$ whenever $f : \kappa \to \boldsymbol{\beta}\omega$ satisfies $\operatorname{ran} f \subseteq A$;
- (iii) $\mathfrak{u} = f(\mathfrak{v})$ for some $\mathfrak{v} \in \boldsymbol{\beta}\kappa$ whenever a continuous $f : \boldsymbol{\beta}\kappa \to \boldsymbol{\beta}\omega$ satisfies $f[\kappa] \subseteq A$;

and if, moreover, $\kappa \leq 2^\omega$, then each of (i)-(iii) is equivalent to the following:

- (iv) $\mathfrak{u} = \widetilde{g}(\mathfrak{v}, \mathfrak{w})$ for some $\mathfrak{v} \in \boldsymbol{\beta}\kappa$ and $\mathfrak{w} \in \boldsymbol{\beta}\omega$ whenever $g : \kappa \times \omega \to \omega$ satisfies $\{\widetilde{g}(\alpha, \mathfrak{w}) : \alpha \in \kappa\} \subseteq A$.

Finally, using the latter, we can generalize Theorem 1 to characterize weak P_κ-points: for every $\kappa < 2^\omega$ and $\mathfrak{u} \in \boldsymbol{\beta}\omega \setminus \omega$ the following are equivalent:

- (i) $\mathfrak{u}$ is a weak P_{κ^+}-point;
- (iia) for all $f : \kappa \to \boldsymbol{\beta}\omega$ and $\mathfrak{v} \in \boldsymbol{\beta}\kappa$ such that $\mathfrak{u} = \widetilde{f}(\mathfrak{v})$ there exists $\alpha \in \kappa$ satisfying $f(\alpha) = \mathfrak{u}$ or $f(\alpha) \in \omega$;
- (iib) for all $f : \kappa \to \boldsymbol{\beta}\omega$ and $\mathfrak{v} \in \boldsymbol{\beta}\kappa$ such that $\mathfrak{u} = \widetilde{f}(\mathfrak{v})$ one of two sets $\{\alpha \in \kappa : f(\alpha) = \mathfrak{u}\}$ and $\{\alpha \in \kappa : f(\alpha) \in \omega\}$ belongs to $\mathfrak{v}$;
- (iiia) for all $g : \kappa \times \omega \to \omega$ and $\mathfrak{v} \in \boldsymbol{\beta}\kappa$, $\mathfrak{w} \in \boldsymbol{\beta}\omega$ such that $\mathfrak{u} = \widetilde{g}(\mathfrak{v}, \mathfrak{w})$ there exists $\alpha \in \kappa$ satisfying $\widetilde{g}(\alpha, \mathfrak{w}) = \mathfrak{u}$ or $\widetilde{g}(\alpha, \mathfrak{w}) \in \omega$;
- (iiib) for all $g : \kappa \times \omega \to \omega$ and $\mathfrak{v} \in \boldsymbol{\beta}\kappa$, $\mathfrak{w} \in \boldsymbol{\beta}\omega$ such that $\mathfrak{u} = \widetilde{g}(\mathfrak{v}, \mathfrak{w})$ one of two sets $\{\alpha \in \kappa : g(\alpha, \mathfrak{w}) = \mathfrak{u}\}$ and $\{\alpha \in \kappa : g(\alpha, \mathfrak{w}) \in \omega\}$ belongs to $\mathfrak{v}$.

Since we will not use these generalizations to obtain our main results, we leave it to the reader to write out simple modifications of the previous proofs.

4. Embedding of $(\mathscr{P}_\omega(2^{\mathfrak{c}}), \subseteq)$

In this section, we prove the first of two our main results, Theorem 3, which isomorphically embeds the lattice of finite subsets of a set of cardinality $2^{\mathfrak{c}}$ ordered by inclusion in $\boldsymbol{\beta}\omega$ with the Rudin–Keisler preorder. Clearly, this solves Problem 61 of [23] in the affirmative way. Moreover, our construction gives an embedding in the Comfort preorder as well, and even in any relation between the two preorders.

We start with the following result, which shows that weak P-points lying $\le_{\mathrm{C}}$-below products of ultrafilters distribute $\le_{\mathrm{RK}}$-below some of their factors; this is vaguely reminiscent of Blass' result on ultrafilters below products of $\le_{\mathrm{RK}}$-minimal ultrafilters (see [2], Theorem 21.1); however, our result is obtained in ZFC alone.

Theorem 2. *Let $\mathfrak{u} \in \boldsymbol{\beta}\omega$ be a weak P-point, $\mathfrak{v}_0, \ldots, \mathfrak{v}_n \in \boldsymbol{\beta}\omega$ arbitrary ultrafiltres. If $\mathfrak{u} \le_{\mathrm{C}} \mathfrak{v}_0 \otimes \ldots \otimes \mathfrak{v}_n$, then $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_k$ for some $0 \le k \le n$.*

Proof. Let us first recall that, whenever $\mathfrak{u}$ is a weak P-point and $\mathfrak{v}$ an arbitrary ultrafilter, then $\mathfrak{u} \le_{\mathrm{C}} \mathfrak{v}$ implies $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}$ (see [19], Theorem 2.10), so we can assume $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_0 \otimes \ldots \otimes \mathfrak{v}_n$. Now we prove the statement by induction on n.

For $n = 0$, the assertion holds trivially. Let $n = 1$, so $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_0 \otimes \mathfrak{v}_1$, and let $g : \omega^2 \to \omega$ be such that $\mathfrak{u} = \widetilde{g}(\mathfrak{v}_0 \otimes \mathfrak{v}_1) = \widetilde{g}(\mathfrak{v}_0, \mathfrak{v}_1)$, where the map g on the left is treated as unary and on the right as binary. Then, by Theorem 1, either the set $A := \{i \in \omega : \widetilde{g}(i, \mathfrak{v}_1) = \mathfrak{u}\}$ or the set $B := \{i \in \omega : \widetilde{g}(i, \mathfrak{v}_1) \in \omega\}$ belongs to $\mathfrak{v}_0$. Let us consider both cases separately.

If $A \in \mathfrak{v}_0$, pick any $i \in A$. Then $\widetilde{g}(i \otimes \mathfrak{v}_1) = \widetilde{g}(i, \mathfrak{v}_1) = \mathfrak{u}$ (where again the map g on the left is treated as unary and on the right as binary), and so, $\mathfrak{u} \le_{\mathrm{RK}} i \otimes \mathfrak{v}_1$. Since $i \otimes \mathfrak{v}_1$ is isomorphic to $\mathfrak{v}_1$, we get $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_1$.

If $B \in \mathfrak{v}_0$, define $f : \omega \to \omega$ by letting

$$f(i) := \begin{cases} \widetilde{g}(i, \mathfrak{v}_1) & \text{if } i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\widetilde{g}$ is right-continuous, the map h defined by letting $h(\mathfrak{v}) := \widetilde{g}(\mathfrak{v}, \mathfrak{v}_1)$ is continuous, hence, coincides with the map $\widetilde{f}$ (since they coincide on all principal ultrafilters). So we get $\widetilde{f}(\mathfrak{v}_0) = h(\mathfrak{v}_0) = \widetilde{g}(\mathfrak{v}_0, \mathfrak{v}_1) = \mathfrak{u}$, and, therefore, $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_0$.

Thus, for $n = 1$, the assertion also holds. Further we argue by induction. Assume that holds for n. Let $\mathfrak{v} := \mathfrak{v}_0 \otimes \ldots \otimes \mathfrak{v}_n$, and assume $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_0 \otimes \ldots \otimes \mathfrak{v}_n \otimes \mathfrak{v}_{n+1} \cong \mathfrak{v} \otimes \mathfrak{v}_{n+1}$. Then either $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}$, and so, $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_k$ for some $0 \le k \le n$ by the induction hypothesis, or $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_{n+1}$.

The proof is complete. □

Remark 3. Theorem 2 admits some generalizations.

A trivial generalization is obtained by replacing the ultrafilter multiplication by arbitrary operation: let $\mathfrak{u} \in \boldsymbol{\beta}\omega$ be a weak P-point, then for any $e : \omega^{n+1} \to \omega$ and arbitrary $\mathfrak{v}_0, \ldots, \mathfrak{v}_n \in \boldsymbol{\beta}\omega$, $\mathfrak{u} \le_{\mathrm{RK}} \widetilde{e}(\mathfrak{v}_0, \ldots, \mathfrak{v}_n)$ implies $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_k$ for some $k \in \omega$ (this is evident from $\widetilde{e}(\mathfrak{v}_0, \ldots, \mathfrak{v}_n) \le_{\mathrm{RK}} \mathfrak{v}_0 \otimes \ldots \otimes \mathfrak{v}_n$).

Also Theorem 2 can be generalized to weak P_κ-points as follows. Suppose $\kappa < 2^\omega$, $\mathfrak{u} \in \boldsymbol{\beta}\omega$ is a weak P_{κ^+}-point, and $\mathfrak{v}_0 \in \boldsymbol{\beta}\kappa$ and $\mathfrak{v}_1, \ldots, \mathfrak{v}_n \in \boldsymbol{\beta}\omega$ arbitrary. If $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_0 \otimes \mathfrak{v}_1 \otimes \ldots \otimes \mathfrak{v}_n$, then $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}_k$ for some $0 \le k \le n$. To prove this, use the characterization of weak P_{κ^+}-points mentioned in Remark 2 and modify the proof above.

Theorem 3 (The First Main Theorem)**.** *There exists a map $e : \mathscr{P}_\omega(\boldsymbol{\beta}\omega) \to \boldsymbol{\beta}\omega$ such that for all $X, Y \in \mathscr{P}_\omega(\boldsymbol{\beta}\omega)$,*

(i) *if $X \subseteq Y$ then $e(X) \le_{\mathrm{RK}} e(Y)$;*
(ii) *if $e(X) \le_{\mathrm{C}} e(Y)$ then $X \subseteq Y$.*

Consequently, e is an isomorphic embedding of the lattice $(\mathscr{P}_\omega(\boldsymbol{\beta}\omega), \subseteq)$ into $(\boldsymbol{\beta}\omega, R)$ whenever $\le_{\mathrm{RK}} \subseteq R \subseteq \le_{\mathrm{C}}$.

Proof. By [49], there exists $A \subseteq \boldsymbol{\beta}\omega$ consisting of weak P-points with the $|A| = |\boldsymbol{\beta}\omega|$. Fix such an A and enumerate its points by $\mathfrak{u}_\alpha$, $\alpha < 2^{\mathfrak{c}}$. We shall construct a required map e whose domain is rather $\mathscr{P}_\omega(A)$

than $\mathscr{P}_\omega(\boldsymbol{\beta}\omega)$. For $X \in \mathscr{P}_\omega(A)$, if $X = \{\mathfrak{u}_{\alpha_0}, \ldots, \mathfrak{u}_{\alpha_n}\}$ and $\alpha_0 < \ldots < \alpha_n < 2^{\mathfrak{c}}$, let

$$e(X) := \mathfrak{u}_{\alpha_0} \otimes \ldots \otimes \mathfrak{u}_{\alpha_n}\,.$$

Let us verify that, for all $X, Y \in \mathscr{P}_\omega(A)$, both (i) and (ii) hold. Let $X := \{\mathfrak{u}_{\alpha_0}, \ldots, \mathfrak{u}_{\alpha_m}\}$, $\alpha_0 < \ldots < \alpha_m < 2^{\mathfrak{c}}$, and $Y := \{\mathfrak{u}_{\beta_0}, \ldots, \mathfrak{u}_{\beta_n}\}$, $\beta_0 < \ldots < \beta_n < 2^{\mathfrak{c}}$.

(i). Clear.

(ii). Assume $e(X) \le_{\mathrm{C}} e(Y)$. A fortiori, we have $\mathfrak{u}_{\alpha_i} \le_{\mathrm{C}} e(Y)$ for all $i \le m$. Therefore, by Theorem 2, for every $i \le m$ there exists $j \le n$ such that $\mathfrak{u}_{\alpha_i} \le_{\mathrm{RK}} \mathfrak{u}_{\beta_j}$. Since A consists of pairwise $\le_{\mathrm{RK}}$-incomparable ultrafilters, we conclude that $\mathfrak{u}_{\alpha_i} = \mathfrak{u}_{\beta_j}$, i.e., that $X \subseteq Y$.

The theorem is proved. □

As an easy consequence, we conclude that $\boldsymbol{\beta}\omega$ endowed with the Rudin–Keisler order, and in fact with any relation between it and the Comfort order, is *universal* for partially ordered sets of a certain class, i.e., contains isomorphic copies of every its elements.

Let us say that a partially ordered set $(X, \le)$ has the *local cardinality* $< \kappa$ iff, for all $x \in X$, the lower cone $\{y \in X : y \le x\}$ has the cardinality $< \kappa$. The expressions like *local cardinality* $\le \kappa$, or *locally finite*, etc., have the expected meaning.

Lemma 4. *Let $\kappa \le \lambda^+$. Then $(\mathscr{P}_\kappa(\lambda), \subseteq)$ is universal for partially ordered sets of cardinality $\le \lambda$ and local cardinality $< \kappa$.*

Proof. A standard argument: if $(X, \le)$ has the local cardinality $< \kappa$, the map $f : X \to \mathscr{P}_\kappa(X)$ defined by letting $f(x) := \{y \in X : y \le x\}$ for all $x \in X$, isomorphically embeds $(X, \le)$ into $(\mathscr{P}_\kappa(X), \subseteq)$. □

Note that, if a partially ordered set $(P, \le)$ is embeddable in $(\boldsymbol{\beta}\omega, R)$, where $R \subseteq {\le_{\mathrm{C}}}$, then it has the cardinality $\le 2^{\mathfrak{c}}$ and local cardinality $\le \mathfrak{c}$, therefore, by Lemma 4, it is embeddable in the lattice $(\mathscr{P}_{\mathfrak{c}^+}(2^{\mathfrak{c}}), \subseteq)$.

Corollary 1. *Let ${\le_{\mathrm{RK}}} \subseteq R \subseteq {\le_{\mathrm{C}}}$. Then $(\boldsymbol{\beta}\omega, R)$ is universal for partially ordered sets that are locally finite and have the cardinality not greater than $2^{\mathfrak{c}}$.*

Proof. Lemma 4 and Theorem 3. □

Remark 4. A variant of this argument shows that there exists an isomorphic embedding e of the tree $((\boldsymbol{\beta}\omega)^{<\omega}, \subseteq)$ into $(\boldsymbol{\beta}\omega, \le_{\mathrm{RF}})$. (Here $X^{<\alpha} := \bigcup_{\beta<\alpha} X^\beta$, so for $s, t \in X^{<\alpha}$, $s \subseteq t$ iff s end-extends t.) Indeed, if A consists of $2^{\mathfrak{c}}$ pairwise $\le_{\mathrm{RF}}$-incomparable ultrafilters (even not necessarily weak P-points), then letting for all $s \in A^{<\omega}$, $e(s) := \mathfrak{u}_{s(0)} \otimes \ldots \otimes \mathfrak{u}_{s(n)}$ where $n+1 := \operatorname{dom} s$, we see that $s \subseteq t$ implies $e(s) \le_{\mathrm{RF}} e(t)$, and the converse implication holds since $\le_{\mathrm{RF}}$ is tree-like. Consequently, $(\boldsymbol{\beta}\omega, \le_{\mathrm{RF}})$ is universal for locally finite trees of cardinality $\le 2^{\mathfrak{c}}$.

5. Embedding of $(\mathscr{P}_{\omega_1}(\omega_1), \subseteq)$

In this section, we prove our second main result, Theorem 4, which embeds the lattice of countable subsets of ω_1 ordered by inclusion in $\boldsymbol{\beta}\omega$ with the Rudin–Keisler preorder and, in fact, with any relation between it and the Comfort preorder. In some ways, the reasoning generalizes those from the previous section.

Let $\mathrm{Lim}(\omega_1)$ denote the set of nonzero limit ordinals $\alpha < \omega_1$. Pick a so-called *ladder system* for $\mathrm{Lim}(\omega_1)$, in which thus for each $\alpha \in \mathrm{Lim}(\omega_1)$ there is fixed a certain ω-sequence $(\alpha_i)_{i\in\omega}$ that is increasing and cofinal in α. Let $\mathbf{u} := (\mathfrak{u}_\alpha)_{\alpha<\omega_1}$ be an ω_1-sequence of ultrafilters in $\boldsymbol{\beta}\omega$. For each $\alpha < \omega_1$ define a map $f_{\mathbf{u},\alpha} : \omega \to \boldsymbol{\beta}\omega$ and the ultrafilter

$$\mathfrak{v}_{\mathbf{u},\alpha} := \widetilde{f_{\mathbf{u},\alpha}}(\mathfrak{u}_\alpha)$$

by recursion on α as follows:

(i) $f_{\mathbf{u},0}$ is any injective map with the values in ω, so $\mathfrak{v}_{\mathbf{u},0} \cong \mathfrak{u}_0$ (it suffices to let $f_{\mathbf{u},0} := \mathrm{id}_\omega$, so $\mathfrak{v}_{\mathbf{u},0} = \mathfrak{u}_0$);

(ii) if $\alpha = \beta+1$, then $f_{\mathbf{u},\alpha}$ is any injective map of ω into $\boldsymbol{\beta}\omega$ with a discrete range satisfying $f_{\mathbf{u},\alpha}(i) \cong \mathfrak{v}_{\mathbf{u},\beta}$ for all $i < \omega$;

(iii) if α is a limit ordinal, then $f_{\mathbf{u},\alpha}$ is any injective map of ω into $\boldsymbol{\beta}\omega$ with a discrete range satisfying $f_{\mathbf{u},\alpha}(i) \cong \mathfrak{v}_{\mathbf{u},\alpha_i}$ for all $i < \omega$.

(Let us emphasize that, although a choice of maps $f_{\mathfrak{u},\alpha}$ satisfying the specified properties can be arbitrary for each sequence $\mathfrak{u}$, our ladder system is fixed in advance for all such functions and sequences.) It is easy to see that for every $\beta < \omega_1$ we have:

$$\mathfrak{v}_{\mathfrak{u},\beta+1} \cong \mathfrak{u}_{\beta+1} \otimes \mathfrak{v}_{\mathfrak{u},\beta} \,. \tag{$*$}$$

In particular, if $n < \omega$, we have $\mathfrak{v}_{\mathfrak{u},n+1} \cong \mathfrak{u}_{n+1} \otimes \ldots \otimes \mathfrak{u}_0$.

Lemma 5. *Let $\mathfrak{u} := (\mathfrak{u}_\alpha)_{\alpha<\omega_1}$ be an ω_1-sequence of ultrafilters in $\boldsymbol{\beta}\omega$ where all the ultrafilters $\mathfrak{u}_\alpha$ with limit indices $\alpha < \omega_1$ are nonprincipal. Then for every $\beta \leq \alpha < \omega_1$ we have:*

$$\mathfrak{u}_\beta \leq_{\mathrm{RK}} \mathfrak{v}_{\mathfrak{u},\alpha} \,.$$

Proof. By induction on α. The induction basis ($\alpha = 0$) is obvious.

Let $\alpha > 0$ and assume the induction hypothesis has already been proved for all $\beta < \alpha$. If $\alpha = \beta + 1$, use ($*$) and the induction hypothesis. Assume now α is a limit ordinal and $\beta < \alpha$. Then there exists an index $i_0 \in \omega$ such that $\beta \leq \alpha_j$ for all j, $i_0 \leq j < \omega$. By the induction hypothesis, for all j, $i_0 \leq j < \omega$, we have:

$$\mathfrak{u}_\beta \leq_{\mathrm{RK}} \mathfrak{v}_{\mathfrak{u},\alpha_j} = f_{\mathfrak{u},\alpha}(j) \,.$$

Since the ultrafilter $\mathfrak{u}_\alpha$ is nonprincipal, we have:

$$\mathfrak{u}_\beta \leq_{\mathrm{RK}} \widetilde{f_{\mathfrak{u},\alpha}}(\mathfrak{u}_\alpha) = \mathfrak{v}_{\mathfrak{u},\alpha} \,.$$

Assume, finally, $\alpha = \beta$ is a limit ordinal. Then for all $i \in \omega$ we have:

$$\mathrm{id}_\omega(i) \leq_{\mathrm{RK}} f_{\mathfrak{u},\alpha}(i) \,,$$

whence (since $\widetilde{\mathrm{id}_\omega} = \mathrm{id}_{\boldsymbol{\beta}\omega}$) we get

$$\mathfrak{u}_\alpha = \widetilde{\mathrm{id}_\omega}(\mathfrak{u}_\alpha) \leq_{\mathrm{RK}} \widetilde{f_{\mathfrak{u},\alpha}}(\mathfrak{u}_\alpha) = \mathfrak{v}_{\mathfrak{u},\alpha} \,.$$

The proof is complete. $\square$

Given an ω_1-sequence $\mathfrak{u} := (\mathfrak{u}_\alpha)_{\alpha<\omega_1}$ of ultrafilters in $\boldsymbol{\beta}\omega$ and $X \subseteq \omega_1$, let $\mathfrak{u}^X := (\mathfrak{u}^X_\alpha)_{\alpha<\omega_1}$ denote its X-*modification* defined by letting

$$\mathfrak{u}^X_\alpha := \begin{cases} \mathfrak{u}_\alpha & \text{if } \alpha \in X \cup \mathrm{Lim}(\omega_1), \\ 0 & \text{otherwise.} \end{cases}$$

For better readability of the notation, let henceforth

$$f_{X,\mathfrak{u},\alpha} := f_{\mathfrak{u}^X,\alpha} \quad \text{and} \quad \mathfrak{v}_{X,\mathfrak{u},\alpha} := \mathfrak{v}_{\mathfrak{u}^X,\alpha} \,.$$

Lemma 6. *Let $\mathfrak{u} := (\mathfrak{u}_\alpha)_{\alpha<\omega_1}$ be an ω_1-sequence of ultrafilters in $\boldsymbol{\beta}\omega \setminus \omega$. Then, for all $\beta \leq \alpha < \omega_1$ and $X, Y \subseteq \omega_1$,*

$$X \subseteq Y \quad \textit{implies} \quad \mathfrak{v}_{X,\mathfrak{u},\beta} \leq_{\mathrm{RK}} \mathfrak{v}_{Y,\mathfrak{u},\alpha} \,.$$

Proof. By induction of α. For the induction basis, let $\alpha = 0$ and $X \subseteq Y \subseteq \omega_1$. Then also $\beta = 0$. If $0 \in X$, then $\mathfrak{v}_{X,\mathfrak{u},0} \cong \mathfrak{u}_0 \cong \mathfrak{v}_{Y,\mathfrak{u},0}$, so the required statement holds. If $0 \notin X$, then $\mathfrak{v}_{X,\mathfrak{u},0} \in \omega$, and the required statement holds as well.

Let now $\alpha > 0$, assume the induction hypothesis has already been proved for all $\beta < \alpha$, and let $X \subseteq Y \subseteq \omega_1$. If $\alpha = \gamma + 1$, then by ($*$) we have:

$$\mathfrak{v}_{Y,\mathfrak{u},\alpha} \cong \mathfrak{u}^Y_\alpha \otimes \mathfrak{v}_{Y,\mathfrak{u},\gamma} \,.$$

Since $\beta \leq \gamma$, by the induction hypothesis we have:

$$\mathfrak{v}_{X,\mathfrak{u},\beta} \leq_{\mathrm{RK}} \mathfrak{v}_{Y,\mathfrak{u},\gamma} \leq_{\mathrm{RK}} \mathfrak{u}^Y_\alpha \otimes \mathfrak{v}_{Y,\mathfrak{u},\gamma} \cong \mathfrak{v}_{Y,\mathfrak{u},\alpha} \,.$$

Let $\alpha > 0$ be a limit ordinal. If $\beta < \alpha$, then there exists $i_0 \in \omega$ such that $\beta \leq \alpha_j$ for all j, $i_0 \leq j < \omega$. By the induction hypothesis, for all j, $i_0 \leq j < \omega$, we have:

$$\mathfrak{v}_{X,\mathfrak{u},\beta} \leq_{\mathrm{RK}} \mathfrak{v}_{Y,\mathfrak{u},\alpha_j} = f_{Y,\mathfrak{u},\alpha}(j) \,.$$

Since the ultrafilter $\mathfrak{u}_\alpha$ is nonprincipal, we have:

$$\mathfrak{v}_{X,\mathbf{u},\beta} \le_{\mathrm{RK}} \widetilde{f_{Y,\mathbf{u},\alpha}}(\mathfrak{u}_\alpha) = \mathfrak{v}_{Y,\mathbf{u},\alpha}\,.$$

And if $\alpha = \beta$, then for each $i < \omega$ by the induction hypothesis we have:

$$f_{X,\mathbf{u},\alpha}(i) = \mathfrak{v}_{X,\mathbf{u},\alpha_i} \le_{\mathrm{RK}} \mathfrak{v}_{Y,\mathbf{u},\alpha_i} = f_{Y,\mathbf{u},\alpha}(i)\,.$$

It follows

$$\mathfrak{v}_{X,\mathbf{u},\alpha} = \widetilde{f_{X,\mathbf{u},\alpha}}(\mathfrak{u}_\alpha) \le \widetilde{f_{Y,\mathbf{u},\alpha}}(\mathfrak{u}_\alpha) = \mathfrak{v}_{Y,\mathbf{u},\alpha}\,.$$

The proof is complete. □

The following lemma can be considered as a generalization of Theorem 2.

Lemma 7. *Let $\mathbf{u} := (\mathfrak{u}_\alpha)_{\alpha<\omega_1}$ be an ω_1-sequence of ultrafilters in $\boldsymbol{\beta}\omega \setminus \omega$. Then for every $\alpha < \omega_1$, $X \subseteq \omega_1$, and weak P-point $\mathfrak{w}$,*

$$\mathfrak{w} \le_{\mathrm{C}} \mathfrak{v}_{X,\mathbf{u},\alpha} \ \textit{ implies } \ \mathfrak{w} \le_{\mathrm{RK}} \mathfrak{u}_\beta$$

for some $\beta \in (X \cup \mathrm{Lim}(\omega_1)) \cap (\alpha + 1)$.

Proof. As in Theorem 2, let us start by recalling that, whenever $\mathfrak{w}$ is a weak P-point and $\mathfrak{v}$ an arbitrary ultrafilter, then $\mathfrak{w} \le_{\mathrm{C}} \mathfrak{v}$ implies $\mathfrak{w} \le_{\mathrm{RK}} \mathfrak{v}$ (see [19], Theorem 2.10). Now prove the lemma by induction on α.

The induction basis ($\alpha = 0$) is evident. Let $\alpha > 0$, assume the induction hypothesis holds for all $\beta < \alpha$, and $\mathfrak{w} \le_{\mathrm{RK}} \mathfrak{v}_{X,\mathbf{u},\alpha} = \widetilde{f_{X,\mathbf{u},\alpha}}(\mathfrak{u}^X_\alpha)$, i.e., for some $g : \omega \to \omega$ we have

$$\mathfrak{w} = \widetilde{g}(\widetilde{f_{X,\mathbf{u},\alpha}}(\mathfrak{u}^X_\alpha)) = \widetilde{g \circ f_{X,\mathbf{u},\alpha}}(\mathfrak{u}^X_\alpha).$$

Since $\mathfrak{w}$ is a weak P-point, it follows from Theorem 1(iib) that one of two sets $A := \{i \in \omega : g \circ f_{X,\mathbf{u},\alpha}(i) = \mathfrak{w}\}$ and $B := \{i \in \omega : g \circ f_{X,\mathbf{u},\alpha}(i) \in \omega\}$ belongs to $\mathfrak{u}^X_\alpha$.

If $A \in \mathfrak{u}^X_\alpha$, pick any $i \in A$. We have

$$\mathfrak{w} = \widetilde{g \circ f_{X,\mathbf{u},\alpha}}(i) = \widetilde{g}(\widetilde{f_{X,\mathbf{u},\alpha}}(i)),$$

where the ultrafilter $\widetilde{f_{X,\mathbf{u},\alpha}}(i)$ is $\mathfrak{v}_{X,\mathbf{u},\beta}$ for some $\beta < \alpha$. Thus, for some $\beta < \alpha$, we have

$$\mathfrak{w} \le_{\mathrm{RK}} \mathfrak{v}_{X,\mathbf{u},\beta}\,,$$

and it remains to use the induction hypothesis.

If $B \in \mathfrak{u}^X_\alpha$, so for all $i \in B$ we have

$$\widetilde{g \circ f_{X,\mathbf{u},\alpha}}(i) = \widetilde{g}(\widetilde{f_{X,\mathbf{u},\alpha}}(i)) \in \omega,$$

define a map $h : \omega \to \omega$ by letting

$$h(i) := \begin{cases} \widetilde{g}(\widetilde{f_{X,\mathbf{u},\alpha}}(i)) & \text{if } i \in B,\\ 0 & \text{otherwise.}\end{cases}$$

The maps $\widetilde{h}$ and $\widetilde{g} \circ \widetilde{f}$ are continuous and coincide on the set $B \in \mathfrak{u}_\alpha$. It follows

$$\widetilde{h}(\mathfrak{u}^X_\alpha) = \widetilde{g}(\widetilde{f_{X,\mathbf{u},\alpha}}(\mathfrak{u}^X_\alpha)) = \mathfrak{w},$$

hence, $\mathfrak{w} \le_{\mathrm{RK}} \mathfrak{u}^X_\alpha$. In particular, it follows that the ultrafilter $\mathfrak{u}^X_\alpha$ is nonprincipal. Therefore, $\alpha \in \mathrm{Lim}(\omega_1)$ or $\mathfrak{u}^X_\alpha = \mathfrak{u}_\alpha$, which thus completes the proof. □

Lemma 8. *Let $\mathbf{u} := (\mathfrak{u}_\alpha)_{\alpha<\omega_1}$ be an ω_1-sequence of weak P-points in $\boldsymbol{\beta}\omega$. Then for all $\alpha, \beta < \omega_1$ and $X, Y \subseteq \omega_1$, if*

$$\mathfrak{v}_{X,\mathbf{u},\alpha} \le_{\mathrm{C}} \mathfrak{v}_{Y,\mathbf{u},\beta}\,,$$

then for each $\gamma \in X \cap (\alpha + 1)$ there exists $\delta \in (Y \cup \mathrm{Lim}(\omega_1)) \cap (\beta + 1)$ such that

$$\mathfrak{u}_\gamma \le_{\mathrm{RK}} \mathfrak{u}_\delta\,.$$

Proof. Note that $\mathfrak{u}^X_\alpha = \mathfrak{u}_\alpha$ for every limit ordinal α. Therefore, in $\mathbf{u}^X$, all ultrafilters $\mathfrak{u}^X_\alpha$ with limit indices α are nonprincipal, and thus we are under the assumption of Lemma 5. It follows that, for each $\gamma \in X \cap (\alpha+1)$, we have:

$$\mathfrak{u}_\gamma = \mathfrak{u}^X_\gamma \leq_{\mathrm{RK}} \mathfrak{v}_{X,\mathbf{u},\alpha} \leq_{\mathrm{C}} \mathfrak{v}_{Y,\mathbf{u},\alpha}$$

(where $\mathfrak{u}_\gamma = \mathfrak{u}^X_\gamma$ holds since $\gamma \in X$ and $\mathfrak{u}_\gamma \leq_{\mathrm{RK}} \mathfrak{v}_{X,\mathbf{u},\alpha}$ by Lemma 5), and so,

$$\mathfrak{u}_\gamma \leq_{\mathrm{C}} \mathfrak{v}_{Y,\mathbf{u},\alpha}\,.$$

But then, since $\mathfrak{u}_\gamma$ is a weak P-point, by using Lemma 7 we get $\mathfrak{u}_\gamma \leq_{\mathrm{RK}} \mathfrak{u}_\delta$ for some $\delta \in (Y \cup \mathrm{Lim}(\omega_1)) \cap (\beta+1)$, as required. ☐

Combining the preceding lemmas, we obtain the main result of this section:

Theorem 4 (The Second Main Theorem)**.** *There exists a map $e : \mathscr{P}_{\omega_1}(\omega_1) \to \beta\omega$ such that for all $X, Y \in \mathscr{P}_{\omega_1}(\omega_1)$,*

(i) *if $X \subseteq Y$ then $e(X) \leq_{\mathrm{RK}} e(Y)$;*
(ii) *if $e(X) \leq_{\mathrm{C}} e(Y)$ then $X \subseteq Y$.*

Consequently, e is an isomorphic embedding of the lattice $(\mathscr{P}_{\omega_1}(\omega_1), \subseteq)$ into $(\beta\omega, R)$ whenever $\leq_{\mathrm{RK}} \subseteq R \subseteq \leq_{\mathrm{C}}$.

Proof. Let $\mathbf{u} := (\mathfrak{u}_\alpha)_{\alpha<\omega_1}$ be an ω_1-sequence of pairwise $\leq_{\mathrm{RK}}$-incomparable weak P-points in $\beta\omega$, and let $A := \omega_1 \setminus \mathrm{Lim}(\omega_1)$. Since $|A| = \aleph_1$, it suffices to define a required map e whose domain is rather $\mathscr{P}_{\omega_1}(A)$ than $\mathscr{P}_{\omega_1}(\omega_1)$.

If $X \subseteq \omega_1$ is at most countable, we let $\sup^+ X := \min(\omega_1 \setminus X)$. Define e by letting, for all $X \in \mathscr{P}_{\omega_1}(A)$,

$$e(X) := \mathfrak{v}_{X,\mathbf{u},\sup^+ X}\,.$$

Let us verify that, for all $X, Y \in \mathscr{P}_{\omega_1}(A)$, both (i) and (ii) hold.

(i). If $X \subseteq Y$, then $\sup^+ X \subseteq \sup^+ Y$, whence it follows $e(X) \leq_{\mathrm{RK}} e(Y)$ by Lemma 6.

(ii). Assume $e(X) \leq_{\mathrm{C}} e(Y)$. Since $Z \subseteq \sup^+ Z$ for every at most countable $Z \subseteq \omega_1$, by Lemma 8, for each $\gamma \in X$ there exists $\delta \in Y \cup \mathrm{Lim}(\omega_1)$ such that $\mathfrak{u}_\gamma \leq_{\mathrm{RK}} \mathfrak{u}_\delta$. But the sequence $\mathbf{u}$ consists of pairwise $\leq_{\mathrm{RK}}$-incomparable ultrafilters, therefore, $\delta = \gamma$, and so $\gamma \in Y \cup \mathrm{Lim}(\omega_1)$. Since γ is a successor ordinal, we conclude that $\gamma \in Y$. Thus, finally, $X \subseteq Y$.

The theorem is proved. ☐

Corollary 2. *Let $\leq_{\mathrm{RK}} \subseteq R \subseteq \leq_{\mathrm{C}}$. Then $(\beta\omega, R)$ is universal for partially ordered sets that are locally at most countable and have the cardinality at most $\aleph_1$.*

Proof. By Lemma 4 and Theorem 4. ☐

Concluding remarks

An obvious necessary condition for an embeddability of a partially ordered set $(P, \leq)$ in $(\beta\omega, \leq_{\mathrm{RK}})$, or more generally, in $(\beta\omega, R)$ with $\leq_{\mathrm{RK}} \subseteq R \subseteq \leq_{\mathrm{C}}$, is that $(P, \leq)$ be of cardinality $\leq 2^{\mathfrak{c}}$ and local cardinality $\leq \mathfrak{c}$, i.e., that it is embeddable in the lattice $(\mathscr{P}_{\mathfrak{c}^+}(2^{\mathfrak{c}}), \subseteq)$. The second author is inclined to believe that this condition is also sufficient, i.e., $(\mathscr{P}_{\mathfrak{c}^+}(2^{\mathfrak{c}}), \subseteq)$ is embeddable in $(\beta\omega, R)$ too, and that this criterion is provable in ZFC alone. The results of this paper are not nearly as strong; we were able to prove in ZFC only that each of two lattices $(\mathscr{P}_\omega(2^{\mathfrak{c}}), \subseteq)$ and $(\mathscr{P}_{\omega_1}(\omega_1), \subseteq)$ is embeddable in $(\beta\omega, R)$. Minor improvements in these results can be made without much effort. E.g., using $\mathfrak{c}^+$-directedness of $\leq_{\mathrm{RK}}$, it is easy to construct by recursion on $\alpha < \mathfrak{c}^+$ an embedding of the lexicographic product of the ordinal $\mathfrak{c}^+$ and the lattice $(\mathscr{P}_{\omega_1}(\omega_1), \subseteq)$ in $(\beta\omega, \leq_{\mathrm{RK}})$ (the same fact could be stated for any $(\beta\omega, R)$ with $\leq_{\mathrm{RK}} \subseteq R \subseteq \leq_{\mathrm{C}}$ if we would know that $\leq_{\mathrm{RK}}$-above each ultrafilter there exist $2^{\mathfrak{c}}$ weak P-points). Perhaps, some modification of the method presented here might allow us to embed a lattice $(\mathscr{P}_{\omega_1}(2^{\mathfrak{c}}), \subseteq)$ in $(\beta\omega, \leq_{\mathrm{RK}})$. However, it seems that new ideas will be needed to move forward even further.

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