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Three-loop banana integrals with three equal masses

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Abstract: We obtain and solve the canonical differential equations for the three-loop banana integrals in dimensional regularisation when three of the four masses are equal. The K3 surface associated with the maximal cuts factorises into a product of two elliptic curves. This allows us to express the differential forms in the canonical differential equations in terms of meromorphic modular forms. We present a rigorous proof that these differential forms only have simple poles and that they define independent cohomology classes. We also present explicit results for all master integrals in terms of iterated integrals of meromorphic modular forms (and integrals thereof). This is the first time that it was possible to express the results of a Feynman integral associated with a K3 geometry and depending on two dimensionless ratios in terms of functions that have previously been studied in the literature.

Contents

1 Introduction

Over the last decade, a lot of progress has been made in analytically evaluating dimensionally-regulated Feynman integrals and in understanding the relevant class of transcendental functions. During this process it has become evident that these functions and their properties are tightly linked to periods of algebraic varieties [1, 2] and iterated integrals [3]. For example, multiple polylogarithms (MPLs) (cf., e.g., refs. [4–7]) are closely related to the geometry of the punctured Riemann sphere. In cases where the underlying geometry involves an elliptic curve, the natural class of functions are elliptic multiple polylogarithms (eMPLs) [8–13] and iterated integrals of modular forms [14–18]. Generalisations of polylogarithms to higher-genus Riemann surfaces have also recently been defined [19–23].

The aforementioned classes of special functions correspond to underlying geometries of complex dimension one. Also higher-dimensional complex manifolds can arise from Feynman integral computations. The most prominent examples are Calabi-Yau (CY) varieties (see for example ref. [24] for a review where they appear). In particular, the maximal cuts of L-loop banana integrals compute the periods of a family of CY $(L-1)$-folds [25–29].[1] However, the resulting class of iterated integrals is in general not understood at the same level

[1] See also refs. [30–33].

as for the aforementioned one-dimensional geometries. It is an interesting question if there are special mass configurations for which the banana integrals can be expressed in terms of transcendental functions that have already been introduced and studied in the mathematics literature. For example, the geometry underlying the two-loop sunrise integral is a family of elliptic curves, and the relevant class of functions can be identified with eMPLs and iterated integrals of modular forms [16, 34–36] (see also refs. [37–39]). At higher loops, this question is closely related to the situation when the underlying CY manifold reduces to a simpler geometry. This is particularly well understood for CY twofolds (also known as K3 surfaces) [40–47]. For example, it is well known that the maximal cuts of the three-loop equal-mass banana integral can be written as the symmetric square of the periods of the sunrise elliptic curve [25, 48, 49]. This result was used in refs. [18, 50, 51] to express the equal-mass three-loop banana integrals in terms of iterated integrals of (meromorphic) modular forms.

Recently, it was shown that also the periods of the K3 surface attached to the three-loop banana integral where three of the four masses are equal can be written as products of modular forms [52, 53]. The main goal of this paper is to present for the first time fully analytic results for these integrals in terms of iterated integrals of meromorphic modular forms (and integrals thereof). We use the method of differential equations [54–58], and our main strategy consists in transforming the differential equations into a canonical form [59]. While there is currently no general consensus what the correct definition of a canonical form is in the presence of general geometries, a lot of progress has recently been made in the case of (hyper)elliptic or CY manifolds [51, 60–67] (see also refs. [68, 69]). In particular, very recently a canonical basis for the three-loop banana integrals with four distinct masses was obtained [70, 71]. The solution to these canonical differential equations involves iterated integrals over kernels that contain the periods of the underlying family of K3 surfaces, and it is expected that these iterated integrals define new classes of transcendental functions. For the case of three equal masses, we can go further: by using the fact that the periods can be cast in the form of a product of modular forms, we can choose a path of integration on which all iterated integrals can be evaluated in terms of iterated integrals of modular forms (and integrals of the latter). This is the first time that it was possible to express the result of a Feynman integral involving a family of CY varieties depending on two (dimensionless) variables in terms of transcendental functions that had already been defined in the literature. We can then use known results for iterated integrals of modular forms to understand their properties. As an example, we show that all the differential forms that appear in the canonical differential equations are independent and only have simple pole, which are properties that canonical differential equations are conjectured to satisfy [64].

Our paper is organised as follows. In section 2 we review known results about (the maximal cuts of) the three-loop banana integrals with three or four equal masses. In section 3 we introduce the canonical differential equations for the three-loop banana integrals with three equal masses and we present the main results of our paper, namely the analytic results for the integrals in terms of iterated integrals of modular forms. In section 4 we draw our conclusions. We include appendices where we review modular forms and where we collect the explicit expressions for the differential forms that appear in the canonical differential equations. There, we also present some proofs omitted in the main text. Together with the arXiv submission, we provide ancillary files containing the canonical differential equations, the differential forms and the solution in terms of iterated integrals.

2 Three-loop banana integral with three equal masses

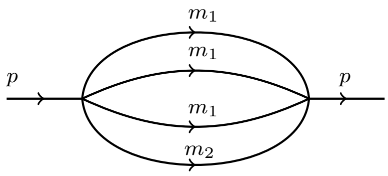

Figure 1. The three-loop banana graph with three equal masses.

In this paper we consider three-loop banana integrals depending on four non-zero propagator masses, with three of the four masses being equal (see figure 1). All integrals of this type are elements of the integral family defined by

$$\begin{aligned} &I_{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9} = \\ &= e^{3\gamma_E\varepsilon} \int \left(\prod_{j=1}^{3} \frac{\mathrm{d}^D k_j}{i\pi^{\frac{D}{2}}} \right) \frac{(k_3^2)^{-a_5}(k_1\cdot p)^{-a_6}(k_2\cdot p)^{-a_7}(k_3\cdot p)^{-a_8}(k_1\cdot k_2)^{-a_9}}{[k_1^2-m_1^2]^{a_1}[k_2^2-m_2^2]^{a_2}[(k_1-k_3)^2-m_1^2]^{a_3}[(k_2-k_3-p)^2-m_1^2]^{a_4}}\,. \end{aligned} \tag{2.1}$$

Here $\gamma_E = -\Gamma'(1)$ is the Euler-Mascheroni constant and we work in dimensional regularisation [72–74] in $D = 2 - 2\varepsilon$ dimensions, and the exponents a_i of the propagators are integers. It is easy to see that the only non-trivial functional dependence of the integrals can be through the ratios

$$z_i = \frac{m_i^2}{p^2}\,, \qquad i = 1, 2\,. \tag{2.2}$$

We therefore set $p^2 = 1$ without loss of generality.

Not all integrals from this family are independent, but they are related by integration-by-parts (IBP) relations [75, 76]. The IBP relations allow us to express every member of this integral family in terms of a finite number of so-called *master integrals*. We solve the IBP relations using `LiteRed2` [77], and we identify the following set of seven master integrals:

$$\begin{aligned} I_1 =& I_{0,1,1,1,0,0,0,0,0}\,, \\ I_2 =& I_{1,0,1,1,0,0,0,0,0}\,, \\ I_3 =& I_{1,1,1,1,0,0,0,0,0}\,, \\ I_4 =& I_{2,1,1,1,0,0,0,0,0}\,, \\ I_5 =& I_{1,2,1,1,0,0,0,0,0}\,, \\ I_6 =& I_{1,1,1,1,-1,0,0,0,0}\,, \\ I_7 =& I_{3,1,1,1,0,0,0,0,0}\,. \end{aligned} \tag{2.3}$$

The computation of any member of the family defined by eq. (2.1) is then reduced to the computation of the master integrals. The master integrals I_1 and I_2 are products of one-loop tadpole integrals, and we will not discuss them any further. The remaining master integrals I_k with $3 \leq k \leq 7$ are genuine three-loop integrals, and their computation is the main focus of this paper. In the following it will be useful to group the master integrals into a vector,

$$\boldsymbol{I} = \big(I_1, \ldots, I_7\big)^T . \tag{2.4}$$

In order to compute the master integrals, we use the method of differential equations [54–56, 58]. If we compute the derivatives of $\boldsymbol{I}$ with respect to z_i, we again obtain a linear combination of master integrals, i.e., we obtain linear systems of equations,

$$\partial_{z_i}\boldsymbol{I}(\boldsymbol{z},\varepsilon) = \boldsymbol{\Omega}_i(\boldsymbol{z},\varepsilon)\boldsymbol{I}(\boldsymbol{z},\varepsilon) , \qquad i = 1, 2 , \tag{2.5}$$

where $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{\Omega}_i(\boldsymbol{z},\varepsilon)$ are matrices of rational functions in $\boldsymbol{z}$ and ε. At this point we note that there is some arbitrariness in our choice of basis in eq. (2.3). Indeed, a judicious choice of master integrals may simplify finding a solution of the differential equations in eq. (2.5). If we define a new basis of master integrals by $\boldsymbol{M}(\boldsymbol{z},\varepsilon) = \boldsymbol{R}(\boldsymbol{z},\varepsilon)\boldsymbol{I}(\boldsymbol{z},\varepsilon)$, then the vector $\boldsymbol{M}(\boldsymbol{z},\varepsilon)$ satisfies differential equations where the matrices $\widetilde{\boldsymbol{\Omega}}_i(\boldsymbol{z},\varepsilon)$ are related to the $\boldsymbol{\Omega}_i(\boldsymbol{z},\varepsilon)$ by a gauge transformation,

$$\widetilde{\boldsymbol{\Omega}}_i = \big(\partial_{z_i}\boldsymbol{R} + \boldsymbol{R}\boldsymbol{\Omega}_i\big)\boldsymbol{R}^{-1} . \tag{2.6}$$

A particularly convenient choice is a so-called *canonical basis*. They were originally introduced in ref. [59] in the context of differential equations for Feynman integrals that evaluate to multiple polylogarithms [4], based on the concepts of uniform transcendental weight [78] and pure functions [79] from $\mathcal{N} = 4$ Super Yang-Mills theory. Here we use the definition of a canonical basis beyond multiple polylogarithms of refs. [51, 60–62, 64–67]. In particular, the method of refs. [60, 64] is based on an extension of the concepts of uniform transcendental weights and pure functions beyond polylogarithms introduced in ref. [80]. A main feature of a canonical basis is that it satisfies an ε-factorised differential equation,

$$\mathrm{d}\boldsymbol{M}(\boldsymbol{z},\varepsilon) = \varepsilon\,\boldsymbol{A}(\boldsymbol{z})\boldsymbol{M}(\boldsymbol{z},\varepsilon) , \tag{2.7}$$

where $\mathrm{d} = \mathrm{d}z_1\,\partial_{z_1} + \mathrm{d}z_2\,\partial_{z_2}$ is the total differential. The solution of an ε-factorised differential equation can be expressed as

$$\boldsymbol{M}(\boldsymbol{x},\varepsilon) = \mathbb{P}_\gamma(\boldsymbol{z},\varepsilon)\boldsymbol{M}_0(\varepsilon) , \tag{2.8}$$

where $\boldsymbol{M}_0(\varepsilon)$ denotes the value of $\boldsymbol{M}(\boldsymbol{z},\varepsilon)$ at some point $\boldsymbol{z} = \boldsymbol{z}_0$ and γ is a path from $\boldsymbol{z}_0$ to a generic point $\boldsymbol{z}$. We denote by $\mathbb{P}_\gamma(\boldsymbol{z},\varepsilon)$ the path-ordered exponential

$$\mathbb{P}_\gamma(\boldsymbol{z},\varepsilon) = \mathbb{P}\exp\left[\varepsilon\int_\gamma \boldsymbol{A}(\boldsymbol{z})\right] , \tag{2.9}$$

which can easily be expanded in ε up to any given order, and the coefficients of this expansion involve iterated integrals [3],

$$\mathbb{P}_\gamma(\boldsymbol{z},\varepsilon) = \mathbb{1} + \sum_{k=1}^{\infty}\varepsilon^k \sum_{1\leq i_1,\ldots,i_k\leq p} \boldsymbol{A}_{i_1}\cdots\boldsymbol{A}_{i_k} I_\gamma(\omega_{i_1},\ldots,\omega_{i_k}) , \tag{2.10}$$

with

$$\boldsymbol{A}(\boldsymbol{z}) = \sum_{i=1}^{p} \boldsymbol{A}_i \omega_i \,, \tag{2.11}$$

and where we defined the iterated integral

$$\begin{aligned} I_\gamma(\omega_{i_1}, \dots, \omega_{i_k}) &= \int_\gamma \omega_{i_1} \cdots \omega_{i_k} \\ &= \int_{0 \le \xi_k \le \cdots \le \xi_1 \le 1} \mathrm{d}\xi_1\, f_{i_1}(\xi_1)\, \mathrm{d}\xi_2\, f_{i_2}(\xi_2) \cdots \mathrm{d}\xi_k\, f_{i_k}(\xi_k) \,, \end{aligned} \tag{2.12}$$

with $\gamma^*\omega_{i_r} = \mathrm{d}\xi_r\, f_{i_r}(\xi_r)$ the pullback of ω_i to the path γ. The ω_i are called the *letters* of the iterated integrals, and the number k of letters is the *length*.

Canonical bases have been obtained for three-loop banana integrals for various mass assignments in refs. [51, 65, 70, 71]. While it is by now well established how to derive a canonical basis for banana integrals with three (and even more) loops, the required class of functions, in particular the iterated integrals that arise in the ε-expansion of the path-ordered exponential, is not always equally well understood. This can be traced back to the fact that in general L-loop banana integrals are associated with a family of Calabi-Yau (CY) $(L-1)$-folds [25–27, 29, 49]. When computed in $D = 2$ dimensions, the maximal cuts of the banana integrals, which are solutions to the associated homogeneous differential equations [81–83], compute the periods of these CY varieties (and their derivatives). The latter also enter the transformation to the canonical basis and the differential equation matrix. As a consequence, the letters ω_i do not only involve rational or algebraic functions of the kinematic variables $\boldsymbol{z}$, but also the periods of the CY $(L-1)$-fold. The periods are themselves computed as the solutions to a system of (partial) differential equations, and they typically define new classes of transcendental, multi-valued functions that have not been studied in the literature.

If we focus on the case of three-loop banana integrals, then there are notable exceptions. For example, in the equal-mass case, the associated CY twofold (also known as a K3 surface) is the symmetric square of an elliptic curve [25, 40–42]. The three periods of this K3 surface can be written as products of periods of the same family of elliptic curves that describe the two-loop equal-mass sunrise integral [48, 49, 84],

$$\begin{aligned} \Psi_1(t) &= \frac{4}{\pi \sqrt[4]{(t-9)(t-1)^3}}\, \mathrm{K}(\Lambda(t)) \,, \\ \Psi_2(t) &= \frac{4}{\pi \sqrt[4]{(t-9)(t-1)^3}}\, \mathrm{K}(1-\Lambda(t)) \,, \end{aligned} \tag{2.13}$$

with

$$\Lambda(t) = \frac{t^2 - 6t + \sqrt{(t-9)(t-1)^3} - 3}{2\sqrt{(t-9)(t-1)^3}} \,, \tag{2.14}$$

and we defined the complete elliptic integral of the first kind,

$$\mathrm{K}(\lambda) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\lambda t^2)}} \,. \tag{2.15}$$

This fact was used in refs. [18, 50, 51] to express the master integrals of this family in terms iterated integrals of meromorphic modular forms [14, 15, 17, 18] for the congruence subgroup $\Gamma_1(6)$ (and integrals thereof), defined by

$$\Gamma_1(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \mod N \text{ and } c \equiv 0 \mod N \right\} . \tag{2.16}$$

We define the elliptic modulus in the usual way as the ratio of the two periods,

$$\tau = i\,\frac{\Psi_2(t)}{\Psi_1(t)} . \tag{2.17}$$

The inverse is a Hauptmodul for $\Gamma_0(6)$ [85],

$$t(\tau) = 9\,\frac{\eta(6\tau)^8\,\eta(\tau)^4}{\eta(2\tau)^8\,\eta(3\tau)^4} , \tag{2.18}$$

with

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\} , \tag{2.19}$$

and $\eta(\tau)$ is the Dedekind eta function,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) , \qquad q = e^{2\pi i \tau} . \tag{2.20}$$

In our conventions the (holomorphic) period takes the form

$$\Pi_0^{\text{e.m.}}(\tau) = \frac{1}{12}\,(t(\tau) - 9)(t(\tau) - 1)\Psi_1(t(\tau))^2 , \tag{2.21}$$

where $\Psi_1(t(\tau))$ is an Eisenstein series of weight one for $\Gamma_1(6)$:

$$\Psi_1(t(\tau)) := \frac{2}{\sqrt{3}}\,\frac{\eta(3\tau)\,\eta(2\tau)^6}{\eta(\tau)^3\,\eta(6\tau)^2} .^{2} \tag{2.22}$$

As a consequence, it is possible to express the master integrals for the three-loop equal-mass banana family fully analytically in terms of a class of special functions that has been studied in the mathematics literature.

In the case of unequal (and non-zero) masses, however, the situation is very different: while it is by now clear from refs. [51, 60–62, 64, 65, 70, 71] how to find canonical bases for banana integrals, in general it is not known how to express the results in terms of a class of functions that is well studied in the literature. In ref. [52] it was shown that the periods associated with the families of K3 surfaces associated with three-loop banana integrals with non-zero propagator masses can always be identified with classes of generalised modular forms. Specifically, for the case of three equal masses studied here, the periods are products of periods of two different elliptic curves. This was made explicit in ref. [53], where concrete expressions for the periods in terms of complete elliptic integrals and modular forms were presented. More precisely, let us denote the vectors of periods of the family of K3 surfaces by

$$\mathbf{\Pi}(\boldsymbol{z}) = \left(\psi_0(\boldsymbol{z}), \psi_1^{(1)}(\boldsymbol{z}), \psi_1^{(2)}(\boldsymbol{z}), \psi_2(\boldsymbol{z})\right)^T . \tag{2.23}$$

[2] In the following, we will drop the τ dependence in $\Psi_1(t(\tau))$ for notational clarity.

It is known that there is a point of maximal unipotent monodromy (MUM) at $\boldsymbol{z}=0$ [27, 28], and we choose the entries of $\boldsymbol{\Pi}(\boldsymbol{z})$ such that $\psi_0(\boldsymbol{z})$ is holomorphic at $\boldsymbol{z}=0$ and the $\psi_i^{(1)}(\boldsymbol{z})$ diverge like a single power of a logarithm,

$$\begin{aligned}\psi_0(\boldsymbol{z}) &= 1+3\,z_1+z_2+12\,z_1\,z_2+15\,(z_1)^2+(z_2)^2+\mathcal{O}(z_i^3)\,,\\ \psi_1^{(1)}(\boldsymbol{z}) &= \log(z_1)\,\psi_0(\boldsymbol{z})+4\,z_1+2\,z_2+28\,z_1\,z_2+26\,(z_1)^2+3\,(z_2)^2+\mathcal{O}(z_i^3)\,,\\ \psi_1^{(2)}(\boldsymbol{z}) &= \log(z_2)\,\psi_0(\boldsymbol{z})+6\,z_1+45\,(z_1)^2+12\,z_1\,z_2+\mathcal{O}(z_i^3)\,.\end{aligned} \tag{2.24}$$

The remaining period is not independent from the first three, and it can be obtained from the quadratic relation

$$\boldsymbol{\Pi}(\boldsymbol{z})^T\boldsymbol{\Sigma}_{31}\boldsymbol{\Pi}(\boldsymbol{z})=0\,, \tag{2.25}$$

with the intersection pairing

$$\boldsymbol{\Sigma}_{31}=\begin{pmatrix}0&0&0&1\\0&-6&-3&0\\0&-3&0&0\\1&0&0&0\end{pmatrix}. \tag{2.26}$$

We also define the *canonical coordinates* close to $\boldsymbol{z}=0$,

$$\sigma_k(\boldsymbol{z})=\frac{\psi_1^{(k)}(\boldsymbol{z})}{\psi_0(\boldsymbol{z})}=\log z_k+\mathcal{O}(z_l)\,,\quad q_k(\boldsymbol{z})=e^{\sigma_k(\boldsymbol{z})}\,. \tag{2.27}$$

The inverse map, which expresses $\boldsymbol{z}$ as a function of the canonical coordinates $\boldsymbol{\sigma}=(\sigma_1,\sigma_2)$ is called the *mirror map*.

In ref. [53] it was shown that the periods of the K3 surface associated with banana integrals with three equal masses can be expressed as products of the periods of the elliptic curve of the equal-mass sunrise integral in eq. (2.13). Translated to our conventions, the result reads,

$$\begin{aligned}\psi_0(\boldsymbol{u}) &= -\frac{u_1\left(u_1^2+2\,u_1-3\right)\left(u_2^3-u_2^2-9\,u_2+9\right)}{12\,(u_1+1)\left(u_1^2-\left(u_2^2-3\right)u_1+u_2^2\right)}\Psi_1(u_1)\,\Psi_1(u_2)\,, \qquad (2.28)\\ \psi_1^{(1)}(\boldsymbol{u}) &= \frac{\pi\,u_1\left(u_1^2+2\,u_1-3\right)\left(u_2^3-u_2^2-9\,u_2+9\right)}{12\,(u_1+1)\left(u_1^2-\left(u_2^2-3\right)u_1+u_2^2\right)}\left[\Psi_2(u_1)\,\Psi_1(u_2)+i\Psi_1(u_1)\,\Psi_1(u_2)\right],\\ \psi_1^{(2)}(\boldsymbol{u}) &= \frac{\pi\,u_1\left(u_1^2+2\,u_1-3\right)\left(u_2^3-u_2^2-9\,u_2+9\right)}{12\,(u_1+1)\left(u_1^2-\left(u_2^2-3\right)u_1+u_2^2\right)}\times\\ &\qquad\times\left[i\Psi_1(u_1)\,\Psi_1(u_2)+2\Psi_1(u_1)\,\Psi_2(u_2)-\Psi_2(u_1)\,\Psi_1(u_2)\right].\end{aligned}$$

The variables $\boldsymbol{u}=(u_1,u_2)$ are related to $\boldsymbol{z}$ by

$$\begin{aligned}z_1(\boldsymbol{u}) &= \frac{\left(u_1^2-u_1\,u_2^2+3\,u_1+u_2^2\right)^2}{(u_1-1)\,u_1\,(u_1+3)(u_2-3)(u_2-1)(u_2+1)(u_2+3)}\,,\\ z_2(\boldsymbol{u}) &= \frac{(u_1-3)^2(u_1+1)^2\,u_2^2}{(u_1-1)\,u_1\,(u_1+3)(u_2-3)(u_2-1)(u_2+1)(u_2+3)}\,.\end{aligned} \tag{2.29}$$

In order to simplify the notations, we use the shorthands $\psi_0(\boldsymbol{u}):=\psi_0(\boldsymbol{z}(\boldsymbol{u}))$ and $\psi_1^{(k)}(\boldsymbol{u}):=\psi_1^{(k)}(\boldsymbol{z}(\boldsymbol{u}))$, and we also defined $\Psi_i(u_j):=\Psi_i(t_j(u_j))$, with

$$t_j(u_j)=\frac{u_j(3-u_j)}{1+u_j}\,. \tag{2.30}$$

Just like for the equal-mass two-loop sunrise and three-loop banana integrals, we then obtain a modular parametrisation for the mirror map and the holomorphic periods. For the mirror map, this is achieved by performing the following identification in eq. (2.29),

$$u_1 = u\left(\tfrac{\tau_1}{6}\right) \quad \text{and} \quad u_2 = u\left(\tau_2\right) \,, \tag{2.31}$$

where $u(\tau)$ is a Hauptmodul for the congruence subgroup $\Gamma_0(12)$:

$$u(\tau) = 3\frac{\eta(2\tau)^2\eta(12\tau)^4}{\eta(4\tau)^4\eta(6\tau)^2} \,, \tag{2.32}$$

and the elliptic moduli $\boldsymbol{\tau} = (\tau_1, \tau_2)$ are related to the canonical coordinates $\boldsymbol{\sigma}$ via[3]

$$\tau_1 = \sigma_1 \quad \text{and} \quad \tau_2 = 3\sigma_1 + 3\sigma_2 \,; \quad \tilde{q}_1 = e^{\tau_1/6} \quad \text{and} \quad \tilde{q}_2 = e^{\tau_2} \,. \tag{2.33}$$

Note that $u(\tau)$ is related to the Hauptmodul $t(\tau)$ for $\Gamma_0(6)$ in eq. (2.18) via the relation (cf. eq. (2.30)) [85],

$$t(\tau) = \frac{u(\tau)(3 - u(\tau))}{1 + u(\tau)}. \tag{2.34}$$

Likewise, the holomorphic period can be identified as a product of Eisenstein series upon inserting eqs. (2.21) and (2.31) into eq. (2.28). We thus see that the periods and the mirror map for the family of K3 surfaces associated with the three-loop banana integral with three equal masses admit a parametrisation in terms of modular forms and functions for the congruence subgroup $\Gamma_1(12)$. In the remainder of this paper we combine these results from refs. [52, 53] with the canonical form of the differential equation, to obtain an analytic result for all master integrals in terms of iterated integrals over modular forms, similar to the equal-mass case.

3 Analytic results for three-loop banana integrals with three equal masses

In the previous section we have reviewed modular parametrisations of the corresponding periods and mirror map. In this section we show how we can combine this result with the canonical differential equations to obtain full analytic expressions for all master integrals in terms of iterated integrals of modular forms.

3.1 The canonical differential equations

We start by defining the rotation from the original basis of master integrals $\boldsymbol{I}$ defined in eq. (2.3) to the canonical basis $\boldsymbol{M}$. We consider the master integrals as functions of the two independent variables $\boldsymbol{\tau}$ defined in eq. (2.33). When working with the method of refs. [60, 64, 65], it is paramount to have a good starting basis of master integrals which reflects as much as possible the underlying geometry. Our starting basis in eq. (2.3) has this property. In particular, the master integrals I_3, I_4, I_5 and I_7 at $\varepsilon = 0$ compute the periods and quasi-periods of the K3 surface and I_6 has a non-vanishing residue at infinity. For more details, we refer to refs. [64, 65].

[3] We have swapped the indices of τ_1 and τ_2 with respect to the choice in ref. [53] for notational convenience.

After applying the method of refs. [60, 64, 65], we find that our original basis from eq. (2.3) and the canonical basis are related by,

$$
\begin{aligned}
M_1 &= \varepsilon^3\, I_{011100000}\,,\\
M_2 &= \varepsilon^3\, I_{101100000}\,,\\
M_3 &= \varepsilon^3\, \frac{I_{111100000}}{\psi_0}\,,\\
M_4 &= \frac{1}{\varepsilon}\partial_{\tau_2} M_3 - F_{11}\, M_3\,,\\
M_5 &= \frac{1}{\varepsilon}\partial_{\tau_1} M_3 - F_{12}\, M_3\,,\\
M_6 &= \varepsilon^3\, I_{1111-10000} - F_{31}\, M_3\,,\\
M_7 &= \frac{1}{\varepsilon}\partial_{\tau_2} M_5 - F_{21}\, M_3 - F_{22}\, M_4 - F_{25}\, M_5 - F_{34}\, M_6 - F_{01}\, M_1 - F_{02}\, M_2\,.
\end{aligned}
\tag{3.1}
$$

The functions $F_{ij}(\boldsymbol{\tau})$ are defined such that the terms in the differential equations that are not ε-factorised vanish. This constraint can be expressed as a set of differential equations satisfied by these functions (for details, see refs. [60, 64, 65]). Using the modular parametrisation for the periods introduced in the previous section, we can solve these differential equations and we obtain closed expressions for all of them in terms of modular forms and functions, as well integrals over them. We find

$$
\begin{aligned}
F_{11}(\boldsymbol{\tau}) &= \mathbb{I}_1(\boldsymbol{\tau}) + \frac{1}{2}(\phi^b_{0,2}(\boldsymbol{\tau}) + \phi^c_{0,2}(\boldsymbol{\tau}))\,,\\
F_{12}(\boldsymbol{\tau}) &= \mathbb{I}_2(\boldsymbol{\tau}) + \frac{1}{2}(\phi^b_{2,0}(\boldsymbol{\tau}) + \phi^c_{2,0}(\boldsymbol{\tau}))\,,\\
F_{25}(\boldsymbol{\tau}) &= -\mathbb{I}_1(\boldsymbol{\tau}) + \frac{1}{2}(\phi^b_{0,2}(\boldsymbol{\tau}) + \phi^c_{0,2}(\boldsymbol{\tau}))\,,\\
F_{22}(\boldsymbol{\tau}) &= -2\,\mathbb{I}_2(\boldsymbol{\tau})\,,\\
F_{31}(\boldsymbol{\tau}) &= \mathbb{I}_3(\boldsymbol{\tau}) + \phi_{1,1}(\boldsymbol{\tau})\,,\\
F_{34}(\boldsymbol{\tau}) &= -18\,\mathbb{I}_3(\boldsymbol{\tau})\,,\\
F_{01}(\boldsymbol{\tau}) &= 21\,\mathbb{I}_3(\boldsymbol{\tau})\,,\\
F_{02}(\boldsymbol{\tau}) &= 3\,\mathbb{I}_3(\boldsymbol{\tau})\,,\\
F_{21}(\boldsymbol{\tau}) &= -2\,\mathbb{I}_1(\boldsymbol{\tau})\,\mathbb{I}_2(\boldsymbol{\tau}) - 9\,\mathbb{I}_3(\boldsymbol{\tau})^2 + \phi_{2,2}(\boldsymbol{\tau})\,.
\end{aligned}
\tag{3.2}
$$

The functions $\phi_{i,j}(\boldsymbol{\tau})$ and $\phi^r_{i,j}(\boldsymbol{\tau})$ are given in appendix B. They are rational functions in $\boldsymbol{u}$ and contain powers of $\Psi_1(\tau_k)$. Note that they are functions of $\boldsymbol{\tau}$ through eqs. (2.21) and (2.31), and so they define modular forms and functions in two variables related to the congruence subgroup $\Gamma^1(12) \times \Gamma_1(12)$, with

$$
\Gamma^1(N) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma^T \in \Gamma_1(N)\right\}. \tag{3.3}
$$

The functions $\mathbb{I}_k$ are integrals of modular forms, and they are given by

$$\begin{aligned}
\mathbb{I}_1(\boldsymbol{\tau}) &= \int_{i\infty}^{\tau_2} \mathrm{d}\tau_2'\, \partial_{\tau_1}\phi_{-2,4}(\tau_1,\tau_2')\,,\\
\mathbb{I}_2(\boldsymbol{\tau}) &= \int_{i\infty}^{\tau_1} \mathrm{d}\tau_1'\, \partial_{\tau_2}\phi_{4,-2}(\tau_1',\tau_2)\,,\\
\mathbb{I}_3(\boldsymbol{\tau}) &= \int_{i\infty}^{\tau_2} \mathrm{d}\tau_2'\, \partial_{\tau_1}\phi_{-1,3}(\tau_1,\tau_2') = \int_{i\infty}^{\tau_1} \mathrm{d}\tau_1'\, \partial_{\tau_2}\phi_{3,-1}(\tau_1',\tau_2)\,.
\end{aligned} \tag{3.4}$$

We see that there are only three functions that are not expressible in terms of rational functions and (derivatives of) periods. These functions were called ε-*functions* in refs. [70, 86]. In refs. [63, 86] a method based on intersection theory [82, 87–106] was introduced to identify relations between ε-functions (see also ref. [107] for a related method). In particular, this method has allowed us to identify $F_{21}(\boldsymbol{\tau})$ as a quadratic polynomial in the $\mathbb{I}_k$ (it can also be seen as a special case of a relation for the banana integral with four unequal masses discussed in ref. [70]). We note that, up to an overall factor, the expansion of the $\mathbb{I}_i$ has integer coefficients when written in the canonical variables q_1, q_2 close to the MUM-point. This is similar to the observations made in the equal-mass case [51] (or more general one-parameter families of K3 surfaces [64]), where the ε-functions are given as primitives of magnetic modular forms [108–111].

Let us discuss some further properties of our ε-functions. We focus on $\mathbb{I}_3(\boldsymbol{\tau})$, but we can easily extend the discussion to the other two ε-functions. First, from the explicit expressions in appendix B, we see that $\phi_{-1,3}$ is a meromorphic modular form in two variables for $\Gamma^1(12)\times\Gamma_1(12)$ with a simple pole on the locus

$$u_1^2 + u_2^2 - u_1(u_2^2-3) = 0\,. \tag{3.5}$$

The derivative $\partial_{\tau_1}\phi_{-1,3}$ then has a double pole on this locus. However, since we have taken the derivative in ∂_{τ_1}, we have lost modularity in τ_1. Hence, $\partial_{\tau_1}\phi_{-1,3}$ is a meromorphic modular form of weight three in τ_2 for $\Gamma_1(12)$ with a double pole on the locus in eq. (3.5). Since $\mathbb{I}_3$ is obtained by taking a primitive in τ_2, we conclude that $\mathbb{I}_3$ has a simple pole on the locus in eq. (3.5). Using the same argument for the other modular forms, we see that all our ε-functions only have simple poles. Second, we find that the following relation among ε-functions holds:

$$\partial_{\tau_1}\mathbb{I}_1(\boldsymbol{\tau}) = \partial_{\tau_2}\mathbb{I}_2(\boldsymbol{\tau})\,. \tag{3.6}$$

This relation allows us to obtain alternative representations for $\mathbb{I}_1$ and $\mathbb{I}_2$ as double integrals in the other variable,

$$\begin{aligned}
\mathbb{I}_1(\boldsymbol{\tau}) &= \int_{i\infty}^{\tau_1} \mathrm{d}\tau_1' \int_{i\infty}^{\tau_1'} \mathrm{d}\tau_1''\, \partial_{\tau_2}^2\phi_{4,-2}(\tau_1'',\tau_2)\,,\\
\mathbb{I}_2(\boldsymbol{\tau}) &= \int_{i\infty}^{\tau_2} \mathrm{d}\tau_2' \int_{i\infty}^{\tau_2'} \mathrm{d}\tau_2''\, \partial_{\tau_1}^2\phi_{-2,4}(\tau_1,\tau_2'')\,.
\end{aligned} \tag{3.7}$$

Said differently, we see that we can interpret $\mathbb{I}_1$ either via eq. (3.4) as the first primitive in τ_2 of the modular form $\partial_{\tau_1}\phi_{-2,4}(\tau_1,\cdot)$ with a double pole, or via eq. (3.7) as the second primitive in τ_1 of the modular form $\partial_{\tau_2}^2\phi_{4,-2}(\cdot,\tau_2)$ with a triple pole.

Let us now discuss the structure of the canonical differential equations. After rotation to the canonical basis, the differential equations take the form

$$\mathrm{d}\boldsymbol{M}(\boldsymbol{\tau},\varepsilon) = \varepsilon\, \boldsymbol{A}(\boldsymbol{\tau})\, \boldsymbol{M}(\boldsymbol{\tau},\varepsilon)\,, \tag{3.8}$$

with

$$\boldsymbol{A}(\boldsymbol{\tau}) = \boldsymbol{A}_1(\boldsymbol{\tau})\,\mathrm{d}\tau_1 + \boldsymbol{A}_2(\boldsymbol{\tau})\,\mathrm{d}\tau_2\,, \tag{3.9}$$

and

$$\boldsymbol{A}_i = \begin{pmatrix} \omega_{11}^i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_{22}^i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_{33}^i & \delta_{i2} & \delta_{i1} & 0 & 0 \\ \omega_{41}^i & \omega_{42}^i & \omega_{43}^i & \omega_{44}^1 & \omega_{45}^i & \omega_{46}^i & \delta_{i1} \\ \omega_{51}^i & \omega_{52}^i & \omega_{53}^i & \omega_{54}^1 & \omega_{55}^i & \omega_{56}^i & \delta_{i2} \\ \omega_{61}^i & \omega_{62}^i & \omega_{63}^i & \omega_{64}^i & \omega_{65}^i & \omega_{66}^i & 0 \\ \omega_{71}^i & \omega_{72}^i & \omega_{73}^i & \omega_{74}^i & \omega_{75}^i & \omega_{76}^i & \omega_{77}^i \end{pmatrix}, \tag{3.10}$$

where δ_{ij} is the Kronecker delta and the ω_{ij}^k are rational functions in u_i, the holomorphic periods $\Psi_1(\tau_k)$ and the ε-functions defined in eq. (3.4). Their explicit expressions are provided in appendix B.

3.2 Properties of the canonical differential equations

In the previous subsection we have presented the canonical differential equations for the three-loop banana integrals with three equal masses, and we have discussed the associated ε-functions. The rotation to the canonical basis in eq. (3.1) was constructed using the method of refs. [60, 64, 65], which aims at constructing a rotation that achieves ε-factorisation. In ref. [64] it was proposed that the resulting differential equations enjoy additional properties that go beyond mere ε-factorisation. In particular, it was argued in ref. [64] that the differential forms ω_{ij} should have at most simple poles and they should define independent cohomology classes. While these properties are manifest for canonical systems involving dlog-forms, checking them for cases with non-trivial geometries is a complicated task. In the following we show how we can leverage the fact that the differential forms ω_{ij}^k are constructed from meromorphic modular forms in two variables to check that the properties conjectured in ref. [64] hold for the three-loop banana integrals with three equal masses.

Simple poles. From the explicit expressions for the functions ω_{ij}^k in appendix B, we can see that the ω_{ij}^k have poles at $\tilde{q}_k = 0$ and/or on the codimension-one loci[4] $L_1(\boldsymbol{u}) = 0$ or $L_2(\boldsymbol{u}) = 0$, where we defined

$$\begin{aligned} L_1(\boldsymbol{u}) &= u_1^2 + u_2^2 - u_1(u_2^2 - 3)\,, \\ L_2(\boldsymbol{u}) &= (u_2u_1 - 3u_1 - 3u_2 - 3)(u_2u_1 + 3u_1 - 3u_2 + 3) \\ &\quad \times (u_2u_1 - u_1 + u_2 + 3)(u_2u_1 + u_1 + u_2 - 3)\,. \end{aligned} \tag{3.11}$$

[4] There are also apparent singularities at $u_k = -1$. However, these singularities are spurious and cancel against a zero in $\Psi_1(u_k)$; see the discussion in appendix A.

Using the explicit expressions in appendix B, we have checked that the differential forms $\mathrm{d}\tau_k\, \omega_{ij}^k$ only have simple poles in τ_k. We stress that this feature arises in a very non-trivial manner, because some of the meromorphic modular forms in appendix B have double or even triple poles. These apparent higher-order poles cancel in the specific combinations that define the ω_{ij}^k. As an example, consider the function

$$\omega_{75}^1 = -2\mathbb{I}_1\mathbb{I}_2 - 9\mathbb{I}_3^2 + \phi_{2,2}\,. \tag{3.12}$$

From the explicit expressions in appendix B, we see that $\phi_{2,2}$ has double poles on the loci $L_1(\boldsymbol{u}) = 0$ and $L_2(\boldsymbol{u}) = 0$. All three ε-functions $\mathbb{I}_k$ $k \in \{1,2,3\}$, have simple poles on the locus $L_1(\boldsymbol{u}) = 0$. $\mathbb{I}_1$ and $\mathbb{I}_2$ also have a simple on the locus $L_2(\boldsymbol{u}) = 0$, while $\mathbb{I}_3$ is regular there. One can easily check that the combination in eq. (3.12) only has simple poles on the loci $L_1(\boldsymbol{u}) = 0$ and $L_2(\boldsymbol{u}) = 0$.

So far we have only discussed the pole structure for codimension-one loci. Since we deal with differential forms in two variables, we also need to investigate the behaviour at poles on codimension-two loci. For example, take a function ω_{ij}^1 that has a simple pole in τ_1 on the codimension locus $L_1(\boldsymbol{u}) = 0$. Explicitly, the poles in the variable τ_1 are given by those values of u_1 such that,

$$u_1 = u_1^{\pm}(u_2) := \frac{1}{2}\left(u_2^2 - 3 \pm \sqrt{\left(u_2^2-9\right)\,\left(u_2^2-1\right)}\right)\,. \tag{3.13}$$

As u_2 varies, we obtain two branches of the codimension-one locus $L_1(\boldsymbol{u}) = 0$, given by $u_1 = u_1^+(u_2)$ and $u_1 = u_1^-(u_2)$. However, on the codimension-two locus inside $L_1(\boldsymbol{u}) = 0$ defined by values of u_2 such that $u_1^+(u_2) = u_1^-(u_2)$, the two zeroes of L_1 coincide, and we obtain a double-pole in τ_1. This happens precisely for $u_2 \in \{\pm 1, \pm 3\}$. Hence, one the codimension-two locus $\boldsymbol{u} \in \{(-1,\pm 1),(3,\pm 3)\}$, the differential forms $\mathrm{d}\tau_1\,\omega_{ij}^1$ have double poles in τ_1, apparently contradicting the expectation that our differential form should only have simple poles. We find that ω_{ij}^1 always has a numerator that vanishes on the codimension-two locus $\boldsymbol{u} \in \{(-1,\pm 1),(3,\pm 3)\}$, thereby reducing the order of the pole.

Independence of cohomology classes. We have also checked that the differential forms ω_{ij} define independent cohomology classes. Loosely speaking, this means that, if we pick a linearly independent set of differential forms ω_{ij}^k, then there is no linear-combination that equals a total derivative (for details, see appendices A and C). The detailed proof is rather lengthy and technical, and it is presented in appendix C. Here it suffices to say that our proof draws heavily on the properties of meromorphic modular forms [17, 18] and the properties of ε-functions, in particular the fact that we can represent them in different ways as primitives in either τ_1 or τ_2 (cf. eqs. (3.4) and (3.7)).

Modular properties of the differential forms. Since our differential forms ω_{ij} are constructed from meromorphic modular forms in two variables for $\Gamma^1(12)\times\Gamma_1(12)$, it is natural to ask what are their properties under modular transformations. This is motivated by the observations of refs. [36, 51, 112, 113], where it was observed in the context of the two-loop sunrise integrals and the three-loop equal-mass banana integrals that the entries of the corresponding canonical differential matrices have a well-defined behaviour under modular transformations. Here we extend these observations to the three-loop banana integral with three equal-masses. This is also the first time that modular properties of Feynman integrals involving modular forms in two variables are studied.

The group $\Gamma^1(12) \times \Gamma_1(12)$ acts on $\boldsymbol{\tau}$ via Möbius transformations,

$$(\tau_1, \tau_2) \mapsto (\gamma_1 \cdot \tau_1, \gamma_2 \cdot \tau_2)\,, \tag{3.14}$$

with $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ and $\gamma_i \cdot \tau_i = \frac{a_i\tau_i+b_i}{c_i\tau_i+d_i}$. A meromorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ (with $\mathbb{H}$ the compelx upper half-plane) is a meromorphic modular form in two variables of weights (k_1, k_2) for $\Gamma^1(12) \times \Gamma_1(12)$ if for all $(\gamma_1, \gamma_2) \in \Gamma^1(12) \times \Gamma_1(12)$ we have

$$f(\gamma_1 \cdot \tau_1, \gamma_2 \cdot \tau_2) = (c_1\tau_1 + d_1)^{k_1}\,(c_2\tau_2 + d_2)^{k_2}\, f(\tau_1, \tau_2)\,. \tag{3.15}$$

The functions $\phi_{l,m}$ and $\phi^r_{l,m}$ defined in appendix B define meromorphic modular forms of weights (l, m) for $\Gamma^1(12) \times \Gamma_1(12)$. The ω^k_{ij} are polynomials in the $\phi_{l,m}$ and $\phi^r_{l,m}$ and our ε-functions. The latter have a more complicated transformation behaviour under modular transformations.[5] Nevertheless, we find that, if we define the weights of $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ to be $(0, 2)$, $(2, 0)$ and $(1, 1)$ respectively, then each ω^k_{ij} has a well-defined weight. We summarise this finding in the following matrices,

$$W_{\boldsymbol{A}_1} = \begin{pmatrix} 2&0&0&0&0&0&0 \\ 0&2&0&0&0&0&0 \\ 0&0&2&0&0&0&0 \\ 1&1&2&2&0&1&0 \\ 3&3&4&4&2&3&0 \\ 2&2&3&3&1&2&0 \\ 3&3&4&4&2&3&2 \end{pmatrix}, \qquad W_{\boldsymbol{A}_2} = \begin{pmatrix} 2&0&0&0&0&0&0 \\ 0&2&0&0&0&0&0 \\ 0&0&2&0&0&0&0 \\ 3&3&4&2&4&3&0 \\ 1&1&2&0&2&1&0 \\ 2&2&3&1&3&2&0 \\ 3&3&4&2&4&3&2 \end{pmatrix}, \tag{3.16}$$

where we collect the weight of ω^k_{ij} in τ_k in the entry (i, j) of $W_{\boldsymbol{A}_k}$. Note that $W_{\boldsymbol{A}_1}$ and $W_{\boldsymbol{A}_2}$ are identical up to exchanging the rows and columns 4 and 5. This is expected, because M_4 and M_5 are respectively derivatives of M_3 with respect to τ_2 and τ_1. We can also zoom in on the matrices and consider the rows and the columns corresponding to M_3, M_4, M_5, M_7, which for $\varepsilon = 0$ are related to the periods and quasi-periods of the underlying K3 surface. We have

$$W^{\mathrm{K3}}_{\boldsymbol{A}_1} = \begin{pmatrix} 2&0&0&0 \\ 2&2&0&0 \\ 4&4&2&0 \\ 4&4&2&2 \end{pmatrix}, \qquad W^{\mathrm{K3}}_{\boldsymbol{A}_2} = \begin{pmatrix} 2&0&0&0 \\ 4&2&4&0 \\ 2&0&2&0 \\ 4&2&4&2 \end{pmatrix}. \tag{3.17}$$

We observe a similar increase in weight from the top right to the bottom left as in other integrals with underlying geometry related to modular forms [36, 51, 112, 113]. In particular, we note that by taking the equal-mass limit we recover the same weight structure as for the equal-mass banana considered in ref. [51].

[5] This transformation behaviour was called *quasi-Eichler* in ref. [51].

So far we have only considered the weights of ω_{ij}^k for fixed values of k. What enters the canonical differential equation in eq. (3.8), however, are the differential one-forms $\omega_{ij} = \mathrm{d}\tau_1\,\omega_{ij}^1 + \mathrm{d}\tau_2\,\omega_{ij}^2$, and we conclude by discussing their modular weights. We start by noting that $\mathrm{d}\tau_1$ and $\mathrm{d}\tau_2$ transform like modular forms of weights $(-2,0)$ and $(0,-2)$, respectively. By inspecting the explicit expressions in appendix B, we see that ω_{ij} transform with well-defined weights (k_1,k_2) for $\Gamma^1(12)\times\Gamma_1(12)$. This is equivalent to the statement that the functions ω_{ij}^1 and ω_{ij}^2 transform with weights (k_1+2,k_2) and (k_1,k_2+2), respectively. The weights of the different non-zero entries of $\boldsymbol{A}$ are summarised in the following matrix (we indicate by – a zero entry):

$$\begin{pmatrix} (0,0) & - & - & - & - & - & - \\ - & (0,0) & - & - & - & - & - \\ - & - & (0,0) & (-,0) & (0,-) & - & - \\ (1,3) & (1,3) & (2,4) & (0,0) & (0,4) & (1,3) & (0,-) \\ (3,1) & (3,1) & (4,2) & (4,0) & (0,0) & (3,1) & (-,0) \\ (2,2) & (2,2) & (3,3) & (3,1) & (1,3) & (0,0) & - \\ (3,3) & (3,3) & (4,4) & (4,2) & (2,4) & (3,3) & (0,0) \end{pmatrix}. \tag{3.18}$$

Closedness of the differential forms. As a consequence of the ε-factorisation, the matrix $\boldsymbol{A}$ in eq. (3.9) satisfies the flatness conditions

$$\mathrm{d}\boldsymbol{A}(\boldsymbol{\tau}) = \boldsymbol{A}(\boldsymbol{\tau})\wedge\boldsymbol{A}(\boldsymbol{\tau}) = 0\,. \tag{3.19}$$

It follows that the differential forms ω_{ij} are closed, $\mathrm{d}\omega_{ij} = 0$. This is equivalent to the statement

$$\partial_{\tau_2}\omega_{ij}^1 = \partial_{\tau_1}\omega_{ij}^2\,. \tag{3.20}$$

This equation implies that ω_{ij}^1 and ω_{ij}^2 are not independent. We now argue that eq. (3.20) implies that at least one of the two functions ω_{ij}^1 or ω_{ij}^2 must contain an ε-function, unless ω_{ij} has modular weights $(0,0)$.

To see that this is true, let us assume that ω_{ij} has weights (k_1,k_2) and that neither ω_{ij}^1 or ω_{ij}^2 contain an ε-function. This implies that ω_{ij}^1 and ω_{ij}^2 are modular forms in two variables for $\Gamma^1(12)\times\Gamma_1(12)$ with weights (k_1+2,k_2) and (k_1,k_2+2), respectively. The derivative of a modular form of weight $k\neq 0$ is not modular (instead it is quasi-modular; see appendix A). Hence, unless the weight is $(k_1,k_2)=(0,0)$, the left-hand side of eq. (3.20) would be modular in τ_1 but not in τ_2, while the right-hand side would be modular in τ_2 but not in τ_1, which is a contradiction. It is easy to check that all the ω_{ij} in appendix B of weights $(k_1,k_2)\neq(0,0)$ involve an ε-function.

3.3 Analytic results for the master integrals

We can easily write down the general solution to the differential equation in eq. (3.8) in terms of the path-ordered exponential in eq. (2.10). In order to fully specify the solution that we are interested in, we also need to determine the initial condition and the path along which we want to integrate. Since the master integrals M_1 and M_2 are just products of one-loop tadpole integrals, we do not discuss them any further, and we focus on the master integrals M_k with $3 \leq k \leq 7$.

A natural choice of boundary condition is the small-mass limit, $m_1^2 = m_2^2 = 0$, which corresponds to $\boldsymbol{\sigma} = 0$, or equivalently $\boldsymbol{\tau} = 0$. We stress that there are logarithmic divergences as the masses tend to zero, and so we compute asymptotic expansions around the point $m_1^2 = m_2^2 = 0$. For the master integral M_3, this is easily done using the Mellin-Barnes representation for the integral (cf., e.g., refs. [29, 50, 51]). We find for the first two orders,

$$\begin{aligned} M_3 =& \varepsilon^3 \Big[16\zeta_3 - \log^2 q_1 (\log q_1 + 3\log q_2) \Big] \\ & + \frac{3}{2}\varepsilon^4 \Big[\log^4 q_1 + 2\log^3 q_1 \log q_2 + \log^2 q_1 \log^2 q_2 + 4\log q_1 (\log q_1 + \log q_2)\zeta_2 \\ & + 12(3\log q_1 + \log q_2)\zeta_3 + 16\zeta_4 \Big] + \mathcal{O}(\varepsilon^5) + \mathcal{O}(q_i) \,, \end{aligned} \tag{3.21}$$

where we recall that $q_i = e^{\sigma_i}$. From eq. (3.21) we can get the asymptotic behaviour for M_4, M_5 and M_7, because those master integrals are proportional to derivatives of M_3 (and to some linear combination of master integrals already determined). Finally, to obtain the asymptotic expansion for $M_6 = \varepsilon^3 I_6 - F_{31} M_3$, we need to compute the expansion of I_6. This is again done using the Mellin-Barnes representation, and we find

$$\begin{aligned} M_6 =& \frac{4}{3} - \varepsilon(3\log q_1 + \log q_2) + \varepsilon^2 \left(\frac{7}{2}\log^2 q_1 + 2\log q_1 \log q_2 + \frac{1}{2}\log^2 q_2 + 2\zeta_2 \right) \\ & - \frac{\varepsilon^3}{6} \Bigg[13\log^3 q_1 + 12\log^2 q_1 \log q_2 + 6\log q_1 \log^2 q_2 + \log^3 q_2 \\ & + 9(3\log q_1 + \log q_2)\zeta_2 + 24\zeta_3 \Bigg] + \mathcal{O}(\varepsilon^4) + \mathcal{O}(q_i) \,. \end{aligned} \tag{3.22}$$

In this way we have obtained the leading behaviour of all master integrals in the small-mass limit.

Next let us discuss our choice of path of integration. As a consequence of the flatness condition in eq. (3.19), the result of the path ordered-integral does not depend on the choice of path (more precisely, the path-ordered exponential depends only on the homotopy class of the path). We consider a path $\gamma = \gamma_2\gamma_1$ which is the concatenation of two straight-line paths γ_1 and γ_2 defined as follows: γ_1 is the line segment in the (τ_1, τ_2)-space from $(i\infty, i\infty)$ to a generic point $(\tau_1, i\infty)$ on the line $\tau_2 = i\infty$, and γ_2 is the line segment from the point $(\tau_1, i\infty)$ to a generic point (τ_1, τ_2). We may use the path-composition formula for the path-ordered exponential to write

$$\mathbb{P}_{\gamma_2\gamma_1}(\boldsymbol{\tau}, \varepsilon) = \mathbb{P}_{\gamma_2}(\boldsymbol{\tau}, \varepsilon)\, \mathbb{P}_{\gamma_1}(\boldsymbol{\tau}, \varepsilon) \,. \tag{3.23}$$

We now discuss the contribution from each line segment in turn.

Let us start from the first segment. On γ_1, we need to solve the equation,

$$\partial_{\tau_1} \boldsymbol{M} = \varepsilon \widetilde{\boldsymbol{A}}_1(\tau_1) \boldsymbol{M} \,, \tag{3.24}$$

where the matrix $\widetilde{\boldsymbol{A}}_1(\tau_1)$ is obtained from eq. (3.9) by pulling it back to the path γ_1

$$\mathrm{d}\tau_1\, \widetilde{\boldsymbol{A}}_1(\tau_1) = \gamma_1^* \boldsymbol{A}(\boldsymbol{\tau})\,. \tag{3.25}$$

We find

$$\widetilde{\boldsymbol{A}} = \begin{pmatrix} \frac{5}{12}h_2^a + \frac{1}{2}h_2^b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_2^a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2^b & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & h_2^b & 0 & 0 & 1 \\ -21\,h_3 & -3\,h_3 & h_4^b & 36\,h_4^a & h_2^b & 18\,h_3 & 0 \\ \frac{1}{2}h_2^b - \frac{11}{12}h_2^a & -\frac{1}{9}h_2^a & \frac{1}{6}h_3 & h_3 & 0 & \frac{4}{3}h_2^a & 0 \\ -\frac{5}{2}h_3 & -\frac{1}{2}h_3 & h_4^a & h_4^b & 0 & 3\,h_3 & h_2^b \end{pmatrix}, \tag{3.26}$$

with

$$\begin{aligned}
h_2^a(\tau_1) =& -\frac{1}{4}(u_1-3)^2\Psi_1(\tau_1)^2\,,\\
h_2^b(\tau_1) =& \frac{9-108u_1+6u_1^2-12u_1^3-7u_1^4}{24(1+u_1)^2}\Psi_1(\tau_1)^2\,,\\
h_3(\tau_1) =& \frac{(u_1-3)^2(u_1-1)}{4\sqrt{3}(1+u_1)}\Psi_1(\tau_1)^3\,,\\
h_4^a(\tau_1) =& \frac{(u_1-1)(9+27u_1+3u_1^2+u_1^3)(9-9u_1+3u_1^2+5u_1^3)}{288(1+u_1)^4}\Psi_1(\tau_1)^4\,,\\
h_4^b(\tau_1) =& \frac{u_1^8+84u_1^7+360u_1^6+2052u_1^5-270u_1^4-5508u_1^3+6048u_1^2-1620u_1-891}{576(1+u_1)^4}\Psi_1(\tau_1)^4\,.
\end{aligned} \tag{3.27}$$

We notice that h_3 is a modular form of weight 3 for the congruence subgroup $\Gamma^1(12)$, while h_k^x ($k=2,4$ and $x=a,b$) are modular forms of weight k (see appendix A for details) for the same congruence subgroup. We also observe that no ε-function appears in the differential equation. The first few orders in the ε-expansion of the integrals read:

$$M_3(\tau_1,i\infty) =\varepsilon^3\left[16\,\zeta_3+\frac{7}{4}I(1,h_3,h_2^a;\tau_1)-\frac{3}{2}I(h_3,h_2^b;\tau_1)\right]+\mathcal{O}(\varepsilon^4)\,, \tag{3.28}$$

$$\begin{aligned}
M_4(\tau_1,i\infty) =&\varepsilon^2 I(1,h_3;\tau_1)+\varepsilon^3\left[\frac{26}{3}\zeta_3+2\zeta_2\log\tilde{q}_1+I(1,h_2^b,h_3;\tau_1)+\frac{17}{24}I(1,h_3,h_2^a;\tau_1)\right.\\
&\left.+\frac{1}{4}I(1,h_3,h_2^b;\tau_1)+I(h_2^b,1,h_3;\tau_1)\right]+\mathcal{O}(\varepsilon^4)\,,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
M_5(\tau_1,i\infty) =&\varepsilon^2\left[\frac{7}{4}I(h_3,h_2^a;\tau_1)-\frac{3}{2}I(h_3,h_2^b;\tau_1)\right]+\varepsilon^3\left[28\zeta_3+\frac{7}{4}I(h_2^b,h_3,h_2^a;\tau_1)\right.\\
&-\frac{3}{2}I(h_2^b,h_3,h_2^b;\tau_1)+\frac{119}{48}I(h_3,h_2^a,h_2^a;\tau_1)-\frac{5}{8}I(h_3,h_2^a,h_2^b;\tau_1)\\
&\left.-\frac{5}{8}I(h_3,h_2^b,h_2^a;\tau_1)-\frac{3}{4}I(h_3,h_2^b,h_2^b;\tau_1)+36I(h_4^a,1,h_3;\tau_1)\right]+\mathcal{O}(\varepsilon^4)\,,
\end{aligned} \tag{3.30}$$

$$M_6(\tau_1,i\infty) =\frac{4}{3}+\varepsilon\left[\frac{3}{4}I(h_2^a;\tau_1)+\frac{1}{2}I(h_2^b;\tau_1)\right]+\varepsilon^2\left[2\,\zeta_2+\frac{73}{144}I(h_2^a,h_2^a;\tau_1)+\frac{5}{24}I(h_2^a,h_2^b;\tau_1)\right.$$

$$
\begin{aligned}
&+\frac{5}{24}I(h_2^b,h_2^a;\tau_1)+\frac{1}{4}I(h_2^b,h_2^b;\tau_1)\Bigg]+\varepsilon^3\Bigg[\frac{9}{8}\zeta_2 I(h_2^a;\tau_1)+\frac{3}{4}\zeta_2 I(h_2^b;\tau_1)-4\zeta_3\\
&+\frac{701}{1728}I(h_2^a,h_2^a,h_2^a;\tau_1)+\frac{25}{288}I(h_2^a,h_2^a,h_2^b;\tau_1)+\frac{25}{288}I(h_2^a,h_2^b,h_2^a;\tau_1)\\
&+\frac{5}{48}I(h_2^a,h_2^b,h_2^b;\tau_1)+\frac{25}{288}I(h_2^b,h_2^a,h_2^a;\tau_1)+\frac{5}{48}I(h_2^b,h_2^a,h_2^b;\tau_1)\\
&+\frac{5}{48}I(h_2^b,h_2^b,h_2^a;\tau_1)+\frac{1}{8}I(h_2^b,h_2^b,h_2^b;\tau_1)+I(h_3,1,h_3;\tau_1)\Bigg]+\mathcal{O}(\varepsilon^4)\,, \qquad (3.31)
\end{aligned}
$$

$$
\begin{aligned}
M_7(\tau_1,i\infty)=&\varepsilon\, I(h_3;\tau_1)+\varepsilon^2\Bigg[2\zeta_2+I(h_2^b,h_3;\tau_1)+\frac{17}{24}I(h_3,h_2^a;\tau_1)+\frac{1}{4}I(h_3,h_2^b;\tau_1)\Bigg]\\
&+\varepsilon^3\Bigg[\frac{35}{3}\zeta_3+2\zeta_2\, I(h_2^b;\tau_1)+\frac{3}{2}\zeta_2\, I(h_3;\tau_1)+I(h_2^b,h_2^b,h_3;\tau_1)\\
&+\frac{17}{24}I(h_2^b,h_3,h_2^a;\tau_1)+\frac{1}{4}I(h_2^b,h_3,h_2^b;\tau_1)+\frac{169}{288}I(h_3,h_2^a,h_2^a;\tau_1)\\
&+\frac{5}{48}I(h_3,h_2^a,h_2^b;\tau_1)+\frac{5}{48}I(h_3,h_2^b,h_2^a;\tau_1)+\frac{1}{8}I(h_3,h_2^b,h_2^b;\tau_1)\\
&+I(h_4^b,1,h_3;\tau_1)\Bigg]+\mathcal{O}(\varepsilon^4)\,. \qquad (3.32)
\end{aligned}
$$

Next we integrate along γ_2. We then need to solve the equation

$$
\partial_{\tau_2}\boldsymbol{M}=\varepsilon\boldsymbol{A}_2(\tau_2)\boldsymbol{M}\,, \qquad (3.33)
$$

where the matrix $\boldsymbol{A}_2$ is given in eq. (3.10). Note that $\boldsymbol{A}_2(\tau_2)$ still depends on τ_1, but we drop this dependence from the argument, because τ_1 is a constant on the line segment γ_2. Since the integration boundaries are different from $i\infty$, ε-functions appear in the result. Putting everything together, we find for the leading term in the ε-expansion for $M_3(\tau_1,\tau_2)$ is

$$
\begin{aligned}
M_3(\tau_1,\tau_2)=&\varepsilon^3\Bigg[16\,\zeta_3+\log\tilde{q}_2\, I(1,h_3;\tau_1)+\frac{7}{4}I(1,h_3,h_2^a;\tau_1)-\frac{3}{2}I(1,h_3,h_2^b;\tau_1)\\
&-\frac{7}{4}I(h_2^a;\tau_1)I(1,\phi_{-1,3};\tau_2)+\frac{3}{2}I(h_2^b;\tau_1)I(1,\phi_{-1,3};\tau_2)\\
&-I(1,\phi_{-1,3},\phi_{0,2}^b;\tau_2)-I(1,\phi_{-1,3},\phi_{0,2}^a;\tau_2)\Bigg]+\mathcal{O}(\varepsilon^4)\,. \qquad (3.34)
\end{aligned}
$$

The ε-expansion for the all the integrals up to $\mathcal{O}(\varepsilon^5)$ can be found in an ancillary file. As a check of our computation, we choose a point that lies within the radius of convergence of the series expansion of the holomorphic period ψ_0 near the MUM-point $\boldsymbol{z}=0$ with negative momentum squared. We expanded all the iterated integrals as series up to order 18. We verified numerically against AMFlow [114] up to $\mathcal{O}(\varepsilon^4)$ and found perfect agreement within 16 digits.

Let us conclude by making a comment about the leading term in the ε-expansion of M_3 shown in eq. (3.34), which corresponds to the master integral evaluated in strictly $D=2$ dimensions. We observe that the iterated integrals only involve the integration kernels h_k, h_k^x and $\phi_{i,j}^k$, but no ε-functions $\mathbb{I}_k$ appear. We have already seen h_k and h_k^x are modular forms for $\Gamma^1(12)$. It is easy to see (cf. appendix A) that for τ_1 fixed also the $\phi_{i,j}^k$

define meromorphic modular forms for $\Gamma_1(12)$. We conclude that in $D = 2$ dimensions, the master integral M_3 can be expressed in terms of genuine iterated integrals of meromorphic modular forms, and no ε-functions appear. This is similar to what happens for the equal-mass three-loop integral [50].

4 Conclusion

In this paper we have presented for the first time fully analytic results for all the master integrals for the three-loop banana integrals with three equal masses in $D = 2 - 2\varepsilon$ dimensions. While analytic results for the three-loop banana integrals with four different masses have recently appeared [70, 71], the main novelty of our result is that the master integrals can be expressed in terms of iterated integrals of meromorphic modular forms (and integrals thereof). While meromorphic modular forms have already appeared in the context of the equal-mass banana integral [18, 50, 51], this is the first time that it was possible to express a Feynman integral depending on two dimensionless ratios in terms of this class of functions.

The appearance of meromorphic modular forms has some practical advantages, because this class of functions is relatively well studied in the literature. In particular, we have used our computation to study the properties of our results. We were able to rigorously prove that our canonical differential equations satisfy the properties conjectured in ref. [64], and we could show that the differential forms in the canonical differential equation have only simple poles and define independent cohomology classes. We stress that all our results rely very profoundly on various recent advances in our understanding of the mathematics underlying Feynman integrals associated with K3 surfaces, in particular their period geometry [52, 53], the corresponding canonical bases [51, 60–62, 64–67, 71] and their properties [86, 107, 115], and the mathematics underlying iterated integrals of meromorphic modular forms [14, 15, 17, 18, 116].

For the future, it would be interesting to see if the approach developed here to solve Feynman integrals associated with K3 surfaces that are products of two elliptic curves can be applied also to other cases. For example, in ref. [117] it was observed that a certain family of integrals appearing in the context of gravitational wave scattering involves a K3 surface whose periods can be written as products of complete elliptic integrals. We expect that it is possible to express those results in terms of iterated integrals of modular forms as well.

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A Review of modular forms

A.1 Modular forms in one variable

Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (of finite index). It acts on the complex upper half-plane $\mathbb{H} = \big\{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\big\}$ via Möbius transformations,

$$\gamma \cdot \tau = \tfrac{a\tau+b}{c\tau+d}\,, \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\,. \tag{A.1}$$

A meromorphic modular form of weight k is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f(\gamma \cdot \tau) = (c\tau + d)^k\, f(\tau)\,, \qquad \text{for all } \gamma \in \Gamma\,. \tag{A.2}$$

A modular function for Γ is a meromorphic modular form of weight 0. We denote by $\mathcal{M}_k(\Gamma)$ the vector space generated by all meromorphic modular forms for Γ, and $M_k(\Gamma)$ is the subspace by all (holomorphic) modular forms. Note that $M_k(\Gamma)$ is non-trivial only for $k > 0$.

We will only be interested in cases where Γ has genus zero, by which we mean that the Riemann surface $Y_\Gamma = \Gamma \setminus \mathbb{H}$ has genus zero. The field of meromorphic functions of Y_Γ can be identified with the field of modular functions $\mathcal{M}_0(\Gamma)$. Since Y_Γ has genus zero, its field of meromorphic functions has a single generator, called a *Hauptmodul* for Γ. Said differently, every modular function can be expressed as a rational function in the Hauptmodul.

Throughout this paper, we encountered modular forms for the congruence subgroups $\Gamma_1(6)$ and $\Gamma_1(12)$.[6] In ref. [118] it was shown that any modular form f of weight k for $\Gamma_1(6)$, i.e., any element of $M_k(\Gamma_1(6))$ can be cast in the form

$$f(\tau) = \psi(\tau)^k \sum_{r=0}^{k} c_k\, t(\tau)^k\,, \tag{A.3}$$

where the c_k are constants, ψ is the modular form of weight one for $\Gamma_1(6)$ defined in eq. (2.22) and t is the Hauptmodul in eq. (2.18).[7] We now provide a similar description of modular forms for $\Gamma_1(12)$. Since $\Gamma_1(12) \subset \Gamma_1(6)$, we have $M_k(\Gamma_1(6)) \subset M_k(\Gamma_1(12))$, and so ψ is also a modular form of weight one for $\Gamma_1(12)$. Let $f \in M_k(\Gamma_1(12))$. From eq. (2.34) and eq. (A.3), we obtain

$$\frac{f(\tau)}{\psi(\tau)^k} = \sum_{r=0}^{k} c_k \left(\frac{u(\tau)(3-u(\tau))}{1+u(\tau)}\right)^r\,. \tag{A.4}$$

The right-hand side has a pole at $u(\tau) = -1$. Since f is holomorphic everywhere, this pole must come from a zero of ψ. We can easily check that this pole is simple. Following exactly the same argument as in the derivation of eq. (A.3) in ref. [118], we find that every modular form $f \in M_k(\Gamma_1(12))$ can be cast in the form

$$f(\tau) = \frac{\psi(\tau)^k}{(1+u(\tau))^k} \sum_{r=0}^{2k} c_k\, u(\tau)^k\,. \tag{A.5}$$

[6] Modular forms for $\Gamma^1(N)$ can be related to those for $\Gamma_1(N)$, cf. ref. [53].

[7] $\Gamma_0(6)$ and $\Gamma_1(6)$ admit the same Hauptmodul, because, even though $\Gamma_0(6)$ and $\Gamma_1(6)$ are different as subgroups of $\mathrm{SL}_2(\mathbb{Z})$, they become identical as subgroups in $\mathrm{PSL}_2(\mathbb{Z})$, so that the Riemann surfaces $Y_{\Gamma_0(6)}$ and $Y_{\Gamma_1(6)}$ are isomorphic.

In the context of canonical differential equations satisfied by Feynman integrals, we are typically interested in differential forms of the type $\mathrm{d}\tau\, f(\tau)$, where f is a meromorphic modular form for Γ (of positive weight). It is possible to identify a finite set of such differential forms that define independent cohomology classes, i.e., a finite set such that every differential form $\mathrm{d}\tau\, f(\tau)$ is a linear combination from this set, up to a total derivative [17, 18, 116]. In order to state the result, it is useful to consider a slightly larger class of functions. A meromorphic quasi-modular form of weight k and depth p for Γ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f(\gamma\cdot\tau) = (c\tau+d)^k \sum_{r=0}^{p} f_r(\tau) \left(\frac{c}{c\tau+d}\right)^r , \tag{A.6}$$

where $f_0, \ldots, f_p$ are meromorphic functions with $f_p \neq 0$. We denote the vector space of all quasi-modular forms of weight k (of any depth) for Γ by $\mathcal{QM}_k(\Gamma)$. Note that quasi-modular forms of depth zero are precisely the modular forms, so that $\mathcal{M}_k(\Gamma) \subseteq \mathcal{QM}_k(\Gamma)$. Unlike $\mathcal{M}_k(\Gamma)$, however, $\mathcal{QM}_k(\Gamma)$ is closed under differentiation (which raises the depth by one unit). We have [17, 18]

$$\mathcal{QM}_k(\Gamma) = \begin{cases} \mathcal{M}_k(\Gamma) \oplus \partial_\tau \mathcal{QM}_{k-2}(\Gamma)\,, & k<2\,,\\ \widetilde{\mathcal{M}}_k(\Gamma) \oplus \mathcal{M}_{2-k}(\Gamma) G_2^{k-1} \oplus \partial_\tau \mathcal{QM}_{k-2}(\Gamma)\,, & k \geq 2\,, \end{cases} \tag{A.7}$$

where G_2 is the Eisenstein series of weight two for $\mathrm{SL}_2(\mathbb{Z})$ and $\widetilde{\mathcal{M}}_k(\Gamma)$ is a finite-dimensional vector space. For the precise definition of $\widetilde{\mathcal{M}}_k(\Gamma)$ we refer to refs. [17, 18]. Here it suffices to say that all elements in $\widetilde{\mathcal{M}}_k(\Gamma)$ have pole orders bounded by $k-1$ at points $\tau_0 \in \mathbb{H}$. The notation $\widetilde{\mathcal{M}}(\Gamma) = \bigoplus_k \widetilde{\mathcal{M}}_k(\Gamma)$ will be useful later on. Equation (A.7) implies that every meromorphic modular form $f \in \mathcal{M}_k(\Gamma)$ with $k \geq 2$ can be written as $f = \tilde{f} + \partial_\tau g$ with $\tilde{f} \in \widetilde{\mathcal{M}}_k(\Gamma)$ and $g \in \mathcal{QM}_{k-2}(\Gamma)$. Note that we have

$$\widetilde{\mathcal{M}}(\Gamma) \cap \partial_\tau \mathcal{QM}(\Gamma) = \{0\}\,. \tag{A.8}$$

A.2 Modular forms in two variables

We now consider two subgroups Γ and Γ' of $\mathrm{SL}_2(\mathbb{Z})$ (each of finite index). A modular form in two variables of weights (k_1, k_2) for $\Gamma \times \Gamma'$ is a meromorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ such that for all $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma'$ we have

$$f(\gamma_1\cdot\tau_1, \gamma_2\cdot\tau_2) = (c_1\tau_1+d_1)^{k_1} (c_2\tau_2+d_2)^{k_2} f(\tau_1,\tau_2)\,. \tag{A.9}$$

We denote by $\mathcal{M}_{k_1,k_2}(\Gamma,\Gamma')$ the vector space of all meromorphic modular forms of weights (k_1, k_2) for $\Gamma \times \Gamma'$. It is easy to check that (see, e.g., ref. [53])

$$\mathcal{M}_{k_1,k_2}(\Gamma,\Gamma') = \mathcal{M}_{k_1}(\Gamma) \otimes \mathcal{M}_{k_2}(\Gamma')\,. \tag{A.10}$$

In other words, all modular forms in two variables for $\Gamma \times \Gamma'$ can be written as linear combinations of products of modular forms in one variable. We can similarly define the algebra of quasi-modular forms for $\Gamma \times \Gamma'$,

$$\mathcal{QM}(\Gamma,\Gamma') = \bigoplus_{k_1,k_2} \mathcal{QM}_{k_1}(\Gamma) \otimes \mathcal{QM}_{k_2}(\Gamma')\,. \tag{A.11}$$

In the context of canonical differential equations for Feynman integrals, one is interested in differential one-forms of the form $\omega_{ij} = \mathrm{d}\tau_1\,\omega_{ij}^1 + \mathrm{d}\tau_2\,\omega_{ij}^2$. We say that such a differential form transforms like a modular form of weights (k_1, k_2) for $\Gamma \times \Gamma'$ if under modular transformations we have

$$\omega_{ij} \mapsto (c_1\tau_1 + d_1)^{k_1}\,(c_2\tau_2 + d_2)^{k_2}\,\omega_{ij}\,. \tag{A.12}$$

This is equivalent to saying that ω_{ij}^1 and ω_{ij}^2 are modular forms in two variables for $\Gamma \times \Gamma'$ with weights $(k_1 + 2, k_2)$ and $(k_1, k_2 + 2)$, respectively.

B Differential forms in the canonical differential equation

The explicit expressions for the differential forms ω_{ij}^k depend on rational functions in u_1, u_2, holomorphic periods and the integrals defined in eq. (3.4). Here we collect the rational functions and the periods in 20 functions $\phi_{k,l}$ shown at the end of this appendix. The ω_{ij}^k read

$$\begin{aligned}
\omega_{11}^1 =& \frac{1}{3}(2\phi_{2,0}^b - \phi_{2,0}^a)\,,\\
\omega_{22}^1 =& \phi_{2,0}^b\,,\\
\omega_{33}^1 =& \frac{1}{2}(2\mathbb{I}_2 + \phi_{2,0}^b + \phi_{2,0}^c)\,,\\
\omega_{41}^1 =& 21\mathbb{I}_3\,,\\
\omega_{42}^1 =& 3\mathbb{I}_3\,,\\
\omega_{43}^1 =& -2\mathbb{I}_1\mathbb{I}_2 - 9\mathbb{I}_3^2 + \phi_{2,2}\,,\\
\omega_{44}^1 =& \frac{1}{2}(\phi_{2,0}^b + \phi_{2,0}^c - 2\mathbb{I}_2)\,,\\
\omega_{45}^1 =& -2\mathbb{I}_1\,,\\
\omega_{46}^1 =& -18\mathbb{I}_3\,,\\
\omega_{51}^1 =& 21\phi_{3,-1}\,,\\
\omega_{52}^1 =& 3\phi_{3,-1}\,,\\
\omega_{53}^1 =& -\mathbb{I}_2^2 - 18\mathbb{I}_3\phi_{3,-1} - 2\mathbb{I}_1\phi_{4,-2} + \phi_{4,0} - \phi_{4,-2}(\phi_{0,2}^b + \phi_{0,2}^c) + \frac{1}{4}(\phi_{2,0}^b + \phi_{2,0}^c)^2\,,\\
\omega_{54}^1 =& -2\phi_{4,-2}\,,\\
\omega_{55}^1 =& \frac{1}{2}(\phi_{2,0}^b + \phi_{2,0}^c - 2\mathbb{I}_2)\,,\\
\omega_{56}^1 =& -18\phi_{3,-1}\,,\\
\omega_{61}^1 =& -\frac{1}{3}(\phi_{2,0}^a + 2\phi_{2,0}^b)\,,\\
\omega_{62}^1 =& -\frac{1}{9}\phi_{2,0}^b\,,\\
\omega_{63}^1 =& \frac{1}{6}(2\phi_{3,1}^a - 3\phi_{3,-1}\phi_{0,2}^b + 5\mathbb{I}_3\phi_{2,0}^b - 4\phi_{3,1}^b - 3\phi_{3,-1}\phi_{0,2}^c - 3\mathbb{I}_3\phi_{2,0}^c - 6\mathbb{I}_2\mathbb{I}_3 - 6\mathbb{I}_1\phi_{3,-1})\,,\\
\omega_{64}^1 =& -\phi_{3,-1}\,,
\end{aligned}$$

$$
\begin{aligned}
\omega^1_{65} =& -\mathbb{I}_2\,,\\
\omega^1_{66} =& \frac{4}{3}\phi^b_{2,0}\,,\\
\omega^1_{71} =& \frac{1}{2}(42\mathbb{I}_2\mathbb{I}_3 + 42\mathbb{I}_1\phi_{3,-1} + 2\mathbb{I}_3\phi^a_{2,0} - 20\phi^a_{3,1} + 21\phi_{3,-1}\phi^b_{0,2} - 31\mathbb{I}_3\phi^b_{2,0} + 32\phi^b_{3,1}\\
&+ 21\phi_{3,-1}\phi^c_{0,2} + 21\mathbb{I}_3\phi^c_{2,0})\,,\\
\omega^1_{72} =& 3\mathbb{I}_2\mathbb{I}_3 + 3\mathbb{I}_1\phi_{3,-1} + \phi^b_{3,1} + \frac{3}{2}\phi_{3,1}(\phi^b_{0,2} + \phi^c_{0,2}) - \frac{1}{2}\mathbb{I}_3(7\phi^b_{2,0} - 3\phi^c_{2,0})\,,\\
\omega^1_{73} =& 2\mathbb{I}_2(\phi_{2,2} - 9\mathbb{I}_3^2) - 2\mathbb{I}_1^2\phi_{4,-2} + \phi_{4,2} + \phi_{4,0}(\phi^b_{0,2} + \phi^c_{0,2}) - \frac{1}{2}\phi_{4,-2}(\phi^b_{0,2} + \phi^c_{0,2})^2\\
&+ 6\mathbb{I}_3(2\phi^a_{3,1} - 4\phi^b_{3,1} - 3\phi_{3,-1}(\phi^b_{0,2} + \phi^c_{0,2})) + 3\mathbb{I}_3^2(5\phi^b_{2,0} - 3\phi^c_{2,0}) + \phi_{2,2}(\phi^b_{2,0} + \phi^c_{2,0})\\
&+ 2\mathbb{I}_1(-\mathbb{I}_2^2 - 18\mathbb{I}_3\phi_{3,-1} + \phi_{4,0} - \phi_{4,-2}(\phi^b_{0,2} + \phi^c_{0,2}) + \frac{1}{4}(\phi^b_{2,0} + \phi^c_{2,0})^2)\,,\\
\omega^1_{74} =& \phi_{4,0} - \phi_{4,-2}(\phi^b_{0,2} + \phi^c_{0,2}) + \frac{1}{4}(\phi^b_{2,0} + \phi^c_{2,0})^2 - \mathbb{I}_2^2 - 18\mathbb{I}_3\phi_{3,-1} - 2\mathbb{I}_1\phi_{4,-2}\,,\\
\omega^1_{75} =& -2\mathbb{I}_1\mathbb{I}_2 - 9\mathbb{I}_3^2 + \phi_{2,2}\,,\\
\omega^1_{76} =& -3(6\mathbb{I}_2\mathbb{I}_3 + 6\mathbb{I}_1\phi_{3,-1} - 2\phi^a_{3,1} + 3\phi_{3,-1}\phi^b_{0,2} - 5\mathbb{I}_3\phi^b_{2,0} + 4\phi^b_{3,1} + 3\phi_{3,-1}\phi^c_{0,2} + 3\mathbb{I}_3\phi^c_{2,0})\,,\\
\omega^1_{77} =& \frac{1}{2}(2\mathbb{I}_2 + \phi^b_{2,0} + \phi^c_{2,0})\,.
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
\omega^2_{11} =& \frac{1}{3}(2\phi^b_{0,2} - \phi^a_{0,2})\,,\\
\omega^2_{22} =& \phi^b_{0,2}\,,\\
\omega^2_{33} =& \mathbb{I}_1 + \frac{1}{2}(\phi^b_{0,2} + \phi^c_{0,2})\,,\\
\omega^2_{41} =& 21\phi_{-1,3}\,,\\
\omega^2_{42} =& 3\phi_{-1,3}\,,\\
\omega^2_{43} =& \phi_{0,4} + \frac{1}{4}(\phi^b_{0,2} + \phi^c_{0,2})^2 - \phi_{-2,4}(\phi^b_{2,0} + \phi^c_{2,0}) - \mathbb{I}_1^2 - 2\mathbb{I}_2\phi_{-2,4} - 18\mathbb{I}_3\phi_{-1,3}\,,\\
\omega^2_{44} =& \frac{1}{2}(\phi^b_{0,2} + \phi^c_{0,2} - 2\mathbb{I}_1)\,,\\
\omega^2_{45} =& -2\phi_{-2,4}\,,\\
\omega^2_{46} =& -18\phi_{-1,3}\,,\\
\omega^2_{51} =& 21\mathbb{I}_3\,,\\
\omega^2_{52} =& 3\mathbb{I}_3\,,\\
\omega^2_{53} =& -2\mathbb{I}_1\mathbb{I}_2 - 9\mathbb{I}_3^2 + \phi_{2,2}\,,\\
\omega^2_{54} =& -2\mathbb{I}_2\,,\\
\omega^2_{55} =& \frac{1}{2}(\phi^b_{0,2} + \phi^c_{0,2} - 2\mathbb{I}_1)\,,\\
\omega^2_{56} =& -18\mathbb{I}_3\,,\\
\omega^2_{61} =& -\frac{1}{3}(\phi^a_{0,2} + 2\phi^b_{0,2})\,,\\
\omega^2_{62} =& -\frac{1}{9}\phi^b_{0,2}\,,
\end{aligned}
$$

$$\begin{aligned}
\omega^2_{63} =& \frac{1}{6}(2\phi^a_{1,3} - 6\mathbb{I}_1\mathbb{I}_3 - 6\mathbb{I}_2\phi_{-1,3} + 5\mathbb{I}_3\phi^b_{0,2} - 4\phi^b_{1,3} - 3\phi_{-1,3}\phi^b_{2,0} - 3\mathbb{I}_3\phi^c_{0,2} - 3\phi_{-1,3}\phi^c_{2,0})\,,\\
\omega^2_{64} =& -\mathbb{I}_3\,,\\
\omega^2_{65} =& -\phi_{-1,3}\,,\\
\omega^2_{66} =& \frac{4}{3}\phi^b_{0,2}\,,\\
\omega^2_{71} =& \frac{1}{2}(42\mathbb{I}_1\mathbb{I}_3 + 42\mathbb{I}_2\phi_{-1,3} + 2\mathbb{I}_3\phi^a_{0,2} - 20\phi^a_{1,3} - 31\mathbb{I}_3\phi^b_{0,2} + 32\phi^b_{1,3} + 21\phi_{-1,3}\phi^b_{2,0}\\
&+ 21\mathbb{I}_3\phi^c_{0,2} + 21\phi_{-1,3}\phi^c_{2,0})\,,\\
\omega^2_{72} =& 3\mathbb{I}_1\mathbb{I}_3 + 3\mathbb{I}_2\phi_{-1,3} + \phi^b_{1,3} - \frac{1}{2}\mathbb{I}_3(7\phi^b_{0,2} - 3\phi^c_{0,2}) + \frac{3}{2}\phi^b_{-1,3}(\phi^b_{2,0} + \phi^c_{0,2})\,,\\
\omega^2_{73} =& \phi_{0,4}(\phi^b_{2,0} + \phi^c_{2,0}) - \frac{1}{2}\phi_{-2,4}(\phi^b_{2,0} + \phi^c_{2,0})^2 + \phi_{2,4} + \phi_{2,2}(\phi^b_{0,2} + \phi^c_{0,2}) - 2\mathbb{I}_1^2\mathbb{I}_2 - 2\mathbb{I}_2^2\phi_{-2,4}\\
&- 36\mathbb{I}_2\mathbb{I}_3\phi_{-1,3} + 2\mathbb{I}_1(\phi_{2,2} - 9\mathbb{I}_3^2) + 24\mathbb{I}_3^2\phi^b_{0,2} + 12\mathbb{I}_3(\phi^a_{1,3} - 2\phi^b_{1,3}) - 9\mathbb{I}_3^2(\phi^b_{0,2} + \phi^c_{0,2})\\
&- 18\mathbb{I}_3\phi_{-1,3}(\phi^b_{2,0} + \phi^c_{2,0}) + 2\mathbb{I}_2(\phi_{0,4} + \frac{1}{4}(\phi^b_{0,2} + \phi^c_{0,2})^2 - \phi_{-2,4}(\phi^b_{2,0} + \phi^c_{2,0}))\,,\\
\omega^2_{74} =& \phi_{2,2} - 2\mathbb{I}_1\mathbb{I}_2 - 9\mathbb{I}_3^2\,,\\
\omega^2_{75} =& \phi_{0,4} + \frac{1}{4}(\phi^b_{0,2} + \phi^c_{0,2})^2 - \phi_{-2,4}(\phi^b_{2,0} + \phi^c_{2,0}) - \mathbb{I}_1^2 - 2\mathbb{I}_2\phi_{-2,4} - 18\mathbb{I}_3\phi_{-1,3}\,,\\
\omega^2_{76} =& -3(6\mathbb{I}_1\mathbb{I}_3 + 6\mathbb{I}_2\phi_{-1,3} - 2\phi^a_{1,3} - 5\mathbb{I}_3\phi^b_{0,2} + 4\phi^b_{1,3} + 3\phi_{-1,3}\phi^b_{2,0} + 3\mathbb{I}_3\phi^c_{0,2} + 3\phi_{-1,3}\phi^c_{2,0})\,,\\
\omega^2_{77} =& \mathbb{I}_1 + \frac{1}{2}(\phi^b_{0,2} + \phi^c_{0,2}).
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
\phi_{4,-2}(\boldsymbol{\tau}) =& \frac{3\Psi_1(\tau_1)^4}{4L_1(\boldsymbol{u})L_2(\boldsymbol{u})(1+u_1)^2\Psi_1(\tau_2)^2}(u_1-3)^2(u_1-1)u_1(3+u_1)(1+u_2)^2\\
&(-81 - 162u_1 + 189u_1^2 - 108u_1^3 - 135u_1^4 - 18u_1^5 - 5u_1^6 + 45u_2^2 - 54u_1u_2^2\\
&+ 135u_1^2u_2^2 - 36u_1^3u_2^2 - 21u_1^4u_2^2 - 6u_1^5u_2^2 + u_1^6u_2^2)\,,\\
\phi_{3,-1}(\boldsymbol{\tau}) =& -\frac{(-3+u_1)^2(-1+u_1)u_1(3+u_1)(1+u_2)\Psi_1(\tau_1)^3}{6(1+u_1)(u_1^2+u_2^2-u_1(-3+u_2^2))\Psi_1(\tau_2)}\,,\\
\phi^a_{2,0}(\boldsymbol{\tau}) =& \frac{(3+u_1^2)(u_1^2+6u_1-3)}{4(1+u_1)^2}\Psi_1(\tau_1)^2\,,\\
\phi^b_{2,0}(\boldsymbol{\tau}) =& -\frac{(-3+u_1)^2(u_1^2-u_2^2+u_1(3+u_2^2))\Psi_1(\tau_1)^2}{4(u_1^2+u_2^2-u_1(-3+u_2^2))}\,,\\
\phi^c_{2,0}(\boldsymbol{\tau}) =& -\frac{12\,\Psi_1(\tau_1)^2}{36L_1(\boldsymbol{u})L_2(\boldsymbol{u})(1+u_1)^2}(u_1^2+u_2^2-u_1(u_2^2-3))(3+u_1^2)(-3+6u_1+u_1^2)(9\\
&+ 6u_1 - 3u_1^2 - 6u_2 - 2u_1^2u_2 - 3u_2^2 - 2u_1u_2^2 + u_1^2u_2^2)(9 + 6u_1 - 3u_1^2 + 6u_2\\
&+ 2u_1^2u_2 - 3u_2^2 - 2u_1u_2^2 + u_1^2u_2^2)\,,\\
\phi_{4,0}(\boldsymbol{\tau}) =& -\frac{\Psi_1(\tau_1)^4}{24(1+u_1)^2(u_1^2+u_2^2-u_1(-3+u_2^2))^2}(-3+u_1)^2(2u_1^8 + 18u_2^4 + u_1^7(21-5u_2^2)\\
&- 9u_1u_2^2(-15+7u_2^2) + u_1^5(198 - 31u_2^2 + u_2^4) + u_1^6(93 - 14u_2^2 + u_2^4)\\
&+ 4u_1^4(36 - 13u_2^2 + 4u_2^4) - 3u_1^3(9 - 31u_2^2 + 22u_2^4) + 3u_1^2(27 - 42u_2^2 + 31u_2^4))\,,
\end{aligned}$$

$$
\begin{aligned}
\phi_{-2,4}(\boldsymbol{\tau}) =& \frac{\Psi_1(\tau_2)^4}{36(1+u_2)^3 L_1(\boldsymbol{u}) L_2(\boldsymbol{u}) \Psi_1(\tau_1)^2}(1+u_1)^2 u_2^2 (9-9u_2-u_2^2+u_2^3)(81-135u_2^2+27u_2^4 \\
&-5u_2^6+2u_1(3+u_2^2)^3+u_1^2(45-27u_2^2+15u_2^4-u_2^6))\,, \\
\phi_{-1,3}(\boldsymbol{\tau}) =& -\frac{(1+u_1)(u_2-3)(u_2-1)u_2^2(3+u_2)\Psi_1(\tau_2)^3}{54(1+u_2)^2(u_1^2+u_2^2-u_1(u_2^2-3))\Psi_1(\tau_1)}\,, \\
\phi^a_{0,2}(\boldsymbol{\tau}) =& -\frac{(u_2^2-3)(u_2^2+3)}{12(1+u_2)^2}\Psi_1(\tau_2)^2\,, \\
\phi^b_{0,2}(\boldsymbol{\tau}) =& -\frac{u_2^2(9-5u_2^2-u_1^2(-5+u_2^2)+2u_1(3+u_2^2))\Psi_1(\tau_2)^2}{6(1+u_2)^2(u_1^2+u_2^2-u_1(-3+u_2^2))}\,, \\
\phi^c_{0,2}(\boldsymbol{\tau}) =& \frac{\Psi_1(\tau_2)^2}{36(1+u_2)^2 L_1(\boldsymbol{u}) L_2(\boldsymbol{u})}(u_1^2+u_2^2-u_1(u_2^2-3))(u_2^2-3)(u_2^2+3)(9+6u_1-3u_1^2-12u_2 \\
&+24u_1u_2+4u_1^2u_2+3u_2^2+2u_1u_2^2-u_1^2u_2^2)(9+6u_1-3u_1^2+12u_2-24u_1u_2-4u_1^2u_2 \\
&+3u_2^2+2u_1u_2^2-u_1^2u_2^2)\,, \\
\phi_{0,4}(\boldsymbol{\tau}) =& \frac{\Psi_1(\tau_2)^4}{216(1+u_2)^4(u_1^2+u_2^2-u_1(-3+u_2^2))^2}u_2^2(81u_2^2-243u_2^4+51u_2^6-17u_2^8 \\
&+u_1^4(-153+51u_2^2-27u_2^4+u_2^6)+u_1^2(-1215+612u_2^2-218u_2^4+68u_2^6-15u_2^8) \\
&+u_1^3(-837+342u_2^2-28u_2^4+10u_2^6+u_2^8) \\
&+u_1(-243-270u_2^2+84u_2^4-114u_2^6+31u_2^8))\,, \\
\phi_{1,1}(\boldsymbol{\tau}) =& -\frac{\Psi_1(\tau_2)\Psi_1(\tau_1)}{36(1+u_1)(1+u_2)(u_1^2+u_2^2-u_1(-3+u_2^2))}(u_1^4(3+u_2^2)+3u_2^2(3+u_2^2) \\
&+u_1^3(27-20u_2^2+u_2^4)-3u_1(9-20u_2^2+3u_2^4)+u_1^2(45-34u_2^2+5u_2^4)),\,, \\
\phi_{2,2}(\boldsymbol{\tau}) =& -\frac{\Psi_1(\tau_2)^2\Psi_1(\tau_2)^2}{72L_1(\boldsymbol{u})^2L_2(\boldsymbol{u})^2(1+u_2)}(u_1-3)^2u_1(-3+2u_1+u_1^2)u_2^2 \\
&(9-9u_2-u_2^2+u_2^3)(u_1^8(513-180u_2^2+70u_2^4-20u_2^6+u_2^8)+8u_1^7(567-72u_2^2 \\
&-30u_2^4+16u_2^6-u_2^8)+216u_1(81-144u_2^2+30u_2^4+8u_2^6-7u_2^8)+4u_1^6(5103 \\
&-1692u_2^2+306u_2^4+196u_2^6-9u_2^8)+8u_1^5(4293+342u_2^4+216u_2^6-19u_2^8) \\
&+27(243-540u_2^2+210u_2^4-60u_2^6+19u_2^8)+24u_1^3(1539-1944u_2^2 \\
&-342u_2^4-53u_2^8)+36u_1^2(-729+1764u_2^2+306u_2^4-188u_2^6+63u_2^8) \\
&+2u_1^4(18387-4860u_2^2-414u_2^4-540u_2^6+227u_2^8))\,, \\
\phi^a_{3,1}(\boldsymbol{\tau}) =& \Psi_1(\tau_1)^3\Psi_1(\tau_2)\frac{(u_1-3)^2(u_1-1)u_1(3+u_1)(u_2^2-3)(3+u_2^2)}{72(1+u_1)(1+u_2)(3u_1+u_1^2+u_2^2-u_1u_2^2)}\,, \\
\phi^b_{3,1}(\boldsymbol{\tau}) =& \frac{\Psi_1(\tau_2)\Psi_1(\tau_1)^3}{72(1+u_1)(1+u_2)(u_1^2+u_2^2-u_1(-3+u_2^2))^2}(u_1-3)^2u_1(-3+2u_1+u_1^2) \\
&(u_2^2(-27+10u_2^2+u_2^4)+u_1^2(-9-10u_2^2+3u_2^4)-u_1(27+3u_2^2+u_2^4+u_2^6))\,, \\
\phi_{4,2}(\boldsymbol{\tau}) =& \frac{\Psi_1(\tau_2)^2\Psi_1(\tau_1)^4}{864(1+u_1)^2(1+u_2)^2(u_1^2+u_2^2-u_1(-3+u_2^2))^3}(-3+u_1)^2(18u_2^6(-63 \\
&+30u_2^2+u_2^4)+2u_1^{10}(-9-30u_2^2+7u_2^4)-27u_1u_2^4(279-291u_2^2+73u_2^4+3u_2^6) \\
&-u_1^9(243+657u_2^2-291u_2^4+31u_2^6)+2u_1^8(-702-1476u_2^2+727u_2^4-90u_2^6+5u_2^8)
\end{aligned}
$$

$$
\begin{aligned}
&+6u_1^2u_2^2(-1215+2430u_2^2-2181u_2^4+492u_2^6+26u_2^8)+2u_1^4u_2^2(-10854+6417u_2^2\\
&-2455u_2^4+611u_2^6+41u_2^8)+2u_1^6(-3321-5499u_2^2+2455u_2^4-713u_2^6+134u_2^8)\\
&-u_1^7(4293+6759u_2^2-3752u_2^4+872u_2^6-45u_2^8+u_2^{10})-3u_1^5(1215+7317u_2^2\\
&-2526u_2^4-842u_2^6+271u_2^8+5u_2^{10})\\
&-3u_1^3(729-3645u_2^2+7848u_2^4-3752u_2^6+751u_2^8+53u_2^{10}))\,,\\
\phi^a_{1,3}(\boldsymbol{\tau}) =&-\Psi_1(\tau_1)\Psi_1(\tau_2)^3\frac{(3+u_1^2)(u_1^2+6u_1-3)(u_2-3)(u_2-1)u_2^2(3+u_2)}{216(1+u_1)(1+u_2)^2(3u_1+u_1^2+u_2^2-u_1u_2^2)}\,,\\
\phi^b_{1,3}(\boldsymbol{\tau}) =&-\frac{\Psi_1(\tau_2)^3\Psi_1(\tau_1)}{108(1+u_1)(1+u_2)^2(u_1^2+u_2^2-u_1(-3+u_2^2))^2}u_2^2(9-9u_2-u_2^2+u_2^3)\\
&(4u_1^5+u_1^6-9u_2^2+12u_1u_2^2+u_1^4(2-5u_2^2)+u_1^2(45-2u_2^2)+4u_1^3(3+u_2^2))\,,\\
\phi_{2,4}(\boldsymbol{\tau}) =&\frac{-\Psi_1(\tau_2)^4\Psi_1(\tau_1)^2}{2592(1+u_1)^2(1+u_2)^4(u_1^2+u_2^2-u_1(-3+u_2^2))^3}u_2^2(u_1^{10}(531-33u_2^2+9u_2^4+5u_2^6)\\
&-9u_2^4(405+81u_2^2-33u_2^4+59u_2^6)-2u_1^9(-2133+531u_2^2+129u_2^4-71u_2^6+8u_2^8)\\
&+18u_1u_2^2(-1944+1917u_2^2-387u_2^4-177u_2^6+79u_2^8)\\
&+u_1^6(25920-3960u_2^2-2138u_2^4+1454u_2^6+594u_2^8-366u_2^{10})\\
&+u_1^4(266814-48114u_2^2-13086u_2^4+2138u_2^6+440u_2^8-320u_2^{10})\\
&+u_1^8(7506-7902u_2^2+2683u_2^4-737u_2^6-17u_2^8+3u_2^{10})\\
&+2u_1^7(-3483-5355u_2^2+5990u_2^4-1554u_2^6+293u_2^8+13u_2^{10})\\
&-6u_1^3(-9477-23733u_2^2+13986u_2^4-5990u_2^6+595u_2^8+43u_2^{10})\\
&+6u_1^5(34749-13635u_2^2+564u_2^4+188u_2^6-505u_2^8+143u_2^{10})\\
&-3u_1^2(6561-4131u_2^2-19899u_2^4+8049u_2^6-2634u_2^8+278u_2^{10}))\,.
\end{aligned}
\tag{B.3}
$$

C Independence of the differential forms

In this appendix we present the details of the proof that the differential forms $\omega_{ij} = \mathrm{d}\tau_1\,\omega^1_{ij} + \mathrm{d}\tau_2\,\omega^2_{ij}$ defined in appendix B define independent cohomology classes. We start by discussing some technical result about connections and gauge transformations before we turn to the proof of the independence of the differential forms.

C.1 Connections and gauge transformations

We consider a differential equation of the form

$$
\mathrm{d}\boldsymbol{I}(x) = \boldsymbol{\Omega}(x)\boldsymbol{I}(x)\,, \tag{C.1}
$$

where x is a complex variable and $\boldsymbol{I}(x)$ is a n-dimensional vector. The differential equation matrix has the form

$$
\boldsymbol{\Omega}(x) = \sum_{p=1}^{N}\boldsymbol{A}_p\,\omega_p\,, \tag{C.2}
$$

where A_p are some non-commutative variables (e.g., matrices). The differential forms are $\omega_p = \mathrm{d}x\,\xi_p(x)$, where the ξ_p are taken from some function field $\mathcal{F}$ that is differentially closed (meaning that it is closed under taking derivatives in x) and whose field of constants is $\mathbb{C}$. We assume that the ω_p define independent cohomology classes *with respect to* $\mathcal{F}$, i.e., we assume that whenever there are $c_1, \ldots, c_N \in \mathbb{C}$ and $f \in \mathcal{F}$ such that

$$\sum_{p=1}^{N} c_p \, \omega_p = \mathrm{d}f \,, \tag{C.3}$$

then necessarily $c_1 = \ldots = c_N = 0$ and f is a constant. The solution to this equation can be written in terms of the path-ordered exponential in eqs. (2.9) and (2.10), and the fact that the differential forms define independent cohomology classes implies that the iterated integrals in eq. (2.10) formed from of the letters $\omega_1, \ldots, \omega_N$ are linearly independent over $\mathcal{F}$ [119].

Consider a path γ from a point x_0 to a generic point x (and which avoids possible singularities from any of the ξ_p), and define

$$\mathbb{I}(x) = \int_\gamma \omega_N = \int_{x_0}^{x} \mathrm{d}x' \, \xi_N(x') \,. \tag{C.4}$$

Due to our assumption that the ω_p define independent cohomology classes, it follows that $\mathbb{I}$ does not lie in $\mathcal{F}$. We can then consider the field $\mathcal{G} := \mathcal{F}(\mathbb{I})$ whose elements are rational functions in $\mathbb{I}$ with coefficients in $\mathcal{F}$. Note that since $\partial_x \mathbb{I} = \xi_N \in \mathcal{F}$, $\mathcal{G}$ is also differentially closed, and its field of constants is still $\mathbb{C}$.

Consider the gauge transformation

$$\boldsymbol{J}(x) = e^{-A_N \mathbb{I}(x)} \boldsymbol{I}(x) \,. \tag{C.5}$$

In this gauge, the differential equation takes the form

$$\mathrm{d}\boldsymbol{J}(c) = e^{-A_N \mathbb{I}(x)} \widetilde{\boldsymbol{\Omega}}(x) e^{A_N \mathbb{I}(x)} \boldsymbol{J}(x) \,, \qquad \widetilde{\boldsymbol{\Omega}}(x) = \sum_{p=1}^{N-1} \boldsymbol{A}_p \, \omega_p \,, \tag{C.6}$$

and the differential equation matrix is given by

$$e^{-A_N \mathbb{I}(x)} \widetilde{\boldsymbol{\Omega}} e^{A_N \mathbb{I}(x)} = \sum_{r=0}^{\infty} \sum_{p=1}^{N-1} \frac{(-1)^r}{r!} \, \mathrm{ad}_{A_N}^r(A_p) \, \omega_p^{(r)} \,, \tag{C.7}$$

with $\mathrm{ad}_{A_N}(A_p) = [A_N, A_p]$ and

$$\omega_p^{(r)} := \mathbb{I}(x)^r \omega_p \,. \tag{C.8}$$

Our goal is to show that the differential forms $\omega_p^{(r)}$, $r \geq 0$ and $1 \leq p < N$, which are the letters that appear in the iterated integrals in the path-ordered exponential that solves eq. (C.5), define independent cohomology classes with respect to $\mathcal{G}$, i.e., whenever there are $c_{rp} \in \mathbb{C}$ and $g \in \mathcal{G}$ such that

$$\sum_{r \geq 0} \sum_{p=1}^{N-1} c_{rp} \, \omega_p^{(r)} = \mathrm{d}g \,, \tag{C.9}$$

then necessarily $c_{rp} = 0$ and g is a constant. To see that this is true, we integrate eq. (C.9) over the path γ used to define $\mathbb{I}$. We obtain

$$\sum_{r\geq 0}\sum_{p=1}^{N-1} c_{rp}\, I_\gamma\big(\omega_p^{(r)}\big) = g(x) - g(x_0) =: \tilde{g}(x)\,, \tag{C.10}$$

and $I_\gamma(\omega)$ denotes the iterated integral of length one, cf. eq. (2.12). Using the fact that iterated integrals form a shuffle algebra, we find

$$\begin{aligned} I_\gamma\big(\omega_p^{(r)}\big) &= \int_\gamma \omega_p^{(r)} = \int_\gamma \omega_p\, \mathbb{I}(x)^r = \int_\gamma \omega_p \left(\int_\gamma \omega_N\right)^r = r! \int_\gamma \omega_p \underbrace{\omega_N \cdots \omega_N}_{r \text{ times}} \\ &= r!\, I_\gamma(\omega_p, \underbrace{\omega_N, \ldots, \omega_N}_{r \text{ times}}) = r!\, I_\gamma(\omega_p, \omega_N^r)\,, \end{aligned} \tag{C.11}$$

where in the last equality we introduced the simplified notation

$$I_\gamma(\omega_p, \omega_N^r) := I_\gamma(\omega_p, \underbrace{\omega_N, \ldots, \omega_N}_{r \text{ times}})\,. \tag{C.12}$$

Since $g \in \mathcal{G}$, also $\tilde{g} \in \mathcal{G}$, and so we can write

$$\tilde{g} = \frac{P(\mathbb{I})}{Q(\mathbb{I})}\,, \tag{C.13}$$

where P and Q are polynomials,

$$\begin{aligned} P(\mathbb{I}) &= \sum_{k=0}^{d_P} \frac{p_k}{k!}\, \mathbb{I}^k = \sum_{k=0}^{d_P} p_k\, I_\gamma(\omega_N^k)\,, \\ Q(\mathbb{I}) &= \sum_{k=0}^{d_Q} \frac{q_k}{k!}\, \mathbb{I}^k = \sum_{k=0}^{d_Q} q_k\, I_\gamma(\omega_N^k)\,, \end{aligned} \tag{C.14}$$

with $p_k, q_k \in \mathcal{F}$. Equation (C.10) can then be cast in the form

$$\sum_{r\geq 0}\sum_{p=1}^{N-1}\sum_{k=0}^{d_Q} r!\, c_{rp}\, q_k\, I_\gamma(\omega_N^k) I_\gamma(\omega_p, \omega_N^r) = \sum_{k=0}^{d_P} p_k\, I_\gamma(\omega_N^k)\,. \tag{C.15}$$

Using the shuffle algebra structure of iterated integrals, we can write $I_\gamma(\omega_N^k)I_\gamma(\omega_p, \omega_N^r)$ again as a linear combination (with integer coefficients) of iterated integrals of length $k + r+1 > 0$. We then see that both sides of eq. (C.15) are linear combinations with coefficients in $\mathcal{F}$ of iterated integrals made from the letters $\omega_1, \ldots, \omega_N$.Every iterated integral on the left-hand side has length at least 1 and involves exactly one letter ω_p with $p \neq N$, while those on the right-hand side only involve the letter ω_N, except for the integral with $k = 0$, which has length 0. Since we know that these iterated integrals are linearly independent over $\mathcal{F}$, both sides must vanish independently. In particular this means $p_0 = \ldots = p_{d_P} = 0$, and so $P = 0$. This implies $\tilde{g} = 0$, or equivalently g must be a constant. Equation (C.10) then takes the form

$$\sum_{r\geq 0}\sum_{p=1}^{N-1} c_{rp}\, I_\gamma\big(\omega_p^{(r)}\big) = \sum_{r\geq 0}\sum_{p=1}^{N-1} r!\, c_{rp}\, I_\gamma(\omega_p, \omega_N^r) = 0\,. \tag{C.16}$$

The linear independence of the $I_\gamma(\omega_p, \omega_N^r)$ then implies $c_{rp} = 0$, which finishes the proof.

C.2 Independence of the differential forms

We now turn to the proof that the differential forms ω_{ij} that appear in the canonical differential equation in section 3 define independent cohomology classes.

We start by assuming that τ_2 has been fixed to some constant generic value, and we only consider the one-forms $\mathrm{d}\tau_1\,\omega^1_{ij}$. We know from appendix B that the ω^1_{ij} are constructed from the ε-functions $\mathbb{I}_k(\cdot,\tau_2)$ in eq. (3.4) and the meromorphic modular forms $\phi_{k,l}(\cdot,\tau_2), \phi^r_{k,l}(\cdot,\tau_2) \in \mathcal{M}_k(\Gamma^1(12))$ from appendix B. From the explicit expression in appendix B, we see that all relevant meromorphic modular forms for $\Gamma^1(12)$ have positive weight (seen as a modular form in τ_1), and we can group them into three classes:

- **Class I:** Functions of class I have at most simple poles:

$$1\,,\,\phi_{3,-1}\,,\,\phi_{4,-2}\,,\,\phi^a_{2,0}\,,\,\phi^a_{3,1}\,,\,\phi^b_{0,2}\,,\,\phi^b_{2,0}\,,\,\phi^c_{0,2}\,,\,\phi^c_{2,0}\,. \tag{C.17}$$

- **Class II:** Functions of class II have at least double poles double or at most triple poles:

$$\phi_{2,2}\,,\,\phi_{4,0}\,,\,\phi_{4,2}\,,\,\phi^b_{3,1}\,. \tag{C.18}$$

- **Class III:** Functions of class III do not enter the ω^1_{ij} directly, but only through the ε-functions in eqs. (3.4) and (3.4):[8]

$$\partial_{\tau_2}\phi_{3,-1}(\cdot,\tau_2)\,,\qquad \partial_{\tau_2}\phi_{4,-2}(\cdot,\tau_2)\,,\qquad \partial^2_{\tau_2}\phi_{4,-2}(\cdot,\tau_2)\,. \tag{C.19}$$

  We know from the discussion in section 3 that these functions have double or triple poles, respectively.

By inspection, we see that the pole orders of these meromorphic modular forms satisfy the criterion to lie in the vector space $\widetilde{\mathcal{M}}(\Gamma^1(12))$ defined in appendix A. We denote by $\widetilde{\mathcal{N}}_1 \subseteq \widetilde{\mathcal{M}}(\Gamma^1(12))$ the linear span of these meromorphic modular forms. From eq. (A.8) we see that

$$\widetilde{\mathcal{N}}_1 \cap \partial_{\tau_1}\mathcal{QM}(\Gamma^1(12)) = \{0\}\,. \tag{C.20}$$

Said differently, if we pick a basis $\{f_p\}$ for $\widetilde{\mathcal{N}}_1$, then the differential forms $\mathrm{d}\tau_1\, f_p(\tau_1)$ define independent cohomology classes (seen as functions of τ_1 only).

So far, we have only considered meromorphic modular forms. However, the functions of class III do not appear directly in the definition of the ω^1_{ij}, but they only enter through the ε-functions $\mathbb{I}_k(\cdot,\tau_2)$. Applying the result from appendix C.1 to the functions of class III one at the time, we can eliminate them one after and replace them with the ε-functions. We consider the vector space $\widetilde{\mathcal{N}}^\varepsilon_1$ generated by products of ε-functions with a function of class I and II.[9] Equation (C.20) together the analysis from appendix C.1 then implies that

$$\widetilde{\mathcal{N}}^\varepsilon_1 \cap \partial_{\tau_1}\mathcal{QM}(\Gamma^1(12))(\mathbb{I}_1,\mathbb{I}_2,\mathbb{I}_3) = \{0\}\,, \tag{C.21}$$

[8] We of course take the derivative in τ_2 before fixing τ_2 to a constant value.

[9] Note that along the way we also introduce products of functions of class III and ε-functions. Those, however, do not appear in the differential forms from section B.

where the elements of $\mathcal{QM}(\Gamma^1(12))(\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3)$ are rational functions in ε-functions with coefficients that are quasi-modular forms.

Let $\widetilde{\mathcal{N}}_1^\omega$ denote the linear span of the functions ω_{ij}^1. By construction, we have $\widetilde{\mathcal{N}}_1^\omega \subseteq \widetilde{\mathcal{N}}_1^\varepsilon$, and so eq. (C.21) implies

$$\widetilde{\mathcal{N}}_1^\omega \cap \partial_{\tau_1} \mathcal{QM}(\Gamma^1(12))(\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3) = \{0\} \,. \tag{C.22}$$

We can of course repeat exactly the same construction starting from the ω_{ij}^2. If we denote their span by $\widetilde{\mathcal{N}}_2^\omega$, we arrive at the conclusion that

$$\widetilde{\mathcal{N}}_2^\omega \cap \partial_{\tau_2} \mathcal{QM}(\Gamma_1(12))(\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3) = \{0\} \,. \tag{C.23}$$

We are now in the position to prove that the differential forms ω_{ij} define independent cohomology classes. Assume that there are constants c_{ij} and a function

$$F \in \mathcal{QM}(\Gamma^1(12), \Gamma_1(12))(\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3) \tag{C.24}$$

such that

$$\sum_{ij} c_{ij}\, \omega_{ij} = \mathrm{d}F(\tau_1, \tau_2) \,, \tag{C.25}$$

where the sum runs over a linearly independent set of the ω_{ij}.[10] Equation (C.25) is equivalent to the system

$$\sum_{ij} c_{ij}\, \omega_{ij}^k = \partial_{\tau_k} F(\tau_1, \tau_2) \,, \qquad k = 1, 2 \,. \tag{C.26}$$

We focus on $k = 1$, and we fix τ_2 to a constant generic value. The left-hand side lies in $\widetilde{\mathcal{N}}_1^\omega$ by construction, while the right-hand side lies in $\partial_{\tau_1} \mathcal{QM}(\Gamma^1(12))(\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3)$. Equation (C.22) then implies $\partial_{\tau_1} F = 0$. Applying exactly the same reasoning to $k = 2$, we conclude that also $\partial_{\tau_2} F = 0$, and so $\mathrm{d}F = 0$. Hence, eq. (C.25) reduces to

$$\sum_{ij} c_{ij}\, \omega_{ij} = 0 \,, \tag{C.27}$$

and so $c_{ij} = 0$ from the linear independence of the ω_{ij}.

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