Every Q-polynomial distance-regular graph is sharp over $\mathbb{R}$

Blas Fernández* Jae-Ho Lee† Jongyook Park‡

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Abstract

Let Γ denote a distance-regular graph with vertex set X and diameter $D \geq 3$. Fix a vertex $x \in X$. Let the field $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathrm{Mat}_X(\mathbb{F})$ denote the $\mathbb{F}$-algebra of matrices whose rows and columns are indexed by X and all entries in $\mathbb{F}$. The Terwilliger algebra $T^{\mathbb{F}} = T^{\mathbb{F}}(x)$ is the subalgebra of $\mathrm{Mat}_X(\mathbb{F})$ generated by the adjacency matrix A of Γ and the dual primitive idempotents $\{E_i^*\}_{i=0}^D$ of Γ with respect to x. Let $\{E_i\}_{i=0}^D$ denote the primitive idempotents of A. Assume that the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial. Let W denote an irreducible $T^{\mathbb{F}}$-module. We say that W is sharp over $\mathbb{F}$ whenever $\dim(E_r^* W) = 1$, where r is the endpoint of W. It is known, by Nomura and Terwilliger (2008), that every irreducible $T^{\mathbb{C}}$-module is sharp. In this paper, we prove that every irreducible $T^{\mathbb{R}}$-module is sharp. Once this is established, we obtain four additional results: (i) if W is an irreducible $T^{\mathbb{R}}$-module, then its complexification $W^{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ is an irreducible $T^{\mathbb{C}}$-module; (ii) two irreducible $T^{\mathbb{R}}$-modules W_1 and W_2 are isomorphic if and only if their complexifications $W_1^{\mathbb{C}}$ and $W_2^{\mathbb{C}}$ are isomorphic as $T^{\mathbb{C}}$-modules; (iii) if $\bigoplus_{i=1}^h \mathrm{Mat}_{n_i}(\mathbb{C})$ is the Wedderburn decomposition of $T^{\mathbb{C}}$, then $\bigoplus_{i=1}^h \mathrm{Mat}_{n_i}(\mathbb{R})$ is the Wedderburn decomposition of $T^{\mathbb{R}}$; (iv) each of the subalgebras $E_1^*TE_1^*$, E_1TE_1, $E_D^*TE_D^*$, and E_DTE_D is commutative and every element of these algebras is a symmetric matrix.

Keywords: Q-polynomial, distance-regular graph, Terwilliger algebra, sharpness, Wedderburn decomposition

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1 Introduction

This paper is concerned with distance-regular graphs [1, 4]. These graphs possess a high degree of combinatorial regularity and a rich algebraic structure [3, 4]. Distance-regular graphs were introduced by Biggs [3]; notable examples include the Hamming graphs, Johnson graphs, and Grassmann graphs [4, Chapter 9]. Since their introduction, distance-regular graphs have been extensively studied, and connections with other areas have been found, such as partially ordered

*IMFM, Jadranska 19, 1000 Ljubljana, Slovenia; University of Primorska, UP FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia; Andrej Marušič Institute, University of Primorska, Muzejski trg 2, 6000 Koper, Slovenia. Email: blas.fernandez@famnit.upr.si

†Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA. Email: jaeho.lee@unf.edu

‡Department of Mathematics, Kyungpook National University, Daegu, 41566, Republic of Korea. Email: jongyook@knu.ac.kr (Corresponding author)

sets, representation theory, quantum groups, and orthogonal polynomials; see [1, 4, 7, 38] and references therein. The Q-polynomial property of a distance-regular graph was introduced by Delsarte in the context of coding theory [8]. The subconstituent algebra (or Terwilliger algebra) was introduced by Terwilliger in his study of Q-polynomial distance-regular graphs [33–35]. Since then, the importance of the Terwilliger algebra has grown significantly; see [2, 5, 9, 10, 12, 14, 20, 23–25, 30–32, 39].

Throughout this paper, let $\mathbb{F}$ denote a field. Let Γ denote a distance-regular graph with vertex set X. Let $\mathbb{F}^X$ denote the $\mathbb{F}$-vector space consisting of the column vectors whose coordinates are indexed by X and all entries in $\mathbb{F}$. Let $\mathrm{Mat}_X(\mathbb{F})$ denote the $\mathbb{F}$-algebra of matrices whose rows and columns are indexed by X and all entries in $\mathbb{F}$. The algebra $\mathrm{Mat}_X(\mathbb{F})$ acts on $\mathbb{F}^X$ by left multiplication. We call $\mathbb{F}^X$ the *standard module*. In the literature on distance-regular graphs, one typically assumes that the field $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Each choice of field has its own advantages and disadvantages depending on the perspective from which Γ is studied. These advantages and disadvantages are discussed in the following paragraphs.

When studying Γ from a geometric perspective, it is convenient to assume $\mathbb{F} = \mathbb{R}$ since the standard module $\mathbb{R}^X$ has a Euclidean space structure. Indeed, working over $\mathbb{R}$ facilitates geometric analysis of the eigenspaces, the cosine sequences corresponding to its eigenvalues, and representations of Γ (cf. [38, Section 4]). Such a geometric analysis is still available over $\mathbb{C}$, but it is generally more cumbersome since the Hermitean dot product on $\mathbb{C}^X$ is more complicated than the dot product on $\mathbb{R}^X$.

When studying Γ from the perspective of the Terwilliger algebra, it is natural to work over $\mathbb{C}$. To describe this perspective, we recall some notation and preliminaries. Fix a vertex $x \in X$. The *Terwilliger algebra* $T = T(x)$ of Γ is the subalgebra of $\mathrm{Mat}_X(\mathbb{F})$ generated by the adjacency matrix of Γ and the dual primitive idempotents of Γ with respect to x; see Section 7. When it is important to emphasize the field $\mathbb{F}$, we will write $T = T^{\mathbb{F}}$. The algebra T is closed under the conjugate-transpose map and hence semisimple [33, Lemma 3.4]. By the Wedderburn theory [6], the algebra T decomposes as a direct sum of full matrix algebras. The precise form of this decomposition depends on whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. The Wedderburn decomposition of $T^{\mathbb{C}}$ is described by the $\mathbb{C}$-algebra isomorphism

$$T^{\mathbb{C}} \cong \bigoplus_{i=1}^{h} \mathrm{Mat}_{n_i}(\mathbb{C}), \tag{1}$$

where $h, n_1, \ldots, n_h$ are positive integers and $\mathrm{Mat}_{n_i}(\mathbb{C})$ denotes the $\mathbb{C}$-algebra of all $n_i \times n_i$ matrices with entries in $\mathbb{C}$. From this decomposition, it follows that every $T^{\mathbb{C}}$-module decomposes as a direct sum of irreducible $T^{\mathbb{C}}$-submodules, each corresponding to one of the simple components $\mathrm{Mat}_{n_i}(\mathbb{C})$ in (1). On the other hand, the Wedderburn decomposition of $T^{\mathbb{R}}$ is described by the $\mathbb{R}$-algebra isomorphism

$$T^{\mathbb{R}} \cong \bigoplus_{i=1}^{\ell} \mathrm{Mat}_{m_i}(\mathbb{D}_i), \tag{2}$$

where $\ell, m_1, \ldots, m_\ell$ are positive integers and each $\mathbb{D}_i$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, with $\mathbb{H}$ denoting the quaternions. Comparing the decompositions (1) and (2), we can see that (2) is relatively more complicated. For this reason, it is natural to work over $\mathbb{C}$.

In the present paper, we assume that Γ is Q-polynomial. In this case, every $\mathbb{D}_i$ in (2) turns out to be $\mathbb{R}$; this will be shown in Theorem 1.5. This result follows from a property of Γ called sharpness. In this paper, we study the sharpness property of Γ and its effect on the structure of $T^{\mathbb{R}}$. We now summarize our main results after several preliminary remarks concerning Γ and its Terwilliger algebra; formal definitions appear in Sections 4–8.

Throughout the paper, we assume that Γ has diameter $D \geq 3$. Assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $A \in \mathrm{Mat}_X(\mathbb{F})$ denote the adjacency matrix of Γ. Let $\{E_i\}_{i=0}^D$ denote the primitive idempotents of A. Assume that the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial. Fix a vertex $x \in X$. Let $A^* = A^*(x) \in \mathrm{Mat}_X(\mathbb{F})$ denote the diagonal matrix with the (y, y)-entry $(A^*)_{y,y} = |X|(E_1)_{x,y}$ for all $y \in X$. We call A^* the dual adjacency matrix of Γ with respect to x. For $0 \leq i \leq D$, let $E_i^* = E_i^*(x) \in \mathrm{Mat}_X(\mathbb{F})$ denote the diagonal matrix with (y, y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1 & \text{if} \quad \partial(x, y) = i, \\ 0 & \text{if} \quad \partial(x, y) \neq i \end{cases} \qquad (y \in X).$$

The matrices $\{E_i^*\}_{i=0}^D$ are the primitive idempotents of A^*. The Terwilliger algebra $T = T(x)$ of Γ is generated by A and A^*. We now recall the notion of a T-module. By a T-module, we mean a T-submodule of $V = \mathbb{F}^X$. Define a form $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ such that for $u, v \in V$,

$$\langle u, v\rangle = \begin{cases} \overline{u}^\top v & \text{if} \quad \mathbb{F} = \mathbb{C}, \\ u^\top v & \text{if} \quad \mathbb{F} = \mathbb{R}, \end{cases}$$

where $\top$ denotes transpose and $-$ denotes complex conjugation. By [33, Lemma 3.4] V decomposes into an orthogonal direct sum of irreducible T-modules. Moreover, every irreducible T-module W is the orthogonal direct sum of the nonvanishing E_iW $(0 \leq i \leq D)$ and the orthogonal direct sum of the nonvanishing E_i^*W $(0 \leq i \leq D)$.

Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d; see Section 7. By [33, Section 3], for $0 \leq i \leq D$ we have $E_i^*W \neq 0$ if and only if $r \leq i \leq r + d$. Moreover, the sum $W = \sum_{i=0}^d E_{r+i}^*W$ is direct and its summands are the eigenspaces of A^* on W. For $0 \leq i \leq D$, we have $E_iW \neq 0$ if and only if $s \leq i \leq s + d$. The sum $W = \sum_{i=0}^d E_{s+i}W$ is direct and its summands are the eigenspaces of A on W. For $0 \leq i \leq d$,

$$AE_{r+i}^*W \subseteq E_{r+i-1}^*W + E_{r+i}^*W + E_{r+i+1}^*W,$$
$$A^*E_{s+i}W \subseteq E_{s+i-1}W + E_{s+i}W + E_{s+i+1}W.$$

By the above comments, A, A^* act on W as a tridiagonal pair; see Definition 3.1. Define a sequence of scalars $\{\rho_i\}_{i=0}^d$ by $\rho_i = \dim(E_{r+i}^*W)$ for $0 \leq i \leq d$. By [15, Corollary 5.7], $\rho_i = \dim(E_{s+i}W)$ for $0 \leq i \leq d$. By [15, Corollaries 5.7, 6.6], the sequence $\{\rho_i\}_{i=0}^d$ satisfies $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq \lfloor d/2 \rfloor$. The sequence $\{\rho_i\}_{i=0}^d$ is called the shape of W. The T-module W is called *sharp* whenever $\rho_0 = 1$ [27, Definition 1.5]. We say Γ is *sharp* over $\mathbb{F}$ whenever, for every vertex y, each irreducible $T(y)$-module is sharp.

In [28], Nomura and Terwilliger showed that every irreducible $T^{\mathbb{C}}$-module is sharp. Hence Γ is sharp over $\mathbb{C}$. Motivated by this result, it is natural to ask whether Γ is also sharp over $\mathbb{R}$. Although the sharpness of Γ over $\mathbb{R}$ has been believed, to the best of our knowledge this fact has not been formally proved or explicitly stated in the literature. We now present the first main result of this paper.

Theorem 1.1. *Every irreducible $T^{\mathbb{R}}$-module is sharp.*

Corollary 1.2. *Every Q-polynomial distance-regular graph is sharp over $\mathbb{R}$.*

The proofs of the theorem and the corollary appear in Section 10.

Building on Theorem 1.1, we now present four additional results. Let W denote an irreducible $T^{\mathbb{R}}$-module. By extending the ground field from $\mathbb{R}$ to $\mathbb{C}$, we turn W into a $T^{\mathbb{C}}$-module. We denote this module by $W^{\mathbb{C}}$ and call it the *complexification* of W; see Section 11 for the formal definition. This raises a natural question: is the $T^{\mathbb{C}}$-module $W^{\mathbb{C}}$ irreducible? We answer this question in the following theorem, which is our second main result.

Theorem 1.3. *Let W denote an irreducible $T^{\mathbb{R}}$-module. Then the complexification $W^{\mathbb{C}}$ is an irreducible $T^{\mathbb{C}}$-module.*

Our third main result concerns the effect of complexification on the isomorphism classes.

Theorem 1.4. *Let W_1, W_2 denote irreducible $T^{\mathbb{R}}$-modules. Then W_1 and W_2 are isomorphic as $T^{\mathbb{R}}$-modules if and only if $W_1^{\mathbb{C}}$ and $W_2^{\mathbb{C}}$ are isomorphic as $T^{\mathbb{C}}$-modules.*

Our fourth main result concerns the Wedderburn decompositions of $T^{\mathbb{C}}$ and $T^{\mathbb{R}}$.

Theorem 1.5. *We refer to the Wedderburn decompositions of $T^{\mathbb{C}}$ in (1) and $T^{\mathbb{R}}$ in (2). Then*

$$\ell = h, \qquad m_i = n_i, \qquad \mathbb{D}_i = \mathbb{R} \qquad (1 \leq i \leq \ell).$$

The proofs of Theorems 1.3, 1.4, and 1.5 appear in Section 11.

Our last main result is motivated by [36] and stated below.

Theorem 1.6. *For either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ the following algebras*

$$E_1^* T E_1^*, \qquad E_1 T E_1, \qquad E_D^* T E_D^*, \qquad E_D T E_D \tag{3}$$

are commutative. Moreover, every element of these algebras is a symmetric matrix.

The proof of Theorem 1.6 appears in Section 13.

This paper is organized as follows. In Section 2, we provide some preliminaries. In Section 3, we review tridiagonal pairs and tridiagonal systems. In Sections 4–8, we discuss distance-regular graphs and related topics, including the Bose-Mesner algebra, the dual Bose-Mesner algebra, the Terwilliger algebra, and the Q-polynomial property. In Section 9, we describe the action of the matrices A, A^* on an irreducible T-module. In Section 10, we prove Theorem 1.1. In Section 11, we prove Theorems 1.3, 1.4, and 1.5. In Section 12, we discuss the algebras $E_1^* T E_1^*$ and $E_1 T E_1$. In Section 13, we examine the algebras $E_D^* T E_D^*$ and $E_D T E_D$ and prove Theorem 1.6.

2 Preliminaries

In this section, we review some notation and basic concepts. In this and next section, we assume that $\mathbb{F}$ is an arbitrary field. All algebras and vector spaces considered in this paper are defined over $\mathbb{F}$. Let V denote a vector space of finite positive dimension. Let $\mathrm{End}(V)$ denote the algebra consisting of the $\mathbb{F}$-linear maps from V to V. For $A \in \mathrm{End}(V)$ and a subspace $W \subseteq V$, we say that W is an *eigenspace* of A whenever W is nonzero and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V \mid Av = \theta v\}$. In this case, θ is called the *eigenvalue* of A associated with W. The map A is said to be *diagonalizable* whenever V is spanned by the eigenspaces of A. Assume that A is diagonalizable. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of A. By linear algebra,

$$V = \sum_{i=0}^{d} V_i \qquad \text{(direct sum)}. \tag{4}$$

For $0 \le i \le d$, define $E_i \in \mathrm{End}(V)$ such that

$$(E_i - I)V_i = 0, \qquad E_i V_j = 0 \quad (0 \le j \le d,\ j \ne i), \tag{5}$$

where I is the identity element in $\mathrm{End}(V)$. We refer to E_i as the *primitive idempotent* of A associated with V_i. Observe that

$$V_i = E_i V \qquad (0 \le i \le d). \tag{6}$$

Moreover,

$$I = \sum_{i=0}^{d} E_i, \qquad E_i E_j = \delta_{i,j} E_i \qquad (0 \le i, j \le d), \tag{7}$$

$$A = \sum_{i=0}^{d} \theta_i E_i, \qquad E_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \le i \le d), \tag{8}$$

$$AE_i = E_i A = \theta_i E_i \qquad (0 \le i \le d). \tag{9}$$

We note that each of $\{E_i\}_{i=0}^d$ and $\{A^i\}_{i=0}^d$ forms a basis for the subalgebra of $\mathrm{End}(V)$ generated by A.

3 Tridiagonal pairs and tridiagonal systems

In this section, we review the concepts of tridiagonal pairs and tridiagonal systems.

Definition 3.1 ([15, Definition 1.1])**.** Let V denote a vector space of finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V, we mean an ordered pair A, A^* of elements in $\mathrm{End}(V)$ that satisfy (i)–(iv) below.

(i) Each of A and A^* is diagonalizable on V.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d),$$

where $V_{-1} = V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \le i \le \delta),$$

where $V_{-1}^* = V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V such that $W \neq 0$, $W \neq V$, $AW \subseteq W$, and $A^*W \subseteq W$.

We say the pair A, A^* is *over* $\mathbb{F}$.

Note 3.2. It is common to denote the conjugate transpose of A by A^*. However, we do not use this notation in this paper. In a tridiagonal pair A, A^*, the $\mathbb{F}$-linear maps A and A^* are arbitrary subject to conditions (i)–(iv) in Definition 3.1.

Let A, A^* be a TD pair on V as in Definition 3.1. By [15, Lemma 4.5] the integers d and δ from (ii), (iii) of Definition 3.1 are equal. We call this common value the *diameter* of the TD pair. An ordering of the eigenspaces of A (resp. A^*) is called *standard* whenever it satisfies (ii) (resp. (iii)) of Definition 3.1. Let $\{V_i\}_{i=0}^d$ be a standard ordering of the eigenspaces of A. Then the inverted ordering $\{V_{d-i}\}_{i=0}^d$ is also standard, and no further ordering is standard [15, Lemma 2.4]. A similar result holds for A^*. An ordering of the primitive idempotents of A (resp. A^*) is called *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

When working with a TD pair, it is convenient to use a closely related concept known as a TD system.

Definition 3.3 ([15, Definition 2.1])**.** Let V denote a vector space of finite positive dimension. By a *tridiagonal system* (or *TD system*) on V, we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \tag{10}$$

that satisfies (i)–(iii) below.

(i) A, A^* is a tridiagonal pair on V.

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

Let Φ be a TD system on V as in Definition 3.3. For $0 \le i \le d$, define

$$\rho_i = \dim(E_i^* V).$$

By construction, $\rho_i \neq 0$ for all $0 \le i \le d$. By [15, Corollaries 5.7, 6.6],

$$\rho_i = \dim(E_i V) \qquad (0 \le i \le d), \tag{11}$$
$$\rho_i = \rho_{d-i} \qquad (0 \le i \le d), \tag{12}$$
$$\rho_{i-1} \le \rho_i \qquad (0 \le i \le d/2). \tag{13}$$

The sequence $\{\rho_i\}_{i=0}^d$ is called the *shape* of Φ. By the shape of the TD pair A, A^*, we mean the shape of Φ. We mention a special case of the shape of a TD pair. A TD pair with shape $(1, 1, \ldots, 1)$ is called a *Leonard pair*; see [37].

Definition 3.4 ([27, Definition 1.5])**.** A TD pair A, A^* is said to be *sharp* whenever $\rho_0 = 1$, where $\{\rho_i\}_{i=0}^d$ is the shape of A, A^*.

Lemma 3.5 ([28, Theorem 1.3])**.** *A TD pair over an algebraically closed field is sharp.*

The sharp TD pairs were studied in [27, 28], and they were classified up to isomorphism [16]. For more information on TD pairs and TD systems, refer to [15–19, 21, 26–28].

4 Distance-regular graphs

We now turn our attention to distance-regular graphs. In this section, we review some definitions and preliminaries concerning distance-regular graphs. In the following sections, we discuss the Bose-Mesner algebra, the dual Bose-Mesner algebra, and the Terwilliger algebra associated with these graphs. For further details on distance-regular graphs, we refer the reader to [1, 4, 7, 38].

Let Γ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X. Let ∂ denote the path-length distance function for Γ. Two vertices $y, z \in X$ are said to be *adjacent* whenever $\partial(y, z) = 1$. Define $D = \max\{\partial(y, z) \mid y, z \in X\}$. We call D the *diameter* of Γ. For $y \in X$ and an integer $i \geq 0$, define

$$\Gamma_i(y) = \{z \in X \mid \partial(y, z) = i\}.$$

We abbreviate $\Gamma(y) = \Gamma_1(y)$. For an integer $k \geq 0$, the graph Γ is said to be *regular with valency* k whenever every vertex is adjacent to exactly k vertices. The graph Γ is said to be *distance-regular* whenever, for all integers $0 \leq h, i, j \leq D$ and all vertices $y, z \in X$ with $\partial(y, z) = h$, the number

$$p^h_{i,j} = |\Gamma_i(y) \cap \Gamma_j(z)| \tag{14}$$

is independent of the choice of y and z, and depends only on h, i, j. The parameter $p^h_{i,j}$ is called an *intersection number* of Γ.

For the rest of this paper, assume that Γ is distance-regular with $D \geq 3$. By construction, $p^h_{i,j} = p^h_{j,i}$ for all $0 \leq h, i, j \leq D$. By the triangle inequality, for $0 \leq h, i, j \leq D$ we have

(i) $p^h_{i,j} = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $p^h_{i,j} \neq 0$ if one of h, i, j is equal to the sum of the other two.

We abbreviate

$$c_i = p^i_{1,i-1} \ (1 \leq i \leq D), \qquad a_i = p^i_{1,i} \ (0 \leq i \leq D), \qquad b_i = p^i_{1,i+1} \ (0 \leq i \leq D-1). \tag{15}$$

Note that $a_0 = 0$, $c_1 = 1$, and

$$c_i > 0 \quad (1 \leq i \leq D), \qquad\qquad b_i > 0 \quad (0 \leq i \leq D-1).$$

Also, abbreviate

$$k_i = p^0_{i,i} \quad (0 \le i \le D). \tag{16}$$

Note that $k_0 = 1$ and $k_i = |\Gamma_i(y)|$ for all $y \in X$. The graph Γ is regular with valency $k = k_1 = b_0$. Moreover,

$$c_i + a_i + b_i = k \qquad\qquad (0 \le i \le D),$$

where $c_0 := 0$ and $b_D := 0$.

5 The Bose-Mesner algebra

We continue our discussion of the distance-regular graph Γ. In this section, we recall the Bose-Mesner algebra. For the rest of this paper, we adopt the following conventions. Assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $\mathrm{Mat}_X(\mathbb{F})$ denote the matrix algebra consisting of the matrices with rows and columns indexed by X and all entries in $\mathbb{F}$. Let $I \in \mathrm{Mat}_X(\mathbb{F})$ denote the identity matrix. Let $\mathbb{F}^X$ denote the vector space consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{F}$. The algebra $\mathrm{Mat}_X(\mathbb{F})$ acts on $\mathbb{F}^X$ by left multiplication. We refer to $\mathbb{F}^X$ as the *standard module*. For notational convenience, we abbreviate $V = \mathbb{F}^X$. We endow V with an inner product defined by

$$\langle u, v\rangle = \overline{u}^\top v \qquad (u, v \in V),$$

where $\top$ denotes transpose and $-$ denotes complex conjugation. Note that $\langle Bu, v\rangle = \langle u, \overline{B}^\top v\rangle$ for $u, v \in V$ and $B \in \mathrm{Mat}_X(\mathbb{F})$. For $y \in X$, let $\hat{y}$ denote the vector in V with a 1 in the y-coordinate and 0 in all other coordinates. We call $\hat{y}$ the *characteristic vector* of y. Note that the set $\{\hat{y} \mid y \in X\}$ forms an orthonormal basis for V. We denote $\mathbb{1} = \sum_{y\in X} \hat{y}$ and let $\mathbb{J} \in \mathrm{Mat}_X(\mathbb{F})$ denote the matrix that has all entries 1.

We recall the Bose-Mesner algebra of Γ. For $0 \le i \le D$, define a matrix $A_i \in \mathrm{Mat}_X(\mathbb{F})$ with (y, z)-entry

$$(A_i)_{y,z} = \begin{cases} 1 & \text{if} \quad \partial(y,z) = i, \\ 0 & \text{if} \quad \partial(y,z) \ne i \end{cases} \qquad (y, z \in X). \tag{17}$$

We call A_i the *i-th distance matrix* of Γ. We observe that

$$A_0 = I, \qquad \sum_{i=0}^{D} A_i = \mathbb{J}, \qquad \overline{A}_i = A_i \ (0 \le i \le D), \qquad A_i^\top = A_i \ (0 \le i \le D). \tag{18}$$

Moreover,

$$A_i A_j = \sum_{h=0}^{D} p^h_{i,j} A_h \qquad (0 \le i, j \le D). \tag{19}$$

By (17)–(19), the matrices $\{A_i\}_{i=0}^D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{F})$. We call M the *Bose-Mesner algebra* of Γ. We abbreviate $A = A_1$ and call this the *adjacency matrix* of Γ. By [38, Corollary 3.4], the matrix A generates M. Moreover, since A is real symmetric, it is diagonalizable and has all real eigenvalues. Consequently, A has exactly $D+1$ mutually distinct eigenvalues, denoted by $\theta_0, \theta_1, \ldots, \theta_D$. We call $\{\theta_i\}_{i=0}^D$ the *eigenvalues of* Γ. The valency k is the maximal eigenvalue of Γ [3, Proposition 3.1]. We set $\theta_0 := k$. For $0 \le i \le D$, let V_i denote the eigenspace of A corresponding to θ_i. Let $E_i \in \mathrm{Mat}_X(\mathbb{F})$ denote the primitive idempotent of A associated with V_i. By linear algebra,

$$\overline{E}_i = E_i, \qquad E_i^\top = E_i \qquad (0 \le i \le D). \tag{20}$$

By (4), (6), we have

$$V = \sum_{i=0}^{D} E_i V \qquad\qquad \text{(orthogonal direct sum)}. \tag{21}$$

Observe that $E_0 V = \mathrm{Span}\{\mathbb{1}\}$ and thus

$$E_0 = |X|^{-1}\mathbb{J}. \tag{22}$$

We call $\{E_i\}_{i=0}^D$ the *primitive idempotents* of Γ. By the comment below (9), the matrices $\{E_i\}_{i=0}^D$ form a basis for M. For $0 \le i \le D$, let m_i denote the dimension of $E_i V$. Observe that $m_i = \mathrm{rank}(E_i)$.

We finish this section with a comment. Let $\circ$ denote the entrywise multiplication in $\mathrm{Mat}_X(\mathbb{F})$. Observe that $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \le i, j \le D$. Therefore, M is closed under $\circ$. Since $\{E_i\}_{i=0}^D$ is a basis for M, there exist scalars $q_{i,j}^h \in \mathbb{F}$ $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{i,j}^h E_h \qquad\qquad (0 \le i, j \le D). \tag{23}$$

The scalars $q_{i,j}^h$ are called the *Krein parameters* of Γ. By construction, $q_{i,j}^h = q_{j,i}^h$ for $0 \le h, i, j \le D$. The Krein parameters are real and nonnegative [1, p. 69]. They are used to define the Q-polynomial property of a distance-regular graph, which will be discussed in Section 8.

6 The dual Bose-Mesner algebra

We continue to discuss the distance-regular graph Γ. In this section, we recall the dual Bose-Mesner algebras of Γ. Fix a vertex $x \in X$. We view x as a "base vertex." For $0 \le i \le D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{F})$ with (y, y)-entry

$$(E_i^*)_{y,y} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = i, \\ 0 & \text{if} \quad \partial(x,y) \ne i \end{cases} \qquad (y \in X). \tag{24}$$

We call E_i^* the *i-th dual primitive idempotent of* Γ *with respect to* x. Observe that

$$\sum_{i=0}^{D} E_i^* = I, \qquad \overline{E_i^*} = E_i^* \ (0 \le i \le D), \qquad E_i^{*\top} = E_i^* \ (0 \le i \le D), \tag{25}$$

$$E_i^* E_j^* = \delta_{i,j} E_i^* \ (0 \le i, j \le D). \tag{26}$$

By (24)–(26), the matrices $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{F})$. We call M^* the *dual Bose-Mesner algebra of* Γ *with respect to* x [33, p. 378]. By (24), we find that for $0 \le i \le D$

$$E_i^* V = \mathrm{Span}\{\hat{y} \mid y \in X, \ \partial(x,y) = i\}. \tag{27}$$

We call E_i^*V the *i-th subconstituent of* Γ *with respect to* x. Observe that $\dim(E_i^*V) = k_i$, where k_i is from (16). Further, observe that $E_0^*V = \mathbb{F}\hat{x}$. By (27),

$$V = \sum_{i=0}^{D} E_i^*V \qquad \text{(orthogonal direct sum).} \tag{28}$$

By construction, the adjacency matrix A satisfies

$$AE_i^*V \subseteq E_{i-1}^*V + E_i^*V + E_{i+1}^*V \qquad (0 \leq i \leq D), \tag{29}$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

Next, we recall the dual distance matrices of Γ. For $0 \leq i \leq D$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{F})$ with $(y, y,)$-entry

$$(A_i^*)_{y,y} = |X|(E_i)_{x,y} \qquad (y \in X). \tag{30}$$

We call A_i^* the *i-th dual distance matrix of* Γ *with respect to* x. By [38, Lemma 5.9], we have

$$A_0^* = I, \qquad \sum_{i=0}^{D} A_i^* = |X|E_0^*. \qquad \overline{A_i^*} = A_i^* \ (0 \leq i \leq D), \qquad A_i^{*\top} = A_i^* \ (0 \leq i \leq D). \tag{31}$$

Moreover,

$$A_i^*A_j^* = \sum_{h=0}^{D} q_{i,j}^h A_h^* \qquad (0 \leq i, j \leq D). \tag{32}$$

By [38, Lemma 5.8], the matrices $\{A_i^*\}_{i=0}^D$ form a basis for M^*.

7 The Terwilliger algebra

We continue our discussion of the distance-regular graph Γ. Recall the base vertex $x \in X$. Recall the Bose-Mesner algebra M and the dual Bose-Mesner algebra $M^* = M^*(x)$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{F})$ generated by M and M^*. The algebra T is called the *Terwilliger algebra of* Γ *with respect to* x [33, Definition 3.3]. Note that T is finite-dimensional and noncommutative. By [33, Lemma 3.2], the following relations hold in T. For $0 \leq h, i, j \leq D$,

$$E_i^*A_hE_j^* = 0 \quad \text{if and only if} \quad p_{i,j}^h = 0, \tag{33}$$

$$E_iA_h^*E_j = 0 \quad \text{if and only if} \quad q_{i,j}^h = 0. \tag{34}$$

By construction, each of M and M^* is closed under the conjugate-transpose map. Therefore, T is also closed under the conjugate-transpose map. In particular, T is semisimple [33, Lemma 3.4].

Next, we recall the T-modules. By a *T-module*, we mean a subspace W of V such that $TW \subseteq W$. For example, V is a T-module. A T-module W is *irreducible* whenever $W \neq 0$ and W does not contain a T-module other than 0 and W. Let W denote a T-module and let U denote a T-module contained in W. Since T is closed under the conjugate-transpose map, the orthogonal complement of U in W is also a T-module. Consequently, W decomposes into an orthogonal direct sum of irreducible T-modules. In particular, V is an orthogonal direct sum of irreducible T-modules.

Example 7.1 ([33, Lemma 3.6]). Recall the vector $\mathbb{1} = \sum_{y \in X} \hat{y}$. This vector satisfies

$$A_i \hat{x} = E_i^* \mathbb{1}, \qquad A_i^* \mathbb{1} = |X| E_i \hat{x} \qquad (0 \leq i \leq D). \tag{35}$$

Moreover, $M\hat{x} = M^*\mathbb{1}$. This common subspace is an irreducible T-module. This T-module is called *primary*.

We recall the notion of isomorphism for T-modules. Let W and W' denote T-modules. An $\mathbb{F}$-linear map $\sigma : W \to W'$ is said to be *T-linear* whenever

$$\sigma(Bw) = B\sigma(w)$$

for all $B \in T$ and $w \in W$. By a *T-module isomorphism* from W to W', we mean a bijective T-linear map from W to W'. We say that W and W' are *isomorphic* whenever there exists a T-module isomorphism from W to W'.

Let W denote an irreducible T-module. By (28), the subspace W is the direct sum of the nonzero spaces among $\{E_i^* W\}_{i=0}^D$. By the *endpoint* of W, we mean $\min\{i \mid 0 \leq i \leq D,\ E_i^* W \neq 0\}$. By the *diameter* of W, we mean $|\{i \mid 0 \leq i \leq D,\ E_i^* W \neq 0\}| - 1$. Let r denote the endpoint of W and let d denote the diameter of W. By [33, Lemma 3.9], we have $r + d \leq D$ and $E_i^* W \neq 0$ if and only if $r \leq i \leq r + d$ $(0 \leq i \leq D)$. By construction, we have

$$AE_i^* W \subseteq E_{i-1}^* W + E_i^* W + E_{i+1}^* W \qquad (r \leq i \leq r + d). \tag{36}$$

For more detailed information about the Terwilliger algebra, refer to [33–35].

8 The Q-polynomial property

We continue to discuss the distance-regular graph Γ. In this section, we discuss the Q-polynomial property. Recall the primitive idempotents $\{E_i\}_{i=0}^D$ of Γ and the Krein parameters $q_{i,j}^h$ from (23). The ordering $\{E_i\}_{i=0}^D$ is called *Q-polynomial* whenever the following hold for $0 \leq h, i, j \leq D$:

(i) $q_{i,j}^h = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $q_{i,j}^h \neq 0$ if one of h, i, j is equal to the sum of the other two.

We abbreviate

$$c_i^* = q_{1,i-1}^i \ (1 \leq i \leq D), \qquad a_i^* = q_{1,i}^i \ (0 \leq i \leq D), \qquad b_i^* = q_{1,i+1}^i \ (0 \leq i \leq D-1). \tag{37}$$

Note that $a_0^* = 0$, $c_1^* = 1$, and

$$c_i^* > 0 \quad (1 \leq i \leq D), \qquad\qquad b_i^* > 0 \quad (0 \leq i \leq D-1).$$

The graph Γ is said to be *Q-polynomial* whenever there exists at least one Q-polynomial ordering of the primitive idempotents.

For the rest of this paper, we assume that the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial. Fix a vertex $x \in X$ and write $M^* = M^*(x)$. Recall the bases $\{A_i^*\}_{i=0}^D$ and $\{E_i^*\}_{i=0}^D$ for M^*. We abbreviate

$A^* = A_1^*$ and call this the *dual adjacency matrix of* Γ *with respect to* x. By [33, Lemma 3.11], the matrix A^* generates M^*. Since $\{E_i^*\}_{i=0}^D$ is a basis for M^*, there exist scalars $\{\theta_i^*\}_{i=0}^D$ such that

$$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*.$$

By [33, Lemma 3.11], the scalars $\{\theta_i^*\}_{i=0}^D$ are real and mutually distinct. For $0 \le i \le D$, we have $A^* E_i^* = E_i^* A^* = \theta_i^* E_i^*$, and $E_i^* V$ is the eigenspace of A^* corresponding to the eigenvalue θ_i^*. We call $\{\theta_i^*\}_{i=0}^D$ the *dual eigenvalues of* Γ. Consider the Terwilliger algebra $T = T(x)$. Note that A, A^* generates T. Recall the relations (33), (34) for T. Setting $h = 1$ in these relations, we find that for $0 \le i, j \le D$,

$$E_i^* A E_j^* = 0 \qquad \text{if} \quad |i - j| > 1, \tag{38}$$

$$E_i A^* E_j = 0 \qquad \text{if} \quad |i - j| > 1. \tag{39}$$

Moreover, setting $j = i$ in (33), (34), we find that for $0 \le h, i \le D$,

$$E_i^* A_h E_i^* = 0 \quad \text{if} \quad h > 2i, \tag{40}$$

$$E_i A_h^* E_i = 0 \quad \text{if} \quad h > 2i. \tag{41}$$

We mention a result for later use.

Lemma 8.1. *For $0 \le i \le D$, consider the subspaces $E_i^* M E_i^*$ and $E_i M^* E_i$. Then the following* (i), (ii) *hold.*

(i) *Every element of $E_i^* M E_i^*$ is symmetric.*

(ii) *Every element of $E_i M^* E_i$ is symmetric.*

Proof. (i): Since $\{A_j\}_{j=0}^D$ forms a basis for M and the matrices E_i^*, $\{A_j\}_{j=0}^D$ are symmetric.
(ii): Similar to (i). ∎

9 Action of A, A^* on an irreducible T-module

Recall the Q-polynomial distance-regular graph Γ. Fix a vertex $x \in X$ and write $T = T(x)$. In this section, we discuss how the matrices A, A^* act on an irreducible T-module. Recall the eigenspaces $\{E_i V\}_{i=0}^D$ of A and the eigenspaces $\{E_i^* V\}_{i=0}^D$ of A^*. In (29), we mentioned the action of A on $\{E_i^* V\}_{i=0}^D$. Using (39), the action of A^* on $\{E_i V\}_{i=0}^D$ is given as follows:

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V \qquad (0 \le i \le D), \tag{42}$$

where $E_{-1} = 0$ and $E_{D+1} = 0$.

Let W denote an irreducible T-module. By (21), the subspace W is the direct sum of the nonzero spaces among $\{E_i W\}_{i=0}^D$. By the *dual endpoint* of W, we mean $\min\{i \mid 0 \le i \le D,\ E_i W \neq 0\}$. By the *dual diameter* of W, we mean $|\{i \mid 0 \le i \le D,\ E_i W \neq 0\}| - 1$. Let s denote the dual endpoint of W and let δ denote the dual diameter of W. By [33, Lemma 3.12], we have $s + \delta \le D$ and $E_i W \neq 0$ if and only if $s \le i \le s + \delta$ $(0 \le i \le D)$. Moreover, the dual diameter of W is equal to the diameter of W [38, Corollary 13.7]. By construction, we have

$$A^* E_i W \subseteq E_{i-1} W + E_i W + E_{i+1} W \qquad (s \le i \le s + d). \tag{43}$$

Lemma 9.1 ([33, Lemmas 3.9, 3.12]). *Let W denote an irreducible T-module. Then the pair A, A^* acts on W as a TD pair.*

Lemma 9.2 ([33, Lemmas 3.9, 3.12]). *Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. Then the sequence*

$$\Phi = (A; \{E_{s+i}\}_{i=0}^{d}; A^*; \{E^*_{r+i}\}_{i=0}^{d}) \tag{44}$$

acts on W as a TD system.

Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. Define a sequence $\{\rho_i\}_{i=0}^{d}$ by

$$\rho_i = \dim(E^*_{r+i}W) \qquad (0 \leq i \leq d). \tag{45}$$

Note that $\rho_i \neq 0$ for $0 \leq i \leq d$. The sequence $\{\rho_i\}_{i=0}^{d}$ is called the *shape* of W. The T-module W is called *sharp* whenever $\rho_0 = 1$. In the next result, we emphasize a point for later use.

Lemma 9.3 ([15, Corollary 5.7]). *Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. For $0 \leq i \leq d$, the following subspaces have the same dimension:*

$$E^*_{r+i}W, \qquad E^*_{r+d-i}W, \qquad E_{s+i}W, \qquad E_{s+d-i}W. \tag{46}$$

We finish this section with some comments on the primary module.

Lemma 9.4 (cf. [33, Lemma 3.6], [9, Proposition 8.3]). *Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. Then the following* (i)–(iv) *are equivalent.*

(i) *W is primary.*

(ii) $r = 0$.

(iii) $s = 0$.

(iv) $d = D$.

Suppose that W is primary. Then

$$\rho_i = 1 \qquad (0 \leq i \leq d), \tag{47}$$

where $\{\rho_i\}_{i=0}^{d}$ is the shape of W.

Proof. By [33, Lemma 3.6]. ■

10 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Recall the Q-polynomial distance-regular graph Γ. Fix $x \in X$ and write $T = T(x)$. We have preliminary comments. For $0 \leq i \leq D$, consider the subspace $E^*_iTE^*_i$. Note that $E^*_iTE^*_i$ is an algebra with multiplicative identity E^*_i. By construction, $E^*_iTE^*_i$ is finite-dimensional and closed under the conjugate transpose map, hence it is semisimple [6]. In general, this algebra is not commutative [35, Theorem 5.1]. Note that the i-th subconstituent E^*_iV is invariant under the action of $E^*_iTE^*_i$.

Lemma 10.1 ([28, Theorem 2.6]). *Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. Then the following* (i)–(iii) *hold on W.*

(i) *The algebra $E_r^*TE_r^*$ is generated by $E_r^*ME_r^*$.*

(ii) *The elements of $E_r^*ME_r^*$ mutually commute.*

(iii) *The algebra $E_r^*TE_r^*$ is commutative.*

Proof. By [28, Theorem 2.6]. ■

For $0 \leq i \leq D$, consider the subspace E_iTE_i. Note that E_iTE_i is an algebra with the multiplicative identity E_i. By construction, E_iTE_i is finite-dimensional and closed under the conjugate transpose map, hence it is semisimple [6]. In general, this algebra is not commutative [35, Theorem 5.1]. Note that the subspace E_iV is invariant under the action of E_iTE_i.

Lemma 10.2 ([28, Theorem 2.6]). *Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. Then the following* (i)–(iii) *hold on W.*

(i) *The algebra E_sTE_s is generated by $E_sM^*E_s$.*

(ii) *The elements of $E_sM^*E_s$ mutually commute.*

(iii) *The algebra E_sTE_s is commutative.*

Proof. By [28, Theorem 2.6]. ■

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Assume $\mathbb{F} = \mathbb{R}$. Let W denote an irreducible T-module with endpoint r, dual endpoint s and diameter d. Let $\{\rho_i\}_{i=0}^{d}$ denote the shape of W. We will show that $\rho_0 = 1$; that is, $\dim(E_r^*W) = 1$. Observe that E_r^*W is non-zero, finite-dimensional, and invariant under $E_r^*TE_r^*$. Consider the restriction of $E_r^*TE_r^*$ to W. By Lemmas 8.1(i) and 10.1, all elements of $E_r^*TE_r^*$ are simultaneously diagonalizable on W. Therefore, there exists a nonzero vector $w \in E_r^*W$ that is a common eigenvector for $E_r^*TE_r^*$.

Now, consider the subspace Tw of V. Observe that Tw is a nonzero T-submodule of W. By the irreducibility of W, we have $Tw = W$. Using this and the fact that $w \in E_r^*W$, applying $E_r^*TE_r^*$ to w gives

$$E_r^*TE_r^*w = E_r^*Tw = E_r^*W.$$

Since w is a common eigenvector for $E_r^*TE_r^*$, we have $E_r^*TE_r^*w = \mathrm{Span}\{w\}$, which is equal to E_r^*W. Therefore, $\dim(E_r^*W) = 1$, and hence $\rho_0 = 1$. The proof is complete. ■

Next, we prove Corollary 1.2.

Proof of Corollary 1.2. Consider a Q-polynomial distance-regular graph with diameter D. If $D = 1$, then the graph is complete, which is trivially sharp. If $D = 2$, then the graph is strongly regular and sharp by the result of [40]. If $D \geq 3$, then the graph is sharp by Theorem 1.1. The result follows. ■

11 The complexification of a T-module

In this section, we prove Theorems 1.3, 1.4, and 1.5. Recall the Q-polynomial distance-regular graph Γ. Fix $x \in X$ and write $T = T(x)$. Recall the ground field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. When it is important to emphasize the ground field, we will write $T = T^{\mathbb{F}}$. By construction, $T^{\mathbb{F}}$ is the subalgebra of $\mathrm{Mat}_X(\mathbb{F})$ generated by A and A^*. We now compare $T^{\mathbb{R}}$ and $T^{\mathbb{C}}$. Recall that $A, A^* \in \mathrm{Mat}_X(\mathbb{R})$. By construction,

$$T^{\mathbb{C}} = T^{\mathbb{R}} + \boldsymbol{i}T^{\mathbb{R}}, \tag{48}$$

where $\boldsymbol{i} = \sqrt{-1}$ and the sum in (48) is direct over $\mathbb{R}$. Note that $\dim_{\mathbb{R}} T^{\mathbb{R}} = \dim_{\mathbb{C}} T^{\mathbb{C}}$. By (48), we have

$$T^{\mathbb{C}} = T^{\mathbb{R}} \otimes \mathbb{C}, \tag{49}$$

where we understand $\otimes = \otimes_{\mathbb{R}}$. Motivated by (49), we call $T^{\mathbb{C}}$ the *complexification of* $T^{\mathbb{R}}$.

Next, let W denote a $T^{\mathbb{R}}$-module. The *complexification of* W is the $\mathbb{C}$-vector space

$$W^{\mathbb{C}} = W \otimes \mathbb{C}, \tag{50}$$

see [29, p. 379]. Then $W^{\mathbb{C}}$ is a $T^{\mathbb{C}}$-module. By construction,

$$W^{\mathbb{C}} = W + \boldsymbol{i}W, \tag{51}$$

where the sum in (51) is direct over $\mathbb{R}$. Note that $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} W^{\mathbb{C}}$.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Recall that W is an irreducible $T^{\mathbb{R}}$-module. Let r denote the endpoint of W. By Theorem 1.1, W is sharp, so $\dim_{\mathbb{R}}(E^*_r W) = 1$. Consider the complexification $W^{\mathbb{C}}$ of W. By construction, $\dim_{\mathbb{C}}(E^*_r W^{\mathbb{C}}) = 1$. Decompose $W^{\mathbb{C}}$ as a direct sum of irreducible $T^{\mathbb{C}}$-modules to get

$$W^{\mathbb{C}} = \sum_j W_j. \tag{52}$$

Applying E^*_r to (52), we obtain a direct sum of $\mathbb{C}$-vector spaces

$$E^*_r W^{\mathbb{C}} = \sum_j E^*_r W_j. \tag{53}$$

Since the left-hand side of (53) has dimension 1, there is a unique nonzero summand on the right-hand side of (53). We denote this summand by $E^*_r W_\xi$. It follows that $E^*_r W^{\mathbb{C}} = E^*_r W_\xi \subseteq W_\xi$. Moreover, $E^*_r W^{\mathbb{C}}$ is the $\mathbb{C}$-span of $E^*_r W$, so

$$E^*_r W \subseteq E^*_r W^{\mathbb{C}} \subseteq W_\xi.$$

Pick $0 \neq z \in E^*_r W \subseteq W_\xi$. Since W_ξ is an irreducible $T^{\mathbb{C}}$-module, we have $W_\xi = T^{\mathbb{C}} z$. Also, by the irreducibility of the $T^{\mathbb{R}}$-module W, we have $W = T^{\mathbb{R}} z$. Since $T^{\mathbb{R}} z \subseteq T^{\mathbb{C}} z$, it follows that $W \subseteq W_\xi$. Furthermore, $\boldsymbol{i}W \subseteq W_\xi$ since W_ξ is closed under multiplication by complex scalars. Therefore,

$$W + \boldsymbol{i}W \subseteq W_\xi. \tag{54}$$

By (51) and (54), we obtain $W^{\mathbb{C}} \subseteq W_\xi$. Since W_ξ is an irreducible $T^{\mathbb{C}}$-module, it follows that $W_\xi = W^{\mathbb{C}}$. Hence, the $T^{\mathbb{C}}$-module $W^{\mathbb{C}}$ is irreducible. ■

We now prove Theorem 1.4.

Proof of Theorem 1.4. Recall that W_1 and W_2 are irreducible $T^{\mathbb{R}}$-modules. Suppose that W_1 and W_2 are isomorphic as $T^{\mathbb{R}}$-modules. By (50), we have an isomorphism of $T^{\mathbb{C}}$-modules

$$W_1^{\mathbb{C}} = W_1 \otimes \mathbb{C} \;\cong\; W_2 \otimes \mathbb{C} = W_2^{\mathbb{C}}.$$

Conversely, suppose $W_1^{\mathbb{C}}$ and $W_2^{\mathbb{C}}$ are isomorphic as $T^{\mathbb{C}}$-modules. Let $\eta : W_1^{\mathbb{C}} \to W_2^{\mathbb{C}}$ be a $T^{\mathbb{C}}$-module isomorphism. For $j \in \{1,2\}$ view $W_j^{\mathbb{C}} = W_j + \boldsymbol{i}W_j$ as in (51). For $w \in W_1 \subseteq W_1^{\mathbb{C}}$, write

$$\eta(w) = \sigma_1(w) + \boldsymbol{i}\sigma_2(w), \tag{55}$$

where σ_1, $\sigma_2 : W_1 \to W_2$ are $\mathbb{R}$-linear maps.

We first show that both σ_1 and σ_2 are $T^{\mathbb{R}}$-linear. Pick $B \in T^{\mathbb{R}}$ and $w \in W_1$. Since $B \in T^{\mathbb{C}}$ and η is $T^{\mathbb{C}}$-linear, we have $\eta(Bw) = B\eta(w)$. Thus,

$$\sigma_1(Bw) + \boldsymbol{i}\sigma_2(Bw) = B\sigma_1(w) + \boldsymbol{i}B\sigma_2(w).$$

Comparing real and imaginary parts, we obtain $\sigma_1(Bw) = B\sigma_1(w)$ and $\sigma_2(Bw) = B\sigma_2(w)$. Therefore, both σ_1 and σ_2 are $T^{\mathbb{R}}$-linear.

Next, we show that at least one of σ_1, σ_2 is nonzero. If $\sigma_1 = \sigma_2 = 0$, then η vanishes on W_1. By $\mathbb{C}$-linearity, $\eta(\boldsymbol{i}w) = \boldsymbol{i}\eta(w) = 0$ for all $w \in W_1$, so η also vanishes on $\boldsymbol{i}W_1$. Hence, $\eta = 0$ on $W_1^{\mathbb{C}} = W_1 + \boldsymbol{i}W_1$. This contradicts that η is an isomorphism. Therefore at least one of σ_1, σ_2 is nonzero. Suppose that $\sigma_1 \neq 0$ (the case $\sigma_2 \neq 0$ is similar). Then $\sigma_1(W_1)$ is a nonzero submodule of the $T^{\mathbb{R}}$-module W_2, and hence $\sigma_1(W_1) = W_2$ by the irreducibility of the $T^{\mathbb{R}}$-module W_2. Moreover, $\ker \sigma_1$ is a submodule of the $T^{\mathbb{R}}$-module W_1. Since the $T^{\mathbb{R}}$-module W_1 is irreducible and $\sigma_1 \neq 0$, we have $\ker \sigma_1 = 0$. Therefore $\sigma_1 : W_1 \to W_2$ is bijective, and hence a $T^{\mathbb{R}}$-module isomorphism. The result follows. ∎

Next, we prove Theorem 1.5. We have some preliminary comments. For an irreducible $T^{\mathbb{R}}$-module W, let $\operatorname{End}_{T^{\mathbb{R}}}(W)$ denote the $\mathbb{R}$-algebra consisting of the $T^{\mathbb{R}}$-linear maps from W to W. Then $\operatorname{End}_{T^{\mathbb{R}}}(W)$ is a finite-dimensional division algebra over $\mathbb{R}$ (cf. [11, §1.2]). Consequently, $\operatorname{End}_{T^{\mathbb{R}}}(W)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ as an $\mathbb{R}$-algebra (cf. [22, Theorem 13.12]).

Lemma 11.1. *Assume that $\mathbb{F} = \mathbb{R}$. Let W denote an irreducible T-module. Then $\operatorname{End}_T(W)$ is isomorphic to $\mathbb{R}$ as an $\mathbb{R}$-algebra.*

Proof. Let r denote the endpoint of W. Set $U_0 = E_r^*W$. Since $U_0 \neq 0$ and W is irreducible, we have $W = TU_0$. Let $\operatorname{End}(U_0)$ denote the $\mathbb{R}$-algebra consisting of the $\mathbb{R}$-linear maps from U_0 to U_0. Since $\dim U_0 = 1$ by Theorem 1.1, every element of $\operatorname{End}(U_0)$ acts on U_0 as multiplication by a real scalar. Hence the $\mathbb{R}$-algebra $\operatorname{End}(U_0)$ is isomorphic to $\mathbb{R}$.

Next, pick $\sigma \in \operatorname{End}_T(W)$. Since σ is T-linear, we have

$$\sigma(U_0) = \sigma(E_r^*W) = E_r^*\sigma(W) \subseteq E_r^*W = U_0.$$

Thus the restriction map

$$\varphi : \operatorname{End}_T(W) \longrightarrow \operatorname{End}(U_0), \qquad \varphi(\sigma) = \sigma|_{U_0}$$

is well-defined. We claim that φ is an $\mathbb{R}$-algebra isomorphism. To prove this claim, we first show that φ is injective. Suppose $\varphi(\sigma) = 0$. Then σ vanishes on U_0. Since $W = TU_0$, for any $w \in W$ we can write $w = Bu$ with some $B \in T$ and $u \in U_0$. By T-linearity,

$$\sigma(w) = \sigma(Bu) = B\,\sigma(u) = B \cdot 0 = 0.$$

Thus $\sigma = 0$ on W, and hence φ is injective. Next, we show that φ is surjective. Since the identity map id_W on W belongs to $\mathrm{End}_T(W)$, we have $\varphi(\mathrm{id}_W) = \mathrm{id}_{U_0} = 1 \in \mathbb{R}$. Thus the image of $\mathrm{End}_T(W)$ under φ is an $\mathbb{R}$-subalgebra of $\mathbb{R}$ containing 1, which must be all of $\mathbb{R}$. Therefore, φ is surjective. By the above comments, φ is an $\mathbb{R}$-algebra isomorphism. The result follows. ∎

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. We first show that $\mathbb{D}_i = \mathbb{R}$ for $1 \le i \le \ell$. Recall the Wedderburn decomposition of $T^{\mathbb{R}}$ from (2). For $1 \le i \le \ell$, we abbreviate $S_i = \mathrm{Mat}_{m_i}(\mathbb{D}_i)$, which is the i-th component of the direct sum in (2). Let W_i denote an irreducible $T^{\mathbb{R}}$-module corresponding to S_i. Let $\mathrm{End}_{S_i}(W_i)$ denote the $\mathbb{R}$-algebra consisting of the S_i-linear maps from $W_i \to W_i$. Since every simple component of $T^{\mathbb{R}}$ other than S_i acts as zero on W_i, the $T^{\mathbb{R}}$-action on W_i coincides with the S_i-action on W_i. It follows that $\mathrm{End}_{T^{\mathbb{R}}}(W_i) = \mathrm{End}_{S_i}(W_i)$. By [22, Theorem 3.3], the $\mathbb{R}$-algebra $\mathrm{End}_{S_i}(W_i)$ is isomorphic to $\mathbb{D}_i$. Therefore, we have an $\mathbb{R}$-algebra isomorphism

$$\mathrm{End}_{T^{\mathbb{R}}}(W_i) \cong \mathbb{D}_i. \tag{56}$$

On the other hand, by Lemma 11.1 we have an $\mathbb{R}$-algebra isomorphism

$$\mathrm{End}_{T^{\mathbb{R}}}(W_i) \cong \mathbb{R}. \tag{57}$$

By (56), (57), we have $\mathbb{D}_i = \mathbb{R}$ for $1 \le i \le \ell$.

Next, we show that $\ell = h$ and $m_i = n_i$ for $1 \le i \le \ell$. We complexify (2) to obtain $\mathbb{C}$-algebra isomorphisms

$$\begin{aligned} T^{\mathbb{C}} = T^{\mathbb{R}} \otimes \mathbb{C} &\cong \left(\bigoplus_{i=1}^{\ell} \mathrm{Mat}_{m_i}(\mathbb{R}) \right) \otimes \mathbb{C} \\ &= \bigoplus_{i=1}^{\ell} \left(\mathrm{Mat}_{m_i}(\mathbb{R}) \otimes \mathbb{C} \right) \\ &\cong \bigoplus_{i=1}^{\ell} \mathrm{Mat}_{m_i}(\mathbb{C}). \end{aligned}$$

Comparing this with (1), we obtain $\ell = h$ and $m_i = n_i$ for $1 \le i \le \ell$. The proof is complete. ∎

We finish this section with some consequences of Theorems 1.3 and 1.4.

Lemma 11.2. *Let W denote an irreducible $T^{\mathbb{R}}$-module. Then the irreducible $T^{\mathbb{C}}$-module $W^{\mathbb{C}}$ has the same endpoint, dual endpoint, diameter, and shape as W.*

Proof. For $0 \leq j \leq D$, we have $E^*_j W^{\mathbb{C}} = E^*_j W + \boldsymbol{i} E^*_j W$. Thus, the subspace $E^*_j W^{\mathbb{C}}$ is the $\mathbb{C}$-span of $E^*_j W$. By this comment, we have

$$\dim_{\mathbb{R}}(E^*_j W) = \dim_{\mathbb{C}}(E^*_j W^{\mathbb{C}}).$$

In particular,

$$E^*_j W = 0 \quad \text{if and only if} \quad E^*_j W^{\mathbb{C}} = 0.$$

Similarly, we have

$$\dim_{\mathbb{R}}(E_j W) = \dim_{\mathbb{C}}(E_j W^{\mathbb{C}}),$$

and

$$E_j W = 0 \quad \text{if and only if} \quad E_j W^{\mathbb{C}} = 0.$$

The result follows from these comments along with the definitions of the endpoint, dual endpoint, diameter, and shape of W. ∎

Corollary 11.3. *Let W denote a $T^{\mathbb{R}}$-module. Consider a direct sum decomposition of W into irreducible $T^{\mathbb{R}}$-modules*

$$W = \sum_i W_i. \tag{58}$$

Then the complexification $W^{\mathbb{C}}$ has a direct sum decomposition

$$W^{\mathbb{C}} = \sum_i W_i^{\mathbb{C}}. \tag{59}$$

Moreover, the following (i), (ii) *hold.*

(i) *Each $W_i^{\mathbb{C}}$ is an irreducible $T^{\mathbb{C}}$-module with the same endpoint, dual endpoint, diameter, and shape as W_i.*

(ii) *The multiplicity of $W_i^{\mathbb{C}}$ in $W^{\mathbb{C}}$ equals the multiplicity of W_i in W.*

Proof. This follows from Theorem 1.3, Theorem 1.4 and Lemma 11.2. ∎

12 The algebras $E^*_1 T E^*_1$ and $E_1 T E_1$

Recall the Q-polynomial distance-regular graph Γ. Recall $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Fix $x \in X$ and write $T = T(x)$. In this section, we discuss the algebras $E^*_1 T E^*_1$ and $E_1 T E_1$. Let W denote an irreducible T-module. Consider the subspaces $E^*_1 W$ and $E_1 W$. If W is primary, then both $E^*_1 W$ and $E_1 W$ have dimension one, as discussed in (47). In the next lemma, we assume that W is nonprimary.

Lemma 12.1. *Let W denote a nonprimary irreducible T-module with endpoint r, dual endpoint s, and diameter d. Then $1 \leq r, s \leq D$. Moreover, the following* (i)–(iv) *hold.*

(i) *If $r = 1$, then $E^*_1 W$ has dimension one.*

(ii) *If $2 \leq r \leq D$, then $E^*_1 W = 0$.*

(iii) *If $s = 1$, then $E_1 W$ has dimension one.*

(iv) *If* $2 \le s \le D$, *then* $E_1 W = 0$.

Proof. We have $1 \le r, s \le D$ by Lemma 9.4.
(i): By Theorem 1.1.
(iii): By (i) and Lemma 9.3
(ii),(iv): By definition of endpoint and dual endpoint. ∎

Proposition 12.2. *The algebras* $E_1^* T E_1^*$ *and* $E_1 T E_1$ *are commutative.*

Proof. We first show that $E_1^* T E_1^*$ is commutative. Since V is an orthogonal direct sum of irreducible T-modules, it suffices to show that the action of $E_1^* T E_1^*$ on any irreducible T-module is commutative. Let W denote an irreducible T-module. By (47) and Lemma 12.1(i),(ii) either $E_1^* W$ has dimension one or $E_1^* W = 0$. In either case, the action of $E_1^* T E_1^*$ on W is commutative. We have shown that $E_1^* T E_1^*$ is commutative. The argument for $E_1 T E_1$ is similar. ∎

Recall the all-ones matrix $\mathbb{J}$ from Section 5.

Lemma 12.3. *The following* (i), (ii) *hold.*

(i) $E_1^* M E_1^*$ *is spanned by* $E_1^*, E_1^* \mathbb{J} E_1^*, E_1^* A E_1^*$.

(ii) $E_1 M E_1$ *is spanned by* $E_1, E_1 E_0^* E_1, E_1 A^* E_1$.

Proof. (i): Since M has a basis $\{A_i\}_{i=0}^D$, the subspace $E_1^* M E_1^*$ is spanned by $\{E_1^* A_i E_1^*\}_{i=0}^D$. Applying (40) to $\{E_1^* A_i E_1^*\}_{i=0}^D$, we find that $E_1^* A_i E_1^* = 0$ for $3 \le i \le D$. Therefore, $E_1^* M E_1^*$ is spanned by E_1^*, $E_1^* A E_1^*$ and $E_1^* A_2 E_1^*$. Recall that $\mathbb{J} = \sum_{i=0}^D A_i$. Using this and (40), we have

$$E_1^* \mathbb{J} E_1^* = E_1^* + E_1^* A E_1^* + E_1^* A_2 E_1^*.$$

From this equation, $E_1^* A_2 E_1^*$ can be expressed in terms of E_1^*, $E_1^* \mathbb{J} E_1^*$ and $E_1^* A E_1^*$. The result follows.
(ii): Since M^* has a basis $\{A_i^*\}_{i=0}^D$, the subspace $E_1 M^* E_1$ is spanned by $\{E_1 A_i^* E_1\}_{i=0}^D$. Applying (41) to $\{E_1 A_i^* E_1\}_{i=0}^D$, we find that $E_1 A_i^* E_1 = 0$ for $3 \le i \le D$. Therefore, $E_1 M^* E_1$ is spanned by E_1, $E_1 A^* E_1$ and $E_1 A_2^* E_1$. Recall that $|X| E_0^* = \sum_{i=0}^D A_i^*$. Using this and (41), we have

$$|X| E_1 E_0^* E_1^* = E_1 + E_1 A^* E_1 + E_1 A_2^* E_1.$$

From this equation, $E_1 A_2^* E_1$ can be expressed in terms of E_1, $E_1 E_0^* E_1$ and $E_1 A^* E_1$. The result follows. ∎

Lemma 12.4. *Let* W *and* W' *denote irreducible* T*-modules with endpoint* 1. *Let* μ *(resp.* μ'*) denote the eigenvalue of* $E_1^* A E_1^*$ *on* $E_1^* W$ *(resp.* $E_1^* W'$*). Then the* T*-modules* W *and* W' *are isomorphic if and only if* $\mu = \mu'$.

Proof. Pick $0 \ne w \in E_1^* W$ and $0 \ne w' \in E_1^* W'$. By Lemma 12.1(i), we have $E_1^* W = \mathrm{Span}\{w\}$ and $E_1^* W' = \mathrm{Span}\{w'\}$. Observe that w (resp. w') is an eigenvector of $E_1^* A E_1^*$ corresponding to the eigenvalue μ (resp. μ').

First, suppose that the T-modules W and W' are isomorphic. There exists a T-module isomorphism $\sigma : W \to W'$. Since w (resp. w') is a basis for E_1^*W (resp. E_1^*W'), we have $\sigma(w) = \beta w'$ for some $0 \neq \beta \in \mathbb{F}$. Then

$$0 = (\sigma E_1^* A E_1^* - E_1^* A E_1^* \sigma)w = \sigma E_1^* A E_1^* w - E_1^* A E_1^* \sigma(w) = \beta(\mu - \mu')w'. \tag{60}$$

Since $\beta \neq 0$ and $w' \neq 0$, it follows that $\mu = \mu'$.

Conversely, suppose that $\mu = \mu'$. We will show that the T-modules W and W' are isomorphic. Note that $W = Tw$ and $W' = Tw'$ by the irreducibility. We claim that for any $B \in T$, we have $Bw = 0$ if and only if $Bw' = 0$. Observe that $w = E_1^* w$, and thus

$$Bw = 0 \quad \iff \quad BE_1^* w = 0 \quad \iff \quad \|BE_1^* w\|^2 = 0 \quad \iff \quad \overline{w}^\top E_1^* \overline{B}^\top B E_1^* w = 0. \tag{61}$$

Consider the matrix $E_1^* \overline{B}^\top B E_1^*$. Since $\overline{B}^\top B \in T$, it follows that $E_1^* \overline{B}^\top B E_1^* \in E_1^* T E_1^*$. By Lemma 10.1(i) with $r = 1$ and Lemma 12.3(i), the algebra $E_1^* T E_1^*$ is generated by $E_1^* \mathbb{J} E_1^*$ and $E_1^* A E_1^*$ on W, where we note that E_1^* is the multiplicative identity. Therefore, the restriction of $E_1^* \overline{B}^\top B E_1^*$ to W can be expressed in terms of the restrictions of $E_1^* \mathbb{J} E_1^*$ and $E_1^* A E_1^*$ to W. We now apply this restriction to w. Since $E_1^* \mathbb{J} E_1^* w = 0$ and $E_1^* A E_1^* w = \mu w$, it follows that

$$E_1^* \overline{B}^\top B E_1^* w = p(\mu) w, \tag{62}$$

for some polynomial $p \in \mathbb{F}[\lambda]$. Taking the inner product of both sides of (62) with w yields

$$\overline{w}^\top E_1^* \overline{B}^\top B E_1^* w = p(\mu)\|w\|^2.$$

Therefore,

$$\overline{w}^\top E_1^* \overline{B}^\top B E_1^* w = 0 \quad \iff \quad p(\mu)\|w\|^2 = 0 \quad \iff \quad p(\mu) = 0. \tag{63}$$

By (61) and (63), it follows that $Bw = 0$ if and only if $p(\mu) = 0$. Similarly, it follows that $Bw' = 0$ if and only if $p(\mu) = 0$. Therefore, $Bw = 0$ if and only if $Bw' = 0$ as claimed. Next, define the map $\sigma : W \to W'$ that sends Bw to Bw' for all $B \in T$. By the claim, this map is well-defined. One verifies that σ is a T-module homomorphism. Moreover, σ is injective by the claim and surjective by irreducibility. Consequently, σ is a bijection, and hence a T-module isomorphism. The result follows. ■

Let $\mathbf{W}$ denote the span of all irreducible T-modules with endpoint 1.

Corollary 12.5. *For the T-module $\mathbf{W}$, the number of mutually non-isomorphic irreducible T-modules with endpoint 1 is equal to the number of distinct eigenvalues of $E_1^* A E_1^*$ on $E_1^* \mathbf{W}$.*

Proof. By Lemma 12.4. ■

Lemma 12.6. *The algebra $E_1^* T E_1^*$ is generated by $E_1^* M E_1^*$.*

Proof. Let ℓ denote the number of mutually non-isomorphic irreducible T-modules with endpoint 1. Let $W_1, W_2, \ldots, W_\ell$ denote mutually non-isomorphic irreducible T-modules with endpoint 1. For $1 \leq i \leq \ell$, let mult_i denote the multiplicity of W_i in V. Then we have a direct sum

$$\mathbf{W} = \sum_{i=1}^{\ell} \sum_{j=1}^{\mathrm{mult}_i} W_i^{(j)}, \tag{64}$$

where each $W_i^{(j)}$ is an irreducible T-module isomorphic to W_i. Let W_0 denote the primary T-module. Note that the multiplicity of W_0 in V is 1. Consider the direct sum $W_0 + \mathbf{W}$. Applying E_1^* to this sum, we get

$$E_1^*V = E_1^*W_0 + E_1^*\mathbf{W} \tag{65}$$

$$= E_1^*W_0 + \sum_{i=1}^{\ell} \sum_{j=1}^{\mathrm{mult}_i} E_1^*W_i^{(j)} \qquad \text{(direct sum)}. \tag{66}$$

By construction, $E_1^*TE_1^*$ acts faithfully on E_1^*V. Moreover, each summand in (66) is an $E_1^*TE_1^*$-submodule and has dimension one by (47) and Lemma 12.1(i). By Wedderburn's theorem [6] we have an $\mathbb{F}$-algebra isomorphism

$$E_1^*TE_1^* \cong \mathbb{F} \oplus \mathbb{F} \oplus \cdots \oplus \mathbb{F} \qquad (\ell + 1 \text{ copies}). \tag{67}$$

In particular, $\dim(E_1^*TE_1^*) = \ell + 1$.

Next, pick $0 \neq w_0 \in E_1^*W_0$, and for $1 \leq i \leq \ell$ and $1 \leq j \leq \mathrm{mult}_i$, pick $0 \neq w_i^{(j)} \in E_1^*W_i^{(j)}$ Observe that

$$\{w_0\} \cup \{w_i^{(j)} \mid 1 \leq i \leq \ell, \quad 1 \leq j \leq \mathrm{mult}_i\} \tag{68}$$

forms a basis for E_1^*V. Define the matrix $K \in \mathrm{Mat}_X(\mathbb{F})$ by

$$K = E_1^*\mathbb{J}E_1^* + E_1^*AE_1^*. \tag{69}$$

Observe that $K \in E_1^*ME_1^*$. We describe the action of K on E_1^*V. Recall the direct sum (65) of E_1^*V. First, consider the action of K on $E_1^*W_0$. Note that w_0 is a scalar multiple of $E_1^*\mathbb{1}$. By construction, we have $Kw_0 = (k + a_1)w_0$, where k is the valency of Γ and a_1 is the intersection number of Γ. We denote $\mu_0 := k + a_1$. Next, consider the action of K on $E_1^*\mathbf{W}$. Note that $E_1^*\mathbf{W}$ is the orthogonal complement of $E_1^*W_0$ in E_1^*V. Since $E_1^*\mathbb{J}E_1^*$ acts as zero on $E_1^*\mathbf{W}$, the action of K on $E_1^*\mathbf{W}$ coincides with the action of $E_1^*AE_1^*$ on $E_1^*\mathbf{W}$. By Corollary 12.5, the matrix $E_1^*AE_1^*$ has ℓ mutually distinct eigenvalues $\{\mu_i\}_{i=1}^{\ell}$ on $E_1^*\mathbf{W}$. For $1 \leq i \leq \ell$, each scalar μ_i is an eigenvalue of the local graph $\Gamma(x)$ and thus $\mu_i < \mu_0$. By these comments, the matrix K has $\ell + 1$ mutually distinct eigenvalues $\{\mu_i\}_{i=0}^{\ell}$ on E_1^*V. Since K is real and symmetric, its action on E_1^*V has minimal polynomial of degree $\ell + 1$. Hence, the matrices

$$E_1^*, \ K, \ K^2, \ \ldots \ , \ K^\ell \tag{70}$$

are linearly independent. By this and since $\dim(E_1^*TE_1^*) = \ell + 1$, the matrices (70) form a basis for $E_1^*TE_1^*$. Therefore, K generates $E_1^*TE_1^*$. By this and since $K \in E_1^*ME_1^*$, the result follows. ∎

Proposition 12.7. *Every element of $E_1^*TE_1^*$ is symmetric.*

Proof. By Lemma 8.1(i), every element of $E_1^*ME_1^*$ is symmetric. By Proposition 12.2 and Lemma 12.6, the algebra $E_1^*TE_1^*$ is commutative and generated by $E_1^*ME_1^*$. The result follows. ∎

We have described the algebra $E_1^*TE_1^*$. We now discuss its structure in more detail.

Lemma 12.8 (Reduction rules, [38, Lemmas 6.3, 6.5])**.** *The following* (i)–(iii) *hold.*

(i) $E_0 E_1^* E_0 = |X|^{-1} k E_0$, *where k is the valency of Γ.*

(ii) $E_0 E_1^* A = \sum_{h=0}^{D} p_{1,1}^h E_0 E_h^*$, *where p_{ij}^h are the intersection numbers of Γ.*

(iii) $A E_1^* E_0 = \sum_{h=0}^{D} p_{1,1}^h E_h^* E_0$.

Proof. Set $i = 1$ and $j = 1$ in [38, Lemma 6.3(ii),(iv), Lemma 6.5(iv)]. ■

Lemma 12.9. *The following* (i)–(iv) *hold.*

(i) *The algebra $E_1^* T E_1^*$ is generated by $E_1^* \mathbb{J} E_1^*$, $E_1^* A E_1^*$.*

(ii) $(E_1^* \mathbb{J} E_1^*)^2 = k E_1^* \mathbb{J} E_1^*$.

(iii) $(E_1^* \mathbb{J} E_1^*)(E_1^* A E_1^*) = a_1 E_1^* \mathbb{J} E_1^* = (E_1^* A E_1^*)(E_1^* \mathbb{J} E_1^*)$, *where a_1 is from* (15).

(iv) *$E_1^* \mathbb{J} E_1^*$ is a basis for a (two-sided) ideal of $E_1^* T E_1^*$.*

Proof. (i): By Lemmas 12.3(i) and 12.6.
(ii): Recall $\mathbb{J} = |X| E_0$. Using this and Lemma 12.8(i), we have

$$(E_1^* \mathbb{J} E_1^*)^2 = |X|^2 E_1^* (E_0 E_1^* E_0) E_1^* = k |X| E_1^* E_0 E_1^* = k E_1^* \mathbb{J} E_1^*.$$

(iii): We have

$$\begin{aligned}
(E_1^* \mathbb{J} E_1^*)(E_1^* A E_1^*) &= |X| E_1^* (E_0 E_1^* A) E_1^* \\
&= |X| E_1^* \left(\sum_{h=0}^{D} p_{1,1}^h E_0 E_h^* \right) E_1^* && \text{(by Lemma 12.8(ii))} \\
&= |X| E_1^* \left(p_{1,1}^0 E_0 E_0^* + p_{1,1}^1 E_0 E_1^* + p_{1,1}^2 E_0 E_2^* \right) E_1^* \\
&= a_1 E_1^* \mathbb{J} E_1^* && (a_1 = p_{1,1}^1 \text{ from (15)}).
\end{aligned}$$

Similarly, using Lemma 12.8(iii) we obtain $(E_1^* A E_1^*)(E_1^* \mathbb{J} E_1^*) = a_1 E_1^* \mathbb{J} E_1^*$.
(iv): Immediate from (i)–(iii). ■

We now discuss the algebra $E_1 T E_1$.

Lemma 12.10. *The algebra $E_1 T E_1$ is generated by $E_1 M^* E_1$.*

Proof. Similar to Lemma 12.6. ■

Proposition 12.11. *Every element of $E_1 T E_1$ is symmetric.*

Proof. Similar to Proposition 12.7. ■

Lemma 12.12 (Reduction rules, [38, Lemmas 6.4, 6.6])**.** *The following* (i)–(iii) *hold.*

(i) $E_0^* E_1 E_0^* = |X|^{-1} m_1 E_0^*$, *where m_1 is the trace of E_1.*

(ii) $E_0^* E_1 A^* = \sum_{h=0}^{D} q_{1,1}^h E_0^* E_h$, *where $q_{i,j}^h$ are the Krein parameters of Γ.*

(iii) $A^* E_1 E_0^* = \sum_{h=0}^{D} q_{1,1}^h E_h E_0^*$.

Proof. Set $i = 1$ and $j = 1$ in [38, Lemma 6.4(ii),(iv), Lemma 6.6(iv)]. ■

Lemma 12.13. *The following* (i)–(iv) *hold.*

(i) *The algebra $E_1 T E_1$ is generated by $E_1 E_0^* E_1$, $E_1 A^* E_1$.*

(ii) $(E_1 E_0^* E_1)^2 = m_1 |X|^{-1} E_1 E_0^* E_1$, *where m_1 is the trace of E_1.*

(iii) $(E_1 E_0^* E_1)(E_1 A^* E_1) = a_1^* E_1 E_0^* E_1 = (E_1 A^* E_1)(E_1 E_0^* E_1)$, *where a_1^* is from* (37).

(iv) *$E_1 E_0^* E_1$ is a basis for a (two-sided) ideal of $E_1 T E_1$*

Proof. (i): By Lemmas 12.3(ii) and 12.10.

(ii): By Lemma 12.12(i), we have

$$(E_1 E_0^* E_1)^2 = E_1 (E_0^* E_1 E_0^*) E_1 = m_1 |X|^{-1} E_1 E_0^* E_1.$$

(iii): We have

$$\begin{aligned}
(E_1 E_0^* E_1)(E_1 A^* E_1) &= E_1 (E_0^* E_1 A^*) E_1 \\
&= E_1 \left(\sum_{i=1}^{D} q_{1,1}^h E_0^* E_h \right) E_1 && \text{(By Lemma 12.12(ii))} \\
&= E_1 (q_{1,1}^0 E_0^* E_0 + q_{1,1}^1 E_0^* E_1 + q_{1,1}^2 E_0^* E_2) E_1 \\
&= a_1^* E_1 E_0^* E_1 && (a_1^* = q_{1,1}^1 \text{ from } (37)).
\end{aligned}$$

Similarly, using Lemma 12.12(iii) we obtain $(E_1 A^* E_1)(E_1 E_0^* E_1) = a_1^* E_1 E_0^* E_1$.
(iv): Immediate from (i)–(iii).

■

13 The algebras $E_D^* T E_D^*$ and $E_D T E_D$

We continue to discuss the Q-polynomial distance-regular graph Γ. Fix $x \in X$ and write $T = T(x)$. In this section, we discuss the algebras $E_D^* T E_D^*$ and $E_D T E_D$.

Lemma 13.1. *Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. Note that $r + d \leq D$ and $s + d \leq D$. The following* (i)–(iv) *hold.*

(i) *If $r + d = D$, then $E_D^* W$ has dimension one.*

(ii) *If $r + d < D$, then $E_D^* W = 0$.*

(iii) *If $s + d = D$, then $E_D W$ has dimension one.*

(iv) *If $s + d < D$, then $E_D W = 0$.*

Proof. (i),(iii): By Theorem 1.1 and Lemma 9.3.
(ii),(iv): By construction. ■

The following lemmas are analogues to Lemmas 10.1 and 10.2.

Lemma 13.2 ([28, Theorem 2.6]). *Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. Then the following* (i)–(iii) *hold on W.*

(i) *The algebra $E^*_{r+d}TE^*_{r+d}$ is generated by $E^*_{r+d}ME^*_{r+d}$.*

(ii) *The elements of $E^*_{r+d}ME^*_{r+d}$ mutually commute.*

(iii) *The algebra $E^*_{r+d}TE^*_{r+d}$ is commutative.*

Proof. By [28, Theorem 2.6] with $\{E^*_{r+i}\}_{i=0}^d$ replaced by $\{E^*_{r+d-i}\}_{i=0}^d$. ∎

Lemma 13.3 ([28, Theorem 2.6]). *Let W denote an irreducible T-module with endpoint r, dual endpoint s, and diameter d. Then the following* (i)–(iii) *hold on W.*

(i) *The algebra $E_{s+d}TE_{s+d}$ is generated by $E_{s+d}M^*E_{s+d}$.*

(ii) *The elements of $E_{s+d}M^*E_{s+d}$ mutually commute.*

(iii) *The algebra $E_{s+d}TE_{s+d}$ is commutative.*

Proof. Similar to Lemma 13.2. ∎

Proposition 13.4. *The algebras $E^*_DTE^*_D$ and E_DTE_D are commutative.*

Proof. We first show that $E^*_DTE^*_D$ is commutative. Since V is an orthogonal direct sum of irreducible T-modules, it suffices to show that the action of $E^*_DTE^*_D$ on any irreducible T-module is commutative. Let W be an irreducible T-module. By Lemma 13.1(i),(ii), either $\dim(E^*_DW) = 1$ or $E^*_DW = 0$. In either case, the action of $E^*_DTE^*_D$ on W is commutative. We have shown that $E^*_DTE^*_D$ is commutative. The argument for E_DTE_D is similar. ∎

Let W denote an irreducible T-module with endpoint r and diameter d satisfying $r + d = D$. Pick $0 \neq w \in E^*_DW$. Since E^*_DW is invariant under $E^*_DME^*_D$ and by Lemma 13.1(i), there exist scalars $\{\varphi_h\}_{h=1}^D$ in $\mathbb{R}$ such that

$$E^*_DA_hE^*_Dw = \varphi_hw \qquad (1 \leq h \leq D). \tag{71}$$

We call $\{\varphi_h\}_{h=1}^D$ the *local eigenvalue sequence* of E^*_DW.

Lemma 13.5. *Let W and W' denote irreducible T-modules with endpoint r and diameter d satisfying $r + d = D$. Then the T-modules W and W' are isomorphic if and only if the local eigenvalue sequences of E^*_DW and E^*_DW' coincide.*

Proof. Let $\{\varphi_h\}_{h=1}^D$ (resp. $\{\varphi'_h\}_{h=1}^D$) denote the local eigenvalue sequence of E^*_DW (resp. E^*_DW'). Pick $0 \neq w \in E^*_DW$ and $0 \neq w' \in E^*_DW'$. By Lemma 13.1(i), we have $E^*_DW = \mathrm{Span}\{w\}$ and $E^*_DW' = \mathrm{Span}\{w'\}$. Observe that w (resp. w') is an eigenvector of $E^*_DA_hE^*_D$ corresponding to the eigenvalue φ_h (resp. φ'_h) for $1 \leq h \leq D$.

First, suppose that the T-modules W and W' are isomorphic. There exists a T-module isomorphism $\sigma : W \to W'$. Since w (resp. w') is a basis for E^*_DW (resp. E^*_DW'), we have $\sigma(w) = \beta w'$ for some $0 \neq \beta \in \mathbb{F}$. Then for $0 \leq h \leq D$

$$0 = (\sigma E^*_DA_hE^*_D - E^*_DA_hE^*_D\sigma)w = \sigma E^*_DA_hE^*_Dw - E^*_DA_hE^*_D\sigma(w) = \beta(\varphi_h - \varphi'_h)w'.$$

Since $\beta \neq 0$ and $w' \neq 0$, it follows that $\varphi_h = \varphi'_h$ for $1 \leq h \leq D$.

Conversely, suppose that $\varphi_h = \varphi'_h$ for $1 \leq h \leq D$. We will show that W and W' are isomorphic as T-modules. Observe that $W = Tw$ and $W' = Tw'$ by the irreducibility. We first claim that for any $B \in T$, we have $Bw = 0$ if and only if $Bw' = 0$. Observe that $w = E^*_D w$, and thus

$$Bw = 0 \iff BE^*_D w = 0 \iff \|BE^*_D w\|^2 = 0 \iff \overline{w}^\top E^*_D \overline{B}^\top BE^*_D w = 0. \tag{72}$$

Consider the matrix $E^*_D \overline{B}^\top BE^*_D$. Since $\overline{B}^\top B \in T$, it follows that $E^*_D \overline{B}^\top BE^*_D \in E^*_D T E^*_D$. By Lemma 13.2(i), and since M is spanned by $\{A_h\}_{h=0}^D$ with $A_0 = I$, the algebra $E^*_D T E^*_D$ is generated on W by the elements $\{E^*_D A_h E^*_D\}_{h=1}^D$. Therefore, the restriction of $E^*_D \overline{B}^\top BE^*_D$ to W can be expressed in terms of the restrictions of $\{E^*_D A_h E^*_D\}_{h=1}^D$ to W. Hence, there exists a polynomial $\phi \in \mathbb{F}[\lambda_1, \lambda_2, \ldots, \lambda_D]$ such that on W,

$$E^*_D \overline{B}^\top BE^*_D = \phi(E^*_D AE^*_D,\ E^*_D A_2 E^*_D,\ \ldots\ ,\ E^*_D A_D E^*_D). \tag{73}$$

We now apply $E^*_D \overline{B}^\top BE^*_D$ to w. Using (71) and (73), we find that on W,

$$E^*_D \overline{B}^\top BE^*_D w = \phi(E^*_D AE^*_D,\ E^*_D A_2 E^*_D,\ \ldots\ ,\ E^*_D A_D E^*_D)w = \phi(\varphi_1, \varphi_2, \ldots, \varphi_D)w. \tag{74}$$

Take the inner product of each term in (74) with w to obtain

$$\overline{w}^\top E^*_D \overline{B}^\top BE^*_D w = \phi(\varphi_1, \varphi_2, \ldots, \varphi_D)\|w\|^2.$$

Therefore,

$$\begin{aligned} \overline{w}^\top E^*_D \overline{B}^\top BE^*_D w = 0 \quad &\iff \quad \phi(\varphi_1, \varphi_2, \ldots, \varphi_D)\|w\|^2 = 0 \\ &\iff \quad \phi(\varphi_1, \varphi_2, \ldots, \varphi_D) = 0. \end{aligned} \tag{75}$$

By (72) and (75), we obtain $Bw = 0$ if and only if $\phi(\varphi_1, \varphi_2, \ldots, \varphi_D) = 0$. Similarly, we obtain $Bw' = 0$ if and only if $\phi(\varphi_1, \varphi_2, \ldots, \varphi_D) = 0$. Therefore, $Bw = 0$ if and only if $Bw' = 0$ as claimed. Next, define the map $\sigma : W \to W'$ that sends Bw to Bw' for all $B \in T$. By the claim, this map is well-defined. One verifies that σ is a T-module homomorphism. Moreover, σ is injective by the claim and surjective by irreducibility. Consequently, σ is a bijection, and hence a T-module isomorphism. The result follows. ■

Lemma 13.6. *The algebra $E^*_D T E^*_D$ is generated by $E^*_D M E^*_D$.*

Proof. Let ℓ denote the number of mutually non-isomorphic irreducible T-modules with endpoint r and diameter d satisfying $r + d = D$. Note that $\ell \geq 1$ since the primary module has endpoint 0 and diameter D. Let $W_1, \ldots, W_\ell$ denote mutually non-isomorphic irreducible T-modules with endpoint r and diameter d satisfying $r + d = D$. For $1 \leq i \leq \ell$, let mult_i denote the multiplicity of W_i in V. By construction, V contains a direct sum

$$\sum_{i=1}^{\ell} \sum_{j=1}^{\text{mult}_i} W_i^{(j)} \tag{76}$$

where each $W_i^{(j)}$ is an irreducible T-module isomorphic to W_i. Applying E_D^* to (76), we get

$$E_D^* V = \sum_{i=1}^{\ell} \sum_{j=1}^{\text{mult}_i} E_D^* W_i^{(j)} \qquad \text{(direct sum)}. \tag{77}$$

By construction, $E_D^* T E_D^*$ acts faithfully on $E_D^* V$. Moreover, each summand in (77) is an $E_D^* T E_D^*$-submodule and has dimension one by Lemma 13.1(i). By Wedderburn's theorem [6] we have an $\mathbb{F}$-algebra isomorphism

$$E_D^* T E_D^* \cong \mathbb{F} \oplus \mathbb{F} \oplus \cdots \oplus \mathbb{F} \qquad (\ell \text{ copies}). \tag{78}$$

In particular, $\dim(E_D^* T E_D^*) = \ell$.

Next, for $1 \leq i \leq \ell$ and $1 \leq j \leq \text{mult}_i$, pick $0 \neq w_i^{(j)} \in E_D^* W_i^{(j)}$. Observe that

$$\{w_i^{(j)} \mid 1 \leq i \leq \ell, \quad 1 \leq j \leq \text{mult}_i\} \tag{79}$$

forms a basis for $E_D^* V$. For each $1 \leq i \leq \ell$, let $\{\varphi_{i,h}\}_{h=1}^D$ denote the local eigenvalue sequence of $E_D^* W_i$; that is, the scalars $\{\varphi_{i,h}\}_{h=1}^D$ satisfy

$$E_D^* A_h E_D^* w_i^{(j)} = \varphi_{i,h} w_i^{(j)} \qquad (1 \leq h \leq D, \ 1 \leq j \leq \text{mult}_i). \tag{80}$$

For $1 \leq i \leq \ell$, we view the local eigenvalue sequence $\{\varphi_{i,h}\}_{h=1}^D$ of $E_D^* W_i$ as a vector

$$u_i := (\varphi_{i,1}, \ \varphi_{i,2}, \ \ldots, \ \varphi_{i,D}) \in \mathbb{F}^D. \tag{81}$$

Since the T-modules $W_1, W_2, \ldots, W_\ell$ are mutually non-isomorphic, it follows from Lemma 13.5 that the vectors $\{u_i\}_{i=1}^\ell$ are mutually distinct. By this fact and since the field $\mathbb{F}$ is infinite, there exists a linear transformation $f : \mathbb{F}^D \to \mathbb{F}$ such that the scalars $\{f(u_i)\}_{i=1}^\ell$ are mutually distinct. Write this linear transformation as

$$f(\lambda_1, \lambda_2, \ldots, \lambda_D) = \sum_{i=1}^D \alpha_i \lambda_i$$

for some $\alpha_1, \ldots, \alpha_D \in \mathbb{F}$. Define the matrix $\Delta \in \text{Mat}_X(\mathbb{F})$ by

$$\Delta = \sum_{h=1}^D \alpha_h E_D^* A_h E_D^*. \tag{82}$$

Observe that $\Delta \in E_D^* M E_D^*$. Moreover, by (80)

$$\Delta w_i^{(j)} = f(u_i) w_i^{(j)} \qquad (1 \leq i \leq \ell, \quad 1 \leq j \leq \text{mult}_i). \tag{83}$$

This implies that the action of Δ on $E_D^* V$ has ℓ distinct eigenvalues $\{f(u_i)\}_{i=1}^\ell$. Since Δ is real and symmetric, its action on $E_D^* V$ has minimal polynomial of degree ℓ. It follows that the matrices

$$E_D^*, \ \Delta, \ \Delta^2, \ldots, \Delta^{\ell-1} \tag{84}$$

are linearly independent. By this and since $\dim(E_D^* T E_D^*) = \ell$, the matrices (84) form a basis for $E_D^* T E_D^*$. Therefore, Δ generates $E_D^* T E_D^*$. By this and since $\Delta \in E_D^* M E_D^*$, the result follows. ■

Proposition 13.7. *Every element of $E^*_D T E^*_D$ is symmetric.*

Proof. By Lemma 8.1(i), every element of $E^*_D M E^*_D$ is symmetric. By Proposition 13.4 and Lemma 13.6, the algebra $E^*_D T E^*_D$ is commutative and generated by $E^*_D M E^*_D$. The result follows. ■

We now discuss the algebra $E_D T E_D$.

Lemma 13.8. *The algebra $E_D T E_D$ is generated by $E_D M^* E_D$.*

Proof. Similar to Lemma 13.6. ■

Proposition 13.9. *Every element of $E_D T E_D$ is symmetric.*

Proof. Similar to the proof of Proposition 13.7. ■

We now prove Theorem 1.6

Proof of Theorem 1.6. By Propositions 12.2, 12.7, 13.4, and 13.9. ■

Recall the definition of association schemes; see [1, Section 2.2]. We conclude this paper with two problems.

Problem 13.10. Find necessary and sufficient conditions under which $E^*_1 T E^*_1$ is isomorphic to the Bose-Mesner algebra of a commutative association scheme with vertex set $\Gamma_1(x)$.

Problem 13.11. Find necessary and sufficient conditions under which $E^*_D T E^*_D$ is isomorphic to the Bose-Mesner algebra of a commutative association scheme with vertex set $\Gamma_D(x)$.

Example 13.12. We present some graphs Γ such that for every vertex x, the algebra $E^*_1 T E^*_1$ is isomorphic to the Bose-Mesner algebra of a commutative association scheme on $\Gamma_1(x)$.

(i) Assume Γ is the Hamming graph $H(D,N)$, where $D \geq 2$ and $N \geq 3$; see [4, Section 9.2]. The vertex set X of Γ consists of D-tuples with entries from the set $\{1, 2, \ldots, N\}$. Two vertices in X are adjacent whenever they differ in exactly one coordinate. For any $y, z \in X$, the distance $\partial(y,z)$ is equal to the number of coordinates at which y and z differ. Fix a vertex $x \in X$ and write $T = T(x)$. Consider the first subconstituent $\Gamma(x)$. The subgraph of Γ induced on $\Gamma(x)$ is a disjoint union of D copies of the complete graph K_{N-1}. This subgraph corresponds to the commutative association scheme $\mathcal{X} = (\Gamma(x), \{R_i\}_{i=0}^2)$, where

$$
\begin{aligned}
R_0 &= \{(y,y) \mid y \in \Gamma(x)\},\\
R_1 &= \{(y,z) \mid y,z \in \Gamma(x),\ \partial(y,z) = 1\},\\
R_2 &= \{(y,z) \mid y,z \in \Gamma(x),\ y \neq z,\ \partial(y,z) \neq 1\}.
\end{aligned}
$$

The Bose-Mesner algebra of $\mathcal{X}$ has dimension 3 and is isomorphic to $E^*_1 T E^*_1$.

(ii) Assume Γ is the Johnson graph $J(N, D)$, where $D \geq 2$ and $N \geq 2D$; see [4, Section 9.1]. The vertex set X of Γ consists of the D-element subsets of the set $\{1, 2, \ldots, N\}$. Two vertices $y, z \in X$ are adjacent whenever $|y \cap z| = D - 1$. For any $y, z \in X$, the distance $\partial(y, z)$ is equal to $D - |y \cap z|$. Fix a vertex $x \in X$ and write $T = T(x)$. Consider the first subconstituent $\Gamma(x)$. The subgraph of Γ induced on $\Gamma(x)$ is isomorphic to the Cartesian product $K_D \times K_{N-D}$. This subgraph corresponds to the commutative association scheme $\mathcal{X} = (\Gamma(x), \{R_i\}_{i=0}^{3})$, where

$$
\begin{aligned}
R_0 &= \{(y, y) \mid y \in \Gamma(x)\}, \\
R_1 &= \{(y, z) \mid y, z \in \Gamma(x),\ x \cap y = x \cap z\}, \\
R_2 &= \{(y, z) \mid y, z \in \Gamma(x),\ x \cup y = x \cup z\}, \\
R_3 &= \{(y, z) \mid y, z \in \Gamma(x),\ \partial(x, y) = 2\}.
\end{aligned}
$$

The Bose-Mesner algebra of $\mathcal{X}$ has dimension 4 and is isomorphic to $E_1^* T E_1^*$.

(iii) Assume Γ is the Grassmann graph $J_q(N, D)$, where $D \geq 2$ and $N > 2D$; see [4, Section 9.3]. Let $\mathbb{V}$ denote an N-dimensional vector space over $\mathbb{F}_q$. The vertex set X of Γ consists of the D-dimensional subspaces of $\mathbb{V}$. Two vertices $y, z \in X$ are adjacent whenever $\dim(y \cap z) = D - 1$. For any $y, z \in X$, the distance $\partial(y, z)$ is equal to $D - \dim(y \cap z)$. Fix a vertex $x \in X$ and write $T = T(x)$. Consider the first subconstituent $\Gamma(x)$. The subgraph of Γ induced on $\Gamma(x)$ is isomorphic to the q-clique extension of the Cartesian product $K_{\mathsf{n}} \times K_{\mathsf{m}}$, where $\mathsf{n} = \left[{D \atop 1}\right]_q$ and $\mathsf{m} = \left[{N-D \atop 1}\right]_q$; see [13, Section 4]. This subgraph corresponds to the commutative association scheme $\mathcal{X} = (\Gamma(x), \{R_i\}_{i=0}^{4})$, where

$$
\begin{aligned}
R_0 &= \{(y, y) \mid y \in \Gamma(x)\}, \\
R_1 &= \{(y, z) \mid y, z \in \Gamma(x),\ x \cap y = x \cap z,\ x + y = x + z\}, \\
R_2 &= \{(y, z) \mid y, z \in \Gamma(x),\ x \cap y = x \cap z,\ x + y \neq x + z\}, \\
R_3 &= \{(y, z) \mid y, z \in \Gamma(x),\ x \cap y \neq x \cap z,\ x + y = x + z\}, \\
R_4 &= \{(y, z) \mid y, z \in \Gamma(x),\ \partial(x, y) = 2\}.
\end{aligned}
$$

The Bose-Mesner algebra of $\mathcal{X}$ has dimension 5 and is isomorphic to $E_1^* T E_1^*$.

Example 13.13. We present some graphs Γ such that for every vertex x, the algebra $E_D^* T E_D^*$ is isomorphic to the Bose-Mesner algebra of a commutative association scheme on $\Gamma_D(x)$.

(i) Assume Γ is the Hamming graph $H(D, N)$ as in Example 13.12(i). Fix a vertex $x \in X$ and write $T = T(x)$. Consider the last subconstituent $\Gamma_D(x)$. The subgraph of Γ induced on $\Gamma_D(x)$ is isomorphic to the Hamming graph $H(D, N-1)$. This subgraph corresponds to the commutative association scheme $\mathcal{X} = (\Gamma_D(x), \{R_i\}_{i=0}^{D})$, where

$$
R_i = \{(y, z) \mid y, z \in \Gamma_D(x),\ \partial(y, z) = i\} \qquad (0 \leq i \leq D).
$$

The restrictions of $E_D^* A_i E_D^*$ $(0 \leq i \leq D)$ to $E_D^* V$ form a basis for the Bose-Mesner algebra of $\mathcal{X}$. By Lemma 13.6, this Bose-Mesner algebra is isomorphic to $E_D^* T E_D^*$.

(ii) Assume Γ is the Johnson graph $\Gamma = J(N, D)$ as in Example 13.12(ii). Fix a vertex $x \in X$ and write $T = T(x)$. Consider the last subconstituent $\Gamma_D(x)$. The subgraph of Γ induced on $\Gamma_D(x)$ is isomorphic to the Johnson graph $J(N-D, D)$. The subgraph corresponds to the commutative association scheme $\mathcal{X} = (\Gamma_D(x), \{R_i\}_{i=0}^{d})$, where $d = \min(D, N-2D)$ and

$$R_i = \{(y, z) \mid y, z \in \Gamma_D(x),\ |y \cap z| = d - i\} \qquad (0 \le i \le d).$$

The restrictions of $E_D^* A_i E_D^*$ $(0 \le i \le d)$ to $E_D^* V$ form a basis for the Bose-Mesner algebra of $\mathcal{X}$. By Lemma 13.6, this Bose-Mesner algebra is isomorphic to $E_D^* T E_D^*$.

(iii) Assume Γ is the Grassmann graph $J_q(2D, D)$ with $D \ge 2$ as in Example 13.12(iii). Fix a vertex $x \in X$ and write $T = T(x)$. Consider the last subconstituent $\Gamma_D(x)$. The subgraph of Γ induced on $\Gamma_D(x)$ is isomorphic to the bilinear forms graph on the set $D \times D$ matrices over $\mathbb{F}_q$; see [4, Section 9.5A]. The subgraph corresponds to the commutative association scheme $\mathcal{X} = (\Gamma_D(x), \{R_i\}_{i=0}^{D})$, where

$$R_i = \{(y, z) \mid y, z \in \Gamma_D(x),\ \dim(y \cap z) = D - i\} \qquad (0 \le i \le D).$$

The restrictions of $E_D^* A_i E_D^*$ $(0 \le i \le D)$ to $E_D^* V$ form a basis for the Bose-Mesner algebra of $\mathcal{X}$. By Lemma 13.6, this Bose-Mesner algebra is isomorphic to $E_D^* T E_D^*$.

Remark 13.14. If $E_1^* T E_1^*$ (resp. $E_D^* T E_D^*$) is the Bose-Mesner algebra of a commutative association scheme with vertex set $\Gamma_1(x)$ (resp. $\Gamma_D(x)$), then this association scheme must be symmetric by Proposition 12.7 (resp. 13.7).

Declaration of competing interest

The authors have no relevant financial or non-financial interests to disclose.

Data availability

All data generated or analyzed during this study are included in this published article.

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References

[1] E. Bannai, T. Ito. Algebraic Combinatorics I: Association Schemes. Benjamin/Cummings, Menlo Park, CA, 1984.

[2] P.-A. Bernard, N. Crampé, L. Vinet. The Terwilliger algebra of symplectic dual polar graphs, the subspace lattices and $U_q(\mathfrak{sl}_2)$. Discrete Math. 345 (2022) no. 12, Paper No. 113169, 19pp.

[3] N. Biggs. Algebraic Graph Theory. Cambridge University Press, London, 1994.

[4] A. E. Brouwer, A. M. Cohen, and A. Neumaier. Distance-Regular Graphs. Springer-Verlag, Berlin, 1989.

[5] J. S. Caughman IV. The Terwilliger algebras of bipartite P- and Q-polynomial association schemes. Discrete Math. 196 (1999) 65–95.

[6] C. Curtis, I. Reiner. Representation Theory of Finite Groups and Associative Algebras. Pure and Applied Mathematics, Vol. XI, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[7] E. R. van Dam, J. H. Koolen, H. Tanaka. Distance-regular graphs. Electron. J. Combin. (2016) DS22.

[8] P. Delsarte. An algebraic approach to the association schemes of coding theory. Philips Res. Rep. Suppl., No.10 (1973).

[9] E. Egge. A generalization of the Terwilliger algebra. J. Algebra 233 (2000) 213–252.

[10] B. Fernández. Certain graphs with exactly one irreducible T-module with endpoint 1, which is thin. J. Algebraic Combin. 56 (2022), no. 4, 1287–1307.

[11] W. Fulton and J. Harris. Representation Theory: A First Course. Graduate Texts in Mathematics, Vol. 129. Springer, New York, 1991.

[12] A. L. Gavrilyuk, J. H. Koolen. The Terwilliger polynomial of a Q-polynomial distance-regular graph and its application to pseudo-partition graphs. Linear Algebra Appl. 466 (2015) 117–140.

[13] A. L. Gavrilyuk, J. H. Koolen. A characterization of the Grassmann graphs. J. Comb. Theory, Ser. B 171 (2025) 1–27.

[14] H.-W. Huang. An imperceptible connection between the Clebsch-Gordan coefficients of $U_q(\mathfrak{sl}_2)$ and the Terwilliger algebras of Grassmann graphs. J. Combin. Theory Ser. A 214 (2025) Paper No. 106028, 83pp.

[15] T. Ito, K. Tanabe, P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes. Codes and Association Schemes (Piscataway NJ, 1999), American Mathematical Society, Providence, RI, 2001, pp. 167–192.

[16] T. Ito, K. Nomura, P. Terwilliger. A classification of sharp tridiagonal pairs. Linear Algebra Appl. 435 (2011) 1857–1884.

[17] T. Ito, P. Terwilliger. The shape of a tridiagonal pair. J. Pure Appl. Algebra 188 (2004) 145–160.

[18] T. Ito, P. Terwilliger. Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. Ramanujan J. 13 (2007) 39–62.

[19] T. Ito, P. Terwilliger. Tridiagonal pairs of Krawtchouk type. Linear Algebra Appl. 427 (2007) 218–233.

[20] T. Ito, P. Terwilliger. Distance-regular graphs of q-Racah type and the q-tetrahedron algebra. Michigan Math. J. 58 (2009), no. 1, 241–254.

[21] T. Ito, P. Terwilliger. How to sharpen a tridiagonal pair. J. Algebra Appl. 9 (2010), no. 4, 543–552.

[22] T. Y. Lam. A First Course in Noncommutative Rings, 2nd edition. Graduate Texts in Mathematics, Vol. 131. Springer, 2001.

[23] J.-H. Lee. Q-polynomial distance-regular graphs and a double affine Hecke algebra of rank one. Linear Algebra Appl. 439 (2013) 3184–3240.

[24] J.-H. Lee. Nonsymmetric Askey-Wilson polynomials and Q-polynomial distance-regular graphs. J. Combin. Theory Ser. A, 147 (2017) 75–118.

[25] J. V. Morales. A rank two Leonard pair in Terwilliger algebras of Doob graphs. J. Combin. Theory Ser. A, 210 (2025) Paper No. 105958, 21pp.

[26] K. Nomura, P. Terwilliger. The split decomposition of a tridiagonal pair. Linear Algebra Appl. 424 (2007) 339–345.

[27] K. Nomura, P. Terwilliger. Sharp tridiagonal pairs. Linear Algebra Appl. 429 (2008) 79–99.

[28] K. Nomura, P. Terwilliger. The structure of a tridiagonal pair. Linear Algebra Appl. 429 (2008) 1647–1662.

[29] S. Roman. Advanced Linear Algebra, 3rd edition. Graduate Texts in Mathematics, Vol. 135. Springer, New York, 2010.

[30] H. Suzuki. The Terwilliger algebra associated with a set of vertices in a distance-regular graph. J. Algebraic Combin. 22 (2005), no. 1, 5–38.

[31] Y.-Y. Tan, J. Koolen, M. Cao, J. Park. Thin Q-polynomial distance-regular graphs have bounded c_2. Graphs Combin. 38 (2022), no. 6, Paper No. 175, 18 pp.

[32] H. Tanaka, T. Wang. The Terwilliger algebra of the twisted Grassmann graph: the thin case. Electron. J. Combin. 27 (2020) no. 4, Paper No. 4.15, 22 pp.

[33] P. Terwilliger. The subconstituent algebra of an association scheme I. J. Algebraic Combin. 1 (1992) 363–388.

[34] P. Terwilliger. The subconstituent algebra of an association scheme II. J. Algebraic Combin. 2 (1993) 73–103.

[35] P. Terwilliger. The subconstituent algebra of an association scheme III. J. Algebraic Combin. 2 (1993) 177–210.

[36] P. Terwilliger. Lecture note on Terwilliger algebra (edited by H. Suzuki) 1993.

[37] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. Linear Algebra Appl. 330 (2001) 149–203.

[38] P. Terwilliger. Distance-regular graphs, the subconstituent algebra, and the Q-polynomial property. London Math. Soc. Lecture Note Ser. 487, Cambridge University Press, London, 2024, 430–491.

[39] P. Terwilliger, A. Žitnik. The quantum adjacency algebra and subconstituent algebra of a graph. J. Combin. Theory Ser. A 166 (2019) 297–314.

[40] M. Tomiyama, N. Yamazaki. The subconstituent algebra of a strongly regular graph. Kyushu J. Math. 48, No. 2 (1994) 323–334.