ON GONALITY-TIGHT GRAPHS

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Abstract. We address a question posed by Fessler–Jensen–Kelsey–Owen regarding graphs whose second gonality is greater than the first by exactly 1. We answer the question affirmatively under a stronger condition, thereby characterising the entire gonality sequence for a large family of graphs. We prove a structure theorem for the graphs satisfying the condition, and show that they are all obtained via an inductive process by gluing together complete and banana graphs under certain rules.

Contents

1. Introduction

The k-th gonality of a graph, denoted $\mathrm{gon}_k(G)$ is the smallest degree of a divisor of rank k. The gonality sequence $\mathrm{gon}_1(G), \mathrm{gon}_2(G), \ldots$, controls much of its geometric properties. For instance, if any element $\mathrm{gon}_k(G)$ equals $2k$ for k strictly smaller than the genus, then the graph is hyperelliptic [Cop16, Len17]. Questions about the gonality of graphs first appeared in the seminal paper of Baker and Norine in which they introduced the rank of divisors and discussed the notion of hyperelliptic graphs [BN07]. The connection between the gonality and covers of trees were then studied in [Cha13, DV21, MZ25] and their moduli space in [CD18]. The first reference to the gonality sequence of a graph as a whole appeared in [CP17], where its values for the complete graphs were determined.

The gonalities of algebraic curves have been studied since the early days of algebraic geometry, but the gonality sequence as a whole first appeared in a paper by Lange and Martens in 2012

[LM12]. For the general curve, its gonality sequence is completely determined by the Brill–Noether theorem. However, there are still open questions regarding the possible gonality sequences of special curves and the interplay between tropical and algebraic geometry has the potential for new results. For instance, the authors of [CDJP19] use the gonality sequence of bipartite graphs to compute those of curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

It is natural to ask which sequences may appear as the gonality sequence of a curve or a graph, and to describe the locus of objects with this sequence (see [ADM⁺21, Question 1.3]). As shown in [ADM⁺21, Theorem 1.4], the gonality sequence is completely determined by the first gonality for graphs of genus smaller than 6. The sequence has been partially determined in various cases of generalised banana graphs [CC12, FJK24, ADM⁺21] and chess graphs [Spe22, CDD⁺24, JMS⁺24], but not much more is known in general.

In the case of algebraic curves, it has recently been shown that much of their geometry is determined whenever the first two elements of the sequence differ by 1 [FJK24, Lemma 7.3]. The authors then go on to ask whether the same holds for graphs [FJK24, Question 1.3]. In this paper, we refer to graphs or curves whose second gonality exceeds the first by 1 as *gonality-tight* (see Definition 2.1). A key example of gonality-tight graphs are the generalised banana graphs $B^*_{a,a}$, for which the second gonality is realised by a divisor with a single chip on each vertex, apart from one, which has two chips [ADM⁺21, Lemma 5.6]. In the current paper, we examine when the second gonality of a gonality-tight graph is realised by a divisor of this form. As we show, the existence of such a divisor imposes strong conditions on the structure of the graph. In fact, this happens exactly for special graphs that we refer to as *quasi-banana graphs* (see Figure 1 for an illustration and Definition 3.12 for the exact definition).

Theorem A. *A graph G is a quasi-banana graph if and only if it is gonality-tight and there exists a vertex $v \in V(G)$ such that the divisor*

$$D = v + \sum_{w \in V(G)} w$$

realises the second gonality.

We then proceed to examine additional properties of quasi-banana graphs and, in particular, fully answer [FJK24, Question 1.3] for this family of graphs.

Theorem B. *Let G be a quasi-banana graph with first gonality $\mathrm{gon}_1(G) = k$. Then the following hold.*

(1) The genus of G is $g = \binom{k}{2}$.

(2) The gonality sequence of G is given by

$$\mathrm{gon}_r(G) = \begin{cases} l(k+1) - h & \text{if } r < g, \\ g + r & \text{if } r \geqslant g, \end{cases}$$

where $1 \leqslant l \leqslant k-2$ and $0 \leqslant h \leqslant l$ are uniquely determined integers such that $r = \frac{l(l+3)}{2} - h$.

(3) If $k \neq 2$, all divisors realising the second gonality are linearly equivalent.

(4) Let D be a divisor realising the second gonality. Then for every integer $-1 \leqslant l \leqslant k$,

$$r(l \cdot D) = \frac{l(l+3)}{2}.$$

In particular, $(k-2) \cdot D$ is equivalent to the canonical divisor K_G.

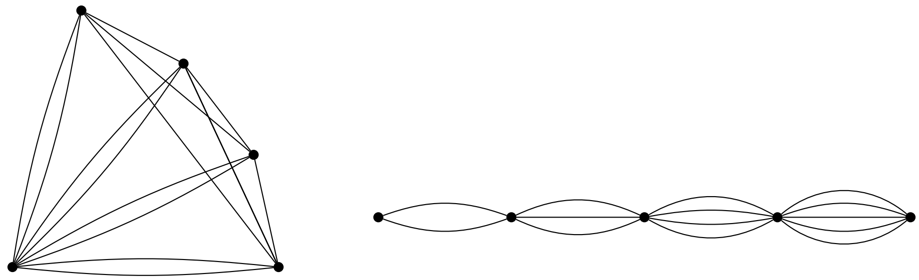


FIGURE 1. Two quasi-banana graphs of gonality 5.

These properties are somewhat surprising as quasi-banana graphs that share the same first gonality can still have significantly different structures (as seen in Figure 1 for instance). Based on a large number of examples obtained computationally, we believe that the results hold in greater generality.

Conjecture. *Let G be a gonality-tight graph with first gonality k for some integer k.*

(1) The genus of G is equal to $g = \binom{k}{2}$.

(2) The gonality sequence of G is given by

$$\mathrm{gon}_r(G) = \begin{cases} l(k+1) - h & \textit{if } r < g, \\ g + r & \textit{if } r \geqslant g, \end{cases}$$

where $1 \leqslant l \leqslant k-2$ and $0 \leqslant h \leqslant l$ are uniquely determined integers such that $r = \frac{l(l+3)}{2} - h$.

(3) If D realises the second gonality, then $(k-2) \cdot D \sim K_G$. Moreover, for every integer $1 \leqslant l \leqslant k-2$, the divisors realising the $\frac{l(l+3)}{2}$-th gonality are all linearly equivalent to $l \cdot D$.

The conjecture is consistent with the known gonality sequence of the complete graphs computed in [CP17]. The first two parts of the conjecture appeared in [FJK24, Question 1.3]. In the same paper, it is also shown that the first two parts hold for algebraic curves [FJK24, Lemma 7.2]. We now verify that the third part also holds for algebraic curves.

Lemma 1.1. *Let C be a smooth curve with first gonality k and second gonality $k+1$ for some integer k. Denote by D a divisor that realises the second gonality. For $1 \leqslant l \leqslant k-2$, a divisor realises the $\frac{l(l+3)}{2}$-th gonality if and only if it is equivalent to $l \cdot D$.*

Proof. The result is an almost immediate consequence of [Har86, Theorem 2.1], which Hartshorne attributes to Noether. By [FJK24, Lemma 7.1], we can assume that C is a plane curve of degree $k+1$ and that the divisor obtained as a line section of C is equivalent to D.

Let Z be a divisor realising the $\frac{l(l+3)}{2}$-th gonality. By [FJK24, Lemma 7.2], the degree of Z is $l \cdot (k+1)$. From Part (2.b) of Noether's theorem, the divisor Z is obtained as the scheme-theoretic intersection of C with another curve C'' of degree l. Therefore Z is equivalent to $l \cdot D$.

Similarly, since D is a line section of C, the divisor $l \cdot D$ is the intersection of C with a curve of degree l, so the only if direction of Noether's theorem implies that the rank of $l \cdot D$ is $\frac{l \cdot (l+3)}{2}$. □

We also ask for the following generalisation of Theorem A.

Question 1.2. Suppose that G is a gonality-tight graph whose first gonality is equal to the number of vertices. Then G is a quasi-banana graph.

We remark that the generalised banana graphs $B^*_{a,b}$ are *not* gonality-tight when $b \neq a$, so they don't form a counter-example to the question.

It would also be interesting to explore the result of equally subdividing each of the edges. While we don't focus on this direction in the current paper, we expect this operation to preserves the gonality-tight property, giving an even larger family of gonality-tight graphs.

Question 1.3. Let G be a quasi-banana graph and let G' be the graph obtained by subdividing each edge G an equal number of times. Then G' is gonality-tight and any divisor on G realising the second gonality, continues to realise the second gonality when viewed as a divisor on G'.

Note that the graph obtained from subdividing the edges of a quasi-banana graph is *no longer* a quasi-banana graph.

1.1. **Strategy of the proof of the main theorems.** The backwards direction of Theorem A is proven in Section 3. As a first step, we prove Lemma 3.1, which imposes conditions on the structure of gonality-tight graphs with a special divisor realising the second gonality. In Section 3.2, we then repeatedly apply this lemma to show that all such graphs must be quasi-banana graphs. The proof of the forward direction is postponed to Section 5.2, as it relies on various tools developed in between. The proof of Theorem B, which describes the properties of quasi-banana graphs, will be carried out in Sections 4 and 5. First, Section 4 is dedicated to proving two lemmas that describe the effect on the gonalities of attaching a complete graph to a given graph. Those lemmas will be then used inductively to prove Theorem B in Section 5.1.

Acknowledgements. We thank Dave Jensen for helpful discussions about the problem. The project was undertaken as part of the Undergraduate Summer Research Programme at the School of Mathematics and Statistics in St Andrews and we thank them for their hospitality. The research was supported by the EPSRC New Investigator Award (grant number EP/X002004/1) and by the Student Research Bursaries of the Edinburgh Mathematical Society.

2. Preliminaries

We assume familiarity with basic terms in the theory of tropical divisors, such as degree, linear equivalence, v-reduced divisors, rank, and the Riemann–Roch theorem. There are many good sources such as [BN07, AC13, CLM15, LU23], but our terminology mostly matches that of [ADM+21, Section 2]. We briefly recall various basic terms and establish some terminology.

Unless stated otherwise, all graphs are assumed to be finite connected loopless multigraphs. If H is a subgraph of G, and $U \subseteq V(G) \setminus V(H)$, we write $H \cup U$ for the subgraph whose vertex set is $V(H) \cup U$ and whose edge set consists of all the edges in H together with all edges that connect either two vertices in U or a vertex in U to a vertex in $V(H)$. Similarly, if $W \subseteq V(H)$, we write $H \setminus W$ for the graph obtained from H by removing the vertices in W and all edges incident to any vertex in W.

A *divisor* on a graph G is a $\mathbb{Z}$-linear sum of the vertices $V(G)$, that is, an element of the free abelian group generated by $V(G)$. If H is a subgraph of G, we regard the group $\mathrm{Div}(H)$ of divisors on H as a subgroup of $\mathrm{Div}(G)$, the group of divisors on G. In particular, if $u, v \in V(H) \subseteq V(G)$, we make no distinction between, e.g., $2u + v$ as a divisor on H, and the same element viewed as a divisor on G. A divisor is *supported* on a subgraph H if it has no chips outside of H, namely $D(u) = 0$ for all vertices u not in H.

Since the rank of a divisor does depend on the ambient graph, we denote by $r_H(D)$ the rank of a divisor D supported on H, thought of as a divisor on the subgraph H. If D is a divisor on G and H is a subgraph on G, then we write $D|_H$ for the divisor D restricted to the vertices of H. That is,

$$D|_H = \sum_{w \in V(H)} D(w) \cdot w,$$

where $D(w)$ denotes the coefficient of w in D. We similarly denote $f|_H$ for the restriction of a function f to the vertices of H. We say that a divisor is *equivalent to effective* or just *winnable* if it is linearly equivalent to an effective divisor. Given a set of vertices U, we will use the phrase *chip-firing from* U or *set-firing* U to refer to chip-firing once from each vertex of U.

We will make repeated use of the Dhar's burning algorithm. Given a divisor D, the algorithm can be described by starting a fire from a vertex v that spreads under certain rules. If the fire doesn't burn the entire graph, there is an unburnt set of vertices U. We then chip-fire from U and repeat the process. This process eventually terminates and the resulting divisor is the unique v-reduced divisor equivalent to D. See [ADM+21, Section 2.3] and [FJK24, Section 2] for a detailed explanation.

The k-th *gonality* of a graph, denoted $\mathrm{gon}_k(G)$, is the lowest degree of a divisor of rank k for $k \geqslant 0$. We say that a divisor D *realises* the k-th gonality if it has rank k and degree $\mathrm{gon}_k(G)$. A priori, there may be more than one divisor class realising any gonality. The *gonality sequence* of a graph is the sequence $\mathrm{gon}_1(G), \mathrm{gon}_2(G), \ldots$. Note that the 0-th gonality is not interesting since it always equals 0. Every gonality sequence is strictly increasing [ADM+21, Lemma 3.1]. Of key interest to us are graphs whose second gonality exceeds the first by 1, so we give them a name.

Definition 2.1. Let G be a graph with first gonality k and second gonality $k+1$ for some k. Then G is called a *gonality-tight* graph.

We denote the canonical divisor on a graph H by K_H. The following is an immediate corollary of Riemann–Roch [BN07] that will useful multiple times.

Corollary 2.2. *Let H be a graph of genus g, and D a divisor on H. Then,*

(1) If $\deg(D) = 2g-2$, then

$$r_H(D) = \begin{cases} g-1 & \textit{if } D \sim K_H, \\ g-2 & \textit{otherwise.} \end{cases}$$

(2) If $\deg(D) \geqslant 2g-1$, $r_H(D) = \deg(D) - g$.

(3) For $r \geqslant g$, $\mathrm{gon}_r(G) = g + r$.

Proof. If $\deg(D) = 2g-2$ and $D \sim K_H$ then from Riemann–Roch,

$$r_H(D) = r_H(K_H) = r_H(K_H) - r_H(0) = \deg(K_H) - g + 1 = g - 1.$$

If instead $D \not\sim K_H$, then

$$r_H(D) - r_H(K_H - D) = g - 1.$$

Since $D \not\sim K_H$, the divisor $K_H - D$ is of degree 0, but not equivalent to the zero divisor. Its rank is therefore -1. It follows that $r_H(D) = g-2$.

If $\deg(D) \geqslant 2g-1$, then

$$r_H(D) - r_H(K_H - D) = \deg(D) - g + 1.$$

Since the degree of $K_H - D$ is negative, it follows that $r_H(K_H - D) = -1$, so $r_H(D) = \deg(D) - g$.

Finally, suppose that $r \geqslant g$. Since $r + g > 2g - 1$, by Part (2), the rank of every divisor of degree $g + r$ equals r. From Part (2), the rank of any divisor of degree $2g - 1$ is strictly smaller than r, so the gonality equals $r + g$.

□

3. Gonality-tight and quasi-banana graphs

In this section, we explore gonality-tight graphs whose second gonality is realised by a special divisor. We show that such graphs always have a certain structure, and conclude that they coincide with quasi-banana graphs.

3.1. **Restricting the structure of gonality-tight graphs.**

Throughout, when v is a vertex of a graph G, and G' is a subgraph of G, we denote by $N_{G'}(v)$ the set of neighbours of v in G'.

Lemma 3.1. *Let G be a gonality-tight graph with a vertex v, and suppose that the graph obtained after removing a vertex v is a disjoint union of two graphs G' and G'' (where G' and G'' are allowed to have multiple components or be empty). Suppose that, for some $k \geqslant 1$, the divisor*

$$D_2 = (k+1) \cdot v + \sum_{w \in V(G')} w$$

is v-reduced and realises the second gonality. Then the following hold.

1. *There are exactly $k + 1$ edges between v and each of its neighbours in G'.*
2. *There is precisely one edge between every pair of neighbours of v in G'.*
3. *If $V(G') \setminus N_{G'}(v)$ is non-empty, then there is a vertex $u \in N_{G'}(v)$ such that every path from $V(G') \setminus N_{G'}(v)$ to $N_{G'}(v)$ to v passes through u (in particular, u is the unique vertex with that property).*
4. *Set $n = |N_{G'}(v)|$. Then the u-reduced divisor equivalent to D_2 is*

$$(k+n+1) \cdot u + \sum_{w \in V(G') \setminus N_{G'}(v)} w.$$

The rest of this subsection is dedicated to proving the lemma. Throughout, we let G be the graph with $\mathrm{gon}_1(G) + 1 = \mathrm{gon}_2(G)$, such that one of the divisors realising the second gonality is a v-reduced divisor

$$D_2 = (k+1) \cdot v + \sum_{w \in V(G')} w,$$

with $k \geqslant 1$. This implies that $\mathrm{gon}_2(G) = \deg(D_2) = (k+1) + |V(G')|$ and $\mathrm{gon}_1(G) = \mathrm{gon}_2(G) - 1 = k + |V(G')|$. Note that,

$$k = \mathrm{gon}_1(G) - |V(G')| \leqslant |V(G)| - |V(G')| = |V(G'')| + 1.$$

We denote by $u_1, \ldots, u_n$ the neighbours of v in G' (see Figure 2). We start by proving two lemmas that will be used throughout.

Lemma 3.2. *Let H be a graph with a cut vertex v whose removal results in the disjoint union of (not necessarily connected) subgraphs H' and H''. Suppose that D is a divisor of rank r with the property that $D|_{H'} - v$ has rank -1 (as a divisor on $H' \cup \{v\}$). Then $D|_{H'' \cup \{v\}}$ has rank at least r as a divisor on $H'' \cup \{v\}$.*

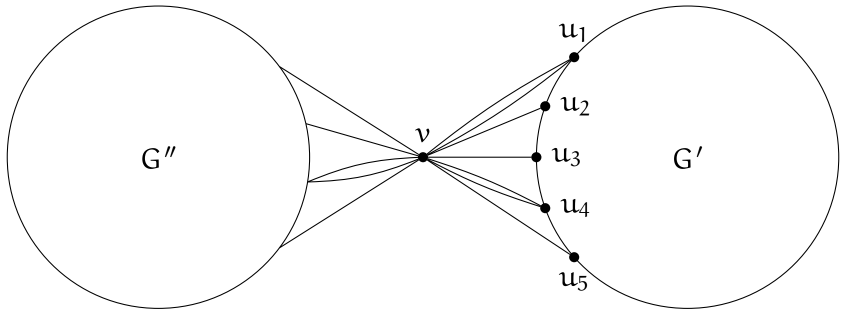


FIGURE 2. The graph G from Lemma 3.1.

Proof. Let E be an effective divisor of degree r on $H'' \cup \{v\}$. As $r_H(D) = r$, let f be the piecewise linear function on H such that $D - E + \mathrm{div}(f) \geqslant 0$. We will show that the divisor $D|_{H''\cup\{v\}} - E + \mathrm{div}(f|_{H''\cup\{v\}})$ is effective. This divisor can only differ from $D - E + \mathrm{div}(f)$ at v, so we only need to check that it is effective at v. The difference between $\mathrm{div}(f)(v)$ and $\mathrm{div}(f|_{H''\cup\{v\}})(v)$ is exactly the sum slopes of f at v incoming from H'. Therefore, if the divisor $D|_{H''\cup\{v\}} - E + \mathrm{div}(f|_{H''\cup\{v\}})$ is not effective at v, the sum of the slopes of f at v incoming from H' must be positive. It then follows that $\mathrm{div}(f|_{H'\cup\{v\}})$ is positive at v, so $-v + D|_{H'} + \mathrm{div}(f|_{H'\cup\{v\}})$ is effective, which is a contradiction.

□

Lemma 3.3. *Let H be a graph with a cut vertex v whose removal results in the disjoint union of (not necessarily connected) subgraphs H' and H''. Let D be a divisor on $H'' \cup \{v\}$ with rank r. Let F be an effective divisor on H, such that $D \leqslant F|_{H''\cup\{v\}}$ Then, whenever E is an effective divisor with degree r supported on $H'' \cup \{v\}$, the divisor $F - E$ is equivalent to an effective divisor on H.*

Proof. Since $r_{H''\cup\{v\}}(D) = r$, there is a piecewise linear function f'' on $H'' \cup \{v\}$ such that $D - E + \mathrm{div}(f'') \geqslant 0$. Let f be the piecewise linear function on G that is constant on $H' \cup \{v\}$ and coincides with f'' on $H'' \cup \{v\}$. Since $D \leqslant F|_{H''\cup\{v\}}$ and f is constant on $H' \cup \{v\}$, it follows that $F - E + \mathrm{div}(f) \geqslant 0$.

□

We now return to our fixed graph G and divisor D_2.

Claim 3.4. The divisor $k \cdot v$ has rank at least 1 as a divisor on $G'' \cup \{v\}$. In particular, if D is an effective divisor on G, such that $D(v) \geqslant k$, the divisor $D - w$ is equivalent to an effective divisor for every vertex $w \in V(G'')$.

Proof. Since our fixed divisor D_2 realises the second gonality, the divisor $D_2 - v$ has rank 1. Moreover, D_2 is v-reduced, and so the divisor

$$D_2|_{G'} - v = -v + \sum_{w \in V(G')} w = (D_2 - v)|_{G'} - v$$

is as well. In particular, it has rank -1 as a divisor on $G' \cup \{v\}$. Applying Lemma 3.2 to $D_2 - v$, we conclude that $k \cdot v = (D_2 - v)|_{G''\cup\{v\}}$ has rank at least 1 as a divisor on $G'' \cup \{v\}$. Now, by Lemma 3.3, for every $w \in V(G'')$, the divisor $k \cdot v - w$ is equivalent to an effective divisor on G. As $D - w \geqslant k \cdot v - w$, the claim is proven.

□

Recall that $u_1, u_2, ...u_n$ are the neighbours of v in G'.

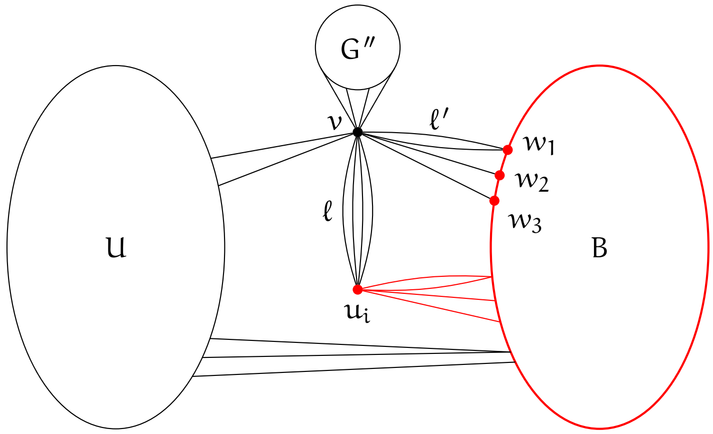


FIGURE 3. The result of starting a fire at u_i with respect to $D_2 - 2u_i$. Parts of the graph that burn down are highlighted in red.

Claim 3.5. There are exactly $k+1$ edges between v and each u_i.

Proof. Fix one of the vertices u_i. As D_2 is v-reduced, if v is ever on fire, the whole graph burns down. Since D_2 has rank 2, the divisor $D_2 - 2 \cdot u_i$ is winnable. In particular, if we start a fire from u_i, it cannot burn v, so there must be fewer than $k+2$ edges between u_i and v. Denote by $\ell \leqslant k+1$ the number of edges between v and u_i.

Now, start a fire from u_i with respect to the divisor D_2. Denote by B the subgraph of $G' \setminus \{u_i\}$ that is burned by the fire and by U the subgraph that isn't. The subgraphs B and U may have multiple components. Let ℓ' be the number of edges between v and the vertices on fire other than u_i (see Figure 3). Denote by $w_1, w_2, \dots w_m$ the vertices at the other end of those edges. Note that the number of those vertices might be smaller than ℓ' due to multiple edges.

We claim that $\ell' = 0$. Otherwise, we will show that the divisor $D_2 - u_i - \sum w_j$ has rank 1, which is a contradiction since $\deg(D_2 - u_i - \sum w_j) < \deg(D_2) - 1 = \mathrm{gon}_1(G)$. Indeed, if $u \in V(G') \cup \{v\}$ is any vertex other than u_i or $w_1, w_2, \dots w_m$, then $D_2 - u_i - \sum w_j$ already has a chip at u. If $u \in V(G'')$ then, by Claim 3.4, the divisor $D_2 - u_i - \sum w_j - u$ is equivalent to an effective divisor on G. Now, suppose that u is either u_i or one of $w_1, w_2 \dots w_m$. The fact that $U \cup \{v\}$ didn't burn by the fire starting from u_i precisely means that chip-firing from this set doesn't create any new debts. But chip-firing from the set also brings a chip to u.

It follows that $\ell' = 0$ and the number of edges between $B \cup \{u_i\}$ and v is ℓ. We will show that, if $\ell < k+1$ then the divisor $D_2 - v - u_i$ has rank 1, which is a contradiction. So suppose that $\ell < k+1$. The divisor $D_2 - v - u_i$ already has a chip on on every vertex of $G' \cup \{v\}$ other than u_i. If $w \in V(G'')$ then by Claim 3.4, the divisor $D_2 - v - u_i - w$ is equivalent to an effective divisor. We are left with checking that the divisor obtained by taking a chip from u_i is winnable. Start a fire from u_i. Since the number of edges between $B \cup \{u_i\}$ and v is $\ell < k+1$, the fire doesn't burn v. Chip-firing the unburnt part results in clearing the debt at u_i. We have reached a contradiction, and the only possibility is that $k+1 \leqslant \ell < k+2$, so the claim is proven. ∎

Claim 3.6. If a fire is started at some $w \in V(G')$ with respect to D_2, then exactly one $u_i \in N_{G'}(v)$ burns down.

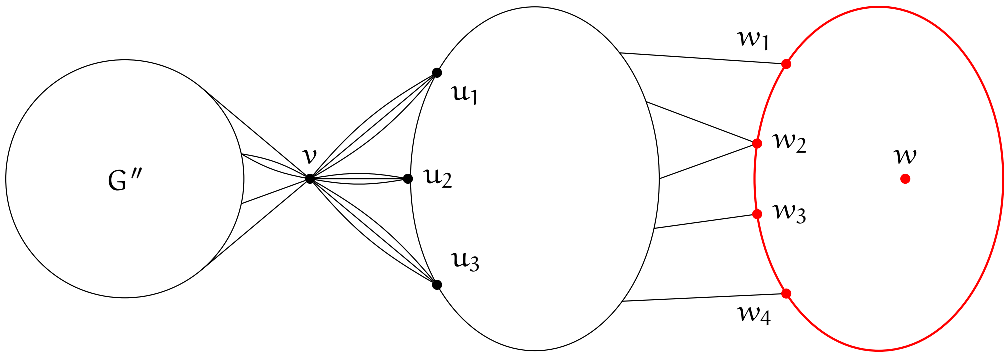


FIGURE 4. Graph G after a fire is started at w with respect to D_2. Parts of the graph on fire are highlighted in red.

Proof. If $w \in N_{G'}(v)$, then it follows from the proof of Claim 3.5 that starting a fire at w does not cause any of the other neighbours of v to catch fire. Hence the claim holds in this case.

Now, let $w \in V(G') \setminus N_{G'}(v)$. Suppose, for contradiction, that no neighbour of v burns down when a fire is started at w with respect to D_2, as illustrated in Figure 4. Let $w_1, w_2, \ldots w_m$ be the vertices that burn, but have at least one unburnt neighbour.

We claim that the divisor $D_2 - v - \sum w_i$ has rank at least 1, which is a contradiction since its degree is smaller than the gonality. If $w' \in V(G'')$, then by Claim 3.4, the divisor $D_2 - v - \sum w_i - w'$ is equivalent to an effective divisor. If w' is one of the $w_1, w_2, \ldots w_m$, then set-firing the unburnt part of G in Figure 4 adds at least one chip to w', without creating any debt elsewhere. $D_2 - v - \sum w_i - w'$ is again equivalent to effective, and so the rank of $D_2 - v - \sum w_i$ is at least 1.

Therefore, there must be some $u_i \in N_{G'}(v)$ that burns when a fire is started at w. By Claim 3.5, there are $k+1$ edges between v and each of its neighbours in G', so if another $u_j \neq u_i$ burns down, the fire will spread to v and then the whole of G. This would imply that $D_2 - 2 \cdot w$ is w-reduced and therefore unwinnable, contradicting that D_2 has rank 2. Hence, exactly one vertex u_i burns, as claimed. □

We say that a vertex u *separates* w from v if every path from v to w passes through u. As we are mostly interested in separating vertices from the fixed vertex v, we will often just say that u separates w, without mentioning v. For our fixed graph G, we will refer to the set of vertices separated by one of the vertices u_i as the *branch* of u_i, and denote it by B_i.

Claim 3.7. Setting fire with respect to D_2 to u_i or any vertex in its branch B_i results in burning precisely $\{u_i\} \cup B_i$.

Proof. Start a fire at u_i with respect to D_2. For the rest of the proof, we will use the notations from Figure 5. Our goal is to show that the vertices that burn are exactly the ones separated by u_i, which is equivalent to showing that $\ell = 0$. So assume that $\ell > 0$. Let $w_1, w_2, \ldots w_m$ be the vertices of the burned locus that are adjacent to the unburnt locus. Similarly to previous arguments, we will reach a contradiction by showing that the divisor $D_2 - u_i - \sum w_j$ has rank 1. If we remove a chip from $w \in V(G'')$, then by Claim 3.4, $D_2 - u_i - \sum w_j - w$ is equivalent to an effective divisor. It suffices to check that, after removing a chip from u_i or one of the vertices w_j, the divisor is equivalent to effective. So take a chip away from one of those vertices, which we denote w, and

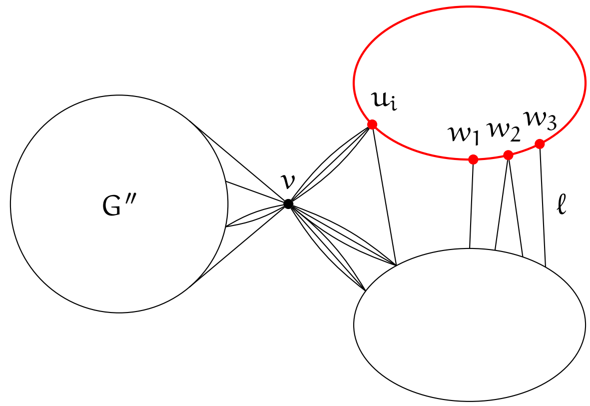


Figure 5. The graph G after starting a fire at u_i with respect to D_2. Here, ℓ is the number of edges between vertices on fire other than u_i and vertices of G not on fire. Burnt regions are highlighted in red.

start a fire from u_i. Since the vertices $u_i, w_1, w_2, \dots w_m$ where burnt when starting a fire from u_i with respect to the divisor D_2, the burnt locus will not change when starting a fire from u_i with respect to $D_2 - u_i - \sum w_j - w$. Now, chip-firing from the complement of the burnt locus results in an effective divisor.

Next, let w be a vertex in B_i. By Claim 3.6, after staring a fire at w with respect to D_2, exactly one neighbour of v will burn down. As u_i separates w from all other neighbours of v, that is precisely the neighbour that catches fire. Once u_i burns, the fire spreads through all of $\{u_i\} \cup B_i$ and no further, as desired.

□

Claim 3.8. There is exactly one edge between u_i and u_j for any i and j.

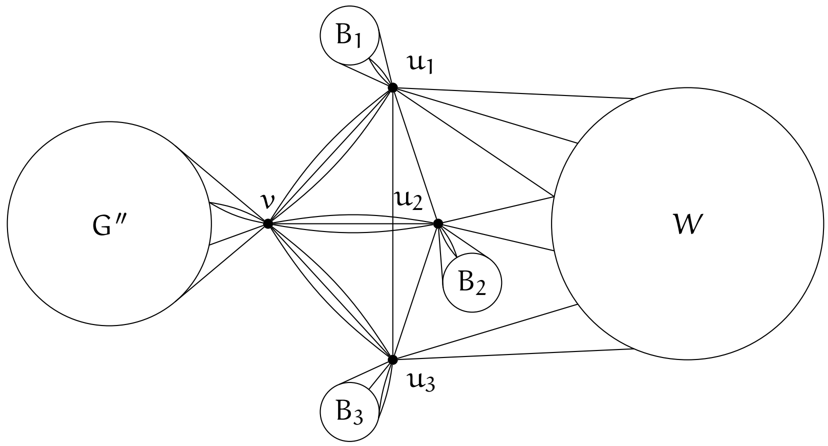
Proof. If there were two or more edges between u_i and u_j, then setting fire from u_i with respect to D_2 would burn u_j, contradicting Claim 3.6. It follows that there is at most one edge between each u_i and u_j.

Now, assume there are vertices u_i and u_j with no edges between them. Consider the divisor $D_2 - u_i - u_j$. If we take a chip away from u_i and set it on fire, since there is no edge between u_i and u_j, by Claim 3.7, the only part of the graph other than u_i that will burn down is B_i. By chip-firing the unburnt vertices, including v, we obtain an effective divisor. Similarly, taking away a chip form u_j does not make the divisor unwinnable. Finally, for any $w \in V(G'')$, $D_2 - u_i - u_j - w$ is equivalent to an effective divisor, by Claim 3.4. Therefore, $D_2 - u_i - u_j$ has rank at least 1. which is a contradiction as its degree is smaller than the gonality of G.

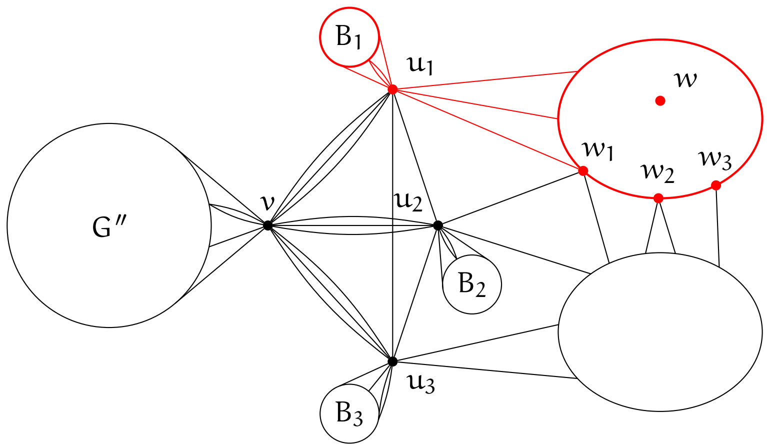
□

Claim 3.9. Let W be the complement in $V(G')$ of the union of the sets $\{u_i\} \cup B_i$, namely the vertices in G' that don't burn when starting a fire from any u_i with respect to D_2 (See Figure 6(a) for an illustration). Then W is empty.

Proof. Assume for a contradiction that W is not empty, so it contains at least one vertex w. Start a fire at w with respect to D_2. By Claim 3.6, the fire will burn exactly one neighbour u_i of v and therefore doesn't reach v, as depicted in Figure 6(b). By assumption, w is not in a branch of u_i,



(a) Sketch of the graph G including W and branches of each u_i.



(b) Graph G after a fire is started at w with respect to D_2.

FIGURE 6. Graph G before and after the vertex w is set on fire. The parts of G that burn down are highlited in red.

so there exist some $w_1, w_2 \ldots, w_m \in W$ such that setting a fire from w burns them, and they are adjacent to some unburnt vertex.

Now, consider the divisor $D_2 - u_i - \sum w_j$. We claim that its rank is at least 1, which is a contradiction, since its degree is lower than the gonality of G. If a chip is taken from some $w \in V(G'')$, then by Claim 3.4, the divisor $D_2 - u_i - \sum w_j - w$ is still winnable. We now only need to check that the divisor is equivalent to effective after taking away a chip from u_i or one of the vertices w_j. So take away a chip from one of those vertices, that we denote by w', and start a fire from it. Since we have only taken away chips from vertices that burnt when starting a fire from u_i with respect to D_2, the set of vertices that catch fire is going to be a subset of the ones that are on fire in Figure 6(b). In particular, the vertex w' will have an unburnt neighbour. Therefore, chip-firing from the set of unburnt vertices will remove the debt from G.

□

Claim 3.10. There is up to one non-empty branch.

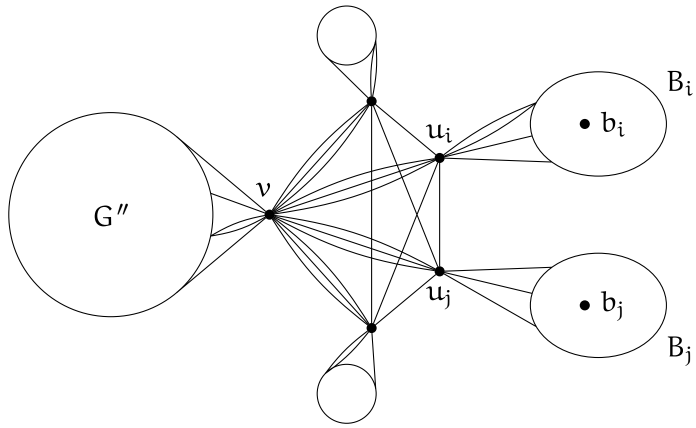


FIGURE 7. Sketch of G with two non-empty branches.

Proof. Assume for a contradiction there are two different non-empty branches B_i and B_j attached to u_i and u_j respectively. So there exist vertices $b_i \in B_i$ and $b_j \in B_j$, as sketched in Figure 7. We will show that the divisor $D_2 - b_i - b_j$ has rank at least 1, which is a contradiction since its degree is smaller than the gonality.

By Claim 3.4, $D_2 - b_i - b_j - w$ is equivalent to an effective divisor for any $w \in V(G'')$. It suffices to show that $D_2 - b_i - b_j$ is equivalent to effective when taking away a chip from either b_i or b_j as there is already a chip everywhere else.

Consider first the divisor $D_2 - 2b_i$. Performing Dhar's burning algorithm at b_i results in an effective divisor since the divisor has rank at least 0. If the algorithm results in any changes to the divisor on B_j, there is an iteration in which the fire burns some, but not all of the vertices of $B_j \cup \{u_j\}$. By Claim 3.7, this can never happen and the configuration of chips on B_j does not change during the algorithm. Since D_2 had a chip at b_j, the resulting divisor has a chip at b_j as well.

It follows that the divisor $D_2 - 2b_i - b_j$ is equivalent to an effective divisor. By a similar argument, $D_2 - 2b_i - b_j$ is equivalent to an effective divisor as well, so we are done. ☐

Let $B := G' \setminus N_v(G')$. Note that the subgraph B is possibly disconnected, or empty. By Claims 3.9 and 3.10, if $V(B)$ is non-empty, it is equal to a single branch B_i of some u_i. We relabel that vertex to u. The existence of D_2 severely restricts the structure of the graph G, as seen in Figure 8.

Claim 3.11. Using the notation from Figure 8, the u-reduced divisor equivalent to D_2 is

$$D_2' := (k+n+1) \cdot u + \sum_{w \in V(B)} w.$$

Proof. We apply Dhar's burning algorithm to D_2, starting a fire at u. By Claim 3.7, all of B burns down, while no other vertices do. Now set-firing the unburnt vertices, namely

$$V(G'') \cup \{v\} \cup N_{G'}(v) \setminus \{u\},$$

gives

$$(k+n+1) \cdot u + \sum_{w \in V(B)} w,$$

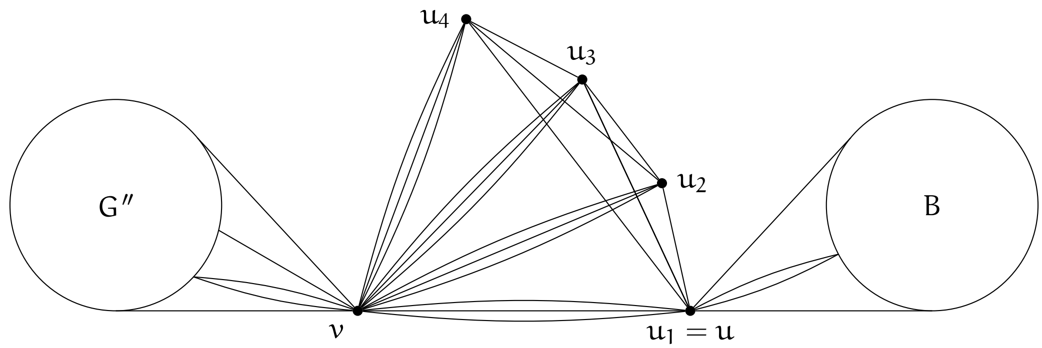


FIGURE 8. Structure of G. Both G'' and B are (possibly disconnected or empty) subgraphs of G. There are $k+1$ edges between v and each u_i and one edge between each distinct pair of u_i.

which is u-reduced by Claim 3.7.

□

This completes the proof of the lemma. Moreover, when B is non-empty we can iterate the argument by taking u as the new vertex v, the divisor D_2' as D_2, B as the subgraph G' and $G'' \cup \big(\{v\} \cup N_v(G') \setminus \{u\}\big)$ as G'', but more on that in Proposition 3.18.

3.2. **Quasi-banana graphs.** In this subsection, we construct a large family of gonality-tight graphs. We show that this family includes all gonality-tight graphs G whose second gonality is realised by a divisor of the form

$$D = v + \sum_{w \in V(G)} w.$$

We refer to these graphs as *quasi-banana graphs*.

Definition 3.12. Let m be a non-negative integer and let $N = (q_1, q_2, \dots q_m)$ be an m-tuple of positive integers. A *quasi-banana graph* $QB_m(N)$ is defined as follows. The vertex set is

$$V\left(QB_m(N)\right) = \bigcup_{i=1}^{m} C_i,$$

where,

$$C_0 = \{u_{0,1}\} \qquad \text{and} \qquad C_i = \{u_{i,j} \mid 1 \leqslant j \leqslant q_i\}.$$

The edge set is specified by the following.

(1) For every fixed $1 \leqslant i \leqslant m$, and distinct $1 \leqslant j, k \leqslant q_i$, there is exactly one edge between $u_{i,j}$ and $u_{i,k}$.
(2) For every fixed $0 \leqslant i < m$ and $1 \leqslant j \leqslant q_{i+1}$, there are exactly $\sum_{k=1}^{i} q_k + 2$ edges between $u_{i,1}$ and $u_{i+1,j}$.
(3) There are no other edges.

Example 3.13. The quasi-banana graph QB_0 consists of a single vertex and no edges.

Example 3.14. The quasi-banana graph $QB_3(4, 2, 3)$ is illustrated in Figure 9.

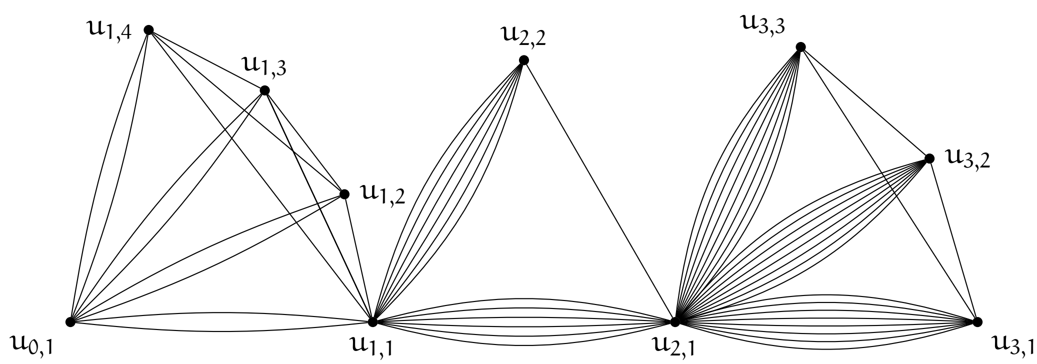


FIGURE 9. The quasi-banana graph $QB_3(4, 2, 3)$.

Example 3.15. The *generalised banana graphs* $B^*_{m,m}$, introduced in [ADM+21] (although initially studied in [CC12] by a different name) and further studied in [FJK24], coincide with the quasi-banana graphs $QB_m(\mathbf{1}_m)$, where $\mathbf{1}_m$ is the m-tuple with all entries equal to one.

Example 3.16. Let $QB_m(q_1, q_2, \dots q_m)$ be a quasi-banana graph with $m \geqslant 1$. Then

$$QB_m(q_1, q_2, \dots q_m) \setminus C_m \cong QB_{m-1}(q_1, q_2, \dots q_{m-1}).$$

If $q_m \geqslant 2$ then

$$QB_m(q_1, q_2, \dots q_m) \setminus \{u_{m,q_m}\} \cong QB_m(q_1, q_2, \dots q_m - 1).$$

Before describing the connection to gonality-tight graphs, we record the following claim.

Claim 3.17. Let H be a graph whose first gonality equals $|V(H)|$ and fix a vertex v of H. Let E be an effective divisor with at most a single chip on every vertex other than v. Then applying the Dhar's burning algorithm from v results in burning the entire graph, namely, E is v-reduced.

Proof. The claim doesn't depend on the number of chips at v, so we may assume that $E(v) = 1$. We may also assume that E has a chip at every vertex. Suppose that the fire doesn't burn the entire graph. Then there is a collection of vertices $\varnothing \neq U \neq V(H)$ such that $E + \mathrm{div}(U)$ in an effective divisor. But then there is a vertex $w \notin U$ such that $(E + \mathrm{div}(U))(w) \geqslant 1$. It follows that the divisor $E - w$ has rank 1, which is a contradiction, since its degree is smaller than $|V(H)|$. □

The following proposition is the main result of the section.

Proposition 3.18. *Let G be a gonality-tight graph with a vertex v such that the divisor*

$$D = v + \sum_{w \in V(G)} w$$

realises the second gonality. Then G is a quasi-banana graph and v is the vertex $u_{0,1}$ in the notation of Definition 3.12.

Proof. Since G is gonality-tight, and

$$\mathrm{gon}_2(G) = \deg(D_0) = |V(G)| + 1,$$

it follows that $\mathrm{gon}_1(G) = |V(G)|$. By Claim 3.17, the divisor D is v-reduced. Relabel the vertex v as $u_{0,1}$, and set $C_0 = \{u_{0,1}\}$. Then

$$D = 2u_{0,1} + \sum_{w \in V(G) \setminus C_0} w.$$

We now apply Lemma 3.1 to G with $k = 1$, taking $u_{0,1}$ as the vertex v, $G' = G \setminus \{u_{0,1}\}$ and $G'' = \varnothing$.

According to the lemma, if $V(G) \setminus \{u_{0,1}\}$ is non-empty, there are exactly $k+1 = 2$ edges between $u_{0,1}$ and each of its neighbours in G'. Denote the set of neighbours of $u_{0,1}$ by

$$C_1 = N_{G'}(u_{0,1}) = \{u_{1,1}, u_{1,2}, \dots u_{1,q_1}\},$$

taking the vertex $u_{1,1}$ as the unique vertex u from Part (3) of the lemma if $V(G) \setminus (C_0 \cup C_1)$ is non-empty. The vertices in C_1 form a complete subgraph, and each one is connected to $u_{0,1}$ by exactly two edges. Part (4) then implies that the $u_{1,1}$-reduced divisor equivalent to D is

$$D' = (q_1 + 2) \cdot u_{1,1} + \sum_{w \in V(G) \setminus (C_0 \cup C_1)} w.$$

If $V(G) \setminus (C_0 \cup C_1)$ is non-empty, we apply Lemma 3.1 again, taking $u_{1,1}$ as the new vertex v, and letting $G' = G \setminus (C_0 \cup C_1)$. The vertices in the next set of neighbours

$$C_2 = N_{G'}(u_{1,1}) = \{u_{2,1}, u_{2,2}, \dots u_{2,q_2}\}$$

are connected with $q_1 + 2$ edges to $u_{1,1}$ each and have precisely one edge between each other. If $V(G) \setminus (C_0 \cup C_1 \cup C_2)$ is non-empty, Part (3) of the lemma again provides a unique vertex $u_{2,1} \in C_2$ separating the remaining vertices.

Continuing inductively, after i applications of the lemma, we have constructed disjoint vertex sets

$$C_0, C_1, \dots C_i,$$

where for each $j \geqslant 1$ we write $C_j = \{u_{j,1}, u_{j,2}, \dots u_{j,q_j}\}$. These satisfy the following.

- For each $j \geqslant 1$ there is exactly one edge between each distinct pair of vertices in C_j.
- For $j \geqslant 1$, each vertex in C_j is connected to $u_{j-1,1}$ with $\sum_{k=1}^{j-1} q_k + 2$ edges, where $q_k = |C_k|$.
- If $V(G) \setminus \bigcup_{k=1}^{i} C_k$ is non-empty, then the vertex $u_{i,1}$ is a cut vertex separating the remaining vertices. The $u_{i,1}$-reduced divisor realising the second gonality is

$$\left(\sum_{k=1}^{i} q_k + 2 \right) + \sum_{w \in V(G) \setminus \bigcup_{k=1}^{i} C_k} w \sim D.$$

  This allows us to apply the lemma again.

As G is finite, this process eventually terminates after m iterations, for some $m \geqslant 0$.

The resulting graph is completely determined by the $q_i = |C_i|$, and satisfies the conditions of Definition 3.12. Hence G is a quasi-banana graph $QB_m(q_1, q_2, \dots q_m)$, with $u_{0,1} = v$. □

We will show in Section 5 that every quasi-banana graph G is gonality-tight and prove Theorem A. At this point, we have only established that the family of gonality-tight graphs with a divisors of the form D realising the second gonality is a subfamily of quasi-banana graphs.

4. Two technical lemmas

In this section, we prove two technical results, namely Lemmas 4.1 and 4.6 that will be used inductively in Section 5 to prove the upper and lower bound respectively of the gonality sequence.

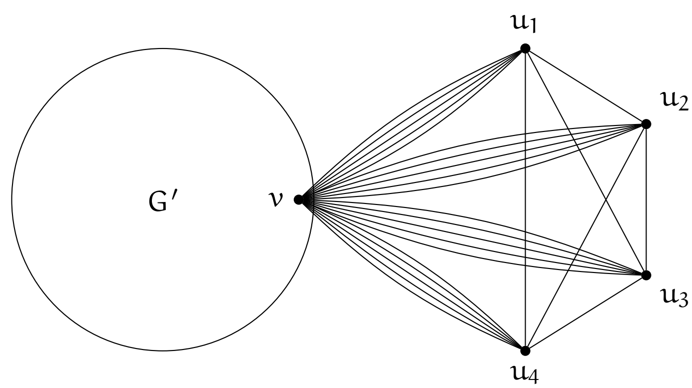


FIGURE 10. Graph G. Here $k = 4$ and $n = 4$.

4.1. **Bounding gonalities from above.** Recall that, when D is a divisor on a subgraph H of G, we denote by $r_H(D)$ and $r_G(D)$ the rank of D as a divisor on H and G respectively. This subsection is devoted to the following lemma.

Lemma 4.1. *Let k be a non-negative integer and suppose that a graph G' of genus $\binom{k}{2}$ has a vertex v such that, for every integer $1 \leqslant l \leqslant k-2$,*

$$r_{G'}\big(l(k+1)\cdot v\big) \geqslant \frac{l(l+3)}{2}$$

Let G be the graph obtained by attaching a complete graph K_n to G' along $k+1$ edges between v and each vertex in K_n (see Figure 10 for an illustration). Then the following hold.

(1) *The genus of G is $g(G) = \binom{k+n}{2}$.*
(2) *For every $u \in V(K_n)$, the divisor $(k+n-2)(k+n+1)\cdot u$ is equivalent to the canonical divisor K_G.*
(3) *For every $-2 \leqslant l \leqslant k+n$ and every $u \in V(K_n)$,*

$$r_G\big(l(k+n+1)\cdot u\big) \geqslant \frac{l(l+3)}{2}.$$

Proof of Lemma 4.1(1). The genus of G can be computed explicitly from the definition:

$$\begin{aligned} g(G) &= |E(G)| - |V(G)| + 1 \\ &= \left(|E(G')| + n(k+1) + \frac{n(n-1)}{2}\right) - \big(|V(G')| + n\big) + 1 \\ &= \big(|E(G')| - |V(G')| + 1\big) + n(k+1) + \frac{n(n-1)}{2} - n \\ &= g(G') + \frac{2nk + n^2 - n}{2} = \frac{k^2 - k + 2nk + n^2 - n}{2} \\ &= \frac{(k+n)^2 - (k+n)}{2} = \binom{k+n}{2}. \end{aligned}$$

$\square$

Proof of Lemma 4.1(2). Now, consider the divisor

$$D_2 = (k+1)\cdot v + \sum_{w\in V(K_n)} w.$$

Claim 4.2. D_2 satisfies the following:

(1) For every $u \in V(K_n)$, $D_2 \sim (k+n+1)\cdot u$.
(2) $(k+1)\cdot D_2 \sim (k+n+1)(k+1)\cdot v$.

Proof. Both statements are consequences of set-firing.

(1) Set-firing $V(G)\setminus\{u\}$ with respect to D_2 gives $(k+n+1)\cdot u$.
(2) Set-firing $V(K_n)$ with respect to $(k+1)\cdot D_2$ produces $(k+n+1)(k+1)\cdot v$.

□

Returning to prove Lemma 4.1(2), notice that

$$r_{G'}\big((k-2)(k+1)\cdot v\big) \geqslant \frac{(k-2)(k+1)}{2} = \binom{k}{2} - 1 = g(G') - 1$$

and

$$\deg\big((k-2)(k+1)\cdot v\big) = (k-2)(k+1) = k(k-1) - 2 = 2g(G') - 2.$$

By Corollary 2.2, we have $(k-2)(k+1)\cdot v \sim K_{G'}$. If u is a vertex in a subgraph H' of H, let $\mathrm{val}_{H'}(u)$ denote the valency of u considered as a vertex of H. The canonical divisor of G can now be computed:

$$\begin{aligned}
K_G &= \sum_{w\in V(G)} \big(\mathrm{val}_G(w)-2\big)\cdot w = \sum_{w\in V(G')} \big(\mathrm{val}_G(w)-2\big)\cdot w + \sum_{w\in V(K_n)} \big(\mathrm{val}_G(w)-2\big)\cdot w \\
&= \sum_{w\in V(G')\setminus\{v\}} \big(\mathrm{val}_{G'}(w)-2\big)\cdot w + \big(\mathrm{val}_{G'}(v) + n(k+1) - 2\big)\cdot v + \sum_{w\in V(K_n)} \big(\mathrm{val}_G(w)-2\big)\cdot w \\
&= \sum_{w\in V(G')} \big(\mathrm{val}_{G'}(w)-2\big)\cdot w + n(k+1)\cdot v + \sum_{w\in V(K_n)} \big((k+n)-2\big)\cdot w \\
&= K_{G'} + n(k+1)\cdot v + \sum_{w\in V(K_n)} (k+n-2)\cdot w \\
&\sim (k+1)(k-2)\cdot v + n(k+1)\cdot v + \sum_{w\in V(K_n)} (k+n-2)\cdot w \\
&= (k+n-2)\cdot D_2
\end{aligned}$$

By Claim 4.2, for any $u \in V(K_n)$,

$$K_G \sim (k+n-2)\cdot D_2 \sim (k+n-2)(k+n+1)\cdot u.$$

This concludes the proof of Part (2).

Proof of Lemma 4.1(3). This is the trickiest and most technical part of the proof of the lemma. We first show that, for every $u \in V(K_n)$ and every integer $-2 \leqslant l \leqslant \frac{k+n-2}{2}$,

$$r_G\big(l(k+n+1)\cdot u\big) = r_G(l\cdot D_2) \geqslant \frac{l(l+3)}{2}.$$

The first equality follows from Claim 4.2, and we prove the inequality by strong induction on l. For the base cases $l = -2, -1$, or 0, the statement holds since

$$r_G(-2 \cdot D_2) = r_G(-1 \cdot D_2) = -1 \quad \text{and} \quad r_G(0 \cdot D_2) = 0.$$

Now fix some $1 \leqslant l \leqslant \frac{k+n-2}{2}$, and assume that

$$r_G(r \cdot D_2) \geqslant \frac{r(r+3)}{2} \qquad \text{for all } -2 \leqslant r \leqslant l-1.$$

Let E be any effective divisor on G with $\deg(E) = \frac{l(l+3)}{2}$. We want to show that $l \cdot D_2 - E$ is equivalent to an effective divisor. First, suppose $1 \leqslant l \leqslant k$. We have the following two cases.

Case (A): $l \cdot D_2 - E$ *is not effective on* K_n.

Here, there must exist $w \in V(K_n)$, such that $(l+1) \cdot w \leqslant E$. Therefore, it suffices to show that

$$r_G\big(l \cdot D_2 - (l+1) \cdot w\big) \geqslant \frac{l(l+3)}{2} - (l+1) = \frac{(l-1)(l+2)}{2}.$$

By Claim 4.2,

$$l \cdot D_2 - (l+1) \cdot w \sim (l-1) \cdot D_2 + \big(k+n+1-(l+1)\big) \cdot w \geqslant (l-1) \cdot D_2.$$

Hence, by the inductive hypothesis,

$$r_G\big(l \cdot D_2 - (l+1) \cdot w\big) \geqslant r_G\big((l-1) \cdot D_2\big) \geqslant \frac{(l-1)(l+2)}{2}.$$

This concludes case (A).

Case (B): $l \cdot D_2 - E$ *is effective on* K_n.

If $l \cdot D_2 - E$ is effective on K_n, then by Lemma 3.3, it suffices to show that

$$r_{G'}(l \cdot D_2|_{G'}) = r_{G'}\big(l(k+1) \cdot v\big) \geqslant \frac{l(l+3)}{2}.$$

This follows from the following claim.

Claim 4.3.

$$r_{G'}\big(r(k+1) \cdot v\big) \geqslant \begin{cases} \frac{r(r+3)}{2} & \text{if } 0 \leqslant r \leqslant k, \\ \frac{r(r+3)}{2} - \frac{(r-k)(r-k+1)}{2} & \text{if } k+1 \leqslant r. \end{cases}$$

Proof. If $r = 0$, then clearly $r_{G'}\big(0(k+1) \cdot v\big) = 0$. For $1 \leqslant r \leqslant k-2$, the inequality follows from the assumption. When $r \geqslant k-1$, we have

$$\deg\big(r(k+1) \cdot v\big) = r(k+1) \geqslant (k-1)(k+1) = k(k-1) + k - 1 \geqslant 2g(G') - 1.$$

Thus, by Corollary 2.2,

$$\begin{aligned} r_{G'}\big(r(k+1) \cdot v\big) &= \deg\big(r(k+1) \cdot v\big) - g(G') = r(k+1) - \frac{k(k-1)}{2} \\ &= \frac{r(r+3)}{2} - \frac{(r-k)(r-k+1)}{2}. \end{aligned}$$

Note that the second term is 0 when $r = k-1$ or $r = k$. This proves the claim.

$\square$

Now, assume that $k+1 \leqslant l \leqslant \frac{k+n-2}{2}$. This time, there are 3 cases to consider:

Case (A): $l \cdot D_2 - E$ *is not effective on* K_n.

The argument is the same as before. There exists $w \in V(K_n)$ such that $(l+1) \cdot w \leqslant E$, and therefore

$$r_G\big(l \cdot D_2 - (l+1) \cdot w\big) \geqslant r_G\big((l-1) \cdot D_2\big) \geqslant \frac{(l-1)(l+2)}{2} = \frac{l(l+3)}{2} - (l+1),$$

as in the previous case (A).

Case (B): $l \cdot D_2 - E$ *is effective on* K_n, *and* $\deg(E|_{K_n}) \geqslant \frac{(l-k)(l-k+1)}{2}$.

As $l \cdot D_2 - E|_{K_n}$ is effective, by Lemma 3.3, it is enough to show that $r_{G'}\big((l \cdot D_2 - E|_{K_n})|_{G'}\big) \geqslant \deg(E|_{G'})$. From Claim 4.3, we have

$$\begin{aligned} r_{G'}\big((l \cdot D_2 - E|_{K_n})|_{G'}\big) = r_{G'}\big(l(k+1) \cdot v\big) &\geqslant \frac{l(l+3)}{2} - \frac{(l-k)(l-k+1)}{2} \\ &\geqslant \deg(E) - \deg(E|_{K_n}) = \deg(E|_{G'}). \end{aligned}$$

The desired inequality holds, completing case (B).

Case (C): $l \cdot D_2 - E$ *is effective on* K_n, *and* $\deg(E|_{K_n}) < \frac{(l-k)(l-k+1)}{2}$.

Let $0 \leqslant r_0 \leqslant l-k-1$ be the unique integer such that

$$\frac{r_0(r_0+1)}{2} \leqslant \deg(E|_{K_n}) \leqslant \frac{(r_0+1)(r_0+2)}{2} - 1 = \frac{r_0(r_0+3)}{2}. \tag{1}$$

Next, set

$$q = \left\lfloor \frac{l-r_0}{k+1} \right\rfloor$$

and let $0 \leqslant s \leqslant k$ be the remainder of the division of $l - r_0$ by $k+1$. So,

$$l - r_0 = q(k+1) + s.$$

Note that $q \geqslant 1$. By Claim 4.2,

$$\begin{aligned} l \cdot D_2 &= r_0 \cdot D_2 + q(k+1) \cdot D_2 + s \cdot D_2 \\ &\sim (r_0+s) \cdot D_2 + q(k+n+1)(k+1) \cdot v \end{aligned} \tag{2}$$

Since $r_0 + s = l - q(k+1) < l$, the inductive hypothesis gives,

$$r_G\big((r_0+s) \cdot D_2\big) \geqslant \frac{(r_0+s)(r_0+s+3)}{2} \geqslant \frac{r_0(r_0+3)}{2} \geqslant \deg(E|_{K_n})$$

Therefore, by (1),

$$\begin{aligned} r_G\big((r_0+s) \cdot D_2 - E|_{K_n}\big) &\geqslant \frac{(r_0+s)(r_0+s+3)}{2} - \deg(E|_{K_n}) \\ &\geqslant \frac{(r_0+s)(r_0+s+3)}{2} - \frac{r_0(r_0+3)}{2} =: \gamma. \end{aligned} \tag{3}$$

Write $E|_{G'}$ as a sum of two effective divisors E_1 and E_2, where $\deg(E_1) = \max\big(\gamma, \deg(E|_{G'})\big)$.

Claim 4.4. The divisor $(r_0+s) \cdot D_2 - E|_{K_n} - E_1$ is equivalent to an effective divisor on G.

Proof. This follows immediately from (3) and the fact that $\deg(E_1) \leqslant \gamma$. □

Claim 4.5. The divisor $q(k+n+1)(k+1) \cdot v - E_2$ is equivalent to an effective divisor on G.

Proof. If $\deg(E_1) = \deg(E|_{G'})$, then E_2 is the zero divisor, and the claim holds. So, we can now assume that $\deg(E_1) = \gamma$. From (1),

$$\begin{aligned}\deg(E_2) &= \deg(E|_{G'}) - \deg(E_1) = \deg(E) - \deg(E_1) - \deg(E|_{K_n}) \\ &\leqslant \frac{l(l+3)}{2} - \gamma - \frac{r_0(r_0+1)}{2}.\end{aligned} \tag{4}$$

As E_2 is supported on G', by Lemma 3.3, it suffices to show that:

$$r_{G'}\big(q(k+n+1)(k+1)\cdot v\big) \geqslant \deg(E_2).$$

Since $q \geqslant 1$, by Claim 4.3,

$$r_{G'}\big(q(k+n+1)(k+1)\cdot v\big) \geqslant \frac{\alpha(\alpha+3)}{2} - \frac{(\alpha-k)(\alpha-k+1)}{2}, \tag{5}$$

where $\alpha := q(k+n+1)$.

Combining (4) and (5), we get:

$$\begin{aligned}&2\big[r_{G'}\big(q(k+n+1)(k+1)\cdot v\big) - \deg(E_2)\big] \geqslant \\ &\qquad \alpha(\alpha+3) - (\alpha-k)(\alpha-k+1) - l(l+3) + 2\gamma + r_0(r_0+1) =: \delta\end{aligned} \tag{6}$$

Set $\beta := q(k+1) \geqslant (k+1)$. Now,

$$l = r_0 + q(k+1) + s = r_0 + \beta + s \quad \text{and} \quad \alpha(k+1) = \beta(k+n+1).$$

A direct computation gives

$$\alpha(\alpha+3) - (\alpha-k)(\alpha-k+1) = 2(k+1)\alpha - k^2 + k = 2\beta(k+n+1) - k^2 + k,$$

and

$$2\gamma = (r_0+s)^2 + 3(r_0+s) - r_0^2 - 3r_0 = 2r_0 s + s^2 + 3s.$$

Next,

$$\begin{aligned}l(l+3) = l^2 + 3l &= (r_0+\beta+s)^2 + 3(r_0+\beta+s) \\ &= r_0^2 + \beta^2 + s^2 + 2r_0 s + 2r_0\beta + 2s\beta + 3r_0 + 3\beta + 3s\end{aligned}$$

Substituting the results above gives

$$\begin{aligned}\delta &= \big(2\beta(k+n+1) - k^2 + k\big) - \big(r_0^2 + \beta^2 + s^2 + 2r_0 s + 2r_0\beta + 2s\beta + 3r_0 + 3\beta + 3s\big) \\ &\qquad + (2r_0 s + s^2 + 3s) + (r_0^2 + r_0) \\ &= -\beta^2 + (2k + 2n - 2r_0 - 2s - 1)\beta - k^2 + k - 2r_0\end{aligned}$$

Since, $2r_0 = 2l - 2\beta - 2s \leqslant k + n - 2 - 2\beta - 2s$, we get

$$\begin{aligned}\delta &= -\beta^2 + (2k + 2n - 2r_0 - 2s - 1)\beta - k^2 + k - 2r_0 \\ &\geqslant -\beta^2 + \big(2k + 2n - (k + n - 2 - 2\beta - 2s) - 2s - 1\big)\beta - k^2 + k - (k + n - 2 - 2\beta - 2s) \\ &= \beta^2 + (k+n+3)\beta - k^2 - n + 2s + 2 \\ &\geqslant (k+1)^2 + (k+n+3)(k+1) - k^2 - n + 2s + 2 \\ &= k^2 + 6k + kn + 2s + 6 > 0.\end{aligned}$$

Finally, from (6) and the computation above, we obtain

$$2\big[r_{G'}\big(q(k+n+1)(k+1)\cdot v\big) - \deg(E_2)\big] \geqslant \delta > 0,$$

which proves the claim.

$\square$

Now, by (2), we have:

$$\begin{aligned}l \cdot D_2 - E &\sim \big((r_0+s)\cdot D_2 + q(k+n+1)(k+1)\cdot v\big) - E|_{K_n} - E_1 - E_2\\ &= \big((r_0+s)\cdot D_2 - E_{K_n} - E_1\big) + \big(q(k+n+1)(k+1)\cdot v - E_2\big).\end{aligned}$$

By Claims 4.4 and 4.5, both terms on the right-hand side are equivalent to effective divisors, concluding case (C).

Therefore, for every integer $-2 \leqslant l \leqslant \frac{k+n-2}{2}$, we have

$$r_G\big(l(k+n+1)\cdot u\big) = r_G(l\cdot D_2) \geqslant \binom{l+2}{2} - 1 = \frac{l(l+3)}{2}. \tag{7}$$

To extend this inequality to all $\frac{k+n-2}{2} < l \leqslant k+n$, we take a different approach. First, note that by Part (2) of the lemma,

$$(k+n-2)\cdot D_2 \sim K_G \qquad \text{and} \qquad K_G - l\cdot D_2 \sim (k+n-l-2)\cdot D_2.$$

Now fix some $u \in V(G')$ and integer $\frac{k+n-2}{2} < l \leqslant k+n$. We have that

$$-2 \leqslant k+n-l-2 < \frac{k+n-2}{2}.$$

From (7) and the Riemann–Roch theorem, after some computation, it now follows that

$$\begin{aligned}r_G\big(l(k{+}n+1)\cdot u\big) &= r_G(l\cdot D_2) = \deg(l\cdot D_2) - g(G) + 1 + r_G(K_G - l\cdot D_2)\\ &= l(k+n+1) - \binom{k+n}{2} + 1 + r_G\big((k+n-l-2)\cdot D_2\big)\\ &\geqslant \frac{2l(k+n)+2l}{2} - \frac{(k+n)^2-(k+n)}{2} + 1 + \frac{(k+n-l-2)(k+n-l+1)}{2}\\ &= \frac{l(l+3)}{2}.\end{aligned}$$

This establishes the inequality for all $-2 \leqslant l \leqslant k+n$, and proves the lemma.

4.2. **Bounding gonalities from below.** This subsection is dedicated to proving the following lemma.

Lemma 4.6. *Let G' be a graph with genus $\binom{k}{2}$ for some $k \geqslant 1$. Assume that for every $1 \leqslant l \leqslant k-2$,*

$$\operatorname{gon}_{\frac{l(l+1)}{2}}(G') \geqslant lk,$$

and for every integer $r \geqslant 1$, the divisor $\operatorname{gon}_r(G')\cdot v$ realises the r-th gonality of G'. Let G be the graph obtained by attaching a complete graph K_n to G' along $k+1$ edges between $v \in V(G')$ and each vertex in K_n (see Figure 10 for an illustration). Then, for every $1 \leqslant l \leqslant k+n-2$,

$$\operatorname{gon}_{\frac{l(l+1)}{2}}(G) \geqslant l(k+n).$$

The lemma builds on the results of Cools and Panizzut [CP17], who compute lower bounds for the gonalities of complete graphs. We summarise their results in the following proposition.

Proposition 4.7 (Lower bound for the gonality sequence of complete graphs, [CP17]). *Let K_n be a complete graph with vertices $v_1, v_2, \dots v_n$. For any $1 \leqslant l \leqslant n-3$, let D be a v_1-reduced divisor with*

$$\deg(D) = l(n-1) - 1$$

satisfying

$$D(v_2) \leqslant D(v_3) \leqslant \cdots \leqslant D(v_n).$$

Then, there exists an integer $d \geqslant 0$ such that firing d times from v_1 produces a divisor D' given by

$$D'(v_i) = \begin{cases} D(v_1) - d(n-1), & \text{if } i = 1, \\ D(v_i) + d, & \text{if } 2 \leqslant i \leqslant n, \end{cases}$$

for which the divisor E defined by

$$E(v_i) = \begin{cases} \min\left(0, D'(v_1) + 1\right) & \text{if } i = 1, \\ \min\left(0,\ D'(v_i) - i + 2\right), & \text{if } 2 \leqslant i \leqslant n, \end{cases}$$

has $\deg(E) \leqslant \frac{l(l+1)}{2}$. Moreover, $D' - E$ is v_1-reduced, has rank -1 and satisfies

$$(D' - E)(v_2) \leqslant (D' - E)(v_3) \leqslant \cdots \leqslant (D' - E)(v_n).$$

For more details, see [CP17, Section 4, Remark 12].

Lemma 4.8 (Lemma 5, [CP17]). *Let D be a divisor on the complete graph K_n. Fix a vertex v_1, and order the remaining vertices $v_2, v_3, \dots v_n$ so that,*

$$D(v_2) \leqslant D(v_3) \leqslant \ldots \leqslant D(v_n).$$

Then D is v_1-reduced if and only if

$$0 \leqslant D(v_i) \leqslant i - 2, \qquad \text{for every } 2 \leqslant i \leqslant n.$$

We now prove Lemma 4.6. Fix $1 \leqslant l \leqslant k+n-2$ and let D be an effective divisor on G with $\deg(D) \leqslant l(k+n)-1$. We aim to show that the rank of D is less than $\frac{l(l+1)}{2}$. Since rank is invariant under linear equivalence, we may assume that D is v-reduced.

Claim 4.9. Let D be an effective v-reduced divisor on G. Define

$$D' = \deg(D|_{G'}) \cdot v + D|_{K_n}.$$

Then,

$$r_G(D) \leqslant r_G(D').$$

Note that the divisors D and D' both have the same degree.

Proof. Let E be an effective divisor on G with $\deg(E) = r_G(D)$. We will show that $D' - E$ is effective. Write

$$E = E|_{G'} + E|_{K_n}.$$

As D is v-reduced, by Lemma 3.2, the divisor $D|_{K_n \cup \{v\}}$ has rank at least $\deg(E)$ as a divisor on $K_n \cup \{v\}$. Therefore $D|_{K_n \cup \{v\}} - E|_{K_n}$ is equivalent to an effective v-reduced divisor F on $K_n \cup \{v\}$.

We claim that the divisors $D|_{K_n \cup \{v\}} - E|_{K_n}$ and F are equivalent as divisors on G as well. Take any sequence of chip-firing moves on $K_n \cup \{v\}$ that transforms the divisor $D|_{K_n \cup \{v\}} - E|_{K_n}$ to F. We can extend it to a sequence of chip-firings on G by replacing every firing of v with a set-firing the whole of $V(G')$. As v is a cut vertex, applying the chip-firing moves in this sequence to the divisor $D|_{K_n\cup\{v\}} - E|_{K_n}$ on G has the exact same effect on $K_n \cup \{v\}$, while keeping the configuration of chips at $G' \setminus \{v\}$ unchanged throughout.

Therefore, we have

$$D - E|_{K_n} = D|_{G'\setminus\{v\}} + D|_{K_n\cup\{v\}} - E|_{K_n} \sim D|_{G'\setminus\{v\}} + F = D|_{G'\setminus\{v\}} + F(v)\cdot v + F|_{K_n} \geqslant 0,$$

and

$$\begin{aligned} D' - E|_{K_n} &= \deg(D|_{G'\setminus\{v\}})\cdot v + D|_{K_n\cup\{v\}} - E|_{K_n} \\ &\sim \deg(D|_{G'\setminus\{v\}})\cdot v + F = \big(\deg(D|_{G'\setminus\{v\}}) + F(v)\big)\cdot v + F|_{K_n} \geqslant 0. \end{aligned}$$

As E has degree equal to $r_G(D)$,

$$r_G\big(D|_{G'\setminus\{v\}} + F(v)\cdot v + F|_{K_n}\big) = r_G\big(D - E|_{K_n}\big) \geqslant r_G(D) - \deg(E|_{K_n}) = \deg(E|_{G'}).$$

Notice that since both F and D are v-reduced, $D|_{G'\setminus\{v\}} + F(v)\cdot v + F|_{K_n}$ is v-reduced as well. So by Lemma 3.2,

$$r_{G'}\big(D|_{G'\setminus\{v\}} + F(v)\cdot v\big) \geqslant \deg(E|_{G'}).$$

Therefore,

$$\mathrm{gon}_{\deg(E|_{G'})}(G') \leqslant \deg\left(D|_{G'\setminus\{v\}} + F(v)\cdot v\right) = \deg(D|_{G'\setminus\{v\}}) + F(v).$$

By assumption,

$$\deg(E|_{G'}) = r_{G'}\big(\mathrm{gon}_{\deg(E|_{G'})}(G')\cdot v\big) \leqslant r_{G'}\big[\big(\deg(D|_{G'\setminus\{v\}}) + F(v)\big)\cdot v\big].$$

Now we can apply Lemma 3.3 to $\big(\deg(D|_{G'\setminus\{v\}}) + F(v)\big)\cdot v + F|_K$, concluding that

$$\big(\deg(D|_{G'\setminus\{v\}}) + F(v)\big)\cdot v + F|_K - E|_{G'}$$

is equivalent to an effective divisor on G. Now,

$$D' - E = D' - E|_K - E|_{G'} \sim \big(\deg(D|_{G'\setminus\{v\}}) + F(v)\big)\cdot v + F|_K - E|_{G'}$$

is equivalent to an effective divisor as well. As E was an arbitrary effective divisor of degree $r_G(D)$, the claim is follows.

□

By Claim 4.9, it is enough to show that the rank of $D' = \deg(D|_{G'})\cdot v + D|_{K_n}$ is less than $\frac{l(l+1)}{2}$. Let u_1 be any vertex in K_n. Perform Dhar's burning algorithm to obtain a u_1-reduced divisor D_G on G equivalent to D'. As there are no chips on $G' \setminus \{v\}$, during each iteration of the algorithm, either all of G' burns down or none of it. So the number of chips on vertices in G' other than v does not change. In particular, D_G is supported on $K_n \cup \{v\}$. Label the other vertices of K_n by $u_2, u_3, \ldots u_n$ so that

$$D_G(u_2) \leqslant D_G(u_3) \leqslant \cdots \leqslant D_G(u_n). \tag{8}$$

Define

$$\alpha = \left\lfloor \frac{D_G(v)}{k+1} \right\rfloor$$

and let $0 \leqslant \beta \leqslant k$ be the remainder of the divison of $D_G(v)$ by $k+1$. So

$$D_G(v) = \alpha(k+1) + \beta.$$

Set

$$N = \max\big(\{1\} \cup \{2 \leqslant i \leqslant n \mid D_G(u_i) \leqslant \alpha\}\big).$$

Construct a divisor D_K on the complete graph K_{n+k+1} with vertices $v_1, v_2, \ldots v_{n+k+1}$ by

$$D_K(v_i) = \begin{cases} D_G(u_i) & \text{if } 1 \leqslant i \leqslant N, \\ \alpha & \text{if } N+1 \leqslant i \leqslant N+k+1-\beta, \\ \alpha+1 & \text{if } N+k+2-\beta \leqslant i \leqslant N+k+1, \\ D_G\left(u_{i-(k+1)}\right) & \text{if } N+k+2 \leqslant i \leqslant n+k+1. \end{cases}$$

Claim 4.10. The divisor D_K is v_1-reduced on K_{n+k+1}, with $\deg(D_K) = \deg(D) = l(k+n)-1$ and

$$D_K(v_2) \leqslant D_K(v_3) \leqslant \cdots \leqslant D_K(v_{n+k+1}).$$

Proof. We first verify the degree of D_K.

$$\begin{aligned} \deg(D_K) &= \sum_{i=1}^{N} D_K(v_i) + \sum_{i=N+1}^{N+k+1-\beta} D_K(v_i) + \sum_{i=N+k+2-\beta}^{N+k+1} D_K(v_i) + \sum_{i=N+k+2}^{n+k+1} D_K(v_i) \\ &= \sum_{i=1}^{N} D_G(u_i) + \sum_{i=N+1}^{N+k+1-\beta} \alpha + \sum_{i=N+k+2-\beta}^{N+k+1} (\alpha+1) + \sum_{i=N+1}^{n} D_G(u_i) \\ &= \sum_{w \in V(K_n)} D_G(w) + \alpha(k+1) + \beta \\ &= \deg(D_G|_{K_n}) + \deg(D_G|_{G'}) = \deg(D_G) = l(k+n)-1. \end{aligned}$$

Since $D_K(v_i) = D_G(u_i)$ for every $2 \leqslant i \leqslant N$, and $D_K(v_i) = D_G(u_{i-(k+1)})$ for every $N+k+2 \leqslant i \leqslant n+k+1$, from (8) we have

$$D_K(v_2) \leqslant D_K(v_3) \leqslant \cdots \leqslant D_K(v_N) \qquad \text{and} \qquad D_K(v_{N+k+2}) \leqslant D_K(v_{N+k+3}) \leqslant \cdots \leqslant D_K(v_{n+k+1}).$$

By construction,

$$D_K(v_{N+1}) \leqslant D_K(v_{N+2}) \leqslant \ldots \leqslant D_K(v_{N+k+1}).$$

and by definition of N,

$$D_K(v_{N+1}) = \alpha \geqslant D_G(u_N) = D_K(v_N), \qquad \text{if } N \geqslant 2,$$

and

$$D_K(v_{N+k+1}) \leqslant \alpha+1 \leqslant D_G(u_{N+1}) = D_K(v_{N+k+2}), \qquad \text{if } N \leqslant n-1.$$

These inequalities give

$$D_K(v_2) \leqslant D_K(v_3) \leqslant \ldots \leqslant D(v_{n+k+1}).$$

By Lemma 4.8, it remains to show that $0 \leqslant D(v_i) \leqslant i-2$ for all $2 \leqslant i \leqslant n+k+1$. Assume for a contradiction that there exists $2 \leqslant j \leqslant n+k+1$ such that $D_K(v_j) \geqslant j-1$. We consider 3 cases.

Case (A): $2 \leqslant j \leqslant N$.

Let $A = V(G') \cup \{u_j, u_{j+1}, \ldots u_n\}$. Then,

$$\operatorname{outdeg}_A(v) = (j-1)(k+1) \leqslant D_K(v_{k+1})(k+1) \leqslant \alpha(k+1) \leqslant D_G(v),$$

and

$$\operatorname{outdeg}_A(u_i) = j-1 \leqslant D_K(v_j) = D_G(u_j) \leqslant D_G(u_i), \qquad \text{for all } j \leqslant i \leqslant n.$$

Therefore, set-firing A with respect to D_G doesn't create any debt on $G \setminus \{u_1\}$, contradicting that D_G is u_1-reduced.

Case (B): $N+1 \leqslant j \leqslant N+k+1$.

Let $A = V(G') \cup \{u_{N+1}, u_{N+2} \dots u_n\}$. Notice that if $j = N+1$, then $\alpha = D_K(v_{N+1}) \geqslant N$, and if $N+2 \leqslant j \leqslant N+k+1$,

$$\alpha = D_K(v_{N+1}) \geqslant D_K(v_j) - 1 \geqslant j-2 \geqslant N.$$

Now,

$$\mathrm{outdeg}_A(v) = N(k+1) \leqslant \alpha(k+1) \leqslant D_G(v),$$

and

$$\begin{aligned}\mathrm{outdeg}_A(u_i) = N &\leqslant j-1 \leqslant D_K(v_j)\\ &\leqslant D_K(v_{N+k+2}) = D_G(u_{N+1}) \leqslant D(u_i), \qquad \text{for all } N+1 \leqslant i \leqslant n.\end{aligned}$$

Again, set-firing A with respect to D_G creates no debt on $G \setminus \{u_1\}$, contradicting that D_G is u_1-reduced.

Case (C): $N+k+2 \leqslant j \leqslant n+k+1$.

Let $A = \{u_{j-(k+1)}, u_{j-(k+1)+1}, \dots u_n\}$.

$$\mathrm{outdeg}_A(u_i) = j-1 \leqslant D_K(v_j) = D_G(u_j) \leqslant D_G(u_i), \qquad \text{for all } j-(k+1) \leqslant i \leqslant n.$$

Set-firing A with respect to D_G doesn't create any debt on $G \setminus \{u_1\}$, again contradicting that D_G is u_1-reduced.

We obtain a contradiction in all cases, showing no such j exists. By Lemma 4.8, D_K is v_1-reduced. □

We now apply Proposition 4.7 to the divisor D_K. This yields a non-negative integer d and divisors D'_K and E_K as described in the proposition. Explicitly,

$$D'_K(v_i) = \begin{cases} D_K(v_i) - d(k+n) & \text{if } i = 1,\\ D_K(v_i) + d & \text{if } 2 \leqslant i \leqslant N,\\ \alpha + d & \text{if } N+1 \leqslant i \leqslant N+k-\beta+1,\\ \alpha + d + 1 & \text{if } N+k-\beta+2 \leqslant i \leqslant N+k+1,\\ D_K(v_i) + d & \text{if } N+k+2 \leqslant i \leqslant n+k+1,\end{cases}$$

and

$$E_K(v_i) = \begin{cases} \min\left(0, D'_K(v_1) + 1\right) & \text{if } i = 1,\\ \min\left(0,\ D'_K(v_i) - i + 2\right) & \text{if } 2 \leqslant i \leqslant n+k+1,\end{cases}$$

with $\deg(E_K) \leqslant \frac{l(l+1)}{2}$. Let

$$r = E_K(v_{N+1}) = \min\left(0, D'(v_{N+1}) - N + 1\right) = \min(0,\ \alpha + d - N + 1) \geqslant 0.$$

Note that if $r > 0$, then $N = \alpha + d - r + 1$.

Claim 4.11. Let H be a subgraph of K_{n+k+1} consisting of vertices $v_{N+1}, v_{N+2} \dots v_{N+k+1}$ and all edges between them. Then

$$\deg(E_K|_H) = \begin{cases} \frac{r(r+1)}{2} & \text{if } 0 \leqslant r \leqslant k-\beta,\\ \frac{r(r+1)}{2} + \beta - r + k & \text{if } k-\beta+1 \leqslant r \leqslant k-1,\\ \frac{k(k+1)}{2} + (k+1)(r-k) + \beta & \text{if } k \leqslant r.\end{cases}$$

Proof. First notice that if $E_K(\nu_i) = 0$ for some $N+1 \leqslant i \leqslant N+k+1$, then for any $i+1 \leqslant j \leqslant N+k+1$ we have

$$0 \geqslant D'_K(\nu_i) - i + 2 \geqslant D'_K(\nu_j) - j + 2 \geqslant E_K(\nu_j) \geqslant 0,$$

hence $E_K(\nu_j) = 0$. We consider the following four cases.

Case (A): $r = 0$.

Then $E_K(\nu_{N+1}) = 0$, and so $E_K(\nu_j) = 0$ for all $N+2 \leqslant j \leqslant N+k+1$. Therefore, $\deg(E_K|_H) = 0$.

Case (B): $1 \leqslant r \leqslant k - \beta$.

For each $N+1 \leqslant i \leqslant N+r+1 \leqslant N+k-\beta+1$,

$$\begin{aligned} E_K(\nu_i) &= \min\big(0, D'_K(\nu_i) - i + 2\big) = \min(0, \alpha + d - i + 2) \\ &= \min(0, N+r+1-i) = N+r+1-i. \end{aligned}$$

In particular, $E(\nu_{N+r+1}) = 0$. Hence,

$$\deg(E_K|_H) = \sum_{i=N+1}^{N+r+1} E_K(\nu_i) = \sum_{i=N+1}^{N+r+1} (N+r+1-i) = \sum_{j=0}^{r} j = \frac{r(r+1)}{2}.$$

Case (C): $k - \beta + 1 \leqslant r \leqslant k - 1$.

For every $N+1 \leqslant i \leqslant N+k-\beta+1 < N+r+1$,

$$\begin{aligned} E_K(\nu_i) &= \min\big(0, D'_K(\nu_i) - i + 2\big) = \min(0, \alpha + d - i + 2) \\ &= \min(0, N+r+1-i) = N+r+1-i, \end{aligned}$$

and for $N+k-\beta+2 \leqslant i \leqslant N+r+2 \leqslant N+k+1$,

$$\begin{aligned} E_K(\nu_i) &= \min\big(0, D'_K(\nu_i) - i + 2\big) = \min(0, \alpha + d + 1 - i + 2) \\ &= \min(0, N+r+2-i) = N+r+2-i. \end{aligned}$$

In particular, $E(\nu_{N+r+2}) = 0$. Then

$$\begin{aligned} \deg(E_K|_H) &= \sum_{i=N+1}^{N+r+2} E_K(\nu_i) = \sum_{i=N+1}^{N+k-\beta+1} E_K(\nu_i) + \sum_{i=N+k-\beta+2}^{N+r+2} E_K(\nu_i) \\ &= \sum_{i=N+1}^{N+k-\beta+1} (N+r+1-i) + \sum_{i=N+k-\beta+2}^{N+r+2} (N+r+2-i) \\ &= \sum_{j=r-k+\beta}^{r} j + \sum_{j=0}^{r-k+\beta-1} (j+1) = \sum_{j=0}^{r} j + (r-k+\beta) = \frac{r(r+1)}{2} + r - k + \beta, \end{aligned}$$

as claimed.

Case (D): $k \leqslant r$.

For all $N+1 \leqslant i \leqslant N+k-\beta+1 \leqslant N+r+1$,

$$\begin{aligned} E_K(\nu_i) &= \min\big(0, D'_K(\nu_i) - i + 2\big) = \min(0, \alpha + d - i + 2) \\ &= \min(0, N+r+1-i) = N+r+1-i, \end{aligned}$$

and for every $N+k-\beta+2 \leqslant i \leqslant N+k+1 < N+r+2$,

$$\begin{aligned} E_K(v_i) &= \min\big(0, D_K'(v_i) - i + 2\big) = \min(0, \alpha + d + 1 - i + 2) \\ &= \min(0, N + r + 2 - i) = N + r + 2 - i. \end{aligned}$$

Therefore,

$$\begin{aligned} \deg(E_K|_H) &= \sum_{i=N+1}^{N+r+2} E_K(v_i) = \sum_{i=N+1}^{N+k-\beta+1} E_K(v_i) + \sum_{i=N+k-\beta+2}^{N+k+1} E_K(v_i) \\ &= \sum_{i=N+1}^{N+k-\beta+1} (N+r+1-i) + \sum_{i=N+k-\beta+2}^{N+k+1} (N+r+2-i) \\ &= \sum_{j=r-k+\beta}^{r} j + \sum_{j=r-k}^{r-k+\beta-1} (j+1) = \sum_{j=\beta}^{k} (j+r-k) + \sum_{j=0}^{\beta-1} (j+r-k+1) \\ &= \sum_{j=0}^{k} j + (k-\beta+1)(r-k) + \beta(r-k+1) = \frac{k(k+1)}{2} + (k+1)(r-k) + \beta, \end{aligned}$$

which completes the proof.

$\square$

We now return to the graph G and divisor D_G. Let D_G' be the divisor obtained by firing d times at the vertex u_1 with respect to D_G. Then

$$D_G'(w) = \begin{cases} D_G(u_1) - d(k+n) & \text{if } w = u_1, \\ D_G(w) + d & \text{if } w \in \{u_2, \dots u_N\}, \\ (\alpha + d)(k+1) + \beta & \text{if } w = v, \\ 0 & \text{if } w \in V(G') \setminus \{v\}, \\ D_G(w) + d & \text{if } w \in \{u_{N+1}, \dots u_n\}. \end{cases}$$

Notice that for $1 \leqslant i \leqslant N$,

$$D_G'(u_i) = D_K'(v_i)$$

and for $N+1 \leqslant i \leqslant n$,

$$D_G'(u_i) = D_K'(v_{i+k+1}).$$

Claim 4.12. There is an effective divisor E_r on G' such that $\deg(E_r) = \deg(E_K|_H)$ and $D_G'|_{G'} - E_r$ is equivalent to a v-reduced divisor D_1 on G' satisfying

$$D_1(v) < N(k+1).$$

Proof. First, note that

$$D_G'|_{G'} = \big((\alpha + d)(k+1) + \beta\big) \cdot v.$$

We regard this as a divisor on G' throughout the proof. If $l = k-1$ or k, then

$$\frac{l(l+1)}{2} \geqslant \frac{(k-1)k}{2} = g(G').$$

So by Corollary 2.2 we have,

$$\mathrm{gon}_{\frac{(k-1)k}{2}}(G') = \mathrm{gon}_{g(G')}(G') = 2g(G') = (k-1)k,$$
$$\mathrm{gon}_{\frac{k(k+1)}{2}}(G') = \mathrm{gon}_{g(G')+k}(G') = 2g(G') + k = k^2.$$

Similarly as in the proof of the previous claim, we consider four cases.

Case (A): $r = 0$.

Since $r = \min(0,\ \alpha + d - N + 1)$, we have $\alpha + d - N + 1 \leqslant 0$. Hence,

$$D'_G|_{G'}(\nu) = (\alpha + d)(k+1) + \beta < (\alpha + d + 1)(k+1) = N(k+1).$$

Taking $D_1 = D'_G|_{G'}$ and $E_r = 0$ concludes this case.

Case (B): $1 \leqslant r \leqslant k - \beta$.

Since $\mathrm{gon}_{\frac{r(r+1)}{2}}(G') \geqslant rk$, there exists an effective divisor E_r on G' with degree

$$\deg(E_r) = \frac{r(r+1)}{2} = \deg(E_K|_H)$$

such that the divisor $(rk-1)\cdot\nu - E_r$ is unwinnable. Let D_0 be the ν-reduced divisor equivalent to $(rk-1)\cdot\nu - E_r$, so $D_0(\nu) \leqslant -1$. Define D_1 as,

$$\begin{aligned} D'_G|_{G'} - E_r &= \big((\alpha+d)(k+1) + \beta\big)\cdot\nu - E_r \\ &= \big((\alpha+d)(k+1) + \beta - rk + 1\big)\cdot\nu + (rk-1)\cdot\nu - E_r \\ &\sim \big((\alpha+d)(k+1) + \beta - rk + 1\big)\cdot\nu + D_0 =: D_1. \end{aligned}$$

Then D_1 is ν-reduced and satisfies

$$D_1(\nu) \leqslant (\alpha+d)(k+1) + \beta - rk.$$

Now,

$$\begin{aligned} N(k+1) - D_1(\nu) &\geqslant (\alpha + d - r + 1)(k+1) - \big((\alpha+d)(k+1) + \beta - rk\big) \\ &= k - r - \beta + 1 \geqslant 1 > 0. \end{aligned}$$

Case (C): $k - \beta + 1 \leqslant r \leqslant k - 1$.

We have

$$\mathrm{gon}_{\frac{r(r+1)}{2}+\beta-r+k}(G') \geqslant \mathrm{gon}_{\frac{r(r+1)}{2}}(G') + \beta - r + k \geqslant rk + \beta - r + k.$$

Therefore, there exists an effective divisor E_r with degree

$$\deg(E_r) = \frac{r(r+1)}{2} + \beta - r + k = \deg(E_K|_H)$$

such that the divisor

$$(rk + \beta - r + k - 1)\cdot\nu - E_r$$

is unwinnable. That is, it is equivalent to a ν-reduced divisor D_0 with $D_0(\nu) \leqslant -1$. Then,

$$\begin{aligned} D'_G|_{G'} - E_r &= \big((\alpha+d)(k+1) + \beta\big)\cdot\nu - E_r \\ &= \big((\alpha+d)(k+1) - rk + r - k + 1\big)\cdot\nu + (rk + \beta - r + k - 1)\cdot\nu - E_r \\ &\sim \big((\alpha+d)(k+1) - rk + r - k + 1\big)\cdot\nu + D_0 =: D_1. \end{aligned}$$

D_1 is also v-reduced with $D_1(v) \leqslant (\alpha+d)(k+1) - rk + r - k$. So

$$\begin{aligned} N(k+1) - D_1(v) &\geqslant (\alpha+d-r+1)(k+1) - \big((\alpha+d)(k+1) - rk + r - k\big) \\ &= 2(k-r) + 1 > 0. \end{aligned}$$

Case (D): $k \leqslant r$.

Since

$$\mathrm{gon}_{\frac{k(k+1)}{2}+(k+1)(r-k)+\beta}(G') \geqslant \mathrm{gon}_{\frac{k(k+1)}{2}}(G') + (k+1)(r-k) + \beta \geqslant k^2 + (k+1)(r-k) + \beta,$$

there exists an effective divisor E_r with

$$\deg(E_r) = \frac{k(k+1)}{2} + (k+1)(r-k) + \beta = \deg(E_K|_H),$$

such that the divisor

$$\big(k^2 + (k+1)(r-k) + \beta - 1\big) \cdot v - E_r$$

is unwinnable. Let D_0 be the corresponding v-reduced divisor with $D_0(v) \leqslant -1$. Then

$$\begin{aligned} D'_G|_{G'} - E_r &= \big((\alpha+d)(k+1) + \beta\big) \cdot v - E_r \\ &= \big((\alpha+d)(k+1) - k^2 - (k+1)(r-k) + 1\big) \cdot v \\ &\quad + \big(k^2 + (k+1)(r-k) + \beta - 1\big) \cdot v - E_r \\ &\sim \big((\alpha+d)(k+1) - k^2 - (k+1)(r-k) + 1\big) \cdot v + D_0 =: D_1. \end{aligned}$$

D_1 is again v-reduced with $D_1(v) \leqslant (\alpha+d)(k+1) - rk + r - k$. Now,

$$\begin{aligned} N(k+1) - D_1(v) &\geqslant (\alpha+d-r+1)(k+1) - \big((\alpha+d)(k+1) - k^2 - (k+1)(r-k)\big) \\ &= (1-k)(k+1) + k^2 = 1 > 0. \end{aligned}$$

This completes the proof of the claim.

□

Define a divisor E_G on G by

$$E_G(w) = \begin{cases} E_r(w) & \text{if } w \in V(G'), \\ E_K(v_i) & \text{if } w = u_i, \text{ for } 1 \leqslant i \leqslant N, \\ E_K(v_{i+k+1}) & \text{if } w = u_i, \text{ for } N+1 \leqslant i \leqslant n. \end{cases}$$

Then $E_G|_{G'} = E_r$ and $E_G = E_r + E_G|_{K_n}$. A direct computation gives

$$\begin{aligned} \deg(E_G) &= \sum_{i=1}^{N} E_K(v_i) + \deg(E_r) + \sum_{i=N+k+2}^{n+k+1} E_K(v_i) \\ &= \sum_{i=1}^{N} E_K(v_i) + \sum_{i=N+1}^{N+k+1} E_K(v_i) + \sum_{i=N+k+2}^{n+k+1} E_K(v_i) \\ &= \deg(E_K) \leqslant \frac{l(l+1)}{2}. \end{aligned}$$

By Claim 4.12, we have $D'_G|_{K_n} - E_r \sim D_1$ as divisors on G'. Take any chip-firing sequence that transforms $D'_G|_{K_n} - E_r$ into D_1. We extend this sequence of chip-firings to G by replacing each firing of v with set-firing all of $V(K_n) \cup \{v\}$. Since v is a cut vertex, applying this sequence to $D'_G|_{K_n} - E_r$ as a divisor on G, has the same effect on G' and doesn't cause any changes on the configuration of chips on K_n. Therefore $D'_G|_{K_n} - E_r$ and D_1 are equivalent as divisors on G. We now have

$$D_G - E_G \sim D'_G - E_G = D'_G|_{G'} + D'_G|_{K_n} - E_r - E_G|_{K_n} \sim D_1 + (D'_G - E_G)|_{K_n} =: F'.$$

Moreover,

$$F'(u_i) = (D'_G - E_G)|_{K_n}(u_i) = \begin{cases} (D'_K - E_K)(v_i) & \text{if } 1 \leqslant i \leqslant N, \\ (D'_K - E_K)(v_{i+k+1}) & \text{if } N+1 \leqslant i \leqslant n. \end{cases}$$

Claim 4.13. Starting a fire from u_1 with respect to F' burns the entire graph G. In particular, $D_G - E_G \sim F'$ is unwinnable and

$$r_G(D_G) < \deg(E_G) \leqslant \frac{l(l+1)}{2}.$$

Proof. Recall that the divisor $D'_K - E_K$ is v_1-reduced, with

$$F'(u_1) = (D'_K - E_K)(v_1) \leqslant -1,$$

and that

$$(D'_K - E_K)(v_2) \leqslant (D'_K - E_K)(v_3) \leqslant \cdots \leqslant (D'_K - E_K)(v_{n+k+1}).$$

By Lemma 4.8, for $2 \leqslant i \leqslant N$,

$$F'(u_i) = (D'_K - E_K)(v_i) \leqslant i - 2, \tag{9}$$

and for $N+1 \leqslant i \leqslant n$,

$$F'(u_i) = (D'_K - E_K)(v_{i+k+1}) \leqslant i + k - 1. \tag{10}$$

By Claim 4.12,

$$F'(v) = D_1(v) < N(k+1), \tag{11}$$

and as $F'|_{G'} = D_1$, a fire at v with respect to F' burns all of G'.

Now start a fire at u_1 with respect to F'. From (9), fire spread s successively through $u_2, u_3, \ldots u_N$. By (11), the vertex v will catches fire, burning down all of G'. Finally, from (10), the fire continues spreading through $u_{N+1}, u_{N+2}, \ldots u_n$. As $F'(u_1) \leqslant -1$, F' is unwinnable, completing the proof. □

Therefore, for any divisor D on G with degree $l(k+n) - 1$, we have constructed a divisor D_G such that

$$r_G(D) \leqslant r_G(D_G) < \frac{l(l+1)}{2}.$$

This shows that $\operatorname{gon}_{\frac{l(l+1)}{2}}(G) \geqslant l(k+n)$, and concludes the proof of the lemma.

5. Proof of the main theorems

In this section, we use the tools developed throughout the paper to prove the two main results: Theorem A and Theorem B. First, we record the following claim about equivalent divisors on quasi-banana graphs.

Claim 5.1. Let $QB_m(N)$ be a quasi-banana graph with notation adapted from Definition 3.12. Let $1 \leqslant L \leqslant K \leqslant m$. Then, for any $1 \leqslant j \leqslant q_L$,

$$u_{0,1} + \sum_{w \in C_{0,K}} w \sim \left(\sum_{i=1}^{L} q_i + 2 \right) \cdot u_{L,j} + \sum_{w \in C_{L+1,K}} w,$$

where

$$C_{A,B} = \bigcup_{i=A}^{B} C_i.$$

Proof. Start with the divisor

$$u_{0,1} + \sum_{w \in C_{1,K}} w.$$

Set-fire the sets

$$C_{0,1} \setminus \{u_{1,1}\}, C_{0,2} \setminus \{u_{2,1}\}, \ldots C_{0,L-1} \setminus \{u_{L-1,1}\}$$

to obtain

$$\left(\sum_{i=1}^{L-1} q_i + 2 \right) \cdot u_{L-1,1} + \sum_{w \in C_{L,K}} w.$$

Now, if $j = 1$, set-firing $C_{0,L} \setminus \{u_{L,1}\}$ produces the required divisor. If $j \geqslant 2$, set-firing $V(QB_m(N)) \setminus \{u_{L,j}\}$ gives the required divisor. □

5.1. **Proof of Theorem B.** We aim to show that every quasi-banana graph $QB_m(N)$ satisfies the conclusions listed in the theorem. In addition to that, we show that the divisor

$$u_{0,1} + \sum_{w \in V(QB_m(N))} w$$

realises the second gonality.

We proceed by induction on m. For the base case $m = 0$, there is only one quasi-banana graph QB_0 consisting of a single vertex and no edges. The genus of QB_0 is 0, and by Corollary 2.2, its gonality sequence is $1, 2, 3, \ldots$. The unique degree 2 divisor $2u_{0,1}$ realises the second gonality, and

$$\big(\operatorname{gon}_1(QB_0) - 2 \big) \cdot 2u_{0,1} = -2u_{0,1} = K_{QB_0}$$

has rank -1. Therefore, the theorem holds for $m = 0$.

Now assume that $m \geqslant 1$ and that for every quasi-banana graph $QB_{m-1}(N)$, the divisor

$$u_{0,1} + \sum_{w \in V(QB_{m-1}(N))} w$$

realises the second gonality, and that all conclusions of the theorem hold. Let

$$G = QB_m(q_1, q_w, \ldots q_m) \qquad \text{and} \qquad G' = QB_{m-1}(q_1, q_w, \ldots q_{m-1}),$$

and denote $v = u_{m-1,1}$. Define

$$k_{m-1} = \sum_{i=1}^{m-1} q_i + 1, \qquad k_m = \sum_{i=1}^{m} q_i + 1.$$

The graph G is created from G' by attaching a complete graph K with vertices $u_{m,1}, u_{m,2}, \dots u_{m,q_m}$ along $k_{m-1}+1$ edges between v and each vertex in K. We can therefore think of G' as a subgraph of G. We define

$$D_{m-1} = u_{0,1} + \sum_{w \in V(G')} w, \qquad \text{and} \qquad D_m = u_{0,1} + \sum_{w \in V(G)} w.$$

By the inductive hypothesis, G' is gonality-tight, so

$$\operatorname{gon}_1(G') = \deg(D_{m-1}) - 1 = |V(G')| = \sum_{i=1}^{m-1} q_i + 1 = k_{m-1}.$$

By Claim 5.1,

$$D_{m-1} \sim \left(\sum_{i=1}^{m-1} q_i + 2\right) \cdot u_{m-1,1} = (k_{m-1}+1) \cdot v$$

and

$$D_m \sim \left(\sum_{i=1}^{m} q_i + 2\right) \cdot u_{m,1} = (k_m+1) \cdot u_{m,1},$$

where D_{m-1} and D_m are divisor on G' and G respectively. Furthermore, by the inductive hypothesis,

$$g(G') = \binom{k_{m-1}}{2},$$

and for all $-1 \leqslant l \leqslant k_{m-1}$,

$$r_{G'}\big(l(k_{m-1}+1) \cdot v\big) = r_{G'}(l \cdot D_{m-1}) = \frac{l(l+3)}{2}.$$

Now we can apply Lemma 4.1 to G' and G, with $k = k_{m-1}$ and $n = q_m$. This gives

$$g(G) = \binom{k_{m-1}+q_m}{2} = \binom{k_m}{2}, \tag{12}$$

and

$$(k_m - 2) \cdot D_m \sim (k_m-2)(k_m+1) \cdot u_{m,1} \sim K_G. \tag{13}$$

We also get that for every integer $-2 \leqslant l \leqslant k_{m-1} + q_m = k_m$,

$$r_G(l \cdot D_m) = r_G\big(l(k_m+1) \cdot u_{m,1}\big) \geqslant \frac{l(l+3)}{2}. \tag{14}$$

The inequality implies that for $1 \leqslant l \leqslant k_m - 2$, we have $\operatorname{gon}_{\frac{l(l+3)}{2}}(G) \leqslant l(k_m-1)$. Because the gonality sequence is strictly increasing, we can extend this bound to the other gonalities. For every $0 \leqslant h \leqslant l$,

$$\operatorname{gon}_{\frac{l(l+3)}{2}-h}(G) \leqslant \operatorname{gon}_{\frac{l(l+3)}{2}}(G) - h = l(k_m+1) - h \tag{15}$$

Next, we derive the matching lower bound. By the inductive hypothesis,

$$\operatorname{gon}_{\frac{l(l+1)}{2}}(G') = \operatorname{gon}_{\frac{l(l+3)}{2}-l}(G') = l(k_{m-1}+1) - l = lk_{m-1}.$$

Moreover, for any $1 \leqslant r \leqslant \frac{(k_{m-1}-2)(k_{m-1}+1)}{2} = g(G') - 1$,

$$r_{G'}\big(\operatorname{gon}_r(G') \cdot v\big) = r_{G'}\big[\big(l(k_{m-1}+1) - h\big) \cdot v\big] \geqslant r_{G'}\big(l(k_{m-1}+1) \cdot v\big) - h = \frac{l(l+3)}{2} - h = \operatorname{gon}_r(G').$$

This means that $\mathrm{gon}_r(G')\cdot v$ realises the r-th gonality of G'. The same conclusion follows for $r \geqslant g(G')$ from Corollary 2.2. Now the assumptions of Lemma 4.6 hold for G' and G. Applying the lemma with $k = k_{m-1}$ and $n = q_m$, gives that for all $1 \leqslant l \leqslant k_{m-1} + q_m - 2 = k_m - 2$,

$$\mathrm{gon}_{\frac{l(l+1)}{2}}(G) \geqslant l(k_{m-1} + q_m) = lk_m,$$

or equivalently,

$$\mathrm{gon}_{\frac{l(l+3)}{2}-l}(G) \geqslant l(k_m + 1) - l.$$

Again, since the gonality sequence is strictly increasing, for any $0 \leqslant h \leqslant l$,

$$\mathrm{gon}_{\frac{l(l+3)}{2}-h}(G) \geqslant \mathrm{gon}_{\frac{l(l+3)}{2}-l}(G) + (l-h) \geqslant l(k_m+1) - h. \tag{16}$$

Combining (15) and (16), gives the equality:

$$\mathrm{gon}_{\frac{l(l+3)}{2}-h}(G) = l(k_m+1) - h, \qquad \text{for any } 1 \leqslant l \leqslant k_m - 2, \text{ and } 0 \leqslant h \leqslant l.$$

For $r \geqslant g(G)$, Corollary 2.2 gives $\mathrm{gon}_r(G) = g(G) + r$. Now as

$$\frac{l(l+3)}{2} - l - 1 = \frac{(l-1)(l+2)}{2}, \qquad \text{and} \qquad \frac{(k_m-2)(k_m+1)}{2} = g(G) - 1,$$

the full gonality sequence is

$$\mathrm{gon}_r(G) = \begin{cases} l(k_m+1) - h & \text{if } r < g(G), \\ g + r & \text{if } r \geqslant g(G), \end{cases}$$

where $1 \leqslant l \leqslant k-2$ and $0 \leqslant h \leqslant l$ are uniquely determined integers such that $r = \frac{l(l+3)}{2} - h$, proving Part (2) of the theorem. In particular,

$$\mathrm{gon}_1(G) = k_m, \qquad \mathrm{gon}_2(G) = k_m + 1,$$

and G is gonality-tight. Combining this with (12) proves Part (1).

Next, we show that if $k_m > 2$, all divisors realising the second gonality are linearly equivalent. We first verify that D_m realises the second gonality.

$$\deg(D_m) = |V(G)| + 1 = \sum_{i=1}^{m} q_i + 2 = k_m + 1 = \mathrm{gon}_2(G) \tag{17}$$

and by (14) with $l = 1$,

$$r_G(D_m) \geqslant 2, \tag{18}$$

as required. By Claim 5.1,

$$D_m = u_{0,1} + \sum_{w\in V(G)} w \sim \left(\sum_{i=1}^{m-1} q_i + 2\right)\cdot u_{m-1,1} + \sum_{w \in V(K)} w = (k_{m-1}+1)\cdot v + \sum_{i=1}^{q_m} u_{m,1} =: D'_m.$$

Because there are two or more edges between v and each vertex of K, starting a fire on $u_{m-1,1}$ with respect to D'_m burns all of G. Therefore, D'_m is v-reduced.

Claim 5.2. The only v-reduced divisor on G which realises the second gonality is D'_m.

Proof. Assume for a contradiction that D''_m is a v-reduced divisor on G of degree $k_m + 1$ and rank 2 with $D''_m \neq D'_m$. By Lemma 3.2

$$r_{G'}(D''_m|_{G'}) \geqslant 2.$$

Since $\mathrm{gon}_2(G') = k_{m-1} + 1$, we must have that $\deg(D''|_{G'}) \geqslant k_{m-1} + 1$, that is

$$M := \deg(D''_m|_K) = \deg(D''_m) - \deg(D''_m|_{G'}) \leqslant k_m + 1 - (k_{m-1} + 1) = q_m.$$

We consider the following three cases.

Case (A): $M = q_m$.

If

$$\deg(D''_m|_K) = M = q_m = V(K), \qquad \text{and} \qquad D''_m \neq D'_m,$$

then there must exist a vertex $u \in V(K)$ such that $D''_m(u) = 0$. Also,

$$D''_m(v) \leqslant \deg(D''_m|_{G'}) = \deg(D''_m) - \deg(D''_m|_K) = k_m + 1 - q_m = k_{m-1} + 1. \tag{19}$$

Consider the divisor $D''_m - u - v$. It is not effective, as there are -1 chips on u. Now, start a fire at u with respect to $D''_m - u - v$. From (19), there are less than $k_{m-1} + 1$ chips on v, so fire spreads to v. Since D''_m is v-reduced, the entire graph burns down. Therefore, $D''_m - u - v$ is not equivalent to an effective divisor, contradicting the assumption that D''_m has rank 2.

Case (B): $1 \leqslant M \leqslant q_m - 1$.

First note that

$$D''_m(v) \leqslant \deg(D''_m|_{G'}) = \deg(D''_m) - \deg(D''_m|_K) = k_m + 1 - M = k_{m-1} + 1 + (q_m - M), \tag{20}$$

and $V(K) = q_m \geqslant 2$. Order the vertices of K so that

$$D''_m(u_{m,1}) \leqslant D''_m(u_{m,2}) \leqslant \cdots \leqslant D''_m(u_{m,q_m}).$$

Now as

$$\sum_{i=1}^{q_m} D(u_{m,i}) = \deg(D''_m|_K) = M \leqslant q_m - 1,$$

we have

$$D''_m(u_{m,1}) = D''_m(u_{m,2}) = \cdots = D''_m(u_{m,q_m - M}) = 0, \qquad \text{and} \qquad D''_m(u_{m,q_m-M+1}) \leqslant 1.$$

Let

$$A = \{u_{m,2}, u_{m,3}, \ldots u_{m,q_m-M+1}\}, \qquad \text{and} \qquad E = u_{m,1} + u_{m,q_m-M+1}.$$

Consider the divisor $D''_m - E$. It has -1 chips on $u_{m,1}$ and 0 or less chips on every vertex in A. Therefore, starting a fire at $u_{m,1}$ with respect to $D''_m - E$ burns all of A. The total number of edges from A to v is equal to

$$|A|(k_{m-1} + 1) = (q_m - M + 1)(k_{m-1} + 1).$$

From (20),

$$(D''_m - E)(v) = D''_m(v) \leqslant k_{m-1} + 1 + (q_m - M) < (q_m - M + 1)(k_{m-1} + 1).$$

Hence the fire reaches v and burns all of G, contradicting the assumption that D''_m has rank 2.

Case (C): $M = 0$.

In this case,

$$\deg(D''_m|_{G'}) = \deg(D''_m) = k_m + 1 = k_{m-1} + q_m + 1.$$

Set-firing $V(G')$ produces the divisor

$$F := D''_m - q_m(k_{m-1}+1)\cdot v + \sum_{w\in V(K)} (k_{m-1}+1)\cdot w.$$

Rank of F is 2 as $F \sim D''_m$. For any $u \in V(K)$, starting a fire at v with respect to $F-u$ burns u and then all of K. Additionally, as D''_m is v-reduced and

$$F|_{G'\setminus\{v\}} = D''_m|_{G'\setminus\{v\}},$$

the fire spreads to the rest of G'. Therefore $F-u$ is v-reduced and has rank at least 1. Now by Lemma 3.2, $F|_{G'}$ has rank at least 1 as a divisor on G'. However, since $k_{m-1}+q_m = k_m > 2$,

$$\deg(F|_{G'}) = \deg(D''_m|_{G'}) - q_m(k_{m-1}+1) = (k_{m-1}+q_m+1) - q_m(k_{m-1}+1) < k_{m-1} = \mathrm{gon}_1(G'),$$

a contradiction.

□

Two divisors are linearly equivalent if and only if they are equivalent to the same v-reduced divisor. As rank is invariant under equivalence, by Claim 5.2, every divisor realising the second gonality of G must be equivalent to D'_m. This proves Part (3) of the theorem.

For Part (4), let D be any divisor realising the second gonality. If $k_m = \mathrm{gon}_1(G) = 2$, then the genus of G is 1, and the result follows by Corollary 2.2. Now assume that $k_m > 2$. As all divisors realising the second gonality are equivalent by Part (3), we have $D \sim D'_m \sim D_m$. Therefore, for every integer $-1 \leqslant l \leqslant k_m$,

$$l\cdot D \sim l\cdot D_m \qquad \text{and} \qquad r_G(l\cdot D) = r_G(l\cdot D_m).$$

From the computed gonality sequence, notice that, for every $1 \leqslant l \leqslant k_m$,

$$\deg(l\cdot D_m) = l\deg(D_m) = l(|V(G)|+1) = l\left(\sum_{i=1}^{m} q_i + 2\right) = l(k_m+1) = \mathrm{gon}_{\frac{l(l+3)}{2}}(G).$$

So the equality in (14) holds for all $1 \leqslant l \leqslant k_m$. Therefore,

$$r_G(l\cdot D) = r_G(l\cdot D_m) = \frac{l(l+3)}{2}, \qquad \text{for all } 1 \leqslant l \leqslant k_m.$$

The cases when $l=-1$ and $l=0$ follow immediately as

$$r_G(-1\cdot D_2) = -1, \quad \text{and} \quad r_G(0\cdot D_2) = 0.$$

Finally, (13) shows that $(k_m-2)\cdot D \sim (k_m-2)\cdot D_m \sim K_G$, completing the proof.

5.2. **Proof of Theorem A.** The following is an immediate corollary of the proof of Theorem B.

Corollary 5.3. *Let G be a quasi-banana graph. Then the divisor*

$$u_{0,1} + \sum_{w\in V(G)} w$$

realises the second gonality.

Proof. In the proof of Theorem B, it is established that for every quasi-banana graph G, the divisor

$$D_m = u_{0,1} + \sum_{w\in V(G)} w$$

has degree equal to the second gonality of G (see Equation (17)) and rank at least 2 (see Equation (18)). □

Moreover, Theorem B implies that every quasi-banana graph is gonality-tight, regardless of the first gonality. This established the forward implication of Theorem A. The converse is precisely Proposition 3.18. Therefore, the theorem follows.

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