GROWTH ESTIMATES FOR SOLUTIONS TO THE WAVE EQUATION ON DAMEK–RICCI SPACES

YUNXIANG WANG, LIXIN YAN, AND HONG-WEI ZHANG

Abstract. Let $\mathcal{L}$ be the left-invariant distinguished Laplacian, and let $\mathrm{d}\rho$ denote the right Haar measure on a Damek–Ricci space S. Let $u(t,x)$ denote the solution to the wave equation $\partial_t^2 u - \mathcal{L}u = 0$ with initial data $(u, \partial_t u)|_{t=0} = (f, g)$. In this paper, we establish the sharp-in-regularity L^p bounds

$$\|u(t,\cdot)\|_{L^p(S,\mathrm{d}\rho)} \lesssim_p (1+|t|)^{2\left|\frac{1}{p}-\frac{1}{2}\right|} \|(\mathrm{Id}+\mathcal{L})^{\frac{\alpha_0}{2}} f\|_{L^p(S,\mathrm{d}\rho)} + (1+|t|)\,\|(\mathrm{Id}+\mathcal{L})^{\frac{\alpha_1}{2}} g\|_{L^p(S,\mathrm{d}\rho)}$$

for all $t \in \mathbb{R}^*$ and $1 < p < \infty$, where the exponents $\alpha_0 = n\,|1/p - 1/2|$ and $\alpha_1 = n\,|1/p-1/2| - 1$ attain their critical values. This result settles, in full generality, the conjecture raised by Müller, Thiele, and Vallarino.

1. Introduction

The wave equation has long played a central role in analysis and partial differential equations on manifolds, where the non-flat geometry strongly influences wave propagation. In particular, on negatively curved manifolds, the long-time dispersive estimates for the wave equation differ significantly from the classical results in the Euclidean setting. This phenomenon has been observed in various contexts, for instance, in [2, 22, 30] on real hyperbolic spaces, in [5, 33] on their higher-rank generalizations, that is, on noncompact symmetric spaces, in [27] on non-trapping asymptotically hyperbolic manifolds, and in [4] on the so-called Damek–Ricci spaces.

Damek–Ricci spaces are solvable extensions $S = N \rtimes \mathbb{R}^+$ of Heisenberg-type groups N, endowed with a left-invariant Riemannian structure. As Riemannian manifolds, these solvable Lie groups include important examples such as real hyperbolic spaces and all other noncompact rank-one symmetric spaces. Most of them, however, are not symmetric, thereby providing counterexamples to the Lichnerowicz conjecture [15]. We refer to Section 2 for more details on the structure of S and the analysis carried out on it.

The dispersive effect of the wave equation, manifested through the $L^{p'}$-L^p estimates of the propagator, describes how the solution spreads out over time. In contrast, the L^p ($p \neq 2$) norm estimates for solutions to the wave equation may not exhibit decay but instead show growth in time. The following Sobolev estimate was first established by Müller and Thiele [24] on the group model of the real hyperbolic space, namely the $ax+b$ group, which corresponds to the case where the nilpotent part N is abelian and thus represents the simplest instance of a Damek–Ricci space. It was later extended to the full class of Damek–Ricci spaces by Müller and Vallarino [25].

Theorem 1.1 (Müller&Vallarino [25])**.** *Let S be a Damek–Ricci space of dimension n, and let $\mathcal{L}$ denote the left-invariant distinguished Laplacian and $\mathrm{d}\rho$ the right Haar measure on S.*

2020 *Mathematics Subject Classification.* 43A85, 22E30, 42B15, 35L20.

Key words and phrases. Wave equation, Damek–Ricci space, sharp L^p-estimate, Hardy space, Fourier integral operators.

Then, for all $1 < p < \infty$, there exists $C_p > 0$ such that the solution to the Cauchy problem

$$\begin{cases} \partial_t^2 u(t,x) - \mathcal{L}u(t,x) = 0, & t \in \mathbb{R},\, x \in S, \\ u(0,t) = f(x),\, \partial_t u(t,x)|_{t=0} = g(x), \end{cases} \tag{1.1}$$

satisfies the estimate

$$\|u(t,\cdot)\|_{L^p(S,\mathrm{d}\rho)} \leq C_p \Big((1+|t|)^{2|\frac{1}{p}-\frac{1}{2}|} \|(\mathrm{Id}+\mathcal{L})^{\frac{\alpha_0}{2}} f\|_{L^p(S,\mathrm{d}\rho)} + (1+|t|)\, \|(\mathrm{Id}+\mathcal{L})^{\frac{\alpha_1}{2}} g\|_{L^p(S,\mathrm{d}\rho)} \Big), \tag{1.2}$$

for all $t \in \mathbb{R}^*$, provided that

$$\alpha_0 > (n-1)\left|\frac{1}{p} - \frac{1}{2}\right| \quad \textit{and} \quad \alpha_1 > (n-1)\left|\frac{1}{p} - \frac{1}{2}\right| - 1.$$

Müller and Thiele on the $ax+b$ group [24, Remark 8.1], and Müller and Vallarino on Damek–Ricci spaces [25, Remark 2], conjectured that estimate (1.2) should remain valid at the endpoint cases $\alpha_0 = (n-1)\,|1/p - 1/2|$ and $\alpha_1 = (n-1)\,|1/p-1/2| - 1$. This conjecture was recently confirmed on the $ax+b$ group by Wang and Yan [32]. The following result extends this conclusion to the entire class of Damek–Ricci spaces.

Theorem 1.2. *Theorem 1.1 remains valid at the endpoint cases*

$$\alpha_0 = (n-1)\left|\frac{1}{p} - \frac{1}{2}\right| \quad \textit{and} \quad \alpha_1 = (n-1)\left|\frac{1}{p} - \frac{1}{2}\right| - 1,$$

and these regularity conditions are sharp.

Let us briefly comment on Theorem 1.2. This endpoint result represents a strengthening of Theorem 1.1. In fact, we now know that estimate (1.2) holds if and only if $\alpha_0 \geq (n-1)\,|1/p-1/2|$ and $\alpha_1 \geq (n-1)\,|1/p-1/2| - 1$, which corresponds to the analogue of the related results in the Euclidean setting, see [23, 26].

However, as pointed out by Müller and Thiele [24, p. 147], the exponent $2|1/p - 1/2|$ of $(1+|t|)$ in estimate (1.2) is independent of the topological dimension n. This contrasts with the corresponding estimates of Miyachi [23] and Peral [26] in the Euclidean case, where the exponent is $(n-1)|1/p-1/2|$. We observe here that the corresponding exponent could be $(\nu-1)|1/p-1/2|$, where ν denotes the pseudo-dimension (or dimension at infinity), and $\nu = 3$ when S is a Damek–Ricci space. This natural conjecture is motivated by previously known results on dispersive estimates in similar settings, such as real hyperbolic spaces, Damek–Ricci spaces, and general noncompact symmetric spaces, where the long-time dispersive decay rate depends on the dimension at infinity rather than on the topological dimension, see, for instance, [2, 4, 5]. Establishing estimate (1.2) under sharp regularity conditions for higher rank groups would therefore be an interesting direction for further investigation on this point.

It is noteworthy that Ionescu proved in [19] a sharp result analogous to estimate (1.2) for the wave equation associated with the Laplace–Beltrami operator on noncompact symmetric spaces of rank one. In that case, the time growth on the right-hand side of the estimate is exponential rather than polynomial as in our setting. See also [25, Remark 3]. Although Damek–Ricci spaces also have exponential volume growth, we still obtain polynomial growth with respect to $|t|$ in estimate (1.2). This reflects the difference between the Laplace–Beltrami operator and the distinguished Laplacian.

Theorem 1.2 follows from an interpolation argument based on the following improved L^1-norm estimate. We denote by $\mathfrak{h}^1(S)$ the local Hardy space on S, and $\|\cdot\|_{\mathfrak{h}^1(S)}$ the corresponding norm (see Subsection 2.3 for more details).

Theorem 1.3. *Following the notation of Theorem 1.1, there exists $C > 0$ such that the solution to the Cauchy problem (1.1) satisfies, for all $t \in \mathbb{R}^*$,*

$$\|u(t,\cdot)\|_{L^1(S,\mathrm{d}\rho)} \leq C\,(1+|t|)\,\big(\|(\mathrm{Id}+\mathcal{L})^{\frac{\alpha_0}{2}} f\|_{\mathfrak{h}^1(S)} + \|(\mathrm{Id}+\mathcal{L})^{\frac{\alpha_1}{2}} g\|_{\mathfrak{h}^1(S)}\big).$$

if and only if

$$\alpha_0 \geq \frac{n-1}{2} \quad \textit{and} \quad \alpha_1 \geq \frac{n-1}{2} - 1.$$

The proof of Theorem 1.3 is based on a careful study of the corresponding multipliers of the wave propagators. It combines refined spherical analysis with techniques inspired by Peral [26] and Tao [29], and the argument relies crucially on the explicit asymptotic expansions of the spherical function and its derivatives. It is well known that extending results from the $ax+b$ group to the full class of Damek–Ricci spaces is far from automatic. The nilpotent part of the group is no longer abelian but an H-type group endowed with a more intricate algebraic structure. Moreover, such a generalization provides the first insights into the study of the higher rank situations mentioned above.

Before presenting the detailed proofs of the main theorems in Section 3, we review in Section 2 the structure of the Damek–Ricci space S and the analysis on it, where we also recall the definition of the local Hardy space on S. Throughout this paper, the notation $A \lesssim B$ between two positive quantities means that there exists a constant $C > 0$, independent of all possible variables, such that $A \leq CB$.

2. Preliminaries

In this section, we begin by reviewing the structure of the Damek–Ricci space, followed by a summary of the analysis on it. Most of these materials have already appeared in the literature. See, for instance, [1, 3, 10, 11, 13, 14, 15, 16, 21]. For the reader's convenience, we recall here some basic facts. In the last part of this section, we review the theory of the local Hardy space on S (see [25] for further details).

2.1. **Damek-Ricci spaces.**

We first define a Heisenberg-type group, and then its solvable extension, the Damek–Ricci space.

Heisenberg-type groups. Let $\mathfrak{n}$ be a real Lie algebra equipped with an inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $|\cdot|$. Let $\mathfrak{v}$ and $\mathfrak{z}$ be complementary orthogonal subspaces of $\mathfrak{n}$ such that $[\mathfrak{n},\mathfrak{z}] = \{0\}$ and $[\mathfrak{v},\mathfrak{v}] \subseteq \mathfrak{z}$. Assume that $\mathfrak{n}$ is of *Heisenberg-type* (or *H-type* for short), meaning that for every z in $\mathfrak{z}$, the linear map $J_z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_z v, v'\rangle = \langle z, [v,v']\rangle \quad \text{for all } v, v' \in \mathfrak{v}$$

is orthogonal. The connected and simply connected Lie group N whose Lie algebra is $\mathfrak{n}$ is called an H-type group. Via the exponential map, we identify N with $\mathfrak{n}$ and write

$$\begin{array}{ccc} \mathfrak{v}\times\mathfrak{z} & \to & N \\ (v,z) & \mapsto & \exp(v+z). \end{array}$$

The multiplication in N is thus given by

$$(v,z)(v',z') = \left(v+v', z+z'+\frac{1}{2}[v,v']\right) \quad \forall\, v,\, v' \in \mathfrak{v},\ \ \forall\, z,\, z' \in \mathfrak{z}.$$

Although our definitions and subsequent analysis can, with minor modifications, also be applied to the case where N is abelian, since this case has essentially already been treated in [32], we shall in this paper focus on the case where N is nonabelian. The group N is a two-step nilpotent group with the unimodular Haar measure $\mathrm{d}v\mathrm{d}z$. We denote by $Q = m_{\mathfrak{z}} + m_{\mathfrak{v}}/2$ the *homogeneous dimension* of N, where $m_{\mathfrak{z}}$ and $m_{\mathfrak{v}}$ are dimensions of $\mathfrak{z}$ and $\mathfrak{v}$, respectively.

Damek–Ricci spaces. Let H be the unit vector generating the one-dimensional subalgebra $\mathfrak{a}$, and define its adjoint action on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ by

$$\mathrm{ad}(H)v = \frac{1}{2}v \quad \forall\, v \in \mathfrak{v} \qquad \text{and} \qquad \mathrm{ad}(H)z = z \quad \forall\, z \in \mathfrak{z}.$$

We extend the inner product on $\mathfrak{n}$ to the direct sum $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ by requiring $\mathfrak{a}$ to be orthogonal to $\mathfrak{n}$. Then $\mathfrak{s}$ is a solvable Lie algebra, and the corresponding simply connected Lie group $S = N \rtimes A$ is called the *Damek–Ricci space*, where $A = \mathbb{R}_+$ acts on N by the dilation

$$\delta_a(v,z) = (a^{\frac{1}{2}}v, az) \quad \forall\, (v,z) \in N,\ \ \forall\, a \in \mathbb{R}_+.$$

The product in S is then defined by the rule

$$(v,z,a)(v',z',a') = \left(v + a^{\frac{1}{2}}v', z + az' + \frac{1}{2}a^{\frac{1}{2}}[v,v'], aa'\right) \qquad \forall\, (v,z,a),\ (v',z',a') \in S.$$

We denote by $n = m_{\mathfrak{v}} + m_{\mathfrak{z}} + 1$ the dimension of S. Recall that the group S is nonunimodular, the right and left Haar measures on S are given, respectively, by

$$\mathrm{d}\rho(v,z,a) = a^{-1}\mathrm{d}v\mathrm{d}z\mathrm{d}a \qquad \text{and} \qquad \mathrm{d}\lambda(v,z,a) = a^{-(Q+1)}\mathrm{d}v\mathrm{d}z\mathrm{d}a$$

The modular function is thus given by $\delta(v,z,a) = a^{-Q}$. Hereafter, all function spaces on S are defined with respect to the right Haar measure $\mathrm{d}\rho$, unless specified otherwise.

We equip S with the left-invariant Riemannian metric d induced by the inner product

$$\langle (v,z,a), (v',z',a') \rangle = \langle v, v' \rangle + \langle z, z' \rangle + aa'$$

on $\mathfrak{s}$. It is known that, for all $(v,z,a) \in S$,

$$\cosh^2\left(\frac{d((v,z,a),e)}{2}\right) = \left(\frac{a^{1/2}+a^{-1/2}}{2} + \frac{a^{-1/2}|v|^2}{8}\right)^2 + \frac{a^{-1}|z|^2}{4} \tag{2.1}$$

$$= \frac{\left(1 + a + \frac{|v|^2}{4}\right)^2 + |z|^2}{4a}, \tag{2.2}$$

where e denotes the identity of the group S, see, for instance, [1, (2.18)]. We shall later use the notation $R(x)$ to denote the distance between the point $x \in S$ and the identity e. Let B_r denote the ball in S centered at e with radius r. We know from [1, (1.18)] that

$$\rho(B_r) \sim \begin{cases} r^n & \text{if } 0 < r \leq 1, \\ \mathrm{e}^{Qr} & \text{if } r > 1, \end{cases}$$

which implies that the group S, equipped with the right Haar measure $\mathrm{d}\rho$, is of exponential growth.

2.2. **Analysis on Damek–Ricci spaces.** In this section, we review the essential analytical tools on the Damek–Ricci space S. We first introduce the spherical harmonic analysis on S, which serves as a key ingredient in our arguments. Then we recall the polar coordinates and differential operators on S.

Spherical harmonic analysis. A radial function on S is a function that depends only on the distance from the identity. According to [1, (1.16)], if κ is radial, then

$$\int_S \kappa \,\mathrm{d}\lambda = \int_0^\infty \kappa(r)\, A(r)\,\mathrm{d}r, \tag{2.3}$$

where

$$A(r) = 2^{m_\mathfrak{v}+m_\mathfrak{z}} \sinh^{m_\mathfrak{v}+m_\mathfrak{z}} \frac{r}{2} \, \cosh^{m_\mathfrak{z}} \frac{r}{2} \qquad \forall\, r \geq 0. \tag{2.4}$$

One easily checks that

$$A(r) \lesssim \left(\frac{r}{1+r}\right)^{n-1} \mathrm{e}^{Qr} \qquad \forall\, r \geq 0. \tag{2.5}$$

For a suitable radial function κ, one can define the spherical Fourier transform by

$$\mathscr{F}(\kappa)(\lambda) = \nu_n \int_0^\infty \kappa(r)\, \varphi_\lambda(r)\, A(r)\,\mathrm{d}r, \tag{2.6}$$

where $\nu_n = 2\pi^{n/2}\Gamma(n/2)^{-1}$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ and φ_λ denotes the *elementary spherical function*, which can be expressed as hypergeometric functions via the formula

$$\varphi_\lambda(r) = {}_2F_1\left(\frac{Q}{2} - \mathrm{i}\lambda, \frac{Q}{2} + \mathrm{i}\lambda; \frac{n}{2}; -\sinh^2 \frac{r}{2}\right), \tag{2.7}$$

according to [6, (3)], with ${}_2F_1(a,b;c;z)$ being the classical hypergeometric function given by

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t. \tag{2.8}$$

The spherical function satisfies the estimate

$$|\varphi_\lambda(r)| \leq \varphi_0(r) \lesssim (1+r)\,\mathrm{e}^{-\frac{Q}{2}r} \qquad \forall\, \lambda \in \mathbb{C},\ \forall\, r \geq 0. \tag{2.9}$$

It also satisfies an explicit asymptotic expansion, known as the Harish–Chandra expansion, which plays a crucial role in our argument. A detailed explanation is provided in Section 3.

Using the ground spherical function, we obtain the following integration formula, which was already mentioned in [25, Lemma 2.1].

Lemma 2.1. *For every radial function $\kappa \in C_c^\infty(S)$,*

$$\int_S \delta(x)^{\frac{1}{2}} \kappa(x)\,\mathrm{d}\rho(x) = \int_0^\infty \varphi_0(r)\,\kappa(r)\,A(r)\,\mathrm{d}r = \int_0^\infty \kappa(r)\,V(r)\,\mathrm{d}r,$$

where

$$V(R) \lesssim \begin{cases} R^{n-1} & \text{if } 0 < R \leq 1, \\ R\,\mathrm{e}^{\frac{Q}{2}R} & \text{if } R \geq 1. \end{cases}$$

For suitable radial functions κ on S, we also have the inversion formula of the above Fourier transform:

$$\kappa(r)=\frac{2^{m_{\mathfrak{z}}-2}\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}+1}}\int_0^\infty \mathscr{F}(\kappa)(\lambda)\,\varphi_\lambda(r)\,|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda,$$

where $\mathbf{c}$ denotes the Harish-Chandra $\mathbf{c}$-function given by

$$\mathbf{c}(\lambda)=\frac{2^{Q-\mathrm{i}\,2\lambda}\Gamma(\mathrm{i}\,2\lambda)}{\Gamma(\frac{Q}{2}+\mathrm{i}\lambda)}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{m_{\mathfrak{v}}}{4}+\frac{1}{2}+\mathrm{i}\lambda)},\tag{2.10}$$

see [1, p. 648]. It is well known that the $\mathbf{c}$-function satisfies

$$\partial_\lambda^k\mathbf{c}(\lambda)^{\pm1}=\mathrm{O}(|\lambda|^{\mp\frac{n-1}{2}-k})\qquad\forall\,k\in\mathbb{N},\ \forall\,|\lambda|\geq1.\tag{2.11}$$

In fact, we have the following asymptotic expansion of the $\mathbf{c}$-function as $|\lambda|\to\infty$:

$$\mathbf{c}(\lambda)=2^{Q-1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)\left((\mathrm{i}\lambda)^{-\frac{n-1}{2}}+O(\lambda^{-\frac{n+1}{2}})\right).\tag{2.12}$$

Polar coordinates. According to [16, Section 4], the group S can be identified with the open unit ball $\mathbb{B}:=\{(V,Z,b)\in\mathfrak{v}\times\mathfrak{z}\times\mathbb{R}:|V|^2+|Z|^2+b^2<1\}$ in $\mathfrak{s}$ via the map $F:S\to\mathbb{B}$ defined by

$$F(v,z,a):=\frac{1}{\left(1+a+\frac{|v|^2}{4}\right)^2+|z|^2}\left(\left(1+a+\frac{|v|^2}{4}-J_z\right)v,2z,-1+\left(a+\frac{|v|^2}{4}\right)^2+|z|^2\right).$$

The inverse map $F^{-1}:\mathbb{B}\to S$ is given by

$$F^{-1}(V,Z,b)=\frac{1}{(1-b)^2+|Z|^2}\big(2(1-b+J_Z)V,2Z,1-r^2\big),\tag{2.13}$$

where $r^2=|V|^2+|Z|^2+b^2$. It follows from (2.1) that

$$|F(v,z,a)|^2=1-\frac{4a}{\left(1+a+\frac{|v|^2}{4}\right)^2+|z|^2}=\tanh^2\frac{R(v,z,a)}{2},$$

where $R(v,z,a)$ denotes the distance between the point $(v,z,a)\in S$ and the identity. The polar coordinates on S are induced by those on $\mathbb{B}$ via the map F^{-1} and are therefore given by

$$x(R,\omega)=F^{-1}\left(\left(\tanh\frac{R}{2}\right)\omega\right),\quad(R,\omega)\in\mathbb{R}_+\times\mathbb{S}^{n-1}.\tag{2.14}$$

Under the polar coordinates, one has the following integration formula:

$$\int_S f(x)\,\mathrm{d}\lambda(x)=\int_0^\infty\left(\int_{\mathbb{S}^{n-1}}f(x(r,\omega))\,\mathrm{d}\omega\right)A(r)\,\mathrm{d}r,\tag{2.15}$$

where $A(r)$ is the volume element given by (2.4). The spherical function given by (2.7) is of the form

$$\varphi_\lambda(r)=\frac{1}{\nu_n}\int_{\mathbb{S}^{n-1}}\delta(x(r,\omega))^{\mathrm{i}\frac{\lambda}{Q}-\frac{1}{2}}\,\mathrm{d}\omega.\tag{2.16}$$

The next lemma is vital to the proof of our main result.

Lemma 2.2. *Let $\ell = 0, 1$. Then, we have,*

$$\int_{\mathbb{S}^{n-1}} \Big| \partial_R^{\ell} \Big[\delta(x(R,\omega))^{-\frac{1}{2}} \Big] \Big| \, \mathrm{d}\omega \lesssim \begin{cases} 1 & \text{if } 0 < R \le 1, \\ R\, \mathrm{e}^{-\frac{Q}{2}R} & \text{if } R > 1. \end{cases}$$

Proof. It follows from (2.16) and (2.9) that

$$\int_{\mathbb{S}^{n-1}} \delta(x(R,\omega))^{-\frac{1}{2}} \, \mathrm{d}\omega = \varphi_0(R) \lesssim (1+R)\, \mathrm{e}^{-\frac{Q}{2}R}, \tag{2.17}$$

which proves the lemma for $\ell = 0$.

Now, let us show that

$$\mathbf{I}(R) := \int_{\mathbb{S}^{n-1}} \Big| \partial_R \Big[\delta(x(R,\omega))^{-\frac{1}{2}} \Big] \Big| \, \mathrm{d}\omega \lesssim (1+R)\, \mathrm{e}^{-\frac{Q}{2}R}. \tag{2.18}$$

In view of (2.13), if $\omega = (V, Z, b) \in \mathbb{S}^{n-1}$, then

$$\begin{aligned} \Big| \partial_R \Big[\delta(x(R,\omega))^{-\frac{1}{2}} \Big] \Big| &= \left| \partial_R \left[\left(\frac{1 - \tanh^2 \frac{R}{2}}{\left(1 - b \tanh \frac{R}{2}\right)^2 + |Z|^2 \tanh^2 \frac{R}{2}} \right)^{\frac{Q}{2}} \right] \right| \\ &\lesssim \left(\tanh \frac{R}{2} \right) \delta(x(R,\omega))^{-\frac{1}{2}} + \frac{\cosh^{-Q-2} \frac{R}{2}}{\left(\left(1 - b \tanh \frac{R}{2}\right)^2 + |Z|^2 \tanh^2 \frac{R}{2} \right)^{\frac{Q+1}{2}}}. \end{aligned}$$

Hence

$$\mathbf{I}(R) \lesssim \underbrace{\int_{\mathbb{S}^{n-1}} \delta(x(R,\omega))^{-\frac{1}{2}} \, \mathrm{d}\omega}_{\mathbf{I}_1(R)} + \underbrace{\int_{\mathbb{S}^{n-1}} \frac{\cosh^{-Q-2} \frac{R}{2}}{\left(\left(1 - b \tanh \frac{R}{2}\right)^2 + |Z|^2 \tanh^2 \frac{R}{2} \right)^{\frac{Q+1}{2}}} \, \mathrm{d}\omega}_{\mathbf{I}_2(R)}.$$

Clearly by (2.17),

$$\mathbf{I}_1(R) \lesssim (1+R)\, \mathrm{e}^{-\frac{Q}{2}R}.$$

Therefore it remains to show that $\mathbf{I}_2(R)$ satisfies the same estimate as above. If $0 < R \le 1$, the integrand inside $\mathbf{I}_2(R)$ is uniformly bounded. Otherwise, by the same change-of-variable argument in the proof of [16, Theorem 5.12], we have

$$\begin{aligned} \mathbf{I}_2(R) &= \cosh^{-Q-2} \frac{R}{2} \int_N \left[\left(1 + \tanh \frac{R}{2} + \frac{|v|^2}{2} + \left(1 - \tanh \frac{R}{2} \right) \left(\frac{|v|^4}{16} + |z|^2 \right) \right)^2 \right. \\ &\qquad \left. + 4|z|^2 \tanh^2 \frac{R}{2} \right]^{-\frac{Q+1}{2}} \left(\left(1 + \frac{|v|^2}{4} \right)^2 + |z|^2 \right) \mathrm{d}v \mathrm{d}z \\ &\lesssim \mathrm{e}^{-\frac{Q+2}{2}R} \int_N \left(1 + |v|^2 + |z| + \mathrm{e}^{-2R} (|v|^4 + |z|^2) \right)^{-(Q+1)} (1 + |v|^4 + |z|^2) \, \mathrm{d}v \mathrm{d}z. \end{aligned}$$

Since there exists a constant $C > 0$ such that the last integral is bounded by $C\mathrm{e}^R$ for all $R > 1$, the proof is therefore completed. ∎

Distinguished Laplacian and spectral multipliers. Let $\{X_0,\dots,X_{n-1}\}$ be an orthonormal basis of the Lie algebra $\mathfrak{s}$, where $X_0=H$, the vectors $X_1,\dots,X_{m_\mathfrak{v}}$ form an orthonormal basis of $\mathfrak{v}$ and $X_{m_\mathfrak{v}+1},\dots,X_{n-1}$ form an orthonormal basis of $\mathfrak{z}$. As usual, we shall identify an element $X\in\mathfrak{s}$ with the corresponding left-invariant differential operator on S given by the Lie derivative $Xf(x)=\frac{\mathrm{d}}{\mathrm{d}t}f(x\exp tX)\big|_{t=0}$. Viewing $X_0,X_1,\dots,X_{n-1}$ in this way as left-invariant vector fields, the Riemannian gradient ∇ on S is defined by

$$\nabla f=\sum_{i=0}^{n-1}(X_i f)\,X_i, \tag{2.19}$$

for any function f in $C^\infty(S)$. Its Riemannian norm, denoted by $|\nabla f|_{\mathbf{g}}$, is given by

$$|\nabla f|_{\mathbf{g}}=\left(\sum_{i=0}^{n-1}|X_i f|^2\right)^{\frac12}.$$

The distinguished Laplacian is the left-invariant operator $\mathcal{L}$ defined by

$$\mathcal{L}=-\sum_{i=0}^{n-1}X_i^2. \tag{2.20}$$

It is self-adjoint on $C_c^\infty(S)\subset L^2(S)$. Let Δ_S be the Laplace–Beltrami operator on S, and denote by Δ_Q the shifted Laplacian $-\Delta_S-Q^2/4$. Recall that the spectra of Δ_Q on $L^2(S,\mathrm{d}\lambda)$ and $\mathcal{L}$ on $L^2(S,\mathrm{d}\rho)$ are both $[0,\infty)$. In particular, we know from [7, Proposition 2] that

$$\mathcal{L}f=\delta^{\frac12}\,\Delta_Q(\delta^{-\frac12}f),$$

for smooth radial functions f on S. It follows from the spectral theorem that, for each bounded Borel measurable function ψ on $\mathbb{R}_+$, both operators $\psi(\sqrt{\Delta_Q})$ and $\psi(\sqrt{\mathcal{L}})$ are L^2-bounded, and satisfy

$$\psi(\sqrt{\mathcal{L}})f=\delta^{\frac12}\psi(\sqrt{\Delta_Q})(\delta^{-\frac12}f),$$

for all radial functions f in $C_c^\infty(S)$.

Let k_ψ and κ_ψ be the convolution kernel of $\psi(\sqrt{\mathcal{L}})$ and $\psi(\sqrt{\Delta_Q})$, respectively, that is

$$\psi(\sqrt{\mathcal{L}})(f)=f*k_\psi\quad\text{and}\quad\psi(\sqrt{\Delta_Q})(f)=f*\kappa_\psi,$$

where "$*$" denotes the convolution on S defined by

$$(f*g)(x)=\int_S f(xy)\,g(y^{-1})\,\mathrm{d}\lambda(y), \tag{2.21}$$

for all functions f and g in $C_c(S)$. According to [25, Proposition 3.1], the kernel κ_ψ is radial and given by

$$\kappa_\psi(x)=\frac{2^{m_\mathfrak{z}-2}\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}+1}}\int_0^\infty\psi(\lambda)\,\varphi_\lambda(R(x))\,|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda. \tag{2.22}$$

Recall that $R(x)$ denotes the distance between the point $x\in S$ and the identity. We deduce from [25, Proposition 3.1] that

$$k_\psi(x)=\delta(x)^{\frac12}\kappa_\psi(x)=\frac{2^{m_\mathfrak{z}-2}\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}+1}}\delta(x)^{\frac12}\int_0^\infty\psi(\lambda)\,\varphi_\lambda(R(x))\,|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda. \tag{2.23}$$

2.3. **The local Hardy space.** We recall in the section the definition of the local atomic Hardy space $\mathfrak{h}^1(S)$, which can be viewed as the analogue, in the setting of Damek–Ricci spaces, of the local Hardy space introduced by Goldberg in the Euclidean case [18]. The local Hardy space was further developed and studied by Meda and Volpi [21] and by Taylor [31] in more general contexts. It is straightforward to verify that Damek–Ricci spaces satisfy the geometric assumptions of [21] and [31], so that the corresponding theory applies to our setting.

A function a in $L^1(S)$ is called a *standard* $\mathfrak{h}^1$*-atom* if a is supported in a ball B of radius less than 1, and satisfies

(i) size condition: $\|a\|_{L^2(S)} \leq \rho(B)^{-1/2}$;
(ii) cancellation condition: $\int a \,\mathrm{d}\rho = 0$.

A *global* $\mathfrak{h}^1$*-atom* is a function a in $L^1(S)$ supported in a ball B of radius 1 such that $\|a\|_{L^2(S)} \leq \rho(B)^{-1/2}$. Standard and global $\mathfrak{h}^1$-atoms will be referred to as *admissible atoms.* The Hardy space $\mathfrak{h}^1(S)$ is the space of functions f in $L^1(S)$ such that $f = \sum_j c_j a_j$, where $\sum_j |c_j| < \infty$ and a_j are admissible atoms. The norm $\|f\|_{\mathfrak{h}^1(S)}$ is defined as the infimum of $\sum_j |c_j| < \infty$ over all atomic decompositions of f. We now introduce several technical lemmas that will be useful in the subsequent analysis.

Lemma 2.3. *Let T be a bounded linear operator on $L^2(S)$ with Schwartz kernel K_T, which satisfies the following two conditions:*

(i) The local Hörmonder condition:

$$\sup_{B} \sup_{y,y' \in B} \int_{(2B)^c} |K_T(x,y) - K_T(x,y')| \,\mathrm{d}\rho(x) < \infty,$$

where the first supremum is taken over all balls B of radii at most 1;

(ii) The size condition:

$$\sup_{y \in S} \int_{(B(y,2))^c} |K_T(x,y)| \,\mathrm{d}\rho(x) < \infty.$$

Then, the operator T is bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$.

Proof. The proof is a combination of [9, Theorem 8.2] and [8, Proposition 4.5 (ii)]. □

Recall that the symbol class is defined by

$$\mathcal{S}^\sigma = \left\{ m \in C^\infty(\mathbb{R}) : \|m\|_{\mathcal{S}^\sigma,k} := \sup_{\lambda \in \mathbb{R}} (1+\lambda^2)^{-\frac{\sigma-k}{2}} |m^{(k)}(\lambda)| \leq C_{\alpha,k} \text{ for all } k \in \mathbb{N} \right\}.$$

Lemma 2.4. *Suppose that $m \in \mathcal{S}^\sigma$ is an even symbol. Then the following statements hold.*

(i) If $\sigma = 0$, then the multiplier $m(\mathcal{L})$ is bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$, and is bounded on $L^p(S)$ for all $1 < p < \infty$.

(ii) If $\sigma = -1$, then the operator $|\nabla m(\mathcal{L}) \cdot|_{\mathbf{g}}$, where ∇ denotes the Riemannian gradient on S, is bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$, and from $L^p(S)$ to $L^p(S)$ for all $1 < p < \infty$.

Proof. We briefly outline the proof of this lemma. Following the argument of [32, Lemma A.1], we obtain that the Schwartz kernel K of any operator in (i) or (ii) satisfies

$$|K(x,y)| + |(\nabla K(x,\cdot))(y)|_{\mathbf{g}} \lesssim \delta(xy)^{\frac{1}{2}} (d(x,y))^{-N} \quad \text{if } d(x,y) \geq 1,$$

for any $N \in \mathbb{N}^*$, and

$$|(\nabla K(x,\cdot))(y)|_{\mathbf{g}} \lesssim \delta(xy)^{\frac{1}{2}} (d(x,y))^{-n-1} \quad \text{if } 0 < d(x,y) \leq 1.$$

Using Lemma 2.1 and reasoning as in the proof of [32, Lemma 2.11], we verify that both the local Hörmander condition and the size condition are satisfied. Finally, a standard interpolation argument yields the L^p-boundedness. $\square$

We conclude this section by recalling the following analytic interpolation property.

Lemma 2.5. *Denote by Σ the closed strip $\{\zeta \in \mathbb{C} : \mathrm{Re}\zeta \in [0,1]\}$. Suppose that $\{T_\zeta\}_{\zeta\in\Sigma}$ is a family of uniformly bounded operators on $L^2(S)$ such that the map $\zeta \mapsto \int_G T_\zeta(f)\, g\, \mathrm{d}\rho$ is continuous on Σ and analytic in the interior of Σ, whenever $f, g \in L^2(S)$. Moreover, assume that*

$$\|T_\zeta f\|_{L^1(S)} \leq A_1 \|f\|_{\mathfrak{h}^1(S)} \quad \text{for all } f \in L^2(S) \cap \mathfrak{h}^1(S) \text{ and } \mathrm{Re}\zeta = 0,$$

and

$$\|T_\zeta f\|_{L^2(S)} \leq A_2 \|f\|_{L^2(S)} \quad \text{for all } f \in L^2(S) \text{ and } \mathrm{Re}\zeta = 1,$$

where A_1 and A_2 are positive constants independent of $\mathrm{Im}\zeta \in \mathbb{R}$. Then for every $\vartheta \in (0,1)$ the operator T_ϑ is bounded on L^{q_ϑ} with $1/q_\vartheta = 1 - \vartheta/2$ and

$$\|T_\vartheta f\|_{L^{q_\vartheta}(S)} \leq C\, A_1^{1-\vartheta} A_2^{\vartheta} \|f\|_{L^{q_\vartheta}(S)} \quad \text{for all } f \in L^2(S) \cap L^{q_\vartheta}(S).$$

Proof. This follows from [21, Theorem 5 (ii)] and [12, Theorem 1]. $\square$

3. Proof of the main theorems

We prove our main theorems in this section, focusing on Theorem 1.3, since Theorem 1.2 follows from an interpolation argument. The proof of the necessity part of Theorem 1.3 is a straightforward adaptation of the argument for the $ax + b$ group given in [32], and will therefore be omitted. The sufficiency part of Theorem 1.3 follows from the results below concerning the corresponding multipliers.

Proposition 3.1. *Let m be an even symbol. Then, for all $t \in \mathbb{R}^*$, the following results hold.*

(a) *If $m \in \mathcal{S}^{-(n-1)/2}$, then*

$$\|m(\sqrt{\mathcal{L}})\, \cos(t\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \lesssim 1 + |t|.$$

(b) *If $m \in \mathcal{S}^{-(n-1)/2+1}$, then*

$$\left\| m(\sqrt{\mathcal{L}})\, \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} \right\|_{\mathfrak{h}^1(S)\to L^1(S)} \lesssim 1 + |t|.$$

The proof of Proposition 3.1 relies on the idea, originating from the critical case on Euclidean space, of relating the wave propagator to the spherical measure. See, for instance, [26, 29]. In our setting, the spherical measure is closely related to the spherical function given by (2.7), which therefore requires a detailed analysis. The following expression will be useful later on.

Lemma 3.2. *Let f be a suitable function on S. Then, for all $t > 0$ and $z \in S$, we have*

$$\varphi_{\sqrt{\mathcal{L}}}(t)(f)(z) = \frac{1}{\nu_n} \int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac{1}{2}} f(z \cdot x(t,\omega))\, \mathrm{d}\omega, \tag{3.1}$$

where ν_n is the surface area of the unit sphere $\mathbb{S}^{n-1}$, δ is the modular function on S, and $x(t,\omega)$ is the polar coordinate given by (2.14).

Proof. Let $A(t)$ be the volume element given by (2.4), and μ_t be the Dirac mass on $\mathbb{R}$ centered at t. We denote by $\mathrm{d}\sigma_t(r) = \nu_n^{-1}A(t)^{-1}\mu_t(r)$ the normalized spherical measure on S. Note that $\mathscr{F}(\mathrm{d}\sigma_t) = \varphi_\lambda(t)$. It follows from (2.23) that

$$\varphi_{\sqrt{\mathcal{L}}}(t)(f)(z) = \big(f * (\delta^{\frac{1}{2}}\mathrm{d}\sigma_t)\big)(z).$$

Combining with (2.21), (2.15) and the fact that the modular function δ defines a group homomorphism from S to $\mathbb{R}_+^*$, we obtain

$$\begin{aligned}\varphi_{\sqrt{\mathcal{L}}}(t)(f)(z) &= \frac{1}{\nu_n}\frac{1}{A(t)}\int_S f(z\cdot x)\,\delta(x^{-1})^{\frac{1}{2}}\mu_t(R(x^{-1}))\,\mathrm{d}\lambda(x)\\ &= \frac{1}{\nu_n}\frac{1}{A(t)}\int_0^\infty\left(\int_{\mathbb{S}^{n-1}}\delta(x(r,\omega))^{-\frac{1}{2}}f(z\cdot x(r,\omega))\,\mathrm{d}\omega\right)\mu_t(r)\,A(r)\,\mathrm{d}r\\ &= \frac{1}{\nu_n}\int_{\mathbb{S}^{n-1}}\delta(x(t,\omega))^{-\frac{1}{2}}f(z\cdot x(t,\omega))\,\mathrm{d}\omega,\end{aligned}$$

as expected. □

We now proceed to the proof of Proposition 3.1. The argument is divided into two parts, corresponding to the short-time and long-time cases. Both require explicit asymptotic expansions of the spherical function and its derivative. We begin with the large-time analysis, for which the following lemma will be needed.

Lemma 3.3. *Let $A(t)$ be the volume element given by (2.4). Suppose that $t > 1$ and $\lambda \geq 1$. Then the following results hold.*

(a) *There exist symbols $a_t \in \mathcal{S}^{-(n+1)/2}$, uniformly in $t > 1$, such that*

$$\varphi_\lambda(t) = c_n\,A(t)^{-\frac{1}{2}}\lambda^{-\frac{n-1}{2}}\cos\left(\lambda t - \frac{n-1}{4}\pi\right) + \mathfrak{r}_t(\lambda),$$

where $c_n = 2^{(n-1)/2}\pi^{-1/2}\Gamma(n/2)$ and

$$\mathfrak{r}_t(\lambda) = A(t)^{-\frac{1}{2}}\left(\mathrm{e}^{\mathrm{i}\lambda t}a_t(\lambda) + \mathrm{e}^{-\mathrm{i}\lambda t}a_t(-\lambda)\right).$$

(b) *There exist symbols $b_t \in \mathcal{S}^{-(n-1)/2}$, uniformly in $t > 1$, such that*

$$\varphi_\lambda'(t) = -c_n\,A(t)^{-\frac{1}{2}}\lambda^{-\frac{n-3}{2}}\sin\left(\lambda t - \frac{n-1}{4}\pi\right) + \mathfrak{s}_t(\lambda),$$

where

$$\mathfrak{s}_t(\lambda) = A(t)^{-\frac{1}{2}}\left(\mathrm{e}^{\mathrm{i}\lambda t}b_t(\lambda) + \mathrm{e}^{-\mathrm{i}\lambda t}b_t(-\lambda)\right).$$

Proof. This lemma is a consequence of the well-known Harish-Chandra expansion of the spherical function. It follows from [3, pp. 735–736] that

$$\varphi_\lambda(t) = 2^{-\frac{m_{\mathfrak{z}}}{2}}A(t)^{-\frac{1}{2}}\left(\mathrm{e}^{\mathrm{i}\lambda t}\mathbf{c}(\lambda)\sum_{\ell=0}^\infty\Gamma_\ell(\lambda)\,\mathrm{e}^{-\ell t} + \mathrm{e}^{-\mathrm{i}\lambda t}\mathbf{c}(-\lambda)\sum_{\ell=0}^\infty\Gamma_\ell(-\lambda)\,\mathrm{e}^{-\ell t}\right), \tag{3.2}$$

where the $\mathbf{c}$-function is given by (2.10). The coefficients Γ_ℓ satisfy $\Gamma_0 \equiv 1$, and for each $\ell \in \mathbb{N}^*$ and $k \in \mathbb{N}$,

$$|\partial_\lambda^k\Gamma_\ell(\lambda)| \leq C\ell^d(1+|\lambda|)^{-k-1} \qquad \forall\,\lambda\in\mathbb{R}^*, \tag{3.3}$$

for some constants $C > 0$ and $d \geq 1$, see [3, Lemma 1]. We rewrite (3.2) as

$$\varphi_\lambda(t) = 2^{-\frac{m_{\mathfrak{z}}}{2}}A(t)^{-\frac{1}{2}}\left(\mathrm{e}^{\mathrm{i}\lambda t}\mathbf{c}(\lambda) + \mathrm{e}^{-\mathrm{i}\lambda t}\mathbf{c}(-\lambda)\right) + A(t)^{-\frac{1}{2}}\left(\mathrm{e}^{\mathrm{i}\lambda t}a_t(\lambda) + \mathrm{e}^{-\mathrm{i}\lambda t}a_t(-\lambda)\right), \tag{3.4}$$

where

$$a_t(\lambda)=2^{-\frac{m_{\mathfrak{z}}}{2}}\mathbf{c}(\lambda)\sum_{\ell=1}^{\infty}\Gamma_\ell(\lambda)\,\mathrm{e}^{-\ell t} \tag{3.5}$$

is a symbol in $S^{-(n+1)/2}$, according to (2.11) and (3.3). We then prove (a) by substituting (2.12) into (3.4).

To prove (b), we differentiate both sides of (3.4) with respect to t, which yields

$$\begin{aligned}\varphi_\lambda'(t)&=\mathrm{i}\lambda\,2^{-\frac{m_{\mathfrak{z}}}{2}}A(t)^{-\frac12}\left(\mathrm{e}^{\mathrm{i}\lambda t}\mathbf{c}(\lambda)-\mathrm{e}^{-\mathrm{i}\lambda t}\mathbf{c}(-\lambda)\right)\\&\quad-2^{-\frac{m_{\mathfrak{z}}}{2}-1}A(t)^{-\frac12}\frac{A'(t)}{A(t)}\left(\mathrm{e}^{\mathrm{i}\lambda t}\mathbf{c}(\lambda)+\mathrm{e}^{-\mathrm{i}\lambda t}\mathbf{c}(-\lambda)+\mathrm{e}^{\mathrm{i}\lambda t}a_t(\lambda)+\mathrm{e}^{-\mathrm{i}\lambda t}a_t(-\lambda)\right)\\&\quad+2^{-\frac{m_{\mathfrak{z}}}{2}}A(t)^{-\frac12}\left(\mathrm{e}^{\mathrm{i}\lambda t}\partial_t(a_t)(\lambda)+\mathrm{e}^{-\mathrm{i}\lambda t}\partial_t(a_t)(-\lambda)\right).\end{aligned}$$

The middle term is easily handled, since $A'(t)/A(t)$ remains bounded for all $t>1$. By (3.5) and (3.3), the last term satisfies

$$\partial_t(a_t)(\lambda)=-2^{-\frac{m_{\mathfrak{z}}}{2}}\mathbf{c}(\lambda)\sum_{\ell=1}^{\infty}\ell\,\Gamma_\ell(\lambda)\,\mathrm{e}^{-\ell t}=\mathrm{O}(\lambda^{-\frac{n-1}{2}})\qquad\forall\,\lambda>1.$$

Therefore, there exist some symbols $b_t\in\mathcal{S}^{-(n-1)/2}$, uniformly in $t>1$, such that

$$\varphi_\lambda'(t)=\mathrm{i}\lambda\,2^{-\frac{m_{\mathfrak{z}}}{2}}A(t)^{-\frac12}\left(\mathrm{e}^{\mathrm{i}\lambda t}\mathbf{c}(\lambda)-\mathrm{e}^{-\mathrm{i}\lambda t}\mathbf{c}(-\lambda)\right)+A(t)^{-\frac12}\left(\mathrm{e}^{\mathrm{i}\lambda t}b_t(\lambda)+\mathrm{e}^{-\mathrm{i}\lambda t}b_t(-\lambda)\right). \tag{3.6}$$

We conclude the proof by substituting (2.11) into (3.6). □

Let us now turn to the proof of Theorem 3.1 in the large-time case.

Proof of Proposition 3.1 for $|t|>1$. Without loss of generality, we may assume that $t>1$. Let $\eta\in C_c^\infty(\mathbb{R})$ be an even cut-off function such that $\eta\equiv1$ on $[-1,1]$. To prove part (a), let us assume that m is an even symbol in $S^{-(n-1)/2}$. It follows from [25, Theorem 6.1 (i)] that

$$\|\eta(\sqrt{\mathcal{L}})\,m(\sqrt{\mathcal{L}})\,\cos(t\sqrt{\mathcal{L}})\|_{L^1(S)\to L^1(S)}\le C\,t,$$

for some constant $C>0$ independent of t. Therefore, it remains to show that

$$\|(1-\eta(\sqrt{\mathcal{L}}))\,m(\sqrt{\mathcal{L}})\,\cos(t\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)}\le C\,t. \tag{3.7}$$

To prove (3.7), we combine the elementary identity

$$\cos(\lambda t)=\cos\Big(\lambda t-\frac{n-1}{4}\pi\Big)\cos\Big(\frac{n-1}{4}\pi\Big)-\sin\Big(\lambda t-\frac{n-1}{4}\pi\Big)\sin\Big(\frac{n-1}{4}\pi\Big) \tag{3.8}$$

with Lemma 3.3 to obtain the following crucial observation: there exist constants c_1, $c_2>0$, independent of $t>1$, and symbols $a_{1,t}(\lambda)$, $a_{2,t}(\lambda)\in\mathcal{S}^{-1}$, uniformly in $t>1$, such that for all $\lambda\in\operatorname{supp}(1-\eta)$,

$$\cos(\lambda t)=c_1\,A(t)^{\frac12}\lambda^{\frac{n-1}{2}}\varphi_\lambda(t)+c_2\,A(t)^{\frac12}\lambda^{\frac{n-3}{2}}\varphi_\lambda'(t)+\mathrm{e}^{\mathrm{i}\lambda t}a_{1,t}(\lambda)+\mathrm{e}^{-\mathrm{i}\lambda t}a_{2,t}(\lambda),$$

where $A(r)$ is the volume element given by (2.4). Therefore, for such λ, we have

$$m(\lambda)\cos(\lambda t)=c_1\,A(t)^{\frac12}m_1(\lambda)\,\varphi_\lambda(t)+c_2\,A(t)^{\frac12}m_2(\lambda)\,\varphi_\lambda'(t)+\mathrm{e}^{\mathrm{i}\lambda t}b_{1,t}(\lambda)+\mathrm{e}^{-\mathrm{i}\lambda t}b_{2,t}(\lambda) \tag{3.9}$$

with $m_1 \in \mathcal{S}^0$, $m_2 \in \mathcal{S}^{-1}$, and $b_{1,t}(\lambda), b_{2,t}(\lambda) \in \mathcal{S}^{-1}$ uniformly in $t > 1$. Note that, on $\operatorname{supp}(1-\eta)$, we can rewrite the remainder terms as

$$B_t(\lambda) = \mathrm{e}^{\mathrm{i}\lambda t} b_{1,t}(\lambda) + \mathrm{e}^{-\mathrm{i}\lambda t} b_{2,t}(\lambda) = \widetilde{b}_{1,t}(\lambda)\cos(\lambda t) + \widetilde{b}_{2,t}(\lambda)\frac{\sin(\lambda t)}{\lambda},$$

with $\widetilde{b}_{1,t} \in \mathcal{S}^{-(n+1)/2}$ and $\widetilde{b}_{2,t} \in \mathcal{S}^{-(n-1)/2}$ uniformly in $t > 1$. We then deduce from [25, Theorem 6.1] that

$$\|(1-\eta(\sqrt{\mathcal{L}}))\, B_t(\sqrt{\mathcal{L}})\|_{L^1(S)\to L^1(S)} \le C\, t.$$

We now consider the principal terms in (3.9). Let us set

$$M_{1,t}(\lambda) := A(t)^{\frac12}\varphi_\lambda(t)\,\widetilde{m}_1(\lambda) \qquad \text{and} \qquad M_{2,t}(\lambda) := A(t)^{\frac12}\varphi_\lambda'(t)\,\widetilde{m}_2(\lambda),$$

with

$$\widetilde{m}_1(\lambda) := (1-\eta(\lambda))\, m_1(\lambda) \in \mathcal{S}^0 \qquad \text{and} \qquad \widetilde{m}_2(\lambda) := (1-\eta(\lambda))\, m_2(\lambda) \in \mathcal{S}^{-1}.$$

Therefore, the proof of (3.7) reduces to showing that

$$\|M_{j,t}(\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \le C\, t, \quad j = 1,2. \tag{3.10}$$

In the case where $j = 1$, we know, on the one hand, from Lemma 2.4 (i) that the operator $\widetilde{m}_1(\sqrt{\mathcal{L}})$ is bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$. On the other hand, combining (3.1) with Lemma 2.2, we have

$$\|A(t)^{\frac12}\varphi_{\sqrt{\mathcal{L}}}(t)\|_{L^1(S)\to L^1(S)} \le C\,\mathrm{e}^{\frac{Q}{2}t}\int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac12}\,\mathrm{d}\omega \le C\, t,$$

from where we obtain

$$\|M_{1,t}(\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \le \|A(t)^{\frac12}\varphi_{\sqrt{\mathcal{L}}}(t)\|_{L^1(S)\to L^1(S)}\|\widetilde{m}_1(\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \le C\, t.$$

For $j = 2$, we apply (3.1) to write

$$\begin{aligned} M_{2,t}(\sqrt{\mathcal{L}})(f)(z) &= A(t)^{\frac12}\partial_t\Big[\varphi_{\sqrt{\mathcal{L}}}(t)\,\widetilde{m}_2(\sqrt{\mathcal{L}})(f)(z)\Big] \\ &= \frac{1}{\nu_n}A(t)^{\frac12}\partial_t\Big[\int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac12}\widetilde{m}_2(\sqrt{\mathcal{L}})(f)(z\cdot x(t,\omega))\,\mathrm{d}\omega\Big]. \end{aligned}$$

We then have $M_{2,t}(\sqrt{\mathcal{L}})(f)(z) = T_{1,t}(f)(z) + T_{2,t}(f)(z)$, with

$$T_{1,t}(f)(z) = \frac{1}{\nu_n}A(t)^{\frac12}\int_{\mathbb{S}^{n-1}} \partial_t[\delta(x(t,\omega))^{-\frac12}]\,\widetilde{m}_2(\sqrt{\mathcal{L}})(f)(z\cdot x(t,\omega))\,\mathrm{d}\omega,$$

and

$$T_{2,t}(f)(z) = \frac{1}{\nu_n}A(t)^{\frac12}\int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac12}\langle \nabla\widetilde{m}_2(\sqrt{\mathcal{L}})(f)(z\cdot x(t,\omega)), \partial_t[z\cdot x(t,\omega)]\rangle_{\mathbf{g}}\,\mathrm{d}\omega.$$

Here $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ denotes the Riemannian metric on S and ∇ is the gradient given by (2.19).

On the one hand, we deduce from Lemmas 2.2 and 2.4 (i) that

$$\|T_{1,t}\|_{\mathfrak{h}^1(S)\to L^1(S)} \le C\left(\mathrm{e}^{\frac{Q}{2}t}\int_{\mathbb{S}^{n-1}} |\partial_t[\delta(x(t,\omega))^{-\frac12}]|\,\mathrm{d}\omega\right)\|\widetilde{m}_2(\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \le C\, t,$$

that is, $T_{1,t}$ is bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$. On the other hand, we know that the factor $|\partial_t[z\cdot x(t,\omega)]|_{\mathbf{g}}\equiv 1$, since $z\cdot x(\cdot,\omega)$ is a geodesic on S. Therefore, by successively applying the Cauchy–Schwarz inequality, and Lemmas 2.2 and 2.4(ii), we obtain

$$\|T_{2,t}(f)\|_{L^1(S)}\leq C\,\left(\mathrm{e}^{\frac{Q}{2}t}\int_{\mathbb{S}^{n-1}}\delta(x(t,\omega))^{-\frac12}\,\mathrm{d}\omega\right)\||\nabla\widetilde{m}_2(\sqrt{\mathcal{L}})(f)|_{\mathbf{g}}\|_{L^1(S)}\leq C\,t\,\|f\|_{\mathfrak{h}^1(S)}.$$

In conclusion, we have proved that (3.10) holds for $j=1,2$. The proof of part (a) of Proposition 3.1 in the long-time case is thus completed. We omit the proof of part (b), as it follows from a similar argument. $\square$

For the short-time case, we need the following lemma, which is similar to Lemma 3.3 but involves different remainder terms. The proof relies on an expansion in terms of Bessel functions, rather than on the Harish-Chandra expansion used in the large-time case.

Lemma 3.4. *Let $A(t)$ be the volume element given by (2.4). Suppose that $0<t\leq 1$ and $\lambda>1$ are such that $\lambda t\geq 1$. Then the following results hold.*

(a) *There exist symbols $a_t\in\mathcal{S}^{-(n+1)/2}$, uniformly in $0<t\leq 1$, such that*

$$\varphi_\lambda(t)=c_n\,A(t)^{-\frac12}\lambda^{-\frac{n-1}{2}}\cos\left(\lambda t-\frac{n-1}{4}\pi\right)\,+\,\mathfrak{r}_t(\lambda),$$

where $c_n=2^{(n-1)/2}\pi^{-1/2}\Gamma(n/2)$ and

$$\mathfrak{r}_t(\lambda)=\mathrm{e}^{\mathrm{i}\lambda t}a_t(\lambda t)+\mathrm{e}^{-\mathrm{i}\lambda t}a_t(-\lambda t).$$

(b) *There exist symbols $b_t\in\mathcal{S}^{-(n-1)/2}$, uniformly in $0<t\leq 1$, such that*

$$\varphi_\lambda'(t)=-c_n\,A(t)^{-\frac12}\lambda^{-\frac{n-3}{2}}\sin\left(\lambda t-\frac{n-1}{4}\pi\right)\,+\,\mathfrak{s}_t(\lambda),$$

where

$$\mathfrak{s}_t(\lambda)=t^{-1}\left(\mathrm{e}^{\mathrm{i}\lambda t}b_t(\lambda t)+\mathrm{e}^{-\mathrm{i}\lambda t}b_t(-\lambda t)\right).$$

Proof. It follows from [6, Theorem 3.1] that, for all $0<t\leq 1$,

$$\varphi_\lambda(t)=\frac{\Gamma(\frac n2)}{\Gamma(\frac{n-1}{2})}\,t^{\frac{n-1}{2}}A(t)^{-\frac12}\sum_{\ell=0}^{\infty}2^{\frac n2-1+\ell}\Gamma\left(\frac{n-1}{2}+\ell\right)c_\ell(t)\,J_{\frac n2-1+\ell}(\lambda t)\,(\lambda t)^{-\frac n2+1-\ell}t^{2\ell},$$

where J_μ denotes the classical Bessel function of order μ, and the coefficients a_ℓ satisfy

$$c_0\equiv 1\qquad\text{and}\qquad |c_\ell(t)|\leq C\,4^{-\ell}\quad\forall\,\ell\in\mathbb{N}^*.$$

It is well known that J_μ has the asymptotic expansion

$$J_\mu(r)=\left(\frac{\pi r}{2}\right)^{-\frac12}\cos\left(r-\frac{\pi\mu}{2}-\frac{\pi}{4}\right)+\mathrm{e}^{\mathrm{i}r}s_\mu(r)+\mathrm{e}^{-\mathrm{i}r}s_\mu(-r)\quad\text{as }r\to\infty,\tag{3.11}$$

where s_μ is a symbol of order $-3/2$, see, for instance, [28, p. 356]. Hence, there exists a symbol $a_t\in\mathcal{S}^{-(n+1)/2}$, uniformly in $0<t\leq 1$, such that

$$\varphi_\lambda(t)=\frac{2^{\frac{n-1}{2}}\Gamma(\frac n2)}{\sqrt{\pi}}\,A(t)^{-\frac12}\lambda^{-\frac{n-1}{2}}\cos\left(\lambda t-\frac{n-1}{4}\pi\right)+\mathrm{e}^{\mathrm{i}\lambda t}a_t(\lambda t)+\mathrm{e}^{-\mathrm{i}\lambda t}a_t(-\lambda t).$$

Thus, we have proved (a).

For (b), we differentiate both sides of (2.7) with respect to t, and using the formula

$$\frac{\partial}{\partial z}\big[{}_2F_1(a,b;c;z)\big] = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z).$$

We obtain

$$\begin{aligned}\varphi_\lambda'(t) &= -\left[{}_2F_1\left(\frac{Q}{2}-\mathrm{i}\lambda,\frac{Q}{2}+\mathrm{i}\lambda;\frac{n}{2};\cdot\right)\right]'\left(-\sinh^2\frac{t}{2}\right)\sinh\frac{t}{2}\cosh\frac{t}{2}\\ &= \left(-\frac{Q^2+4\lambda^2}{4n}\sinh t\right){}_2F_1\left(\frac{Q+2}{2}-\mathrm{i}\lambda,\frac{Q+2}{2}+\mathrm{i}\lambda;\frac{n+2}{2};-\sinh^2\frac{t}{2}\right). \qquad (3.12)\end{aligned}$$

Another key observation is that, by (2.7), the function ${}_2F_1((Q+2)/2-\mathrm{i}\lambda,(Q+2)/2+\mathrm{i}\lambda;(n+2)/2;-\sinh^2(t/2))$ is the spherical function on the Damek–Ricci space with $\widetilde{m}_{\mathfrak{v}} = m_{\mathfrak{v}}$ and $\widetilde{m}_{\mathfrak{z}} = m_{\mathfrak{z}}+2$ so that $\widetilde{Q} = Q+2$ and $\widetilde{n} = n+2$. Therefore, we deduce from (a) and (2.4) that

$$\begin{aligned}&{}_2F_1\left(\frac{Q+2}{2}-\mathrm{i}\lambda,\frac{Q+2}{2}+\mathrm{i}\lambda;\frac{n+2}{2};-\sinh^2\frac{t}{2}\right)\\ &= c_{n+2}\,(\sinh t)^{-1}A(t)^{-\frac12}\lambda^{-\frac{n+1}{2}}\sin\left(\lambda t-\frac{n-1}{4}\pi\right)+\mathrm{e}^{\mathrm{i}\lambda t}\widetilde{a}_t(\lambda t)+\mathrm{e}^{-\mathrm{i}\lambda t}\widetilde{a}_t(-\lambda t),\end{aligned}$$

where $\widetilde{a}_t \in \mathcal{S}^{-(n+3)/2}$, uniformly in $0<t\le 1$. Substituting this equality into (3.12), we obtain

$$\varphi_\lambda'(t) = -\frac{2^{\frac{n-1}{2}}\Gamma(\frac{n}{2})}{\sqrt{\pi}}A(t)^{-\frac12}\lambda^{-\frac{n-3}{2}}\sin\left(\lambda t-\frac{n-1}{4}\pi\right)+t^{-1}\left(\mathrm{e}^{\mathrm{i}\lambda t}b_t(\lambda t)+\mathrm{e}^{-\mathrm{i}\lambda t}b_t(-\lambda t)\right),$$

for some symbols $b_t\in\mathcal{S}^{-(n-1)/2}$, uniformly in $0<t\le 1$. The proof is therefore completed. □

Let us use the above lemma to prove Proposition 3.1 in the short-time case.

Proof of Proposition 3.1 for $|t|\le 1$. We assume, without loss of generality, that $0<t\le 1$. Let us assume that m is an even symbol in $S^{-(n-1)/2}$ in order to prove part (a). Recall that $\eta\in C_c^\infty(\mathbb{R})$ is an even cut-off function satisfying $\eta\equiv 1$ on $[-1,1]$. Define

$$m_{t,0}(\lambda) := \eta(\lambda t)\,m(\lambda)\,\cos(\lambda t),$$

which also belongs to $S^{-(n-1)/2}$, and hence to S^0, uniformly in $0<t\le 1$. It follows from Lemma 2.4 (i) that the operator $m_{t,0}(\sqrt{\mathcal{L}})$ is bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$. The core of this proof therefore lies in establishing

$$\|(1-\eta(t\sqrt{\mathcal{L}}))\,m(\sqrt{\mathcal{L}})\,\cos(t\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)}\le C \qquad (3.13)$$

for some constants $C>0$ independent of t.

In view of (3.8) and Lemma 3.4, there exist constants c_1, $c_2>0$ such that, on $\mathrm{supp}(1-\eta(t\,\cdot))$,

$$\cos(\lambda t) = c_1\,A(t)^{\frac12}\lambda^{\frac{n-1}{2}}\varphi_\lambda(t)+c_2\,A(t)^{\frac12}\lambda^{\frac{n-3}{2}}\varphi_\lambda'(t)+\mathrm{e}^{\mathrm{i}\lambda t}a_{1,t}(\lambda t)+\mathrm{e}^{-\mathrm{i}\lambda t}a_{2,t}(\lambda t) \qquad (3.14)$$

with $A(t)$ given by (2.4) and $a_{1,t},a_{2,t}\in\mathcal{S}^{-1}$ uniformly in $0<t\le 1$. Note that we may write the remainder term as

$$E_t(\lambda) = \mathrm{e}^{\mathrm{i}\lambda t}a_{1,t}(\lambda t)+\mathrm{e}^{-\mathrm{i}\lambda t}a_{2,t}(\lambda t) = \cos(\lambda t)\,\widetilde{a}_{1,t}(\lambda t)+\frac{\sin(\lambda t)}{\lambda t}\widetilde{a}_{2,t}(\lambda t),$$

with $\widetilde{a}_{1,t} \in \mathcal{S}^{-1}$ and $\widetilde{a}_{2,t} \in \mathcal{S}^{0}$, uniformly in $0 < t \leq 1$.

Note that the remainder term involves only subcritical symbols, so the desired estimate can be obtained by using the method in the proof of [25, Theorem 6.1]. Following their arguments, there exists a cut-off function $\beta \in C_c^\infty(\mathbb{R}^n)$, vanishing on $[-1,1]$, such that

$$(1-\eta(\lambda t))\, m(\lambda)\, \widetilde{a}_{1,t}(\lambda t) = \sum_{\ell=[-\log t]}^{\infty} M_{\ell,t}(2^{-\ell}\lambda),$$

where

$$M_{\ell,t}(\lambda) := \beta(\lambda)\, m(2^\ell \lambda)\, \widetilde{a}_{1,t}(2^\ell \lambda t),$$

satisfies, for all $N = 1, 2, \ldots,$

$$\|M_{\ell,t}\|_{C^N} \leq C_N\, t^{-1} 2^{-\frac{n+1}{2}\ell},$$

uniformly in $\ell \geq [-\log t]$ and $0 < t \leq 1$. We deduce from [25, (6.3)] that

$$\|M_{\ell,t}(2^{-\ell}\sqrt{\mathcal{L}})\, \cos(t\sqrt{\mathcal{L}})\|_{L^1(S)\to L^1(S)} \leq C\, t^{-1} 2^{-\frac{n+1}{2}\ell} 2^{\frac{n-1}{2}\ell} = C\, t^{-1} 2^{-\ell}.$$

It follows that

$$\begin{aligned} &\|(1-\eta(t\sqrt{\mathcal{L}}))\, m(\sqrt{\mathcal{L}})\, \widetilde{a}_{1,t}(t\sqrt{\mathcal{L}})\, \cos(t\sqrt{\mathcal{L}})\|_{L^1(S)\to L^1(S)} \\ &\leq \sum_{\ell=[-\log t]}^{\infty} \|M_{\ell,t}(2^{-\ell}\sqrt{\mathcal{L}})\, \cos(t\sqrt{\mathcal{L}})\|_{L^1(S)\to L^1(S)} \leq C, \end{aligned}$$

and a similar argument as above yields

$$\left\| (1-\eta(t\sqrt{\mathcal{L}}))\, m(\sqrt{\mathcal{L}})\, \widetilde{a}_{2,t}(t\sqrt{\mathcal{L}}) \frac{\sin(t\sqrt{\mathcal{L}})}{t\sqrt{\mathcal{L}}} \right\|_{L^1(S)\to L^1(S)} \leq C.$$

The operator $(1-\eta(t\sqrt{\mathcal{L}}))m(\sqrt{\mathcal{L}})E_t(\sqrt{\mathcal{L}})$, associated with the remainder term, is therefore bounded from $L^1(S)$ to $L^1(S)$.

However, the above method does not allow us to estimate the principal terms in (3.9). Let us define

$$\widetilde{m}_{1,t}(\lambda) := \varphi_\lambda(t)\, m_{1,t}(\lambda) \quad \text{and} \quad \widetilde{m}_{2,t}(\lambda) := \varphi'_\lambda(t)\, m_{2,t}(\lambda),$$

with

$$m_{1,t}(\lambda) := (1-\eta(\lambda t))\, m(\lambda)\, \lambda^{\frac{n-1}{2}} \in \mathcal{S}^0 \quad \text{and} \quad m_{2,t}(\lambda) := (1-\eta(\lambda t))\, m(\lambda)\, \lambda^{\frac{n-3}{2}} \in \mathcal{S}^{-1}$$

uniformly in $0 < t \leq 1$. Given the estimate of the remainder term, the proof of (3.13) reduces to showing that both the operators $\widetilde{m}_{1,t}(\sqrt{\mathcal{L}})$ and $\widetilde{m}_{2,t}(\sqrt{\mathcal{L}})$ are bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$.

On the one hand, since $m_{1,t} \in \mathcal{S}^0$, we know from Lemma 2.4 (i) that the operator $m_{1,t}(\mathcal{L})$ is bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$. On the other hand, by combining (3.1) with Lemma 2.2, we obtain

$$\|\varphi_{\sqrt{\mathcal{L}}}(t)\|_{L^1(S)\to L^1(S)} \leq C \int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac{1}{2}}\, \mathrm{d}\omega \leq C.$$

Therefore, we have

$$\|\widetilde{m}_{1,t}(\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \leq \|\varphi_{\sqrt{\mathcal{L}}}(t)\|_{L^1(S)\to L^1(S)} \|m_{1,t}(\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \leq C.$$

In other words, the operator $\widetilde{m}_{1,t}(\sqrt{\mathcal{L}})$ is also bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$.

For the operator $\widetilde{m}_{2,t}(\sqrt{\mathcal{L}})$, we apply (3.1) once again to write

$$\begin{aligned}\widetilde{m}_{2,t}(\sqrt{\mathcal{L}})(f)(z) &= \partial_t\Big[\varphi_{\sqrt{\mathcal{L}}}(t)\, m_{2,t}(\sqrt{\mathcal{L}})(f)(z)\Big]\\ &= \frac{1}{\nu_n}\partial_t\left[\int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac{1}{2}} m_{2,t}(\sqrt{\mathcal{L}})(f)(z\cdot x(t,\omega))\,\mathrm{d}\omega\right],\end{aligned}$$

from where we split up $\widetilde{m}_{2,t}(\sqrt{\mathcal{L}}) = T_{1,t}+T_{2,t}+T_{3,t}$, where the operators T_j, with $j=1,2,3$, are defined by

$$T_{1,t}(f)(z) = \frac{1}{\nu_n}\int_{\mathbb{S}^{n-1}} \partial_t[\delta(x(t,\omega))^{-\frac{1}{2}}]\, m_{2,t}(\sqrt{\mathcal{L}})(f)(z\cdot x(t,\omega))\,\mathrm{d}\omega,$$

$$T_{2,t}(f)(z) = \frac{1}{\nu_n}\int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac{1}{2}}\, (\partial_t m_{2,t})(\sqrt{\mathcal{L}})(f)(z\cdot x(t,\omega))\,\mathrm{d}\omega,$$

and

$$T_{3,t}(f)(z) = \frac{1}{\nu_n}\int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac{1}{2}} \langle \nabla m_{2,t}(\sqrt{\mathcal{L}})(f)(z\cdot x(t,\omega)), \partial_t[z\cdot x(t,\omega)]\rangle_{\mathbf{g}}\,\mathrm{d}\omega.$$

On the one hand, we deduce straightforwardly from Lemma 2.2 and Lemma 2.4 (i) that

$$\|T_{1,t}\|_{\mathfrak{h}^1(S)\to L^1(S)} \le \left(\int_{\mathbb{S}^{n-1}} |\partial_t[\delta(x(t,\omega))^{-\frac{1}{2}}]|\,\mathrm{d}\omega\right) \|m_{2,t}(\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \le C.$$

On the other hand, since $m_{2,t}(\lambda) = (1-\eta(\lambda t))m(\lambda)\lambda^{n/2-1}$, we have $\partial_t m_{2,t} \in \mathcal{S}^0$ uniformly in $0<t\le 1$. Again, using Lemma 2.2 and Lemma 2.4 (i), we obtain

$$\|T_{2,t}\|_{\mathfrak{h}^1(S)\to L^1(S)} \le \left(\int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac{1}{2}}\,\mathrm{d}\omega\right) \|(\partial_t m_{2,t})(\sqrt{\mathcal{L}})\|_{\mathfrak{h}^1(S)\to L^1(S)} \le C.$$

By arguments similar to those in the proof of the long-time case, we deduce from Lemmas 2.2 and 2.4 (ii) that

$$\|T_{3,t}(f)\|_{L^1(S)} \le \left(\int_{\mathbb{S}^{n-1}} \delta(x(t,\omega))^{-\frac{1}{2}}\,\mathrm{d}\omega\right) \||\nabla \mathfrak{m}_{2,t}(\sqrt{\mathcal{L}})(f)|_{\mathbf{g}}\|_{L^1(S)} \le C\,\|f\|_{\mathfrak{h}^1(S)}.$$

Combining these estimates, we obtain that the operator $\widetilde{m}_{2,t}(\sqrt{\mathcal{L}})$ is bounded from $\mathfrak{h}^1(S)$ to $L^1(S)$. The estimate (3.13) is therefore proved, and the proof of part (a) of Proposition 3.1 is complete. The proof of part (b) is analogous to that of part (a) and is therefore omitted. □

Once Proposition 3.1 is proved, the sufficient parts of Theorems 1.2 and 1.3 can be regarded as its corollaries. We now sketch their proofs below.

Proof of Theorem 1.3 (sufficient part). According to the spectral theorem, the solution to the Cauchy problem (1.1) is given by

$$u(t,x) = \cos(t\sqrt{\mathcal{L}})(f)(x) + \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}(g)(x).$$

We deduce from Proposition 3.1 that the estimate

$$\|u(t,\cdot)\|_{L^1(S)} \lesssim (1+|t|)\big(\|(\mathrm{Id}+\mathcal{L})^{\frac{\alpha_0}{2}} f\|_{\mathfrak{h}^1(S)} + \|(\mathrm{Id}+\mathcal{L})^{\frac{\alpha_1}{2}} g\|_{\mathfrak{h}^1(S)}\big)$$

holds in the critical case where

$$\alpha_0 = (n-1)\left|\frac{1}{p}-\frac{1}{2}\right| \quad \text{and} \quad \alpha_1 = (n-1)\left|\frac{1}{p}-\frac{1}{2}\right| - 1,$$

and therefore also in the supercritical cases. □

Proof of Theorem 1.2 (sufficient part). By using an interpolation argument based on Lemma 2.5 (see, for instance, [32, Proposition 4.3]), we know that for $1 < p < \infty$ and for the even symbols $m_1 \in \mathcal{S}^{-(n-1)|1/p-1/2|}$ and $m_2 \in \mathcal{S}^{-(n-1)|1/p-1/2|-1}$,

$$\|m_1(\sqrt{\mathcal{L}})\,\cos(t\sqrt{\mathcal{L}})\|_{L^p(S)\to L^p(S)} \lesssim (1+|t|)^{2|1/p-1/2|},$$

and

$$\left\| m_2(\sqrt{\mathcal{L}})\,\frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} \right\|_{L^p(S)\to L^p(S)} \lesssim 1+|t|,$$

which directly implies the sufficient part of Theorem 1.2. □

We have proved the sufficient parts of Theorems 1.2 and 1.3. As mentioned above, the proofs of their necessary parts are straightforward adaptations of the arguments for the $ax+b$ group given in [32], and are therefore omitted.

Acknowledgements. Y. Wang and L. Yan are supported by National Key R&D Program of China 2022YFA1005700. L. Yan is supported by NNSF of China 12571111. H.-W. Zhang receives funding from the Deutsche Forschungsgemeinschaft (DFG) through SFB-TRR 358/1 2023 —Project 491392403.

References

[1] J.-P. Anker, E. Damek and C. Yacoub, Spherical analysis on harmonic AN groups. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **23** (1996), no. 4, 643–679.
[2] J.-P. Anker and V. Pierfelice, Wave and Klein-Gordon equations on hyperbolic spaces. *Anal. PDE* **7** (2014), no. 4, 953–995.
[3] J.-P. Anker, V. Pierfelice and M. Vallarino, The wave equation on hyperbolic spaces. *J. Differential Equations* **252** (2012), no. 10, 5613–5661.
[4] J.-P. Anker, V. Pierfelice, and M. Vallarino, The wave equation on Damek-Ricci spaces. *Ann. Mat. Pura Appl. (4)* **194** (2015), no. 3, 731–758.
[5] J.-P. Anker and H.-W. Zhang, Wave equation on general Riemannian noncompact symmetric spaces. *Amer. J. Math.* **146** (2024), no. 4, 983–1031.
[6] F. Astengo, A class of L^p convolutors on harmonic extensions of H-type groups. *J. Lie Theory* **5** (1995), no. 2, 147–164.
[7] F. Astengo, The maximal ideal space of a hear algebra on Solvable extensions of H-type groups. *Boll. Unione Mat. Ital. A (7)* **9** (1995), no. 1, 157–165.
[8] A. Carbonaro, G. Mauceri and S. Meda, Comparison of spaces of Hardy type for the Ornstein-Uhlenbeck operator. *Potential Anal.* **33** (2010), no. 1, 85–105.
[9] A. Carbonaro, G. Mauceri and S. Meda, H^1 and BMO for certain locally doubling metric measure spaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **8** (2009), no. 3, 543–582.
[10] M. Cowling, A. H. Dooley, A. Korányi and F. Ricci, H-type groups and Iwasawa decompositions. *Adv. Math.* **87** (1991), 1–41.
[11] M. Cowling, A. H. Dooley, A. Korányi and F. Ricci, An approach to symmetric spaces of rank one via groups of Heisenberg-type. *J. Geom. Anal.* **8** (1998), 199–237.
[12] M. Cwikel and S. Janson, Interpolation of analytic families of operators. *Studia Math.* **79** (1984), no. 1, 61–71.

[13] E. Damek, Curvature of a semidirect extension of a Heisenberg-type nilpotent group. *Colloq. Math.* **53** (1987), 249–253.

[14] E. Damek, The geometry of a semidirect extension of a Heisenberg type nilpotent group. *Colloq. Math.* **53** (1987), 255–268.

[15] E. Damek and F. Ricci, A class of nonsymmetric harmonic Riemannian spaces. *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 139–142.

[16] E. Damek and F. Ricci, Harmonic analysis on solvable extensions of H-type groups. *J. Geom. Anal.* **2** (1992), no. 3, 213–248.

[17] A. Erdélyi et al., *Higher transcendental functions.* Vol. II, McGraw-Hill, New York, 1953.

[18] D. Goldberg, A local version of real Hardy spaces. *Duke Math. J.* **46** (1979), 27–42.

[19] A.D. Ionescu, Fourier integral operators on noncompact symmetric spaces of real rank one. *J. Funct. Anal.* **174** (2000), no. 2, 274–300.

[20] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.* **258** (1975), 145–159.

[21] S. Meda and S. Volpi, Spaces of Goldberg type on certain measured metric spaces. *Ann. Mat. Pura Appl. (4)* **196** (2017), no. 3, 947–981.

[22] J. Metcalfe and M. Taylor, Nonlinear waves on 3D hyperbolic space. *Trans. Amer. Math. Soc.* **363** (2011), no. 7, 3489–3529.

[23] A. Miyachi, On some estimates for the wave equation in L^p and H^p. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), no. 2, 331–354.

[24] D. Müller and C. Thiele, Wave equation and multiplier estimates on $ax + b$ groups. *Studia Math.* **179** (2007), no. 2, 117–148.

[25] D. Müller and M. Vallarino, Wave equation and multiplier estimates on Damek-Ricci spaces. *J. Fourier Anal. Appl.* **16** (2010), 204–232.

[26] J.C. Peral, L^p estimates for the wave equation. *J. Funct. Anal.* **36** (1980), no. 1, 114–145.

[27] Y. Sire, C.D. Sogge, C. Wang, and J. Zhang, Strichartz estimates and Strauss conjecture on non-trapping asymptotically hyperbolic manifolds. *Trans. Amer. Math. Soc.* **373** (2020), no. 11, 7639–7668.

[28] E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals.* Princeton Math. Ser. **43**. Monographs in Harmonic Analysis **III**. Princeton Univ. Press, Princeton, NJ, 1993.

[29] T. Tao, The weak-type $(1, 1)$ of Fourier integral operators of order $-(n-1)/2$. *J. Aust. Math. Soc.* **76** (2004), no. 1, 1–24.

[30] D. Tataru, Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation. *Trans. Amer. Math. Soc.* **353** (2001), no. 2, 795–807.

[31] M. Taylor, Hardy spaces and BMO on manifolds with bounded geometry. *J. Geom. Anal.* **19** (2009), 137–190.

[32] Y. Wang and L. Yan, Sharp L^p-estimates for wave equation on $ax + b$ groups. Available at arxiv.org/abs/2506.17531.

[33] H.-W. Zhang, Wave equation on certain noncompact symmetric spaces. *Pure Appl. Anal.* **3** (2021), no. 2, 363–386.