Local-in-time existence of strong solutions to a class of compressible Power-Law flows

Fang Li[1*] and Chang Mengge[1†]

[1*]Department of Mathematics and CNS, Northwest University, Xi'an, 710000, People's Republic of China.

*Corresponding author(s). E-mail(s): fangli@nwu.edu.cn;
Contributing authors: changmg@stumail.nwu.edu.cn;
†These authors contributed equally to this work.

Abstract

We consider a model of the compressible non-Newtonian fluids for power-law flow fulfilling a periodic domain in $\mathbb{R}^3$, in which the extra stress tensor is induced by a potential with $p(t,x)$-structure. The local-in-time existence of strong solution is proved for all $\frac{7}{5} < \inf p(t,x) \leqslant \sup p(t,x) \leqslant 2$. Further, an improved blow-up criterion for strong solutions is given in terms of the $L^\infty(0,T;L^3(\Omega))$-norm of the gradient of the velocity.

Keywords: Strong solutions, Blow-up criterion, Compressible non-Newtonian fluid, Power-Law model

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain occupied by a fluid. Let ρ and $\mathbf{u}$ be the density and velocity of the fluid, respectively. Denote by t and x the time and spatial variables. Then the Navier-Stokes equations, describing the flow of a compressible non-Newtonian fluid, read as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \quad in \quad (0,T)\times\Omega, \\ \partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathcal{A}(\mathbb{D}\mathbf{u}) + \nabla(\rho)^\gamma = \rho\mathbf{f} \quad in \quad (0,T)\times\Omega. \end{cases} \tag{1}$$

Here $T>0$ is the time of evolution, $\mathbf{f}$ is the external force, $\gamma \geqslant \frac{3}{2}$, $\mathbb{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$, and the non-Newtonian tensor $\mathcal{A}(\mathbb{D}\mathbf{u})$ is given by the following power law

$$\mathcal{A}(\mathbb{D}\mathbf{u}) = (1 + |\mathbb{D}\mathbf{u}|^2)^{\frac{p-2}{2}}\mathbb{D}\mathbf{u} \tag{2}$$

with $1 < p < +\infty$.

In the last decades, the theoretical and numerical analysis of non-Newtonian fluids has seen a renew of interest, stimulated by the wide spectrum of applied problems, ranging from biological applications to industrial processes. In particular, many processes deal with materials exhibiting non-Newtonian behaviors. For instance, food processing, polymer manufacturing, tribology, injection molding of foams, rubber extrusion. First existence results concerning incompressible non-Newtonian fluid with

$$\mathcal{A}(\mathbb{D}\mathbf{u}) = \nu_1\mathbb{D}\mathbf{u} + \nu_2|\mathbb{D}\mathbf{u}|^{p-2}\mathbb{D}\mathbf{u},$$

were proved for constant $p \geqslant \frac{11}{5}$ by Lions and Ladyzhenskaya in [18, 19]. Since then, many improvements have appeared. Bellout, Bloom and Nečas [4] proved that there exist Young measure valued solutions to the incompressible non-Newtonian fluids for space periodic problems under some conditions. Málek, Nečas, Rokyta and Růžička proved the existence and uniqueness of global strong solutions for incompressible non-Newtonian fluids of power-law type when $p > \frac{3d-4}{d}$ in [20]. Kaplický, Málek and Stará [15] proved the existence of $C^{1,\alpha}-$solutions of the nonlinear system that describes plane stationary flow of a class of non-Newtonian fluids, when $p > \frac{3}{2}$, a global $C^{1,\alpha}-$solution is constructed, and when $p > \frac{6}{5}$, a solution with interior $C^{1,\alpha}$ regularity is obtained. Later, Zhang, Guo and Guo [31] proved the existence of weak solutions for steady-state incompressible non-Newtonian fluids with spatial dimension $d \geqslant 2$ under slip boundary conditions when $p \geqslant 2$. Burczak, Modena and Székelyhidi [5] proved the non-uniqueness of Leray-Hopf solutions for three dimensional incompressible non-Newtonian fluids of power-law type when $1 < p < \frac{2d}{d+2}$ with $d \geqslant 3$. Motivated by the model for the motion of electrorheological fluids in [24, 25, 27], where p is a function in space and time. Electrorheological fluids are a special type of smart fluids which change their material properties due to the application of an electric field. In the model in [25], p is not a constant but a function of the electric field $\mathbf{E}$, i.e. $p = p(|\mathbf{E}|^2)$. The electric field itself is a solution to the quasi-static Maxwell equations and is not influenced by the motion of the fluid. Růžička proved that there exists a strong solution to three dimensional incompressible non-Newtonian fluids of power-law type for large time and data as long as $p(t,x) \in W^{1,\infty}((0,T)\times\Omega)$ in [26] and $\frac{11}{5} < p^- \leqslant p^+ < p^- + \frac{4}{3}$ in [27], where

$$p^- = \inf_{(0,T)\times\Omega} p(t,x) \text{ and } p^+ = \sup_{(0,T)\times\Omega} p(t,x). \tag{3}$$

Diening and Růžička [7] proved the local existence of strong solutions in a three dimensional bounded periodic domain when $p(t,x)$ satisfies $\frac{7}{5} < p^- \leqslant p(t,x) \leqslant p^+ \leqslant 2$.

The research on compressible non-Newtonian fluids with constant power-law exponents can be traced back to 1994, such as, Nečasová and Novotný proved the existence of measure-valued solutions in [23]. Later, Mamontov [21, 22] illustrated the global existence of weak solution under the assumptions of an exponentially growing viscosity and isothermal pressure. Recently, Abbatiello, Feireisl and Novotńy proved the existence of so-called dissipative solution in [1]. For the existence of the strong solution, the local-in-time existence were established for the absence of vacuum in by Kalousek, Macha and Necasova in [16]. Fang and Guo discussed the analytical solution to a class of non-Newtonian fluids with free boundaries in [10]. Xu and Yuan [28] proved the local-in-time existence and uniqueness in one space dimension with singularity and vacuum. While the initial energy is small, Yuan, Si and Feng [29] established the global well-posedness of strong solutions for the initial boundary value problems of the one dimensional model (1). Recently, Bilal Al Taki [3] proved the existence and uniqueness of strong solutions for compressible non-Newtonian fluids of power-law type in a local time sense when $1 < p < \infty$. And, Al Baba, Al Taki and Hussein [2] show the existence, uniqueness, and continuous dependence on the data of local-in-time strong solutions to the Navier-Stokes equations describing non-Newtonian compressible fluids when there is some vacuum initially. Fang and Zang [11] proved the global existence and uniqueness of strong solutions to the Cauchy problem for one dimensional compressible non-Newtonian fluids of with $p \in (1,2)$. More results on the mathematical theory of non-Newtonian fluids, we can refer [6, 8, 9, 13, 14, 16, 30].

However, there are few results to our knowledge on the compressible non-Newtonian fluids of power-law type with $p(t,x) \in W^{1,\infty}((0,T)\times\Omega)$. Motived by Diening and Růžička's work in [7], we aim to establish the existence of strong solutions to the following system

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0 \quad in \quad (0,T)\times\Omega, \\ \partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}((1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u}) + \nabla(\rho)^\gamma = \rho\mathbf{f} \quad in \quad (0,T)\times\Omega, \end{cases} \tag{4}$$

subjected to the initial condition

$$\rho(x,0) = \rho_0(x), \quad \rho\mathbf{u}(x,0) = \mathbf{m}_0(x), \quad x \in \Omega, \tag{5}$$

where Ω is the torus $\Omega = \mathbb{T}^3$.

Local strong solutions to the problem (4)-(5), is defined as following.

Definition 1 Let $p(t,x) \in W^{1,\infty}((0,T)\times\Omega)$ with $\frac{7}{5} < p^- \leqslant p(t,x) \leqslant p^+ \leqslant 2$. Given a positive time $T \in (0,+\infty)$. The pair $(\rho,\mathbf{u})$ is called a strong solution to the problem (4)-(5) on $(0,T)\times\Omega$, if $(\rho,\mathbf{u})$ has the properties

$$\begin{cases} \rho \in L^\infty(0,T;W^{1,3p^-}(\Omega)), \\ \mathbf{u} \in L^\infty(0,T;W^{1,\frac{12}{5}}(\Omega)), \quad \mathbf{u} \in L^{\frac{p^-(5p^- -6)}{2-p^-}}(0,T;W^{2,\frac{3p^-}{p^-+1}}(\Omega)) \cap L^{p^-}(0,T;W^{2,3p^-}(\Omega)), \\ \sqrt{\rho}\partial_t\mathbf{u} \in L^\infty(0,T;L^2(\Omega)), \quad \partial_t\mathbf{u} \in L^{p^-}(0,T;W^{1,\frac{3p^-}{p^-+1}}(\Omega)). \end{cases} \tag{6}$$

Remark 1 *Thanks to the regularities of the strong solutions stated in Definition 1, by the system (4), one can show that the strong solutions have the following additional regularities*

$$\partial_t \rho \in L^\infty(0,T;L^{3p^-}(\Omega)).$$

The first main result of this paper is the theorem concerning the local existence of strong solutions to the problem (4)-(5), which is stated as follows.

Theorem 1 *Let $T>0$, $\Omega=\mathbb{T}^3$ and $\frac{7}{5}<p^-\leqslant p(t,x)\leqslant p^+\leqslant 2$ with $p(t,x)\in W^{1,\infty}((0,T)\times\Omega)$. Assume that the initial data $(\rho_0,\mathbf{u}_0)$ and the external force $\mathbf{f}$ satisfy the following regularity*

$$\begin{aligned} &0\leqslant \rho_0\in W^{1,3p^-}(\Omega), \quad \sqrt{\rho_0}\mathbf{u}_0\in L^2(\Omega), \quad \nabla\mathbf{u}_0\in L^{min\{6(p^--1),2p^-\}}(\Omega),\\ &\mathbf{f}\in L^\infty(0,T;W^{1,2}(\Omega)), \quad \partial_t\mathbf{f}\in L^2(0,T;L^2(\Omega)), \end{aligned} \tag{7}$$

and the compatibility condition

$$-\operatorname{div}((1+|\mathbb{D}\mathbf{u}_0|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u}_0)+\nabla(\rho_0)^\gamma=\rho_0^{1/2}\mathbf{g}, \tag{8}$$

for some $\mathbf{g}\in L^2(\Omega)$. Then, there exist a small time $T^*\in(0,T)$ and a strong solution $(\rho,\mathbf{u})$ in the sense of Definition 1 to the problem (4)-(5) with the compatibility conditions (8).

Remark 2 *Basically, the condition $\frac{7}{5}<p^-\leqslant p(t,x)\leqslant p^+\leqslant 2$ covers the constant $p\in(\frac{7}{5},2]$. Indeed, Yuan and Xu proved the local-in-time existence and uniqueness of strong solutions to the problem (4)-(5) in [28] for $1<p<2$ in the case of*

$$\mathcal{A}(\mathbb{D}\mathbf{u})=|\mathbb{D}\mathbf{u}|^{p-2}\mathbb{D}\mathbf{u},$$

which is limited to the one-dimensional space.

Remark 3 *The uniqueness of the strong solution obtained in Theorem 1 has not been shown. There is a technique challenge from the property of $\mathbf{u}$.*

Next, we state a blow-up criterion for the problem (4)-(5) with the compatibility conditions (8).

Theorem 2 *Let $T>0$, $\Omega=\mathbb{T}^3$ and $\frac{7}{5}<p^-\leqslant p(t,x)\leqslant p^+\leqslant 2$ with $p(t,x)\in W^{1,\infty}((0,T)\times\Omega)$. Assume that $(\rho_0,\mathbf{u}_0)$ and extend force $\mathbf{f}$ satisfy (7) and (8). Let $(\rho,\mathbf{u})$ be a strong solution to the problem (4)-(5) satisfying the regularity (6). If $T^*\in(0,+\infty)$ is the maximal times of existence, then*

$$\lim_{T\to T^*}(\|\rho\|_{L^\infty(0,T;L^\infty(\Omega))}+\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^3(\Omega))})=\infty. \tag{9}$$

Remark 4 *The blow up criterion (9) involves both the density and velocity. It may be natural to expect the higher regularity of velocity if the blow up criterion without the density. Taking similar argument for (9), one can get the following blow up criterion*

$$\lim_{T\to T^*}\int_0^T\|\nabla\mathbf{u}\|^4_{L^\infty(\Omega)}dt=\infty. \tag{10}$$

The main difficulty in the blow up criterions (9) and (10) is to control the gradient of density, which is not known and coupled with the second derivative of velocity.

Remark 5 *The blow up criterion (9) and (10) hold for the case that the power is constant $p\in(\frac{7}{5},2)$.*

2 Preliminaries

Motivated by [7], we are going to give a definition of a $p(t,x)$-potential and derive basic properties of it. Assume that the extra stress tensor $\mathcal{A}(\mathbb{D}\mathbf{u})$ is induced by $p-$potential.

The functions discussed in this section is from $\Omega\times\mathbb{R}^{d\times d}$ to $\mathbb{R}$. We distinguish the partial derivatives by ∂_i and ∂_{jk}, a single index means a partial derivative with respect to the i-th space coordinate, while a double index represents a partial derivative with respect to the (j,k)-component of the underlying space of $d\times d$-matrices. Since we are dealing with functions from $\Omega\times\mathbb{R}^{d\times d}$ to $\mathbb{R}$, we will distinguish the partial derivatives by ∂_i and ∂_{jk}, By ∇ we denote the space gradient, while $\nabla_{d\times d}$ denotes the matrix consisting of the partial derivatives with respect to the space of matrices. In a few cases we use d_i instead of ∂_i to indicate a total derivative. Note that by $\mathbf{B}^{\mathrm{sym}}$ we denote the symmetric part of a matrix $\mathbf{B}\in\mathbb{R}^{d\times d}$, i.e. $\mathbf{B}^{\mathrm{sym}}=\frac{1}{2}(\mathbf{B}+\mathbf{B}^{\top})$. $\mathbf{B}_{jk}$ denotes the (j,k)-component of the matric $\mathbf{B}$, and $|\mathbf{B}|$ denotes $(\sum\limits_{j,k=1}^{3}|\mathbf{B}_{jk}|^2)^{\frac{1}{2}}$. Further let $\mathbb{R}^{\mathrm{sym}}_{d\times d}$ be the subspace of $\mathbb{R}^{d\times d}$ consisting of the symmetric matrices. Moreover we use C as a positive constant which is generic but does not depend on the ellipticity constants. $\mathbb{R}$ is the real space and $\mathbb{R}^+$ denotes the space of positive real numbers. We assume that $1<p(t,x)\leqslant 2$.

Let $I=(0,T)$ with $T>0$ and $p(t,x):I\times\Omega\to(1,2]$ be a $W^{1,\infty}(I\times\Omega)$ function with

$$1<p^-:=\inf_{I\times\Omega}p(t,x)\leqslant p^+:=\sup_{I\times\Omega}p(t,x)\leqslant 2.$$

Adopting a computational concept from work in [8], we introduce some auxiliary functions as follows and state the properties of the functions Let $F:I\times\Omega\times(\mathbb{R}^+\cup\{0\})\to\mathbb{R}^+\cup\{0\}$ be such that the function $F(t,x,\cdot)$ is a $p(t,x)$-potential for a.e. $(t,x)\in I\times\Omega$, and the ellipticity constants do not depend on (t,x), that is, the function $\Phi:\mathbb{R}^{d\times d}\to\mathbb{R}^+\cup\{0\}$ defined through $\Phi(t,x,\mathbf{B})=F(t,x,|\mathbf{B}^{\mathrm{sym}}|)$ satisfies

$$\sum\nolimits_{jklm}(\partial_{jk}\partial_{lm}\Phi)(t,x,\mathbf{B})\mathbf{C}_{jk}\mathbf{C}_{lm}\geqslant\gamma_1(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{C}^{\mathrm{sym}}|^2, \tag{11}$$

$$\left|(\nabla^2_{d\times d}\Phi)(t,x,\mathbf{B})\right|\leqslant\gamma_2(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-2}{2}}, \tag{12}$$

for all $\mathbf{B},\mathbf{C}\in\mathbb{R}^{d\times d}$ with constants $\gamma_1,\gamma_2>0$.

Further we assume that F is continuously differentiable with respect to t and x and that $(\partial_tF)(t,x,\cdot):\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$, resp. $(\partial_jF)(t,x,\cdot):\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$, is a C^1-function on $\mathbb{R}^+\cup\{0\}$, resp. a C^2-function on $\mathbb{R}^+$, for all $t\in I$, $x\in\Omega$.

Moreover, assume that for $j=1,2,3$

$$\begin{aligned}
&(\partial_tF)(t,x,0)=0,\\
&(\partial_tF)(t,x,R)>0\quad for\ all\ R>0,\\
&(\partial_jF)(t,x,0)=0,\\
&(\partial_jF)(t,x,R)>0\quad for\ all\ R>0,\\
&|(\partial_t\nabla_{d\times d}\Phi)(t,x,\mathbf{B})|\leqslant\gamma_3(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}^{\mathrm{sym}}|),\\
&|(\nabla\nabla_{d\times d}\Phi)(t,x,\mathbf{B})|\leqslant\gamma_3(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}^{\mathrm{sym}}|),
\end{aligned}\tag{13}$$

with $\gamma_3>0$. Such functions F and Φ are called a time and space dependent $p(t,x)$-potential. We define the extra stress $\mathbf{S}$ induced by F and Φ as follows

$$\mathbf{S}(\mathbf{B}):=\nabla_{d\times d}\Phi(\mathbf{B})=F'(|\mathbf{B}^{\mathrm{sym}}|)\frac{\mathbf{B}^{\mathrm{sym}}}{|\mathbf{B}^{\mathrm{sym}}|}$$

for all $\mathbf{B}$ in $\mathbb{R}^{d\times d}\setminus\{\mathbf{0}\}$.

Remark 6 *The standard example of such* $\mathbf{S}$ *is*

$$\mathbf{S}(\mathbb{D}\mathbf{u}):=\mathcal{A}(\mathbb{D}\mathbf{u})=(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u},$$

where $p\in W^{1,\infty}(I\times\Omega)$.

Remark 7 *([8]) Let* $d\geqslant 2$, *for all* $\mathbf{B}\in\mathbb{R}^{d\times d}\setminus\{\mathbf{0}\}$, *then*

$$(\partial_{jk}\Phi)(\mathbf{B})=F'(|\mathbf{B}^{\mathrm{sym}}|)\frac{\mathbf{B}^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|},$$

$$(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B})=F'(|\mathbf{B}^{\mathrm{sym}}|)\left(\frac{\delta^{\mathrm{sym}}_{jk,lm}}{|\mathbf{B}^{\mathrm{sym}}|}-\frac{\mathbf{B}^{\mathrm{sym}}_{jk}\mathbf{B}^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|^3}\right)+F''(|\mathbf{B}^{\mathrm{sym}}|)\frac{\mathbf{B}^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|}\frac{\mathbf{B}^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|},$$

$$\gamma_1(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-2}{2}}\leqslant F''(|\mathbf{B}^{\mathrm{sym}}|)\leqslant\gamma_2(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-2}{2}},$$

$$\lim_{|\mathbf{B}|\to 0}\mathbf{S}_{jk}(\mathbf{B})=\lim_{|\mathbf{B}|\to 0}(\partial_{jk}\Phi)(\mathbf{B})=0,$$

where $\delta^{\mathrm{sym}}_{jk,lm}:=\frac{1}{2}(\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl})$.

Remark 8 *([8]) Let* $\mathbf{B},\mathbf{C} \in \mathbb{R}^{d\times d}$*. Due to* $\Phi(\mathbf{B}) = F(|\mathbf{B}^{\rm sym}|)$*, then* $\Phi(\mathbf{B}) = \Phi(\mathbf{B}^{\rm sym})$ *implies that*

$$\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B})\mathbf{C}_{jk}\mathbf{C}_{lm} = \sum_{jklm}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}^{\rm sym})\mathbf{C}^{\rm sym}_{jk}\mathbf{C}^{\rm sym}_{lm},$$

$$(\nabla_{d\times d}\Phi)(\mathbf{B}) = (\nabla_{d\times d}\Phi)(\mathbf{B}^{\rm sym}),$$
$$(\nabla^2_{d\times d}\Phi)(\mathbf{B}) = (\nabla^2_{d\times d}\Phi)(\mathbf{B}^{\rm sym}).$$

Since later we will mostly deal with symmetric matrices, we will use $\mathbf{B}$ *instead of* $\mathbf{B}^{\rm sym}$*.*

As in [20] and [8], it is deduced from (11) and (12) that the following properties of $\mathbf{S}$ hold.

Lemma 3 *([8]) There exist constants* $c_1, c_2 > 0$ *independent of* γ_1 *and* γ_2 *such that*

$$\mathbf{S}(\mathbf{0}) = 0, \tag{14}$$
$$\sum_{ij}(\mathbf{S}_{ij}(\mathbf{B}) - \mathbf{S}_{ij}(\mathbf{C}))(\mathbf{B}_{ij} - \mathbf{C}_{ij}) \geqslant c_1\gamma_1(1 + |\mathbf{B}|^2 + |\mathbf{C}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{B} - \mathbf{C}|^2, \tag{15}$$
$$\sum_{ij}\mathbf{S}_{ij}(\mathbf{B})\mathbf{B}_{ij} \geqslant c_1\,\gamma_1(1 + |\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{B}|^2, \tag{16}$$
$$|\mathbf{S}(\mathbf{B}) - \mathbf{S}(\mathbf{C})| \leqslant c_2\,\gamma_2(1 + |\mathbf{B}|^2 + |\mathbf{C}|^2)^{\frac{p(t,x)-2}{2}}\,|\mathbf{B} - \mathbf{C}|, \tag{17}$$
$$|\mathbf{S}(\mathbf{B})| \leqslant c_2\,\gamma_2(1 + |\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}\,|\mathbf{B}|, \tag{18}$$

hold for all $\mathbf{B},\ \mathbf{C} \in \mathbb{R}^{d\times d}_{\rm sym}$*.*

Let Φ be a $p(t,x)$-potential and $\mathbf{u}$ denote a sufficiently smooth function over the space time cylinder. The brackets $\langle\cdot,\cdot\rangle$ denote integration over the space domain Ω. We will encounter the following important expressions

$$\mathcal{I}_\Phi(t,\mathbf{u}) := \Big\langle \sum_r \sum_{jk\alpha\beta}(\partial_{\alpha\beta}\partial_{jk}\Phi)\big(\mathbb{D}\mathbf{u}\big)\partial_r(\mathbb{D}\mathbf{u})_{\alpha\beta}, \partial_r(\mathbb{D}\mathbf{u})_{jk}\Big\rangle(t), \tag{19}$$
$$\mathcal{J}_\Phi(t,\mathbf{u}) := \Big\langle \sum_{jk\alpha\beta}(\partial_{\alpha\beta}\partial_{jk}\Phi)\big(\mathbb{D}\mathbf{u}\big)\partial_t(\mathbb{D}\mathbf{u})_{\alpha\beta}, \partial_t(\mathbb{D}\mathbf{u})_{jk}\Big\rangle(t), \tag{20}$$
$$\mathcal{G}_\Phi(t,\mathbf{w},\mathbf{v}) := \Big\langle \sum_{jk\alpha\beta}(\partial_{\alpha\beta}\partial_{jk}\Phi)(\mathbb{D}\mathbf{w})(\mathbb{D}\mathbf{v})_{\alpha\beta}, (\mathbb{D}\mathbf{v})_{jk}\Big\rangle(t), \tag{21}$$

where $\mathbf{w}: I\times\Omega \to \mathbb{R}^3$ and $\mathbf{v}: I\times\Omega \to \mathbb{R}^3$ (or $\mathbf{v}: I\times\Omega \to \mathbb{R}^{3\times 3}$) are sufficiently smooth functions.

For the sake of simplicity, we write $\mathcal{I}_\Phi(\mathbf{u})$, $\mathcal{J}_\Phi(\mathbf{u})$ and $\mathcal{G}_\Phi(\mathbf{w},\mathbf{v})$ instead $\mathcal{I}_\Phi(t,\mathbf{u})$, $\mathcal{J}_\Phi(t,\mathbf{u})$ and $\mathcal{G}_\Phi(t,\mathbf{w},\mathbf{v})$. It is easy to find that

$$\mathcal{I}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{u},\mathbb{D}\mathbf{u}) \text{ and } \mathcal{J}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{u},\partial_t\mathbf{u}).$$

Due to the properties of Φ, there exits some $\gamma_1 > 0$ such that

$$\mathcal{G}_\Phi(\mathbf{w},\mathbf{v}) \geqslant \gamma_1 \int_\Omega (1 + |\mathbb{D}\mathbf{w}|^2)^{\frac{p(t,x)-2}{2}}|\mathbb{D}\mathbf{v}|^2\,dx. \tag{22}$$

Since $(1 + |\mathbb{D}\mathbf{w}|^2)^{\frac{1}{2}}$ appears frequently, it is essential to introduce the shortcut

$$\widetilde{D}\mathbf{w} := (1 + |\mathbb{D}\mathbf{w}|^2)^{\frac{1}{2}}. \tag{23}$$

As a consequence,

$$\mathcal{I}_\Phi(\mathbf{u}) \geqslant C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{u})^{p(t,x)-2}|\nabla\mathbb{D}\mathbf{u}|^2\,dx, \tag{24}$$
$$\mathcal{J}_\Phi(\mathbf{u}) \geqslant C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{u})^{p(t,x)-2}|\partial_t\mathbb{D}\mathbf{u}|^2\,dx. \tag{25}$$

Note that

$$|\nabla^2\mathbf{u}| \leqslant 3|\nabla\mathbb{D}\mathbf{u}| \leqslant 6|\nabla^2\mathbf{u}|.$$

Thus, $|\nabla\mathbb{D}\mathbf{u}|$ can always be replaced by $|\nabla^2\mathbf{u}|$ by increasing the multiplicative constant.

Next, we introduce the following two lemmas. The first one is a special form Gagliardo-Nirenberg's interpolation inequality(see [17]), the second one describes local version of Gronwall's lemma(see [7]).

Lemma 4 *(Gagliardo-Nirenberg Inequality[17]) Let $\Omega \subset \mathbb{R}^d (d \geqslant 2)$ be a bounded domain and j, k be positive integers. For $1 \leqslant p,\ r \leqslant \infty$ and $0 \leqslant j < k$, $j/k \leqslant \theta \leqslant 1$, there exists some constant $C = C(d, k, p, r, j, \theta, \Omega)$ such that*

$$\|\nabla^j \mathbf{u}\|_{L^q(\Omega)} \leqslant C \|\nabla^k \mathbf{u}\|^{\theta}_{L^p(\Omega)} \|\mathbf{u}\|^{1-\theta}_{L^r(\Omega)},$$

holds for any $\mathbf{u} \in W^{k,p}_0(\Omega)$, where $\frac{1}{q} = \frac{j}{d} + \theta(\frac{1}{p} - \frac{k}{d}) + \frac{1-\theta}{r}$.

Lemma 5 *(Local Version of Gronwall's Lemma [7]) Let $T, \alpha, c_0 > 0$ be given constants and let $h \in L^1(0,T)$ with $h \geqslant 0$ a.e. in $[0,T]$, $f \in C^1[0,T]$ satisfy $f \geqslant 0$ and*

$$f'(t) \leqslant h(t) + c_0 f(t)^{1+\alpha}. \tag{26}$$

Let $t_0 \in [0,T]$ be such that $\alpha c_0 H(t_0)^\alpha t_0 < 1$, where

$$H(t) := f(0) + \int_0^t h(s) ds.$$

Then

$$f(t) \leqslant H(t) + H(t) \left(\left(1 - \alpha c_0 H(t)^\alpha t\right)^{-\frac{1}{\alpha}} - 1 \right)$$

holds for all $t \in [0, t_0]$.

Then, we introduce the following auxiliary result, which is given in [12].

Lemma 6 *Let $\mathbf{v} \in W^{1,2}(\Omega)$, and let ρ be a non-negative function such that*

$$0 < M \leqslant \int_\Omega \rho dx, \quad \int_\Omega \rho^\gamma dx \leqslant E_0,$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, M, E_0 are positive constants, and $\gamma > 1$.

Then there exists a constant c depending solely on M, E_0 such that

$$\|\mathbf{v}\|^2_{L^2(\Omega)} \leqslant c(E_0, M) \big(\|\nabla \mathbf{v}\|^2_{L^2(\Omega)} + \big(\int_\Omega \rho |\mathbf{v}| dx \big)^2 \big).$$

At the end of this section, we take idea [3] to establish some regularity properties of a solution to the following nonlinear elliptic system

$$\begin{cases} -\operatorname{div} \mathcal{A}(\mathbb{D}\mathbf{u}) = \mathbf{f}, \\ \mathcal{A}(\mathbb{D}\mathbf{u}) = (1 + |\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}} \mathbb{D}\mathbf{u}, \end{cases} \tag{27}$$

where $p(t,x) \in W^{1,\infty}((0,T) \times \Omega)$ with $1 < p^- \leqslant p(t,x) \leqslant p^+ \leqslant 2$.

Lemma 7 *Let Ω be a periodic domain in $\mathbb{R}^d$. Given a function $\mathbf{f} \in L^s(\Omega)$ $(1 < s < \infty)$ such that*

$$\int_\Omega \mathbf{f}\, dx = 0.$$

Assume that $\mathbf{u}$ is a unique solution to the system (27). If $d = 2$ (or 3), then the linearized operator associated to equation (27) at a reference solution $\mathbf{u}^* \in W^{2-\frac{2}{s},s}(\Omega)$ $(2s > d+2)$, denoted by $\mathcal{A}(\mathbf{u}^*, \mathbb{D})$, still yield maximal L^s-regularity provided that $p(t,x) \geqslant 1$. Moreover, there exists a constant $C > 0$ such that

$$\|\mathbf{u}\|_{W^{2,s}(\Omega)} \leqslant \|\mathcal{A}(\mathbf{u}^*, \mathbb{D})\mathbf{u}\|_{L^s(\Omega)} \leqslant C(\|\mathbf{f}\|_{L^s(\Omega)} + \|\mathbb{D}\mathbf{u}\|^{p^-}_{L^{p^-}(\Omega)} + 1). \tag{28}$$

Proof It is important to prove that the linearized operator $\mathcal{A}(\mathbf{u}^*, \mathbb{D})$ satisfies the strong ellipticity condition. To start, let us rewrite the elliptic operator as follows

$$\begin{aligned} \operatorname{div} \mathcal{A}(\mathbb{D}\mathbf{u}) &= \operatorname{div} \left((1 + |\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}} \mathbb{D}\mathbf{u} \right) \\ &= \left(\frac{1}{2} (1 + |\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}} (\Delta \mathbf{u} + \nabla \operatorname{div} \mathbf{u}) \right) \\ &+ \frac{p(t,x) - 2}{2} (1 + |\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}} \nabla (|\mathbb{D}\mathbf{u}|^2) \mathbb{D}\mathbf{u} \end{aligned}$$

$$+\frac{1}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\ln(1+|\mathbb{D}\mathbf{u}|^2)\nabla p(t,x)\mathbb{D}\mathbf{u}$$
$$\triangleq L_1+L_2+L_3. \tag{29}$$

Let $\overline{\operatorname{div}\mathcal{A}(\mathbb{D}\mathbf{u})}=L_1+L_2$ and $\overline{\mathbf{f}}=L_3+\mathbf{f}$, then we can rewrite $(27)_1$ as

$$-\overline{\operatorname{div}\mathcal{A}(\mathbb{D}\mathbf{u})}=\overline{\mathbf{f}}.$$

Since $\mathbb{D}\mathbf{u}$ is symmetric, the i-th entry of $\overline{\operatorname{div}\mathcal{A}(\mathbb{D}\mathbf{u})}$ becomes

$$\begin{aligned}
[\overline{\operatorname{div}\mathcal{A}(\mathbb{D}\mathbf{u})}]_i=&\sum_{k=1}^{3}\frac{1}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}(\partial_k^2\mathbf{u}_i+\partial_i\partial_k\mathbf{u}_k)\\
&+2\sum_{j,k,l=1}^{3}\frac{p(t,x)-2}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}(\mathbb{D}\mathbf{u})_{ij}(\mathbb{D}\mathbf{u})_{kl}\,\partial_j(\mathbb{D}\mathbf{u})_{kl}\\
=&\sum_{k=1}^{3}\frac{1}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}(\partial_k^2\mathbf{u}_i+\partial_i\partial_k\mathbf{u}_k)\\
&+\sum_{j,k,l=1}^{3}(p(t,x)-2)(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}(\mathbb{D}\mathbf{u})_{ik}(\mathbb{D}\mathbf{u})_{jl}\,\partial_k\partial_l\mathbf{u}_j\\
=&\sum_{j,k,l=1}^{3}a_{ij}^{kl}\partial_k\partial_l\mathbf{u}_j.
\end{aligned}$$

Set

$$a_{i,j}^{k,l}=\frac{1}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}(\delta_{kl}\delta_{ij}+\delta_{il}\delta_{jk})+(p(t,x)-2)(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}(\mathbb{D}\mathbf{u})_{ik}\,(\mathbb{D}\mathbf{u})_{jl},$$

where δ_{kl} denotes the Kronecker symbol. Define the quasi-linear differential operator $\mathcal{A}(\mathbf{u},\mathbb{D})$ as

$$\mathcal{A}(\mathbf{u},\mathbb{D})=\sum_{k,l=1}A^{k,l}\,\mathbb{D}_k\,\mathbb{D}_l,$$

where the matrix-valued coefficients

$$A^{k,l}(\mathbf{u})=(a_{i,j}^{k,l}).$$

Now, if $\mathbf{u}\in W^{2,3}(\Omega)$, then $\mathbf{u}\in C^1(\overline{\Omega})$ by Sobolev embedding, said the coefficients of the differential operator $\mathcal{A}(x,\mathbb{D})=\mathcal{A}(\mathbf{u}(x),\mathbb{D})$ are uniformly bounded continuous. Freezing $\mathcal{A}$ at a reference solution $\mathbf{u}^*$, we obtain the linear operator

$$\mathcal{A}(\mathbf{u}^*,\mathbb{D})=\sum_{k,l=1}A_*^{k,l}\,\mathbb{D}_k\,\mathbb{D}_l\,.$$

Recalling the definition of a_{jl}^{kl} and using the sum convention, we have that for $\xi\in\mathbb{R}^d,\ \eta\in\mathbb{C}^d$,

$$\begin{aligned}
(\mathcal{A}(x,\xi)\eta,\eta)=&\sum_{i,j,k,1=1}^{n}a_{ij}^{kl}\xi_k\eta_l\xi_i\bar{\eta}_j\\
=&\frac{1}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}(\sum_{i,k=1}^{n}\xi_k^2\eta_i\bar{\eta}_i+\sum_{i,k=1}^{n}\xi_i\eta_k\xi_k\bar{\eta}_i)\\
&+(p(t,x)-2)(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}\sum_{i,k=1}^{n}d_{ik}\xi_k\eta_i\sum_{j,l=1}^{n}d_{jl}\xi_l\bar{\eta}_j\\
=&\frac{1}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}(|\xi|^2|\eta|^2+(\xi,\eta)\overline{(\eta,\xi)})\\
&+(p(t,x)-2)(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}(\mathbb{D}\xi,\eta)\overline{(\mathbb{D}\xi,\eta)}\\
=&\frac{1}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}(|\xi|^2|\eta|^2+|(\xi,\eta)|^2)\\
&+(p(t,x)-2)(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}|(\mathbb{D}\xi,\eta)|^2\\
\geqslant&\frac{1}{2}(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}(|\xi|^2|\eta|^2+|(\xi,\eta)|^2)\\
&+(p(t,x)-2)(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}|\mathbb{D}\mathbf{u}|^2|(\xi,\eta)|^2\\
\geqslant&(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}(p(t,x)-1)(1+|\mathbb{D}\mathbf{u}|^2),
\end{aligned}\tag{30}$$

which together with Cauchy-Schwarz inequality and the fact that $|\xi|=|\eta|=1$. Obviously, the operator $\mathcal{A}(\mathbf{u}^*,\mathbb{D})$ is strongly elliptic, we have

$$\|\mathbf{u}\|_{W^{2,s}(\Omega)}\leqslant\|\mathcal{A}(\mathbf{u}^*,\mathbb{D})\mathbf{u}\|_{L^s(\Omega)}\leqslant C\|\overline{\mathbf{f}}\|_{L^s(\Omega)}. \tag{31}$$

Note that

$$\|\overline{\mathbf{f}}\|_{L^s(\Omega)}\ =\frac{1}{2}\|(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\ln(1+|\mathbb{D}\mathbf{u}|^2)\nabla p(t,x)\mathbb{D}\mathbf{u}+\mathbf{f}\|_{L^s(\Omega)}$$

$$\leqslant \frac{1}{2}\|(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\ln(1+|\mathbb{D}\mathbf{u}|^2)\nabla p(t,x)\mathbb{D}\mathbf{u}\|_{L^s(\Omega)}+\|\mathbf{f}\|_{L^s(\Omega)}$$

and it follows from the Gagliardo-Nirenberg inequality and Lemma 4 that

$$\begin{aligned}
&\|(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\ln(1+|\mathbb{D}\mathbf{u}|^2)\nabla p(t,x)\mathbb{D}\mathbf{u}\|_{L^s(\Omega)}\\
\leqslant& C\|(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbb{D}\mathbf{u}|^2)\|_{L^s(\Omega)}\\
\leqslant& C\|(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-1+\varepsilon}{2}}\|_{L^s(\Omega)}\\
\leqslant& C(1+\|\mathbb{D}\mathbf{u}\|^{(p^+-1+\varepsilon)}_{L^{s(p^+-1+\varepsilon)}(\Omega)})\\
\leqslant& C(1+\|\mathbb{D}\mathbf{u}\|^{(1-\theta)(p^+-1+\varepsilon)}_{L^{p^-}(\Omega)}\|\nabla^2\mathbf{u}\|^{\theta(p^+-1+\varepsilon)}_{L^s(\Omega)})\\
\leqslant& \delta\|\nabla^2\mathbf{u}\|_{L^s(\Omega)}+C(\delta)(1+\|\mathbb{D}\mathbf{u}\|^{\frac{(1-\theta)(p^+-1+\varepsilon)}{1-\theta(p^+-1+\varepsilon)}}_{L^{p^-}})
\end{aligned}\tag{32}$$

for any fixed $\delta\in(0,1)$, where

$$\frac{1}{(p^+-1+\varepsilon)s}=\theta(\frac{1}{s}-\frac{1}{3})+\frac{1-\theta}{p^-}.$$

Moreover, $\theta=\frac{3(s(p^+-1+\varepsilon)-p^-)}{(p^-s+3s-3p^-)(p^+-1+\varepsilon)}$. Thus, one can choose δ small enough and suitable ε to derive that

$$\frac{(1-\theta)(p^+-1+\varepsilon)}{1-\theta(p^+-1+\varepsilon)}\leqslant p^-$$

and so

$$\|\mathbf{u}\|_{W^{2,s}(\Omega)}\leqslant\|\mathcal{A}(\mathbf{u}^*,\mathbb{D})\mathbf{u}\|_{L^s(\Omega)}\leqslant C(\|\mathbf{f}\|_{L^s(\Omega)}+\|\mathbb{D}\mathbf{u}\|^{p^-}_{L^{p^-}(\Omega)}+1).$$

□

3 A priori estimates

First, we derive the following lemma which is important when examining the regularity of solutions.

Lemma 8 *Let $T>0$, Φ be a $p(t,x)$-potential and $1<q\leqslant 2$. For all (sufficiently smooth) $\mathbf{u}$ and $\mathbf{v}$, there*

$$\|\mathbb{D}\mathbf{v}\|_{L^q(\Omega)}\leqslant C(\mathcal{G}_\Phi(t,\mathbf{w},\mathbf{v}))^{\frac{1}{2}}\|(\widetilde{D}\mathbf{w})^{\frac{2-p^-}{2}}\|_{L^{\frac{2q}{2-q}}(\Omega)},\tag{33}$$

holds for almost every $t\in(0,T)$, where $\frac{2q}{2-q}=\infty$ for $q=2$.

Proof Observe that $1\leqslant\frac{2}{q}<\infty$ and $1<\frac{2}{2-q}\leqslant\infty$. For $1\leqslant q<2$, one finds that

$$\begin{aligned}
\|\mathbb{D}\mathbf{v}\|^q_{L^q(\Omega)}&=\int_\Omega\left((\widetilde{D}\mathbf{w})^{p^--2}|\mathbb{D}\mathbf{v}|^2\right)^{\frac{q}{2}}(\widetilde{D}\mathbf{w})^{\frac{(2-p^-)q}{2}}dx\\
&\leqslant\left(\int_\Omega(\widetilde{D}\mathbf{w})^{p^--2}|\mathbb{D}\mathbf{v}|^2dx\right)^{\frac{q}{2}}\left\|(\widetilde{D}\mathbf{w})^{\frac{(2-p^-)q}{2}}\right\|_{L^{\frac{q}{2-q}}(\Omega)}\\
&=\left(\int_\Omega(\widetilde{D}\mathbf{w})^{p^--2}|\mathbb{D}\mathbf{v}|^2dx\right)^{\frac{q}{2}}\left\|(\widetilde{D}\mathbf{w})^{\frac{2-p^-}{2}}\right\|^q_{L^{\frac{2q}{2-q}}(\Omega)}.
\end{aligned}$$

Taking into account of (22), one proves the lemma for $1<q<2$. The case $q=2$ is similar. □

Next, we derive some estimates for $\mathcal{I}_\Phi$ and $\mathcal{J}_\Phi$.

Lemma 9 *Let $T>0$. For all sufficiently smooth $\mathbf{u}$,*

$$\|\mathbf{u}\|^{p^-}_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}\leqslant C\left(\mathcal{I}_\Phi(t,\mathbf{u})+1\right),\tag{34}$$

$$\begin{aligned}
\|\partial_t\mathbf{u}\|^{p^-}_{W^{1,\frac{3p^-}{p^-+1}}(\Omega)}&\leqslant C\mathcal{J}_\Phi(t,\mathbf{u})^{\frac{p^-}{2}}\left(\mathcal{I}_\Phi(t,\mathbf{u})+1\right)^{\frac{2-p^-}{2}}\\
&\leqslant C(\mathcal{J}_\Phi(t,\mathbf{u})+\mathcal{I}_\Phi(t,\mathbf{u})+1).
\end{aligned}\tag{35}$$

hold for almost every $t\in(0,T)$.

Proof According to Lemma 8, one finds that

$$\begin{aligned}
\|\nabla\mathbb{D}\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)} &\leqslant C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12}\|(\widetilde{D}\mathbf{u})^{\frac{2-p^-}{2}}\|_{L^{\frac{6p^-}{2-p^-}}(\Omega)}\\
&\leqslant C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12}\|\widetilde{D}\mathbf{u}\|_{L^{3p^-}(\Omega)}^{\frac{2-p^-}{2}}\\
&\leqslant C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12}\left(1+\|\mathbb{D}\mathbf{u}\|_{L^{3p^-}(\Omega)}\right)^{\frac{2-p^-}{2}}\\
&\leqslant C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12}\left(1+C\,\|\nabla\mathbb{D}\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\right)^{\frac{2-p^-}{2}},
\end{aligned}$$

which implies that

$$\|\nabla\mathbb{D}\mathbf{u}\|^{p^-}_{L^{\frac{3p^-}{p^-+1}}(\Omega)} \leqslant C\left(\mathcal{I}_\Phi(\mathbf{u})+1\right).$$

Since $|\nabla^2\mathbf{u}|\leqslant 3\,|\nabla\mathbb{D}\mathbf{u}|$, one deduces from Sobolev embedding and Korn's inequality that

$$\|\mathbf{u}\|^{p^-}_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)} \leqslant C\left(\mathcal{I}_\Phi(\mathbf{u})+1\right).$$

Analogously, it is deduce from Lemma 8 to get that

$$\begin{aligned}
\|\partial_t\mathbb{D}\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)} &\leqslant C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\|(\widetilde{D}\mathbf{u})^{\frac{2-p^-}{2}}\|_{L^{\frac{6p^-}{2-p^-}}(\Omega)}\\
&\leqslant C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\left(1+C\|\nabla\mathbb{D}\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\right)^{\frac{2-p^-}{2}}\\
&\leqslant C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\left(1+C\left(\mathcal{I}_\Phi(\mathbf{u})+1\right)^{\frac{1}{p^-}}\right)^{\frac{2-p^-}{2}}\\
&\leqslant C\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\left(1+\mathcal{I}_\Phi(\mathbf{u})\right)^{\frac{2-p^-}{2p^-}}.
\end{aligned}$$

According to Korn's inequality, one obtains that

$$\|\partial_t\mathbf{u}\|_{W^{1,\frac{3p^-}{p^-+1}}(\Omega)} \leqslant C\,\|\partial_t\mathbb{D}\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)} \leqslant C\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\left(1+\mathcal{I}_\Phi(\mathbf{u})\right)^{\frac{2-p^-}{2p^-}}.$$

Thus, (36) follow from Young's inequality. □

Next, we give the standard energy estimate for $(\rho,\mathbf{u})$.

Lemma 10 *Let $(\rho,\mathbf{u})$ be a smooth solution to the problem (4)-(5) on $[0,T)\times\Omega$ for some $T>0$. Then there exists a positive constant $C>0$ such that*

$$\sup_{0\leqslant t\leqslant T}\int_\Omega(\rho|\mathbf{u}|^2+\rho^\gamma)dx+\int_0^T\int_\Omega|\mathbb{D}\mathbf{u}|^{p(t,x)}\,dxdt\leqslant C. \tag{36}$$

Proof It is easy to deduce from $(4)_1$ that

$$\int_\Omega\rho(t)\,dx=\int_\Omega\rho_0\,dx\quad(t\geqslant 0).$$

Next, multiplying $(4)_2$ by $\mathbf{u}$ and integrating by parts over Ω, one obtains that

$$\frac12\frac{d}{dt}\int_\Omega(\rho|\mathbf{u}|^2+\frac{1}{\gamma-1}\rho^\gamma)\,dx+\int_\Omega(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}|\mathbb{D}\mathbf{u}|^2\,dx=\int_\Omega\rho\mathbf{f}\cdot\mathbf{u}\,dx.$$

Moreover, it follows from Hölder inequality that

$$\left|\int_\Omega\rho\mathbf{f}\cdot\mathbf{u}\,dx\right|\leqslant\|\mathbf{f}\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}\|\rho\|^{1/2}_{L^\gamma(\Omega)}\|\sqrt{\rho}\mathbf{u}\|_{L^2(\Omega)}\leqslant C\|\mathbf{f}\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}(1+\|\rho\|^\gamma_{L^\gamma(\Omega)}+\|\sqrt{\rho}\mathbf{u}\|^2_{L^2(\Omega)}).$$

Since $\gamma\geqslant\frac32$, $\frac{2\gamma}{\gamma-1}\leqslant 6$, one applys (7) and Gronwall's inequality to deduce that

$$\sup_{0\leqslant t\leqslant T}\int_\Omega(\rho|\mathbf{u}|^2+\rho^\gamma)\,dx+\int_0^T\int_\Omega|\mathbb{D}\mathbf{u}|^{p(t,x)}\,dxdt\leqslant C.$$

□

Next, we establish the estimates for velocity.

Lemma 11 *Let* $\frac{12}{5} \leqslant q \leqslant min\{6(p^- - 1), 2p^-\}$. *Suppose that* $(\rho, \mathbf{u})$ *is a smooth solution to the problem* (4)-(5) *on* $[0,T) \times \Omega$ *for some* $T > 0$. *Then there exist positive constants* $m_1 = m_1(p^-)$, $m_2 = m_2(p^-)$, $m_3 = m_3(p^-)$ *and* $m_4 = m_4(p^-)$ *such that*

$$\begin{aligned}
&\frac{d}{dt}\|\widetilde{D}\mathbf{u}\|^q_{L^q(\Omega)} + \mathcal{I}^r_\Phi(\mathbf{u}) \\
&\leqslant \eta \mathcal{J}_\Phi(\mathbf{u}) + C(1 + \|\nabla\rho\|^{m_1}_{L^{3p^-}(\Omega)} + \|\widetilde{D}\mathbf{u}\|^{m_2}_{L^q(\Omega)} + \|\rho\|^{m_3}_{L^\infty(\Omega)} + \|\sqrt{\rho}\partial_t\mathbf{u}\|^{m_4}_{L^2(\Omega)}),
\end{aligned} \tag{37}$$

where $1 \leqslant r < \frac{5p^- - 6}{2 - p^-}$.

Proof Multiplying $(4)_2$ by $-\Delta\mathbf{u}$ and integrating the resulting equation over Ω, one obtains that

$$\begin{aligned}
\int_\Omega \operatorname{div} \mathcal{A}(\mathbb{D}\mathbf{u}) \cdot \Delta\mathbf{u} dx &= \int_\Omega \operatorname{div}\left((1 + |\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}} \mathbb{D}\mathbf{u}\right) \cdot \Delta\mathbf{u} dx \\
&= \int_\Omega \rho\mathbf{u} \cdot \nabla\mathbf{u} \cdot \Delta\mathbf{u} + \int_\Omega \nabla(\rho)^\gamma \cdot \Delta\mathbf{u} + \int_\Omega \rho\partial_t\mathbf{u} \cdot \Delta\mathbf{u} - \int_\Omega \rho\mathbf{f} \cdot \Delta\mathbf{u} \\
&\triangleq \sum_{i=1}^4 J_i.
\end{aligned}$$

The first term on the right-hand side of equation (38) is estimated as follows

$$\begin{aligned}
J_1 &= \sum_{i,j,k} \int_\Omega \rho\mathbf{u}^j \partial_j \mathbf{u}^i \partial_k\partial_k \mathbf{u}^i dx \\
&= -\sum_{i,j,k} \int_\Omega (\partial_k\rho\mathbf{u}^j\partial_j\mathbf{u}^i\partial_k\mathbf{u}^i + \rho\partial_k\mathbf{u}^j\partial_j\mathbf{u}^i\partial_k\mathbf{u}^i + \rho\mathbf{u}^j\partial_k\partial_j\mathbf{u}^i\partial_k\mathbf{u}^i)dx \\
&\leqslant \int_\Omega |\nabla\rho||\mathbf{u}||\nabla\mathbf{u}|^2 dx + \int_\Omega \rho|\nabla\mathbf{u}|^3 dx + \int_\Omega \rho|\mathbf{u}||\nabla^2\mathbf{u}||\nabla\mathbf{u}|dx \\
&\triangleq J_{11} + J_{12} + J_{12}.
\end{aligned} \tag{38}$$

Using Lemma 9 and Sobolev embedding inequality, one gets that

$$\begin{aligned}
J_{11} &\leqslant C\|\nabla\rho\|_{L^{3p^-}(\Omega)}\|\mathbf{u}\|_{L^{12}(\Omega)}\|\nabla\mathbf{u}\|^2_{L^3(\Omega)} \\
&\leqslant C(\|\nabla\rho\|^6_{L^{3p^-}(\Omega)} + \|\nabla\mathbf{u}\|^6_{L^{\frac{12}{5}}(\Omega)} + \|\nabla\mathbf{u}\|^3_{L^3(\Omega)}) \\
&\leqslant \varepsilon\mathcal{I}_\Phi + C(1 + \|\nabla\rho\|^6_{L^{3p^-}(\Omega)} + \|\nabla\mathbf{u}\|^6_{L^{\frac{12}{5}}(\Omega)} + C\|\widetilde{D}\mathbf{u}\|^{R_1}_{L^q(\Omega)})
\end{aligned} \tag{39}$$

holds for any fixed $\varepsilon \in (0,1)$, where $R_1 = \frac{3q(p^- - 1)}{2q + 3p^- - 9}$. In fact, in the last inequality of (39) we have used

$$\begin{aligned}
\|\nabla\mathbf{u}\|^3_{L^3(\Omega)} &\leqslant \|\nabla\mathbf{u}\|^{\frac{3q(p^- - 1)}{3p^- - q}}_{L^q(\Omega)} \|\nabla\mathbf{u}\|^{\frac{3p^-(3-q)}{3p^- - q}}_{L^{3p^-}(\Omega)} \\
&\leqslant C_\varepsilon\|\nabla\mathbf{u}\|^{\frac{3q(p^- - 1)}{2q + 3p^- - q}}_{L^q(\Omega)} + \varepsilon\|\nabla\mathbf{u}\|^{p^-}_{L^{3p^-}(\Omega)} \\
&\leqslant C_\varepsilon\|\nabla\mathbf{u}\|^{\frac{3q(p^- - 1)}{2q + 3p^- - q}}_{L^q(\Omega)} + C\varepsilon\|\nabla\mathbf{u}\|^{p^-}_{W^{1,\frac{3p^-}{p^- + 1}}(\Omega)} \\
&\leqslant C_\varepsilon\|\nabla\mathbf{u}\|^{\frac{3q(p^- - 1)}{2q + 3p^- - q}}_{L^q(\Omega)} + C\varepsilon(\mathcal{I}_\Phi(\mathbf{u}) + 1) \\
&\leqslant C_\varepsilon\|\widetilde{D}\mathbf{u}\|^{\frac{3q(p^- - 1)}{2q + 3p^- - q}}_{L^q(\Omega)} + C\varepsilon(\mathcal{I}_\Phi(\mathbf{u}) + 1)
\end{aligned} \tag{40}$$

holds for any fixed $\varepsilon \in (0,1)$, when $\frac{12}{5} \leqslant q < 3$, the estimate (40) is also true when $3 \leqslant q \leqslant min\{6(p^- - 1), 2p^-\}$ is obviously true. For J_{12} and J_{13}, one uses Lemma 4 and Lemma 9 to get that

$$\begin{aligned}
J_{12} &\leqslant \|\rho\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^3_{L^3(\Omega)} \\
&\leqslant \|\rho\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^{\frac{12(p^- - 1)}{5p^- - 4}}_{L^{\frac{12}{5}}(\Omega)} \|\mathbf{u}\|^{\frac{3p^-}{(5p^- - 4)}}_{L^{3p^-}(\Omega)} \\
&\leqslant \|\rho\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^{\frac{12(p^- - 1)}{5p^- - 4}}_{L^{\frac{12}{5}}(\Omega)} \|\mathbf{u}\|^{\frac{3p^-}{(5p^- - 4)}}_{W^{2,\frac{3p^-}{3p^- + 1}}(\Omega)} \\
&\leqslant \|\rho\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^{\frac{12(p^- - 1)}{5p^- - 4}}_{L^{\frac{12}{5}}(\Omega)} (\mathcal{I}_\Phi(\mathbf{u}) + 1)^{\frac{3}{5p^- - 4}} \\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u}) + C(1 + \|\rho\|^{R_2}_{L^\infty(\Omega)} + \|\nabla\mathbf{u}\|^{R_3}_{L^{\frac{12}{5}}(\Omega)})
\end{aligned} \tag{41}$$

and

$$J_{13} \leqslant \|\rho\|_{L^\infty(\Omega)}\|\nabla^2\mathbf{u}\|_{L^{\frac{3p^-}{p^- + 1}}(\Omega)}\|\nabla\mathbf{u}\|_{L^{\frac{12}{5}}(\Omega)}\|\mathbf{u}\|_{L^{84}(\Omega)}$$

$$\begin{aligned}
&\leqslant \|\rho\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}\|\nabla\mathbf{u}\|_{L^{\frac{12}{5}}(\Omega)}\|\nabla^2\mathbf{u}\|^{\frac{6p^-}{7(5p^--4)}}_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\|\mathbf{u}\|^{\frac{29p^--28}{7(5p^--4)}}_{L^{12}(\Omega)}\\
&\leqslant \|\rho\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^{\frac{64p^--56}{7(5p^--4)}}_{L^{\frac{12}{5}}(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{41p^--28}{7p^-(5p^--4)}}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+C(1+\|\nabla\mathbf{u}\|^{R_4}_{L^{\frac{12}{5}}(\Omega)}+\|\rho\|^{R_5}_{L^\infty(\Omega)}) \qquad (42)
\end{aligned}$$

hold for any fixed $\varepsilon\in(0,1)$, where $R_2=\frac{2(5p^--4)}{5p^--7}$, $R_3=\frac{24(p^--1)}{5p^--7}$, $R_4=\frac{2p^-(64p^--56)}{35(p^-)^2-69p^-+28}$ and $R_5=\frac{14p^-(5p^--4)}{35(p^-)^2-69p^-+28}$. The second term and the fourth term on the right-hand side of equation (38) are estimated as

$$\begin{aligned}
&|J_2|+|J_4|\\
&\leqslant C\|\rho\|_{L^\infty(\Omega)}\|f\|_{L^6(\Omega)}\|\mathbf{u}\|_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}+C\|\mathbf{u}\|_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}\|\rho\|^{\gamma-1}_{L^\infty(\Omega)}\|\nabla\rho\|_{L^{3p^-}(\Omega)}\\
&\leqslant C\|\rho\|_{L^\infty(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}+C\|\rho\|^{\gamma-1}_{L^\infty(\Omega)}\|\nabla\rho\|_{L^{3p^-}(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+C(1+\|\rho\|^{R_6}_{L^\infty(\Omega)}+\|\nabla\rho\|^{2R_6}_{L^{3p^-}(\Omega)}+\|\rho\|^{2(\gamma-1)R_6}_{L^\infty(\Omega)}) \qquad (43)
\end{aligned}$$

holds for any fixed $\varepsilon\in(0,1)$, where $R_6=\frac{p^-}{p^--1}$. The third term on the right-hand side of equation (38) is estimated as

$$\begin{aligned}
|J_3| &\leqslant \|\rho\|^{\frac12}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{2p^--1}}(\Omega)}\|\Delta\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant \|\rho\|^{\frac12}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{4p^--4}{3p^--2}}_{L^2(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{2-p^-}{3p^--2}}_{L^{3p^-}(\Omega)}\|\mathbf{u}\|_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant \|\rho\|^{\frac{p^-}{3p^--2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{4p^--4}{3p^--2}}_{L^2(\Omega)}\|\partial_t\mathbf{u}\|^{\frac{2-p^-}{3p^--2}}_{W^{1,\frac{3p^-}{p^-+1}}(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant \|\rho\|^{\frac{p^-}{3p^--2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{4p^--4}{3p^--2}}_{L^2(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})^{\frac12}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2-p^-}{2p^-}})^{\frac{2-p^-}{3p^--2}}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant \|\rho\|^{\frac{p^-}{3p^--2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{4p^--4}{3p^--2}}_{L^2(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{p^-+2}{2(3p^--2)}}\mathcal{J}_\Phi(\mathbf{u})^{\frac{2-p^-}{2(3p^--2)}}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+\varepsilon+\|\rho\|^{R_7}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{R_8}_{L^2(\Omega)}\mathcal{J}_\Phi(\mathbf{u})^{R_9} \qquad (44)
\end{aligned}$$

holds for any fixed $\varepsilon\in(0,1)$, where $R_7=\frac{2p^-}{5p^--6}$, $R_8=\frac{8(p^--1)}{5p^--6}$ and $R_9=\frac{2-p^-}{5p^--6}$. In addition, the nonlinear main part is estimated as follows

$$\begin{aligned}
&\int_\Omega \operatorname{div}\mathcal{A}(\mathbb{D}\mathbf{u})\cdot\Delta\mathbf{u}dx\\
=&\sum_{ijklm}\int_\Omega(\partial_{ij}\partial_{kl}\Phi)(\mathbb{D}\mathbf{u})\partial_m(\mathbb{D}\mathbf{u})_{ij}\partial_m(\mathbb{D}\mathbf{u})_{kl}dx+\sum_{klm}\int_\Omega(\partial_m\partial_{kl}\Phi)(\mathbb{D}\mathbf{u})\partial_m(\mathbb{D}\mathbf{u})_{kl}dx\\
\geqslant&\mathcal{I}_\Phi(\mathbf{u})-C\int_\Omega(\widetilde{D}\mathbf{u})^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbb{D}\mathbf{u}|)|\nabla^2\mathbf{u}|dx\\
\geqslant&\mathcal{I}_\Phi(\mathbf{u})-\varepsilon\int_\Omega(\mathbb{D}\mathbf{u})^{p(t,x)-2}|\nabla^2\mathbf{u}|^2dx-C_\varepsilon\int_\Omega(\widetilde{D}\mathbf{u})^{p(t,x)}\ln^2(1+|\mathbb{D}\mathbf{u}|)dx\\
\geqslant&\frac{C}{2}\mathcal{I}_\Phi(\mathbf{u})-C_\varepsilon\int_\Omega(\widetilde{D}\mathbf{u})^{p(t,x)}\left(1+\ln^2(\widetilde{D}\mathbf{u})\right)dx \qquad (45)
\end{aligned}$$

due to (13), (19) and (24). Note that

$$(\widetilde{D}\mathbf{u})^{p(t,x)}\left(1+\ln^2(\widetilde{D}\mathbf{u})\right)\leqslant C_\delta(\widetilde{D}\mathbf{u})^{p(t,x)+\sigma}$$

holds for every $\sigma\in(0,1)$. Especially, with the help of (38)-(45) we choose suitable σ such that $p(t,x)+\sigma\leqslant q$ and obtain that

$$\begin{aligned}
\mathcal{I}_\Phi(\mathbf{u})\leqslant&C(1+\|\rho\|^{R_{10}}_{L^\infty(\Omega)}+\|\nabla\rho\|^{R_{11}}_{L^{3p^-}(\Omega)}+\|\nabla\mathbf{u}\|^{R_{12}}_{L^{\frac{12}{5}}(\Omega)}\\
&+\|\widetilde{D}\mathbf{u}\|^{R_{13}}_{L^q(\Omega)}+\|\rho\|^{R_7}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{R_8}_{L^2(\Omega)}\mathcal{J}_\Phi(\mathbf{u})^{R_9}), \qquad (46)
\end{aligned}$$

where $R_{10}=max\{R_2,R_5,R_6,2(\gamma-1)R_6\}$, $R_{11}=max\{2R_6,6\}$, $R_{12}=max\{6,R_3,R_4\}$ and $R_{13}=max\{R_1,q\}$. Thus, for any fixed $r\in[1,\frac{5p^--6}{2-p^-})$, one gets that

$$\begin{aligned}
&\mathcal{I}^r_\Phi(\mathbf{u})\\
&\leqslant C(1+\|\rho\|^{rR_{10}}_{L^\infty(\Omega)}+\|\nabla\rho\|^{rR_{11}}_{L^{3p^-}(\Omega)}+\|\nabla\mathbf{u}\|^{rR_{12}}_{L^{\frac{12}{5}}(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{rR_{13}}_{L^q(\Omega)}+\|\rho\|^{rR_7}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{rR_8}_{L^2(\Omega)}\mathcal{J}_\Phi(\mathbf{u})^{rR_9})
\end{aligned}$$

$$\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\nabla\rho\|^{rR_{11}}_{L^{3p^-}(\Omega)}+\|\nabla\mathbf{u}\|^{rR_{12}}_{L^{\frac{12}{5}}(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{rR_{13}}_{L^q(\Omega)}+\|\rho\|^{R_{14}}_{L^\infty(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{R_{15}}_{L^2(\Omega)}), \tag{47}$$

where $R_{14}=max\{R_{10},\frac{2rR_7(5p^- -6)}{5p^- -6-(2-p^-)r}\}$ and $R_{15}=\frac{2rR_8(5p^- -6)}{5p^- -6-(2-p^-)r}$.

Note that for $q>1$

$$\partial_t\left((\widetilde{D}\mathbf{u})^q\right)=q(\widetilde{D}\mathbf{u})^{q-2}\sum_{jk=1}^{3}(\mathbb{D}\mathbf{u})_{jk}\partial_t(\mathbb{D}\mathbf{u})_{jk}.$$

The assumption $\frac{12}{5}\leqslant q\leqslant min\{6(p^- -1),2p^-\}$, implies $\frac{12}{5}\leqslant q\leqslant 2q-p^-\leqslant 3p^-$ and $1<\frac{3(q-p^-)}{3p^- -q}<r$. So, one uses Lemma 4 and Lemma 9 to derive that

$$\begin{aligned}
\frac{d}{dt}(\|\widetilde{D}\mathbf{u}\|^q_{L^q(\Omega)}) &\leqslant \int_\Omega(\widetilde{D}\mathbf{u})^{q-1}|\partial_t\mathbb{D}\mathbf{u}|dx\\
&=q\int_\Omega(\widetilde{D}\mathbf{u})^{\frac{p^- -2}{2}}|\partial_t\mathbb{D}\mathbf{u}|(\widetilde{D}\mathbf{u})^{q-\frac{p^-}{2}}dx\\
&\leqslant C\mathcal{J}_\Phi^{\frac{1}{2}}(\mathbf{u})\left(\|\widetilde{D}\mathbf{u}\|^{2q-p^-}_{L^{2q-p^-}}\right)^{\frac{1}{2}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C\|\widetilde{D}(\mathbf{u})\|^{2q-p^-}_{L^{2q-p^-}(\Omega)}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C\|\widetilde{D}(\mathbf{u})\|^{\frac{2q(2p^- -q)}{3p^- -q}}_{L^q(\Omega)}\|\widetilde{D}(\mathbf{u})\|^{\frac{3p^-(q-p^-)}{3p^- -q}}_{L^{3p^-}(\Omega)}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C\|\widetilde{D}(\mathbf{u})\|^{\frac{2q(2p^- -q)}{3p^- -q}}_{L^q(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{3(q-p^-)}{3p^- -q}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+\varepsilon\mathcal{I}^r_\Phi(\mathbf{u})+\varepsilon C\|\widetilde{D}(\mathbf{u})\|^{R_{16}}_{L^q(\Omega)}
\end{aligned} \tag{48}$$

holds for any fixed $\varepsilon\in(0,1)$, where $R_{16}=\frac{2qp^- r(2p^- -q)}{3p^- r(3p-q)-9p^-(q-p^-)}$. Taking account of (47) and (48), one obtains that

$$\begin{aligned}
&\frac{d}{dt}\|\widetilde{D}\mathbf{u}\|^q_{L^q(\Omega)}+\mathcal{I}^r_\Phi(\mathbf{u})\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\nabla\rho\|^{m_1}_{L^{3p^-}(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{m_2}_{L^q(\Omega)}+\|\rho\|^{m_3}_{L^\infty(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{m_4}_{L^2(\Omega)})
\end{aligned}$$

holds for any fixed $\varepsilon\in(0,1)$, where $m_1=rR_{11},\ m_2=max\{rR_{12},rR_{13},R_{16}\},\ m_3=R_{14}$ and $m_4=R_{15}$. □

Lemma 12 *Let $max\{1,\frac{5}{5p^- -4}\}\leqslant r<\frac{5p^- -6}{2-p^-}$ and $\frac{12}{5}\leqslant q\leqslant min\{6(p^- -1),2p^-\}$. Suppose that $(\rho,\mathbf{u})$ is a smooth solution to the problem (4)-(5) on $[0,T)\times\Omega$ for some $T>0$. Then there exists positive constant $m_5=m_5(p^-)$, $m_6=m_6(p^-)$, $m_7=m_7(p^-)$ and $m_8=m_8(p^-)$ such that*

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^2(\Omega)}+\mathcal{J}_\Phi(\mathbf{u})\\
&\leqslant \varepsilon\mathcal{I}^r_\Phi(\mathbf{u})+C(1+\|\nabla\rho\|^{m_5}_{L^{3p^-}(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{m_6}_{L^q(\Omega)}+\|\rho\|^{m_7}_{L^\infty(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{m_8}_{L^2(\Omega)})
\end{aligned} \tag{49}$$

hods for any fixed $\varepsilon\in(0,1)$.

Proof Differentiating $(4)_2$ with respect to time, one gets that

$$\begin{aligned}
&\rho\partial_t(\partial_t\mathbf{u})+\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}-\operatorname{div}\partial_t\left((1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u}\right)+\nabla\partial_t(\rho^\gamma)\\
=&\partial_t(\rho\mathbf{f})-\partial_t\rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})-\rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}.
\end{aligned}$$

Multiplying the above equation by $\partial_t\mathbf{u}$ and integrating the resulting equation over Ω, one obtains that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\rho|\partial_t\mathbf{u}|^2dx-\int_\Omega\operatorname{div}\partial_t\left((1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u}\right)\cdot\partial_t\mathbf{u}\,dx\\
=&\int_\Omega\partial_t(\rho\mathbf{f})\cdot\partial_t\mathbf{u}-\partial_t\rho\partial_t\mathbf{u}\cdot\partial_t\mathbf{u}-\partial_t\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\partial_t\mathbf{u}-\rho(\partial_t\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}-\nabla\partial_t(\rho^\gamma)\cdot\partial_t\mathbf{u}\,dx\\
=&\sum_{i=1}^{5}I_i.
\end{aligned} \tag{50}$$

For I_1, one finds that

$$\begin{aligned}
I_1&=\int_\Omega\partial_t\rho\mathbf{f}\cdot\partial_t\mathbf{u}+\rho\partial_t\mathbf{f}\cdot\partial_t\mathbf{u}\,dx\\
&=-\int_\Omega\operatorname{div}(\rho\mathbf{u})\mathbf{f}\cdot\partial_t\mathbf{u}+\rho\partial_t\mathbf{f}\cdot\partial_t\mathbf{u}\,dx\\
&=\int_\Omega\rho\mathbf{u}\cdot\nabla\mathbf{f}\cdot\partial_t\mathbf{u}+(\rho\mathbf{u})\cdot(\mathbf{f}\cdot\nabla\partial_t\mathbf{u})+\rho\partial_t\mathbf{f}\cdot\partial_t\mathbf{u}\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\nabla \mathbf{f}\|_{L^2(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\\
&+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{f}\|_{L^6(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\|\mathbf{u}\|_{L^{12}(\Omega)}+\|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\partial_t\mathbf{f}\|_{L^2(\Omega)}\\
&\leqslant C\|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{u}\|^{1-\theta}_{L^{12}(\Omega)}\|\nabla^2\mathbf{u}\|^{\theta}_{L^{\frac{3p^-}{3p^-+1}}(\Omega)}\\
&+C\|\rho\|_{L^\infty(\Omega)}\|\partial_t\mathbf{u}\|_{W^{1,\frac{3p^-}{p^-+1}}(\Omega)}\|\nabla\mathbf{u}\|_{L^{\frac{12}{5}}(\Omega)}+C\|\rho\|_{L^\infty(\Omega)}+C\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^2(\Omega)}\\
&\leqslant C\|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\widetilde{D}\mathbf{u}\|^{1-\theta}_{L^q(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{\theta}{p^-}}\\
&+C\|\rho\|_{L^\infty(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})+\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}+C\|\rho\|_{L^\infty(\Omega)}+C\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^2(\Omega)}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{R_{17}}_{L^2(\Omega)}+\|\rho\|^{R_{18}}_{L^\infty(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{R_{19}}_{L^q(\Omega)})
\end{aligned}\tag{51}
$$

hold for any fixed $\varepsilon\in(0,1)$, where $\theta=\frac{p^-}{5p^--4}$, $R_{17}=max\{\frac{3(5p^--4)}{5p^--5},2\}$, $R_{18}=max\{\frac{3(5p^--4)}{10(p^--1)},2R_6\}$ and $R_{19}=max\{\frac{12}{5},2R_6\}$.

For I_2, one takes similar argument to get that

$$
\begin{aligned}
I_2&=\int_\Omega \mathrm{div}(\rho\mathbf{u})|\partial_t\mathbf{u}|^2dx\\
&=-\int_\Omega \rho\mathbf{u}\partial_t\mathbf{u}\nabla\partial_t\mathbf{u}\;dx\\
&\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{2p^--1}}(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\mathbf{u}\|^{1-\theta}_{L^{12}(\Omega)}\|\nabla^2\mathbf{u}\|^{\theta}_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{1-\theta_1}_{L^2(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\theta_1}_{L^{3p^-}(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2-p^-}{2p^-}})\\
&\leqslant \|\rho\|^{\frac{1+\theta_1}{2}}_{L^\infty(\Omega)}\|\widetilde{D}\mathbf{u}\|^{1-\theta}_{L^q(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{1-\theta_1}_{L^2(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{\theta}{p^-}}(\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2-p^-}{2p^-}})^{\theta_1+1}\\
&\leqslant \|\rho\|^{\frac{1+\theta_1}{2}}_{L^\infty(\Omega)}\|\widetilde{D}\mathbf{u}\|^{1-\theta}_{L^q(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{1-\theta_1}_{L^2(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2\theta+(2-p^-)(\theta_1+1)}{2p^-}}\mathcal{J}_\Phi(\mathbf{u})^{\frac{\theta_1+1}{2}}\\
&\leqslant \varepsilon\mathcal{I}^r_\Phi(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\rho\|^{R_{20}}_{L^\infty(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{R_{21}}_{L^q(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{R_{22}}_{L^2(\Omega)})
\end{aligned}\tag{52}
$$

hold for any fixed $\varepsilon\in(0,1)$, where $\theta_1=\frac{2-p^-}{3p^--2}$, $\frac{1}{\delta_1}=\frac{p^-r(5p^--4)-5(p^-)^2+17p^--10}{r(5p^--4)(3p^--2)}$,$R_{20}=\frac{3\delta_1p^-}{(3p^--2)(\delta_1-1)}$, $R_{21}=\frac{12\delta_1(p^--1)}{(5p^--4)(\delta_1-1)}$, $R_{22}=\frac{12\delta_1(p^--1)}{(3p^--2)(\delta_1-1)}$. Hence, it is to be used that the following standard fact

$$
max\{1,\frac{5}{5p^--4}\}\leqslant r<\frac{5p^--6}{2-p^-},
$$

to deduce

$$
\frac{\theta_1+1}{2}+\frac{2\theta+(2-p^-)(\theta_1+1)}{2p^-r}<1.
$$

Now, we are in a position to estimate the third term on the righthand side of (50) which indeed plays a crucial role in the analysis

$$
\begin{aligned}
I_3&=\int_\Omega \mathrm{div}(\rho\mathbf{u})(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\partial_t\mathbf{u})\;dx\\
&=-\int_\Omega\rho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot(\nabla\mathbf{u}\cdot\partial_t\mathbf{u})\;dx-\int_\Omega\rho\mathbf{u}\otimes\mathbf{u}\cdot\nabla^2\mathbf{u}\cdot\partial_t\mathbf{u}\;dx-\int_\Omega\rho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot(\mathbf{u}\cdot\nabla\partial_t\mathbf{u})\;dx\\
&\triangleq I_{31}+I_{32}+I_{33}.
\end{aligned}\tag{53}
$$

For I_{31}, one finds that

$$
\begin{aligned}
|I_{31}|&\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|^2_{L^4(\Omega)}\\
&\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\mathbf{u}\|^{1-\theta}_{L^{12}(\Omega)}\|\mathbf{u}\|^{\theta}_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|^{2(1-\theta_2)}_{L^{\frac{12}{5}}(\Omega)}\|\nabla\mathbf{u}\|^{2\theta_2}_{L^{3p^-}(\Omega)}\\
&\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\widetilde{D}\mathbf{u}\|^{3-2\theta_2-\theta}_{L^q(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{\theta+2\theta_2}{p^-}}\\
&\leqslant \varepsilon\mathcal{I}^r_\Phi(\mathbf{u})+C(1+\|\rho\|^{R_{23}}_{L^\infty(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{R_{24}}_{L^2(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{R_{25}}_{L^q(\Omega)})
\end{aligned}\tag{54}
$$

hold for any fixed $\varepsilon\in(0,1)$, where $\theta_2=\frac{2p^-}{5p^--4}$, $\frac{\theta+2\theta_2}{p^-r}<1$, $R_{23}=\frac{3r(5p^--4)}{2(r(5p^--4)-5)}$, $R_{24}=\frac{3r(5p^--4)}{r(5p^--4)-5}$ and $R_{25}=\frac{6r(5p^--6)}{r(5p^--4)-5}$. For I_{32}, one gets that

$$
|I_{32}|\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{2p^--1}}(\Omega)}\|\nabla^2\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}}\|\mathbf{u}\|^2_{L^\infty(\Omega)}
$$

$$
\begin{aligned}
&\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{1-\theta_1}_{L^2(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\theta_1}_{L^{3p^-}(\Omega)}\|\mathbf{u}\|_{W^{2,\frac{3p^-}{p^-+1}}}\|\mathbf{u}\|^{2(1-\theta)}_{L^{12}(\Omega)}\|\nabla^2\mathbf{u}\|^{2\theta}_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant \|\rho\|^{\frac{1+\theta_1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{1-\theta_1}_{L^2(\Omega)}\|\widetilde{D}\mathbf{u}\|^{2(1-\theta)}_{L^q(\Omega)}\|\mathbf{u}\|^{2\theta+1}_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}\|\partial_t\mathbf{u}\|^{\theta_1}_{W^{1,\frac{3p^-}{p^-+1}}}\\
&\leqslant \|\rho\|^{\frac{1+\theta_1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{1-\theta_1}_{L^2(\Omega)}\|\widetilde{D}\mathbf{u}\|^{2(1-\theta)}_{L^q(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{(2-p^-)\theta_1+4\theta+2}{2p^-}}\mathcal{J}_\Phi(\mathbf{u})^{\frac{\theta}{2}}\\
&\leqslant \varepsilon\mathcal{I}^r_\Phi(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\rho\|^{R_{26}}_{L^\infty(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{R_{27}}_{L^2(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{R_{28}}_{L^q(\Omega)})
\end{aligned}
\tag{55}
$$

where $\frac{(2-p^-)\theta_1+4\theta+2+\theta p^-}{2p^-}<1$, $\frac{1}{\delta_2}=\frac{5(p^-)^2-16+18p^-+(2-p^-)(5p^--4)r}{2r(3p^--2)(5p^--4)}$, $R_{26}=\frac{3p^-\delta_2}{(3p^--2)(\delta_2-1)}$, $R_{27}=\frac{12\delta_2(p^--1)}{(3p^--2)(\delta_2-1)}$ and $R_{28}=\frac{12\delta_2(p^--1)}{(5p^--4)(\delta_2-1)}$, hold for any fixed $\varepsilon\in(0,1)$. For I_{33}, one has that

$$
\begin{aligned}
|I_{33}| &\leqslant \|\rho\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^{84}(\Omega)}\|\nabla\mathbf{u}\|_{L^{\frac{12}{5}}(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant \|\rho\|_{L^\infty(\Omega)}\|\mathbf{u}\|^{2-\theta-\theta_3}_{L^{12}(\Omega)}\|\nabla\mathbf{u}\|_{L^{\frac{12}{5}}(\Omega)}\|\nabla^2\mathbf{u}\|^{\theta+\theta_3}_{L^{\frac{3p^-}{p^-+1}}(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2-p^-}{2p^-}})\\
&\leqslant \|\rho\|_{L^\infty(\Omega)}\|\widetilde{D}\mathbf{u}\|^{3-\theta-\theta_3}_{L^q(\Omega)}\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2-p^-+2\theta+2\theta_3}{2p^-}}\\
&\leqslant \varepsilon\mathcal{I}^r_\Phi(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\rho\|^{R_{29}}_{L^\infty(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{R_{30}}_{L^q(\Omega)})
\end{aligned}
\tag{56}
$$

hold for any fixed $\varepsilon\in(0,1)$, where $\theta_3=\frac{6p^-}{7(5p^--4)}$, $\frac{1}{\delta_3}=\frac{-35(p^-)^2(1+r)-56+124p^--28p^-r}{14p^-r(5p^--4)}$, $R_{29}=2\frac{\delta_3}{\delta_3-1}$ and $R_{30}=\frac{(92p^--84)\delta_3}{7(5p^--4)(\delta_3-1)}$.

Note that

$$
\begin{aligned}
\|\mathbb{D}\partial_t\mathbf{u}\|_{L^{\frac{2q}{2-p^-+q}}} &\leqslant C\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}\|(\widetilde{D}\mathbf{u})^{\frac{2-p^-}{2}}\|_{L^{\frac{2q}{2-p^-}}(\Omega)}\\
&\leqslant C\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}\|(\widetilde{D}\mathbf{u})\|^{\frac{2-p^-}{2}}_{L^q(\Omega)}.
\end{aligned}
$$

It is deduced from Lemma 8 that

$$
\begin{aligned}
I_4 &\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^q(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^{\frac{2q}{q-1}}}\\
&\leqslant \|\rho\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^q(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{2(1-\theta_4)}_{L^2(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{2\theta_4}_{L^{\frac{6q}{6-3p^-+q}}}\\
&\leqslant \|\rho\|^{\frac{1+2\theta_4}{2}}_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^q(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{2(1-\theta_4)}_{L^2(\Omega)}\|\nabla\partial_t\mathbf{u}\|^{2\theta_4}_{L^{\frac{2q}{2-p^-+q}}}\\
&\leqslant \|\rho\|^{\frac{1+2\theta_4}{2}}_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^q(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{2(1-\theta_4)}_{L^2(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}\|\widetilde{D}\mathbf{u}\|^{\frac{2-p^-}{2}}_{L^q(\Omega)})^{2\theta_4}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C(\|\rho\|^{R_{31}}_{L^\infty(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{R_{32}}_{L^2(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{R_{33}}_{L^q(\Omega)})
\end{aligned}
\tag{57}
$$

hold for any fixed $\varepsilon\in(0,1)$, where $\theta_4=\frac{3}{2q+3p^--6}$, $R_{31}=\frac{3(2q+3p^-)}{2(2q+3p^--9)}$, $R_{32}=6$ and $R_{33}=\frac{6q}{2q+3p^--9}$.

In addition, it follows from $(4)_1$ that

$$
\partial_t(\rho^\gamma)+\operatorname{div}((\rho)^\gamma\mathbf{u})+(\gamma-1)(\rho)^\gamma\operatorname{div}\mathbf{u}=0.
\tag{58}
$$

One estimate I_5 as follows

$$
\begin{aligned}
|I_5| &= |\int_\Omega \partial_t(\rho)^\gamma\operatorname{div}\partial_t\mathbf{u}\,dx|\\
&= -\int_\Omega(\nabla(\rho)^\gamma\cdot\mathbf{u}+\gamma(\rho)^\gamma\operatorname{div}\mathbf{u})\operatorname{div}\partial_t\mathbf{u}\,dx\\
&\leqslant C\|\rho\|^{\gamma-1}_{L^\infty(\Omega)}\|\nabla\rho\|_{L^{3p^-}(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\|\mathbf{u}\|_{L^{12}(\Omega)}+\|\rho\|^\gamma_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^{\frac{3p^-}{2p^--1}}(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}}\\
&\leqslant C\|\rho\|^{\gamma-1}_{L^\infty(\Omega)}\|\nabla\rho\|_{L^{3p^-}(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})+\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}\\
&+\|\rho\|^\gamma_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^{1-\theta_1}_{L^2(\Omega)}\|\nabla\mathbf{u}\|^{\theta_1}_{L^{3p^-}(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})+\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant C\|\rho\|^{\gamma-1}_{L^\infty(\Omega)}\|\nabla\rho\|_{L^{3p^-}(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})+\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}\\
&+\|\rho\|^\gamma_{L^{\infty(\Omega)}}\|\widetilde{D}\mathbf{u}\|^{1-\theta_1}_{L^q(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{\theta_1}{p^-}}(\mathcal{J}_\Phi(\mathbf{u})+\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\rho\|^{R_{34}}_{L^\infty(\Omega)}+\|\nabla\rho\|^{R_{35}}_{L^{3p^-}(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{R_{36}}_{L^q(\Omega)})
\end{aligned}
\tag{59}
$$

hold for any fixed $\varepsilon\in(0,1)$, where $R_{34}=max\{3(\gamma-1)R_6,\frac{2\gamma(3p^--2)}{3p^--4}\}$, $R_{35}=3R_6$ and $R_{36}=max\{3R_6,\frac{8p^-}{3p^--4}\}$.

Moreover, one uses the strategy argument as for (45) to find that

$$-\int_\Omega \operatorname{div}\partial_t\left((1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u}\right)\cdot\partial_t\mathbf{u}\,dx \geqslant \mathcal{J}_\Phi(\mathbf{u})-\|\widetilde{D}\mathbf{u}\|^q_{L^q(\Omega)}. \tag{60}$$

Therefore, it deduces from (50)-(60) that

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^2(\Omega)}+\mathcal{J}_\Phi(\mathbf{u})\\ &\leqslant \varepsilon\mathcal{I}^r_\Phi(\mathbf{u})+C(1+\|\nabla\rho\|^{m_5}_{L^{3p^-}(\Omega)}+\|\widetilde{D}\mathbf{u}\|^{m_6}_{L^q(\Omega)}+\|\rho\|^{m_7}_{L^\infty(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{m_8}_{L^2(\Omega)}),\end{aligned} \tag{61}$$

hold for any fixed $\varepsilon\in(0,1)$, and the proof of (49) is completed, where $m_5=R_{35}$, $m_6=max\{R_{19},R_{21},R_{25},R_{28},R_{30},$
$R_{33},R_{36},q\}$, $m_7=max\{R_{18},R_{20},R_{23},R_{26},R_{29},R_{31},R_{34}\}$ and $m_8=max\{R_{17},R_{22},R_{24},$
$R_{27},R_{32}\}$. □

Finally, we are going to derive the estimates for density.

Lemma 13 *Let $max\{1,\frac{5}{5p^- -4}\}\leqslant r<\frac{5p^- -6}{2-p^-}$ and $\frac{12}{5}\leqslant q\leqslant min\{6(p^- -1),2p^-\}$. Suppose that $(\rho,\mathbf{u})$ is a smooth solution to (4)-(5) on $[0,T)\times\Omega$ for some $T>0$. Then there exists a positive constant $m_9=m_9(p^-)$, $m_{10}=m_{10}(p^-)$ and $m_{11}=m_{11}(p^-)$ such that*

$$\frac{d}{dt}\|\rho\|_{W^{1,3p^-}(\Omega)}\leqslant \varepsilon\mathcal{I}^r_\Phi(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\widetilde{D}\mathbf{u}\|^{m_9}_{L^q(\Omega)}+\|\rho\|^{m_{10}}_{W^{1,3p^-}}) \tag{62}$$

holds for any fixed $\varepsilon\in(0,1)$.

Proof The equation $(4)_1$ implies that

$$\rho_t+\rho\operatorname{div}\mathbf{u}+\nabla\rho\cdot\mathbf{u}=0. \tag{63}$$

Differentiating it with respect to x_k yields

$$\partial_t(\rho_{x_k})+\mathbf{u}\cdot\nabla\rho_{x_k}+\nabla\rho\cdot\mathbf{u}_{x_k}+\rho_{x_k}\operatorname{div}\mathbf{u}+\rho\operatorname{div}\mathbf{u}_{x_k}=0. \tag{64}$$

Now, multiplying the equation (64) by $\rho_{x_k}|\rho_{x_k}|^{3p^- -2}$, integrating the resulting equation over Ω and summing over k, we get that

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|^{3p^-}_{L^{3p^-}(\Omega)}&\leqslant C\int_\Omega|\nabla\mathbf{u}||\nabla\rho|^{3p^-}\,dx+\int_\Omega\rho|\nabla\operatorname{div}\mathbf{u}||\nabla\rho|^{3p^- -1}\,dx\\ &\leqslant C\big(\|\nabla\mathbf{u}\|_{L^\infty(\Omega)}\|\nabla\rho\|^{3p^-}_{L^{3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}\|\nabla\operatorname{div}\mathbf{u}\|_{L^{3p^-}(\Omega)}\|\nabla\rho\|^{3p^- -1}_{L^{3p^-}(\Omega)}\big),\end{aligned}$$

which implies that

$$\frac{d}{dt}\|\nabla\rho\|_{L^{3p^-}(\Omega)}\leqslant C(\|\nabla\mathbf{u}\|_{L^\infty(\Omega)}\|\nabla\rho\|_{L^{3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}\|\nabla^2\mathbf{u}\|_{L^{3p^-}(\Omega)}).$$

Similarly, we can derive a bound on $\|\rho\|_{L^{3p^-}(\Omega)}$ with multiplying (63) by ρ and integrating the resultant over Ω. Thus, it is to obtain that

$$\frac{d}{dt}\|\rho\|_{W^{1,3p^-}(\Omega)}\leqslant C\|\mathbf{u}\|_{W^{2,3p^-}(\Omega)}\|\rho\|_{W^{1,3p^-}(\Omega)}. \tag{65}$$

In order to complete the estimate of (65), one has to show the L^s-estimates on the elliptic operator associated to the system based on the results stated in Lemma 7. Indeed, we write the momentum equation as follows

$$-\mathcal{A}(\mathbf{u}^*,\mathbb{D})\mathbf{u}=-\rho\mathbf{u}_t-\rho(\mathbf{u}\cdot\nabla)\mathbf{u}-\nabla(\rho)^\gamma+\rho\mathbf{f}+(\mathcal{A}(\mathbf{u},\mathbb{D})\mathbf{u}-\mathcal{A}(\mathbf{u}^*,\mathbb{D})\mathbf{u}) \tag{66}$$

where $\mathbf{u}^*$ is a reference solution. Regarding (66) as a quasi-linear elliptic equation, we know that the linearized operator still yield maximal L^s-regularity according to Lemma 7. Since

$$\begin{aligned}&\mathcal{A}(\mathbf{u},\mathbb{D})\mathbf{u}-\mathcal{A}(\mathbf{u}^*,\mathbb{D})\mathbf{u}\\ &=\frac{1}{2}((1+|\mathbb{D}\mathbf{u}|^2)-(1+|\mathbb{D}\mathbf{u}^*|^2))(\Delta\mathbf{u}+\nabla\operatorname{div}\mathbf{u})\\ &+\sum_{j,k,l=1}^{3}(p(t,x)-2)((1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-4}{2}}(\mathbb{D}\mathbf{u})_{ik}(\mathbb{D}\mathbf{u})_{jl}-(1+|\mathbb{D}\mathbf{u}^*|^2)^{\frac{p(t,x)-4}{2}}(\mathbb{D}\mathbf{u}^*)_{ik}(\mathbb{D}\mathbf{u}^*)_{jl})\partial_k\partial_l\mathbf{u}_j,\end{aligned}$$

one finds that

$$\begin{aligned}&\|\mathcal{A}(\mathbf{u},\mathbb{D})\mathbf{u}-\mathcal{A}(\mathbf{u}^*,\mathbb{D})\mathbf{u}\|_{L^s(\Omega)}\\ &\leqslant C\||\mathbb{D}\mathbf{u}|^2-|\mathbb{D}\mathbf{u}^*|^2\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{W^{2,s}(\Omega)}\\ &+C\sum_{k,l=1}^{3}\||(\mathbb{D}\mathbf{u})_{ik}(\mathbb{D}\mathbf{u})_{jl}-(\mathbb{D}\mathbf{u}^*)_{ik}(\mathbb{D}\mathbf{u}^*)_{jl}|\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{W^{2,s}(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{k,l=1}^{3}\|(|\mathbb{D}\mathbf{u}|^2-|\mathbb{D}\mathbf{u}^*|^2)(\mathbb{D}\mathbf{u}^*)_{ik}(\mathbb{D}\mathbf{u}^*)_{jl}\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{W^{2,s}(\Omega)}\\
&\leqslant C\||\mathbb{D}\mathbf{u}|-|\mathbb{D}\mathbf{u}^*|\|_{L^\infty(\Omega)}\||\mathbb{D}\mathbf{u}|+|\mathbb{D}\mathbf{u}^*|\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{W^{2,s}(\Omega)}\\
&+C\sum_{k,l=1}^{3}(\|(\mathbb{D}\mathbf{u})_{ik}|_{L^\infty(\Omega)}\||(\mathbb{D}\mathbf{u})_{jl}|-|(\mathbb{D}\mathbf{u}^*)_{jl}|\|_{L^\infty(\Omega)}\\
&+C\sum_{k,l=1}^{3}\|(\mathbb{D}\mathbf{u}^*)_{jl}\|_{L^\infty(\Omega)}\||(\mathbb{D}\mathbf{u})_{ik}|-|(\mathbb{D}\mathbf{u}^*)_{ik}|\|_{L^\infty(\Omega)})\|\mathbf{u}\|_{W^{2,s}(\Omega)}\\
&+C\sum_{k,l=1}^{3}\||\mathbb{D}\mathbf{u}|-|\mathbb{D}\mathbf{u}^*|\|_{L^\infty(\Omega)}\|(\mathbb{D}\mathbf{u}^*)_{ik}(\mathbb{D}\mathbf{u}^*)_{jl}\|_{L^\infty(\Omega)}\||\mathbb{D}\mathbf{u}|+|\mathbb{D}\mathbf{u}^*|\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{W^{2,s}(\Omega)}\\
&\leqslant C\|\nabla(\mathbf{u}-\mathbf{u}^*)\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{W^{2,s}(\Omega)}.
\end{aligned}
$$

So, one can choose the reference solution $\mathbf{u}^*$ of (66) such that $\mathbf{u}$ represents a small perturbation of $\mathbf{u}^*$, and deduces from (66) that

$$
\|\mathbf{u}\|_{W^{2,s}(\Omega)}\leqslant C\left(\|-\rho\partial_t\mathbf{u}-\rho(\mathbf{u}\cdot\nabla)\mathbf{u}-\nabla(\rho)^\gamma+\rho\mathbf{f}\|_{L^s(\Omega)}+\|\nabla\mathbf{u}\|_{L^{p^-}(\Omega)}^{p^-}+1\right)+\frac{1}{2}\|\mathbf{u}\|_{W^{2,s}(\Omega)}. \tag{67}
$$

Thus, it follows that

$$
\begin{aligned}
&\|\mathbf{u}\|_{W^{2,3p^-}(\Omega)}\\
&\leqslant C(\|\rho\partial_t\mathbf{u}\|_{L^{3p^-}(\Omega)}+\|\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^{3p^-}(\Omega)}+\|\nabla(\rho)^\gamma\|_{L^{3p^-}(\Omega)}+\|\rho\mathbf{f}\|_{L^{3p^-}(\Omega)}\|\nabla\mathbf{u}\|_{L^{p^-}(\Omega)}^{p^-}+1)\\
&\leqslant C(\|\rho\|_{L^\infty(\Omega)}\|\partial_t\mathbf{u}\|_{L^{3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^{3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}^{\gamma-1}\|\nabla\rho\|_{L^{3p^-}(\Omega)}\\
&+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{f}\|_{L^{3p^-}}+\|\nabla\mathbf{u}\|_{L^{p^-}(\Omega)}^{p^-}+1)\\
&\leqslant C(\|\rho\|_{L^\infty(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+\mathcal{J}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^{12}(\Omega)}^{1-\theta}\|\nabla^2\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}^{1+\theta}\\
&+\|\rho\|_{L^\infty(\Omega)}^{\gamma-1}\|\nabla\rho\|_{L^{3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{f}\|_{L^{3p^-}}+\|\nabla\mathbf{u}\|_{L^{p^-}(\Omega)}^{p^-}+1)\\
&\leqslant C(\|\rho\|_{L^\infty(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+\mathcal{J}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}+\|\rho\|_{L^\infty(\Omega)}\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}^{1-\theta}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1+\theta}{p^-}}\\
&+\|\rho\|_{L^\infty(\Omega)}^{\gamma-1}\|\nabla\rho\|_{L^{3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{f}\|_{L^{3p^-}}+\|\nabla\mathbf{u}\|_{L^{p^-}(\Omega)}^{p^-}+1)\\
&\leqslant \varepsilon\mathcal{I}_\Phi^r(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\rho\|_{L^\infty(\Omega)}^{R_{37}}+\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}^{R_{38}}+\|\nabla\rho\|_{L^{3p^-}(\Omega)}^2)
\end{aligned}\tag{68}
$$

holds for any fixed $\varepsilon\in(0,1)$, where $R_{37}=max\{\frac{2p^-(5p^--4)r}{p^-(5p^--4)r-6p^-+4},2(\gamma-1)\}$, $R_{38}=max\{R_6,\frac{8p^-r(p^--1)}{p^-(5p^--4)r-6p^-+4}\}$. Combining (65) and (68), we conclude with

$$
\begin{aligned}
&\frac{d}{dt}\|\rho\|_{W^{1,3p^-}(\Omega)}\\
&\leqslant C(\|\rho\|_{L^\infty(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+\mathcal{J}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\|\rho\|_{W^{1,3p^-}(\Omega)} &(69)\\
&+\|\rho\|_{L^\infty(\Omega)}^{\gamma-1}\|\rho\|_{W^{1,3p^-}(\Omega)}^2+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{f}\|_{L^{3p^-}}\|\rho\|_{W^{1,3p^-}(\Omega)}\\
&+\|\nabla\mathbf{u}\|_{L^{p^-}(\Omega)}^{p^-}\|\rho\|_{W^{1,3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}^{\theta}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1+\theta}{p^-}}\|\rho\|_{W^{1,3p^-}(\Omega)})\\
&\leqslant \varepsilon\mathcal{I}_\Phi^r(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(1+\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}^{m_9}+\|\rho\|_{W^{1,3p^-}(\Omega)}^{m_{10}}) &(70)
\end{aligned}
$$

holds for any fixed $\varepsilon\in(0,1)$, where $m_9=max\{\frac{12p^-r(p^--1)}{p^-(5p^--4)r-6p^-+4},2p^-\}$ and $m_{10}=max\{\frac{6p^-(5p^--4)r}{p^-(5p^--4)r-6p^-+4},2R_6,4,2(\gamma-1)\}$. $\square$

Consequently, the following proposition is deduced from Lemma 11 - Lemma 13 directly.

Proposition 14 *Let $max\{1,\frac{5}{5p^--4}\}\leqslant r<\frac{5p^--6}{2-p^-}$ and $\frac{12}{5}\leqslant q\leqslant min\{6(p^--1),2p^-\}$. Suppose that $(\rho,\mathbf{u})$ is a smooth solution to (4)-(5) on $[0,T)\times\Omega$ for some $T>0$. Then there exists a positive constant m_{11}, m_{12} and m_{13} such that*

$$
\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^2+\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}^q+\|\rho\|_{W^{1,3p^-}(\Omega)})+\|\mathbf{u}\|_{W^{2,3p^-}(\Omega)}+\mathcal{I}_\Phi^r(\mathbf{u})+\mathcal{J}_\Phi(\mathbf{u})\\
&\leqslant C(1+\|\rho\|_{W^{1,3p^-}(\Omega)}^{m_{11}}+\|\widetilde{D}\mathbf{u}\|_{L^q(\Omega)}^{m_{12}}+\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{m_{13}}).
\end{aligned}\tag{71}
$$

4 Proof of Theorem 1

4.1 The Faedo-Galerkin approximations

Based on the necessary a priori estimates established in the previous section, the existence of strong solutions can be established by a standard argument: construct approximate solutions via a semi-discrete Galerkin scheme, derive uniform bounds and obtain solutions by passing to the limit, see ([3], [6]). Precisely, we construct approximate solutions in an appropriate finite-dimensional space by a fixed-point argument combined with the characteristics method and the Faeodo-Galerkin method. The existence of solutions will be extended to the complete space with the help of the a priori estimates established in the previous section. Take $X = H^2(\Omega)$ as the basic function space as $X = H_0^1(\Omega) \cap H^2(\Omega)$ and its finite-dimensional subspaces as

$$X^N = \mathrm{span}\{\phi^1, ..., \phi^N\} \subset X \cap C^2(\bar{\Omega}) \quad (N = 1, 2, ...),$$

where $\{\phi^k\}$ denotes the set consisting of the eigenvectors of the Stokes operator. Thus, $\{\phi^k\}$ is an orthogonal basis of $L^2(\Omega)$ (is also an orthogonal in $H^2(\Omega)$).

Suppose $\rho_0, \mathbf{u}_0$ and $\mathbf{f}$ be the data satisfying the hypotheses of Theorem 1. We assume that $\rho_0 \in C^1(\overline{\Omega})$ satisfying $\rho_0 \geqslant \delta$ for some $\delta > 0$. Let us define $\mathbf{u}_\delta^N = \sum\limits_{r=1}^{N} c_{r,\delta}^N(t)\phi^r(x)$, then we are looking for solution $(\rho_\delta^N, \mathbf{u}_\delta^N)$ to the following initial problem

$$\begin{cases} \rho_t^N + \operatorname{div}(\rho_\delta^N \mathbf{u}_\delta^N) = 0, \\ \int_\Omega (\rho_\delta^N \partial_t \mathbf{u}_\delta^N + \rho_\delta^N (\mathbf{u}_\delta^N \cdot \nabla)\mathbf{u}_\delta^N + \operatorname{div}((1 + |\mathbb{D}\mathbf{u}_\delta^N|^2)^{\frac{p(t,x)-2}{2}} \mathbb{D}\mathbf{u}_\delta^N) + \nabla(\rho_\delta^N)^\gamma) \cdot \mathbf{v}\, dx \\ \qquad\qquad = \int_\Omega \rho_\delta^N \mathbf{f} \cdot \mathbf{v}\, dx \quad (\mathbf{v} \in X^N), \\ \rho_\delta^N(0) = \rho_0, \quad \mathbf{u}_\delta^N(0) = \sum\limits_{r=1}^{N} \langle \mathbf{u}_0, \phi^r \rangle \phi^r. \end{cases} \tag{72}$$

Actually, for any given $\mathbf{u}_\delta^N \in C^1([0,T); X^N)$, one can deduce from the classical theory of transport equation that there exists a classical solution $\rho_\delta^N(t,x) \in C^1([0,T); X^N)$ to $(72)_1$ with initial data $\rho_\delta^N(0) \geqslant \delta > 0$. Moreover,

$$\inf_{x\in\Omega} \rho_0(x) \exp\Big(-\int_0^t \|\operatorname{div} \mathbf{u}_\delta^N\|_{L^\infty(\Omega)}\, ds\Big) \leqslant \rho_\delta^N(t,x) \leqslant \sup_{x\in\Omega} \rho_0(x) \exp\Big(\int_0^t \|\operatorname{div} \mathbf{u}_\delta^N\|_{L^\infty(\Omega)}\, ds\Big). \tag{73}$$

So, there exists a positive constant C such that

$$\rho_\delta^N(t,x) > C\delta > 0 \quad \text{for all} \quad (t,x) \in [0,T) \times \Omega.$$

Introduce the operator

$$\begin{aligned} &\mathcal{T} : C^1([0,T]; X^N) \mapsto C^1([0,T]; C^1(\Omega)), \\ &\mathbf{u}_\delta^N \mapsto \rho_\delta^N[\mathbf{u}_\delta^N], \end{aligned}$$

that is, $\rho_\delta^N[\mathbf{u}_\delta^N] = \mathcal{T}(\mathbf{u}_\delta^N)$. Taking argument in [Chapter 7, [12]], one finds that the mapping $\mathcal{T}$ is continuous.

Next, we turn to our attention to the approximated problem represented by the integral equation $(72)_2$. Suppose that $\rho_\delta^N \in C^1([0,T]; C^1(\Omega))$ is given. Following argument in [Chapter 7, [12]], we consider a family of linear operators

$$\mathcal{M}[\rho] : X^N \mapsto (X^N)^*, \quad \langle \mathcal{M}[\rho]\mathbf{v}, \mathbf{w} \rangle = \int_\Omega \varrho \mathbf{v} \cdot \mathbf{w}\, dx \ (\forall \mathbf{v}, \mathbf{w} \in X^N),$$

where $(X^N)^*$ stands for the dual space of X^N. It is easy to see that

$$\inf_{\|w\|_{L^2(\Omega)}=1} \langle \mathcal{M}[\rho]w, w \rangle = \inf_{\|w\|_{L^2(\Omega)}=1} \int_\Omega \varrho |w|^2\, dx \geqslant \inf_{(t,x)\in[0,T]\times\Omega} \varrho(t,x) \geqslant C\delta > 0.$$

So, the operator $\mathcal{M}$ is invertible and

$$\|\mathcal{M}^{-1}[\rho]\|_{\mathcal{L}((X^N)^*,X^N)} \leqslant (C\delta)^{-1},$$

where $\mathcal{L}((X^N)^*, X^N)$ denotes the set of bounded linear mappings from $(X^N)^*$ to X^N. Moreover, $\mathcal{M}^{-1}$ is Lipschitz continuous in the sense

$$\|\mathcal{M}^{-1}[\varrho_1] - \mathcal{M}^{-1}[\varrho_2]\|_{\mathcal{L}((X^N)^*,X^N)} \leqslant C(N,\delta)\|\varrho_1 - \varrho_2\|_{L^1(\Omega)}, \tag{74}$$

for $\varrho_1, \varrho_2 \in L^1(\Omega)$ such that $\varrho_1, \varrho_2 \geqslant C\delta > 0$. Thus, the integral equation $(72)_2$ can be rephrased as

$$\mathbf{u}_\delta^N(t) = \mathcal{M}^{-1}[\mathcal{T}(\mathbf{u}_\delta^N)(t)]\Big(\mathcal{M}[\rho_0]\mathbf{u}_0^N + \int_0^t \mathcal{N}[T(\mathbf{u}_\delta^N), \mathbf{u}_\delta^N(s)]\,ds\Big) \quad \text{in} \quad X^N,$$

where

$$\begin{aligned}
&\mathcal{N} : X^N \mapsto (X^N)^* \\
&\langle \mathcal{N}[\mathcal{T}(\mathbf{u}_\delta^N), \mathbf{u}_\delta^N], \eta\rangle \equiv \int_\Omega \mathcal{T}(\mathbf{u}_\delta^N)\mathbf{f}\cdot\eta dx + \int_\Omega (\mathcal{T}(\mathbf{u}_\delta^N))^\gamma \mathrm{div}\eta dx \\
&\qquad + \int_\Omega [\mathcal{T}(\mathbf{u}_\delta^N)\mathbf{u}_\delta^N \otimes \mathbf{u}_\delta^N - ((1+|\mathbb{D}\mathbf{u}_\delta^N|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u}_\delta^N)] : \nabla\eta dx.
\end{aligned}$$

Thus, the integral equation $(72)_2$ can be rewritten as a system of ordinary differential equations. This in turn fulfills the Carathéodory conditions and is therefore solvable locally in time, i.e. on a small time interval $[0, T^*)$ for some $T^* < T$. To ensure solvability for large times at least for this finite dimensional problem we have to establish a first a priori estimate.

Since $\mathbf{f} \in L^\infty(0,T;W^{1,2}(\Omega))$ and $\partial_t\mathbf{f} \in L^2(0,T;L^2(\Omega))$, one deduces that

$$\sup_{0\leqslant t\leqslant T}\int_\Omega (\rho_\delta^N|\mathbf{u}_\delta^N|^2 + (\rho_\delta^N)^\gamma)\,dx + \int_0^T\int_\Omega |\mathbb{D}\mathbf{u}_\delta^N|^{p(t,x)}\,dxdt \leqslant C(T,\rho_0,\mathbf{u}_0,\mathbf{f}). \tag{75}$$

Moreover, all norms on X^N are equivalent. One finds that the velocity $\mathbf{u}_\delta^N$ is bounded in $L^1(0,T(n);W^{1,\infty}(\Omega))$. So, it follows from (73) that the density ρ_δ^N is bounded both from below and from above by a constant independent of $T(n)$. As a consequence, $\mathbf{u}_\delta^N$ and ρ_δ^N remain bounded in X^N independently of $T(n)$. Thus, one is allowed to iterate the previous local existence result to construct a solution $(\rho_\delta^N, \mathbf{u}_\delta^N) \in C^1([0,T];X^N) \times C^1([0,T];X^N)$ to (72).

4.2 Estimates independent of N and δ

It easy to see that the energy equation (75) implies that

$$\|\sqrt{\rho_\delta^N}\mathbf{u}_\delta^N\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \tag{76}$$

$$\|\rho_\delta^N\|_{L^\infty(0,T;L^\gamma(\Omega))} \leqslant C, \tag{77}$$

$$\|\mathbb{D}\mathbf{u}_\delta^N\|_{L^{p^-}((0,T)\times\Omega)} \leqslant C. \tag{78}$$

Evoking Lemma 6, there exist a positive constant C independent N and δ, such that

$$\|\mathbf{u}_\delta^N\|_{L^{p^-}(0,T;W^{1,p^-}(\Omega))} \leqslant C. \tag{79}$$

Moreover, one uses the same arguments as developed in the previous section, obtains that there exists positive constants m_{11}, m_{12}, m_{13}, independent of N and δ, and $\frac{12}{5} \leqslant q \leqslant min\{6(p^- - 1), 2p^-\}$ such that

$$\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho_\delta^N}\partial_t\mathbf{u}_\delta^N\|_{L^2(\Omega)}^2 + \|\widetilde{D}\mathbf{u}_\delta^N\|_{L^q(\Omega)}^q + \|\rho_\delta^N\|_{W^{1,3p^-}(\Omega)}) \\
&+ \|\mathbf{u}_\delta^N\|_{W^{2,3p^-}(\Omega)} + \mathcal{I}_\Phi(\mathbf{u}_\delta^N)^r + \mathcal{J}_\Phi(\mathbf{u}_\delta^N) \\
&\leqslant C(1 + \|\rho_\delta^N\|_{W^{1,3p^-}(\Omega)}^{m_{11}} + \|\widetilde{D}\mathbf{u}_\delta^N\|_{L^q(\Omega)}^{m_{12}} + \|\sqrt{\rho}\partial_t\mathbf{u}_\delta^N\|_{L^2(\Omega)}^{m_{13}}).
\end{aligned} \tag{80}$$

Thus, one deduces from Lemma 5 that there exists a small time $T^* \in (0,T)$ such that

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T^*}(\|\sqrt{\rho_\delta^N}\partial_t\mathbf{u}_\delta^N\|_{L^2(\Omega)}^2+\|\widetilde{D}\mathbf{u}_\delta^N\|_{L^q(\Omega)}^q+\|\rho_\delta^N\|_{W^{1,3p^-}(\Omega)})\\
&+\int_0^{T^*}(\|\mathbf{u}_\delta^N\|_{W^{2,3p^-}(\Omega)}+\mathcal{I}_\Phi^r(\mathbf{u}_\delta^N)+\mathcal{J}_\Phi(\mathbf{u}_\delta^N))\,dt\\
&\leqslant C,
\end{aligned} \tag{81}$$

where C is independent of N and δ. So, one integrates (68) over $(0,T^*)$ to obtain that

$$\int_0^{T^*}\|\mathbf{u}_\delta^N\|_{W^{2,3p^-}(\Omega)}^{p^-}\,dt\leqslant C, \tag{82}$$

where C is independent of N and δ.

Recalling the definition of $\mathcal{I}_\Phi(\mathbf{u}_\delta^N)$ and $\mathcal{J}_\Phi(\mathbf{u}_\delta^N)$, one follows from Lemma 9 that the approximation solutions $(\rho_\delta^N,\mathbf{u}_\delta^N)$ have the following properties

$$\begin{cases}
\|\sqrt{\rho_\delta^N}\mathbf{u}_\delta^N\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C, & \|\rho_\delta^N\|_{L^\infty(0,T;L^\gamma(\Omega))}\leqslant C,\\
\|\rho_\delta^N\|_{L^\infty(0,T;W^{1,3p^-}(\Omega))}\leqslant C, & \|\mathbf{u}_\delta^N\|_{L^\infty(0,T;W^{1,\frac{12}{5}}(\Omega))}\leqslant C,\\
\|\mathbf{u}_\delta^N\|_{L^{p^-}(0,T;W^{2,p^-}(\Omega))}\leqslant C, & \|\mathbf{u}_\delta^N\|_{L^{\frac{p^-(5p^--6)}{2-p^-}}(0,T^*;W^{2,\frac{3p^-}{p^-+1}}(\Omega))}\leqslant C,\\
\|\sqrt{\rho_\delta^N}\partial_t\mathbf{u}_\delta^N\|_{L^\infty(0,T^*;L^2(\Omega))}\leqslant C, & \|\partial_t\mathbf{u}_\delta^N\|_{L^{p^-}(0,T^*;W^{1,\frac{3p^-}{p^-+1}}(\Omega))}\leqslant C
\end{cases} \tag{83}$$

where C is independent of N and δ.

4.3 Passage to the limit

In this subsection, we are ready to pass the limit for $N\to\infty$ and $\delta\to 0^+$ in the sequence of approximate solution $(\rho_\delta^N,\mathbf{u}_\delta^N)$. We first consider the limit for $N\to\infty$. Over we can pick a subsequence (still denoted by $(\rho_\delta^N,\mathbf{u}_\delta^N)$) with

$$\begin{aligned}
&\rho_\delta^N\overset{*}{\rightharpoonup}\rho_\delta \text{ in } L^\infty(0,T^*;W^{1,3p^-}(\Omega)), && \mathbf{u}_\delta^N\overset{*}{\rightharpoonup}\mathbf{u}_\delta \text{ in } L^\infty(0,T^*;W^{1,\frac{12}{5}}(\Omega)),\\
&\mathbf{u}_\delta^N\rightharpoonup\mathbf{u}_\delta \text{ in } L^{\frac{p^-(5p^--6)}{2-p^-}}(0,T^*;W^{2,\frac{3p^-}{p^-+1}}(\Omega)), && \mathbf{u}_\delta^N\rightharpoonup\mathbf{u}_\delta \text{ in } L^{p^-}(0,T^*;W^{2,3p^-}(\Omega)),\\
&\sqrt{\rho_\delta^N}\partial_t\mathbf{u}_\delta^N\rightharpoonup\sqrt{\rho}\partial_t\mathbf{u}_\delta \text{ in } L^\infty(0,T^*;L^2(\Omega)), && \partial_t\mathbf{u}_\delta^N\rightharpoonup\partial_t\mathbf{u}_\delta \text{ in } L^{p^-}(0,T^*;W^{1,\frac{3p^-}{p^-+1}}(\Omega)).
\end{aligned}$$

with the help of the Aubin-Lions lemma. These convergence properties are sufficient to pass to the limit in N. Let us just give some details about convergence of the extra term that

$$\mathbf{S}(\mathbb{D}\mathbf{u})=(1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u}.$$

Observe that due to the (83), we know that

$$|\mathbf{S}(\mathbb{D}\mathbf{u}_\delta^N(x))|\leqslant C\quad\text{for all } x\in\Omega.$$

On the other hand, we have

$$\mathbb{D}\mathbf{u}_\delta^N\to\mathbb{D}\mathbf{u}_\delta\quad\text{a.e. in } (0,T)\times\Omega.$$

Thus, by the continuity of $\mathbf{S}$, ones obtain

$$\mathbf{S}(\mathbb{D}\mathbf{u}_\delta^N)\to\mathbf{S}(\mathbb{D}\mathbf{u}_\delta)\quad\text{a.e. in }\ \times\Omega.$$

In addition, one takes the lower semi-continuity of norm and finds that $(\rho_\delta,\mathbf{u}_\delta)$ satisfies the following estimate

$$\sup_{0\leqslant t\leqslant T}\int_\Omega(\rho_\delta|\mathbf{u}_\delta|^2+(\rho_\delta)^\gamma)\,dx+\int_0^{T^*}\int_\Omega|\mathbb{D}\mathbf{u}_\delta|^{p(t,x)}\,dxdt\leqslant C \tag{84}$$

and

$$\sup_{0\leqslant t\leqslant T^*}\left(\|\sqrt{\rho_\delta}\partial_t\mathbf{u}_\delta\|^2_{L^2(\Omega)}+\|\mathbf{u}_\delta\|_{W^{1,\frac{12}{5}}(\Omega)}+\|\rho_\delta\|_{W^{1,3p^-}(\Omega)}\right)$$
$$+\int_0^{T^*}\left(\|\mathbf{u}_\delta\|^{p^-}_{W^{2,3p^-}(\Omega)}+\|\partial_t\mathbf{u}_\delta\|^{p^-}_{W^{1,\frac{3p^-}{p^-+1}}(\Omega)}+\|\mathbf{u}_\delta\|^{\frac{p^-(5p^--6)}{2-p^-}}_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}\right)dt\leqslant C, \tag{85}$$

where C is dependent of $T^*,\rho_0,\mathbf{u}_0,\mathbf{f}$ and independent of δ.

Our next goal is to carry out the limit process when $\delta\to 0^+$. For each $\delta\in(0,1)$, we take $\rho_{\delta,0}\in C^1(\overline{\Omega})$ with $\rho_{\delta,0}>\delta$ and

$$\rho_{\delta,0}\to\rho_0 \text{ in } W^{1,3p^-}(\Omega) \text{ (as } \delta\to 0^+).$$

Recall that T^* is independent of δ and $\mathbf{u}_\delta(0,x)\to\mathbf{u}(0,x)\in W^{2,p^-}(\Omega)$ (as $\delta\to 0^+$). One takes the same argument employed in the passage for $N\to\infty$ to get a subsequence of $(\rho_\delta,\mathbf{u}_\delta)$ converges to a strong solution $(\rho,\mathbf{u})$ to the problem (4)-(5) with compatibility conditions (8).

5 Proof of Theorem 2

The aim of this section is to establish a blow-up criterion for the problem (4)-(5) with compatible conditions (8). Let $(\rho,\mathbf{u})$ be a strong solution to the problem (4)-(5) on $[0,T^*)\times\Omega$, which satisfies the regularity (6) on the time interval $[0,T^*)$. We take to proof by contradiction to complete the proof of Theorem 2. Suppose that

$$\lim_{T\to T^*}\|\rho\|_{L^\infty(0,T;L^\infty(\Omega))}+\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^3(\Omega))}\leqslant M<\infty, \tag{86}$$

then it is going to deduce a contradiction to the maximality of T^*.

To begin with, one finds that the following energy inequality

$$\sup_{0\leqslant t\leqslant T}\int_\Omega(\rho|\mathbf{u}|^2+(\rho)^\gamma)\,dx+\int_0^{T^*}\int_\Omega|\mathbb{D}\mathbf{u}|^{p(t,x)}\,dxdt\leqslant C \tag{87}$$

holds for any $T\in(0,T^*)$.

Next, we drive bounds for time and second order derivatives of $\mathbf{u}$, by taking similar argument in Section 3.

Lemma 15 *Under the condition of Theorem 2 and (86), there exists a positive constant C such that*

$$\sup_{0\leqslant t\leqslant T}(\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^2(\Omega)}+\|\rho\|_{W^{1,3p^-}(\Omega)})+\int_0^T\left(\mathcal{I}^r_\Phi(\mathbf{u})+\mathcal{J}_\Phi(\mathbf{u})+\|\mathbf{u}\|^{p^-}_{W^{2,3p^-}(\Omega)}\right)dt\leqslant C \tag{88}$$

holds for any $T\in(0,min\{\frac{1}{(R_{39}-1)Cf^{\alpha-1}(0)},T^*\})$, where C depends on T continuously, $max\{1,\frac{5}{5p^--6}\}\leqslant r<\frac{5p^--6}{2-p^-}$, $\alpha=max\{\frac{p^-r}{p^--1},\frac{4r(p^--1)}{5p^--6-r(2-p^-)}\}$ and $f(0)\triangleq 1+\|\sqrt{\rho_0}\partial_t(\mathbf{u}_0)\|^2_{L^2(\Omega)}+\|\rho_0\|_{W^{1,3p^-}(\Omega)}$.

Proof Recalling the definition of (19), taking similar argument to derive (38) and (45), and selecting appropriate coefficient, ones gets that

$$\begin{aligned}\mathcal{I}_\Phi(\mathbf{u})&\leqslant\int_\Omega|\nabla\rho||\mathbf{u}||\nabla\mathbf{u}|^2\,dx+\int_\Omega\rho|\nabla\mathbf{u}|^3\,dx+\int_\Omega\rho|\mathbf{u}||\nabla^2\mathbf{u}||\nabla\mathbf{u}|\,dx\\&\quad+\int_\Omega\rho|\partial_t\mathbf{u}||\Delta\mathbf{u}|\,dx+\int_\Omega|\nabla(\rho)^\gamma||\Delta\mathbf{u}|dx+\int_0^T\rho|\mathbf{f}||\Delta\mathbf{u}|\,dx+C\int_\Omega(\widetilde{D}\mathbf{u})^{p(t,x)+\sigma}\,dx\\&\leqslant\sum_{i=1}^7L_i,\end{aligned} \tag{89}$$

where $\sigma\in(0,1)$. Now, one takes into account the definition of (20) and estimates each term on the right hand of (89) under the condition (86) as follows

$$\begin{aligned}L_1&\leqslant C\|\nabla\rho\|_{L^{3p^-}(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^2_{L^3(\Omega)}\\&\leqslant C\|\nabla\rho\|_{L^{3p^-}(\Omega)}\|\nabla\mathbf{u}\|^3_{L^3(\Omega)}\\&\leqslant C\|\nabla\rho\|_{L^{3p^-}(\Omega)},\\L_2&\leqslant C\|\rho\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^3_{L^3(\Omega)}\leqslant C,\end{aligned}$$

$$
\begin{aligned}
L_3 &\leqslant C\|\rho\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^3(\Omega)}\|\nabla^2\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant C\|\nabla\mathbf{u}\|^2_{L^3(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+C,\\
L_4 &\leqslant C\|\rho\|^{\frac12}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^{\frac{3p^-}{2p^--1}}(\Omega)}\|\Delta\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{4p^--4}{3p^--2}}_{L^2(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{2-p^-}{3p^--2}}_{L^{3p^-}(\Omega)}\|\mathbf{u}\|_{W^{2,\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant C\|\rho\|^{\frac{p^-}{3p^--2}}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{4p^--4}{3p^--2}}_{L^2(\Omega)}\|\partial_t\mathbf{u}\|^{\frac{2-p^-}{3p^--2}}_{W^{1,\frac{3p^-}{p^-+1}}(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{4p^--4}{3p^--2}}_{L^2(\Omega)}(\mathcal{J}_\Phi(\mathbf{u})^{\frac12}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2-p^-}{2p^-}})^{\frac{2-p^-}{3p^--2}}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{4p^--4}{3p^--2}}_{L^2(\Omega)}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{p^-+2}{2(3p^--2)}}\mathcal{J}_\Phi(\mathbf{u})^{\frac{2-p^-}{2(3p^--2)}}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+C(1+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{8(p^--1)}{5p^--6}}_{L^2(\Omega)}\mathcal{J}_\Phi(\mathbf{u})^{\frac{2-p^-}{5p^--6}}),\\
L_5 &\leqslant C\|\rho\|^{\gamma-1}_{L^\infty(\Omega)}\|\nabla\rho\|_{L^{3p^-}(\Omega)}\|\Delta\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant C\|\nabla\rho\|_{L^{3p^-}}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+C(1+\|\nabla\rho\|^{\frac{p^-}{p^--1}}_{L^{3p^-}(\Omega)}),\\
L_6 &\leqslant C\|\rho\|_{L^\infty(\Omega)}\|\mathbf{f}\|_{L^6(\Omega)}\|\Delta\mathbf{u}\|_{L^{\frac{3p^-}{p^-+1}}(\Omega)}\\
&\leqslant C(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\\
&\leqslant \varepsilon\mathcal{I}_\Phi(\mathbf{u})+C,\\
L_7 &\leqslant C\|\nabla\mathbf{u}\|^3_{L^3(\Omega)}\leqslant C,
\end{aligned}
$$

for any fixed $\varepsilon\in(0,1)$. Substituting the above estimates into (89), one gets that

$$
\mathcal{I}_\Phi(\mathbf{u})\leqslant C(1+\|\nabla\rho\|^{\frac{p^-}{p^--1}}_{L^{3p^-}(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{8(p^--1)}{5p^--6}}_{L^2(\Omega)}\mathcal{J}_\Phi(\mathbf{u})^{\frac{2-p^-}{5p^--6}}).
$$

So, for $r\in[max\{1,\frac{5}{5p^--6}\},\frac{5p^--6}{2-p^-})$

$$
\begin{aligned}
\mathcal{I}^r_\Phi(\mathbf{u}) &\leqslant C(1+\|\nabla\rho\|^{\frac{p^-r}{p^--1}}_{L^{3p^-}(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{8(p^--1)r}{5p^--6}}_{L^2(\Omega)}\mathcal{J}_\Phi(\mathbf{u})^{\frac{(2-p^-)r}{5p^--6}})\\
&\leqslant C(1+\eta\mathcal{J}_\Phi(\mathbf{u})+\|\nabla\rho\|^{\frac{p^-r}{p^--1}}_{L^{3p^-}(\Omega)}+\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{8r(p^--1)}{5p^--6-r(2-p^-)}}_{L^2(\Omega)})
\end{aligned}\tag{90}
$$

hold for fixed $\eta\in(0,1)$. Moreover, one obtains from (50) that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\rho|\partial_t\mathbf{u}|^2dx+\mathcal{J}_\Phi(\mathbf{u})\\
&\leqslant C\int_\Omega\rho|\partial_t\mathbf{u}|^2|\nabla\mathbf{u}|\ dx+\int_\Omega\rho|\mathbf{u}||\nabla\mathbf{f}||\partial_t\mathbf{u}|\ dx+\int_\Omega\rho|\mathbf{u}||\mathbf{f}||\nabla\partial_t\mathbf{u}|\ dx+\int_\Omega\rho|\partial_t\mathbf{f}||\partial_t\mathbf{u}|\ dx\\
&+\int_\Omega\gamma\rho^\gamma|\operatorname{div}\mathbf{u}||\operatorname{div}\partial_t\mathbf{u}|\ dx+\int_\Omega|\nabla P||\mathbf{u}||\operatorname{div}\partial_t\mathbf{u}|\ dx+\int_\Omega\rho|\mathbf{u}||\partial_t\mathbf{u}||\nabla\partial_t\mathbf{u}|\ dx+\int_\Omega\rho|\mathbf{u}||\nabla\mathbf{u}|^2|\partial_t\mathbf{u}|\ dx\\
&+\int_\Omega\rho|\mathbf{u}|^2|\nabla^2\mathbf{u}||\partial_t\mathbf{u}|\ dx+\int_\Omega\rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\partial_t\mathbf{u}|\ dx+\int_\Omega|\widetilde{D}\mathbf{u}|^{p(t,x)+\sigma}\ dx\\
&\leqslant\sum_{i=1}^{11}A_i.
\end{aligned}\tag{91}
$$

where $\sigma\in(0,1)$. Based on (24)-(25) and Sobolev embedding, one estimates A_i $(i=1,2,\cdots,11)$ as follows

$$
\begin{aligned}
A_1 &\leqslant C\|\rho\|^{\frac12}_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^3(\Omega)}\|\nabla\mathbf{u}\|_{L^3(\Omega)}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac47}_{L^2(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac{10}{7}}_{L^{\frac{15}{4}}(\Omega)}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac47}_{L^2(\Omega)}\|\nabla\partial_t\mathbf{u}\|^{\frac{10}{7}}_{L^{\frac53}(\Omega)}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|^{\frac47}_{L^2(\Omega)}(\int_\Omega(|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2})^{\frac56}|\widetilde{D}\mathbf{u}|^{\frac{5(2-p^-)}{6}}\ dx)^{\frac67}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{4}{7}}(\int_\Omega |\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2}\,dx)^{\frac{5}{7}}(\int_\Omega |\widetilde{D}\mathbf{u}|^{5(2-p^-)}\,dx)^{\frac{1}{7}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{4}{7}}\mathcal{J}_\Phi(\mathbf{u})^{\frac{5}{7}}(\int_\Omega |\widetilde{D}\mathbf{u}|^{5(2-p^-)}\,dx)^{\frac{1}{7}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C\|\sqrt{\rho}\partial_t\mathbf{u}|_{L^2(\Omega)}^2(\int_\Omega |\widetilde{D}\mathbf{u}|^{5(2-p^-)}\,dx)^2\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^2,\\
A_2 &\leqslant C\|\rho\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{f}\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\\
&\leqslant C\|\nabla\mathbf{u}\|_{L^3(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)},\\
A_3 &\leqslant C\|\rho\|_{L^\infty(\Omega)}\|\mathbf{f}\|_{L^6(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{6}{5}}(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\\
&\leqslant C\|\nabla\mathbf{u}\|_{L^3(\Omega)}(\int_\Omega (|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2})^{\frac{3}{5}}|\widetilde{D}\mathbf{u}|^{\frac{3(2-p^-)}{5}}\,dx)^{\frac{5}{6}}\\
&\leqslant C(\int_\Omega |\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2}\,dx)^{\frac{1}{2}}(\int_\Omega |\widetilde{D}\mathbf{u}|^{\frac{3(2-p^-)}{2}}\,dx)^{\frac{1}{3}}\\
&\leqslant C\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}\|\widetilde{D}\mathbf{u}\|_{L^3(\Omega)}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C,\\
A_4 &\leqslant C\|\rho\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\partial_t\mathbf{f}\|_{L^2(\Omega)}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^2+\|\partial_t\mathbf{f}\|_{L^2(\Omega)}^2,\\
A_5 &\leqslant C\|\rho\|_{L^\infty(\Omega)}^{\gamma}\|\nabla\mathbf{u}\|_{L^3(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega)}\\
&\leqslant C\|\nabla\mathbf{u}\|_{L^3(\Omega)}(\int_\Omega (|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2})^{\frac{3}{4}}|\widetilde{D}\mathbf{u}|^{\frac{3(2-p^-)}{4}}\,dx)^{\frac{2}{3}}\\
&\leqslant C(\int_\Omega |\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2}\,dx)^{\frac{1}{2}}(\int_\Omega |\widetilde{D}\mathbf{u}|^{3(2-p^-)}\,dx)^{\frac{1}{6}}\\
&\leqslant C\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}\|\widetilde{D}\mathbf{u}\|_{L^3(\Omega)}^{\frac{1}{2}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C,\\
A_6 &\leqslant C\|\rho\|_{L^\infty(\Omega)}^{\gamma-1}\|\nabla\rho\|_{L^{3p^-}(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\\
&\leqslant C\|\nabla\rho\|_{L^{3p^-}(\Omega)}\|\nabla\mathbf{u}\|_{L^3(\Omega)}(\int_\Omega (|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2})^{\frac{3}{4}}|\widetilde{D}\mathbf{u}|^{\frac{3(2-p^-)}{4}}\,dx)^{\frac{2}{3}}\\
&\leqslant C\|\nabla\rho\|_{L^{3p^-}(\Omega)}(\int_\Omega |\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2}\,dx)^{\frac{1}{2}}(\int_\Omega |\widetilde{D}\mathbf{u}|^{3(2-p^-)}\,dx)^{\frac{1}{6}}\\
&\leqslant C\|\nabla\rho\|_{L^{3p^-}(\Omega)}\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}\|\widetilde{D}\mathbf{u}\|_{L^3(\Omega)}^{\frac{1}{2}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C\|\nabla\rho\|_{L^{3p^-}(\Omega)}^2,\\
A_7 &\leqslant C\|\mathbf{u}\|_{L^\infty(\Omega)}\|\rho\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^3(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega)}\\
&\leqslant C\|\nabla\mathbf{u}\|_{L^3(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{7}}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega)}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}(\int_\Omega (|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2})^{\frac{5}{6}}|\widetilde{D}\mathbf{u}|^{5(2-p^-)}\,dx)^{\frac{3}{7}}\\
&\quad(\int_\Omega (|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2})^{\frac{3}{4}}|\widetilde{D}\mathbf{u}|^{\frac{3(2-p^-)}{4}}\,dx)^{\frac{2}{3}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}(\int_\Omega |\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^--2}\,dx)^{\frac{6}{7}}(\int_\Omega |\widetilde{D}\mathbf{u}|^{5(2-p^-)}\,dx)^{\frac{3}{14}}(\int_\Omega |\widetilde{D}\mathbf{u}|^{3(2-p^-)}\,dx)^{\frac{1}{6}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}\mathcal{J}_\Phi(\mathbf{u})^{\frac{6}{7}}\|\widetilde{D}\mathbf{u}\|_{L^3(\Omega)}^{\frac{5}{7}}\\
&\leqslant C\mathcal{J}_\Phi(\mathbf{u})^{\frac{6}{7}}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^2,\\
A_8 &\leqslant C\|\rho\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^3(\Omega)}\|\nabla\mathbf{u}\|_{L^3(\Omega)}^2\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^{\frac{15}{4}}(\Omega)}^{\frac{5}{7}}\|\nabla\mathbf{u}\|_{L^3(\Omega)}^3
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{7}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}(\int_\Omega(|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^- -2})^{\frac{5}{6}}|\widetilde{D}\mathbf{u}|^{\frac{5(2-p^-)}{6}}\,dx)^{\frac{3}{7}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}(\int_\Omega|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^- -2}\,dx)^{\frac{5}{14}}(\int_\Omega|\widetilde{D}\mathbf{u}|^{5(2-p^-)}\,dx)^{\frac{1}{14}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}\mathcal{J}_\Phi(\mathbf{u})^{\frac{5}{14}}\|\widetilde{D}\mathbf{u}\|_{L^3(\Omega)}^{\frac{3}{14}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C(\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^2+1),\\
A_9&\leqslant C\|\rho\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\mathbf{u}\|_{L^\infty(\Omega)}^2\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^3(\Omega)}\|\nabla^2\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega)}\\
&\leqslant C\|\nabla\mathbf{u}\|_{L^3(\Omega)}^2\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{7}}\|\nabla^2\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega)}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}(\int_\Omega(|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^- -2})^{\frac{5}{6}}|\widetilde{D}\mathbf{u}|^{\frac{5(2-p^-)}{6}}\,dx)^{\frac{3}{7}}(\int_\Omega(|\nabla^2\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^- -2})^{\frac{3}{4}}|\widetilde{D}\mathbf{u}|^{\frac{3(2-p^-)}{4}}\,dx)^{\frac{2}{3}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}(\int_\Omega|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^- -2}\,dx)^{\frac{5}{14}}(\int_\Omega|\widetilde{D}\mathbf{u}|^{5(2-p^-)}\,dx)^{\frac{1}{14}}\\
&\quad(\int_\Omega(|\nabla^2\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^- -2})^{\frac{3}{4}}\,dx)^{\frac{1}{2}}(\int_\Omega|\widetilde{D}\mathbf{u}|^{\frac{3(2-p^-)}{4}}\,dx)^{\frac{1}{6}}\\
&\leqslant C\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{2}{7}}\mathcal{J}_\Phi(\mathbf{u})^{\frac{5}{14}}\mathcal{I}_\Phi(\mathbf{u})^{\frac{1}{2}}\|\widetilde{D}\mathbf{u}\|_{L^3(\Omega)}^{\frac{5}{7}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+\varepsilon\mathcal{I}_\Phi(\mathbf{u})+C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2(\Omega)}^2,\\
A_{10}&\leqslant C\|\rho\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}^2\|\nabla\mathbf{u}\|_{L^3(\Omega)}\|\nabla\partial_t\mathbf{u}\|_{L^{\frac{3}{2}}(\Omega)}\\
&\leqslant C\|\nabla\mathbf{u}\|_{L^3(\Omega)}^3(\int_\Omega(|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^- -2})^{\frac{3}{4}}|\widetilde{D}\mathbf{u}|^{\frac{3(2-p^-)}{4}}\,dx)^{\frac{2}{3}}\\
&\leqslant C(\int_\Omega|\nabla\partial_t\mathbf{u}|^2|\widetilde{D}\mathbf{u}|^{p^- -2}\,dx)^{\frac{1}{2}}(\int_\Omega|\widetilde{D}\mathbf{u}|^{3(2-p^-)}\,dx)^{\frac{1}{6}}\\
&\leqslant C\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}\|\widetilde{D}\mathbf{u}\|_{L^3(\Omega)}^{\frac{1}{2}}\\
&\leqslant \eta\mathcal{J}_\Phi(\mathbf{u})+C,\\
A_{11}&\leqslant C\|\nabla\mathbf{u}\|_{L^3(\Omega)}^3\leqslant C
\end{aligned}
$$

for any fixed $\eta\in(0,1)$. Adding above estimate into (91) and choosing η sufficiently small, one obtains that

$$
\frac{d}{dt}\int_\Omega\rho|\partial_t\mathbf{u}|^2\,dx+\mathcal{J}_\Phi(\mathbf{u})\leqslant\varepsilon\mathcal{I}_\Phi(\mathbf{u})+C(\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^2+\|\nabla\rho\|_{L^{3p^-}(\Omega)}^2+\|\partial_t\mathbf{f}\|_{L^2(\Omega)}^2+1) \tag{92}
$$

for any fixed $\varepsilon\in(0,1)$. Note that

$$
\|\mathbf{u}\|_{W^{2,s}(\Omega)}\leqslant C\|-\rho\partial_t\mathbf{u}-\rho(\mathbf{u}\cdot\nabla)\mathbf{u}-\nabla\rho^\gamma\|_{L^s(\Omega)}+\|\nabla\mathbf{u}\|_{L^{p^-}(\Omega)}^{p^-}.
$$

It follows from Lemma 9 that

$$
\begin{aligned}
\|\mathbf{u}\|_{W^{2,3p^-}(\Omega)}&\leqslant C\left(\|\rho\partial_t\mathbf{u}\|_{L^{3p^-}(\Omega)}+\|\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^{3p^-}(\Omega)}+\|\rho\mathbf{f}\|_{L^{3p^-}(\Omega)}+\|\nabla\rho^\gamma\|_{L^{3p^-}(\Omega)}+\|\mathbb{D}\mathbf{u}\|_{L^{p^-}(\Omega)}^{p^-}\right)\\
&\leqslant C\Big(\|\rho\|_{L^\infty(\Omega)}\|\partial_t\mathbf{u}\|_{L^{3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^{3p^-}(\Omega)}+\|\rho\|_{L^\infty(\Omega)}\|\mathbf{f}\|_{L^{3p^-}(\Omega)}\\
&\quad+\|\rho\|_{L^\infty(\Omega)}^{\gamma-1}\|\nabla\rho\|_{L^{3p^-}(\Omega)}+1\Big)\\
&\leqslant C(\mathcal{I}_\Phi(\mathbf{u})+\mathcal{J}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}+C\mathcal{I}_\Phi(\mathbf{u})^{\frac{1}{p^-}}+C\|\nabla\rho\|_{L^{3p^-}(\Omega)}+C.
\end{aligned} \tag{93}
$$

Since

$$
\frac{d}{dt}\|\rho\|_{W^{1,3p^-}(\Omega)}\leqslant C\|\mathbf{u}\|_{W^{2,3p^-}(\Omega)}\|\rho\|_{W^{1,3p^-}(\Omega)},
$$

one deduces from (93) and the Young's inequality that

$$
\begin{aligned}
\frac{d}{dt}\|\rho\|_{W^{1,3p^-}(\Omega)}&\leqslant C\|\mathbf{u}\|_{W^{2,3p^-}(\Omega)}\|\rho\|_{W^{1,3p^-}(\Omega)}\\
&\leqslant C(\mathcal{I}_\Phi(\mathbf{u})+\mathcal{J}_\Phi(\mathbf{u})+1)^{\frac{1}{p^-}}\|\rho\|_{W^{1,3p^-}(\Omega)}+C\mathcal{I}_\Phi(\mathbf{u})^{\frac{1}{p^-}}\|\rho\|_{W^{1,3p^-}(\Omega)}\\
&\quad+C\|\nabla\rho\|_{L^{3p^-}(\Omega)}\|\rho\|_{W^{1,3p^-}(\Omega)}+C\|\rho\|_{W^{1,3p^-}(\Omega)}\\
&\leqslant\varepsilon\mathcal{I}_\Phi(\mathbf{u})+\eta\mathcal{J}_\Phi(\mathbf{u})+C(\|\rho\|_{W^{1,3p^-}(\Omega)}^{\frac{p^-}{p^- -1}}+1),
\end{aligned} \tag{94}
$$

for any fixed $\varepsilon,\eta\in(0,1)$. Summary, it follows from (90), (92) and (94) that

$$
\frac{d}{dt}(\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^2+\|\rho\|_{W^{1,3p^-}(\Omega)})+\mathcal{I}_\Phi^r(\mathbf{u})+\mathcal{J}_\Phi(\mathbf{u})
$$

$$\leqslant C(1+\|\rho\|_{W^{1,3p^-}(\Omega)}^{\frac{p^-r}{p^--1}}+\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(\Omega)}^{\frac{8r(p^--1)}{5p^--6-r(2-p^-)}}+\|\partial_t\mathbf{f}\|^2_{L^2(\Omega)}). \tag{95}$$

Hence,

$$\frac{d}{dt}(\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^2(\Omega)}+\|\rho\|_{W^{1,3p^-}(\Omega)})\leqslant C(1+\|\sqrt{\rho}\mathbf{u}_t\|^2_{L^2(\Omega)}+\|\rho\|_{W^{1,3p^-}(\Omega)}+\|\partial_t\mathbf{f}\|^2_{L^2(\Omega)})^{\alpha}, \tag{96}$$

where $\alpha=max\{\frac{p^-r}{p^--1},\frac{4r(p^--1)}{5p^--6-r(2-p^-)}\}>1$. Setting

$$f(t)\triangleq 1+\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L(\Omega)}+\|\rho\|_{W^{1,3p^-}(\Omega)},$$

one rewrites (96) as

$$f'(t)\leqslant Cf^{\alpha}(t).$$

Thanks to the compatibility condition

$$\sqrt{\rho_0(x)}(\sqrt{\rho_0(x)}\partial_t\mathbf{u}(t=0,x)+\sqrt{\rho_0(x)}\mathbf{u}_0\cdot\nabla\mathbf{u}_0-\sqrt{\rho_0(x)}\mathbf{g})=0,$$

it holds that

$$\sqrt{\rho_0(x)}\sqrt{\rho_0(x)}\partial_t\mathbf{u}(t=0,x)=-\sqrt{\rho_0(x)}\mathbf{u}_0\cdot\nabla\mathbf{u}_0+\sqrt{\rho_0(x)}\mathbf{g}\in L^2(\Omega).$$

So,

$$f(t)\leqslant Cf(0)(1-(\alpha-1)Ctf^{\alpha-1}(0))^{-\frac{1}{\alpha-1}},$$

holds for any $t\in(0,min\{\frac{1}{(\alpha-1)Cf^{\alpha-1}(0)},T^*\})$, with

$$f(0)\triangleq 1+\|\sqrt{\rho_0}\partial_t\mathbf{u}(t=0,\cdot)\|^2_{L^2(\Omega)}+\|\rho_0\|_{W^{1,3p^-}(\Omega)}.$$

Therefore,

$$\sup_{0\leqslant t\leqslant T}(\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^2(\Omega)}+\|\rho\|_{W^{1,3p^-}(\Omega)})+\int_0^T\mathcal{I}^r_{\Phi}(\mathbf{u})+\mathcal{J}_{\Phi}(\mathbf{u})\,dt\leqslant C \tag{97}$$

holds for any $T\in(0,min\{\frac{1}{(\alpha-1)Cf^{\alpha-1}}(0),T^*\})$. In addition, one integrates (93) over $(0,T)$ to deduce that

$$\int_0^T\|\mathbf{u}\|^{p^-}_{W^{2,3p^-}(\Omega)}\,dt\leqslant C, \tag{98}$$

holds for any $T\in(0,min\{\frac{1}{(\alpha-1)Cf^{\alpha-1}(0)},T^*\})$. □

In view of Lemma 15, one takes $t_1\in(0,min\{\frac{1}{(\alpha-1)Cf^{\alpha-1}(0)},T^*\})$ and finds that

$$(\rho,\mathbf{u})(x,t_1)\triangleq\lim_{t\to t_1}(\rho,\mathbf{u}),$$

satisfy the conditions imposed on the initial data (7) at the time $t=t_1$. Furthermore,

$$\Big(-\operatorname{div}((1+|\mathbb{D}\mathbf{u}|^2)^{\frac{p(t,x)-2}{2}}\mathbb{D}\mathbf{u})+\nabla(\rho)^{\gamma}\Big)|_{t=t_1}=\lim_{t\to t_1}\rho^{\frac12}(x,t_1)g(x),$$

with $g(x)\triangleq\lim_{t\to t_1}(\rho^{\frac12}(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}))(x,t_1)\in L^{3p^-}(\Omega)$. Thus, $(\rho,\mathbf{u})(x,t_1)$ satisfies (8) also. So, one can take $(\rho,\mathbf{u})(x,t_1)$ as the initial data and apply Theorem 1 to extend the local strong solution beyond T^*. This contradicts the assumption on T^*.

Acknowledgements. Fang's research is supported by the National Natural Science Foundation of China(11501445).

References

[1] Abbatiello, A., Feireisl, E., Novotný, A.: Generalized solutions to models of compressible viscous fluids. Discrete Contin. Dyn. Syst. **41**, 1-28 (2021)

[2] Al Baba, H., Al Taki, B., Hussein, A.: Remark on the local well-posedness of compressible non-Newtonian fluids with initial vacuum. J. Math. Fluid Mech. **26**, 18pp (2024)

[3] Al Taki, B.: Well-posedness for a class of compressible non-Newtonian fluids equations. J. Differential Equations. **349**, 138-175 (2023)

[4] Bellout, H., Bloom, F., Nečas, J.: Young measure-valued solutions for non-Newtonian incompressible fluids. Comm. Partial Differential Equations. **19**, 1763-1803 (1994)

[5] Burczak, J., Modena, S., Székelyhidi, L.: Non Uniqueness of Power-Law Flows. Commun. Math. Phys. **388**, 199-243 (2021)

[6] Choe, H.J, Kim, H.: Strong solutions of the Navier-Stokes equations for isentropic compressible fluids. Comm. Partial Differential Equations. **28**, 1183-1201 (2003)

[7] Diening, L., Růžička, M.: Strong solutions for generalized Newtonian fluids. J. Math. Fluid Mech. **7**, 413-450 (2005)

[8] Diening, L., Růžička, M., Wolf, J.: Existence of weak solutions for unsteady motion of generalized Newtonian fluids. Ann. Sc. Norm. Super. Pisa Cl. Sci. **9**, 1-46 (2010)

[9] Fang, L., Guo, J.Y., Ling, C.X., Guo, Z.H.: Local strong solutions to a class of compressible non-Newtonian fluid equations without initial compatibility conditions. Commun. Math. Sci. **23**, 1859-1880 (2025)

[10] Fang, L., Z.H. Guo, Analytical solutions to a class of non-Newtonian fluids with free boundaries. J. Math. Phys. **53**, 103701 (2012)

[11] Fang, L., Zang, A.B.: Global existence of strong solutions to the Cauchy problem for a one dimensional compressible non-Newtonian fluid. J. Math. Fluid Mech. **25**, 37 pp (2023)

[12] Feireisl, E.: Dynamics of viscous compressible fluids. Oxford University Press: Oxford (2004)

[13] Guo, B.L., Zhu, P.C.: Partial regularity of suitable weak solutions to the system of the incompressible non-Newtoniana fluids. J. Differential Equations. **178**, 281-297 (2002)

[14] Guo, B.L., Shang, Y.: Dynamics of non-Newtonian fluid. National Defence Indu-stry Press, Beijing (2006)

[15] Kaplický, P., Málek, J., Stará, J.: $C^{1,\alpha}$- solutions to a class of nonlinear fluids in the 2D stationary Dirichlet problem. J. Math. Sci. **109**, 1867-1893 (2002)

[16] Kalousek, M., Mácha, V., Nečasová, Š.: Local-in-time existence of strong solutions to a class of compressible non-Newtonian Navier-Stokes equations. Math. Ann. **384**, 1057-1089 (2022)

[17] Ladyzhenskaya, O.A., Solonnikov V.A.; Ural'tseva N.N.: Linear and quasi-linear equations of parabolic type. Amer. Math. Soc., New York (1968)

[18] Ladyzhenskaya, O.A.: The mathematical theory of viscous incompressible flow, Vol. 2. Gordon and Breach, New York (1969)

[19] Lions, J.L.: Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod; Gauthier-Villars, Paris (1969)

[20] Málek, J., Neĉas, J., Rokyta, M., Růžička, M.: Weak and Measure-Valued Solution to Evolutionary PDEs. Appl. Math. Math. Comput. Chapman and Hall, London (1996)

[21] Mamontov, A.E.: On the global solvability of the multidimensional Navier-Stokes equations of a nonlinearly viscous fluid I. Sibirsk. Mat. Zh., **40**, 408-420 (1999)

[22] Mamontov, A.E.: On the global solvability of the multidimensional Navier-Stokes equations of a nonlinearly viscous fluid II. Sibirsk. Mat. Zh., **40**, 635-649 (1999)

[23] Nečasová, M., Novotný, A.: Measure-valued solution for non-Newtonian compressible isothermal monopolar fluid. Acta Appl. Math. **37**, 109-128 (1994)

[24] Rajagopal, K.R., Růžička, M.: On the modeling of electrorheological materials. Mech. Res. Commun., **23**, 401-407 (1996)

[25] Rajagopal, K.R., Růžička, M.: Mathematical modeling of electrorheological materials. Cont. Mech. and Thermodyn., **12**, 59-78 (2001)

[26] Růžička, M.: Flow of shear dependent electrorheological fluids: unsteady space periodic case. In Applied nonlinear analysis, Kluwer/Plenum, New York (1999)

[27] Růžička, M.: Electrorheological fluids: modeling and mathematical theory, volume 1748 of Lecture Notes in Mathematics. Springer-Verlag, Berlin (2000).

[28] Yuan, H.J., Xu, X.J.: Existence and uniqueness of solutions for a class of non-Newtonian fluids with singularity and vacuum. J. Differ. Equ. **245**, 2871-2916 (2008)

[29] Yuan, H.J., Si, X., Feng, Z.S.: Global strong solutions of a class of non-Newtonian fluids with small initial energy. J. Math. Anal. Appl. **474**, 72-93 (2019)

[30] Zang, A.B.: Existence of weak solutions for non-stationary flows of fluids with shear thinning dependent viscosities under slip boundary conditions in half space. Sci. China Math. **61**, 727-744 (2018)

[31] Zhang, J.J., Guo, C.X., Guo, B.L.: Existence of weak solutions for steady viscous fluids with general slip boundary conditions. Nonlinear Anal. **73**, 2476-2488 (2010)