ON FINITENESS PROPERTIES OF SEPARATING SEMIGROUP OF REAL CURVE

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Abstract. A real morphism f from a real algebraic curve X to $\mathbb{P}^1$ is called *separating* if $f^{-1}(\mathbb{RP}^1) = \mathbb{R}X$. A separating morphism defines a covering $\mathbb{R}X \to \mathbb{RP}^1$. Let $X_1, \dots, X_r$ denote the components of $\mathbb{R}X$. M. Kummer and K. Shaw [KS20] defined the separating semigroup of a curve X as the set of all vectors $d(f) = (d_1(f), \dots, d_r(f)) \in \mathbb{N}^r$ where f is a separating morphism $X \to \mathbb{P}^1$ and $d_i(f)$ is the degree of the restriction of f to X_i.

Let us call an additive subsemigroup of $\mathbb{N}^r$ *finitely covered* if it can be written as $S = S_0 \cup \bigcup_{i=1}^m (s_i + \mathbb{N}_0^r)$, where S_0 is a finite set.

In the present paper, we prove that the separating semigroup of a real curve X is finitely covered, but not finitely generated when $\mathbb{R}X$ has at least two connected components.

§1. Introduction

By a *real curve* we mean a complex algebraic curve X equipped with an antiholomorphic involution $\mathrm{conj}\colon X \to X$. Its real points set is $\mathbb{R}X := \{p \in X \mid \mathrm{conj}(p) = p\}$. All curves considered here are smooth and irreducible.

A real curve X is called *separating*, if the space $X \setminus \mathbb{R}X$ is disconnected. In this case $X \setminus \mathbb{R}X$ consists of two connected components which are interchanged by an anti-holomorphic involution.

Ahlfors [Ahl50] proved that a curve X is separating if and only if there exists a *separating morphism* $f\colon X \to \mathbb{P}^1$, that is a morphism such that $f^{-1}(\mathbb{RP}^1) = \mathbb{R}X$. Such a morphism defines a covering map $f|_{\mathbb{R}X}\colon \mathbb{R}X \to \mathbb{RP}^1$. Let $X_1, \dots, X_r$ denote the connected components of $\mathbb{R}X$. Denoting by $d_i(f)$ the degree of the restriction of f to X_i we may associate every separating morphism f with a vector $d(f) := (d_1(f), \dots, d_r(f))$. M. Kummer and K. Shaw [KS20] defined the *separating semigroup* of X as the set of all such vectors:

$$\mathrm{Sep}(X) := \{d(f) \mid f\colon X \to \mathbb{P}^1 \text{ — separating morphism}\}.$$

It is easy to check that $\mathrm{Sep}(X)$ is indeed an additive semigroup, see [KS20, Prop. 2.1].

Without going into details, let us note that the separating semigroup has now been described for the following cases: *M-curves*, i.e. real curves of genus g with maximal number of components $b_0(\mathbb{R}X) = g + 1$, (see [KS20, Theorem 1.7]), curves of genus

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≤ 4 (see [KS20, Theorem 1.7], [Ore19], [Ore21, Example 3.3], [Ore25]), hyperelliptic curves (see [Ore19] and [Ore25, §4]) and plane quintics (see [MO25a]).

In the context of studying the structure of the semigroup $\mathrm{Sep}(X)$, it seems natural to ask whether it is finitely generated in some sense. This is the subject of the present note.

Notation. Let us denote:

$$\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 1\}, \quad \mathbb{N}_0 = \{n \in \mathbb{Z} \mid n \geq 0\}.$$

Below g always denotes the genus of a real curve X and r denotes the number of connected components of $\mathbb{R}X$. We always equip the sets $\mathbb{N}^r$ a $\mathbb{N}_0^r$ with the standard structure of an additive semigroup. For $d \in \mathbb{N}^r$ denote $|d| := \sum_{i=1}^r d_i$.

Main results. We begin with the following definition, which will be needed to state the main results.

Definition 1. An additive subsemigroup $S \subset \mathbb{N}^r$ is called *finitely covered*, if S can be represented as

$$S = S_0 \cup \bigcup_{i=1}^{m} (s_i + \mathbb{N}_0^r),$$

where S_0 is a finite set. In this case, we call $S_0 \cup \{s_1, \ldots, s_m\}$ a *covering set* of the semigroup S.

The principal result of this note is the following theorem.

Theorem 1. The semigroup $\mathrm{Sep}(X)$ is finitely covered for any separating curve X. Moreover, it admits a covering set with finite part of cardinality at most $\binom{2g-3}{r} + \binom{g-1}{r-1}$[1].

It also turns out that the standard notion of finitely generated semigroup is somewhat meaningless in our context. In Theorem 2 we show that $\mathrm{Sep}(X)$ is not finitely generated when $\mathbb{R}X$ has at least 2 connected components (the case of $b_0(\mathbb{R}X) = 1$ is not interesting since any subsemigroup in $\mathbb{N}$ is finitely generated).

§2. Proofs

Often it is more convenient to speak not of separating morphisms themselves, but of divisors corresponding to their fibers. Namely, let us call a totally real[2] divisor $P = p_1 + \ldots + p_n$ *separating* if there exists a separating morphism $f \colon X \to \mathbb{P}^1$ and a point $p_0 \in \mathbb{RP}^1$ such that $P = f^{-1}(p_0)$. In this case we say that P *has the degree partition* $d(P) := d(f) \in \mathrm{Sep}(X)$.

The proof of the main theorem requires several lemmas, so let us introduce them.

Lemma 1 (Prop. 3.2. in [KS20])**.** Let P be a separating divisor on a real separating curve X and for some $p \notin P$ we have $h^0(P+p) > h^0(P)$. Then $P+p$ is separating.

In particular, if P is non-special (that is, $h^1(P) = 0$), then we have $d(P) + \mathbb{N}_0^r \subset \mathrm{Sep}(X)$. □

[1] We assume that $\binom{n}{k} = 0$ if $n < k$ or $n < 0$.

[2] It means that $p_i \in \mathbb{R}X$ for all $i = 1, \ldots, n$.

The following elementary lemma is proven, for instance, in [Hui03, Prop. 2.1]. For the reader's convenience, we provide here a proof which we find illustrative.

Lemma 2. Let X be a real algebraic curve and P be a divisor from the canonical class. Then P has even degree on every component of $\mathbb{R}X$.

Proof. As P belongs to the canonical class, it is a divisor of some meromorphic 1-form ω on X. We may choose this form to be real. The form ω is non-degenerate on $\mathbb{R}X \setminus \operatorname{supp}(P)$, so it defines an orientation on this space. Let X_i be a component of $\mathbb{R}X$. It is clear that two adjacent segments of $X_i \setminus \operatorname{supp}(P)$ are oriented similarly if and only if the multiplicity of their common boundary point in P is even. Therefore, since the total number of orientation changes on X_i must be even, the multiplicity of P on X_i must be even as well. □

Now we move to the proof of the main theorem.

Theorem 1. The semigroup $\operatorname{Sep}(X)$ is finitely covered for any separating curve X. Moreover, it admits a covering set with finite part of cardinality at most $\binom{2g-3}{r} + \binom{g-1}{r-1}$[3].

Proof. First, note that we have the following decomposition:

$$\operatorname{Sep}(X) = \{d(P) \mid P \text{ — separating}\} = \operatorname{Sep}_s(X) \sqcup \operatorname{Sep}_n(X), \text{ where}$$

- $\operatorname{Sep}_s(X) := \{d(P) \mid P \text{ — separating and special}\}$,
- $\operatorname{Sep}_n(X) := \{d(P) \mid P \text{ — separating and non-special}\}$.

It follows from the Riemann-Roch Theorem that every divisor of degree greater than $2g-2$ is non-special. Therefore, for every $d \in \operatorname{Sep}_s(X)$ we have $|d| \leq 2g-2$. Moreover, any special divisor of degree $2g-2$ belongs to the canonical class[4]. By Lemma 2 every divisor from the canonical class has even degree on every component of $\mathbb{R}X$. From that we obtain the following inequality:

$$(\diamond) \qquad |\operatorname{Sep}_s(X)| \leq \binom{2g-3}{r} + \binom{g-1}{r-1}.$$

Indeed, the first term in the right-hand-side of $(\diamond)$ counts the number of tuples $(d_1, \ldots, d_r) \in \mathbb{N}^r$ with sum at most $2g-3$, while the second term counts the number of tuples $(d_1, \ldots, d_r) \in \mathbb{N}^r$ with sum exactly $2g-2$ and even d_i.

Now consider the set $\operatorname{Sep}_n(X)$. Let us introduce "coordinate-wise" partial order on $\mathbb{N}^r$. That is, we say that $u \preceq v$ if $u_i \leq v_i$ for all $i = 1, \ldots, n$. Then, consider minimal elements in $\operatorname{Sep}_n(X)$ with respect to the relation $\preceq$. By Dickson's lemma [Dic13], there are finitely many of them. Let $P_1, \ldots, P_m$ denote the corresponding separating divisors. As every P_i is non-special, by Lemma 1 we have $d(P_i) + \mathbb{N}_0^r \subset \operatorname{Sep}(X)$ for all $i = 1, \ldots, m$. Then,

$$\operatorname{Sep}_n(X) = \bigcup_{i=1}^{m} (d(P_i) + \mathbb{N}_0^r).$$

[3]We assume that $\binom{n}{k} = 0$ if $n < k$ or $n < 0$.

[4]Indeed, let $\deg D = 2g-2$ and $h^0(K_X - D) > 0$. Then, $K_X - D$ is effective (as $h^0(K_X - D) > 0$) divisor of degree zero, so $K_X = D$.

Thus, we have shown that

$$\operatorname{Sep}(X) = \operatorname{Sep}_s(X) \cup \bigcup_{i=1}^{m} (d(P_i) + \mathbb{N}_0^r), \tag{$\dagger$}$$

and as $\operatorname{Sep}_s(X)$ is finite, so $\operatorname{Sep}(X)$ is finitely covered. The bound on the size of the covering set follows from the above.

□

Remark 1. Note that the bound on the size of the covering set is sharp, for example, for a rational curve. For such a curve we have $g = 0$, $r = 1$ and $\operatorname{Sep}(X) = \mathbb{N} = 1 + \mathbb{N}_0$.

However, the covering set built in the proof of Theorem 1 is very inefficient. The reason is that for certain *special* divisors P one has $d(P) + \mathbb{N}_0^r \subset \operatorname{Sep}(X)$. For instance, in [MO25b] the authors provide a wide range of such examples: they prove that if P is a separating divisor of degree g with $h^1(P) = 1$, then $d(P) + \mathbb{N}_0^r \subset \operatorname{Sep}(X)$.

Question 1. Is it possible to estimate the number m in the representation ($\dagger$)?

Question 2. Consider the separating divisor P from the canonical class. What are the restrictions on the partition degree $d(P)$? We know that $|d(P)| = 2g - 2$ and d_i are even.

The following theorem illustrates that the standard notion of a finitely generated semigroup is somewhat meaningless in our context.

Theorem 2. For any separating real curve X with $r = b_0(\mathbb{R}X) \geq 2$ the semigroup $\operatorname{Sep}(X)$ is not finitely generated.

Remark 2. The case of $r = 1$ is not interesting since any additive subsemigroup in $\mathbb{N}$ is finitely generated.

Proof of Theorem 2. By Theorem 1 we have

$$\operatorname{Sep}(X) = \operatorname{Sep}_s(X) \cup \bigcup_{i=1}^{m} \left(d^{(i)} + \mathbb{N}_0^r \right).$$

Let us pick among $d^{(j)}$ a vector with the minimal first coordinate. Without loss of generality, we may assume that it is $d^{(1)}$. Consider $d = d^{(1)} + (0, k, 0 \ldots, 0)$, where $k \in \mathbb{N}$ will be chosen later. Evidently, d is not representable as a sum whose terms include vectors from $\operatorname{Sep}_n(X)$.

For $a \in \mathbb{N}^r$ denote $\mathbf{r}(a) := a_2/a_1$. Note that the following elementary lemma holds.

Lemma 3. $\mathbf{r}(a + b) \leq \max(\mathbf{r}(a), \mathbf{r}(b))$.

Denote $\operatorname{Sep}_s(X) = \{e^{(1)}, \ldots, e^{(m)}\}$ and suppose that $d = \sum_{j=1}^{m} \alpha_j \cdot e^{(j)}$ for some $\alpha_j \in \mathbb{N}$. Then, by Lemma 3 we have

$$\mathbf{r}(d) = \mathbf{r}\left(\sum_{j=1}^{m} \alpha_j \cdot e^{(j)}\right) \leq \max_{j=1,\ldots,m} \left(\mathbf{r}\left(\alpha_j \cdot e^{(j)}\right)\right) = \max_{j=1,\ldots,m} \mathbf{r}\left(e^{(j)}\right) := C. \tag{$\star$}$$

On the other hand, by choosing k large enough, we can make $\mathbf{r}(d)$ arbitrarily large, which contradicts ($\star$). □

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