ON ZERO-SUM PROBLEMS OVER METACYCLIC GROUPS $C_n \rtimes_s C_2$

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Abstract. Let G be a finite group. A finite collection of elements from G, where the order is disregarded and repetitions are allowed, is said to be a product-one sequence if its elements can be ordered such that their product in G equals the identity element of G. Then, the Gao's constant $\mathsf{E}(G)$ of G is the smallest integer ℓ such that every sequence of length at least ℓ has a product-one subsequence of length $|G|$. For a positive integer n, we denote by C_n a cyclic group of order n. Let $G = C_n \rtimes_s C_2$ with $s^2 \equiv 1 \pmod n$ be a metacyclic group. The direct and inverse problems of $\mathsf{E}(G)$ were settled recently, except for the case that $G = C_{3n_2} \rtimes_s C_2$ with $n_2 \neq 1$, $\gcd(n_2, 6) = 1$, $s \equiv -1 \pmod 3$, and $s \equiv 1 \pmod{n_2}$. In this paper, we complete the remaining case and hence for all metacyclic groups of the form $G = C_n \rtimes C_2$, the Gao's constant and the associated inverse problem are now fully settled (see Theorem 1.2).

1. Introduction

Let G be a finite group. By a sequence S over G, we mean a finite collection of elements from G, where the order is disregarded and repetitions are allowed. The *zero-sum problems* study the conditions that ensure a given sequence over G contains a subsequence whose product (or sum, in the abelian case) is the identity of G. Such subsequences are called *product-one* (or *zero-sum*). Originating in classical works of Erdős, Ginzburg and Ziv [10], van Emde Boas and Kruyswijk [11], and Olson [19, 20] during the 1960s, these problems connect to several areas (see, for instance, [1, 21, 14]), and have been extended from abelian to non-abelian groups in the 1980s. This explains why it is common to use multiplicative notation and the term *product-one* instead of additive notation and the term *zero-sum*.

The Gao's constant $\mathsf{E}(G)$ is the smallest positive integer ℓ such that every sequence of length at least ℓ has a product-one subsequence of length $|G|$. This invariant has been extensively studied for finite abelian groups, and it is closely related to $\mathsf{d}(G)$, the small Davenport constant of G, which is the maximal length of a sequence over G that do not have product-one subsequences (see [16, Chapter 16]). In fact, it is known that $\mathsf{E}(G) \geq \mathsf{d}(G) + |G|$ and equality holds for abelian groups [12] and several others (see [2, 3, 4, 5, 17, 23, 15] for non-abelian groups). Zhuang and Gao [26] conjectured that this equality holds for every finite group.

The *direct problem* associated to $\mathsf{E}(G)$ is to determine the exact value of $\mathsf{E}(G)$, while the *inverse problem* seeks to describe the structure of sequences with maximal length that do not have product-one subsequences of length $|G|$ (see also [6, 7, 8, 13, 18, 22] for other recent developments on the direct and inverse problems for non-abelian groups).

Let C_n be a cyclic group of order n, and let $G = C_n \rtimes_s C_2 = \langle x, y \colon x^2 = y^n = 1_G, yx = xy^s \rangle$ with $s^2 \equiv 1 \pmod n$, be a metacyclic group of order $2n$. If $s = 1$ or $s = -1$, then $G \cong C_n \times C_2$ or $G \cong D_{2n}$, a dihedral group of order $2n$. If $s \not\equiv \pm 1 \pmod n$, then $n = n_1 n_2$ for some $n_1, n_2 \in \mathbb{Z}$ with $\gcd(n_1, n_2) \in \{1, 2\}$ for which $s \equiv -1 \pmod{n_1}$ and $s \equiv 1 \pmod{n_2}$ (see Remark 2.1). With these notation, the direct and inverse problems for $\mathsf{E}(G)$ have been settled for almost all metacyclic groups of the given form except for a "small" (yet infinite) family of cases for which $n_1 = 3$ and n_2 is odd. Note that D_{2n_1} is isomorphic to some quotient group of G. By using the inductive method, the inverse structural theorem for D_{2n_1} (see [18, Theorem 1.2]) is needed. Since the structure of extremal

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sequences for the group D_6 has more possibilities than other groups D_{2n_1} with $n_1 > 3$, the method using in [3] does not work.

In this paper, we consider that exceptional case, and with the help of DeVos-Goddyn-Mohar Theorem (see Lemma 3.1), we obtain the following result.

Theorem 1.1. *Let $G = C_{3n_2} \rtimes_s C_2$ be a metacyclic group of order $6n_2$, where $n_2 \neq 1$, $\gcd(6, n_2) = 1$, $s \equiv -1 \pmod 3$, and $s \equiv 1 \pmod{n_2}$.*

1. $\mathsf{E}(G) = 9n_2$.
2. *Let $S \in \mathcal{F}(G)$ be a sequence of length $9n_2 - 1$. Then, S has no product-one subsequence of length $6n_2$ if and only if there exist $\alpha, \tau \in G$ such that $G = \langle \alpha, \tau \colon \tau^2 = \alpha^n = 1_G, \alpha\tau = \tau\alpha^s \rangle$, and $S = (\alpha^{t_1})^{[6n_2-1]} \boldsymbol{\cdot} (\alpha^{t_2})^{[3n_2-1]} \boldsymbol{\cdot} (\tau\alpha^{t_3})$, where $t_1, t_2, t_3 \in \mathbb{Z}$ with $\gcd(t_1 - t_2, 3n_2) = 1$.*

In conjunction with [3, Theorems 1.6 and 1.7], the Gao's constant and the associated inverse problem are completely settled for all metacyclic groups of the form $C_n \rtimes C_2$ as follows.

Theorem 1.2. *Let $G = C_n \rtimes_s C_2$ be a metacyclic group of order $2n$, where $n \geq 3$ and $s^2 \equiv 1 \pmod n$.*

1. $\mathsf{E}(G) = 3n$.
2. *If $S \in \mathcal{F}(G)$ with $|S| = 3n - 1$, then S has no product-one subsequence of length $2n$ if and only if there exist $\tau, \alpha \in G$ for which $G = \langle \tau, \alpha \colon \tau^2 = \alpha^n = 1_G, \alpha\tau = \tau\alpha^s \rangle$ and S has one of the following forms;*
   - *for $n = 3$, either $S = 1_G^{[5]} \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \tau\alpha \boldsymbol{\cdot} \tau\alpha^2$, or $S = (\alpha^{k_1})^{[5]} \boldsymbol{\cdot} (\alpha^{k_2})^{[2]} \boldsymbol{\cdot} \tau\alpha^{k_3}$, where $k_1, k_2, k_3 \in \mathbb{Z}$ with $\gcd(k_1 - k_2, n) = 1$.*
   - *for $n \geq 4$, $S = (\alpha^{t_1})^{[2n-1]} \boldsymbol{\cdot} (\alpha^{t_2})^{[n-1]} \boldsymbol{\cdot} \tau\alpha^{t_3}$, where $t_1, t_2, t_3 \in \mathbb{Z}$ and $\gcd(t_1 - t_2, n) = 1$.*

The paper is organized as follows. In Section 2, we introduce our notation and the preliminary results. In Section 3, we present the results from Additive Theory needed for the proof of our main theorem. In Section 4, we prove Theorem 1.1 after establishing a result on sequences over G of length $9n_2 - 1$ containing at least two terms from $x\langle y \rangle$.

2. Preliminaries

For integers $a, b \in \mathbb{Z}$, $[a, b] = \{x \in \mathbb{Z} \colon a \leq x \leq b\}$ is the discrete interval. Let G be a finite group with identity element 1_G. For a subset $A \subset G$ and an element $g \in G$, we let $gA := \{ga \in G \colon a \in A\}$, and denote by $\langle A \rangle$ the subgroup of G generated by A, and by $\mathsf{H}(A) := \mathrm{stab}(A) = \{g \in G \colon gA = A\}$ the *stabilizer* of A. Then, $\mathsf{H}(A)$ is a subgroup of G, and it is easy to see that $\mathsf{H}(A) = G$ if and only if $A = G$. For a positive integer n, we denote by C_n a cyclic group of order n, and by D_{2n} a dihedral group of order $2n$.

We consider sequences over G as elements of the free abelian monoid $\mathcal{F}(G)$ with basis G, under concatenation as the operation (denoted by $\boldsymbol{\cdot}$). In order to avoid confusion between multiplication in G and multiplication in $\mathcal{F}(G)$, we denote multiplication in G without an explicit symbol and use brackets for all exponentiation in $\mathcal{F}(G)$. Thus, a sequence $S \in \mathcal{F}(G)$ has the form

$$S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_k = \prod\nolimits^{\bullet}_{i \in [1,k]} g_i = \prod\nolimits^{\bullet}_{g \in G} g^{[\mathsf{v}_g(S)]},$$

where $g_1, \ldots, g_k \in G$, $\mathsf{v}_g(S)$ is the *multiplicity* of g in S, and $k = |S| = \sum_{g \in G} \mathsf{v}_g(S)$ is the *length* of S. A *subsequence* of S is a divisor T of S in $\mathcal{F}(G)$, which is equivalent to $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for every $g \in G$. In this case, we write $T \mid S$, and $S \boldsymbol{\cdot} T^{[-1]} = \prod^{\bullet}_{g \in G} g^{[\mathsf{v}_g(S) - \mathsf{v}_g(T)]}$ denote the subsequence of S obtained by deleting the elements of T from S. For a subset $H \subset G$, we denote by S_H the subsequence of S consisting of all elements from H.

For a sequence $S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_k$, the *set of products* and the *set of n-subproducts* of S are denoted by

$$\pi(S) = \{g_{f(1)} \ldots g_{f(k)} \in G \colon f \text{ is a permutation of } [1,k]\} \quad \text{and} \quad \Pi_n(S) = \bigcup_{\substack{T \mid S \\ |T|=n}} \pi(T)\,,$$

respectively. A generic element of $\pi(S)$ will be denoted by $\sigma(S)$. In particular, if G is abelian, then $\pi(S) = \{\sigma(S)\}$ is a singleton set. The sequence $S \in \mathcal{F}(G)$ is called

- *product-one* if $1_G \in \pi(S)$, and
- *product-one free* if $1_G \notin \Pi_n(S)$.

In particular, a product-one sequence of length n (resp., a product-one free sequence of length n) is referred to as an *n-product-one sequence* (resp., *n-product-one free sequence*).

If $S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_k$ is a product-one sequence with $1_G = g_1 \ldots g_k$, then $1_G = g_i \ldots g_k g_1 \ldots g_{i-1}$ for all $i \in [1,k]$. For a normal subgroup H of G, let $\phi_H \colon G \to G/H$ denote the natural homomorphism. By abuse of notation, we also write $\phi_H \colon \mathcal{F}(G) \to \mathcal{F}(G/H)$ for its extension to sequences, that is, $\phi_H\left(\prod^{\bullet}_{g\in G} g^{[\mathsf{v}_g(S)]}\right) = \prod^{\bullet}_{g\in G} \phi_H(g)^{[\mathsf{v}_g(S)]}$. In this case, we say that S is an *n-product-H sequence* if $\phi_H(S)$ is a product-one sequence (equivalently, $\pi(S) \cap H \neq \emptyset$) of length n, and S is an *n-product-H free sequence* if $\phi_H(S)$ has no a product-one sequence of length n.

With this notation, we have

$$\mathsf{E}(G) = \min\{\ell > 0 \colon \text{every } S \in \mathcal{F}(G) \text{ with } |S| \geq \ell \text{ has a } |G|\text{-product-one subsequence}\}\,.$$

Let $G = C_n \rtimes_s C_2$ be a metacyclic group of order $3n$ with $s^2 \equiv 1 \pmod{n}$. Then n can be factored into a product of two integers as follows.

Remark 2.1. *Let $n \geq 3$ be an integer and let $s \in \mathbb{Z}$ such that $s^2 \equiv 1 \pmod{n}$. An argument similar to that in* [3, Lemma 1.4] *shows n admits a factorization of the form $n = n_1 n_2$ for which $s \equiv -1 \pmod{n_1}$, $s \equiv 1 \pmod{n_2}$ and $\gcd(n_1, n_2) \in \{1, 2\}$. If n_1, n_2 are coprime (this occurs, for instance, when n is odd), then the group $G = C_n \rtimes_s C_2 = \langle x, y \colon x^2 = y^n = 1_G, yx = xy^s \rangle$ is isomorphic to $C_{n_2} \times D_{2n_1} = \langle y^{n_1} \rangle \times \langle x, y^{n_2} \rangle$.*

In [3], the authors proved the direct and inverse problems for $\mathsf{E}(G)$ except for a "small" (yet infinite) family of cases; indeed, the only potential exception occurs when $n_1 = 3$ and n_2 is odd (see [3, Theorems 1.6 and 1.7]), since the arguments used for larger values of n_1 do not work in this case.

We present here the results related to the Gao's constant, which will be referred to frequently throughout this paper. The following lemma can be found in [10], [3, Lemma 2.2], and [18, Lemma 2.5].

Lemma 2.2. *Let $n \geq 2$ be an integer and $S \in \mathcal{F}(C_n)$ be a sequence.*

1. $\mathsf{E}(C_n) = 2n - 1$.
2. *If $|S| = 3n - 1$, then S has two disjoint n-product-one subsequences. In particular, S has a $2n$-product-one subsequence.*
3. *If $|S| = 3n - 2$, then S is $2n$-product-one free if and only if there exist a generator g of C_n and $t_1, t_2 \in \mathbb{Z}$ with $\gcd(t_1 - t_2, n) = 1$ such that $S = (g^{t_1})^{[2n-1]} \boldsymbol{\cdot} (g^{t_2})^{[n-1]}$.*

Although the direct and inverse problems concerning the Gao's constant for all dihedral groups have been completely solved, we present here only the case of the dihedral group D_6 of order 6, which we need for our main theorem, and it can be found in [4, Theorem 8] and [18, Theorem 1.2].

Lemma 2.3. *Let $S \in \mathcal{F}(D_6)$ be a sequence.*

1. $\mathsf{E}(D_6) = 9$.

2. *If* $|S| = 8$, *then* S *is 6-product-one free if and only if there exist* $\alpha, \tau \in D_6$ *and* $t_1, t_2, t_3 \in \mathbb{Z}$ *such that* $D_6 = \langle \alpha, \tau \colon \tau^2 = \alpha^3 = 1_G, \alpha\tau = \tau\alpha^{-1} \rangle$, $\gcd(t_1 - t_2, 3) = 1$ *and either* $S = (\alpha^{t_1})^{[5]} \boldsymbol{\cdot} (\alpha^{t_2})^{[2]} \boldsymbol{\cdot} (\tau\alpha^{t_3})$ *or* $S = 1_G^{[5]} \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \tau\alpha \boldsymbol{\cdot} \tau\alpha^2$.

The following lemma is essentially contained in Claim A in the proof of [3, Proposition 5.1], but we include its proof here for completeness.

Lemma 2.4. *Let* $G = C_n \rtimes_s C_2 = \langle x, y \colon x^2 = y^n = 1_G, yx = xy^s \rangle$, *where* $n \ge 3$ *is odd such that* $s^2 \equiv 1 \pmod n$, *and let* $S \in \mathcal{F}(G)$ *with* $|S| = n_2$ *such that* $\pi(S)$ *is a singleton set, and let* n_1, n_2 *be defined as in Remark 2.1.*

1. *If* $\pi(S) \subset \langle y^{n_2} \rangle$ *and* $|S_{x\langle y \rangle}| \ge 1$, *then* $\pi(S) = \{1_G\}$.
2. *If* $\pi(S) \subset x\langle y^{n_2} \rangle$, *then* $S = xy^{\beta_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} xy^{\beta_{2k-1}} \boldsymbol{\cdot} y^{\beta_{2k}} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} y^{\beta_{n_2}}$ *for some* $k \ge 1$, *where* $\beta_1 \equiv \cdots \equiv \beta_{2k-1} \pmod{n_1}$ *and* $\beta_{2k} \equiv \cdots \equiv \beta_{n_2} \equiv 0 \pmod{n_1}$. *Moreover,* $\pi(S) = \{xy^{\beta_1 + tn_1}\}$ *for some* $t \in \mathbb{Z}$ *with* $\beta_1 + tn_1 \equiv 0 \pmod{n_2}$.

Proof. 1. Note that $|S_{x\langle y \rangle}|$ is even, so we may write

$$S = xy^{\beta_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} xy^{\beta_{2k}} \boldsymbol{\cdot} y^{\beta_{2k+1}} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} y^{\beta_{n_2}} .$$

Note that $\pi(S) = \{\sigma(S)\}$ is a singleton. Since $s \equiv -1 \pmod{n_1}$, it follows that

$$\sigma(S) = y^{\beta_1 + \beta_2 s + \ldots + s^{2k}\beta_{2k} + \beta_{2k+1} + \ldots + \beta_{n_2}} = y^{\beta_1 - \beta_2 + \ldots + \beta_{2k-1} - \beta_{2k} + \beta_{2k+1} + \ldots + \beta_{n_2}} .$$

Since reordering the product must yield the same value, we have that $xy^{\beta_u}xy^{\beta_v} = xy^{\beta_v}xy^{\beta_u}$ if and only if $\beta_u \equiv \beta_v \pmod{n_1}$, and $y^{\beta_u}xy^{\beta_v} = xy^{\beta_v}y^{\beta_u}$ if and only if $\beta_u \equiv 0 \pmod{n_1}$. It follows that $\sigma(S) \in \langle y^{n_1} \rangle \cap \langle y^{n_2} \rangle = \{1_G\}$, and hence $\pi(S) = \{1_G\}$.

2. In this case, $|S_{x\langle y \rangle}|$ is odd, and the assertion follows by a similar argument as above. □

3. A tool from Additive Theory

Let us consider a metacyclic group of order $6n_2$, which is the exceptional case in [3],

$$C_{3n_2} \rtimes_s C_2 = \langle x, y \colon x^2 = y^{3n_2} = 1_G, yx = xy^s \rangle = \langle y^3 \rangle \times \langle x, y^{n_2} \rangle \cong C_{n_2} \times D_6 ,$$

where $\gcd(6, n_2) = 1$, $s \equiv -1 \pmod 3$, and $s \equiv 1 \pmod{n_2}$. This potential exceptional case is handled by using the DeVos-Goddyn-Mohar Theorem (see Lemma 3.1), which is a generalization of the Cauchy-Davenport inequality and Kneser Theorem (see [16, Theorem 6.1 and Theorem 6.2]). The following bound is essential for our purpose.

Lemma 3.1 (DeVos-Goddyn-Mohar [9, Corollary 1.5] or [16, Theorem 13.1])**.** *Let* G *be a multiplicatively written finite abelian group, let* $S \in \mathcal{F}(G)$ *be a sequence, let* $n \in [1, |S|]$, *and let* $H = \mathsf{H}\big(\Pi_n(S)\big)$. *Then*

$$|\Pi_n(S)| \ge \left(\sum_{g \in G/H} \min\{n, \mathsf{v}_g\big(\phi_H(S)\big)\} - n + 1 \right) |H| .$$

The following is the main result of this section.

Proposition 3.2. *Let* $G = \langle y \rangle = \langle y^3 \rangle \times \langle y^{n_2} \rangle$ *be a cyclic group of order* $3n_2$ *with* $\gcd(6, n_2) = 1$, *let* $G_1 = \langle y^3 \rangle$, $G_2 = \langle y^{n_2} \rangle$, *and* $\varphi_i \colon G \to G_i$ *be projections for* $i \in [1, 2]$, *and let* S *be a* $7n_2$*-product-*G_2 *sequence over* G. *If, for any decomposition*

$$S = T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_7 \quad \text{with } T_i \text{ } n_2\text{-product-}G_2 \text{ subsequences},$$

the sequence $\sigma(T_1) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \sigma(T_7)$ *has no 6-product-one subsequence, then* $\Pi_{n_2 - 1}\big(\varphi_1(S)\big) = G_1$.

Proof. If $n_2 = 1$, then $G = \langle y \rangle \cong C_3$, $G_1 = \langle y^3 \rangle = \{1_G\}$, and $\Pi_0(S) = \{1_G\} = G_1$. Thus we may suppose that $n_2 \ge 2$ with $\gcd(6, n_2) = 1$, so G_1 is non-trivial. Let $H = \mathsf H\big(\Pi_{n_2-1}\big(\varphi_1(S)\big)\big)$, which is a subgroup of $G_1 = \langle y^3 \rangle$. Then $H = \langle y^{3r} \rangle$ for some $r \in [1, n_2]$ with $r \mid n_2$ and $m := |H| = \frac{n_2}{r}$.

If $r = 1$, then we are done. Suppose $r \ge 2$. Since $\gcd(6, n_2) = 1$ and $r \mid n_2$, we may assume that $r \ge 5$. If $\mathsf v_g(\phi_H(\varphi_1(S))) \le n_2 - 1$ for all $g \in G_1/H$, then Lemma 3.1 implies that

$$n_2 \,=\, |G_1| \,\ge\, \big|\Pi_{n_2-1}\big(\varphi_1(S)\big)\big| \,\ge\, (|S| - (n_2 - 1) + 1)|H| \,\ge\, 6n_2 + 2\,,$$

a contradiction. If there exist two distinct elements $g_1, g_2 \in G_1/H$ such that $\mathsf v_{g_1}\big(\phi_H\big(\varphi_1(S)\big)\big) \ge n_2$ and $\mathsf v_{g_2}\big(\phi_H\big(\varphi_1(S)\big)\big) \ge n_2$, then Lemma 3.1 implies that

$$n_2 \,=\, |G_1| \,\ge\, \big|\Pi_{n_2-1}\big(\varphi_1(S)\big)\big| \,\ge\, \big(2(n_2 - 1) - (n_2 - 1) + 1\big)|H| \,=\, n_2|H| \ge n_2\,,$$

whence H is trivial and $\Pi_{n_2-1}\big(\varphi_1(S)\big) = G_1$, which implies that $H = G_1$ is non-trivial, a contradiction. Therefore there exists exactly one $g \in G_1/H$ such that $\mathsf v_g(\phi_H(\varphi_1(S))) \ge n_2$, and it follows from Lemma 3.1 that

$$\begin{aligned} n_2 \,=\, |G_1| \,&\ge\, \big|\Pi_{n_2-1}\big(\varphi_1(S)\big)\big| \\ &\ge\, \big(|S| - \mathsf v_g\big(\phi_H\big(\varphi_1(S)\big)\big) + (n_2 - 1) - (n_2 - 1) + 1\big)|H| \\ &\ge\, \big(7n_2 + 1 - \mathsf v_g\big(\phi_H\big(\varphi_1(S)\big)\big)\big)\frac{n_2}{r}\,. \end{aligned}$$

Thus $\mathsf v_g\big(\phi_H\big(\varphi_1(S)\big)\big) \ge 7n_2 + 1 - r$, which implies that there exists $t \in [0, r-1]$ such that $\varphi_1(S)$ has at least $7n_2 + 1 - r$ terms lying in the coset $y^{3t}H$. Set

$$S \,=\, S_1 \boldsymbol{\cdot} S_2\,,$$

where $\varphi_1(S_1)$ are those terms that belong to the coset $y^{3t}H$ and $\varphi_1(S_2)$ are those terms that do not belong to the coset $y^{3t}H$. Let $\psi\colon G_1 \to H$ be a map such that $\varphi_1(f) = y^{3t}\psi(f)$ for every $f \in \mathrm{supp}(S_1)$. Then $\psi\big(\varphi_1(S_1)\big)$ is a sequence over H of length at least $7n_2 + 1 - r \ge 2m - 1$. By applying $\mathsf E(C_m) = 2m-1$ (Lemma 2.2.1), we can repeatedly choose subsequences $V_1, \ldots, V_\nu$ of S_1 such that $\psi\big(\varphi_1(V_i)\big)$ is m-product-one for each $i \in [1, \nu]$, where

$$\nu \,=\, \left\lceil \frac{7n_2 + 1 - r - (2m-2)}{m} \right\rceil \,=\, 7r - 2 + \left\lceil \frac{3-r}{m} \right\rceil\,.$$

Since $m \ge 1$ and $r \ge 5$, it follows that $\nu - (6r+1) = (r-3) + \lceil \frac{3-r}{m} \rceil \ge \frac{r-3}{m} + \lceil \frac{3-r}{m} \rceil \ge 0$, whence

$$\nu \,\ge\, 6r + 1\,.$$

Since $\sigma\big(\varphi_1(V_i)\big) = y^{3tm}$ for every $i \in [1, \nu]$, we have that $\prod^\bullet_{i \in I} V_i$ is an n_2-product-G_2 sequence for every $I \subset [1, \nu]$ with $|I| = r$. Let $[1, \nu] = I_1 \cup \cdots \cup I_7$ be a disjoint decomposition with $|I_1| = \cdots = |I_6| = r$ and $|I_7| = \nu - 6r \ge 1$. Now we set

$$T_i \,=\, \prod\nolimits^\bullet_{j \in I_i} V_j \quad \text{for every} \quad i \in [1, 6]\,, \qquad \text{and} \qquad T_7 \,=\, (T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_6)^{[-1]} \boldsymbol{\cdot} S\,.$$

Since S is a $7n_2$-product-G_2 sequence, we obtain that $T_1, \ldots, T_7$ are all n_2-product-G_2 sequences, and that $\prod^\bullet_{j \in I_7} V_j$ divides T_7. In view of our assumption that $\sigma(T_1) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \sigma(T_7)$ has no 6-product-one subsequence, it follows from Lemma 2.2.3 that

$$\sigma(T_1) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \sigma(T_7) \,=\, (y^{t_1 n_2})^{[5]} \boldsymbol{\cdot} (y^{t_2 n_2})^{[2]} \quad \text{for some distinct } t_1, t_2 \in [0, 2]. \tag{3.1}$$

After renumbering if necessary, we may assume that

$$\sigma(T_1) \,=\, \cdots \,=\, \sigma(T_5) \,=\, y^{t_1 n_2}\,.$$

Let $\lambda, \mu \in [1, 7]$ be distinct, and let $k_\lambda \in I_\lambda, k_\mu \in I_\mu$. Then, by swapping two subsequences V_{k_λ} and V_{k_μ}, we obtain a new decomposition

$$S \,=\, T_1' \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_7'\,,$$

where

$$T'_\lambda = V_{k_\mu} \boldsymbol{\cdot} T_\lambda \boldsymbol{\cdot} V_{k_\lambda}^{[-1]}, \quad T'_\mu = V_{k_\lambda} \boldsymbol{\cdot} T_\mu \boldsymbol{\cdot} V_{k_\mu}^{[-1]}, \quad \text{and} \quad T'_j = T_j \text{ for all } j \in [1,7] \setminus \{\lambda, \mu\}. \tag{3.2}$$

Since each sequence T'_j is again an n_2-product-G_2 sequence, it follows from Lemma 2.2.3 and our assumption that

$$\prod\nolimits^{\bullet}_{j\in[1,7]} \sigma(T'_j) = (y^{t_3 n_2})^{[5]} \boldsymbol{\cdot} (y^{t_4 n_2})^{[2]}$$

for some distinct $t_3, t_4 \in [0,2]$. It follows from (3.2) that $t_3 = t_1$, whence

$$\prod\nolimits^{\bullet}_{j\in[1,7]} \sigma(T'_j) = (y^{t_1 n_2})^{[5]} \boldsymbol{\cdot} (y^{t_4 n_2})^{[2]} \text{ for some } t_4 \in [0,2] \setminus \{t_1\}. \tag{3.3}$$

We proceed by the following claim.

A1. We have

1. If $\lambda, \mu \in [1,5]$, then $\sigma(V_{k_\lambda}) = \sigma(V_{k_\mu})$.
2. If $\{\lambda, \mu\} = \{6,7\}$, then $\sigma(V_{k_\lambda}) = \sigma(V_{k_\mu})$.
3. If $\lambda, \mu \in [1,7]$, then $\sigma(V_{k_\lambda}) = \sigma(V_{k_\mu})$.

If **A1**.3 holds, then $\sigma(V_i) = \sigma(V_j)$ for any distinct $i, j \in [1, \nu] = I_1 \cup \ldots \cup I_7$, which implies that $\sigma(T_1) = \cdots = \sigma(T_6)$, a contradiction to (3.1). Thus, to finish the proof, it suffices to show **A1**.

*Proof of **A1**.* 1. Suppose $\lambda, \mu \in [1,5]$. Then (3.2) and (3.3) imply that $\sigma(T'_\lambda) = \sigma(T'_\mu) = \sigma(T_\lambda) = \sigma(T_\mu) = y^{t_1 n_2}$, which implies $\sigma(V_{k_\lambda}) = \sigma(V_{k_\mu})$ as desired.

2. Suppose $\{\lambda, \mu\} = \{6,7\}$. Then (3.2) and (3.3) imply that $\sigma(T'_\lambda) = \sigma(T'_\mu) = y^{t_4 n_2}$ and $\sigma(T_\lambda) = \sigma(T_\mu) = y^{t_2 n_2}$. It follows from $\sigma(T'_\lambda)\sigma(T_\mu) = \sigma(T_\lambda)\sigma(T'_\mu)$ that

$$\sigma(V_{k_\lambda})^2 = \sigma(V_{k_\mu})^2, \text{ and hence } \sigma\big(\varphi_2(V_{k_\lambda})\big)^2 = \sigma\big(\varphi_2(V_{k_\mu})\big)^2 \in G_2 \cong C_3.$$

Thus $\sigma\big(\varphi_2(V_{k_\lambda})\big) = \sigma\big(\varphi_2(V_{k_\mu})\big)$, together with $\sigma\big(\varphi_1(V_{k_\lambda})\big) = \sigma\big(\varphi_1(V_{k_\mu})\big) = y^{3tm}$, implies that $\sigma(V_{k_\lambda}) = \sigma(V_{k_\mu})$ as desired.

3. By Items 1 and 2, we may assume that $\lambda \in [1,5]$ and $\mu \in [6,7]$. By Item 1 again, we have $\sigma(V_i) = \sigma(V_j)$ for any $i, j \in I_1 \cup \cdots \cup I_5$. Now applying Item 1 to the new decomposition $S = T'_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T'_7$ instead of $S = T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_7$, we infer that $\sigma(V_i) = \sigma(V_j)$ for any $i, j \in \{k_\mu\} \cup I_1 \cup \cdots \cup I_5 \setminus \{k_\lambda\}$, whence $\sigma(V_{k_\lambda}) = \sigma(V_{k_\mu})$ as desired. □[**A1**]

□

4. Proof of Theorem 1.1

We first show the following proposition, which plays a key role in both parts of the proof of Theorem 1.1.

Proposition 4.1. *Let $G = \langle x, y \colon x^2 = y^{3n_2} = 1_G, yx = xy^s \rangle \cong C_{n_2} \times D_6$, where $n_2 \geq 5$ with $\gcd(6, n_2) = 1$, $s \equiv -1 \pmod 3$, and $s \equiv 1 \pmod{n_2}$. If S is a sequence over G of length $9n_2 - 1$ with $|S_{x\langle y \rangle}| \geq 2$, then S has a product-one subsequence of length $6n_2$.*

Proof. Note that $\langle y^3 \rangle \cong C_{n_2}$ and $\langle x, y^{n_2} \rangle \cong D_6$. Let $\varphi \colon G \to \langle y^3 \rangle$ be the projection. Then $\varphi(S)$ is a sequence over $\langle y^3 \rangle$ of length $9n_2 - 1$. It follows from $\mathsf{E}(C_{n_2}) = 2n_2 - 1$ (Lemma 2.2.1) that we can repeatedly choose n_2-product-$\langle x, y^{n_2} \rangle$ subsequences $T_1, \ldots, T_8$ of S, that is, every $\varphi(T_i)$ is an n_2-product-one sequence. Among all the possible choices of $T_1, \ldots, T_8$, we may assume that

$$\big|\{i \in [1,8] \colon |(T_i)_{x\langle y \rangle}| > 0\}\big| \text{ is maximal.} \tag{4.1}$$

Then

$$T^* := \sigma(T_1)\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}\sigma(T_8)$$

is a sequence over $\langle x, y^{n_2}\rangle$, where $\sigma(T_i) \in \pi(T_i)$ is chosen arbitrarily. If T^* has a 6-product-one subsequence, then we are done. Thus, we can assume that

$$T^* \text{ is 6-product-one free for any choice of } \sigma(T_i) \in \pi(T_i). \tag{4.2}$$

We distinguish two cases.

Case 1 : $|(T^*)_{x\langle y\rangle}| \ge 2$.

Then Lemma 2.3.2 implies that $T^* = 1_G^{[5]}\boldsymbol{\cdot}x\boldsymbol{\cdot}xy^{n_2}\boldsymbol{\cdot}xy^{2n_2}$. After renumbering if necessary, we may assume that $\sigma(T_i) = 1_G$ for all $i \in [1,5]$. Then $(T_i)_{x\langle y\rangle}$ is nontrivial for $i \in [6,8]$. Note that $\pi(T_i) \subseteq \{x, xy^{n_2}, xy^{2n_2}\}$ for each $i \in [6,8]$. If $|\pi(T_i)| \ge 2$ for some $i \in [6,8]$, then $\pi(T_i) \cap \big\{\sigma(T_j)\colon j \in [6,8]\setminus\{i\}\big\} \ne \emptyset$, and by choosing $\sigma^* \in \pi(T_i) \cap \big\{\sigma(T_j)\colon j \in [6,8]\setminus\{i\}\big\}$, we obtain $\sigma^*\boldsymbol{\cdot}T^*\boldsymbol{\cdot}\sigma(T_i)^{[-1]}$ has a 6-product-one sequence $1_G^{[4]}\boldsymbol{\cdot}(\sigma^*)^{[2]}$, a contradiction to (4.2). Thus

$$|\pi(T_i)| = 1 \quad \text{for all} \quad i \in [6,8]. \tag{4.3}$$

In view of Lemma 2.4.2 and $\sigma(T_6)\boldsymbol{\cdot}\sigma(T_7)\boldsymbol{\cdot}\sigma(T_8) = x\boldsymbol{\cdot}xy^{n_2}\boldsymbol{\cdot}xy^{2n_2}$, we may assume that there exist positive integers u, v, w such that

$$\begin{aligned}
T_6 &= xy^{\beta_1}\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}xy^{\beta_{2u-1}}\boldsymbol{\cdot}y^{\beta_{2u}}\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}y^{\beta_{n_2}}\,,\\
&\quad \text{where } \beta_1 \equiv \cdots \equiv \beta_{2u-1} \pmod 3 \text{ and } \beta_{2u} \equiv \cdots \equiv \beta_{n_2} \equiv 0 \pmod 3,\\
T_7 &= xy^{\gamma_1}\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}xy^{\gamma_{2v-1}}\boldsymbol{\cdot}y^{\gamma_{2v}}\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}y^{\gamma_{n_2}}\,,\\
&\quad \text{where } \gamma_1 \equiv \cdots \equiv \gamma_{2v-1} \pmod 3 \text{ and } \gamma_{2v} \equiv \cdots \equiv \gamma_{n_2} \equiv 0 \pmod 3,\\
T_8 &= xy^{\delta_1}\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}xy^{\delta_{2w-1}}\boldsymbol{\cdot}y^{\delta_{2w}}\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}y^{\delta_{n_2}}\,,\\
&\quad \text{where } \delta_1 \equiv \cdots \equiv \delta_{2w-1} \pmod 3 \text{ and } \delta_{2w} \equiv \cdots \equiv \delta_{n_2} \equiv 0 \pmod 3.
\end{aligned} \tag{4.4}$$

Furthermore, we have $\sigma(T_6) = xy^{\beta_1+3t_6} = xy^{\epsilon_6 n_2}$, $\sigma(T_7) = xy^{\gamma_1+3t_7} = xy^{\epsilon_7 n_2}$, and $\sigma(T_8) = xy^{\delta_1+3t_8} = xy^{\epsilon_8 n_2}$, where $t_6, t_7, t_8 \in \mathbb{Z}$ and $\{\epsilon_6, \epsilon_7, \epsilon_8\} = [0,2]$. Since $n_2 \not\equiv 0 \pmod 3$, after renumbering if necessary, we may assume that

$$\beta_1 \equiv 0 \pmod 3, \quad \gamma_1 \equiv 1 \pmod 3, \quad \text{and} \quad \delta_1 \equiv 2 \pmod 3. \tag{4.5}$$

Since $(xy^{\beta_1}\boldsymbol{\cdot}xy^{\gamma_1}\boldsymbol{\cdot}xy^{\delta_1})^{[-1]}\boldsymbol{\cdot}T_6\boldsymbol{\cdot}T_7\boldsymbol{\cdot}T_8$ has length $3n_2 - 3 \ge 2n_2 - 1$, there exists an n_2-product-$\langle x, y^{n_2}\rangle$ subsequence T_6'. Since $\varphi\big((T_6')^{[-1]}\boldsymbol{\cdot}T_6\boldsymbol{\cdot}T_7\boldsymbol{\cdot}T_8\big)$ is a $2n_2$-product-one sequence over $\langle y^3\rangle$, it is a product of two n_2-product-one subsequences, say T_7' and T_8', i.e.,

$$T_6\boldsymbol{\cdot}T_7\boldsymbol{\cdot}T_8 = T_6'\boldsymbol{\cdot}T_7'\boldsymbol{\cdot}T_8' \text{ for } n_2\text{-product-}\langle x, y^{n_2}\rangle \text{ subsequences } T_i'.$$

We obtain a new sequence

$$(T^*)' = 1_G^{[5]}\boldsymbol{\cdot}\sigma(T_6')\boldsymbol{\cdot}\sigma(T_7')\boldsymbol{\cdot}\sigma(T_8')$$

over $\langle x, y^{n_2}\rangle$, where $\sigma(T_i') \in \pi(T_i')$ is chosen arbitrarily. If $(T^*)'$ has a 6-product-one subsequence, then we are done, whence we may assume that

$$(T^*)' \text{ has no 6-product-one subsequence for any choice of } \sigma(T_i') \in \pi(T_i'). \tag{4.6}$$

If $\big|(T^*)'_{x\langle y\rangle}\big| \ge 2$, then it follows from Lemma 2.3.2 that $(T^*)' = T^*$. Since $xy^{\beta_1}\boldsymbol{\cdot}xy^{\gamma_1}\boldsymbol{\cdot}xy^{\delta_1} \mid T_7'\boldsymbol{\cdot}T_8'$, we infer that either T_7' or T_8', say T_7', must contain at least two terms among $xy^{\beta_1}, xy^{\gamma_1}, xy^{\delta_1}$. By symmetry, we may assume that $xy^{\beta_1}\boldsymbol{\cdot}xy^{\gamma_1} \mid T_7'$. Applying (4.3) to the new sequence $(T^*)'$ instead of T^*, we obtain $|\pi(T_7')| = 1$. By Lemma 2.4.2 again, we obtain $\beta_1 \equiv \gamma_1 \pmod 3$, a contradiction to (4.5).

Therefore we must have that $\big|(T^*)'_{x\langle y\rangle}\big| = 1$, and hence it follows again by Lemma 2.3.2 that $\sigma(T_6')\boldsymbol{\cdot}\sigma(T_7')\boldsymbol{\cdot}\sigma(T_8') = (y^{sn_2})^{[2]}\boldsymbol{\cdot}xy^{kn_2}$ for some $s \in [1,2]$ and $k \in [0,2]$. We may assume that $\sigma(T_8') =$

xy^{kn_2}. Note that $\pi(T_i') \subseteq \{1_G, y^{n_2}, y^{2n_2}\}$ for $i \in [6,7]$. If $|\pi(T_i')| \ge 2$ for some $i \in [6,7]$, then letting $\sigma^* \in \pi(T_i') \setminus \{\sigma(T_i')\}$ implies that either $\sigma^* = 1_G$ or $\sigma^*\sigma(T_j') = 1_G$, where $j \in [6,7] \setminus \{i\}$. Both cases imply that $\sigma^*{\cdot}(T^*)'{\cdot}(\sigma(T_i'))^{[-1]}$ has a 6-product-one subsequence, a contradiction to (4.6). Then Lemma 2.4.1 implies

$$\big|(T_6')_{x\langle y\rangle}\big| = \big|(T_7')_{x\langle y\rangle}\big| = 0\,, \quad \text{and hence } xy^{\beta_1}{\cdot}xy^{\gamma_1}{\cdot}xy^{\delta_1} \mid T_8'.$$

In view of (4.4), we have that

$$\big|(T_6{\cdot}T_7{\cdot}T_8)_{\langle y^3\rangle}\big| = \big|(T_6'{\cdot}T_7'{\cdot}T_8')_{\langle y\rangle}\big| \ge \big|(T_6'{\cdot}T_7')_{\langle y\rangle}\big| = 2n_2\,,$$

which ensures by $\mathsf{E}(C_{n_2}) = 2n_2 - 1$ that $(T_6{\cdot}T_7{\cdot}T_8)_{\langle y^3\rangle}$ has an n_2-product-$\langle x, y^{n_2}\rangle$ subsequence T_0. Thus $\sigma(T_0) \in \langle y^3\rangle \cap \langle x, y^{n_2}\rangle = \{1_G\}$, whence $T_1{\cdot}\ldots{\cdot}T_5{\cdot}T_0$ is a $6n_2$-product-one subsequence of S.

Case 2 : $\big|(T^*)_{x\langle y\rangle}\big| = 1$.

By Lemma 2.3.2, we have $T^* = (y^{t_1n_2})^{[5]}{\cdot}(y^{t_2n_2})^{[2]}{\cdot}(xy^{t_3n_2})$ for $t_1, t_2, t_3 \in [0,3]$ with $\gcd(t_1 - t_2, 3) = 1$. After renumbering if necessary, we may assume that $\sigma(T_i) = y^{t_1n_2}$ for all $i \in [1,5]$ and $\sigma(T_j) = y^{t_2n_2}$ for all $j \in [6,7]$. Note that $\pi(T_k) \subseteq \{1_G, y^{n_2}, y^{2n_2}\}$ for all $k \in [1,7]$. If there exists $k \in [1,7]$ such that $|\pi(T_k)| \ge 2$, then, letting $\sigma^* \in \pi(T_i) \setminus \{\sigma(T_i)\}$, it follows from Lemma 2.3.2 and (4.2) that $\sigma^*{\cdot}T^*{\cdot}(\sigma(T_i))^{[-1]} = T^*$, whence $\sigma^* = \sigma(T_i)$, a contradiction. Thus

$$|\pi(T_k)| = 1 \quad \text{for all } \ k \in [1,7]\,.$$

Subcase 2.1 : $\big|(T_1{\cdot}\ldots{\cdot}T_7)_{x\langle y\rangle}\big| \ge 1$.

If $\big|(T_1{\cdot}\ldots{\cdot}T_5)_{x\langle y\rangle}\big| \ge 1$, say $|(T_1)_{x\langle y\rangle}| \ge 1$, then Lemma 2.4.1 implies that $\sigma(T_1) = 1_G$ and hence

$$\sigma(T_i) = 1_G \quad \text{for all } \ i \in [1,5]\,.$$

Let h be a term of $(T_1)_{x\langle y\rangle}$. Then $h\,\sigma(h^{[-1]}{\cdot}T_1) = 1_G$ and hence it follows from $s \equiv -1 \pmod 3$ that

$$\begin{aligned}
&\sigma(T_6)\,h\,\sigma(T_7)\,\sigma(h^{[-1]}{\cdot}T_1)\sigma(T_2)\sigma(T_3)\sigma(T_4)\\
&= y^{t_2n_2}\,h\,y^{t_2n_2}\,\sigma(h^{[-1]}{\cdot}T_1)\\
&= h\,y^{t_2n_2s + t_2n_2}\,\sigma(h^{[-1]}{\cdot}T_1)\\
&= h\,\sigma(h^{[-1]}{\cdot}T_1)\\
&= 1_G\,,
\end{aligned}$$

whence $T_1{\cdot}\ldots{\cdot}T_4{\cdot}T_6{\cdot}T_7$ is a $6n_2$-product-one subsequence of S. If $\big|(T_6T_7)_{x\langle y\rangle}\big| \ge 1$, say $\big|(T_6)_{x\langle y\rangle}\big| \ge 1$, then Lemma 2.4 again implies that $\sigma(T_i) = 1_G$ for all $i \in [6,7]$. Let h be a term of $(T_6)_{x\langle y\rangle}$. It follows from $s \equiv -1 \pmod 3$ that

$$\begin{aligned}
&\sigma(T_1)\sigma(T_2)\,h\,\sigma(T_3)\sigma(T_4)\,\sigma(h^{[-1]}{\cdot}T_6)\,\sigma(T_7)\\
&= y^{2t_1n_2}\,h\,y^{2t_1n_2}\,\sigma(h^{[-1]}{\cdot}T_6)\\
&= h\,y^{2t_1n_2s + 2t_1n_2}\,\sigma(h^{[-1]}{\cdot}T_6)\\
&= h\,\sigma(h^{[-1]}{\cdot}T_6)\\
&= 1_G\,,
\end{aligned}$$

whence $T_1{\cdot}\ldots{\cdot}T_4{\cdot}T_6{\cdot}T_7$ is a $6n_2$-product-one subsequence of S.

Subcase 2.2 : $\big|(T_1{\cdot}\ldots{\cdot}T_7)_{x\langle y\rangle}\big| = 0$.

Then $Y := T_1{\cdot}\ldots{\cdot}T_7$ is a sequence over $\langle y\rangle$ of length $7n_2$ and

$$\big|\{i \in [1,8] \colon |(T_i)_{x\langle y\rangle}| > 0\}\big| = 1\,.$$

If there exists a decomposition $Y = Z_1{\cdot}\ldots{\cdot}Z_7$ into a product of n_2-product-$\langle x, y^{n_2}\rangle$ subsequences Z_i such that $\sigma(Z_1){\cdot}\ldots{\cdot}\sigma(Z_7)$ has a 6-product-one subsequence, then we are done. Thus we may suppose

that $\sigma(Z_1)\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}\sigma(Z_7)$ has no 6-product-one subsequence for any decomposition of $Y = Z_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}Z_7$ into a product of n_2-product-$\langle x, y^{n_2}\rangle$ subsequences Z_i. In view of Proposition 3.2, we infer that

$$\Pi_{n_2-1}\big(\varphi(T_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}T_7)\big) \,=\, \langle y^3\rangle\,. \tag{4.7}$$

Let $E = S\boldsymbol{\cdot}(T_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}T_8)^{[-1]}$. Suppose $\big|E_{x\langle y\rangle}\big| \ge 1$. Let h be a term of $E_{x\langle y\rangle}$. Since $\varphi(h) \in \langle y^3\rangle$, it follows from (4.7) that $T_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}T_7$ has a subsequence T_0 of length $n_2 - 1$ such that $\varphi(h\boldsymbol{\cdot}T_0)$ is a product-one sequence, whence $T_8^{[-1]}\boldsymbol{\cdot}S$ has an n_2-product-$\langle x, y^{n_2}\rangle$ subsequence T_1' with $h \mid T_1'$. By applying $\mathsf{E}(C_{n_2}) = 2n_2 - 1$, we obtain a new decomposition of S

$$S \,=\, T_1'\boldsymbol{\cdot}T_8\boldsymbol{\cdot}T_2'\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}T_7'\boldsymbol{\cdot}E'\,,$$

where both T_1' and T_8 have terms from $x\langle y\rangle$, a contradiction to (4.1).

Thus $|E_{x\langle y\rangle}| = 0$, whence $\big|S_{x\langle y\rangle}\big| = \big|(T_8)_{x\langle y\rangle}\big| \ge 2$. Let h and h' be two terms with $h\boldsymbol{\cdot}h' \mid (T_8)_{x\langle y\rangle}$. In view of (4.7), we infer that $T_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}T_7$ has a subsequence T_0 of length $n_2 - 1$ such that $\varphi(h\boldsymbol{\cdot}T_0)$ is a product-one sequence, whence $(h')^{[-1]}\boldsymbol{\cdot}S$ has an n_2-product-$\langle x, y^{n_2}\rangle$ subsequence T_1' with $h \mid T_1'$. By applying $\mathsf{E}(C_{n_2}) = 2n_2 - 1$, we obtain a new decomposition of S

$$S \,=\, T_1'\boldsymbol{\cdot}T_2'\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}T_8'\boldsymbol{\cdot}E'\,,$$

where $h \mid T_1'$ and $h' \mid T_2'\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}T_8'\boldsymbol{\cdot}E'$. By (4.1), we must have $h' \mid E'$ and go back to the previous case that $\big|E'_{x\langle y\rangle}\big| \ge 1$. This completes our proof. □

We are now able to prove the main theorem of this paper.

Proof of Theorem 1.1. Since $n_2 \ne 1$ and $\gcd(n_2, 6) = 1$, we have that $n_2 \ge 5$.

1. It is known that $\mathsf{E}(G) \ge 9n_2$ (see [26, Lemma 4]). Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = 9n_2$. If $\big|S_{x\langle y\rangle}\big| \le 1$, then $S_{\langle y\rangle}$ is a sequence over $\langle y\rangle$ of length at least $9n_2 - 1$ and hence Lemma 2.2.2 ensures that $S_{\langle y\rangle}$ has a $6n_2$-product-one subsequence. If $\big|S_{x\langle y\rangle}\big| \ge 2$, then Proposition 4.1 ensures that S has a $6n_2$-product-one subsequence. Therefore $\mathsf{E}(G) \le 9n_2$ and hence $\mathsf{E}(G) = 9n_2$.

2. Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = 9n_2 - 1$. If $S_{\langle y\rangle} = S$, then Lemma 2.2.2 implies that S has a $6n_2$-product-one subsequence. Thus we may assume that $|S_{x\langle y\rangle}| \ge 1$. If $\big|S_{x\langle y\rangle}\big| \ge 2$, then Proposition 4.1 implies that S have a $6n_2$-product-one subsequence. Thus

$$|S_{x\langle y\rangle}| \,=\, 1\,.$$

Hence $\big|S_{\langle y\rangle}\big| = 9n_2 - 2$ and it follows from Lemma 2.2.3 that there exist $t_1, t_2 \in \mathbb{Z}$ with $\gcd(t_1 - t_2, 3n_2) = 1$ such that $S_{\langle y\rangle} = (y^{t_1})^{[6n_2-1]}\boldsymbol{\cdot}(y^{t_2})^{[3n_2-1]}$. The assertion now follows. □

Concluding remark: It is worth mentioning that the direct and inverse problems related to $\mathsf{E}(G)$ are already solved for the non-split metacyclic groups of the form $\langle x, y\colon x^2 = y^n, y^{2n} = 1_G, yx = xy^s\rangle$, where $s^2 \equiv 1 \pmod{2n}$ (see [24, 25]). The remaining case to complete the direct and inverse problems related to $\mathsf{E}(G)$ for all groups having a cyclic subgroup of index 2 is when $G = \langle x, y\colon x^2 = y^m, y^n = 1_G, yx = xy^s\rangle$, where $n = 2^t m$ with $t \ge 2$ and m odd.

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References

[1] W.R. Alford, A. Granville, and C. Pomerance, *There are infinitely many Carmichael numbers.* Ann. of Math. **139** (1994), 703–722.

[2] D.V. Avelar, F.E. Brochero Martínez, and S. Ribas, *A note on Bass' conjecture.* J. Number Theory **249** (2023), 462–469.

[3] ———, *On the direct and inverse zero-sum problems over $C_n \rtimes_s C_2$*, J. Combin. Theory Ser. A **197** (2023), 105751.

[4] J. Bass, *Improving the Erdős-Ginzburg-Ziv theorem for some non-abelian groups*, J. Number Theory **126** (2007), 217–236.

[5] F.E. Brochero Martínez, A. Lemos, B.K. Moriya, and S. Ribas, *The main zero-sum constants over $D_{2n} \times C_2$*, SIAM J. Discrete Math. **37** (2023), 1496–1508.

[6] F.E. Brochero Martínez and S. Ribas, *Extremal product-one free sequences in $C_q \rtimes_s C_m$*, J. Number Theory **204** (2019), 334–353.

[7] ———, *Extremal product-one free sequences in dihedral and dicyclic groups*, Discrete Math. **341** (2018), 570–578.

[8] ———, *Extremal product-one free sequences over $C_n \rtimes_s C_2$*, Discrete Math. **345** (2022), 113062.

[9] M. DeVos, L. Goddyn, and B. Mohar, *A generalization of Kneser's addition theorem*, Adv. Math. **220** (2009), 1531–1548.

[10] P. Erdős, A. Ginzburg, and A. Ziv, *Theorem in the additive number theory*, Bull. Res. Council Israel **10** (1961), 41–43.

[11] P. van Emde Boas and D. Kruyswijk, *A combinatorial problem on finite abelian groups III*, Report ZW-1969-008, Math. Centre, Amsterdam, 1969.

[12] W.D. Gao, *A combinatorial problem on finite abelian groups*, J. Number Theory **58** (1996), 100–103.

[13] W. Gao, Y. Li, and Y. Qu, *On the invariant $\mathsf{E}(G)$ for groups of odd order*, Acta Arith. **201** (2021), 255-267.

[14] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations: algebraic, combinatorial and analytic theory*, Pure and Applied Mathematics 278, Chapman & Hall/CRC, 2006.

[15] N.K. Godara and S. Sarkar, *A note on a conjecture of Gao and Zhuang for groups of order* 27, J. Algebra Appl. **24** (2025), 2550268.

[16] D.J. Grynkiewicz, *Structural Additive Theory*, Developments in Mathematics, vol. 30, Springer, 2013.

[17] D.C. Han and H.B. Zhang, *Erdős-Ginzburg-Ziv theorem and Noether number for $C_m \ltimes_\varphi C_{mn}$*, J. Number Theory **198** (2019), 159–175.

[18] J.S. Oh and Q. Zhong, *On Erdős-Ginzburg-Ziv inverse theorems for dihedral and dicyclic groups*, Israel J. Math **238** (2020), 715–743.

[19] J.E. Olson, *A combinatorial problem on finite Abelian groups I*, J. Number Theory **1** (1969), 8–10.

[20] ———, *A combinatorial problem on finite Abelian groups II*, J. Number Theory **1** (1969), 195–199.

[21] A. Plagne and W.A. Schmid, *An application of coding theory to estimating Davenport constants*, Des. Codes Crypt., **61** (2011), 105–118.

[22] Y. Qu and Y. Li, *Extremal product-one free sequences and $|G|$-product-one free sequences of a meta-cyclic group*, Discrete Math. **345** (2022), 112938.

[23] ———, *On a conjecture of Zhuang and Gao*, Colloq. Math. **171** (2023), 113–126.

[24] S. Ribas, *Some zero-sum problems over $\langle x, y \mid x^2 = y^{n/2}, y^n = 1, yx = xy^s \rangle$*, Bull. Braz. Math. Soc. (N.S.) **56** (2025), Article 12.

[25] J. Yang, X. Zhang, and L. Feng, *On the direct and inverse zero-sum problems over non-split metacyclic groups*, Discrete Math. **347** (2024), 114213.

[26] J.J. Zhuang and W.D. Gao, *Erdős-Ginzburg-Ziv theorem for dihedral groups of large prime index*, Europ. J. Combin. **26** (2005), 1053–1059.

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