$\mathcal{A}$-COMPACT HOLOMORPHIC LIPSCHITZ MAPPINGS ON THE UNIT BALL OF A BANACH SPACE

A. JIMÉNEZ-VARGAS AND D. RUIZ-CASTERNADO

Abstract. Let X and Y be complex Banach spaces, B_X be the open unit ball of X and $\mathcal{H}L_0(B_X, Y)$ be the Banach space of all holomorphic Lipschitz maps $f\colon B_X \to Y$ such that $f(0) = 0$, endowed with the Lipschitz norm. Given a Banach operator ideal $\mathcal{A}$, we use the property of $\mathcal{A}$-compactness by Carl and Stephani to introduce and study the subclass of those functions in $\mathcal{H}L_0(B_X, Y)$ for which its Lipschitz image is a relatively $\mathcal{A}$-compact subset of Y. We focus our attention on its structure as a composition Banach holomorphic Lipschitz ideal by using its connection with $\mathcal{A}$-compact linear operators through linearization/transposition techniques.

Introduction and preliminaries

Let X and Y be complex Banach spaces and let B_X be the open unit ball of X. If $\mathrm{Lip}_0(B_X, Y)$ denotes the set of all zero-preserving Lipschitz maps from B_X into Y and $\mathcal{H}^\infty(B_X, Y)$ stands for the set of all bounded holomorphic maps from B_X into Y, let

$$\mathcal{H}L_0(B_X, Y) := \mathrm{Lip}_0(B_X, Y) \cap \mathcal{H}^\infty(B_X, Y)$$

be the Banach space of all holomorphic Lipschitz maps $f\colon B_X \to Y$ such that $f(0) = 0$, endowed with the Lipschitz norm

$$L(f) = \sup\left\{\frac{\|f(x) - f(y)\|}{\|x - y\|}\colon x, y \in B_X,\ x \neq y\right\}.$$

The little holomorphic Lipschitz space $\mathcal{H}l_0(B_X, Y)$ is the closed subspace of $\mathcal{H}L_0(B_X, Y)$ consisting of all those mappings $f\colon B_X \to Y$ which satisfy the following property:

$$\forall\varepsilon > 0, \quad \exists\delta > 0\colon x, y \in B_X,\ 0 < \|x - y\| < \delta \quad \Rightarrow \quad \frac{\|f(x) - f(y)\|}{\|x - y\|} < \varepsilon.$$

We will write $\mathcal{H}L_0(B_X)$ and $\mathcal{H}l_0(B_X)$ instead of $\mathcal{H}L_0(B_X, \mathbb{C})$ and $\mathcal{H}l_0(B_X, \mathbb{C})$, respectively.

Recently, the linearization of such holomorphic Lipschitz mappings has been addressed by Aron et al. in [5]. Using the Dixmier–Ng Theorem, they not only showed that $\mathcal{H}L_0(B_X, Y)$ is indeed a dual space, but that there exist a complex Banach space $\mathcal{G}_0(B_X)$ and a map $\delta_X \in \mathcal{H}L_0(B_X, \mathcal{G}_0(B_X))$ satisfying the following universal property: for every complex Banach space Y and every map $f \in \mathcal{H}L_0(B_X, Y)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$ such that $T_f \circ \delta_X = f$. In fact, $\mathcal{G}_0(B_X) = \overline{\mathrm{lin}}(\delta_X(B_X)) \subseteq \mathcal{H}L_0(B_X)^*$ and $\delta_X\colon B_X \to \mathcal{G}_0(B_X)$ is the isometry defined by $\delta_X(x)(g) =$

Date: November 25, 2025.

2020 *Mathematics Subject Classification.* 46T25, 46E50, 46T20, 47L20.

Key words and phrases. Vector-valued holomorphic Lipschitz mapping, linearization, $\mathcal{A}$-compact operator, operator ideal, factorization theorems.

This research was partially supported by Junta de Andalucía grant FQM194. The first author was supported by Ministerio de Ciencia e Innovación grant PID2021-122126NB-C31 (funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU); and the second author by an FPU predoctoral fellowship of the Spanish Ministry of Universities (FPU23/03235).

$g(x)$ for all $x \in B_X$ and $g \in \mathcal{H}L_0(B_X)$. Furthermore, the map $f \mapsto T_f$ is an isometric isomorphism from $\mathcal{H}L_0(B_X, Y)$ onto $\mathcal{L}(\mathcal{G}_0(B_X), Y)$.

Given a Banach operator ideal $\mathcal{A}$, our objective in this paper is to study the $\mathcal{A}$-compactness and the $\mathcal{A}$-boundedness of holomorphic Lipschitz maps from B_X into Y.

Carl and Stephani [8] introduced a refinement of the notion of compactness involved to a given Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$. For a Banach space X, a set $K \subseteq X$ is said to be relatively $\mathcal{A}$-compact if there exist a Banach space Y, an operator $T \in \mathcal{A}(Y, X)$ and a relatively compact set $M \subseteq Y$ such that $K \subseteq T(M)$. We denote by $\mathcal{K}_{\mathcal{A}}(X)$ the collection of all relatively $\mathcal{A}$-compact subsets of X. Lassalle and Turco [19] proposed a measure of the size of $\mathcal{A}$-compactness of $K \in \mathcal{K}_{\mathcal{A}}(X)$ defining $m_{\mathcal{A}}(K) = \inf\{\|T\|_{\mathcal{A}}\}$, where the infimum is taken over all such Y, T and M as above.

The $\mathcal{A}$-compactness yields the so-called $\mathcal{A}$-compact operators that are those $T \in \mathcal{L}(X, Y)$ satisfying that $T(\overline{B}_X)$ is a relatively $\mathcal{A}$-compact subset of Y, where $\overline{B}_X$ denotes the closed unit ball of X. The set of such operators is denoted as $\mathcal{K}_{\mathcal{A}}(X, Y)$, and $[\mathcal{K}_{\mathcal{A}}, \|\cdot\|_{\mathcal{K}_{\mathcal{A}}}]$ is a Banach operator ideal with the norm

$$\|T\|_{\mathcal{K}_{\mathcal{A}}} = m_{\mathcal{A}}(T(\overline{B}_X)) \qquad (T \in \mathcal{K}_{\mathcal{A}}(X, Y))$$

(see [19, Section 2]). Some examples of operator ideals $\mathcal{K}_{\mathcal{A}}$ are particularly interesting. For $p \in [1, \infty)$, the ideal of p-compact operators, – introduced by Sinha and Karn [24] – , coincides with the ideal of $\mathcal{N}_p$-compact operators, where $\mathcal{N}_p$ is the ideal of right p-nuclear operators (see [23]). Moreover, Ain, Lillemets and Oja [2, 3] generalized the ideal of p-compact operators introducing the ideal of (p, q)-compact operators for $p \in [1, \infty]$ and $q \in [1, p^*]$, which in this case coincides with the ideal of $\mathcal{N}_{(p,1,q^*)}$-compact operators, where $\mathcal{N}_{(p,1,q^*)}$ stands for the ideal of $(p, 1, q^*)$-nuclear operators (see [22, 18.1.1]).

The Banach ideals of $\mathcal{A}$-compact operators, polynomials and holomorphic functions and their related approximation properties have been studied in [17, 19, 20, 26]. In particular, the ideals of p-compact and (p, q)-compact operators have been extended to different non-linear contexts as, for example, holomorphic [6, 18], bounded-holomorphic [13, 14], polynomial [7], and Lipschitz [1].

We have divided this paper into three sections. Section 1 contains some preliminary results where we introduce the concept of ideal of holomorphic Lipschitz maps (Subsection 1.1) and present both procedures to generate new Banach holomorphic Lipschitz ideals by composition (Subsection 1.2) and duality (Subsection 1.3). In particular, the transposition of functions permit us to identify in Corollary 1.11 the space $\mathcal{H}L_0(B_X, Y)$ with the space of all weak*-to-weak* continuous linear operators from Y^* into $\mathcal{H}L_0(B_X)$.

Given an operator ideal $\mathcal{A}$, our objective in Section 2 is to introduce a variant for holomorphic Lipschitz maps of the concept of $\mathcal{A}$-compact linear operators between Banach spaces [8, 19]. It is important to note that we do not use the global range of a map $f \in \mathcal{H}L_0(B_X, Y)$, but its Lipchitz image

$$\mathrm{Im}_L(f) := \left\{ \frac{f(x) - f(y)}{\|x - y\|} : x, y \in B_X,\ x \neq y \right\} \subseteq Y.$$

So, a map $f \in \mathcal{H}L_0(B_X, Y)$ is said to be $\mathcal{A}$-compact holomorphic Lipschitz if $\mathrm{Im}_L(f)$ is a relatively $\mathcal{A}$-compact subset of Y. In Propositions 2.2 and 2.3, we obtain some immediate characterizations of $\mathcal{A}$-compact holomorphic Lipschitz maps applying the different descriptions of $\mathcal{A}$-compacts sets obtained in [8, 19]. Theorem 2.4 provides a useful characterization of such maps in terms of their linearizations. In particular, interesting descriptions of $\mathcal{K}$-compact and $\mathcal{K}_p$-compact holomorphic Lipschitz maps in term of the continuity and compactness of their transposes are stated (see Theorem 2.8, and Propositions 2.9-2.10).

The $\mathcal{A}$-compactness is generalized by the property of $\mathcal{A}$-boundedness due to Stephani [25]. An extension of this property to the holomorphic Lipschitz setting is presented in Section 3. This approach permits us to study some new classes of holomorphic Lipschitz maps that are not covered by the $\mathcal{A}$-compactness such as weakly compact, Rosenthal, Banach–Saks, Asplund, finite-rank and approximable holomorphic Lipschitz maps. Some variants of classical results on ideals of factorizable operators are established for the associate ideals of holomorphic Lipschitz mappings (see Theorems 3.3-3.5). By transposition, we identify certain spaces of holomorphic (little) Lipschitz maps from B_X into Y with some distinguished spaces of linear operators between Y^* and $\mathcal{H}L_0(B_X)$ equipped both with suitable topologies (see Propositions 2.9-2.10-3.6 and Theorems 3.8-3.10). For future reference, we conclude the paper with an appeal to the approximation property of $\mathcal{H}L_0(B_X)$.

1. Preliminary results

Notation. Throughout this paper, X and Y will be complex Banach spaces. We denote by $\mathcal{L}(X,Y)$, $\mathcal{F}(X,Y)$, $\mathcal{W}(X,Y)$, $\mathcal{K}(X,Y)$ and $\overline{\mathcal{F}}(X,Y)$ the spaces of all bounded, finite-rank bounded, weakly compact, compact and approximable linear operators from X to Y, respectively, endowed with the canonical operator norm $\|\cdot\|$. As usual, X^* and X^{**} stand for the topological dual and the topological bidual of X, respectively. We denote by B_X and $\overline{B}_X$ the open unit ball and the closed unit ball of X, respectively. Given $A \subseteq X$, $\mathrm{co}(A)$, $\mathrm{abco}(A)$ and $\overline{\mathrm{abco}}(A)$ represent the convex hull, the absolutely convex hull and the norm-closed absolutely convex hull of A in X. For any $x \in X$ and $x^* \in X^*$, we will use the notation $\langle x^*, x\rangle = x^*(x)$. We denote by $\mathcal{H}L_0(B_X, B_Y)$ the set of all zero-preserving holomorphic Lipschitz maps from B_X into B_Y.

If E and F are locally convex Hausdorff spaces, $\mathcal{L}(E;F)$ stands for the linear space of all continuous linear operators from E into F. Unless stated otherwise, if E and F are Banach spaces, we will understand that they are endowed with the norm topology. We refer to the monograph [21] for definitions and main properties of the weak* topology w^*, the weak topology w and the bounded weak* topology bw^*. The rest of the necessary notation will be introduced throughout the paper.

1.1. **Ideals of holomorphic Lipschitz mappings.** Based on the notion of operator ideal introduced by Pietsch [22, 1.1], we present a version of this concept for holomorphic Lipschitz maps.

Definition 1.1. *An ideal of holomorphic Lipschitz maps (also called holomorphic Lipschitz ideal) is a subclass $\mathcal{A}^{\mathcal{H}L_0}$ of $\mathcal{H}L_0$ such that for any complex Banach spaces X and Y, the components*

$$\mathcal{A}^{\mathcal{H}L_0}(B_X, Y) := \mathcal{H}L_0(B_X, Y) \cap \mathcal{A}^{\mathcal{H}L_0}$$

satisfy the following conditions:

(I1) *$\mathcal{A}^{\mathcal{H}L_0}(B_X, Y)$ is a linear subspace of $\mathcal{H}L_0(B_X, Y)$.*
(I2) *Given $h \in \mathcal{H}L_0(B_X)$ and $y \in Y$, the map $h \cdot y : B_X \to Y$, defined as $(h \cdot y)(x) = h(x)y$ for all $x \in B_X$, is in $\mathcal{A}^{\mathcal{H}L_0}(B_X, Y)$.*
(I3) *The ideal property: if W and Z are complex Banach spaces, $f \in \mathcal{A}^{\mathcal{H}L_0}(B_X, Y)$, $T \in \mathcal{L}(Y, W)$ and $g \in \mathcal{H}L_0(B_Z, B_X)$, then we have that $T \circ f \circ g \in \mathcal{A}^{\mathcal{H}L_0}(B_Z, W)$.*

If we endow this subclass $\mathcal{A}^{\mathcal{H}L_0}$ with a function $\|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}} : \mathcal{A}^{\mathcal{H}L_0} \to \mathbb{R}_0^+$ satisfying the properties:

(N1) *$(\mathcal{A}^{\mathcal{H}L_0}(B_X, Y), \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}})$ is a (s-Banach) s-normed space with $s \in (0, 1]$ and $L(f) \leq \|f\|_{\mathcal{A}^{\mathcal{H}L_0}}$ for all $f \in \mathcal{A}^{\mathcal{H}L_0}(B_X, Y)$,*
(N2) *$\|h \cdot y\|_{\mathcal{A}^{\mathcal{H}L_0}} = L(h)\|y\|$ for all $h \in \mathcal{H}L_0(B_X)$ and $y \in Y$,*
(N3) *$\|T \circ f \circ g\|_{\mathcal{A}^{\mathcal{H}L_0}} \leq \|T\|\|f\|_{\mathcal{A}^{\mathcal{H}L_0}} L(g)$ if $f \in \mathcal{A}^{\mathcal{H}L_0}(B_X, Y)$, $T \in \mathcal{L}(Y, W)$ and $g \in \mathcal{H}L_0(B_Z, B_X)$,*

then we say that $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]$ is a (s-Banach) s-normed holomorphic Lipschitz ideal.

If $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]$ *and* $[\mathcal{B}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{B}^{\mathcal{H}L_0}}]$ *are s-normed holomorphic Lipschitz ideals, we will write*

$$[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}] \leq [\mathcal{B}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{B}^{\mathcal{H}L_0}}]$$

whenever $\mathcal{A}^{\mathcal{H}L_0} \subseteq \mathcal{B}^{\mathcal{H}L_0}$ *and* $\|f\|_{\mathcal{B}^{\mathcal{H}L_0}} \leq \|f\|_{\mathcal{A}^{\mathcal{H}L_0}}$ *for all* $f \in \mathcal{A}(B_X, Y)$*.*

Note that this concept of holomorphic Lipschitz ideal is closely related to the notion of hyper-ideal of polynomials since it contains finite-rank holomorphic Lipschitz mappings but this does not happen with polynomial ideals in general.

Influenced by [22, 8.1], we now introduce a method to construct new s-normed holomorphic Lipschitz ideals from given ones.

Definition 1.2. *A holomorphic Lipschitz procedure is a correspondence*

$$[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}] \mapsto [(\mathcal{A}^{\mathcal{H}L_0})^{new}, \|\cdot\|^{new}_{\mathcal{A}^{\mathcal{H}L_0}}]$$

that maps an s-normed holomorphic Lipschitz ideal $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]$ *to an s-normed holomorphic Lipschitz ideal* $[(\mathcal{A}^{\mathcal{H}L_0})^{new}, \|\cdot\|^{new}_{\mathcal{A}^{\mathcal{H}L_0}}]$*. If we note*

$$[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]^{new} := [(\mathcal{A}^{\mathcal{H}L_0})^{new}, \|\cdot\|^{new}_{\mathcal{A}^{\mathcal{H}L_0}}],$$

a holomorphic Lipschitz procedure is called:

- *(M) Monotone if* $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}] \leq [\mathcal{B}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{B}^{\mathcal{H}L_0}}]$ *implies* $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]^{new} \leq [\mathcal{B}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{B}^{\mathcal{H}L_0}}]^{new}$*.*
- *(I) Idempotent if* $([\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]^{new})^{new} = [\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]^{new}$*.*
- *(H) Hull if it is monotone, idempotent and* $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}] \leq [\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]^{new}$*.*

Definition 1.3. *Let X and Y be complex Banach spaces. For a normed holomorphic Lipschitz ideal* $[\mathcal{A}^{\mathcal{H}L_0}, L(\cdot)]$*, we set*

$$(\mathcal{A}^{\mathcal{H}L_0})^{clos}(B_X, Y) = \left\{ f \in \mathcal{H}L_0(B_X, Y) \mid \exists (f_n)_{n=1}^{\infty} \subseteq \mathcal{A}^{\mathcal{H}L_0}(B_X, Y) \colon \lim_{n\to\infty} L(f_n - f) = 0 \right\};$$

and for an s-normed holomorphic Lipschitz ideal $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]$*, we define*

$$(\mathcal{A}^{\mathcal{H}L_0})^{reg}(B_X, Y) = \left\{ f \in \mathcal{H}L_0(B_X, Y) \colon \kappa_Y \circ f \in \mathcal{A}^{\mathcal{H}L_0}(B_X, Y^{**}) \right\},$$

where $\kappa_Y : Y \to Y^{**}$ *is the canonical isometric linear embedding.*

The following result can be proved easily.

Proposition 1.4. *The correspondences*

$$[\mathcal{A}^{\mathcal{H}L_0}, L(\cdot)] \mapsto [(\mathcal{A}^{\mathcal{H}L_0})^{clos}, L(\cdot)]$$

and

$$[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}] \mapsto [\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]^{reg}$$

are hull holomorphic Lipschitz procedures. The holomorphic Lipschitz spaces $(\mathcal{A}^{\mathcal{H}L_0})^{clos}(B_X, Y)$ *and* $(\mathcal{A}^{\mathcal{H}L_0})^{reg}(B_X, Y)$ *are called the closed hull and regular hull of* $\mathcal{A}^{\mathcal{H}L_0}(B_X, Y)$*, respectively.* □

The preceding result motivates the introduction of the following concepts.

Definition 1.5. *A normed holomorphic Lipschitz ideal* $[\mathcal{A}^{\mathcal{H}L_0}, L(\cdot)]$ *is called closed if* $[\mathcal{A}^{\mathcal{H}L_0}, L(\cdot)] = [(\mathcal{A}^{\mathcal{H}L_0})^{clos}, L(\cdot)]$*, that is, every component* $\mathcal{A}^{\mathcal{H}L_0}(B_X, Y)$ *is a closed subspace of* $\mathcal{H}L_0(B_X, Y)$ *with the topology induced by the Lipschitz norm.*

an s-normed holomorphic Lipschitz ideal $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]$ *is called regular if* $[\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}] = [\mathcal{A}^{\mathcal{H}L_0}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0}}]^{reg}$*, that is, if* $f \in \mathcal{H}L_0(B_X, Y)$ *and* $\kappa_Y \circ f \in \mathcal{A}^{\mathcal{H}L_0}(B_X, Y^{**})$*, then* $f \in \mathcal{A}^{\mathcal{H}L_0}(B_X, Y)$ *with* $\|f\|_{\mathcal{A}^{\mathcal{H}L_0}} = \|\kappa_Y \circ f\|_{\mathcal{A}^{\mathcal{H}L_0}}$*.*

1.2. **Composition ideals of holomorphic Lipschitz mappings.** We now introduce the composition procedure to generate s-normed holomorphic Lipschitz ideals.

Definition 1.6. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal. For any complex Banach spaces X and Y, define*

$$\mathcal{A} \circ \mathcal{H}L_0(B_X, Y) = \{f \in \mathcal{H}L_0(B_X, Y) \mid \exists S \in \mathcal{A}(Z, Y),\ \exists h \in \mathcal{H}L_0(B_X, Z)\colon f = S \circ h\},$$

and

$$\|f\|_{\mathcal{A}\circ\mathcal{H}L_0} = \inf\{\|S\|_{\mathcal{A}} L(h)\},$$

where the infimum extends over all factorizations of f as above.

The belonging of a holomorphic Lipschitz map to $\mathcal{A}\circ\mathcal{H}L_0$ may be characterized by means of its linearization as follows. This result can be compared to [4, Theorem 3.2] for bounded holomorphic functions.

Theorem 1.7. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal and $f \in \mathcal{H}L_0(B_X, Y)$. The following statements are equivalent:*

(i) f belongs to $\mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$.
(ii) T_f belongs to $\mathcal{A}(\mathcal{G}_0(B_X), Y)$.

In such a case, $\|f\|_{\mathcal{A}\circ\mathcal{H}L_0} = \|T_f\|_{\mathcal{A}}$.

Proof. $(i) \Rightarrow (ii)$: Assume that $f \in \mathcal{A}\circ\mathcal{H}L_0(B_X, Y)$. Then $f = S \circ h$, where Z is a complex Banach space, $S \in \mathcal{A}(Z, Y)$ and $h \in \mathcal{H}L_0(B_X, Z)$. By [5, Proposition 2.3 (c)], there exist $T_f \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$ and $T_h \in \mathcal{L}(\mathcal{G}_0(B_X), Z)$ with $\left\|T_f\right\| = L(f)$ and $\|T_h\| = L(h)$ such that $f = T_f \circ \delta_X$ and $h = T_h \circ \delta_X$. It follows that $T_f \circ \delta_X = S \circ T_h \circ \delta_X$, and since $\mathrm{lin}(\delta_X(B_X))$ is a dense linear subspace of $\mathcal{G}_0(B_X)$ by [5, Proposition 2.3 (b)], we deduce that $T_f = S \circ T_h$ and $\left\|T_f\right\|_{\mathcal{A}} \leq \|S\|_{\mathcal{A}} L(h)$. By the ideal property of $\mathcal{A}$, we conclude that $T_f \in \mathcal{A}(\mathcal{G}_0(B_X), Y)$, and passing to the infimum over all the factorizations of f as above, we obtain that $\left\|T_f\right\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}\circ\mathcal{H}L_0}$.

$(ii) \Rightarrow (i)$: Assume that $T_f \in \mathcal{A}(\mathcal{G}_0(B_X), Y)$. Notice that $f = T_f \circ \delta_X \in \mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$ since $\delta_X \in \mathcal{H}L_0(B_X, \mathcal{G}_0(B_X))$ with $L(\delta_X) = 1$ by [5, Proposition 2.3 (a)]. Moreover,

$$\|f\|_{\mathcal{A}\circ\mathcal{H}L_0} = \|T_f \circ \delta_X\|_{\mathcal{A}\circ\mathcal{H}L_0} \leq \|T_f\|_{\mathcal{A}} L(\delta_X) = \|T_f\|_{\mathcal{A}}.$$

□

Some properties of an operator ideal $\mathcal{A}$ can be inherited by $\mathcal{A} \circ \mathcal{H}L_0$ as we see below.

Corollary 1.8. *If $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is an s-normed operator ideal, then $[\mathcal{A}\circ\mathcal{H}L_0, \|\cdot\|_{\mathcal{A}\circ\mathcal{H}L_0}]$ is an s-normed holomorphic Lipschitz ideal called the holomorphic Lipschitz composition ideal of $\mathcal{A}$. Moreover, $[\mathcal{A} \circ \mathcal{H}L_0, \|\cdot\|_{\mathcal{A}\circ\mathcal{H}L_0}]$ is closed (resp., s-Banach, regular) whenever $[\mathcal{A}, \|\cdot\|]$ (resp., $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$) is so.*

Proof. We first prove that $\mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$ is a linear subspace of $\mathcal{H}L_0(B_X, Y)$. Let $\lambda \in \mathbb{C}$ and $f, g \in \mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$. Then $T_f, T_g \in \mathcal{A}(\mathcal{G}_0(B_X), Y)$ by Theorem 1.7. Hence $T_{\lambda f} = \lambda T_f$ and $T_{f+g} = T_f + T_g$ belongs to $\mathcal{A}(\mathcal{G}_0(B_X), Y)$ by [5, Proposition 2.3 (c)] and the ideal property of $\mathcal{A}$. This implies that $\lambda f, f + g \in \mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$ again by Theorem 1.7, as required.

We now prove the conditions required in Definition 1.1.

(I1-N1): Using Theorem 1.7 and [5, Proposition 2.3 (c)], we obtain that $\|\cdot\|_{\mathcal{A}\circ\mathcal{H}L_0}$ is a s-norm on $\mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$ since

$$\|\lambda f\|_{\mathcal{A}\circ\mathcal{H}L_0} = \left\|T_{\lambda f}\right\|_{\mathcal{A}} = \left\|\lambda T_f\right\|_{\mathcal{A}} = |\lambda| \left\|T_f\right\|_{\mathcal{A}} = |\lambda| \, \|f\|_{\mathcal{A}\circ\mathcal{H}L_0},$$

$$\|f + g\|^s_{\mathcal{A}\circ\mathcal{H}L_0} = \left\|T_{f+g}\right\|^s_{\mathcal{A}} = \left\|T_f + T_g\right\|^s_{\mathcal{A}} \leq \left\|T_f\right\|^s_{\mathcal{A}} + \left\|T_g\right\|^s_{\mathcal{A}} = \|f\|^s_{\mathcal{A}\circ\mathcal{H}L_0} + \|g\|^s_{\mathcal{A}\circ\mathcal{H}L_0},$$

$$\|f\|_{\mathcal{A}\circ\mathcal{H}L_0} = 0 \Rightarrow \left\|T_f\right\|_{\mathcal{A}} = 0 \Rightarrow T_f = 0 \Rightarrow f = T_f \circ \delta_X = 0.$$

Using [5, Proposition 2.3 (c)], the fact that $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is an s-normed operator ideal and Theorem 1.7, we deduce that $L(f) = \left\|T_f\right\| \leq \left\|T_f\right\|_{\mathcal{A}} = \|f\|_{\mathcal{A}\circ\mathcal{H}L_0}$ for all $f \in \mathcal{A} \circ \mathcal{H}L_0$.

(I2-N2): Let $h \in \mathcal{H}L_0(B_X)$ and $y \in Y$. It is easy to prove that $h \cdot y \in \mathcal{H}L_0(B_X, Y)$ with $L(h \cdot y) = L(h)\,\|y\|$. Note that $T_{h\cdot y} = T_h \cdot y$ since $T_h \cdot y \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$ and

$$(h \cdot y)(x) = h(x)y = T_h(\delta_X(x))y = (T_h \cdot y)(\delta_X(x)) = ((T_h \cdot y) \circ \delta_X)(x)$$

for all $x \in B_X$. By the ideal property of $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, we have that $T_{h\cdot y} \in \mathcal{A}(\mathcal{G}_0(B_X), Y)$ with $\left\|T_{h\cdot y}\right\|_{\mathcal{A}} = \|T_h\|\,\|y\| = L(h)\,\|y\|$. Hence $h \cdot y \in \mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$ with $\|h \cdot y\|_{\mathcal{A}\circ\mathcal{H}L_0} = L(h)\,\|y\|$ by Theorem 1.7.

(I3-N3): Let W, Z be complex Banach spaces, $f \in \mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$, $T \in \mathcal{L}(Y, W)$ and $g \in \mathcal{H}L_0(B_Z, B_X)$. In light of [5, p. 3033], we can take $\hat{g} := T_{\delta_Y \circ g} \in \mathcal{L}(\mathcal{G}_0(B_Z), \mathcal{G}_0(B_X))$ such that $\hat{g} \circ \delta_Z = \delta_X \circ g$ with $\|\hat{g}\| = L(g)$. Note that

$$T \circ f \circ g = T \circ (T_f \circ \delta_X) \circ g = (T \circ T_f \circ \hat{g}) \circ \delta_Z,$$

where $T \circ T_f \circ \hat{g} \in \mathcal{A}(\mathcal{G}_0(B_Z), W)$ by the ideal property of $\mathcal{A}$ and $\delta_Z \in \mathcal{H}L_0(B_Z, \mathcal{G}_0(B_Z))$. Hence, $T \circ f \circ g \in \mathcal{A} \circ \mathcal{H}L_0(B_Z, W)$ and

$$\|T \circ f \circ g\|_{\mathcal{A}\circ\mathcal{H}L_0} \leq \|T \circ T_f \circ \hat{g}\|_{\mathcal{A}} L(\delta_Z) \leq \|T\|\|T_f\|_{\mathcal{A}}\|\hat{g}\| = \|T\|\|f\|_{\mathcal{A}\circ\mathcal{H}L_0} L(g).$$

Hence, $[\mathcal{A} \circ \mathcal{H}L_0, \|\cdot\|_{\mathcal{A}\circ\mathcal{H}L_0}]$ is an s-normed holomorphic Lipschitz ideal. To see that the s-norm $\|\cdot\|_{\mathcal{A}\circ\mathcal{H}L_0}$ is complete whenever $\|\cdot\|_{\mathcal{A}}$ is so, it suffices to note that $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{A} \circ \mathcal{H}L_0(B_X, Y), \|\cdot\|_{\mathcal{A}\circ\mathcal{H}L_0})$ onto $(\mathcal{A}(\mathcal{G}_0(B_X), Y), \|\cdot\|_{\mathcal{A}})$ by [5, Proposition 2.3 (c)] and Theorem 1.7.

Assume that $[\mathcal{A}, \|\cdot\|]$ is closed and let us show that so is $[\mathcal{A}\circ\mathcal{H}L_0, \|\cdot\|_{\mathcal{A}\circ\mathcal{H}L_0}]$. Let $f \in \mathcal{H}L_0(B_X, Y)$ and let (f_n) be a sequence in $\mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$ satisfying $\lim_{n\to\infty} L(f_n - f) = 0$. By Theorem 1.7 and [5, Proposition 2.3 (c)], we have that $T_{f_n} \in \mathcal{A}(\mathcal{G}_0(B_X), Y)$ and $\|T_{f_n} - T_f\| = \|T_{f_n - f}\| = L(f_n - f) \to 0$ as $n \to \infty$. Hence $T_f \in \mathcal{A}(\mathcal{G}_0(B_X), Y)$ and reapplying Theorem 1.7 we conclude that $f \in \mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$.

Finally, to prove the regularity, let us suppose that $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is regular. Let $f \in \mathcal{H}L_0(B_X, Y)$ and assume that $\kappa_Y \circ f \in \mathcal{A} \circ \mathcal{H}L_0(B_X, Y^{**})$. Clearly, $T_{\kappa_Y\circ f} = \kappa_Y \circ T_f$. Since $T_{\kappa_Y\circ f} \in \mathcal{A}(\mathcal{G}_0(B_X), Y^{**})$ by Theorem 1.7 and $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is regular, it follows that $T_f \in \mathcal{A}(\mathcal{G}_0(B_X), Y)$ with $\|T_f\|_{\mathcal{A}} = \|\kappa_Y \circ T_f\|_{\mathcal{A}}$. Hence, $f \in \mathcal{A} \circ \mathcal{H}L_0(B_X, Y)$ and

$$\|f\|_{\mathcal{A}\circ\mathcal{H}L_0} = \|T_f\|_{\mathcal{A}} = \|\kappa_Y \circ T_f\|_{\mathcal{A}} = \|T_{\kappa_Y\circ f}\|_{\mathcal{A}} = \|\kappa_Y \circ f\|_{\mathcal{A}\circ\mathcal{H}L_0}.$$

□

1.3. **Dual ideal of holomorphic Lipschitz mappings.** Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal. Following [22, 4.2], $[\mathcal{A}^{\mathrm{dual}}, \|\cdot\|_{\mathcal{A}^{\mathrm{dual}}}]$ determines an s-normed operator ideal, – called the dual ideal of $\mathcal{A}$ – , whose components are the spaces

$$\mathcal{A}^{\mathrm{dual}}(X, Y) = \{T \in \mathcal{L}(X, Y) : T^* \in \mathcal{A}(Y^*, X^*)\}$$

for any Banach spaces X and Y, equipped with the s-norm given by

$$\|T\|_{\mathcal{A}^{\mathrm{dual}}} = \|T^*\|_{\mathcal{A}} \qquad (T \in \mathcal{A}^{\mathrm{dual}}(X, Y)).$$

The introduction of a version of this concept for ideals of holomorphic Lipschitz maps requires an equivalent of the notion of adjoint operator in this setting, as follows.

Definition 1.9. *Given $f \in \mathcal{H}L_0(B_X, Y)$, the holomorphic Lipschitz transpose of f is the map $f^t\colon Y^* \to \mathcal{H}L_0(B_X)$ given by $f^t(y^*) = y^* \circ f$ for all $y^* \in Y^*$.*

To show that f^t satisfies the desired properties, let us recall that $\mathcal{H}L_0(B_X)$ is isometrically isomorphic to $\mathcal{G}_0(B_X)^*$ throughout the map $\Lambda_X : \mathcal{H}L_0(B_X) \to \mathcal{G}_0(B_X)^*$ defined as

$$\Lambda_X(g) = T_g \qquad (g \in \mathcal{H}L_0(B_X)).$$

Proposition 1.10. *Let $f \in \mathcal{H}L_0(B_X, Y)$. Then $f^t \in \mathcal{L}(Y^*, \mathcal{H}L_0(B_X))$ with $\|f^t\| = L(f)$ and $f^t = \Lambda_X^{-1} \circ (T_f)^*$.*

Proof. Given $y^* \in Y^*$, note that $(y^* \circ f)(0) = 0$ and $y^* \circ f \in \mathcal{H}^\infty(B_X, Y)$ with $(y^* \circ f)' = y^* \circ f'$ and $\|y^* \circ f\|_\infty \leq \|y^*\| \|f\|_\infty$. Moreover, for all $x, y \in B_X$ with $x \neq y$, we have

$$\frac{|(y^* \circ f)(x) - (y^* \circ f)(y)|}{\|x - y\|} \leq \|y^*\| \frac{\|f(x) - f(y)\|}{\|x - y\|} \leq \|y^*\| L(f),$$

and thus $y^* \circ f \in \mathcal{H}L_0(B_X, Y)$ with $L(y^* \circ f) \leq \|y^*\| L(f)$. It follows that $f^t \in \mathcal{L}(Y^*, \mathcal{H}L_0(B_X))$ with $\|f^t\| \leq L(f)$. On the other hand, let $\varepsilon \in (0, L(f))$ and consider $x, y \in B_X$ satisfying $\|f(x) - f(y)\|/\|x - y\| > L(f) - \varepsilon$. By the Hahn–Banach Theorem, there is $z^* \in Y^*$ with $\|z^*\| = 1$ such that $|z^*(f(x) - f(y))| = \|f(x) - f(y)\|$. Thus,

$$\begin{aligned}\|f^t\| &\geq \sup_{y^* \neq 0} \frac{L(f^t(y^*))}{\|y^*\|} \geq \frac{L(z^* \circ f)}{\|z^*\|} \geq \frac{|z^*(f(x)) - z^*(f(y))|}{\|x - y\|} \\ &= \frac{\|f(x) - f(y)\|}{\|x - y\|} > L(f) - \varepsilon.\end{aligned}$$

Just doing $\varepsilon \to 0$ we conclude that $\|f^t\| = L(f)$. Furthermore,

$$\begin{aligned}\langle (\Lambda_X \circ f^t)(y^*), \delta_X(x) \rangle &= \langle \Lambda_X(y^* \circ f), \delta_X(x) \rangle = \langle T_{y^* \circ f}, \delta_X(x) \rangle \\ &= \langle y^* \circ T_f, \delta_X(x) \rangle = \langle (T_f)^*(y^*), \delta_X(x) \rangle\end{aligned}$$

for all $y^* \in Y^*$ and $x \in B_X$. Since $\mathcal{G}_0(B_X) = \overline{\mathrm{lin}}(\delta_X(B_X))$, we conclude that $f^t = \Lambda_X^{-1} \circ (T_f)^*$. □

$\mathcal{H}L_0(B_X, Y)$ can be identified with the subspace of $\mathcal{L}(Y^*, \mathcal{H}L_0(B_X))$ consisting of all weak*-to-weak* continuous linear operators from Y^* into $\mathcal{H}L_0(B_X)$).

Corollary 1.11. *The correspondence $f \mapsto f^t$ is an isometric isomorphism from $\mathcal{H}L_0(B_X, Y)$ onto $\mathcal{L}((Y^*, w^*); (\mathcal{H}L_0(B_X), w^*))$.*

Proof. Let $f \in \mathcal{H}L_0(B_X, Y)$. Hence $f^t = \Lambda_X^{-1} \circ (T_f)^* \in \mathcal{L}((Y^*, w^*); (\mathcal{H}L_0(B_X)), w^*)$ by Proposition 1.10 and [21, Theorem 3.1.11]. We have $\|f^t\| = L(f)$, and in order to prove that the map in the statement is onto, let $T \in \mathcal{L}((Y^*, w^*); (\mathcal{H}L_0(B_X)), w^*)$. Then $\Lambda_X \circ T \in \mathcal{L}((Y^*, w^*); (\mathcal{G}_0(B_X)^*, w^*))$ and therefore $S^* = \Lambda_X \circ T$ for some $S \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$. By [5, Proposition 2.3 (c)], there exists $f \in \mathcal{H}L_0(B_X, Y)$ such that $T_f = S$ and we conclude that $T = \Lambda_X^{-1} \circ (T_f)^* = f^t$. □

We now introduce a notion of holomorphic Lipschitz dual ideal of an operator ideal as desired.

Definition 1.12. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal. Given complex Banach spaces X and Y, define*

$$\mathcal{A}^{\mathcal{H}L_0\text{-dual}}(B_X, Y) = \{f \in \mathcal{H}L_0(B_X, Y) \colon f^t \in \mathcal{A}(Y^*, \mathcal{H}L_0(B_X))\},$$

and

$$\|f\|_{\mathcal{A}^{\mathcal{H}L_0\text{-dual}}} = \|f^t\|_{\mathcal{A}} \qquad (f \in \mathcal{A}^{\mathcal{H}L_0\text{-dual}}(B_X, Y)).$$

The pair $[\mathcal{A}^{\mathcal{H}L_0\text{-dual}}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0\text{-dual}}}]$ is an s-normed operator ideal of composition type called the holomorphic Lipschitz dual ideal of $\mathcal{A}$. To be more precise, we have the following.

Theorem 1.13. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal. Then*

$$[\mathcal{A}^{\mathcal{H}L_0\text{-dual}}, \|\cdot\|_{\mathcal{A}^{\mathcal{H}L_0\text{-dual}}}] = [\mathcal{A}^{\mathrm{dual}} \circ \mathcal{H}L_0, \|\cdot\|_{\mathcal{A}^{\mathrm{dual}} \circ \mathcal{H}L_0}].$$

Proof. If $f \in \mathcal{A}^{\mathcal{H}L_0\text{-dual}}(B_X, Y)$, then $f^t \in \mathcal{A}(Y^*, \mathcal{H}L_0(B_X))$. By [5, Proposition 2.3 (c)], there exists $T_f \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$ such that $f = T_f \circ \delta_X$. Since $(T_f)^* = \Lambda_X \circ f^t$ by Proposition 1.10, the ideal property of $\mathcal{A}$ allows us to ensure that $(T_f)^* \in \mathcal{A}(Y^*, \mathcal{G}_0(B_X)^*)$ with $\|(T_f)^*\|_{\mathcal{A}} \leq \|\Lambda_X\|\|f^t\|_{\mathcal{A}} = \|f^t\|_{\mathcal{A}}$. Hence, $T_f \in \mathcal{A}^{\mathrm{dual}}(\mathcal{G}_0(B_X), Y)$ with $\|T_f\|_{\mathcal{A}^{\mathrm{dual}}} = \|(T_f)^*\|_{\mathcal{A}}$. We conclude that $f \in \mathcal{A}^{\mathrm{dual}} \circ \mathcal{H}L_0(B_X, Y)$ with $\|f\|_{\mathcal{A}^{\mathrm{dual}} \circ \mathcal{H}L_0} = \|T_f\|_{\mathcal{A}^{\mathrm{dual}}}$ by Theorem 2.4 and thus $\|f\|_{\mathcal{A}^{\mathrm{dual}} \circ \mathcal{H}L_0} \leq \|f\|_{\mathcal{A}^{\mathcal{H}L_0\text{-dual}}}$.

Conversely, if $f \in \mathcal{A}^{\mathrm{dual}} \circ \mathcal{H}L_0(B_X, Y)$, then $f = T \circ h$, where $T \in \mathcal{A}^{\mathrm{dual}}(Z, Y)$ and $h \in \mathcal{H}L_0(B_X, Z)$ for some complex Banach space Z. Notice that $f^t = h^t \circ T^*$ since

$$\begin{aligned} f^t(y^*) &= (T \circ h)^t(y^*) = y^* \circ (T \circ h) = (y^* \circ T) \circ h \\ &= T^*(y^*) \circ h = h^t(T^*(y^*)) = (h^t \circ T^*)(y^*) \end{aligned}$$

for all $y^* \in Y^*$. Hence $f^t \in \mathcal{A}(Y^*, \mathcal{H}L_0(B_X))$ by the ideal property of $\mathcal{A}$ since $T^* \in \mathcal{A}(Y^*, Z^*)$. Therefore $f \in \mathcal{A}^{\mathcal{H}L_0\text{-dual}}(B_X, Y)$, and from the inequality

$$\|f\|_{\mathcal{A}^{\mathcal{H}L_0\text{-dual}}} = \|f^t\|_{\mathcal{A}} = \|h^t \circ T^*\|_{\mathcal{A}} \leq \|h^t\|\|T^*\|_{\mathcal{A}} = L(h)\|T\|_{\mathcal{A}^{\mathrm{dual}}},$$

we infer that $\|f\|_{\mathcal{A}^{\mathcal{H}L_0\text{-dual}}} \leq \|f\|_{\mathcal{A}^{\mathrm{dual}} \circ \mathcal{H}L_0}$ by taking the infimum over all such representations of f. □

2. $\mathcal{A}$-Compact holomorphic Lipschitz mappings

Following the comments in [20, p. 1094], the original definitions of $\mathcal{A}$-compactness and its measure $m_{\mathcal{A}}$ were given in [8, 19] for normed operator ideals, but they can be easily extended for s-normed operator ideals with $s \in (0, 1]$ and their properties also hold with slight modifications.

Given an s-normed operator ideal $\mathcal{A}$, our aim in this section is to introduce a variant for holomorphic Lipschitz maps of the concept of $\mathcal{A}$-compact linear operators between Banach spaces [8, 19]. Towards this end, we denote the Lipschitz image of a map $f\colon B_X \to Y$ by

$$\mathrm{Im}_L(f) := \left\{ \frac{f(x) - f(y)}{\|x - y\|} \colon x, y \in B_X,\ x \neq y \right\} \subseteq Y.$$

Notice that f is Lipschitz if and only if $\mathrm{Im}_L(f)$ is a bounded subset of Y. This motivates the following.

Definition 2.1. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal. Given complex Banach spaces X and Y, a map $f \in \mathcal{H}L_0(B_X, Y)$ is said to be $\mathcal{A}$-compact holomorphic Lipschitz if* $\mathrm{Im}_L(f)$ *is a relatively $\mathcal{A}$-compact subset of Y. If we denote by $\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}(B_X, Y)$ the set of all $\mathcal{A}$-compact holomorphic Lipschitz maps from B_X into Y, we define*

$$\|f\|_{\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}} = m_{\mathcal{A}}(\mathrm{Im}_L(f)) \qquad (f \in \mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}(B_X, Y)).$$

The following characterizations of $\mathcal{A}$-compact holomorphic Lipschitz maps are immediate in light of [8, Lemma 1.1 and Theorem 1.1].

Let us recall (see [8, Definition 1.2 and Lemma 1.2]) that a sequence $(x_n)_{n=1}^\infty$ in X is called $\mathcal{A}$-null if there is a Banach space Y, an operator $T \in \mathcal{A}(Y, X)$ and a null sequence $(y_n)_{n=1}^\infty$ in Y satisfying that $x_n = T(y_n)$ for all $n \in \mathbb{N}$.

Proposition 2.2. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal. The following are equivalent:*

(i) *$f \in \mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}(B_X, Y)$.*
(ii) *There are a complex Banach space Z and an operator $T \in \mathcal{A}(Z, Y)$ so that for every $\varepsilon > 0$, there exists $(y_j^\varepsilon)_{j=1}^{k_\varepsilon}$ in Y such that* $\mathrm{Im}_L(f) \subseteq \bigcup_{j=1}^{k_\varepsilon}\{y_j^\varepsilon + \varepsilon T(\overline{B}_Z)\}$.
(iii) *There is an $\mathcal{A}$-null sequence $(y_n)_{n=1}^\infty$ in Y so that* $\mathrm{Im}_L(f) \subseteq \mathrm{co}((y_n)_{n=1}^\infty)$.

□

We also omit the proof of the following since it is a direct application of [19, Theorem 1.2, Proposition 1.8 and Corollary 1.9].

Proposition 2.3. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal. The following are equivalent:*

(i) $f \in \mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}(B_X, Y)$.

(ii) *There are a complex Banach space Z, two operators $T \in \mathcal{A}(Z, Y)$ and $R \in \overline{\mathcal{F}}(\ell_1, Z)$ and a relatively compact set $K \subseteq \overline{B}_{\ell_1}$ so that* $\mathrm{Im}_L(f) \subseteq (T \circ R)(K)$.

(iii) *There are an operator $T \in \mathcal{A}(\ell_1, Y)$ and a relatively compact set $K \subseteq \overline{B}_{\ell_1}$ so that* $\mathrm{Im}_L(f) \subseteq T(K)$.

(iv) $f \in \mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}\circ\overline{\mathcal{F}}}}(B_X, Y)$.

In such a case, we have

$$\begin{aligned}\|f\|_{\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}} &= \inf\{\|T\|_{\mathcal{A}}\,\|R\| : T \in \mathcal{A}(Z, Y),\ R \in \overline{\mathcal{F}}(\ell_1, Z),\ K \subseteq \overline{B}_{\ell_1},\ \mathrm{Im}_L(f) \subseteq (T \circ R)(K)\} \\ &= \inf\{\|T\|_{\mathcal{A}} : T \in \mathcal{A}(\ell_1, Y),\ K \subseteq \overline{B}_{\ell_1},\ \mathrm{Im}_L(f) \subseteq T(K)\} \\ &= m_{\mathcal{A}\circ\overline{\mathcal{F}}}(\mathrm{Im}_L(f)).\end{aligned}$$

□

Another useful characterization of $\mathcal{A}$-compact holomorphic Lipschitz mappings can be established in terms of their linearizations. Compare it to its polynomial version [26, Proposition 1].

Theorem 2.4. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal and $f \in \mathcal{H}L_0(B_X, Y)$. The following are equivalent:*

(i) f belongs to $\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}(B_X, Y)$.

(ii) T_f belongs to $\mathcal{K}_{\mathcal{A}}(\mathcal{G}_0(B_X), Y)$.

In such a case, $\|f\|_{\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}} = \|T_f\|_{\mathcal{K}_{\mathcal{A}}}$. As a consequence, $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}(B_X, Y), \|\cdot\|_{\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}})$ onto $(\mathcal{K}_{\mathcal{A}}(\mathcal{G}_0(B_X), Y), \|\cdot\|_{\mathcal{K}_{\mathcal{A}}})$.

Proof. Note first that an application of [5, Propositions 2.3 and 2.6] yields the relations:

$$\begin{aligned}\mathrm{Im}_L(f) = T_f(\mathrm{Im}_L(\delta_X)) \subseteq T_f(\overline{\mathrm{abco}}(\mathrm{Im}_L(\delta_X))) &= T_f(\overline{B}_{\mathcal{G}_0(B_X)}) \\ &\subseteq \overline{\mathrm{abco}}(T_f(\mathrm{Im}_L(\delta_X))) = \overline{\mathrm{abco}}(\mathrm{Im}_L(f)).\end{aligned}$$

$(i) \Rightarrow (ii)$: If $f \in \mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}(B_X, Y)$, then $\mathrm{Im}_L(f)$ is a relatively $\mathcal{A}$-compact subset of Y. A look at [20, p. 1094] shows that $\overline{\mathrm{abco}}(\mathrm{Im}_L(f))$ is relatively $\mathcal{A}$-compact in Y with $m_{\mathcal{A}}(\overline{\mathrm{abco}}(\mathrm{Im}_L(f))) = m_{\mathcal{A}}(\mathrm{Im}_L(f))$. Hence there exist a complex Banach space Z, an operator $T \in \mathcal{A}(Z, X)$ and a compact set $M \subseteq Z$ such that $\overline{\mathrm{abco}}(\mathrm{Im}_L(f)) \subseteq T(M)$. By the second inclusion in the above chain of relations, we obtain that $T_f(\overline{B}_{\mathcal{G}_0(B_X)}) \subseteq T(M)$ and thus $T_f(\overline{B}_{\mathcal{G}_0(B_X)})$ is relatively $\mathcal{A}$-compact in Y and $m_{\mathcal{A}}(T_f(\overline{B}_{\mathcal{G}_0(B_X)})) \leq m_{\mathcal{A}}(\overline{\mathrm{abco}}(\mathrm{Im}_L(f)))$. Therefore $T_f \in \mathcal{K}_{\mathcal{A}}(\mathcal{G}_0(B_X), Y)$ and

$$\|T_f\|_{\mathcal{K}_{\mathcal{A}}} = m_{\mathcal{A}}(T_f(\overline{B}_{\mathcal{G}_0(B_X)})) \leq m_{\mathcal{A}}(\mathrm{Im}_L(f)) = \|f\|_{\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}}.$$

$(ii) \Rightarrow (i)$: Assume that $T_f \in \mathcal{K}_{\mathcal{A}}(\mathcal{G}_0(B_X), Y)$, that is, $T_f(\overline{B}_{\mathcal{G}_0(B_X)})$ is a relatively $\mathcal{A}$-compact subset of Y. Now, by the first inclusion in the cited chain, it follows that $\mathrm{Im}_L(f)$ is relatively $\mathcal{A}$-compact in Y with $m_{\mathcal{A}}(\mathrm{Im}_L(f)) \leq m_{\mathcal{A}}(T_f(\overline{B}_{\mathcal{G}_0(B_X)}))$, that is, $f \in \mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}(B_X, Y)$ and $\|f\|_{\mathcal{H}L_0^{\mathcal{K}_{\mathcal{A}}}} \leq \|T_f\|_{\mathcal{K}_{\mathcal{A}}}$.

Applying what we have just proved and [5, Proposition 2.3 (c)], the last assertion of the statement is deduced. □

The combination of Theorems 1.7 and 2.4 shows that $[\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}, \|\cdot\|_{\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}}]$ is an s-normed holomorphic Lipschitz ideal of composition type which inherits the properties of $[\mathcal{K}_\mathcal{A}, \|\cdot\|_{\mathcal{K}_\mathcal{A}}]$ by Corollary 1.8.

Corollary 2.5. *Let $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$ be an s-normed operator ideal. Then*

$$[\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}, \|\cdot\|_{\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}}] = [\mathcal{K}_\mathcal{A} \circ \mathcal{HL}_0, \|\cdot\|_{\mathcal{K}_\mathcal{A} \circ \mathcal{HL}_0}].$$

Moreover, $[\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}, \|\cdot\|_{\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}}]$ is s-Banach (resp., regular) whenever $[\mathcal{K}_\mathcal{A}, \|\cdot\|_{\mathcal{K}_\mathcal{A}}]$ is so. □

Let $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$ be an s-normed operator ideal. Recall by [22, 4.7 and 8.5] that the surjective hull of $\mathcal{A}$ is defined for any Banach spaces X and Y by

$$\mathcal{A}^{sur}(X, Y) = \{T \in \mathcal{L}(X, Y) \colon T \circ Q_X \in \mathcal{A}(\ell_1(\overline{B}_X), Y)\},$$

where $Q_X \colon \ell_1(\overline{B}_X) \to X$ is the canonical metric surjection defined by

$$Q_X((\lambda_x)_{x\in\overline{B}_X}) = \sum_{x\in\overline{B}_X} \lambda_x x \qquad \left((\lambda_x)_{x\in\overline{B}_X} \in \ell_1(\overline{B}_X)\right).$$

Moreover, $[\mathcal{A}^{sur}, \|\cdot\|_{\mathcal{A}^{sur}}]$ generates an s-normed operator ideal if we set

$$\|T\|_{\mathcal{A}^{sur}} = \|T \circ Q_X\|_\mathcal{A} \qquad (T \in \mathcal{A}^{sur}(X, Y)).$$

More descriptions of $\mathcal{A}$-compact holomorphic Lipschitz mappings can be given by composition with both $\mathcal{K}_{\mathcal{A}\circ\overline{\mathcal{F}}}$ and $(\mathcal{A} \circ \overline{\mathcal{F}})^{sur}$.

Corollary 2.6. *Let $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$ be a s-Banach operator ideal. Then*

$$\begin{aligned}[\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}, \|\cdot\|_{\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}}] &= [\mathcal{K}_{\mathcal{A}\circ\overline{\mathcal{F}}} \circ \mathcal{HL}_0, \|\cdot\|_{\mathcal{K}_{\mathcal{A}\circ\overline{\mathcal{F}}} \circ \mathcal{HL}_0}] \\ &= [(\mathcal{A} \circ \overline{\mathcal{F}})^{sur} \circ \mathcal{HL}_0, \|\cdot\|_{(\mathcal{A}\circ\overline{\mathcal{F}})^{sur} \circ \mathcal{HL}_0}].\end{aligned}$$

Proof. Let $f \in \mathcal{HL}_0(B_X, Y)$. Just applying Theorem 2.4, Theorem 1.7 and [19, Proposition 2.1], we have

$$\begin{aligned} f \in \mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}(B_X, Y) &\Leftrightarrow T_f \in \mathcal{K}_\mathcal{A}(\mathcal{G}_0(B_X), Y) \\ &\Leftrightarrow f \in \mathcal{K}_\mathcal{A} \circ \mathcal{HL}_0(B_X, Y) \\ &\Leftrightarrow f \in \mathcal{K}_{\mathcal{A}\circ\overline{\mathcal{F}}} \circ \mathcal{HL}_0(B_X, Y) \\ &\Leftrightarrow f \in (\mathcal{A} \circ \overline{\mathcal{F}})^{sur} \circ \mathcal{HL}_0(B_X, Y), \end{aligned}$$

and

$$\|f\|_{\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}} = \|T_f\|_{\mathcal{K}_\mathcal{A}} = \|f\|_{\mathcal{K}_\mathcal{A}\circ\mathcal{HL}_0} = \|f\|_{\mathcal{K}_{\mathcal{A}\circ\overline{\mathcal{F}}}\circ\mathcal{HL}_0} = \|f\|_{(\mathcal{A}\circ\overline{\mathcal{F}})^{sur}\circ\mathcal{HL}_0}.$$

□

Applying Corollary 2.5, [19, Corollary 2.4] and Theorem 1.13, we also can describe the holomorphic Lipschitz ideal $\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}$ as a dual ideal as follows. A version of this result was established in the polynomial setting in [26, Proposition 2].

Corollary 2.7. *Let $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$ be an s-normed operator ideal. Then*

$$\begin{aligned}[\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}, \|\cdot\|_{\mathcal{HL}_0^{\mathcal{K}_\mathcal{A}}}] &= [\mathcal{K}_\mathcal{A} \circ \mathcal{HL}_0, \|\cdot\|_{\mathcal{K}_\mathcal{A}\circ\mathcal{HL}_0}] \\ &= [\left(\mathcal{K}_\mathcal{A}^{\mathrm{dual}}\right)^{\mathrm{dual}} \circ \mathcal{HL}_0, \|\cdot\|_{(\mathcal{K}_\mathcal{A}^{\mathrm{dual}})^{\mathrm{dual}}\circ\mathcal{HL}_0}] \\ &= [(\mathcal{K}_\mathcal{A}^{\mathrm{dual}})^{\mathcal{HL}_0\text{-dual}}, \|\cdot\|_{(\mathcal{K}_\mathcal{A}^{\mathrm{dual}})^{\mathcal{HL}_0\text{-dual}}}].\end{aligned}$$

□

We now provide some characterizations of $\mathcal{K}$-compact holomorphic Lipschitz maps in terms of the continuity and compactness of their transposes.

From [19, p. 2455], it is well known that the relatively $\mathcal{K}$-compact and the relatively $\overline{\mathcal{F}}$-compact sets coincide with the relatively compact sets and, consequently, $[\mathcal{K}_{\mathcal{A}}, \|\cdot\|_{\mathcal{K}_{\mathcal{A}}}] = [\mathcal{K}, \|\cdot\|]$ if $\mathcal{A} = \mathcal{K}, \overline{\mathcal{F}}$. For this reason, the elements of

$$[\mathcal{H}L_0^{\mathcal{K}_\mathcal{K}}, \|\cdot\|_{\mathcal{H}L_0^{\mathcal{K}_\mathcal{K}}}] := [\mathcal{H}L_0^{\mathcal{K}}, L(\cdot)]$$

will be referred to as compact holomorphic Lipschitz maps. The characterizations given below should be compared with [19, Proposition 2.5]. Note that the first equivalence in the next result provides a version of Schauder Theorem for holomorphic Lipschitz maps.

Theorem 2.8. *Let $f \in \mathcal{H}L_0(B_X, Y)$. The following statements are equivalent:*

(i) $f\colon B_X \to Y$ is compact holomorphic Lipschitz.
(ii) $f^t\colon Y^* \to \mathcal{H}L_0(B_X)$ is compact.
(iii) $f^t\colon Y^* \to \mathcal{H}L_0(B_X)$ is bounded-weak-to-norm continuous.*
(iv) $f^t\colon Y^* \to \mathcal{H}L_0(B_X)$ is compact and bounded-weak-to-weak continuous.*
(v) $f^t\colon Y^* \to \mathcal{H}L_0(B_X)$ is compact and weak-to-weak continuous.*

Proof. $(i) \Leftrightarrow (ii)$: Applying Theorem 2.4, the Schauder Theorem, Proposition 1.10 and the ideal property of $\mathcal{K}$, one has

$$\begin{aligned} f \in \mathcal{H}L_0^{\mathcal{K}}(B_X, Y) &\Leftrightarrow T_f \in \mathcal{K}(\mathcal{G}_0(B_X), Y) \\ &\Leftrightarrow (T_f)^* \in \mathcal{K}(Y^*, \mathcal{G}_0(B_X)^*) \\ &\Leftrightarrow f^t = \Lambda_X^{-1} \circ (T_f)^* \in \mathcal{K}(Y^*, \mathcal{H}L_0(B_X)). \end{aligned}$$

$(i) \Leftrightarrow (iii)$: Similarly,

$$\begin{aligned} f \in \mathcal{H}L_0^{\mathcal{K}}(B_X, Y) &\Leftrightarrow T_f \in \mathcal{K}(\mathcal{G}_0(B_X), Y) \\ &\Leftrightarrow (T_f)^* \in \mathcal{L}((Y^*, bw^*); \mathcal{G}_0(B_X)^*) \\ &\Leftrightarrow f^t = \Lambda_X^{-1} \circ (T_f)^* \in \mathcal{L}((Y^*, bw^*); \mathcal{H}L_0(B_X)), \end{aligned}$$

by Theorem 2.4 and [21, Theorem 3.4.16].

$(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$: It follows from [16, Proposition 3.1]. □

In the line of Corollary 1.11, the transposition now identifies $\mathcal{H}L_0^{\mathcal{K}}(B_X, Y)$ with the space of all bounded-weak*-to-norm continuous linear operators from Y^* into $\mathcal{H}L_0(B_X)$.

Proposition 2.9. *The correspondence $f \mapsto f^t$ is an isometric isomorphism from $\mathcal{H}L_0^{\mathcal{K}}(B_X, Y)$ onto $\mathcal{L}((Y^*, bw^*); \mathcal{H}L_0(B_X))$.*

Proof. Let $f \in \mathcal{H}L_0^{\mathcal{K}}(B_X, Y)$. Then $f^t \in \mathcal{L}((Y^*, bw^*); \mathcal{H}L_0(B_X))$ by Theorem 2.8 and $\|f^t\| = L(f)$ by Proposition 1.10. To prove the surjectivity, take $T \in \mathcal{L}((Y^*, bw^*); \mathcal{H}L_0(B_X))$. Then $\Lambda_X \circ T \in \mathcal{L}((Y^*, bw^*); \mathcal{G}_0(B_X)^*)$. Then $\kappa_{\mathcal{G}_0(B_X)}(\gamma) \circ \Lambda_X \circ T \in \mathcal{L}((Y^*, bw^*); \mathbb{C})$ for all $\gamma \in \mathcal{G}_0(B_X)$ and, by [21, Theorem 2.7.8], $\kappa_{\mathcal{G}_0(B_X)}(\gamma) \circ \Lambda_X \circ T \in \mathcal{L}((Y^*, w^*); \mathbb{C})$ for all $\gamma \in \mathcal{G}_0(B_X)$, that is, $\Lambda_X \circ T \in \mathcal{L}((Y^*, w^*); (\mathcal{G}_0(B_X)^*, w^*))$ by [21, Corollary 2.4.5]. Hence $\Lambda_X \circ T = S^*$ for some $S \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$. Note that $S^* \in \mathcal{L}((Y^*, bw^*); \mathcal{G}_0(B_X)^*)$ and this means that $S \in \mathcal{K}(\mathcal{G}_0(B_X), Y)$. Now, $S = T_f$ for some $f \in \mathcal{H}L_0^{\mathcal{K}}(B_X, Y)$ by Theorem 2.4. Finally, we have $T = \Lambda_X^{-1} \circ S^* = \Lambda_X^{-1} \circ (T_f)^* = f^t$. □

For $p \in [1, \infty)$, consider the ideals $\mathcal{K}_p$, $\mathcal{N}_p$ and $\mathcal{QN}_p$ of p-compact operators, right p-nuclear operators and quasi p-nuclear operators between Banach spaces, respectively (see [9, 10, 23, 24] for definitions and properties).

From [19, Remarks 1.3 and 1.7], note that the relatively $\mathcal{N}_p$-compact sets and the relatively $\mathcal{K}_p$-compact sets coincide with the relatively p-compact sets, and thus $[\mathcal{K}_{\mathcal{A}}, \|\cdot\|_{\mathcal{K}_{\mathcal{A}}}] = [\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$ if $\mathcal{A} = \mathcal{N}_p, \mathcal{K}_p$.

The application of Theorem 2.4, [10, Corollary 2.7] and Proposition 1.10 with the ideal property of $\mathcal{QN}_p$ provides the following characterizations of $\mathcal{N}_p$-compact holomorphic Lipschitz maps.

Proposition 2.10. *Let $f \in \mathcal{H}L_0(B_X, Y)$. The following assertions are equivalent:*

(i) $f \in \mathcal{H}L_0^{\mathcal{K}_p}(B_X, Y)$.
(ii) $T_f \in \mathcal{K}_p(\mathcal{G}_0(B_X), Y)$.
(iii) $(T_f)^* \in \mathcal{QN}_p(Y^*, \mathcal{G}_0(B_X)^*)$.
(iv) $f^t \in \mathcal{QN}_p(Y^*, \mathcal{H}L_0(B_X))$.

In such a case, $\|f\|_{\mathcal{H}L_0^{\mathcal{K}_p}} = \|T_f\|_{\mathcal{K}_p} = \|(T_f)^*\|_{\mathcal{QN}_p} = \|f^t\|_{\mathcal{QN}_p}$ *for all* $f \in \mathcal{H}L_0^{\mathcal{K}_p}(B_X, Y)$. □

3. $\mathcal{A}$-Bounded holomorphic Lipschitz mappings

Given a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, the property of $\mathcal{A}$-compactness due to Carl and Stephani [8] corresponds to a particular case of the property of $\mathcal{A}$-boundedness introduced by Stephani [25] (see also [11]).

For a Banach space X, a set $B \subseteq X$ is said to be $\mathcal{A}$-bounded if there exist a Banach space Z and an operator $T \in \mathcal{A}(Z, Y)$ such that $B \subseteq T(\overline{B}_Z)$. We denote by $C_{\mathcal{A}}(X)$ the collection of all $\mathcal{A}$-bounded subsets B of X. In such a case, a measure of the size of $\mathcal{A}$-boundedness of B can be given by $n_{\mathcal{A}}(K) = \inf\{\|T\|_{\mathcal{A}}\}$, where the infimum is taken over all such Z and T as above.

Furthermore, an operator $T \in \mathcal{L}(X, Y)$ is said to be $\mathcal{A}$-bounded whenever $T(\overline{B}_X)$ is an $\mathcal{A}$-bounded subset of Y. The set of all such operators is denoted as $C_{\mathcal{A}}(X, Y)$, and $[C_{\mathcal{A}}, \|\cdot\|_{C_{\mathcal{A}}}]$ is a Banach operator ideal with the norm

$$\|T\|_{C_{\mathcal{A}}} = n_{\mathcal{A}}(T(\overline{B}_X)) \qquad (T \in C_{\mathcal{A}}(X, Y)).$$

The version of $\mathcal{A}$-bounded linear operators for holomorphic Lipschitz maps reads as follows.

Definition 3.1. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be an s-normed operator ideal. Given complex Banach spaces X and Y, a map $f \in \mathcal{H}L_0(B_X, Y)$ is said to be $\mathcal{A}$-bounded holomorphic Lipschitz if* $\mathrm{Im}_L(f)$ *is an $\mathcal{A}$-bounded subset of Y. If $\mathcal{H}L_0^{C_{\mathcal{A}}}(B_X, Y)$ stands for the set of all $\mathcal{A}$-bounded holomorphic Lipschitz maps from B_X into Y, we set*

$$\|f\|_{\mathcal{H}L_0^{C_{\mathcal{A}}}} = n_{\mathcal{A}}(\mathrm{Im}_L(f)) \qquad (f \in \mathcal{H}L_0^{C_{\mathcal{A}}}(B_X, Y)).$$

A proof similar to that Theorem 2.4 provides us the following result. See [11] for a complete list of closed operator ideals which are surjective.

Theorem 3.2. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a closed (in the topology of the operator norm) s-normed surjective ideal and $f \in \mathcal{H}L_0(B_X, Y)$. The following are equivalent:*

(i) f belongs to $\mathcal{H}L_0^{C_{\mathcal{A}}}(B_X, Y)$.
(ii) T_f belongs to $C_{\mathcal{A}}(\mathcal{G}_0(B_X), Y)$.

In such a case, $\|f\|_{\mathcal{H}L_0^{C_{\mathcal{A}}}} = \|T_f\|_{C_{\mathcal{A}}}$. As a consequence, $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}L_0^{C_{\mathcal{A}}}(B_X, Y), \|\cdot\|_{\mathcal{H}L_0^{C_{\mathcal{A}}}})$ onto $(C_{\mathcal{A}}(\mathcal{G}_0(B_X), Y), \|\cdot\|_{C_{\mathcal{A}}})$. □

Proof. From the proof of Theorem 2.4, one has that

$$\mathrm{Im}_L(f) \subseteq T_f(\overline{B}_{\mathcal{G}_0(B_X)}) \subseteq \overline{\mathrm{abco}}(\mathrm{Im}_L(f)).$$

$(i) \Rightarrow (ii)$: If $f \in \mathcal{H}L_0^{C_{\mathcal{A}}}(B_X, Y)$, then $\mathrm{Im}_L(f)$ is an $\mathcal{A}$-bounded subset of Y. By [11, Proposition 3], $\overline{\mathrm{abco}}(\mathrm{Im}_L(f))$ is $\mathcal{A}$-bounded in Y with $n_{\mathcal{A}}(\overline{\mathrm{abco}}(\mathrm{Im}_L(f))) = n_{\mathcal{A}}(\mathrm{Im}_L(f))$. Hence we can take a complex Banach space Z and an operator $T \in \mathcal{A}(Z, X)$ such that $\overline{\mathrm{abco}}(\mathrm{Im}_L(f)) \subseteq T(B_Z)$. Now, the second inclusion above yields that $T_f(\overline{B}_{\mathcal{G}_0(B_X)}) \subseteq T(B_Z)$, and thus $T_f(\overline{B}_{\mathcal{G}_0(B_X)})$ is $\mathcal{A}$-bounded in Y with $n_{\mathcal{A}}(T_f(\overline{B}_{\mathcal{G}_0(B_X)})) \leq n_{\mathcal{A}}(\overline{\mathrm{abco}}(\mathrm{Im}_L(f)))$. We conclude that $T_f \in C_{\mathcal{A}}(\mathcal{G}_0(B_X), Y)$ and

$$\|T_f\|_{C_{\mathcal{A}}} = n_{\mathcal{A}}(T_f(\overline{B}_{\mathcal{G}_0(B_X)})) \leq n_{\mathcal{A}}(\mathrm{Im}_L(f)) = \|f\|_{\mathcal{H}L_0^{C_{\mathcal{A}}}}.$$

$(ii) \Rightarrow (i)$: If $T_f \in C_{\mathcal{A}}(\mathcal{G}_0(B_X), Y)$, then $T_f(\overline{B}_{\mathcal{G}_0(B_X)})$ is an $\mathcal{A}$-bounded subset of Y. Now, the first inclusion above gives that $\mathrm{Im}_L(f)$ is $\mathcal{A}$-bounded in Y with $n_{\mathcal{A}}(\mathrm{Im}_L(f)) \leq n_{\mathcal{A}}(T_f(\overline{B}_{\mathcal{G}_0(B_X)}))$, that is, $f \in \mathcal{H}L_0^{C_{\mathcal{A}}}(B_X, Y)$ and $\|f\|_{\mathcal{H}L_0^{C_{\mathcal{A}}}} \leq \|T_f\|_{C_{\mathcal{A}}}$. □

Since $\mathcal{W}$-bounded sets coincide with weakly compact sets, the elements of $\mathcal{H}L_0^{C_{\mathcal{W}}}(B_X, Y) := \mathcal{H}L_0^{\mathcal{W}}(B_X, Y)$ may be called weakly compact holomorphic Lipschitz.

The following result provides versions of Gantmacher, Gantmacher–Nakamura and Davis–Figiel–Johnson–Pełczyński Theorems for weakly compact holomorphic Lipschitz mappings.

Theorem 3.3. *Let $f \in \mathcal{H}L_0(B_X, Y)$. The following are equivalent:*

(i) $f\colon B_X \to Y$ is weakly compact holomorphic Lipschitz.
(ii) $T_f\colon \mathcal{G}_0(B_X) \to Y$ is weakly compact.
(iii) $f^t\colon Y^* \to \mathcal{H}L_0(B_X)$ is weakly compact.
(iv) $f^t\colon Y^* \to \mathcal{H}L_0(B_X)$ is weak-to-weak continuous.*
(v) There exist a reflexive Banach space Z, an operator $T \in \mathcal{L}(Z, Y)$ and a mapping $g \in \mathcal{H}L_0(B_X, Z)$ such that $f = T \circ g$.

Proof. $(i) \Leftrightarrow (ii)$ follows from Theorem 3.2; $(ii) \Leftrightarrow (iii)$ by Gantmacher Theorem, Proposition 1.10 and the ideal property of W; and $(iii) \Leftrightarrow (iv)$ by Gantmacher–Nakamura Theorem.

If (ii) holds, then the Davis–Figiel–Johnson–Pełczyński Theorem provides a reflexive Banach space Z and operators $R \in \mathcal{L}(Z, Y)$ and $S \in \mathcal{L}(\mathcal{G}_0(B_X), Z)$ such that $T_f = R \circ S$. By [5, Proposition 2.3 (c)], we can find a map $g \in \mathcal{H}L_0(B_X, Z)$ such that $S = T_g$. Hence $T_f = R \circ T_g = T_{R \circ g}$, this implies that $f = R \circ g$, and this proves (v). Finally, (v) implies (i) because $\mathrm{Im}_L(f) = R(\mathrm{Im}_L(g))$, where R is weak-to-weak continuous by [21, Theorem 2.5.11] and $\mathrm{Im}_L(g)$ is relatively weakly compact in Y since it is a bounded subset of the reflexive Banach space Y. □

Motivated by the closed surjective ideals of Rosenthal, Banach–Saks or Asplund operators (see [11] and the references therein for the definitions and main properties), we introduce the associate holomorphic Lipschitz maps.

Definition 3.4. *Given complex Banach spaces X and Y, a map $f \in \mathcal{H}L_0(B_X, Y)$ is said to be Rosenthal (resp., Banach–Saks, Asplund) if* $\mathrm{Im}_L(f)$ *is a Rosenthal (resp., Banach–Saks, Asplund) subset of Y.*

With a proof similar to that of Theorem 3.3, we may obtain the following.

Theorem 3.5. *Let $f \in \mathcal{H}L_0(B_X, Y)$. The following are equivalent:*

(i) $f\colon B_X \to Y$ is a Rosenthal (resp., Banach–Saks, Asplund) holomorphic Lipschitz map.
(ii) $T_f\colon \mathcal{G}_0(B_X) \to Y$ is a Rosenthal (resp., Banach–Saks, Asplund) operator.
(iii) There exist a Banach space Z which contains a copy of ℓ_1 (resp., has the Banach–Saks property, is Asplund), an operator $T \in \mathcal{L}(Z, Y)$ and a mapping $g \in \mathcal{H}L_0(B_X, Z)$ such that $f = T \circ g$.

□

We next identify $\mathcal{H}L_0^{\mathcal{W}}(B_X, Y)$ with the space of all weak*-to-weak continuous linear operators from Y^* into $\mathcal{H}L_0(B_X)$.

Proposition 3.6. *The correspondence $f \mapsto f^t$ is an isometric isomorphism from $\mathcal{H}L_0^{\mathcal{W}}(B_X, Y)$ onto $\mathcal{L}((Y^*, w^*); (\mathcal{H}L_0(B_X), w))$.*

Proof. We only need to show that such map is surjective. Let $T \in \mathcal{L}((Y^*, w^*); (\mathcal{H}L_0(B_X), w))$. Then $\Lambda_X \circ T \in \mathcal{L}((Y^*, w^*); (\mathcal{G}_0(B_X)^*, w))$ and this last set is contained in $\mathcal{L}((Y^*, w^*); (\mathcal{G}_0(B_X)^*, w^*))$. It follows that $\Lambda_X \circ T = S^*$ for some $S \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$. Hence $S^* \in \mathcal{L}((Y^*, w^*); (\mathcal{G}_0(B_X)^*, w))$ and, by the Gantmacher–Nakamura Theorem, $S \in \mathcal{W}(\mathcal{G}_0(B_X), Y)$. Now, $S = T_f$ for some $f \in \mathcal{H}L_0^{\mathcal{W}}(B_X, Y)$ by Theorem 3.3. Finally, $T = \Lambda_X^{-1} \circ S^* = \Lambda_X^{-1} \circ (T_f)^* = f^t$, as desired. □

In some cases we also can identify the holomorphic little Lipschitz space $\mathcal{H}l_0(B_X, Y)$ with the space of bounded-weak*-to-norm continuous linear operators from Y^* to $\mathcal{H}l_0(B_X)$. In its proof we will apply the following criterion for compactness in $\mathcal{H}l_0(B_X)$.

Lemma 3.7. *Let X be a finite dimensional complex Banach space. A set $K \subseteq \mathcal{H}l_0(B_X)$ is compact if and only if it is closed, bounded, and satisfies*

$$\lim_{\|x-y\|\to 0} \sup_{f\in K} \frac{|f(x) - f(y)|}{\|x - y\|} = 0.$$

Proof. Suppose that $K \subseteq \mathcal{H}l_0(B_X)$ is compact. Clearly, K is closed and bounded. Let $\varepsilon > 0$ and we can assume that $K \subseteq \cup_{k=1}^n B(f_k, \varepsilon/2)$ with $f_1, \ldots, f_n \in K$ for some $n \in \mathbb{N}$. There exists $\delta > 0$ such that

$$\frac{|f_k(x) - f_k(y)|}{\|x - y\|} < \frac{\varepsilon}{2}$$

whenever $x, y \in B_X$ with $0 < \|x - y\| < \delta$ and $k \in \{1, \ldots, n\}$. If $f \in K$, we have that $L(f - f_k) < \varepsilon/2$ for some $k \in \{1, \ldots, n\}$ and thus

$$\frac{|f(x) - f(y)|}{\|x - y\|} \leq L(f - f_k) + \frac{|f_k(x) - f_k(y)|}{\|x - y\|} < \varepsilon$$

if $x, y \in B_X$ with $0 < \|x - y\| < \delta$, and this establishes that the limit in the statement is 0.

Conversely, assume that K is a closed bounded subset of $\mathcal{H}l_0(B_X)$ such that

$$\lim_{\|x-y\|\to 0} \sup_{f\in K} \frac{|f(x) - f(y)|}{\|x - y\|} = 0.$$

Let (f_n) be a sequence in K. By Montel's Theorem and [15, Lemma 2.5] there exists a subsequence (f_{n_k}) which converges pointwise on B_X (in fact, uniformly on compact subsets of B_X) to some function $f \in \mathcal{H}L_0(B_X)$. Let $\varepsilon > 0$. There is a $\delta > 0$ such that for all $k \in \mathbb{N}$,

$$\frac{|f_{n_k}(x) - f_{n_k}(y)|}{\|x - y\|} < \frac{\varepsilon}{2}$$

if $x, y \in B_X$ with $0 < \|x - y\| < \delta$, taking limits with $k \to \infty$ yields

$$\frac{|f(x) - f(y)|}{\|x - y\|} \leq \frac{\varepsilon}{2}$$

if $x, y \in B_X$ with $0 < \|x - y\| < \delta$, and thus

$$\frac{|(f_{n_k} - f)(x) - (f_{n_k} - f)(y)|}{\|x - y\|} < \varepsilon$$

if $x, y \in B_X$ with $0 < \|x - y\| < \delta$. On the other hand, since the set

$$K_\delta = \{(x, y) \in B_X \times B_X \colon \|x - y\| \geq \delta\}$$

is compact in X^2 with the product topology, there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then

$$\left|f_{n_k}(x) - f(x)\right| < \delta\frac{\varepsilon}{2}$$

for all $x \in B_X$ such that $(x, y) \in K_\delta$ for some $y \in B_X$. Therefore, for all $k \geq k_0$, we have that

$$\frac{|(f_{n_k} - f)(x) - (f_{n_k} - f)(y)|}{\|x - y\|} < \varepsilon$$

if $x, y \in B_X$ and $\|x - y\| \geq \delta$. It follows that $L(f_{n_k} - f) \leq \varepsilon$ for all $k \geq k_0$. By the arbitrariness of ε, we deduce that $\lim_{k\to\infty} L(f_{n_k} - f) = 0$. Since K is closed, it follows that $f \in K$. This proves that K is compact. □

Theorem 3.8. *If X is a finite dimensional complex Banach space, then $f \mapsto f^t$ is an isometric isomorphism from $\mathcal{H}l_0(B_X, Y)$ onto $\mathcal{L}((Y^*, bw^*); \mathcal{H}l_0(B_X))$.*

Proof. Let $f \in \mathcal{H}l_0(B_X, Y)$. We claim that $f^t \in \mathcal{L}((Y^*, bw^*); \mathcal{H}l_0(B_X))$. Indeed, note first that

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} = \sup_{y^* \in B_{Y^*}} \frac{|y^*(f(x) - f(y))|}{\|x - y\|} = \sup_{y^* \in B_{Y^*}} \frac{|f^t(y^*)(x) - f^t(y^*)(y)|}{\|x - y\|}$$

for every $x, y \in X$ with $x \neq y$. For each $y^* \in B_{Y^*}$, it follows that $f^t(y^*) \in \mathcal{H}l_0(B_X)$ with $L(f^t(y^*)) \leq L(f)$. Hence $f^t(B_{Y^*})$ is a bounded subset of $\mathcal{H}l_0(B_X)$, and since

$$\lim_{\|x-y\|\to 0} \sup_{y^* \in B_{Y^*}} \frac{|f^t(y^*)(x) - f^t(y^*)(y)|}{\|x - y\|} = \lim_{\|x-y\|\to 0} \frac{\|f(x) - f(y)\|}{\|x - y\|} = 0,$$

Lemma 3.7 assures that $f^t(B_{Y^*})$ is relatively compact in $\mathcal{H}l_0(B_X)$, that is, $f^t \in \mathcal{K}(Y^*, \mathcal{H}l_0(B_X))$. Moreover, f^t belongs to $\mathcal{L}((Y^*, w^*); (\mathcal{H}L_0(B_X), w^*))$ by Corollary 1.11. Hence

$$\Lambda_X \circ f^t \in \mathcal{K}(Y^*, \mathcal{G}_0(B_X)^*) \cap \mathcal{L}((Y^*, w^*); (\mathcal{G}_0(B_X)^*, w^*)).$$

Then there is an $S \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$ for which $S^* = \Lambda_X \circ f^t$. Thus $S^* \in \mathcal{K}(Y^*, \mathcal{G}_0(B_X)^*)$ and, by the Schauder Theorem, $S \in \mathcal{K}(\mathcal{G}_0(B_X), Y)$. Then $f^t = \Lambda_X^{-1} \circ S^* \in \mathcal{L}((Y^*, bw^*); \mathcal{H}L_0(B_X))$. Since $f^t(Y^*) \subseteq \mathcal{H}l_0(B_X)$, our claim is proved.

By Proposition 1.10, the mapping of the statement is a linear isometry. To prove that it is onto, let $T \in \mathcal{L}((Y^*, bw^*); \mathcal{H}l_0(B_X))$. By Proposition 2.9, there is a $f \in \mathcal{H}L_0^{\mathcal{K}}(B_X, Y)$ such that $T = f^t$. We now show $f \in \mathcal{H}l_0(B_X, Y)$. Since $T \in \mathcal{K}(Y^*, \mathcal{H}L_0(B_X))$ by Theorem 2.8 and $T(Y^*) \subseteq \mathcal{H}l_0(B_X)$, we deduce that $T \in \mathcal{K}(Y^*, \mathcal{H}l_0(B_X))$. Then Lemma 3.7 yields

$$\begin{aligned}\lim_{\|x-y\|\to 0} \frac{\|f(x) - f(y)\|}{\|x - y\|} &= \lim_{\|x-y\|\to 0} \sup_{y^* \in B_{Y^*}} \frac{|y^*(f(x) - f(y))|}{\|x - y\|} \\ &= \lim_{\|x-y\|\to 0} \sup_{y^* \in B_{Y^*}} \frac{|T(y^*)(x) - T(y^*)(y)\|}{\|x - y\|} = 0,\end{aligned}$$

and thus $f \in \mathcal{H}l_0(B_X, Y)$. □

We finish this paper with other elementary subclasses of holomorphic Lipschitz maps.

Definition 3.9. *A map $f \in \mathcal{H}L_0(B_X, Y)$ is said to have finite-rank if* $\mathrm{lin}(\mathrm{Im}_L(f))$ *is a finite dimensional subspace of Y. The dimension of this space will be denoted by $\mathcal{H}L_0$*-rank(f)*, and the set of all such maps by $\mathcal{H}L_0^{\mathcal{F}}(B_X, Y)$.*

A map $f \in \mathcal{H}L_0(B_X, Y)$ is said to be approximable if it is the limit in the Lipschitz norm $L(\cdot)$ of a sequence of finite-rank holomorphic Lipschitz maps of $\mathcal{H}L_0(B_X, Y)$.

We denote by $\mathcal{H}L_0^{\mathcal{F}}(B_X,Y)$ and $\mathcal{H}L_0^{\overline{\mathcal{F}}}(B_X,Y)$ the spaces of finite-rank holomorphic Lipschitz maps and approximable holomorphic Lipschitz maps f from B_X into Y such that $f(0)=0$, respectively.

Applying both linearization and transposition, we describe finite-rank holomorphic Lipschitz maps as follows.

Theorem 3.10. *Let $f\in\mathcal{H}L_0(B_X,Y)$. The following assertions are equivalent:*

(i) $f\in\mathcal{H}L_0^{\mathcal{F}}(B_X,Y)$.
(ii) $T_f\in\mathcal{F}(\mathcal{G}_0(B_X),Y)$.
(iii) $f^t\in\mathcal{F}(Y^*,\mathcal{H}L_0(B_X))$.

In such a case, we have

$$\mathcal{H}L_0\text{-rank}(f)=\mathrm{rank}(T_f)=\mathrm{rank}(f^t).$$

Moreover, the maps $f\mapsto T_f$ and $f\mapsto f^t$ are isometric isomorphisms from $(\mathcal{H}L_0^{\mathcal{F}}(B_X,Y),L(\cdot))$ onto $(\mathcal{F}(\mathcal{G}_0(B_X),Y),\|\cdot\|)$ and onto $(\mathcal{F}(Y^*,\mathcal{H}L_0(B_X)),\|\cdot\|)$, respectively.

Proof. First note that $\mathrm{lin}(\mathrm{Im}_L(f))=\mathrm{lin}(f(B_X))$.

$(i)\Rightarrow(ii)$: If $f\in\mathcal{H}L_0^{\mathcal{F}}(B_X,Y)$, then $\mathrm{lin}(\mathrm{Im}_L(f))$ is finite dimensional. Applying [5, Proposition 2.3 (b)-(c)], we deduce that

$$\begin{aligned}T_f(\mathcal{G}_0(B_X))&=T_f(\overline{\mathrm{lin}}(\delta_X(B_X)))\subseteq\overline{T_f(\mathrm{lin}(\delta_X(B_X)))}\\&=\overline{\mathrm{lin}}(T_f(\delta_X(B_X)))=\overline{\mathrm{lin}}(f(B_X))=\mathrm{lin}(f(B_X)),\end{aligned}$$

and thus $T_f\in\mathcal{F}(\mathcal{G}_0(B_X),Y)$ with $\mathrm{rank}(T_f)\le\mathcal{H}L_0\text{-rank}(f)$.

$(ii)\Rightarrow(i)$: If $T_f\in\mathcal{F}(\mathcal{G}_0(B_X),Y)$, then

$$\begin{aligned}\mathrm{lin}(f(B_X))=\mathrm{lin}(T_f(\delta_X(B_X)))&=T_f(\mathrm{lin}(\delta_X(B_X)))\\&\subseteq T_f(\overline{\mathrm{lin}}(\delta_X(B_X)))=T_f(\mathcal{G}_0(B_X)),\end{aligned}$$

and so $f\in\mathcal{H}L_0^{\mathcal{F}}(B_X,Y)$ with $\mathcal{H}L_0\text{-rank}(f)\le\mathrm{rank}(T_f)$.

$(ii)\Leftrightarrow(iii)$ and $\mathrm{rank}(T_f)=\mathrm{rank}(f^t)$ follow from Proposition 1.10 and the fact that the operator ideal $[\mathcal{F},\|\cdot\|]$ is completely symmetric [22, 4.4.7]. □

An application of Theorems 1.7, 1.13 and 3.10 yields the following.

Corollary 3.11. *We have*

$$[\mathcal{H}L_0^{\mathcal{F}},L(\cdot)]=[\mathcal{F}\circ\mathcal{H}L_0,\|\cdot\|_{\mathcal{F}\circ\mathcal{H}L_0}]=[\mathcal{F}^{\mathrm{dual}}\circ\mathcal{H}L_0,\|\cdot\|_{\mathcal{F}^{\mathrm{dual}}\circ\mathcal{H}L_0}]=[\mathcal{F}^{\mathcal{H}L_0\text{-dual}},\|\cdot\|_{\mathcal{F}^{\mathcal{H}L_0\text{-dual}}}].$$

□

Theorem 3.10 shows the connection of an approximable holomorphic Lipschitz map with its linearization:

Corollary 3.12. *Let $f\in\mathcal{H}L_0(B_X,Y)$. Then $f\in\mathcal{H}L_0^{\overline{\mathcal{F}}}(B_X,Y)$ if and only if $T_f\in\overline{\mathcal{F}}(\mathcal{G}_0(B_X),Y)$. Moreover, the map $f\mapsto T_f$ is an isometric isomorphism from $\mathcal{H}L_0^{\overline{\mathcal{F}}}(B_X,Y)$ onto $\overline{\mathcal{F}}(\mathcal{G}_0(B_X),Y)$.* □

Note that $\mathcal{H}L_0^{\mathcal{F}}(B_X,Y)$ is a linear subspace of $\mathcal{H}L_0^{\mathcal{K}}(B_X,Y)$. In fact, we have a little more:

Corollary 3.13. *Every approximable holomorphic Lipschitz map from B_X to Y is compact holomorphic Lipschitz.*

Proof. Let $f \in \mathcal{H}L_0^{\overline{\mathcal{F}}}(B_X, Y)$. Then there is a sequence (f_n) in $\mathcal{H}L_0^{\mathcal{F}}(B_X, Y)$ such that $L(f_n - f) \to 0$ as $n \to \infty$. Since $T_{f_n} \in \mathcal{F}(\mathcal{G}_0(B_X), Y)$ by Theorem 3.10 and $\left\|T_{f_n} - T_f\right\| = \left\|T_{f_n - f}\right\| = L(f_n - f)$ for all $n \in \mathbb{N}$, we deduce that $T_f \in \mathcal{K}(\mathcal{G}_0(B_X), Y)$ since $[\mathcal{K}, \|\cdot\|]$ is a closed ideal of $[\mathcal{L}, \|\cdot\|]$, and so $f \in \mathcal{H}L_0^{\mathcal{K}}(B_X, Y)$ by Theorem 2.8. □

Let us recall that a Banach space X has the approximation property if given a compact set $K \subseteq X$ and $\varepsilon > 0$, there is an operator $T \in \mathcal{F}(X, X)$ such that $\|T(x) - x\| < \varepsilon$ for every $x \in K$. We refer to [5, Section 3] for some results on the approximation property of holomorphic Lipschitz maps.

Grothendieck [12] proved that a dual Banach space X^* has the approximation property if and only if given a Banach space Y, an operator $S \in \mathcal{K}(X, Y)$ and $\varepsilon > 0$, there is an operator $T \in \mathcal{F}(X, Y)$ such that $\|T - S\| < \varepsilon$. Since $\mathcal{H}L_0(B_X)$ is a dual space, combining the Grothendieck's result with Theorems 2.8 and 3.10 , we propose the next characterization.

Corollary 3.14. *$\mathcal{H}L_0(B_X)$ has the approximation property if and only if, for each complex Banach space Y, $\varepsilon > 0$ and $f \in \mathcal{H}L_0^{\mathcal{K}}(B_X, Y)$, there exists $g \in \mathcal{H}L_0^{\mathcal{F}}(B_X, Y)$ such that $L(f - g) < \varepsilon$.* □

References

[1] D. Achour, E. Dahia and P. Turco, Lipschitz p-compact mappings, Monatsh. Math. **189** (2019), no. 4, 595–609.

[2] K. Ain, R. Lillemets and E. Oja, Compact operators which are defined by ℓ_p-spaces, Quaest. Math. **35** (2012), no. 2, 145–159.

[3] K. Ain and E. Oja, On (p, r)-null sequences and their relatives, Math. Nachr. **288** (2015), 1569–1580.

[4] R. M. Aron, G. Botelho, D. Pellegrino and P. Rueda, Holomorphic mappings associated to composition ideals of polynomials, Atti Accad. Naz. Lincei Red. Lincei Mat. Appl. **21** (2010), no. 3, 261–274.

[5] R. M. Aron, Verónica Dimant, Luis C. García-Lirola and Manuel Maestre, Linearization of holomorphic Lipschitz functions, Math. Nachr. **297** (2024), no. 8, 3024–3051.

[6] R. M. Aron, Manuel Maestre and Pilar Rueda, p-Compact holomorphic mappings, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **104** (2010), no. 2, 353–364.

[7] R. M. Aron and Pilar Rueda, p-Compact homogeneous polynomials from an ideal point of view, Contemp. Math. **547** (2011), 61–71.

[8] B. Carl and I. Stephani, On A-compact operators, generalized entropy numbers and entropy ideals, Math. Nachr. **199** (1984), 77–95.

[9] J. M. Delgado, C. Piñeiro and E. Serrano, Operators whose adjoints are quasi p-nuclear, Stud. Math. **197** (2010) 291–304.

[10] D. Galicer, S. Lassalle and P. Turco, The ideal of p-compact operators: a tensor product approach, Stud. Math. **2828** (2012) 269–286.

[11] M. González and J. Gutiérrez, Surjective factorization of holomorphic mappings, Comment. Math. Univ. Carolin. **41** (3), 469–476 (2000).

[12] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc. 16 (1955).

[13] A. Jiménez-Vargas, On holomorphic mappings with relatively p-compact range, Filomat **37** (2023), no. 24, 8067–8077.

[14] A. Jiménez-Vargas and D. Ruiz-Casternado, (p, q)-Compactness in spaces of holomorphic mappings, Open Math. **22** (2024), no. 1, article number: 20230183.

[15] A. Jiménez-Vargas and Moisés Villegas-Vallecillos, Compact composition operators on noncompact Lipschitz spaces, J. Math. Anal. Appl. **398** (2013) 221–229

[16] J. M. Kim, On spaces of weak* to weak continuous compact operators, Bull. Korean Math. Soc. **50** (1) (2013), 161–173.

[17] J. M. Kim, Some results of the $\mathcal{K}_{\mathcal{A}}$-approximation property for Banach spaces, Glasgow Math. J. **61** (2019) 545–555

[18] S. Lassalle and P. Turco, On p-compact mappings and the p-approximation property, J. Math. Anal. Appl. **389** (2012), no. 2, 1204–1221.

[19] S. Lassalle and P. Turco, The Banach ideal of $\mathcal{A}$-compact operators and related approximation properties, J. Funct. Anal. **265** (2013), no. 10, 2452–2464.

[20] S. Lassalle and P. Turco, Polynomials and holomorphic functions on $\mathcal{A}$-compact sets in Banach spaces, J. Math. Anal. Appl. **463** (2018), no. 2, 1092–1108.
[21] R. E. Megginson, An introduction to Banach space theory, Springer-Verlag, New York, 1998.
[22] A. Pietsch, Operator ideals, North-Holland Mathematical Library, vol. 20, North-Holland Publishing Co., Amsterdam-New York, 1980. Translated from German by the author.
[23] R. Ryan, Introduction to tensor products of Banach spaces, Springer Monographs in Mathematics ed., Springer-Verlag, London Ltd, London, 2002.
[24] D. P. Sinha and A. K. Karn, Compact operators whose adjoints factor through subspaces of ℓ_p, Studia. Math. **151** (2002), 17–33.
[25] I. Stephani, Generating systems of sets and quotients of surjective operator ideals. Math. Nachr. **99**, 13–27 (1980).
[26] P. Turco, $\mathcal{A}$-compact mappings, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **110** (2016), 863–880.

(A. Jiménez-Vargas) Departamento de Matemáticas, Universidad de Almería, Ctra. de Sacramento s/n, 04120 La Cañada de San Urbano, Almería, Spain
Email address: `ajimenez@ual.es`

(D. Ruiz-Casternado) Departamento de Matemáticas, Universidad de Almería, Ctra. de Sacramento s/n, 04120 La Cañada de San Urbano, Almería, Spain
Email address: `drc446@ual.es`