THE UNIVERSAL CONTINUOUS SIX FUNCTOR FORMALISM ON LIGHT CONDENSED ANIMA

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ABSTRACT. After the universal property of the six functor formalism $\mathrm{Shv}(-;\mathrm{Sp})$ on locally compact Hausdorff spaces given by Zhu, we show that the six functor formalism $\mathrm{Shv}(-;\mathrm{Sp})$ on light condensed anima in the sense of Heyer-Mann is initial among all six functor formalisms D satisfying some mild conditions, and then we present some applications.

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1. INTRODUCTION

In mathematics, the six functor formalisms are fundamental to understand topology and geometry. Recent years, people started to study the six functor formalisms from higher categorical perspective. For example, Liu-Zheng and Mann ([Man22, Appendix A.5]) gave a very concise definition of six functor formalisms.

Heyer-Mann in [HM24] says that a *geometric setup* is a pair $(\mathcal{C}, E)$, where $\mathcal{C}$ is a category has all finite limits, and E is a family of morphisms in $\mathcal{C}$, which is closed under composition, base change, and contains all isomorphisms, and is closed under passing to diagonals.

Given a geometric setup $(\mathcal{C}, E)$, there is a *category of correspondences* $\mathrm{Corr}(\mathcal{C}, E)$ associated with $(\mathcal{C}, E)$, which can be described as follows:

(a) objects in $\mathrm{Corr}(\mathcal{C}, E)$ are objects in $\mathcal{C}$.

(b) morphisms from Y to X are correspondences $Y \xleftarrow{f} Z \xrightarrow{g} X$, where g lies in E.

One can promote the category $\mathrm{Corr}(\mathcal{C}, E)$ to a symmetric monoidal category $\mathrm{Corr}(\mathcal{C}, E)^{\otimes}$ whose underlying category is $\mathrm{Corr}(\mathcal{C}, E)$. See [HM24, Section 2.3].

Liu-Zheng and Mann says that a *3-functor formalism* is a lax symmetric monoidal functor

$$D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Pr}^{L,\otimes},$$

where the symmetric monoidal structure on $\mathrm{Pr}^{L,\otimes}$ is given by the Lurie's tensor product.

Now, given a 3-functor formalism $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Pr}^{L,\otimes}$, we are able to obtain the following functors:

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(1) For any map $f : Y \to X$, the correspondence $Y \xleftarrow{f} X = X$ gives the pullback functor $f^* : D(X) \to D(Y)$.
(2) For any map $g : Y \to X$ in E, the correspondence $Y = Y \xrightarrow{g} X$ gives the functor $g_! : D(Y) \to D(X)$.
(3) The tensor product functor $- \otimes - : D(X) \otimes D(X) \to D(X)$ is given by the composition $D(X) \otimes D(X) \to D(X \times X) \to D(X)$, where the first functor is given by the lax symmetric monoidal functor D, and the second functor is Δ^*, where $\Delta : X \to X \times X$ is the diagonal.

Then one can make the following definition:

Definition 1.1. A *6-functor formalism* is a 3-functor formalism $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Pr}^{L,\otimes}$, such that the functors $- \otimes A$, f^* and $f_!$ admit right adjoints.

A fundamental example of six functor formalisms is the six functor formalism on locally compact Hausdorff spaces. Given a locally compact Hausdorff space X, we can construnction the category $\mathrm{Shv}(X; \mathrm{Sp})$ of Sp-valued sheaves on X. And we know that the assignment $X \mapsto \mathrm{Shv}(X; \mathrm{Sp})$ promotes to a six functor formalism

$$\mathrm{Shv}(-; \mathrm{Sp}) : \mathrm{Corr}(\mathrm{LCH}, \mathrm{all})^{\otimes} \to \mathrm{Pr}^{L,\otimes},$$

where LCH is the category of locally compact Hausdorff spaces. See [Sch25, 7.4].

In his work [Zhu25], Qingchong Zhu showed that the six functor formalism $\mathrm{Shv}(-; \mathrm{Sp}) : \mathrm{Corr}(\mathrm{LCH})^{\otimes} \to \mathrm{Pr}^{L,\otimes}$ is initial among all *continuous* six functor formalisms (See [Zhu25, 4.18]). Here, a six functor formalism $D : \mathrm{Corr}(\mathrm{LCH})^{\otimes} \to \mathrm{Pr}^{L,\otimes}$ is called continuous, if it satisfies the following conditions:

(1) For any locally compact Hausdorff space X, the category $D(X)$ is dualizable.
(2) D satisfies canonical descent. That is, for any locally compact Hausdorff space X, the assignment $\mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Pr}^L; U \mapsto D(U)$ is a sheaf.
(3) D sasisfies profinite descent. That is, for a cofiltered limit $X \simeq \lim_i X_i$ of compact Hausdorff spaces X_i, the induced functor $D(X) \to \lim_i D(X_i)$ is an equivalence.
(4) For any hypercomplete locally compact Hausdorff space X, the family $\{x^* : D(X) \to D(*)\}_{x \in X}$ is jointly conservative.

In other words, if D is a six functor formalism on locally compact Hausdorff spaces satisfying conditions (1)-(4), then there exists uniquely a morphism

$$\alpha : \mathrm{Shv}(-; \mathrm{Sp}) \to D(-)$$

between six functor formalisms.

Given a six functor formlism D on a geometric setup $(\mathcal{C}, E)$, Heyer-Mann gave an extension of D so that we get a six functor formalism on $(\mathrm{HypShv}(\mathcal{C}), E')$, where E' is a family in $\mathrm{HypShv}(\mathcal{C})$. See [HM24, Section 3.4]. In particular, one can extend the six functor formalism

$$\mathrm{Shv}(-; \mathrm{Sp}) : \mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}) \to \mathrm{Pr}^L$$

on light profinite sets to the six functor formalism

$$\mathrm{Shv}(-; \mathrm{Sp}) : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L$$

on light condensed anima.

Now, we can state our main result:

Theorem 1.2. *The six functor formalism*

$$\mathrm{Shv}(-; \mathrm{Sp}) : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_{\kappa}$$

is initial among all six functor formalisms $D : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_{\kappa}$ *satisfying the following conditions:*

(a) *For any light profinite set S, the ∞-category $D(S)$ is dualizable, and the family $\{s^* : D(S) \to D(*)\}_{s \in S}$ is jointly conservative.*

(b) *The functor* $D : \mathrm{CondAn}^{\mathrm{light,op}} \to \mathrm{Pr}^L_\kappa$ *preserves limits.*

We denote the unique morphism by $\alpha : \mathrm{Shv}(-;\mathrm{Sp}) \to D(-)$.

Given a six functor formalism $D : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Pr}^L$, Heyer-Mann introduced the concepts of D-suave maps and D-prim maps. See [HM24, Section 4.4, Section 4.5]. Using these concepts, we are able to show the following result.

Theorem 1.3. *Let* $f : Y \to X$ *be a map in* E *and* $\alpha : D \simeq \mathrm{Shv}(-;\mathrm{Sp}) \to D'$ *the unique morphism from Theorem 1.2. We have:*

(1) *If* $f : Y \to X$ *is* D*-suave, then the commutative square*

$$\begin{tikzcd} D(Y) \arrow[r, "\alpha_Y"] \arrow[d, "f_!"] & D'(Y) \arrow[d, "f_!"] \\ D(X) \arrow[r, "\alpha_X"] & D'(X) \end{tikzcd}$$

is vertically right adjointable. That is, the canonical map $\alpha_Y f^! \to f^! \alpha_X$ *is an equivalence.*

(2) *If* $f : Y \to X$ *is* D*-prim, then the commutative square*

$$\begin{tikzcd} D(X) \arrow[r, "\alpha_X"] \arrow[d, "f^*"] & D'(X) \arrow[d, "f^*"] \\ D(Y) \arrow[r, "\alpha_Y"] & D'(Y) \end{tikzcd}$$

is vertically right adjointable. That is, the canonical map $\alpha_X f_* \to f_* \alpha_Y$ *is an equivalence.*

With the morphism $\alpha : \mathrm{Shv}(-;\mathrm{Sp}) \to D(-)$ in hand, We can give the following application.

Theorem 1.4. *Let* $D : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_\kappa$ *be a 6-functor formalism satisfying conditions (a) and (b) in Theorem 1.2. For any light condensed anima* X *with projection* $f_X : X \times \mathbb{R} \to X$*, the pullback functor*

$$f_X^* : D(X) \to D(X \times \mathbb{R})$$

is fully faithful.

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2. Six functor formalisms on light profinite sets

In this section, we present the universal property of six functor formalisms on light profinite sets.

We always fix a cardinal $\kappa > \omega$. And let Pr^L_κ be the subcategory of Pr^L spanned by κ-presentable ∞-categories and colimit-preserving functors which preserves κ-compact objects.

We start with the definition of light profinite sets and light condensed anima from [Cla25, Section 4].

Definition 2.1. (1) A *light profinite set* is a topological space S which can be written as $S = \varprojlim_i S_i$, the sequence inverse limit of finite discrete topological spaces S_i. The full subcategory of Top spanned by light profinite sets is denoted by $\mathrm{ProFin}^{\mathrm{light}}$.

(2) A *light condensed anima* is a functor $X : \mathrm{ProFin}^{\mathrm{light,op}} \to \mathrm{An}$ such that:

(a) $X(\sqcup_{i\in I} S_i) \xrightarrow{\sim} \prod_{i\in I} X(S_i)$, where I is a finite set and each S_i is a light profinite set.

(b) If $S_\bullet \to S$ is a hypercover of S in $\mathrm{ProFin}^{\mathrm{light}}$, then the canonical map $X(S) \to \lim_{[n]\in\Delta} X(S_n)$ is an equivalence.

The full subcategory of $\mathrm{Fun}(\mathrm{ProFin}^{\mathrm{light},\mathrm{op}}, \mathrm{An})$ spanned by light condensed anima is denoted by $\mathrm{CondAn}^{\mathrm{light}}$.

Notation. Given a geometric setup $(\mathcal{C}, E)$, we use $6\mathrm{FF}(\mathcal{C}, E)$ to denote the ∞-category of all six functor formalisms on $(\mathcal{C}, E)$. That is, we have

$$6\mathrm{FF}(\mathcal{C}, E) \simeq \mathrm{Fun}^{\mathrm{lax}}(\mathrm{Corr}(\mathcal{C}, E)^{\otimes}, \mathrm{Pr}_{\kappa}^{L,\otimes}) \simeq \mathrm{CAlg}(\mathrm{Fun}(\mathrm{Corr}(\mathcal{C}, E), \mathrm{Pr}_{\kappa}^{L})^{\otimes}).$$

Remark 2.2. Recall from [HM24, 2.3.5] that every morphism $(\mathcal{C}, \mathcal{C}_0) \to (\mathcal{D}, \mathcal{D}_0)$ of geometric setups will induce a lax symmetric monoidal functor

$$\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^{\otimes} \to \mathrm{Corr}(\mathcal{D}, \mathcal{D}_0)^{\otimes}.$$

(1) Since $(\mathrm{ProFin}^{\mathrm{light}}, \mathrm{all}) \to (\mathrm{LCH}, \mathrm{all})$ is a morphism of geometric setups, we get a lax symmetric monoidal functor

$$\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}})^{\otimes} \to \mathrm{Corr}(\mathrm{LCH})^{\otimes},$$

which is indeed symmetric monoidal by [HM24, 2.3.6].

(2) Since $(\mathrm{ProFin}^{\mathrm{light}}, \mathrm{all}) \to (\mathrm{CondAn}^{\mathrm{light}}, E)$ is a morphism of geometric setups by [HM24, 3.4.11], we get a lax symmetric monoidal functor

$$\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}, \mathrm{all})^{\otimes} \to \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\otimes},$$

which is indeed symmetric monoidal by [HM24, 2.3.6].

Recall that a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is called *presentably symmetric monoidal*, if the underlying ∞-category $\mathcal{C}$ is presentable, and the tensor functor $-\otimes- : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits separately in each variable.

Proposition 2.3 ([Nik16, 3.8]). *Let $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$. For any presentably symmetric monoidal ∞-category $\mathcal{E}^{\otimes}$, the functor $q^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\otimes} \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\otimes}$ induced by pre-composition with q admits a symmetric monoidal left adjoint $q_! : \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\otimes} \to \mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\otimes}$.*

Proposition 2.4. *There is an adjunction*

$$6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \rightleftarrows 6\mathrm{FF}(\mathrm{LCH}),$$

where the right adjoint is given by restriction. And the left adjoint $6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \to 6\mathrm{FF}(\mathrm{LCH})$ is fully faithful.

Proof. By Remark 2.2, we have the symmetric monoidal functor $\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}})^{\otimes} \to \mathrm{Corr}(\mathrm{LCH})^{\otimes}$. Since $\mathrm{Pr}_{\kappa}^{L,\otimes}$ is a presentably symmetric monoidal ∞-category, which is from [Aok25, 2.3], by Proposition 2.3, we know that the functor

$$\mathrm{Fun}(\mathrm{Corr}(\mathrm{LCH}), \mathrm{Pr}_{\kappa}^{L})^{\otimes} \to \mathrm{Fun}(\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}), \mathrm{Pr}_{\kappa}^{L})^{\otimes},$$

admits a symmetric monoidal left adjoint. Therefore, we have the adjunction

$$\mathrm{Fun}(\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}), \mathrm{Pr}_{\kappa}^{L})^{\otimes} \rightleftarrows \mathrm{Fun}(\mathrm{Corr}(\mathrm{LCH}), \mathrm{Pr}_{\kappa}^{L})^{\otimes},$$

where the left adjoint is symmetric monoidal, and the right adjoint is lax symmetric monoidal. Passing to commutative algebras, we get the adjunction

$$\mathrm{CAlg}(\mathrm{Fun}(\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}), \mathrm{Pr}_{\kappa}^{L})^{\otimes}) \rightleftarrows \mathrm{CAlg}(\mathrm{Fun}(\mathrm{Corr}(\mathrm{LCH}), \mathrm{Pr}_{\kappa}^{L})^{\otimes}),$$

which is exactly the adjunction $6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \rightleftarrows 6\mathrm{FF}(\mathrm{LCH})$. It's clear that the left adjoint $6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \to 6\mathrm{FF}(\mathrm{LCH})$ is fully faithful. $\square$

The following result is given by [Zhu25, 4.18]. But here we can relax some conditions.

Theorem 2.5. *The six functor formalism*

$$\mathrm{Shv}(-;\mathrm{Sp}) : \mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}) \to \mathrm{Pr}^L_\kappa$$

is initial among all six functor formalisms $D : \mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}) \to \mathrm{Pr}^L_\kappa$ satisfying:

(a) *For any light profinite set S, the ∞-category $D(S)$ is dualizable, and the family $\{s^* : D(S) \to D(*)\}_{s\in S}$ is jointly conservative.*
(b) *$D : \mathrm{ProFin}^{\mathrm{light,op}} \to \mathrm{Pr}^L_\kappa$ is a hypercomplete sheaf.*

In other words, if we let $\mathrm{6FF}(\mathrm{ProFin}^{\mathrm{light}})^{\mathrm{cont}}$ be the full subcategory of $\mathrm{6FF}(\mathrm{ProFin}^{\mathrm{light}})$ spanned by those D satisfying (a) and (b), then $\mathrm{Shv}(-;\mathrm{Sp}) \in \mathrm{6FF}(\mathrm{ProFin}^{\mathrm{light}})^{\mathrm{cont}}$ is initial. We denote the unique morphism by $\alpha : \mathrm{Shv}(-;\mathrm{Sp}) \to D(-)$.

Proof. We will use the notation in [Zhu25, Theorem 4.18]. Let $D \in \mathrm{6FF}(\mathrm{ProFin}^{\mathrm{light}})$ satisfies the assumptions, whose image under the functor $\mathrm{6FF}(\mathrm{ProFin}^{\mathrm{light}}) \to \mathrm{6FF}(\mathrm{LCH})$ is still denoted by D. Now since $\mathrm{Shv}(-;\mathrm{Sp}) \in \mathrm{CoSys}^c$ is initial, there exists a unique morphism $\alpha : \mathrm{Shv}(-;\mathrm{Sp}) \to D(-)$ in CoSys^c. In the proof of [Zhu25, Theorem 4.18], Zhu showed that for any proper map $p : X \to Y$ of locally compact Hausdorff spaces, the commutative square

$$\begin{tikzcd} \mathrm{Shv}(Y;\mathrm{Sp}) \arrow[r, "\alpha_Y"] \arrow[d, "p^*"] & D(Y) \arrow[d, "p^*"] \\ \mathrm{Shv}(X;\mathrm{Sp}) \arrow[r, "\alpha_X"] & D(X) \end{tikzcd}$$

is vertically right adjointable. However, by Proposition 2.4, it suffices to show that for any map $p : X \to Y$ of light profinite sets, the above commutative square is vertically right adjointable. Now, if we drop the profinite descent condition in [Zhu25, Definition 4.11], all things in the proof of [Zhu25, Theorem 4.18] still work. $\square$

Remark 2.6. If we let $\mathrm{6FF}(\mathrm{LCH})^{\mathrm{cont}}$ be the essential image of $\mathrm{6FF}(\mathrm{ProFin}^{\mathrm{light}})^{\mathrm{cont}}$ under the fully faithful functor $\mathrm{6FF}(\mathrm{ProFin}^{\mathrm{light}}) \hookrightarrow \mathrm{6FF}(\mathrm{LCH})$, then there is an equivalence

$$\mathrm{6FF}(\mathrm{ProFin}^{\mathrm{light}})^{\mathrm{cont}} \xrightarrow{\sim} \mathrm{6FF}(\mathrm{LCH})^{\mathrm{cont}}.$$

Remark 2.7. Let $f : S \to T$ be a map between light profinite sets. There are two commutative squares

$$\begin{tikzcd} \mathrm{Shv}(T;\mathrm{Sp}) \arrow[r, "\alpha_T"] \arrow[d, "f^*"] & D(T) \arrow[d, "f^*"] \\ \mathrm{Shv}(S;\mathrm{Sp}) \arrow[r, "\alpha_S"] & D(S) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Shv}(S;\mathrm{Sp}) \arrow[r, "\alpha_S"] \arrow[d, "f_!"] & D(S) \arrow[d, "f_!"] \\ \mathrm{Shv}(T;\mathrm{Sp}) \arrow[r, "\alpha_T"] & D(T). \end{tikzcd}$$

Next, we recall the concrete construction of the functor $\alpha_S : \mathrm{Shv}(S;\mathrm{Sp}) \to D(S)$, for any light profinite set S.

Construction 2.8. Let S be a light profinite set. Note that the ∞-category $\mathrm{Shv}(S;\mathrm{Sp})$ is generated under colimits by objects of the form $i_{U!}1_{\mathrm{Shv}(U;\mathrm{Sp})}$, where $i_U : U \hookrightarrow S$ is an open subset of S. Recall from [Zhu25, 3.10] that the functor

$$\alpha_S : \mathrm{Shv}(S;\mathrm{Sp}) \to D(S)$$

is a colimit-preserving functor, which sends $i_{U!}1_{\mathrm{Shv}(U;\mathrm{Sp})}$ to $i_{U!}1_{D(U)}$, where $i_U : U \hookrightarrow S$ is an open subset of S.

With this construction, one has the following basic observation.

Remark 2.9. Let S be a light profinite set. By the construction of the functor α_S, we have:

(1) $\alpha_S 1_{\mathrm{Shv}(S;\mathrm{Sp})} \simeq 1_{D(S)}$.
(2) For any $A, B \in \mathrm{Shv}(S;\mathrm{Sp})$, we have $\alpha_S(A) \otimes \alpha_S(B) \simeq \alpha_S(A \otimes B)$. Indeed, it suffices to show this equivalence on generators. Note that for open subsets $i_U : U \hookrightarrow S$ and $i_V : V \hookrightarrow S$, we have

$$i_{U!}1_{\mathrm{Shv}(U;\mathrm{Sp})} \otimes i_{V!}1_{\mathrm{Shv}(V;\mathrm{Sp})} \simeq i_{U!}i_U^* i_{V!}1_{\mathrm{Shv}(V;\mathrm{Sp})} \simeq i_{U\cap V!}1_{\mathrm{Shv}(U\cap V;\mathrm{Sp})},$$

thus we are able to show the equivalence

$$\alpha_S(i_{U!}1_{\mathrm{Shv}(U;\mathrm{Sp})} \otimes i_{V!}1_{\mathrm{Shv}(V;\mathrm{Sp})}) \simeq i_{U!}1_{D(U)} \otimes i_{V!}1_{D(V)}.$$

3. Six functor formalisms on light condensed anima

In this section, we show that the six functor formalism

$$\mathrm{Shv}(-;\mathrm{Sp}) : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_\kappa$$

is initial among all six functor formalisms satisfying some conditions, see Theorem 3.3.

We recall the extension result of six functor formalisms given by Heyer-Mann.

Theorem 3.1 ([HM24, 3.4.11])**.** *There is a geometric class E of maps of light condensed anima with the following properties:*

(1) *There exists a unique six functor formalism*

$$\mathrm{Shv}(-;\mathrm{Sp}) : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L$$

extending the six functor formalism $\mathrm{Shv}(-;\mathrm{Sp}) : \mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}) \to \mathrm{Pr}^L$*.*
(2) *If $f : Y \to X$ is a map of light condensed anima such that for any map $S \to X$ from a light profinite set S, the pullback $Y \times_X S \to S$ lies in E, then f lies in E.*
(3) *If $f : Y \to X$ is a map of light condensed anima which is !-locally on source or target in E, then f lies in E.*
(4) *If $f : X \to S$ lies in E and S is a light profinite set, then X admits a* $\mathrm{Shv}(-;\mathrm{Sp})^!$*-cover by light profinite sets.*

The maps in E are called !-able maps.

Corollary 3.2. *There is an adjunction*

$$6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \rightleftarrows 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E),$$

where the right adjoint is given by restriction. And the left adjoint $6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \to 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)$ *is fully faithful.*

Proof. By Remark 2.2, we have the symmetric monoidal functor $\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}, \mathrm{all})^\otimes \to \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E)^\otimes$. Since $\mathrm{Pr}^{L,\otimes}_\kappa$ is a presentably symmetric monoidal ∞-category, which is from [Aok25, 2.3], by Proposition 2.3, we know that the functor

$$\mathrm{Fun}(\mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E), \mathrm{Pr}^L_\kappa)^\otimes \to \mathrm{Fun}(\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}), \mathrm{Pr}^L_\kappa)^\otimes,$$

admits a symmetric monoidal left adjoint. Therefore, we have the adjunction

$$\mathrm{Fun}(\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}), \mathrm{Pr}^L_\kappa)^\otimes \rightleftarrows \mathrm{Fun}(\mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E), \mathrm{Pr}^L_\kappa)^\otimes,$$

where the left adjoint is symmetric monoidal, and the right adjoint is lax symmetric monoidal. Passing to commutative algebras, we get the adjunction

$$\mathrm{CAlg}(\mathrm{Fun}(\mathrm{Corr}(\mathrm{ProFin}^{\mathrm{light}}), \mathrm{Pr}^L_\kappa)^\otimes) \rightleftarrows \mathrm{CAlg}(\mathrm{Fun}(\mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E), \mathrm{Pr}^L_\kappa)^\otimes),$$

which is exactly the adjunction $6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \rightleftarrows 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)$. □

Theorem 3.3. *The six functor formalism*
$$\mathrm{Shv}(-;\mathrm{Sp}) : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_\kappa$$
is initial among all six functor formalisms $D : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_\kappa$ *satisfying:*

(a) *For any light profinite set S, the ∞-category $D(S)$ is dualizable, and the family $\{s^* : D(S) \to D(*)\}_{s\in S}$ is jointly conservative.*

(b) *The functor* $D : \mathrm{CondAn}^{\mathrm{light,op}} \to \mathrm{Pr}^L_\kappa$ *preserves limits.*

In other words, if we let $6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$ *be the full subcategory of* $6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)$ *spanned by those D satisfying (a) and (b), then* $\mathrm{Shv}(-;\mathrm{Sp}) \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$ *is initial. We denote the unique morphism by* $\alpha : \mathrm{Shv}(-;\mathrm{Sp}) \to D(-)$.

Proof. By Corollary 3.2, we have the fully faithful functor
$$6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \hookrightarrow 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E).$$
Note that six functor formalisms $D : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_\kappa$ satisfying (a) and (b) exactly lie in the essential image of $6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}})^{\mathrm{cont}}$ under this fully faithful functor. By Theorem 2.5, we know that $\mathrm{Shv}(-;\mathrm{Sp}) \in 6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}})^{\mathrm{cont}}$ is initial. Thus we can conclude that $\mathrm{Shv}(-;\mathrm{Sp}) \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)$ is initial among all six functor formalisms satisfying (a) and (b). □

Remark 3.4. From the proof of Theorem 3.3, we know that there is an equivalence
$$6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}})^{\mathrm{cont}} \xrightarrow{\sim} 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}.$$

Corollary 3.5. *Let $\mathcal{D}$ be a dualizable ∞-category. The six functor formalism*
$$\mathrm{Shv}(-;\mathcal{D}) : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_\kappa$$
is initial among all six functor formalisms $D : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_\kappa$ *satisfying:*

(a) *For any light profinite set S, the ∞-category $D(S)$ is dualizable and the family $\{s^* : D(S) \to D(*)\}_{s\in S}$ is jointly conservative.*

(b) *The functor* $D : \mathrm{CondAn}^{\mathrm{light,op}} \to \mathrm{Pr}^L_\kappa$ *preserves limits.*

(c) *For each light condensed anima X, the category $D(X)$ is an algebra over $\mathcal{D}$.*

Proof. Combine the proof in Theorem 3.3 and [Zhu25, 4.21]. □

Let $\mathrm{Pr}^{\mathrm{dual}} \subset \mathrm{Pr}^L$ be the full subcategory of Pr^L spanned by dualizable ∞-categories.

Proposition 3.6. *There is an adjunction*
$$\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{dual}}) \rightleftarrows 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}},$$
where the left adjoint is given by $\mathcal{C} \mapsto \mathrm{Shv}(-;\mathcal{C})$ *and the right adjoint is given by* $D \mapsto D(*)$.

Proof. It follows from Theorem 3.3 and [Zhu25, 4.19]. □

Remark 3.7. From the adjunction $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{dual}}) \rightleftarrows 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$, we get the counit $\epsilon : \mathrm{Shv}(-;D(*)) \to D(-)$ in $6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$.

Corollary 3.8. *Let* $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$. *The map* $\alpha : \mathrm{Shv}(-;\mathrm{Sp}) \to D(-)$ *factors through*
$$\begin{tikzcd} \mathrm{Shv}(-;\mathrm{Sp}) \arrow[rr, "\alpha"] \arrow[dr] & & D(-) \\ & \mathrm{Shv}(-;D(*)). \arrow[ur, "\epsilon"] & \end{tikzcd}$$

Proof. Applying counit to the map $\alpha : \mathrm{Shv}(-;\mathrm{Sp}) \to D(-)$, we get the commutative square

$$
\begin{tikzcd}
\mathrm{Shv}(-;\mathrm{Sp}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Shv}(-;\mathrm{Sp}) \arrow[d, "\alpha"] \\
\mathrm{Shv}(-;D(*)) \arrow[r, "\epsilon"] & D(-).
\end{tikzcd}
$$

Then it follows from the above square. $\square$

Corollary 3.9. *Let $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$. The map $\epsilon : \mathrm{Shv}(-;D(*)) \to D(-)$ factors as*

$$\mathrm{Shv}(-;D(*)) \simeq \mathrm{Shv}(-;\mathrm{Sp}) \otimes D(*) \xrightarrow{\alpha\otimes\mathrm{id}} D(-) \otimes D(*) \longrightarrow D(-)$$

in $6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)$.

Proof. Restricting D to light profinite sets, by [Zhu25, 3.11] and [Zhu25, 4.19], we know that the counit $\epsilon : \mathrm{Shv}(-;D(*)) \to D(-)$ in $6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}})^{\mathrm{cont}}$ has the factorization

$$\mathrm{Shv}(-;D(*)) \simeq \mathrm{Shv}(-;\mathrm{Sp}) \otimes D(*) \xrightarrow{\alpha\otimes\mathrm{id}} D(-) \otimes D(*) \longrightarrow D(-).$$

Thus what we want to show is followed by applying the functor $6\mathrm{FF}(\mathrm{ProFin}^{\mathrm{light}}) \to 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)$ to this factorization. $\square$

Remark 3.10. If $\alpha : D \to D'$ is a morphism in $6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)$, then:

(1) for any map $f : Y \to X$ of light condensed anima, there is a commutative square

$$
\begin{tikzcd}
D(X) \arrow[r, "\alpha_X"] \arrow[d, "f^*"] & D'(X) \arrow[d, "f^*"] \\
D(Y) \arrow[r, "\alpha_Y"] & D'(Y).
\end{tikzcd}
$$

(2) for any map $f : Y \to X$ in E, there is a commutative square

$$
\begin{tikzcd}
D(Y) \arrow[r, "\alpha_Y"] \arrow[d, "f_!"] & D'(Y) \arrow[d, "f_!"] \\
D(X) \arrow[r, "\alpha_X"] & D'(X).
\end{tikzcd}
$$

Remark 3.11. Let X be a light condensed anima. We write $X \simeq \mathrm{colim}_i S_i$ as the colimit of light profinite sets S_i. Suppose the maps $f_i : S_i \to X$ and transition maps $f_{ij} : S_i \to S_j$. We have the commutative square

$$
\begin{tikzcd}
\mathrm{Shv}(X;\mathrm{Sp}) \arrow[r, "\alpha_X"] \arrow[d, "f_i^*"] & D(X) \arrow[d, "f_i^*"] \\
\mathrm{Shv}(S_i;\mathrm{Sp}) \arrow[r, "\alpha_{S_i}"] & D(S_i).
\end{tikzcd}
$$

Corollary 3.12. *Let X be a light condensed anima and $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$.*

(1) *The functor $\alpha_X : \mathrm{Shv}(X;\mathrm{Sp}) \to D(X)$ preserves all colimits.*
(2) *We have $\alpha_X 1_{\mathrm{Shv}(X;\mathrm{Sp})} \simeq 1_{D(X)}$.*
(3) *For any $A, B \in \mathrm{Shv}(X;\mathrm{Sp})$, we have $\alpha_X(A) \otimes \alpha_X(B) \simeq \alpha_X(A \otimes B)$.*

Proof. We write $X \simeq \mathrm{colim}_i S_i$ as the colimit of light profinite sets S_i, and denote maps $f_i : S_i \to X$.

For (1), since the family $\{f_i^* : D(X) \to D(S_i)\}_i$ is jointly conservative, it suffices to show that the functor $f_i^*\alpha_X$ preserves all colimits, for each i. Indeed, by Remark 3.11, we have $f_i^*\alpha_X \simeq \alpha_{S_i} f_i^*$, which preserves all colimits, since both f_i^* and α_{S_i} preserve all colimits.

For (2), note that we have

$$\alpha_X 1_{\mathrm{Shv}(X;\mathrm{Sp})} \simeq \lim_i f_{i*} f_i^* \alpha_X 1_{\mathrm{Shv}(X;\mathrm{Sp})} \simeq \lim_i f_{i*} \alpha_{S_i} f_i^* 1_{\mathrm{Shv}(X;\mathrm{Sp})}$$
$$\simeq \lim_i f_{i*} 1_{D(S_i)} \simeq \lim_i f_{i*} f_i^* 1_{D(X)} \simeq 1_{D(X)},$$

where the third equivalence is by Remark 2.9.

For (3), again, it suffices to show that $f_i^*(\alpha_X(A) \otimes \alpha_X(B)) \simeq f_i^* \alpha_X(A \otimes B)$, for each i. Indeed, we have

$$f_i^*(\alpha_X(A) \otimes \alpha_X(B)) \simeq f_i^* \alpha_X(A) \otimes f_i^* \alpha_X(B) \simeq \alpha_{S_i} f_i^*(A) \otimes \alpha_{S_i} f_i^*(B)$$
$$\simeq \alpha_{S_i} f_i^*(A \otimes B) \simeq f_i^* \alpha_X(A \otimes B),$$

where the third equivalence is by Remark 2.9.

□

4. Applications

In this section, we give some applications. We will show that for any light condensed anima X and any presentable 6-functor formalism $D : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L_\kappa$ satisfying conditions (a) and (b) in Theorem 3.3, we have:

(1) The functor $f_X^* : D(X) \to D(X \times [0,1])$ is fully faithful, where $f_X : X \times [0,1] \to X$ is the projection,
(2) The functor $f_X^* : D(X) \to D(X \times \mathbb{R})$ is fully faithful, where $f_X : X \times \mathbb{R} \to X$ is the projection.

We first recall the category of kernels.

Remark 4.1. Let D be a 3-functor formalism on a geometric setup $(\mathcal{C}, E)$. Recall from [HM24, 4.1.3], there is a 2-*category of kernels* $K_{D,S}$ associated with the 3-functor formalism D. For this category of kernels $K_{D,S}$, we have the following descriptions:

- The objects in $K_{D,S}$ are the morphisms $X \to S$ in E.
- Given two objects $X, Y \in K_{D,S}$, the category $\mathrm{Fun}_{K_{D,S}}(Y, X)$ of homomorphisms from Y to X is given by
$$\mathrm{Fun}_{K_{D,S}}(Y, X) \simeq D(X \times_S Y).$$
- Given three objects X, Y, Z in $K_{D,S}$, and morphisms $M : Y \to X$, $N : Z \to Y$ in $K_{D,S}$, the composition $M \circ N : Z \to X$ is given by
$$M \circ N \simeq \pi_{XZ!}(\pi_{XY}^* M \otimes \pi_{YZ}^* N) \in D(X \times_S Z).$$

We then recall the concept of suave and prim objects from [HM24, Section 4.4].

Definition 4.2. Let D be a 3-functor formalism on a geometric setup $(\mathcal{C}, E)$ and $f : X \to S$ a morphism in E. We fix an object $P \in D(X)$.

(1) We say P is f-*suave*, if $P \in D(X)$, viewed as a morphism $X \to S$ in $K_{D,S}$, is a left adjoint. We denote its right adjoint $S \to X$ by $\mathrm{SD}_f(P) \in D(X)$, which is called the f-*suave dual* of P.
(2) We say P is f-*prim*, if $P \in D(X)$, viewed as a morphism $X \to S$ in $K_{D,S}$, is a right adjoint. We denote its left adjoint $S \to X$ by $\mathrm{PD}_f(P) \in D(X)$, which is called the f-*prim dual* of P.

Remark 4.3. Let $\alpha : D \to D'$ be a morphism of six functor formalisms on a geometric setup $(\mathcal{C}, E)$. Recall from [HM24, 4.2.1] that for any $S \in \mathcal{C}$, there is an induced 2-functor $\varphi_\alpha : K_{D,S} \to K_{D',S}$, which acts on objects as the identity, and on morphisms categories as

$$\alpha_{X \times_S Y} : D(X \times_S Y) \to D'(X \times_S Y).$$

Recall from [HM24, 4.1.5] that there is a 2-functor $\Psi_{D,S} : \mathrm{K}_{D,S} \to \mathrm{Cat}_2$, which sends $X \in K_{D,S}$ to $D(X) \in \mathrm{Cat}_2$, and a morphism $M \in \mathrm{Fun}_S(Y, X) \simeq D(X \times_S Y)$ to the

functor
$$\Psi_{D,S}(M) = \pi_{X!}(M \otimes \pi_Y^*(-)) : D(Y) \to (X),$$
where $\pi_X : X \times_S Y \to X$ and $\pi_Y : X \times_S Y \to Y$.

Now, for any morphism $\alpha : D \to D'$ between six functor formalisms on a geometric setup $(\mathcal{C}, E)$, there is an induced morphism
$$\Phi_{D,S} \to \Psi_{D',S} \circ \varphi_\alpha$$
between the 2-functors from $K_{D,S}$ to Cat_2.

Lemma 4.4. *Let $f : X \to S$ be a map in E and $\alpha : D \to D'$ a morphism between six functor formalisms. Fix an object $P \in D(X)$. We have:*

1. *If P is f-suave, then $\alpha_X(P) \in D'(X)$ is f-suave, and $\alpha_X(\mathrm{SD}_f(P)) \simeq \mathrm{SD}_f(\alpha_X(P))$.*
2. *If P is f-prim, then $\alpha_X(P) \in D'(X)$ is f-prim, and $\alpha_X(\mathrm{PD}_f(P)) \simeq \mathrm{PD}_f(\alpha_X(P))$.*

Proof. The morphism $\alpha : D \to D'$ induces a 2-functor $\varphi_\alpha : K_{D,S} \to K_{D',S}$, which preserves adjunctions. If P is f-suave, then $P \in D(X)$, viewed as a morphism $X \to S$ in $K_{D,S}$, is left adjoint. Thus by Remark 4.3, the 2-functor φ_α sends the adjunction $X \underset{\mathrm{SD}_f(P)}{\overset{P}{\rightleftarrows}} S$ to the adjunction $X \underset{\alpha_X(\mathrm{SD}_f(P))}{\overset{\alpha_X(P)}{\rightleftarrows}} S$. Hence $\mathrm{SD}_f(\alpha_X(P)) \simeq \alpha_X(\mathrm{SD}_f(P))$. The case P being f-prim is similar. □

Next, we recall the definition of suave maps and prim maps from [HM24, Section 4.5].

Definition 4.5. Let D be a three functor formalism on a geometric setup $(\mathcal{C}, E)$, and $f : X \to S$ a morphism in E.

1. We say f is *D-suave*, if $1_{D(X)} \in D(X)$ is f-suave. We denote $\omega_f := \mathrm{SD}_f(1_{D(X)}) \simeq f^! 1_{D(S)} \in D(X)$, which is called the *dualizing complex* of f.
2. We say f is *D-prim*, if $1_{D(X)} \in D(X)$ is f-prim. We denote $D_f := \mathrm{PD}_f(1_{D(X)}) \simeq p_{1*}\Delta_! 1_{D(X)} \in D(X)$, which is called the *codualizing complex* of f.

In Lemma 4.4, if we take $P \simeq 1_{D(X)} \in D(X)$, and take $\alpha : D \simeq \mathrm{Shv}(-;\mathrm{Sp}) \to D'$ to be the unique morphism from Theorem 3.3, then we get:

Corollary 4.6. *Let $f : X \to S$ be a map in E and $\alpha : D \simeq \mathrm{Shv}(-;\mathrm{Sp}) \to D'$ the unique morphism from Theorem 3.3. We have:*

1. *If f is D-suave, then f is D'-suave, and $\alpha_X \omega_f \simeq \omega'_f$, where $\omega_f \simeq f^! 1_{D(S)}$ and $\omega'_f \simeq f^! 1_{D'(S)}$.*
2. *If f is D-prim, then f is D'-prim, and $\alpha_X D_f \simeq D'_f$, where $D_f \simeq p_{1*}\Delta_! 1_{D(X)}$ and $D'_f \simeq p_{1*}\Delta_! 1_{D'(X)}$.*

Proof. By Corollary 3.12, we have $\alpha_X(1_{D(X)}) \simeq 1_{D'(X)}$. Now, it follows from Lemma 4.4. □

In the proof of Theorem 4.8, we will use the concept of 2-natural transformation. So we recall this concept.

Definition 4.7 (2-natural transformation)**.** Let $D, E : \mathcal{A} \to \mathcal{B}$ be 2-functors between 2-categories. A *2-natural transformation* $\alpha : D \implies E : \mathcal{A} \to \mathcal{B}$ between D and E assigns to each object $A \in \mathcal{A}$, an arrow $\alpha_A : DA \to EA$ in $\mathcal{B}$, such that:

1. for each map $f : A_1 \to A_2$ in $\mathcal{A}$, we have $\alpha_{A_2} \circ Df \simeq Ef \circ \alpha_{A_1}$ in $\mathcal{B}$.
2. for each 2-cell $\mu : f \implies g$ from A_1 to A_2 in $\mathcal{A}$, we have $\alpha_{A_2} \circ D\mu \simeq E\mu \circ \alpha_{A_1}$ in $\mathcal{B}$.

Theorem 4.8. *Let $f : Y \to X$ be a map in E and $\alpha : D \simeq \mathrm{Shv}(-;\mathrm{Sp}) \to D'$ the unique morphism from Theorem 3.3. We have:*

1. *If $f : Y \to X$ is D-suave, then the commutative square*

$$\begin{tikzcd} D(Y) \arrow[r, "\alpha_Y"] \arrow[d, "f_!"] & D'(Y) \arrow[d, "f_!"] \\ D(X) \arrow[r, "\alpha_X"] & D'(X) \end{tikzcd}$$

is vertically right adjointable. That is, the canonical map $\alpha_Y f^! \to f^! \alpha_X$ is an equivalence.

(2) *If $f : Y \to X$ is D-prim, then the commutative square*

$$\begin{tikzcd} D(X) \arrow[r, "\alpha_X"] \arrow[d, "f^*"] & D'(X) \arrow[d, "f^*"] \\ D(Y) \arrow[r, "\alpha_Y"] & D'(Y) \end{tikzcd}$$

is vertically right adjointable. That is, the canonical map $\alpha_X f_* \to f_* \alpha_Y$ is an equivalence.

Proof. In each of the cases, we can construct some isomorphism between the two functors. For (1), we have

$$\alpha_Y f^! \simeq \alpha_Y(f^! 1_{D(X)} \otimes f^*(-)) \simeq \alpha_Y f^! 1_{D(X)} \otimes \alpha_Y f^*(-) \simeq f^! 1_{D'(X)} \otimes f^* \alpha_X(-) \simeq f^! \alpha_X,$$

where the first equivalence is because f is D-suave, the second equivalence is by Corollary 3.12, the third equivalence is by Corollary 4.6, and the last equivalence is because f is D'-suave.

And for (2), we have

$$\alpha_X f_* \simeq \alpha_X f_!(- \otimes D_f) \simeq f_! \alpha_Y(- \otimes D_f) \simeq f_!(\alpha_Y(-) \otimes \alpha_Y D_f) \simeq f_!(\alpha_Y(-) \otimes D'_f) \simeq f_* \alpha_Y,$$

where the first equivalence is because f is D-prim, the second equivalence is by Remark 3.10, the third equivalence is by Corollary 3.12, the fourth equivalence is by Corollary 4.6, and the last equivalence is because f is D'-prim.

However, with this strategy it is not clear why the constructed isomorphism is the natural one.

Now, we mimic a conceptual proof from [HM24, 4.5.13] that the isomorphism is indeed the correct one, using the 2-category of kernels.

We first prove (1). We claim that $\Psi_{D,X} \to \Psi_{D',X} \circ \varphi_\alpha$ is a 2-natural transformation between 2-functors from $K_{D,X}$ to Cat_2. This is ensured by (a) and (b) in the following:

(a). For any map $M : Y_1 \to Y_2$ in $K_{D,X}$, i.e. any object $M \in D(Y_2 \times_X Y_1)$, there is a commutative square

$$\begin{tikzcd} D(Y_1) \arrow[r, "\alpha_{Y_1}"] \arrow[d, "\pi_{2!}(\pi_1^*(-)\otimes M)"'] & D'(Y_1) \arrow[d, "\pi_{2!}(\pi_1^*(-)\otimes \alpha_{Y_2\times_X Y_1}(M))"] \\ D(Y_2) \arrow[r, "\alpha_{Y_2}"] & D'(Y_2). \end{tikzcd}$$

Indeed, this square can be written as

$$\begin{tikzcd} D(Y_1) \arrow[r, "\pi_1^*"] \arrow[d, "\alpha_{Y_1}"] & D(Y_2 \times_X Y_1) \arrow[r, "-\otimes M"] \arrow[d, "\alpha_{Y_2\times_X Y_1}"] & D(Y_2 \times_X Y_1) \arrow[r, "\pi_{2!}"] \arrow[d, "\alpha_{Y_2\times_X Y_1}"] & D(Y_2) \arrow[d, "\alpha_{Y_2}"] \\ D'(Y_1) \arrow[r, "\pi_1^*"] & D'(Y_2 \times_X Y_1) \arrow[r, "-\otimes \alpha_{Y_2\times_X Y_1}(M)"] & D'(Y_2 \times_X Y_1) \arrow[r, "\pi_{2!}"] & D'(Y_2), \end{tikzcd}$$

where the above squares are commutative by Remark 3.10 and Corollary 3.12.

(b). Let M and N be any two morphisms from Y_1 to Y_2 in $K_{D,X}$. For any 2-morphism $\mu : M \to N$ in $K_{D,X}$, i.e. a morphism $\mu : M \to N$ in $D(Y_2 \times_X Y_1)$, it is easy to show that

$$\alpha_{Y_2}\pi_{2!}(\pi_1^*(-) \otimes \mu) \simeq \pi_{2!}(\pi_1^* \alpha_{Y_1}(-) \otimes \alpha_{Y_2\times_X Y_1}(\mu)).$$

Now if $f : Y \to X$ is D-suave, then after applying the natural transformation $\Psi_{D,X} \to \Psi_{D',X} \circ \varphi_\alpha : K_{D,X} \to \mathrm{Cat}_2$ to the adjunction $Y \underset{\omega_f}{\overset{1_{D(Y)}}{\rightleftarrows}} X$ in $K_{D,X}$, we get the commutative squares

$$\begin{tikzcd} D(Y) \arrow[r, "\alpha_Y"] \arrow[d, "f_!"] & D'(Y) \arrow[d, "f_!"] \\ D(X) \arrow[r, "\alpha_X"] & D'(X) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D(X) \arrow[r, "\alpha_X"] \arrow[d, "\omega_f \otimes f^*"] & D'(X) \arrow[d, "\omega'_f \otimes f^*"] \\ D(Y) \arrow[r, "\alpha_Y"] & D'(Y) \end{tikzcd}$$

in Cat_2. By [KS74, 2.5], the above two squares are mates to each other. Thus the commutative square

$$\begin{tikzcd} D(Y) \arrow[r, "\alpha_Y"] \arrow[d, "f_!"] & D'(Y) \arrow[d, "f_!"] \\ D(X) \arrow[r, "\alpha_X"] & D'(X) \end{tikzcd}$$

is vertically right adjointable.

The proof of (2) is similar if we run the argument in $K_{D,X}^{\mathrm{co,op}}$. □

Lemma 4.9. *Suppose a commutative square*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h"] \arrow[d, "f"] & \mathcal{C}' \arrow[d, "f'"] \\ \mathcal{D} \arrow[r, "g"] & \mathcal{D}' \end{tikzcd}$$

in Pr^L*, which is vertically right adjointable. If both* f^R *and* f'^R *preserve colimits, then for any* $\mathcal{E} \in \mathrm{Pr}^L$*, the square*

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{E} \arrow[r, "h\otimes \mathrm{id}_{\mathcal{E}}"] \arrow[d, "f\otimes \mathrm{id}_{\mathcal{E}}"] & \mathcal{C}' \otimes \mathcal{E} \arrow[d, "f'\otimes \mathrm{id}_{\mathcal{E}}"] \\ \mathcal{D} \otimes \mathcal{E} \arrow[r, "g\otimes \mathrm{id}_{\mathcal{E}}"] & \mathcal{D}' \otimes \mathcal{E} \end{tikzcd}$$

is also vertically right adjointable.

Proof. Since f^R and f'^R preserve colimits, we have $(f\otimes \mathrm{id}_{\mathcal{E}})^R \simeq f^R \otimes \mathrm{id}_{\mathcal{E}}$ and $(f'\otimes \mathrm{id}_{\mathcal{E}})^R \simeq f'^R \otimes \mathrm{id}_{\mathcal{E}}$. Thus, applying $-\otimes \mathrm{id}_{\mathcal{E}}$ to the canonical equivalence $hf^R \xrightarrow{\sim} f'^R g$, we get the canonical equivalence

$$(h \otimes \mathrm{id}_{\mathcal{E}}) \circ (f \otimes \mathrm{id}_{\mathcal{E}})^R \xrightarrow{\sim} (f' \otimes \mathrm{id}_{\mathcal{E}})^R \circ (g \otimes \mathrm{id}_{\mathcal{E}}),$$

which finishes the proof. □

Corollary 4.10. *Let* $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$*. Let* $f : Y \to X$ *be a map in* E *and consider the counit* $\epsilon : \mathrm{Shv}(-; D(*)) \to D(-)$*. We have:*

(1) *If* $f : Y \to X$ *is* $\mathrm{Shv}(-;\mathrm{Sp})$*-suave, then the commutative square*

$$\begin{tikzcd} \mathrm{Shv}(Y; D(*)) \arrow[r, "\epsilon_Y"] \arrow[d, "f_!"] & D(Y) \arrow[d, "f_!"] \\ \mathrm{Shv}(X; D(*)) \arrow[r, "\epsilon_X"] & D(X) \end{tikzcd}$$

is vertically right adjointable.

(2) *If* $f : Y \to X$ *is* $\mathrm{Shv}(-;\mathrm{Sp})$*-prim, then the commutative square*

$$\begin{tikzcd}
\mathrm{Shv}(X;D(*)) \arrow[r, "\epsilon_X"] \arrow[d, "f^*"] & D(X) \arrow[d, "f^*"] \\
\mathrm{Shv}(Y;D(*)) \arrow[r, "\epsilon_Y"] & D(Y)
\end{tikzcd}$$

is vertically right adjointable.

Proof. Note that the map $\sigma : D'(-) \simeq D(-) \otimes D(*) \to D(-)$ is a morphism of six functor formalisms, which induces a 2-functor $\varphi_\sigma : K_{D',X} \to K_{D,X}$ by Remark 4.3. Thus we get a map $\Psi_{D',X} \to \Psi_{D,X} \circ \varphi_\sigma$ between 2-functors from $K_{D',X}$ to Cat_2. We claim that $\Psi_{D',X} \to \Psi_{D,X} \circ \varphi_\sigma$ is a 2-natural transformation, which is ensured by (a) and (b) in the following:

(a). For any map $M : Y_1 \to Y_2$ in $K_{D',X}$, i.e. an object $M \in \mathrm{Fun}_{K_{D',X}}(Y_1, Y_2) \simeq D'(Y_2 \times_X Y_1)$, there is a commutative square

$$\begin{tikzcd}
D'(Y_1) \arrow[r, "\sigma_{Y_1}"] \arrow[d, "\pi_{2!}(\pi_1^*(-)\otimes M)"'] & D(Y_1) \arrow[d, "\pi_{2!}(\pi_1^*(-)\otimes \sigma_{Y_2\times_X Y_1}(M))"] \\
D'(Y_2) \arrow[r, "\sigma_{Y_2}"] & D(Y_2).
\end{tikzcd}$$

Note that this square can be written as

$$\begin{tikzcd}
D'(Y_1) \arrow[r, "\pi_1^*"] \arrow[d, "\sigma_{Y_1}"] & D'(Y_2 \times_X Y_1) \arrow[r, "-\otimes M"] \arrow[d, "\sigma_{Y_2\times_X Y_1}"] & D'(Y_2 \times_X Y_1) \arrow[r, "\pi_{2!}"] \arrow[d, "\sigma_{Y_2\times_X Y_1}"] & D'(Y_2) \arrow[d, "\sigma_{Y_2}"] \\
D(Y_1) \arrow[r, "\pi_1^*"] & D(Y_2 \times_X Y_1) \arrow[r, "-\otimes \sigma_{Y_2\times_X Y_1}(M)"] & D(Y_2 \times_X Y_1) \arrow[r, "\pi_{2!}"] & D(Y_2)
\end{tikzcd}$$

where each square is commutative.

(b). Let M and N be any two morphisms from Y_1 to Y_2 in $K_{D',X}$. For any 2-morphism $\mu : M \to N$ in $K_{D',X}$, i.e. a morphism $\mu : M \to N$ in $D'(Y_2 \times_X Y_1)$, it is easy to show that

$$\sigma_{Y_2}\pi_{2!}(\pi_1^*(-) \otimes \mu) \simeq \pi_{2!}(\pi_1^*\sigma_{Y_1}(-) \otimes \sigma_{Y_2\times_X Y_1}(\mu)).$$

Now, by Corollary 3.9, the square in (1) can be written as

$$\begin{tikzcd}
\mathrm{Shv}(Y;\mathrm{Sp}) \otimes D(*) \arrow[r, "\alpha_Y\otimes \mathrm{id}"] \arrow[d] & D(Y) \otimes D(*) \arrow[r] \arrow[d] & D(Y) \arrow[d] \\
\mathrm{Shv}(X;\mathrm{Sp}) \otimes D(*) \arrow[r, "\alpha_X\otimes \mathrm{id}"] & D(X) \otimes D(*) \arrow[r] & D(X).
\end{tikzcd}$$

In order to show the square in (1) is vertically right adjointable, it suffices to show the left square and right square in the above are both vertically right adjointable.

If $f : Y \to X$ is $\mathrm{Shv}(-;\mathrm{Sp})$-suave, then $f^! : \mathrm{Shv}(X;\mathrm{Sp}) \to \mathrm{Shv}(Y;\mathrm{Sp})$ and $f^! : D(X) \to D(Y)$ preserve all colimits. By Theorem 4.8 and Lemma 4.9, we know that the left square

$$\begin{tikzcd}
\mathrm{Shv}(Y;\mathrm{Sp}) \otimes D(*) \arrow[r, "\alpha_Y\otimes \mathrm{id}"] \arrow[d, "f_!\otimes \mathrm{id}"] & D(Y) \otimes D(*) \arrow[d, "f_!\otimes \mathrm{id}"] \\
\mathrm{Shv}(X;\mathrm{Sp}) \otimes D(*) \arrow[r, "\alpha_X\otimes \mathrm{id}"] & D(X) \otimes D(*)
\end{tikzcd}$$

is vertically right adjointable. For the right square to be vertically right adjointable, after applying the natural transformation $\Psi_{D',X} \to \Psi_{D,X} \circ \varphi_\sigma : K_{D',X} \to \mathrm{Cat}_2$ to the adjunction $Y \underset{\omega_f}{\overset{1_{D'(Y)}}{\rightleftarrows}} X$ in $K_{D',X}$, we get two commutative squares

$$\begin{tikzcd} D(Y)\otimes D(*) \arrow[r] \arrow[d, "f_!\otimes \mathrm{id}"] & D(Y) \arrow[d, "f_!"] \\ D(X)\otimes D(*) \arrow[r] & D(X) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D(X)\otimes D(*) \arrow[r] \arrow[d, "\omega_f\otimes f^*\otimes \mathrm{id}"] & D(X) \arrow[d, "\omega_f\otimes f^*"] \\ D(Y)\otimes D(*) \arrow[r] & D(Y), \end{tikzcd}$$

which are mates to each other by [KS74, 2.5]. Thus we can conclude that the square

$$\begin{tikzcd} D(Y)\otimes D(*) \arrow[r] \arrow[d, "f_!\otimes \mathrm{id}"] & D(Y) \arrow[d, "f_!"] \\ D(X)\otimes D(*) \arrow[r] & D(X) \end{tikzcd}$$

is vertically right adjointable, which finishes the proof of (1).

The proof of (2) is similar.

□

Recall that given a six functor formalism $D : \mathrm{Corr}(\mathrm{CondAn}^{\mathrm{light}}, E) \to \mathrm{Pr}^L$, for any !-able map $f : X \to *$ and any object $E \in D(*)$, one has:

(1) (sheaf cohomology) $H^*(X, E)_D := f_*f^*E$.
(2) (compactly supported cohomology) $H^*_c(X, E)_D := f_!f^*E$.
(3) (sheaf homology) $H_*(X, E)_D := f_!f^!E$.
(4) (Borel-Moore homology) $H^{\mathrm{BM}}_*(X, E)_D := f_*f^!E$.

Here, we write down the six functor formalism D to dedicate the dependence on it.

Proposition 4.11. *Let $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$. For the map $\epsilon : \mathrm{Shv}(-; D(*)) \to D(-)$, we have:*

(1) *If $f : X \to *$ is !-able, then $H^*_c(X, E)_{\mathrm{Shv}(-;D(*))} \simeq H^*_c(X, E)_D$.*
(2) *If $f : X \to *$ is !-able, then $H^{\mathrm{BM}}_*(X, E)_{\mathrm{Shv}(-;D(*))} \simeq H^{\mathrm{BM}}_*(X, E)_D$.*
(3) *If $f : X \to *$ is $\mathrm{Shv}(-; D(*))$-prim, then $H^*(X, E)_{\mathrm{Shv}(-;D(*))} \simeq H^*(X, E)_D$.*
(4) *If $f : X \to *$ is $\mathrm{Shv}(-; D(*))$-suave, then $H_*(X, E)_{\mathrm{Shv}(-;D(*))} \simeq H_*(X, E)_D$.*

Proof. Note that by Remark 3.10, the map $\epsilon : \mathrm{Shv}(-; D(*)) \to D(-)$ gives the commutative diagram

$$\begin{tikzcd} \mathrm{Shv}(*; D(*)) \arrow[r, "\sim"] \arrow[d, "f^*"] & D(*) \arrow[d, "f^*"] \\ \mathrm{Shv}(X; D(*)) \arrow[r] \arrow[d, "f_!"] & D(X) \arrow[d, "f_!"] \\ \mathrm{Shv}(*; D(*)) \arrow[r, "\sim"] & D(*). \end{tikzcd}$$

It follows that (1) holds. Passing to right adjoints, we know that (2) holds. For (3), if $f : X \to *$ is $\mathrm{Shv}(-; D(*))$-prim, by Corollary 4.10, we get the commutative diagram

$$\begin{tikzcd} \mathrm{Shv}(*; D(*)) \arrow[r, "\sim"] \arrow[d, "f^*"] & D(*) \arrow[d, "f^*"] \\ \mathrm{Shv}(X; D(*)) \arrow[r] \arrow[d, "f_*"] & D(X) \arrow[d, "f_*"] \\ \mathrm{Shv}(*; D(*)) \arrow[r, "\sim"] & D(*). \end{tikzcd}$$

Thus we have $H^*(X, E)_{\mathrm{Shv}(-;D(*))} \simeq H^*(X, E)_D$. Using Corollary 4.10, we can also prove (4) similarly. □

Proposition 4.12. *Let $I = [0,1] \subset \mathbb{R}$ be the unit interval and $f : I \to *$ the projection. Let $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$, and X a light condensed anima with the projection $f_X : X \times I \to X$. Then the pullback functor $f_X^* : D(X) \to D(X \times I)$ is fully faithful. In particular, $f_{X*}1_{D(X\times I)} \simeq 1_{D(X)}$.*

Proof. By [HM24, 4.8.7], the functor $f_X^* : \mathrm{Shv}(X; \mathrm{Sp}) \to \mathrm{Shv}(X \times I; \mathrm{Sp})$ is fully faithful. Since $D(*)$ is a dualizable category, the functor $- \otimes D(*)$ preserves fully faithful functors. Thus the functor $f_X^* : \mathrm{Shv}(X; D(*)) \to \mathrm{Shv}(X \times I; D(*))$ is also fully faithful. Since $f : I \to *$ is $\mathrm{Shv}(-; \mathrm{Sp})$-prim, by Corollary 4.10, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}(*; D(*)) \arrow[r, "\sim"] \arrow[d, "f^*"] & D(*) \arrow[d, "f^*"] \\
\mathrm{Shv}(I; D(*)) \arrow[r] \arrow[d, "f_*"] & D(I) \arrow[d, "f_*"] \\
\mathrm{Shv}(*; D(*)) \arrow[r, "\sim"] & D(*).
\end{tikzcd}$$

Thus we have $f_*1_{D(I)} \simeq f_*f^*1_{D(*)} \simeq 1_{D(*)}$. Since $f : I \to *$ is $\mathrm{Shv}(-; \mathrm{Sp})$-prim, by Corollary 4.6, we know that $f : I \to *$ is D-prim. Applying base change to the cartesian square

$$\begin{tikzcd}
X \times I \arrow[r, "p_I"] \arrow[d, "f_X"] & I \arrow[d, "f"] \\
X \arrow[r, "p"] & *,
\end{tikzcd}$$

we get

$$f_{X*}1_{D(X\times I)} \simeq f_{X*}p_I^*1_{D(I)} \simeq p^*f_*1_{D(I)} \simeq p^*1_{D(*)} \simeq 1_{D(X)}.$$

Note that as the pullback of $f : I \to *$, the map $f_X : X \times I \to X$ is also D-prim. Now, for any $E \in D(X)$, since $f_X : X \times I \to X$ is D-prim, by the projection formula, we get

$$f_{X*}f_X^*E \simeq f_{X*}(f_X^*E \otimes 1_{D(X\times I)}) \simeq E \otimes f_{X*}1_{D(X\times I)} \simeq E \otimes 1_{D(X)} \simeq E.$$

Thus the functor $f_X^* : D(X) \to D(X \times I)$ is fully faithful. $\square$

Lemma 4.13. *Let $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$. Let X be a locally compact Hausdorff space, $i : Z \hookrightarrow X$ a closed subset and $j : U = X \setminus Z \hookrightarrow X$ the complement open subset. We have the fiber sequence*

$$j_!1_{D(U)} \to 1_{D(X)} \to i_*1_{D(Z)}$$

in $D(X)$.

Proof. By Corollary 4.6, since j is $\mathrm{Shv}(-; \mathrm{Sp})$-étale and i is $\mathrm{Shv}(-; \mathrm{Sp})$-proper, we know that j is D-étale and i is D-proper, which by [HM24, 4.6.4] shows that $i_* \simeq i_!$ and $j^* \simeq j^!$. By proper base change, since $U \times_X Z = 0$, we have $j^*i_* \simeq 0$, hence $j^*\mathrm{fib}(1_{D(X)} \to i_*1_{D(Z)}) \simeq 1_{D(U)}$, which gives the map

$$u : j_!1_{D(U)} \to \mathrm{fib}(1_{D(X)} \to i_*1_{D(Z)})$$

via adjunction. Since D satisfies condition (a) in Theorem 3.3, the functors i^* and j^* are jointly conservative. It is clear that i^*u and j^*u are equivalences. Thus the map $j_!1_{D(U)} \to \mathrm{fib}(1_{D(X)} \to i_*1_{D(Z)})$ is an equivalence. $\square$

Lemma 4.14. *Let $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$. If $f : \mathbb{R} \to *$ is the projection, then*

1. $f^!1_{D(*)} \simeq 1_{D(\mathbb{R})}[1]$.
2. $f_!1_{D(\mathbb{R})} \simeq 1_{D(*)}[-1]$.

Proof. For (1), consider the functor $\alpha_{\mathbb{R}} : \mathrm{Shv}(\mathbb{R};\mathrm{Sp}) \to D(\mathbb{R})$. Since $f : \mathbb{R} \to *$ is $\mathrm{Shv}(-;\mathrm{Sp})$-suave, by Corollary 4.6, the functor $\alpha_{\mathbb{R}}$ sends $f^!1_{\mathrm{Shv}(\mathbb{R};\mathrm{Sp})}$ to $f^!1_{D(*)}$. By [HM24, 4.8.9], we have $f^!1_{\mathrm{Shv}(\mathbb{R};\mathrm{Sp})} \simeq 1_{\mathrm{Shv}(\mathbb{R};\mathrm{Sp})}[1]$. Since the functor $\alpha_{\mathbb{R}}$ preserves colimits, we get

$$f^!1_{D(*)} \simeq \alpha_{\mathbb{R}}(f^!1_{\mathrm{Shv}(\mathbb{R};\mathrm{Sp})}) \simeq \alpha_{\mathbb{R}}(1_{\mathrm{Shv}(\mathbb{R};\mathrm{Sp})}[1]) \simeq \alpha_{\mathbb{R}}(1_{\mathrm{Shv}(\mathbb{R};\mathrm{Sp})})[1] \simeq 1_{D(\mathbb{R})}[1].$$

For (2), condider the diagram

$$\begin{tikzcd}
\mathbb{R} \simeq (0,1) \arrow[r, hook, "j"] \arrow[dr, "f"'] & I \arrow[d, "p"] & \{0,1\} \arrow[l, hook', "i"'] \arrow[dl] \\
& *. &
\end{tikzcd}$$

By Proposition 4.12, we have $p_*1_{D(I)} \simeq 1_{D(*)}$, thus we get

$$\begin{aligned}
f_!1_{D(\mathbb{R})} &\simeq p_!j_!1_{D(\mathbb{R})} \simeq p_*j_!1_{D(\mathbb{R})} \simeq \mathrm{fib}(p_*1_{D(I)} \to p_*i_*1_{D(\{0,1\})}) \\
&\simeq \mathrm{fib}(1_{D(*)} \to 1_{D(*)} \oplus 1_{D(*)}) \simeq 1_{D(*)}[-1],
\end{aligned}$$

where the third equivalence is by Lemma 4.13. ☐

Theorem 4.15. *Let $D \in 6\mathrm{FF}(\mathrm{CondAn}^{\mathrm{light}}, E)^{\mathrm{cont}}$. For any light condensed anima X with projection $f_X : X \times \mathbb{R} \to X$, the pullback functor $f_X^* : D(X) \to D(X \times \mathbb{R})$ is fully faithful.*

Proof. The proof here is the same as the proof in [Zhu25, 4.28]. Consider the square

$$\begin{tikzcd}
X \times \mathbb{R} \arrow[r, "p_{\mathbb{R}}"] \arrow[d, "f_X"] & \mathbb{R} \arrow[d, "f"] \\
X \arrow[r, "p"] & *.
\end{tikzcd}$$

By Lemma 4.14, we have $f_!1_{D(\mathbb{R})} \simeq 1_{D(*)}[-1]$, thus by base change, we have

$$f_{X!}1_{D(X\times\mathbb{R})} \simeq f_{X!}p_{\mathbb{R}}^*1_{D(\mathbb{R})} \simeq p^*f_!1_{D(\mathbb{R})} \simeq p^*(1_{D(*)}[-1]) \simeq 1_{D(X)}[-1].$$

The map $f : \mathbb{R} \to *$ is $\mathrm{Shv}(-;\mathrm{Sp})$-suave, and hence is D-suave, we get

$$f_X^!1_{D(X)} \simeq f_X^!p^*1_{D(*)} \simeq p_{\mathbb{R}}^*f^!1_{D(*)} \simeq p_{\mathbb{R}}^*(1_{D(\mathbb{R})}[1]) \simeq 1_{D(X\times\mathbb{R})}[1],$$

where the third equivalence is by Lemma 4.14. Thus we have

$$f_X^!(-) \simeq f_X^!1_{D(X)} \otimes f_X^*(-) \simeq 1_{D(X\times\mathbb{R})}[1] \otimes f_X^*(-) \simeq f_X^*(-)[1].$$

Now, for any $E \in D(X)$, we have

$$\begin{aligned}
f_{X*}f_X^*E &\simeq f_{X*}f_X^!E[-1] \simeq f_{X*}\underline{\mathrm{Hom}}(1_{D(X\times\mathbb{R})}, f_X^!E)[-1] \\
&\simeq \underline{\mathrm{Hom}}(f_{X!}1_{D(X\times\mathbb{R})}, E)[-1] \simeq \underline{\mathrm{Hom}}(1_{D(X)}[-1], E)[-1] \simeq E.
\end{aligned}$$

Thus $f_X^* : D(X) \to D(X \times \mathbb{R})$ is fully faithful. ☐

References

[Aok25] Ko Aoki, *Higher presentable categories and limits*, https://arxiv.org/pdf/2510.13503 (2025).

[Cla25] Dustin Clausen, *Duality and linearization for p-adic lie groups*, https://arxiv.org/pdf/2506.18174 (2025).

[HM24] Claudius Heyer and Lucas Mann, *6-functor formalisms and smooth representations*, https://arxiv.org/pdf/2410.13038 (2024).

[KS74] G. M. Kelly and Ross Street, *Review of the elements of 2-categories*, Category Seminar (1974).

[Man22] Lucas Mann, *A -adic 6-functor formalism in rigid-analytic geometry*, https://arxiv.org/pdf/2206.02022 (2022).

[Nik16] Thomas Nikolaus, *Stable ∞-operads and the multiplicative yoneda lemma*, https://arxiv.org/pdf/1608.02901 (2016).

[Sch25] Peter Scholze, *Six-functor formalisms*, https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf (2025).

[Zhu25] Qingchong Zhu, *Continuous six-functor formalism on locally compact hausdorff spaces*, https://arxiv.org/pdf/2507.13537 (2025).

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