THE H-PRINCIPLE FAILS FOR PRELEGENDRIANS IN CORANK 2 FAT DISTRIBUTIONS

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ABSTRACT. We investigate the h-principle problem for fat distributions. These are maximally non-integrable distributions with natural symplectisations and contactisations, that generalize contact distributions to higher corank. We focus on the corank-2 case, where we study a natural class of submanifolds, which we call prelegendrians. Their key feature is that they admit a canonical Legendrian lift to the contactisation. Our main results state that the h-principle fails for these submanifolds in all dimensions. This is the first example of rigidity in the study of maximally non-integrable distributions, outside of contact topology.

First, we find an infinite family of $(2n+1)$-tori in the standard fat $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$, with the following two properties: (1) They all represent the same formal prelegendrian class, (2) but they are not prelegendrian isotopic because they are distinguished by pseudoholomorphic curve invariants of their Legendrian lift.

Secondly, we define the notion of prelegendrian stabilization in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$. This allows us to take an arbitrary prelegendrian and produce another one, in the same formal class, whose Legendrian lift is loose.

In order to prove these results we also develop the fundamentals of the theory of prelegendrians. This includes: (1) introducing the notion of front projection in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$, (2) proving that pseudoholomorphic curve invariants are robust under perturbations of the fat structure, allowing us to transport our results to non-standard fat structures, (3) introducing a zooming argument showing that any fat structure in dimension 6 admits prelegendrians.

1. Introduction

1.1. **Context.** Let M^n be a smooth manifold. An r-distribution $\mathcal{D}$ on M is a smooth section of the Grassmannian bundle $\mathrm{Gr}_r(TM) \to M$. It is a classical problem in Differential Topology to study the h-principle [Gro86, CEM24] for classes of maximally non-integrable distributions. That is, whether distributions in a certain class can be classified, up to homotopy, by bundle-theoretic data, hence becoming *flexible;* or whether, by contrast, they cannot, hence becoming *rigid* and relevant from a geometric point of view.

The case most studied is $(r, n) = (2m, 2m+1)$, corresponding to *contact* distributions, in which there is a rich interplay between rigidity and flexibility as crystallized through the tight/overtwisted dichotomy [Ben83, Eli89, BEM15]. For the case $(r, n) = (2m-1, 2m)$, known as *even-contact* distributions, a full h-principle was established by McDuff through Gromov's convex integration technique [McD87]. The case $(r, n) = (2, 4)$ corresponds to *Engel* distributions. Here, several results have appeared in the direction flexibility [Vog09, CPdPP17, CdPP20, dPV20], but the presence of rigidity is still unknown. Other instances of flexibility have been established for several other classes in [Ada03, MAdP21, MA23]. In particular, outside the contact realm the presence of rigidity remains mysterious.

In this article, we work with *fat distributions*, a notion originating in Weinstein's theory of *fat bundles* [Wei80, Mon02]. These generalize contact distributions to higher-corank. In the case

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$(4,6)$, fat distributions are precisely the *elliptic* distributions conjectured to present rigidity in [MAdP21]. Fat distributions have also been studied in [Bho24, BD23, Bho22].

Following the "Legendrian route" from contact topology to detect rigidity [Ben83, Eli92, CMP19], we introduce the class of *prelegendrian* submanifolds in fat distributions, and study the h-principle problem for them. The main contribution of this article is that, for corank 2 fat distributions, the h-principle fails for prelegendrian submanifolds, hence rigidity is present at the submanifold level in this geometry.

1.2. **Fat manifolds and prelegendrians.** Let M^n be a smooth manifold. The space of co-oriented hyperplanes in M is $\mathbb{S}T^*M$, while the space of hyperplanes in M is $\mathbb{P}T^*M$. These manifolds carry canonical contact structures $(\mathbb{S}T^*M, \xi_{\mathrm{can}})$ and $(\mathbb{P}T^*M, \xi_{\mathrm{can}})$ known as the space of co-oriented contact elements, and contact elements. These contact manifolds are typical examples of *Legendrian fibrations* and fit in a commuting diagram

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[rr] \arrow[d, hook] & & \mathbb{RP}^{n-1} \arrow[d, hook] \\
(\mathbb{S}T^*M, \xi_{\mathrm{can}}) \arrow[rr] \arrow[dr] & & (\mathbb{P}T^*M, \xi_{\mathrm{can}}) \arrow[dl] \\
& M &
\end{tikzcd}$$

Here, the map $(\mathbb{S}T^*M, \xi_{\mathrm{can}}) \to (\mathbb{P}T^*M, \xi_{\mathrm{can}})$ is a contact double cover induced by the antipodal map on each Legendrian fibre.

Let $\mathcal{D}$ be a distribution on M. The *annihilator* of $\mathcal{D}$ is the subbundle

$$\mathcal{D}^\perp \coloneqq \{\alpha \in T^*M : \alpha_{|\mathcal{D}} = 0\} \subseteq T^*M.$$

The space of co-oriented hyperplanes on M containing $\mathcal{D}$ is denoted by $\mathbb{S}\mathcal{D}^\perp \subseteq \mathbb{S}T^*M$, and the space of hyperplanes containing $\mathcal{D}$ by $\mathbb{P}\mathcal{D}^\perp \subseteq \mathbb{P}T^*M$.

In this article, we are interested in the following geometric class of maximally non-integrable distributions that generalize contact structures to higher corank:

Definition 1.1. Let $\mathcal{D}$ be a distribution in a smooth manifold M. We say that $\mathcal{D}$ is *fat*, and $(M, \mathcal{D})$ is a *fat manifold*, if one of the following equivalent conditions holds:

- For every local non-zero section $\alpha : U \to \mathcal{D}^\perp$, $\mathrm{d}\alpha|_{\mathcal{D}}$ is a symplectic form on $\mathcal{D}$.
- $\mathcal{C}(M, \mathcal{D}) \coloneqq (\mathbb{S}\mathcal{D}^\perp, \xi_{\mathrm{can}}) \subseteq (\mathbb{S}T^*M, \xi_{\mathrm{can}})$ is a contact submanifold, called the *contactisation* of $(M, \mathcal{D})$.
- $\mathcal{C}_{\mathbb{P}}(M, \mathcal{D}) \coloneqq (\mathbb{P}\mathcal{D}^\perp, \xi_{\mathrm{can}}) \subseteq (\mathbb{P}T^*M, \xi_{\mathrm{can}})$, is a contact submanifold, called the *projectivised contactisation* of $(M, \mathcal{D})$.
- $\mathcal{S}(M, \mathcal{D}) \coloneqq (\mathcal{D}^\perp \setminus 0_M, \mathrm{d}\lambda_{\mathrm{can}}) \subseteq (T^*M \setminus 0_M, \mathrm{d}\lambda_{\mathrm{can}})$ is a symplectic submanifold, called the *symplectisation* of $(M, \mathcal{D})$. △

The equivalence between all these conditions will be proved later in the manuscript, but is a well-known fact; see [Mon02].

From the definition it follows that the rank of $\mathcal{D}$ is $2n$ even, and that both contactisations fit in a commuting diagram of *isotropic fibrations*

$$\begin{tikzcd}
\mathbb{S}^{k-1} \arrow[r] \arrow[d, hook] & \mathbb{RP}^{k-1} \arrow[d, hook] \\
\mathcal{C}(M,\mathcal{D}) \arrow[r] \arrow[dr] & \mathcal{C}_{\mathbb{P}}(M,\mathcal{D}) \arrow[d] \\
& (M,\mathcal{D})
\end{tikzcd}$$

where k is the corank of $\mathcal{D}$. The map $\mathcal{C}(M,\mathcal{D}) \to \mathcal{C}_{\mathbb{P}}(M,D)$ is a contact double cover induced by the antipodal map on the isotropic fibres.

Remark 1.2. For a distribution ξ of corank 1 on a manifold M, being fat is the same as being contact. In this case, $\mathcal{C}_{\mathbb{P}}(M,\xi) = (M,\xi)$. $\mathcal{C}(M,\xi)$ is the co-orientable double cover of (M,ξ), which is disconnected if (M,ξ) is co-orientable. $\mathcal{S}(M,\xi)$ is the (complete)[1] symplectisation of (M,ξ). △

Remark 1.3. In the degenerate case $(M,\mathcal{D}) = (M,\{0\})$, we recover the space of co-oriented contact elements, resp. contact elements, as the contactisation, resp. projectivised contactisation, of $(M,\{0\})$. △

It follows that the contactisations of a fat manifold $(M,\mathcal{D})$ are compact whenever M is compact. In particular, *Gray stability* and its variants, such as the contact isotopy extension theorem, apply to them, yielding effective fat invariants:

Lemma 1.4. *Let $(M,\mathcal{D})$ be a compact fat manifold. The contactomorphism class of $\mathcal{C}(M,\mathcal{D})$ and $\mathcal{C}_{\mathbb{P}}(M,\mathcal{D})$ is invariant under homotopies of the underlying fat structure.*

This indicates that it is reasonable to pursue the use of symplectic and contact invariants—e.g., those arising from pseudoholomorphic curves—in order to establish *rigidity phenomena* for fat manifolds. In this article, we successfully follow this program to find rigidity for a certain class submanifolds of fat manifolds, called prelegendrians:

Definition 1.5. Let $(M^{2n+k},\mathcal{D}^{2n})$ be a fat manifold. A $(n+k-1)$-submanifold $\Lambda^{n+k-1} \subseteq (M,\mathcal{D})$ is said to be *prelegendrian* if $\dim T\Lambda \cap \mathcal{D} = n$. △

Sometimes, when dealing with a prelegendrian submanifold Λ, we will abuse notation and treat it as an embedding by fixing some inclusion. This will be clear from context.

Remark 1.6. A generic Legendrian germ $L \subseteq \mathcal{C}(M,\mathcal{D})$ projects to a germ of a prelegendrian in $(M,\mathcal{D})$. This follows from the fact that a generic germ is transverse to the projection to $(M,\mathcal{D})$. △

Remark 1.7. Prelegendrian submanifolds of contact manifolds are precisely Legendrian submanifolds. △

The terminology *prelegendrian* is justified by the following lifting property (Proposition 2.13). Let $\Lambda : S \hookrightarrow (M^{2n+k},\mathcal{D}^{2n})$ be a prelegendrian embedding. Then, there exists a unique[2] Legendrian embedding $\mathcal{L}(\Lambda) : N \hookrightarrow \mathcal{C}(M,\mathcal{D})$ making the following diagram commute:

[1] In contact topology it is customary to assume that $\xi = \ker\alpha$ is co-oriented. The choice of contact form α defines a splitting $\mathcal{S}(M,\xi) \cong (M \times \mathbb{R}^*, d(t\alpha))$, and it is common to treat only the positive half $(M \times (0,+\infty), d(t\alpha))$ as the symplectisation of (M,ξ).

[2] Being precise one should fix a *prelegendrian co-orientation.* We will discuss this in detail later in the article. However, for all the prelegendrians treated in this article there will be always an obvious prelegendrian co-orientation.

$$\begin{tikzcd} & \mathcal{C}(M,\mathcal{D}) \arrow[d] \\ S \arrow[ur, "\mathcal{L}(\Lambda)"] \arrow[r, "\Lambda"] & (M,\mathcal{D}) \end{tikzcd}$$

An analogous property holds for the projective contactisation. We will refer to $\mathcal{L}(\Lambda)$ as *Legendrian lift.* The construction of the Legendrian lift is canonical. In particular, if $\mathcal{L}(\Lambda_0)$ and $\mathcal{L}(\Lambda_1)$ are not Legendrian isotopic, then Λ_0 and Λ_1 are not prelegendrian isotropic. The prelegendrian isotopy invariant that we will use to build our exotic prelegendrians is precisely the Legendrian lift.

Remark 1.8. The previously mentioned approach is somewhat reminiscent to the theory of Legendrian co-normal lifts to provide invariants of submanifolds, e.g. knot contact homology [OV00, Ng05, Ng14, She19]. In the degenerate case $(M,\{0\})$ prelegendrian submanifolds are just hypersurfaces, and its Legendrian lifts are obtained by treating them as *fronts* of Legendrians in $(\mathbb{S}T^*M,\xi_{\text{can}})$ and $(\mathbb{P}T^*M,\xi_{\text{can}})$, i.e. taking the co-normal lifts. △

1.3. **Main results.** Even though we will ultimately prove results for general fat manifolds of corank 2, many of our constructions deal with a concrete class thereof. Namely, the almost complex analogue of the space of (co-oriented) contact elements.

1.3.1. *Almost complex grasssmannians.* Let (X^{2n+2},J) be an almost complex manifold, with $n\geq 1$. The space of J-complex hyperplanes on (X,J) is denoted by $\mathrm{Gr}(X,J)$ and referred as the *complex Grassmannian* of (X,J). There is a natural structure of $\mathbb{CP}^n$-bundle $\pi_{\mathrm{Gr}}:\mathrm{Gr}(X,J)\to(X,J)$.

According to Proposition 4.5, there is a natural corank 2 fat distribution $\mathcal{D}_{\text{can}}$ on $\mathrm{Gr}(X,J)$, defined as

$$\mathcal{D}_{\text{can}}(W)\coloneqq(\,\mathrm{d}\pi_{\mathrm{Gr}})^{-1}(W),\qquad W\in\mathrm{Gr}(X,J).$$

The fat manifold $(\mathrm{Gr}(X,J),\mathcal{D}_{\text{can}})$ is referred to as the space of *J-complex contact elements* in (X,J).

The *standard fat distribution* $\mathcal{D}_{\text{std}}$ on $\mathbb{C}^{2n+1}$ is defined as follows. We consider $\mathrm{Gr}(\mathbb{C}^{n+1},J_{\text{std}})=\mathbb{C}^{n+1}\times\mathbb{CP}^n$ and restrict ourselves to those elements in $\mathbb{CP}^n$ representing hyperplanes graphical over $\mathbb{C}^n\times\{0\}$. This subspace can be canonically identified with $\mathbb{C}^{n+1}\times\mathbb{C}^n$ by considering the slopes of such a hyperplane. Identically, we are considering the complement of a constant hyperplane section in $\mathbb{C}^{n+1}\times\mathbb{CP}^n$ and thus passing to an affine chart in each fibre. The fat manifold $(\mathbb{C}^{2n+1},\mathcal{D}_{\text{std}})$ will be called the *standard (corank 2) fat space.*

By construction, the contactisation of $(\mathrm{Gr}(X,J),\mathcal{D}_{\text{can}})$ is the space of co-oriented contact elements $\mathcal{C}(\mathrm{Gr}(X,J),\mathcal{D}_{\text{can}})=(\mathbb{S}T^*X,\xi_{\text{can}})$, and there is a commuting diagram

$$\begin{tikzcd} \mathbb{S}^{2n+1} \arrow[rr] \arrow[d, hook] & & \mathbb{CP}^n \arrow[d, hook] \\ (\mathbb{S}T^*X,\xi_{\text{can}}) \arrow[rr, "\text{Hopf}"] \arrow[dr, "\pi_{\mathrm{F}}"'] & & (\mathrm{Gr}(X,J),\mathcal{D}_{\text{can}}) \arrow[dl, "\pi_{\mathrm{Gr}}"] \\ & (X,J) & \end{tikzcd}$$

where the *Hopf map* $(\mathbb{S}T^*X,\xi_{\text{can}})\to(\mathrm{Gr}(X,J),\mathcal{D}_{\text{can}})$ is defined by taking the Hopf fibration in each fibre.

1.3.2. *Prelegendrian fronts.* Let $\Lambda \subseteq (\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ be a prelegendrian submanifold. Its *prelegendrian front* is the singular hypersurface $\pi_{\mathrm{Gr}}(\Lambda) \subseteq (X,J)$. It follows from the construction that the prelegendrian front is nothing but the front of the Legendrian lift:

$$\pi_{\mathrm{Gr}}(\Lambda) = \pi_{\mathrm{F}}(\mathcal{L}(\Lambda)).$$

Our first contribution is the characterisation of a class of prelegendrian fronts among all legendrian fronts:

Theorem 1.9. *Let $\mathcal{F} \subseteq (X,J)$ be a front with only self-intersections and cusp singularities. Then, $\mathcal{F}$ is a prelegendrian front if and only if its singularity loci are* co-real *in (X,J).*

This structural result allows for the study prelegendrians from a topological perspective, via their fronts. In particular, it allows us to prove that the class of compact prelegendrians $\Lambda \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{can}})$ is non-empty, which is something not obvious just from the definitions.

1.3.3. *Co-real spinning.* In view of Theorem 1.9, finding prelegendrians amounts to finding compact Legendrian fronts in $\mathcal{F} \subseteq \mathbb{C}^{n+1} = J^0\mathbb{R}^{2n+1}$ whose singularity loci are co-real. To achieve this, we will implement a (co-real) version of the front spinning construction introduced by Ekholm, Etnyre and Sullivan [EES05b] (see also [EES09, Gol14, LC13]).

Co-real spinning is surprisingly flexible and will allow us to go beyond the existence of compact prelegendrians. Indeed, the main result of this manuscript reads:

Theorem 1.10. *There exist infinitely many prelegendrian embeddings*

$$\Lambda_s : \mathbb{T}^{2n+1} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}}), \qquad s \in \mathbb{Z}_{>0},$$

all formally prelegendrian isotopic. These prelegendrians are pairwise not prelegendrian isotopic, since their Legendrian lifts $\mathcal{L}(\Lambda_s)$ to the contactisation $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ are pairwise not Legendrian isotopic.

The Legendrian lifts $\mathcal{L}(\Lambda_s)$ are distinguished by their Legendrian Contact Homology [Che02, EES05a, EES07, EGH00, Eli98]. That is, the h-principle fails for prelegendrian submanifolds in corank 2 fat distributions, and this failure can be detected by means of symplectic and contact invariants. To the best of our knowledge this is the first instance of rigidity in the study of maximally non-integrable distributions beyond the contact case.

Remark 1.11. In fact, we prove more: there exist infinitely many corank 2 fat manifolds that have infinitely many prelegendrians that are pairwise not prelegendrian isotopic but are formally prelegendrian isotopic. See Remark 11.5 for further details. △

1.3.4. *Prelegendrian N-pushing.* To realize the formal prelegendrian isotopy between the tori from Theorem 1.10, we will define an *N-pushing operation* along a co-real submanifold N of a prelegendrian front. This is an adaptation of the analogous Legendrian construction treated in [Eli90, EES05b][3].

Recall that, given a Legendrian $L \subseteq (Y,\xi)$, with $\dim L \geq 2$, there is a notion of stabilized Legendrian $s(L)$, which is loose in the sense of Murphy, and formally equivalent to L, see [Eli90, EES05b, Mur19]. This is obtained by performing an N-pushing front move along a submanifold with $\chi(N) = 0$ near a cusp chart.

[3]In [EES05b] this operation is referred as stabilization, but we preferred to call it N-pushing and leave the word "stabilization" for the actual front move producing loose Legendrians, i.e. when is performed near a cusp chart.

It follows that a by-product of introducing prelegendrian N-pushing is that we can establish a stabilization procedure for prelegendrians in $\mathrm{Gr}(X, J)$. This is the second main result of this article:

Theorem 1.12. *Let $\Lambda \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ be a prelegendrian submanifold. Then, there exists a prelegendrian $s(\Lambda) \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$, such that*

- *Λ and $s(\Lambda)$ are formally prelegendrian isotopic by a C^0-small formal isotopy.*
- *Their Legendrian lifts satisfy $\mathcal{L}(s(\Lambda)) \simeq s(\mathcal{L}(\Lambda))$.*

We refer to $s(\Lambda)$ as a prelegendrian stabilization *of Λ.*

Remark 1.13. The construction in Theorem 1.12 is a prelegendrian front-move and can be performed locally: one can arrange that Λ and $s(\Lambda)$ differ only inside an arbitrarily small ball $\mathcal{O}p(p)$ around a point $p \in \Lambda$. △

This construction suggests that there could be a class of loose prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{can}})$ that is governed by an h-principle. Investigating this question will be part of future work of the authors.

On the other side, the prelegendrian stabilization gives a clear procedure to find exotic pairs of prelegendrians: build prelegendrians Λ so that $\mathcal{L}(\Lambda)$ is non-loose. This follows the same line of reasoning as in Theorem 1.10, since proving the non-looseness of a Legendrian is a non-trivial task that usually requires input from pseudoholomorphic curve theory (or its variants in generating functions, microlocal sheaf theory, etc).

Corollary 1.14. *There exists a prelegendrian embedding*

$$\Lambda : \mathbb{T}^{n+1} \times \mathbb{S}^{j_0} \times \cdots \times \mathbb{S}^{j_k} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{can}}),$$

where $j_0 + \cdots + j_k = n$, such that $\mathcal{L}(\Lambda) \subseteq \mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ is non-loose. In particular, Λ and $s(\Lambda)$ are formally prelegendrian isotopic but not prelegendrian isotopic.

1.3.5. *Conormal lifts.* We were unable to find exotic pairs of prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{can}})$ whose underlying smooth manifold is not a (non-trivial) product. However, we can achieve this in the more general case of the space of complex contact elements $(\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$. A natural prelegendrian submanifold is the one associated with a smooth real co-oriented hypersurface $M \subseteq (X, J)$, namely

$$\Lambda_M = (M, TM \cap JTM) \subseteq (\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}}).$$

The Legendrian lift of $\mathcal{L}(\Lambda_M) = (M, TM) \subseteq (\mathbb{S}T^*X, \xi_{\mathrm{can}})$ is the co-normal lift of the hypersurface M associated to the given co-orientation.

Therefore, our prelegendrian stabilization from Theorem 1.12 implies:

Corollary 1.15. *Let $Y \subseteq (X, J)$ be a codimension 0 compact submanifold with connected boundary $M = \partial Y$. Then, Λ_M and $s(\Lambda_M)$ are formally prelegendrian isotopic but not prelegendrian isotopic.*

Remark 1.16. If we apply Corollary 1.15 to the case in which (X, J) is $\mathbb{C}^{n+1}$, we will produce prelegendrians in $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ that are homologically essential and thus not contained in $\mathbb{C}^{2n+1}$. The reason is that M is likely to have a complex Gauss map $M \to \mathbb{CP}^n$ that is homotopically non-trivial. Finding prelegendrians contained in the standard fat $\mathbb{C}^{2n+1}$ was precisely the motivation behind Theorem 1.10. △

1.3.6. *General corank-2 fat distributions.* It is tempting to think of $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ as the *model* fat distribution of corank-2, but this is not correct. It turns out that there is no Darboux theorem in this setting. Nonetheless, $(\mathbb{C}^3, \mathcal{D}_{\mathrm{std}})$ serves as an *approximate* model when $n = 1$. This is explained in Section 12, where we show:

Theorem 1.17. *Fix a fat distribution $(M^6, \mathcal{D}^4)$. Then:*

- *It contains compact prelegendrians.*
- *The prelegendrians it contains can be stabilised.*

We also prove a similar (but weaker) statement in higher dimensions.

In a similar direction, consider the following problem: we fix a path of corank-2 fat distributions $(\mathcal{D}_s)_{s\in[0,1]}$, as well as a prelegendrian Λ for $\mathcal{D}_0$. There is no Gray stability in this setting, so there is no reason for a path of prelegendrians $(\Lambda_s)_{s\in[0,1]}$ in $\mathcal{D}_s$ to exist with $\Lambda_0 = \Lambda$. However:

Lemma 1.18. *If Λ is compact, a path Λ_s does exist for small s. In dimension 6 a path exists for all time if $\mathcal{D}_0 = \mathcal{D}_{\mathrm{std}}$.*

On the side of rigidity, we moreover obtain:

Theorem 1.19. *Let Λ and Λ' be two distinct prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$, as produced by Theorem 1.10 or Corollary 1.14. Then there is no path $(\mathcal{D}_s, \Lambda_s)_{s\in[0,1]}$ of corank-2 fat distributions and prelegendrians such that:*

- $\mathcal{D}_0 = \mathcal{D}_1 = \mathcal{D}_{\mathrm{std}}$.
- *$\mathcal{D}_s$ agrees with $\mathcal{D}_{\mathrm{std}}$ outside of a compact.*
- *$\Lambda_0 = \Lambda$ and $\Lambda_1 = \Lambda'$.*

That is, rigidity persists even if we allow homotopies of the fat structures themselves.

1.4. Open questions.

We finish the introduction with a set of open questions about fat manifolds and prelegendrians, that should be relevant for the further development of the theory.

Question 1.20. Does there exist a prelegendrian embedding $\Lambda : \mathbb{S}^{2n+1} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ of the sphere? △

In this article, we build prelegendrians by the spinning construction. In particular, our prelegendrians are topologically a product. It could be that understanding higher order singularities of prelegendrian fronts is required to answer this question.

In a different direction:

Question 1.21. Does the h-principle fail for prelegendrian embeddings in corank $k > 2$ fat manifolds? △

We are currently working on the case $k = 4$, using ideas and constructions similar to the ones presented here. For other coranks we still lack a geometric understanding of how fat structures look (making it difficult to manipulate prelegendrians topologically).

Analogously to the case of contact structures, there is a class of subcritical preisotropic submanifolds in fat manifolds (see Definitions 2.8 and 2.12). The h-principle for subcritical preisotropic immersions, and prelegendrian immersions of open manifolds, has been established by Bhowmick, building on Gromov's work [Bho22, Gro96]. The following question is natural:

Question 1.22. Does the h-principle hold for subcritical preisotropic embeddings and prelegendrian immersions in corank $k > 1$ fat manifolds? $\triangle$

The case $k = 1$ corresponds to the usual contact case in which the answer is known to be true by work of Gromov [CEM24, Gro86].

Our prelegendrian stabilization from Theorem 1.12 naturally leads to the following:

Question 1.23. Does there exist a class of *loose* prelegendrians (containing our stabilized prelegendrians), that is governed by an h-principle? $\triangle$

If the answer to this question is negative, then it is not hard to see that the following question also has a negative answer:

Question 1.24. Is the isotopy class of the prelegendrian lift $\mathcal{L}(\Lambda)$ a complete prelegendrian invariant? Is it a complete prelegendrian invariant among the class of formally equivalent prelegendrians? $\triangle$

This question is a concrete incarnation of Eliashberg's "pseudoholomorphic curves or nothing" philosophy. The problem is also analogous to the fundamental question of whether the Legendrian isotopy class of the co-normal lift is a complete isotopy invariant. This is known to be true for knots in $\mathbb{R}^3$ by a celebrated result of Shende [ENS18, She19]. The study of singularities of Legendrian submanifolds with respect to an isotropic foliation [AG18] seems relevant in studying this problem.

We finish with the most fundamental question that motivates this work:

Question 1.25. Recall that fat distributions satisfy the h-principle on open manifolds, according to Gromov's h-principle for open and diff-invariant relations [CEM24, Gro86]. Then, focusing on closed manifolds:

(a) Does the h-principle fail for corank $k > 1$ fat distributions themselves?
(b) If the answer to (a) is positive, does there exist an *overtwisted* class of fat distributions?
(c) Is there a fat manifold with overtwisted contactisation? $\triangle$

We conjecture that the answer to (a) should be positive, at least for $k = 2$. This aligns with the conjecture of Martínez-Aguinaga and the second author about $(4, 6)$ fat distributions in [MAdP21]. A possible approach for problem (c) would be to define a "preovertwisted" object lifting to an overtwisted disc in the contactisation. Note moreover that all the fat manifolds appearing in this article (e.g. complex grassmannians) have tight contactisations (spaces of contact elements, which are fillable).

1.5. **Outline of the article.** The article is organized as follows. In Section 2 we introduce *fat manifolds* and *prelegendrians*. We prove all the basic properties of these, such as the existence of contactisations and Legendrian lifts. Finally, we introduce *formal prelegendrian embeddings*. Section 3 reviews co-real submanifolds in almost complex manifolds. These will play a crucial role in our constructions of prelegendrians. In Section 4 we introduce several examples of corank-2 fat manifolds, including the *space of complex contact elements*; we also determine their contactisations. This will be the main class of fat manifolds to be studied throughout the article. In Section 5 we give a quick recap of Legendrian fibrations and introduce *simple* fronts. Section 6 reviews the *stabilization* of higher dimensional Legendrians and *loose Legendrians*. In Section 7 we review the Legendrian front-spinning operation. In Section 8 we introduce the notion of *prelegendrian front*, and establish Theorem 1.9 and the existence of basic prelegendrian Reidemeister moves.

Afterwards, we define a *prelegendrian spinning* operation along certain co-real submanifolds as a method to produce prelegendrian fronts in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$. In Section 9 we adapt the Legendrian front operations from Section 6 to the prelegendrian setting. In Section 10 we prove the existence of special co-real submanifolds. By our prelegendrian spinning operation this implies the existence of compact prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$. These submanifolds are also used to prove the existence of prelegendrian stabilizations, i.e. Theorem 1.12. In Section 11 we combine the previous constructions to produce our exotic prelegendrians: the proofs of Theorem 1.10, Corollary 1.14 and Corollary 1.15 are provided. Our results about general fat manifolds of corank-2 are proven in Section 12.

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2. Fat manifolds

In this Section we introduce the main characters of the article: *fat manifolds* and *prelegendrian submanifolds.* We prove the key properties about them: existence of contactisations (Theorem 2.6) and Legendrian lifts (Proposition 2.13). We conclude the section by introducing formal prelegendrian embeddings, which is necessary to set up the h-principle problem.

2.1. **Fat Distributions.**

Let $\mathcal{D}^r \subseteq TM$ be a distribution of rank r on a smooth manifold M. The *Lie-bracket* of $\mathcal{D}$ with itself is the $C^\infty(M)$-module of vector fields

$$[\mathcal{D}, \mathcal{D}] := \{[u_1, u_2] \mid u_1, u_2 \in \Gamma(\mathcal{D})\}.$$

Throughout this article we will always assume that $\mathcal{D}$ is *regular,* meaning that the module $[\mathcal{D}, \mathcal{D}]$ has constant rank, hence defines a distribution that we denote by the same symbol.

The *curvature map* of $\mathcal{D}$ is the bundle morphism

$$\Omega : \mathcal{D} \wedge \mathcal{D} \to TM/\mathcal{D}, \quad (X, Y) \mapsto -[X, Y] \mod \mathcal{D}.$$

For each distribution $H \subseteq TM$ such that $\mathcal{D} \subseteq H$, there is also an associated H*-directional curvature*

$$\begin{aligned} \Omega_H : \mathcal{D} \times \mathcal{D} &\to TM/H \\ (X, Y) &\mapsto -[X, Y] \mod H. \end{aligned}$$

It is often convenient to dualize this picture. The annihilator of $\mathcal{D}$ is the subbundle

$$(TM/\mathcal{D})^* \simeq \mathcal{D}^\perp = \{\alpha \in T^*M : \alpha_{|\mathcal{D}} = 0\},$$

consisting of all covectors vanishing on $\mathcal{D}$. Dualizing Ω and using Cartan's formula, we obtain the *dual curvature* map

$$\Omega^* : \mathcal{D}^\perp \to \wedge^2 \mathcal{D}^*, \quad \alpha \mapsto -\alpha \circ \Omega = -\alpha \circ \Omega_{\ker \alpha} = \mathrm{d}\alpha|_{\mathcal{D}}.$$

This map encodes the same information as Ω, but from the perspective of 1-forms rather than vector fields.

The basic link between curvature and bracket-generating properties is well known:

Lemma 2.1. *Let $\mathcal{D}$ be a distribution on a manifold M. Then the following statements are equivalent:*

1. *$[\mathcal{D},\mathcal{D}] = TM$, i.e. $\mathcal{D}$ is bracket-generating of step 2.*
2. *The dual curvature Ω^* is a monomorphism.*
3. *The curvature Ω is an epimorphism.*

We are interested in a special subclass of step-2 distributions whose curvature exhibits maximal non-degeneracy.

Definition 2.2. A distribution $\mathcal{D}$ of corank k on a manifold M is said to be *fat* at a point $p \in M$ if, for every non-vanishing 1-form $\alpha \in \mathcal{D}^\perp$ at p, the 2-form $\Omega^*(\alpha)$ defines a symplectic form on $\mathcal{D}_p$.

A distribution $\mathcal{D}$ is *fat* if it is fat at every point $p \in M$. The pair $(M, \mathcal{D})$ is called a *fat manifold*, whenever $\mathcal{D}$ is fat. △

In other words, $(M, \mathcal{D})$ is a fat manifold if and only if for every (local) non-zero section of $\alpha : U \to \mathcal{D}^\perp$, the vector bundle $(\mathcal{D}_{|U}, d\alpha)$ is symplectic. In particular, fat distributions of corank 1 are precisely contact distributions. It follows that the rank of a fat distribution $\mathcal{D}$ is always an even number $2n$. We will usually write $(M^{2n+k}, \mathcal{D}^{2n})$ to denote fat manifolds of corank k.

Remark 2.3. A fat distribution $\mathcal{D}^{2n}$ is necessarily bracket-generating of step 2, by Lemma 2.1. Moreover, fat distributions are maximally non-integrable in the following sense: The distribution $\mathcal{D}^{2n}$ is fat if and only if for every hyperplane distribution $H \subseteq TM$ with $\mathcal{D} \subseteq H$, the H-directional curvature

$$\Omega_H : \mathcal{D} \wedge \mathcal{D} \to TM/H, (X, Y) \mapsto -[X, Y] \mod H,$$

is a symplectic form with values in the line bundle TM/H. In the case $H = \ker(\alpha)$, $\Omega_H = \Omega^*(\alpha)$. In particular, fat distributions came equipped, at least locally, with a $\mathbb{S}^{k-1}$-family of conformal symplectic structures. △

Inspired by [BD23], we adopt their construction and refer to it as the *fatization*.

Example 2.4. Let $(W^{2n}, \lambda_1, \dots, \lambda_k)$ be a smooth manifold W^{2n} equipped with $(k-1)$-sphere of Liouville forms, i.e. a k-tuple $\lambda_1, \dots, \lambda_k \in \Omega^1 W$ of 1-forms such that

1. $\mathrm{d}\lambda_i$ is a symplectic form for all $i = 1, \dots, k$.
2. For every $(a_1, \dots, a_k) \in \mathbb{S}^{k-1}$ the combination $a_1 \,\mathrm{d}\lambda_1 + a_2 d\lambda_2 + \cdots + a_k \,\mathrm{d}\lambda_k$ is symplectic.

The *fatization of* $(W^{2n}, \lambda_1, \dots, \lambda_k)$ *is the corank k fat manifold*

$$(W \times \mathbb{R}^k, \mathcal{D} = \ker(\mathrm{d}z_1 - \lambda_1) \cap \cdots \cap \ker(\mathrm{d}z_k - \lambda_k)),$$

where $(z_1, \dots, z_k)$ are the standard coordinates on $\mathbb{R}^k$. To produce compact examples one can moreover quotient $\mathbb{R}^k$ to a torus $\mathbb{T}^k$.

The fatization is a fat manifold. Indeed, every non-zero form $\alpha \in \mathcal{D}^\perp$, can be uniquely expressed as

$$\alpha = ta_1(\mathrm{d}z_1 - \lambda_1) + ta_2(\mathrm{d}z_2 - \lambda_2) + \cdots ta_2(\mathrm{d}z_k - \lambda_k), (t, (a_1, \dots, a_k)) \in (0, \infty) \times \mathbb{S}^{k-1}.$$

From here it follows that $\mathrm{d}\alpha = t(a_1 \,\mathrm{d}\lambda_1 + \cdots a_k \,\mathrm{d}\lambda_k)$ is symplectic on $\mathcal{D} \cong TW$.

A concrete example is $\mathbb{R}^4$. It admits a natural 2-sphere of Liouville forms coming from the complex structures i, j and k given by the unit quaternions. More generally, exact hyperkähler manifolds provide a natural source of examples.

In the non-exact setting, one can consider a k-tuple of symplectic forms such that any combination is symplectic. If all the forms in the tuple are of integral class, one can consider the associated $\mathbb{S}^1$-bundles. Taking their product produces a $\mathbb{T}^k$-bundle with fat connection. △

Beyond the contact case, i.e. corank 1, fatness is rare and highly constrained. The following result provides a precise set of numerical restrictions:

Theorem 2.5 (Rayner [Ray68])**.** *Let $\mathcal{D}^r$ be a distribution of positive rank r and corank k on a manifold M^{r+k}. If $\mathcal{D}^r$ is fat, then the following numerical constraints hold:*

- *r is even and $r \geq k+1$.*
- *If $k \geq 2$, then 4 divides r.*
- *The $(r-1)$-sphere S^{r-1} admits k-many linearly independent vector fields.*

Conversely, given any pair (r,k) satisfying the above conditions, there is a germ of fat distribution of rank r and corank k.

2.2. **Contactisation and symplectisation of a fat manifold.** Given a manifold M, we denote by λ_{can} the canonical Liouville 1-form in T^*M. The manifold T^*M carries a natural $\mathbb{R}^*$-action (resp. $\mathbb{R}^+$-action), induced by the associated Liouville vector field. The quotient of $T^*M \setminus 0$ under this action is the space $\mathbb{P}T^*M$ of hyperplanes in M (resp. the space $\mathbb{S}T^*M$ of co-oriented hyperplanes in M). Since the action conformally expands the symplectic form, it defines a canonical contact structure $(\mathbb{P}T^*M, \xi_{\mathrm{can}})$ known as *space of contact elements on M (*resp. $(\mathbb{S}T^*M, \xi_{\mathrm{can}})$ known as *space of co-oriented contact elements*).

Consider the bundle projections $\pi_{\mathbb{P}} : \mathbb{P}T^*M \to M$ and $\pi : \mathbb{S}T^*M \to M$. Then, the canonical contact structure on $\mathbb{P}T^*M$ is

$$\xi_{\mathrm{can}}(\alpha) = (\,\mathrm{d}_\alpha \pi_{\mathbb{P}})^{-1}(\ker \alpha), \quad \alpha \in \mathbb{P}T^*M.$$

The canonical contact structure on $\mathbb{S}T^*M$ is $\xi_{\mathrm{can}} = \ker \lambda_{\mathrm{can}}$, hence at a point $\alpha \in \mathbb{S}T^*M$ is also given by $\xi_{\mathrm{can}}(\alpha) = (\,\mathrm{d}_\alpha \pi)^{-1}(\ker \alpha)$. The canonical double covering $(\mathbb{S}T^*M, \xi_{\mathrm{can}}) \to (\mathbb{P}T^*M, \xi_{\mathrm{can}})$ is the co-oriented double contact cover of $(\mathbb{P}T^*M, \xi_{\mathrm{can}})$, and commutes with the bundle projections.

Fix a distribution $\mathcal{D} \subseteq TM$ on M of rank k. Its annihilator $\mathcal{D}^\perp \subseteq T^*M$ is preserved by the previous actions. Thus, we define the $\mathbb{RP}^{k-1}$-subbundle $\mathbb{P}\mathcal{D}^\perp \subseteq \mathbb{P}T^*M$ (resp. the $\mathbb{S}^{k-1}$-subbundle $\mathbb{S}\mathcal{D}^\perp \subseteq \mathbb{S}T^*M$). Notice that $\mathbb{P}\mathcal{D}^\perp$ (resp. $\mathbb{S}\mathcal{D}^\perp$) is nothing but the space of hyperplanes (resp. co-oriented hyperplanes) in M that contain $\mathcal{D}$. The bundle projections will be denoted as before.

The following characterisation of fatness is the key input that we will use in this article:

Theorem 2.6. *Let $\mathcal{D} \subseteq TM$ be a distribution on M. The following are equivalent:*

(1) $\mathcal{D}$ is fat.
(2) $\mathcal{D}^\perp \setminus 0_M \subseteq (T^*M, \mathrm{d}\lambda_{\mathrm{can}})$ is a symplectic submanifold.
(3) $\mathbb{P}(\mathcal{D}^\perp) \subseteq (\mathbb{P}T^*M, \xi_{\mathrm{can}})$ is a contact submanifold.
(4) $\mathbb{S}T^*M \subseteq (\mathbb{S}T^*M, \xi_{\mathrm{can}})$ is a contact contact submanifold.

Proof. The equivalence between (2), (3) and (4) is a standard fact in contact and symplectic geometry, c.f. [CEM24, Gei08]. We prove the equivalence between (1) and (2): $\mathcal{D}^\perp \setminus 0_M \subseteq T^*M$ is a symplectic if and only if $\mathcal{D}$ is fat.

Write $m = \dim(M)$, $k = \operatorname{corank}(\mathcal{D})$ and $r = \operatorname{rank}(\mathcal{D})$. Then, $\dim(\mathcal{D}^\perp) = m + k = r + 2k$ is even if and only if r is even. From now on, we will assume that $r = 2n$ is even, hence $\dim(\mathcal{D}^\perp) = 2(n+k)$.

Let $\mathbf{q} = (q_1, \dots, q_m)$ be a local coordinate chart on an open set $U \subseteq M$. Trivialize the distribution $\mathcal{D}$ on U by a collection of 1-forms $\{\alpha_1, \dots, \alpha_k\}$ such that:

$$\mathcal{D} \overset{\text{loc.}}{=} \ker \alpha_1 \cap \dots \cap \ker \alpha_k.$$

That is, the collection $\{\alpha_1, \dots, \alpha_k\}$ forms a local frame for $\mathcal{D}^\perp$. Denote the corresponding fibre coordinates by $(a_1, \dots, a_k)$.

Hence, $(\mathbf{q}; a_1, \dots, a_k)$ gives a local coordinates on $\mathcal{D}^\perp$. The canonical Liouville 1-form on T^*M restricts to:

$$\lambda = \sum_{i=1}^{k} a_i \alpha_i,$$

and its differential is:

$$\omega := \mathrm{d}\lambda = \mathrm{d}\Big(\sum_{i=1}^{k} a_i \alpha_i\Big) = \sum_{i=1}^{k} \Big(\mathrm{d}a_i \wedge \alpha_i + a_i\, \mathrm{d}\alpha_i\Big)$$

Since the 1-forms α_i are independent of the fibre coordinates a_i, the maximal wedge power of ω has the form:

$$\omega^{(n+k)} = c \bigwedge_{i=1}^{k} \big(\mathrm{d}a_i \wedge \alpha_i\big) \wedge \big(\sum_{i=1}^{k} a_i\, \mathrm{d}\alpha_i\big)^n. \tag{1}$$

Realize that $T\mathcal{D}^\perp \cong TM \oplus \mathrm{Vert} \cong \mathcal{D} \oplus TM/\mathcal{D} \oplus \mathrm{Vert}$. The first term in (1) is always volume form on the subspace $TM/\mathcal{D} \oplus \mathrm{Vert}$. Therefore, ω is symplectic in $\mathcal{D}^\perp \setminus 0_M$ if and only if the second term in (1) is a volume form on $\mathcal{D}^{2n}$ for all $(a_1, \dots, a_k) \neq (0, \dots, 0)$, which is precisely the definition of fatness. This concludes the proof. □

Definition 2.7. Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold.

(1) The *projectivised contactisation* of $(M, \mathcal{D})$ is $\mathcal{C}_{\mathbb{P}}(M, \mathcal{D}) := (\mathbb{P}\mathcal{D}^\perp, \xi_{\mathrm{can}})$,
(2) The (co-oriented) *contactisation* of $(M, \mathcal{D})$ is $\mathcal{C}(M, \mathcal{D}) := (\mathbb{S}\mathcal{D}^\perp, \xi_{\mathrm{can}})$,
(3) The *symplectisation* of $(M, \mathcal{D})$ is $\mathcal{S}(M, \mathcal{D}) := (\mathcal{D}^\perp \setminus 0_M,\, \mathrm{d}\lambda_{\mathrm{can}})$. △

2.3. **Preisotropic and prelegendrian submanifolds.** Let us introduce a class of submanifolds of fat manifolds that is specially adapted to the geometry:

Definition 2.8. Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold. A submanifold $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ is said to be *preisotropic* if

$$(T\Lambda + \mathcal{D})_{|\Lambda} \subseteq TM_{|\Lambda}$$

has constant corank 1. △

Equivalently, $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ is preisotropic if

$$\dim(T\Lambda \cap \mathcal{D}) = \dim(\Lambda) - k + 1.$$

A *preisotropic immersion* is a parametrization of a possibly immersed preisotropic submanifold, and a *preisotropic embedding* is a parametrization of an embedded preisotropic submanifold.

Remark 2.9. In the contact case $(M^{2n+1}, \mathcal{D}^{2n})$, preisotropic submanifolds are precisely isotropic submanifolds. △

Every preisotropic submanifold $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ carries a hyperplane distribution

$$H_\Lambda \coloneqq (T\Lambda + \mathcal{D})_{|\Lambda},$$

that we will refer to as *preisotropic hyperplane.* A *preisotropic co-orientation* is a co-orientation of H_Λ, i.e. an orientation of the line bundle $l_\Lambda \coloneqq TM_{|\Lambda}/H_\Lambda$. In the examples of preisotropics treated in this article there will be always an obvious preisotropic co-orientation unless otherwise stated.

Remark 2.10. In the contact case $(M^{2n+1}, \mathcal{D}^{2n})$, the preisotropic hyperplane is the contact structure, $H_\Lambda = \mathcal{D}_{|\Lambda}$. In particular, in co-oriented contact manifolds every preisotropic carries a natural preisotropic co-orientation. $\triangle$

Since $(M^{2n+k}, \mathcal{D}^{2n})$ is fat, the curvature of $\mathcal{D}$ along the preisotropic hyperplane H_Λ defines a symplectic form

$$\Omega_{H_\Lambda} : D_{|\Lambda} \wedge \mathcal{D}_{|\Lambda} \mapsto l_\Lambda$$

with values on the line bundle l_Λ.

Lemma 2.11. *Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold and $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ a preisotropic submanifold. Then, $T\Lambda \cap \mathcal{D}_{|\Lambda} \subseteq (\mathcal{D}_{|\Lambda}, \Omega_{H_\Lambda})$ is an isotropic sub-bundle. In particular,* $\mathrm{rk}(T\Lambda \cap \mathcal{D}) \leq n$*, equivalently,* $\dim(\Lambda) \leq n + k - 1$*.*

Proof. The computation is local so we may assume that Λ has a fixed preisotropic co-orientation, i.e $H_\Lambda = \ker \alpha$ for some 1-form α, and identify $\Omega_{H_\Lambda} = \mathrm{d}\alpha$. Since $\iota^*\alpha = 0$, for the inclusion map $\iota : \Lambda \hookrightarrow M$, we conclude that $\iota^* \mathrm{d}\alpha = 0$ on $T\Lambda \cap \mathcal{D}$, from which the result follows. $\square$

This motivates the following definition:

Definition 2.12. A preisotropic submanifold $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ is said to be *subcritical* if $\dim(\Lambda) < n + k - 1$. A *prelegendrian submanifold* is a preisotropic submanifold Λ of the maximal dimension $\dim(\Lambda) = n + k - 1$. $\triangle$

2.3.1. *Isotropic and Legendrian lifts to the contactisation.* The following lifting property justifies the terminologies *preisotropic* and *prelegendrian:*

Proposition 2.13. *Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold.*

- *If $u : S \to (M^{2n+k}, \mathcal{D}^{2n})$ is a preisotropic immersion, then there exists a unique isotropic immersion $\widetilde{u} : S \to \mathcal{C}_{\mathbb{P}}(M, \mathcal{D})$, that makes the following diagram commute:*

(2)
$$\begin{tikzcd} & \mathcal{C}_{\mathbb{P}}(M,\mathcal{D}) \arrow[d] \\ S \arrow[ur, dashed, "\hat{u}"] \arrow[r, "u"] & (M,\mathcal{D}) \end{tikzcd}$$

- *If $u : S \to (M^{2n+k}, \mathcal{D}^{2n})$ is a co-oriented preisotropic immersion, then there exists a unique isotropic immersion $\widetilde{u} : S \to \mathcal{C}(M, \mathcal{D})$, that makes the following diagram commute:*

(3)
$$\begin{tikzcd} & \mathcal{C}(M,\mathcal{D}) \arrow[d] \\ & \mathcal{C}_{\mathbb{P}}(M,\mathcal{D}) \arrow[d] \\ S \arrow[uur, dashed, "\tilde{u}"] \arrow[ur, "\hat{u}"] \arrow[r, "u"] & (M,\mathcal{D}) \end{tikzcd}$$

Moreover, if u is an embedding, then $\hat{u}$ and $\tilde{u}$ are also embeddings.

Proof. Recall that $\mathcal{C}_{\mathbb{P}}(M,\mathcal{D}) = (\mathbb{P}\mathcal{D}^{\perp},\xi_{\mathrm{can}}) \subseteq (\mathbb{P}T^*M,\xi_{\mathrm{can}})$ while $\mathcal{C}(M,\mathcal{D}) = (\mathbb{S}\mathcal{D}^{\perp},\xi_{\mathrm{can}}) \subseteq (\mathbb{S}T^*M,\xi_{\mathrm{can}})$. We denote the projections by $\pi_{\mathbb{P}} : \mathcal{C}_{\mathbb{P}}(M,\mathcal{D}) \to (M,\mathcal{D})$ and $\pi : \mathcal{C}(M,\mathcal{D}) \to (M,\mathcal{D})$ respectively.

For each $p \in S$, since u is preisotropic there is a unique 1-form $\hat{\alpha}_p \in \mathcal{D}^{\perp} = \mathrm{Ann}(\mathcal{D})$ defined up to $\mathbb{R}^*$-scaling, such that

$$\mathrm{d}u(T_pN) + \mathcal{D}_p = \ker\tilde{\alpha}_p.$$

In the case where u is co-oriented preisotropic, there is a unique $\tilde{\alpha}_p$, defined up to positive $\mathbb{R}^+$-scaling, that induces the given co-orientation satisfying the same property.

The lifts are defined by $\hat{u}(p) := [\hat{\alpha}_p]$ and $\tilde{u}(p) := [\tilde{\alpha}_p]$, respectively. Differentiating the identities $\pi_{\mathbb{P}} \circ \hat{u} = u$ and $\pi \circ \tilde{u} = u$, we obtain:

$$\mathrm{d}\pi_{\mathbb{P}}(\mathrm{Im}\ \mathrm{d}\hat{u}_p) = \mathrm{Im}\ \mathrm{d}u_p \subseteq \ker\hat{\alpha}_p$$

and

$$\mathrm{d}\pi(\mathrm{Im}\ \mathrm{d}\tilde{u}_p) = \mathrm{Im}\ \mathrm{d}u_p \subseteq \ker\tilde{\alpha}_p.$$

From here it follows that both maps are isotropic immersions. Indeed, the immersion property is obvious. For the isotropic property, realize that at the point $[\hat{\alpha}_p] \in \mathbb{P}\mathcal{D}^{\perp}$ the canonical contact structure is defined as $\xi_{\mathrm{can}}([\hat{\alpha}_p]) = d\pi_{\mathbb{P}}^{-1}\ker(\hat{\alpha}_p)$, the same argument applies in the co-oriented case. □

Definition 2.14. Let $(M^{2n+k},\mathcal{D}^{2n})$ be a fat manifold and $u : S \to (M,\mathcal{D})$ a (co-oriented) preisotropic immersion. The isotropics $\mathcal{I}_{\mathbb{P}}(u) := \hat{u} : S \to \mathcal{C}_{\mathbb{P}}(M,\mathcal{D})$ and $\mathcal{I}(u) := \tilde{u} : S \to \mathcal{C}(M,\mathcal{D})$, from the proposition above, are called the *projective isotropic lift* and the *isotropic lift* of u, respectively. △

In the prelegendrian case, we will use the notation $\mathcal{L}_{\mathbb{P}}(u) = \mathcal{I}_{\mathbb{P}}(u)$ and $\mathcal{L}(u) = \mathcal{I}(u)$ to denote the lifts, and we will refer to them as *Legendrian lifts.* When dealing with submanifolds (not maps) we will use the same notation to denote the lifts, i.e. if $\Lambda \subseteq (M,\mathcal{D})$ is preisotropic, resp. prelegendrian, we will write $\mathcal{I}(\Lambda)$, resp. $\mathcal{L}(\Lambda)$, to denote the lifts, and similarly for the projective ones.

The construction of the lift is continuous for families of preisotropics. In particular, the isotropic lifts produce effective invariants to distinguish preisotropics. For later use, we highlight the following:

Corollary 2.15. *Let $(M^{2n+k},\mathcal{D}^{2n})$ be a fat manifold, and $\Lambda_0,\Lambda_1 : S \hookrightarrow (M,\mathcal{D})$ two (co-oriented) prelegendrian embeddings. Then,*

- *If $\mathcal{L}_{\mathbb{P}}(\Lambda_0)$ and $\mathcal{L}_{\mathbb{P}}(\Lambda_1)$ are not Legendrian isotopic, then Λ_0 and Λ_1 are not prelegendrian isotopic.*
- *If $\mathcal{L}(\Lambda_0)$ and $\mathcal{L}(\Lambda_1)$ are not Legendrian isotopic, then Λ_0 and Λ_1 are not co-oriented prelegendrian isotopic.*

Remark 2.16. In practice, we will want to distinguish prelegendrians without the co-orientation information by their lift to the (co-oriented) contactisations. For this, one would need to distinguish the set of all possible co-oriented Legendrian lifts. When the prelegendrian is connected there are just two different lifts, and both are related by the co-orientation reversing contactomorphism of $\mathcal{C}(M,\mathcal{D})$ given the antipodal action in the $\mathbb{S}^{k-1}$-fibres. Therefore, it is enough to distinguish $\mathcal{L}(\Lambda_0)$ and $\mathcal{L}(\Lambda_1)$ by means of a Legendrian invariant preserved by this action, to conclude that Λ_0 and Λ_1 are not prelegendrian isotopic. All the Legendrian invariants treated in this article satisfy that property. △

2.4. **Formal preisotropics and prelegendrians.** The study of preisotropic immersions in fat distributions was first carried out by Gromov [Gro96], who referred to them as *partially horizontal immersions*, and was subsequently developed by Bhowmick [Bho22], who completed the h-principle analysis in certain cases via the Hamilton–Moser implicit function technique.

To state the h-principle and, at the same time, to prepare for the exotic prelegendrian examples to which we return later, we introduce the following notion, which encodes the formal (i.e. topological) data of being preisotropic.

Definition 2.17. Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold, and let S be a smooth manifold.

(1) A *formal preisotropic immersion* is a pair (f, F), such that:
- $f : S \to M$ is a smooth map,
- $F : TS \to TM$ is a bundle monomorphism covering f such that
  (a) $H_p = F(T_pS) + D_{f(p)}$ has corank 1, for all $p \in S$, and
  (b) $F(T_pS) \cap \mathcal{D}_{f(p)} \subseteq (\mathcal{D}_{f(p)}, \Omega_{H_p})$ is an isotropic subspace, for all $p \in S$.

(2) A *formal preisotropic embedding* is a pair (f, F_s) such that:
- $f : S \to M$ is an embedding,
- $F_s : TN \to TM$ with $s \in [0,1]$ is a 1-parameter family of bundle monomorphisms covering f such that $F_0 = \mathrm{d}f$,
- (f, F_1) is a formal preisotropic immersion.

(3) A *formal prelegendrian immersion (resp. embedding)* is a formal preisotropic immersion (resp. embedding) of dimension $n + k - 1$. △

Regarding preisotropic immersions there is the following h-principle:

Theorem 2.18. [Bho22, Theorem 5.3] *Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold and let S be a smooth manifold. Then, for any subcritical formal preisotropic immersion (f, F) from S to $(M, \mathcal{D})$, there exists a homotopy of formal preisotropic immersions (f_t, F_t) such that:*

- *$f_0 = f$ and $F_0 = F$.*
- *f_1 is a preisotropic immersion.*
- *$F_1 = \mathrm{d}f_1$.*

In the prelegendrian case, the same statement holds provided that S is an open manifold.

Remark 2.19. Recall that subcritical isotropic embeddings in contact manifolds satisfy an h-principle [CEM24], via the h-principle for isocontact embeddings in codimension ≥ 4. By contrast, Bhowmick establishes the result above using microflexibility and microextension arguments, which yield immersions rather than embeddings. By analogy with the rigid/flexible dichotomy in contact topology, we pursue the failure of the h-principle for prelegendrian embeddings. The h-principle problem for subcritical preisotropic embeddings and prelegendrian immersions remains to be explored. △

3. Co-real submanifolds in almost complex manifolds

In this section we recall basic facts about linear complex algebra, almost-complex manifolds, and co-real submanifolds that we will use in subsequent sections. We need them to 1) construct examples of corank-2 fat manifolds, and 2) build examples of compact prelegendrians in these fat manifolds. This section also serves to fix notation.

3.1. **Complex linear algebra.** Throughout this subsection we fix a real vector space V of dimension $2n$ endowed with a *complex structure* $J : V \to V$, thereby turning V into a complex vector space. There is a dual complex structure J^* on V^* defined as $J^*(\alpha) = \alpha \circ J$.

Definition 3.1. Given (V, J):

- A subspace $W \subseteq V$ is called J-invariant if $JW = W$.
- A $\mathbb{C}$-valued $\mathbb{R}$-linear map $\beta : V \to \mathbb{C}$ is called J-linear if $\beta \circ J = i\,\beta$.
- For a subspace $H \subseteq V$, its complex part is the J-invariant subspace $H^J := H \cap JH$.
- A subspace $S \subseteq V$ is called *real* if $S^J = \{0\}$.
- A subspace $S \subseteq V$ called *co-real* if $S + JS = V$; equivalently, the annihilator $S^\perp \subseteq V^*$ is real.
- The complexification of a linear map $\alpha : V \to \mathbb{R}$ is the J-linear map $\alpha^J := \alpha - i\,\alpha \circ J$. △

Note that if S is real then $\dim S \leq n$, while if S is co-real then $\dim S \geq n$.

The following lemma will be used later to build co-real submanifolds.

Lemma 3.2. *Let $S \subseteq V$ be co-real, and $H \subseteq V$ be any subspace. Then, $S + H$ is co-real.*

Proof. Since $S + JS = V$, we have

$$(S + H) + J(S + H) = S + JS + H + JH = V,$$

so $S + H$ is co-real. □

We denote the real and complex Grassmannians of V, respectively, by:

$$\mathrm{Gr}_k(V) := \{H \subseteq V : H \text{ is a } k\text{-dimensional subspace}\},$$
$$\mathrm{Gr}_{2k}(V, J) := \{W \subseteq V : W \text{ is } J\text{-invariant}\} \subseteq \mathrm{Gr}_{2k}(V).$$

Since we will mostly work with (real or complex) hyperplane Grassmannians, we will write $\mathrm{Gr}(V) := \mathrm{Gr}_{2n-1}(V)$ and $\mathrm{Gr}(V, J) = \mathrm{Gr}_{2n-2}(V, J)$, whenever there is no possible confusion. Sometimes it is more convenient to work with the dual picture, called Grassmannian duality:

Lemma 3.3. *There are canonical bijections:*

$$\mathrm{Gr}_k(V) \to \mathrm{Gr}_{2n-k}(V^*)$$
$$H \mapsto H^\perp,$$

and

$$\mathrm{Gr}_{2k}(V, J) \to \mathrm{Gr}_{2n-2k}(V^*, J^*)$$
$$W \mapsto W^\perp.$$

The *Hopf map* is defined as:

$$\mathrm{Hopf} : \mathrm{Gr}_{2n-1}(V) \to \mathrm{Gr}_{2n-2}(V, J)$$
$$H \mapsto H^J.$$

Equivalently, $\mathrm{Hopf}(H)$ is the maximal J-invariant subspace contained in H. This is well defined since every real hyperplane is co-real. Applying Grassmannian duality and identifying with (real/complex) projectivisation, we obtain the dual Hopf map:

$$\mathrm{Hopf}^* : \mathrm{Gr}_1(V^*) \simeq \mathbb{P}(V^*) \to \mathbb{P}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})) \simeq \mathrm{Gr}_2(V^*, J^*)$$
$$[\alpha]_{\mathbb{R}} \mapsto [\alpha^J]_{\mathbb{C}}$$

In the case $V = \mathbb{R}^{2n} \cong \mathbb{C}^n$ (i.e. $J = J_{\text{std}}$), the above dual Hopf map yields the standard Hopf fibration:

$$\begin{aligned}\pi_{\text{Hopf}} &: \mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}\\ [p_1 : \cdots : p_{2n}]_{\mathbb{R}} &\mapsto [p_1 + i\,p_2 : \cdots : p_{2n-1} + i\,p_{2n}]_{\mathbb{C}}\end{aligned}$$

With the notation above we recall the following linear-algebra lemma, that will be used later.

Lemma 3.4. *Let $W \in \mathrm{Gr}_{2n-2}(V)$ be a subspace of codimension 2, and let $\alpha_1, \alpha_2 \in V^*$ be linear maps with $W = \ker\alpha_1 \cap \ker\alpha_2$. Then W is J-invariant if and only if α_1^J and α_2^J are $\mathbb{C}$-linearly dependent in $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.*

Proof. By Grassmannian duality (Lemma 3.3), W is J-invariant if and only if the plane $\mathrm{span}_{\mathbb{R}}\{\alpha_1, \alpha_2\} \subseteq V^*$ is J^*-invariant (i.e. a complex line), which is equivalent to α_1^J and α_2^J being $\mathbb{C}$-linearly dependent. $\square$

3.2. **Co-real submanifolds.** Let (X, J) be an almost complex manifold of real dimension $2n$, i.e. $J : TX \to TX$ is a bundle isomorphism with $J_p : T_pX \to T_pX$ making T_pX a complex vector space for each $p \in X$.

Definition 3.5. An immersion $f : S \to (X, J)$ is called *co-real* if for every $q \in S$ the subspace $\mathrm{d}f_q(T_qS) \subseteq T_{f(q)}X$ is co-real (with respect to $J_{f(q)}$).

A submanifold $S \subseteq (X, J)$ is called *co-real* if $TS \subseteq TX$ is a co-real subbundle. △

The class of co-real immersions/embeddings fits into the framework of *directed* immersions/embeddings [Gro86, CEM24], and the corresponding open subset of the Grassmannian bundle is ample. Consequently, Gromov's convex integration technique yields the following h-principle:

Theorem 3.6 (Gromov)**.** *Co-real embeddings satisfy the C^0-dense, parametric and relative to the domain, h-principle.*

Next, we will state several results about co-real submanifolds. Some of them will seem a bit unmotivated or technical, but will become relevant in our construction of prelegendrian stabilizations and compact prelegendrians. The first one is a (formal) topological obstruction to the existence of co-real embeddings in $\mathbb{C}^n$:

Proposition 3.7 ([Wel69])**.** *If $N \subseteq \mathbb{C}^n$ is an oriented, co-real submanifold, then its Euler characteristic satisfies $\chi(N) = 0$.*

Proof. Since N is co-real in $\mathbb{C}^n$, there is a splitting

$$T\mathbb{C}^n|_N \simeq (TN)^J \oplus \eta \ \oplus \nu_{N\backslash\mathbb{C}^n},$$

with ν_N the normal bundle, $\eta \subseteq TN$ and $J(\eta) = \nu_N$. In particular,

$$TN \simeq (TN)^J \oplus \nu_N.$$

Hence $e(TN) = e((TN)^J) \cup e(\nu_N)$. Since $e(\nu_N) = 0$ (see [MS74]) it follows that:

$$\chi(N) = \langle e(TN), [N]\rangle = 0.$$

$\square$

The following result will be fundamental in our construction of exotic prelegendrians (Theorem 1.10) via prelegendrian spinning along a co-real torus. In particular, it will allow us to find the required torus:

Lemma 3.8. *Let (X, J) be an almost complex manifold such that $\pi : X \to N$ is a vector bundle over a compact manifold N. Assume that the zero section $0_N \subseteq X$ is co-real. Then,*

- *For every connection $\mathrm{Hor} \subseteq TX$ of X, then there exists a sufficiently small tubular neighbourhood U of 0_N such that $\mathrm{Hor}_{|U} \subseteq TX_{|U}$ is a co-real subbundle.*
- *Every submanifold $W \subseteq X$ fibred over N is fibred isotopic to a co-real submanifold.*

Proof. The first part is a consequence of the compactness of N and the fact that co-reality is an open condition, since $\mathrm{Hor}_p = T_pN$ for $p \in N \equiv 0_N$.

For the second part, fix a connection Hor of X that restricts to a connection of W, and consider the dilation $\delta_t : X \to X$, $t > 0$, along the fibre directions. It follows from the previous part and Lemma 3.2 that $\delta_t(W)$ is co-real for $t > 0$ sufficiently small. □

3.2.1. *Sections of the complex tangencies.* The following result will have a relevant role in proving that prelegendrian stabilization does not change the formal prelegendrian isotopy class. The N appearing in the statement will later be the co-real submanifold along which we perform the stabilization and $\mathcal{F}$ will be a prelegendrian front.

Proposition 3.9. *Let $N \subseteq \mathcal{F} \subseteq (X, J)$, where $\mathcal{F}$ is an oriented hypersurface and N is a co-real submanifold, and suppose the normal bundle $\nu_{N\setminus\mathcal{F}}$ admits a nowhere-vanishing section. Set $M := \overline{\mathcal{O}p_{\mathcal{F}}(N)} \subseteq \mathcal{F}$. Then every nowhere-vanishing, inward-pointing section $Z \in \Gamma(TM^J_{|\partial M})$ extends to a nowhere-vanishing section $\tilde{Z} \in \Gamma(TM^J)$.*

Proof. Consider the tubular projection $\pi : M \to N$, together with a connection $\mathrm{Hor} \subseteq TM$. Define $S := \mathrm{Hor}$. Notice that since $N \subseteq X$ is co-real we may assume that $S \subseteq TX_{|M}$ is a co-real subbundle after possibly shrinking M. Define $F := TM \subseteq E := TX|_M$, so we have a flag of vector bundles $S \subseteq F \subseteq E$ over M.

Observe that

$$F = TM \cong S \oplus \mathrm{Vert} \cong S \oplus \pi^*\nu_{N\setminus\mathcal{F}}.$$

Since $\nu_{N\setminus\mathcal{F}} \to N$ has a nowhere vanishing section by hypothesis, we conclude that F/S has a nowhere vanishing section. Therefore, we are in the situation of Lemma 3.10 (below) and there exists a non-zero section $s : M \to F^J \cap S = TM^J \cap \mathrm{Hor}$.

Finally, we will interpolate between the "almost vertical" section Z and s near ∂M to conclude. First, extend the boundary section $Z : \partial M \to TM^J_{|\partial M}$ to a collar neighbourhood, and denote the extension by $\hat{Z} : \partial M \times [0, \varepsilon) \to TM^J_{|M\times[0,\varepsilon)}$. Since Z is inward-pointing, we may assume that $\hat{Z}$ is nowhere zero neither horizontal.

Then define

$$\tilde{Z}(p) = (1 - \beta(p))s(p) + \beta(p)\hat{Z}(p), \quad \text{for } p \in M,$$

where $\beta : M \to [0, 1]$ is a bump function supports on $\partial M \times [0, \varepsilon)$, and $\beta|_{\partial M\times[0,\varepsilon/2]} \equiv 1$. Then $\tilde{Z} \in \Gamma(TM^J)$ is the required extension of Z. Indeed, it is nowhere zero because $\hat{Z}(p)$ is not horizontal while $s(p) \in \mathrm{Hor}$. □

We now prove the claimed auxiliary result.

Lemma 3.10. *Let $S \subseteq F \subseteq E$ be a flag of vector bundles over a manifold M such that*

- *$E \to M$ is a complex vector bundle, with fiberwise complex structure $J : E \to E$,*

- *$F \subseteq E$ has real corank one,*
- *$S \subseteq E$ is a co-real subbundle, and*
- *The normal bundle F/S admits a nowhere zero section.*

Then, there exists a nowhere zero section of $F^J \cap S$ over M.

Proof. Since $S \subseteq E$ is a co-real subbundle there is an isomorphism of vector bundles:

$$E \simeq S^J \oplus J(E/S) \oplus E/S$$

with $J(E/S) \subseteq S$.

Moreover, $E/S \cong E/F \oplus F/S$ hence, the previous isomorphism gives

$$E \simeq S^J \oplus J(F/S) \oplus J(E/F) \oplus F/S \oplus E/F.$$

On the other hand, since $F \subseteq E$ has corank 1, it is automatically co-real, hence

$$E \simeq F^J \oplus J(E/F) \oplus E/F$$

Taking the quotient by the complex line $J(E/F) \oplus E/F$ in the both expressions, we conclude that

$$F^J \simeq S^J \oplus J(F/S) \oplus F/S,$$

with $J(F/S) \subseteq S$, from which the result follows. □

4. Examples of corank 2 Fat Distributions

In this section we present examples of corank-2 fat distributions. The most relevant one is the *space of complex contact elements* over an almost complex manifold. We will show that the contactisations of the later are canonically contactomorphic to the spaces of contact elements of the underlying almost complex manifold.

4.1. An alternative characterisation of fatness for corank 2.

To introduce the examples of corank 2 fat distributions, we will need an alternative characterisation of of fatness in corank 2, which can be found in [Bho24, Rut23]. We begin with an algebraic definition:

Definition 4.1. Let (ω_1, ω_2) be a pair of symplectic forms on a real vector space W. Then there exists a unique automorphism $A : W \to W$ such that:

$$\omega_1(w_1, w_2) = \omega_2(w_1, Aw_2), \quad w_1, w_2 \in W.$$

The automorphism $A : W \to W$ is called the *connecting isomorphism* of the pair (ω_1, ω_2). △

More precisely, the connecting isomorphism is defined by $A := I_1^{-1} \circ I_2$, where $I_i : W \to W^*$ is the isomorphism induced via $I_i(u) = \omega_i(u, -)$.

Let $\mathcal{D}^{2n}$ be a distribution on M^{2n+2}, then the dual curvature at each point $p \in M$ is

$$\Omega_p^* : \mathcal{D}_p^\perp \to \wedge^2 \mathcal{D}_p^*$$

Choosing a basis $\{e_1, e_2\}$ of $\mathcal{D}_p^\perp$, we obtain a pair of 2-forms (ω_1, ω_2) on $\mathcal{D}_p$ given by

$$\omega_i := \Omega_p^*(e_i), \quad i = 1, 2.$$

With the above notation, we have the following:

Lemma 4.2. *Let M^{2n+2} be a manifold equipped with a distribution $\mathcal{D}^{2n}$. Then the following are equivalent:*

(1) $\mathcal{D}$ is fat at the point $p \in M$.
(2) The pair of 2-forms (ω_1, ω_2) associated with any basis of $\mathcal{D}_p^\perp$, are symplectic on $\mathcal{D}_p$, and the corresponding connecting isomorphism A has no real eigenvalues.

Proof. ($\Longrightarrow$) Suppose that $\mathcal{D}$ is fat. Then ω_1 and ω_2 are symplectic by definition. If A has a real eigenvalue, there would exist $X \in \mathcal{D}_p$ and $\lambda \in \mathbb{R}$ with $AX = \lambda X$, hence

$$\omega_1(\cdot, X) = \omega_2(\cdot, AX) = \lambda\omega_2(\cdot, X),$$

The linear combination $\omega := \lambda\omega_1 - \omega_2$ would then be degenerate, contradicting fatness.

($\Longleftarrow$) Conversely, suppose some non-zero linear combination $\omega := c_1\omega_1 + c_2\omega_2$ is not symplectic. Then there exists $X \in \mathcal{D}$ such that:

$$0 = \omega(\cdot, X) = c_1\omega_1(\cdot, X) + c_2\omega_2(\cdot, X).$$

Without loss of generality, assuming $c_1 \neq 0$, we obtain

$$\omega_1(\cdot, X) = \omega_2(\cdot, \lambda X), \qquad \lambda := -\frac{c_2}{c_1},$$

which shows that A has a real eigenvalue. $\square$

4.2. **Holomorphic contact manifolds.**

Definition 4.3. Let M be a complex manifold of complex dimension $2n+1$. A holomorphic vector subbundle $\mathscr{L} \subseteq TM$ of complex corank 1 is called a *holomorphic contact structure*, if for every point $p \in M$, there exists an open neighbourhood $U \subseteq M$ and such that $\mathscr{L}|_U = \ker\Theta$ is defined by a holomorphic 1-form Θ satisfying:

$$\mathrm{d}\Theta \wedge (\Theta)^n \neq 0.$$

The pair $(M, \mathscr{L})$ is called a *holomorphic contact manifold*. $\triangle$

A fundamental example is $\mathbb{C}^{2n+1}$ with complex coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$, equipped with the *standard holomorphic contact form*:

$$\Theta_{\mathrm{std}} = \mathrm{d}z + \sum_{i=1}^{n} y_i \,\mathrm{d}x_i,$$

whose associated holomorphic contact structure is $\ker(\Theta_{\mathrm{std}})$.

The following classical result, which exhibits a family of fat distributions of corank 2, appears in [BD23] and [Rut23].

Lemma 4.4. *Let $(M, \mathscr{L})$ be a holomorphic contact manifold. We view M as a smooth manifold and $\mathscr{L}$ as a real distribution of corank 2, then $\mathscr{L}$ is a fat distribution of corank 2 on M.*

Proof. By the Holomorphic Darboux Theorem [AFL17, Theorem A.2], it suffices to prove the statement for $(\mathbb{C}^{2n+1}, \ker\Theta_{\mathrm{std}})$. Identify $\mathbb{C}^{2n+1}$ with $\mathbb{R}^{4n+2}$ and and write the associated real coordinates as

$$z = z_1 + iz_2, \quad x_j = x_{j1} + ix_{j2}, \quad y_j = y_{j1} + iy_{j2}.$$

Then, we can express the holomorphic contact form as $\Theta_{\mathrm{std}} = \lambda_1 + i\lambda_2$, where

$$\lambda_1 = \mathrm{d}z_1 + \sum_j (y_{j1}\,\mathrm{d}x_{j1} - y_{j2}\,\mathrm{d}x_{j2}), \quad \lambda_2 = \mathrm{d}z_2 + \sum_j (y_{j2}\,\mathrm{d}x_{j1} + y_{j1}\,\mathrm{d}x_{j2}).$$

It is straightforward to check that $\mathscr{L}$ is spanned by the vector fields

$$X_{j1} \coloneqq \partial_{x_{j1}} - y_{j1}\partial_{z_1} - y_{j2}\partial_{z_2}, \quad X_{j2} \coloneqq \partial_{x_{j2}} + y_{j2}\partial_{z_1} - y_{j1}\partial_{z_2}$$
$$Y_{j1} \coloneqq \partial_{y_{j1}}, \quad Y_{j2} \coloneqq \partial_{y_{j2}},$$

and we compute

$$\mathrm{d}\lambda_1 = \sum_j \big(\mathrm{d}y_{j1} \wedge \mathrm{d}x_{j1} - \mathrm{d}y_{j2} \wedge \mathrm{d}x_{j2} \big),$$
$$\mathrm{d}\lambda_2 = \sum_j \big(\mathrm{d}y_{j2} \wedge \mathrm{d}x_{j1} + \mathrm{d}y_{j1} \wedge \mathrm{d}x_{j2} \big).$$

Note that each summand of $\mathrm{d}\lambda_1$ and $\mathrm{d}\lambda_2$ depends only on the block $\{X_{j1}, X_{j2}, Y_{j1}, Y_{j2}\}$; hence a direct computation shows that

$$\mathrm{d}\lambda_1(u, v) = \mathrm{d}\lambda_2(u, Jv),$$

for all $u, v \in \mathscr{L}$, where J is denoted by the standard complex structure.

This shows that the connecting isomorphism $A : \mathscr{L} \to \mathscr{L}$ associated to $(\mathrm{d}\lambda_1, \mathrm{d}\lambda_2)$ is the restriction of J to $\mathscr{L}$. In particular, A has no real eigenvalues, and hence $\mathscr{L}$ is fat by Lemma 4.2. □

4.3. **The space of complex contact elements.** Another class of corank-2 fat distributions arises from the study of almost-complex grassmannians, generalising the space of contact elements to the almost-complex setting. The study of such structures in the holomorphic setting was carried out in [FL22].

Let (X, J) be an almost-complex manifold of real dimension $2n$. The *almost-complex Grassmannian* of complex codimension one is the fibre bundle

$$\pi : \mathrm{Gr}_{2n-2}(X, J) \to (X, J),$$

whose fibre over $x \in X$ is

$$\mathrm{Gr}_{2n-2}(X, J)_x = \mathrm{Gr}_{2n-2}(T_x X, J_x).$$

For brevity we write $\mathrm{Gr}(X, J)$ for $\mathrm{Gr}_{2n-2}(X, J)$ and refer to it as the *almost-complex Grassmannian* of (X, J). Analogously to the construction of the space of contact elements in contact geometry, we have:

Proposition 4.5. *Let (X, J) be an almost complex manifold. The distribution $\mathcal{D}_{\mathrm{can}} \subseteq T\mathrm{Gr}(X, J)$ defined by:*

$$\mathcal{D}_{\mathrm{can}}(W) \coloneqq \mathrm{d}\pi^{-1}(W), W \in \mathrm{Gr}(X, J),$$

is fat. The distribution $\mathcal{D}_{\mathrm{can}}$ is called the canonical fat distribution*, and the fat manifold* $(\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$ *is referred to as the* space of complex contact elements *over* (X, J)*.*

The thesis [Rut23] treats the case $\dim X = 4$. The general case can be established with an analogous computation. Instead of pursuing this route, we just remark that this will follow from the fact that its contactisation is the space of contact elements, which will be shown below.

It is straightforward to check that the canonical fat distribution is invariant under any diffeomorphism preserving the almost-complex structure. In particular, consider (X, J) an almost complex manifold and $g : X \to X$ a diffeomorphism. Then, g induces a pushforward almost complex structure $g_* J$ on X, and the map $\mathrm{Gr}(g) : \mathrm{Gr}(X, J) \to \mathrm{Gr}(X, g_* J)$ is a diffeomorphism that preserves $\mathcal{D}_{\mathrm{can}}$.

Remark 4.6. Fat structures of corank at least 2 have local invariants. The ones we are considering are rather special. Namely, there is a submersion $\pi_{\mathrm{Gr}} : (\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}}) \to (X,J)$ whose fibres are horizontal submanifolds (i.e., tangent to $\mathcal{D}_{\mathrm{can}}$). It is at the moment unknown whether a general corank-2 fat distribution admits such a submersion locally.

A certain local invariant of corank-2 fat distributions was introduced in [BD23]: the degree. The distributions $\mathcal{D}_{\mathrm{can}}$ have degree 2. A germ of fat distribution of type $(8,10)$ and degree different from 2 was constructed in [BD23].

We expect our constructions of prelegendrians to extend to arbitrary fat distributions of corank 2, but this remains an open question. △

4.3.1. *Contactisation of the space of complex contact elements.* The contactisation of the space of complex contact elements $(\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ has a particularly nice and geometric form:

Theorem 4.7. *Let (X,J) be an almost complex manifold. Then,*

- *The projectivised contactisation $\mathcal{C}_{\mathbb{P}}(\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ is canonically contactomorphic to the space of contact elements $(\mathbb{P}T^*X, \xi_{\mathrm{can}})$*
- *The contactisation $\mathcal{C}(\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ is canonically contactomorphic to the space co-oriented contact elements $(\mathbb{S}T^*X, \xi_{\mathrm{can}})$.*

Moreover, the following diagram

$$\begin{tikzcd}
\mathbb{RP}^1 \arrow[d, hook] & & \mathbb{S}^1 \arrow[ll] \arrow[d, hook] \\
(\mathbb{P}T^*X, \xi_{\mathrm{can}}) \arrow[dr, "\mathrm{Hopf}"'] & & (\mathbb{S}T^*X, \xi_{\mathrm{can}}) \arrow[ll] \arrow[dl, "\mathrm{Hopf}"] \\
& (\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}}) \arrow[d, "\pi_{\mathrm{Gr}}"] & \\
& (X,J) &
\end{tikzcd}$$

in which the diagonal maps are the fiberwise geometric Hopf maps described in the Section 3, commutes.

Proof. Recall that $\mathcal{C}(\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}}) = (\mathbb{P}(\mathcal{D}_{\mathrm{can}}^{\perp}), \xi_{\mathrm{can}})$ and denote by $\pi_{\mathrm{Gr}} : \mathrm{Gr}(X,J) \to (X,J)$ the Grassmannian bundle projection.

An element of $\mathbb{P}(\mathcal{D}_{\mathrm{can}}^{\perp})$ is a pair $((x,W);H)$ with $(x,W) \in \mathrm{Gr}(X,J)$ and $H \subsetneq T_{(x,W)}\mathrm{Gr}(X,J)$ a real hyperplane. Define

$$\begin{aligned}
\Phi : P(\mathcal{D}_{\mathrm{can}}^{\perp}) &\longrightarrow \mathbb{P}T^*X \\
((x,W);H) &\mapsto \tilde{H} \coloneqq (\,\mathrm{d}\pi_{\mathrm{Gr}})_{(x,W)}(H).
\end{aligned}$$

Then $\tilde{H}$ is a real hyperplane in T_xX with $W \subsetneq \tilde{H}$ (equivalently, $\tilde{H} \cap J\tilde{H} = W$), so Φ is a bundle map over (X,J). Choosing a connection on $\pi_{\mathrm{Gr}} : \mathrm{Gr}(X,J) \to (X,J)$ yields an inverse bundle diffeomorphism, and the resulting square commutes.

$$\begin{tikzcd}
\mathbb{P}(\mathcal{D}^{\perp}) \arrow[rr, "\Phi"] \arrow[d, "\pi_{\mathbb{P}}"'] & & \mathbb{P}T^*X \arrow[ddll, "\rho"] \\
\mathrm{Gr}(X, J) \arrow[d, "\pi_{\mathrm{Gr}}"'] & & \\
(X, J) & &
\end{tikzcd}
\qquad
\begin{tikzcd}
((x, W); H) \arrow[rr, mapsto] \arrow[d, mapsto] & & (x, \tilde{H}) \arrow[ddll, mapsto] \\
(x, W) \arrow[d, mapsto] & & \\
x & &
\end{tikzcd}$$

To see that Φ is a contactomorphism, differentiate the diagram. By definition of the canonical distribution ξ_{can} on $\mathbb{P}(\mathcal{D}^{\perp}_{\mathrm{can}})$,

$$v \in \xi_{\mathrm{can}}(H) \iff (\,\mathrm{d}\pi_{\mathbb{P}})_H(v) \in H \implies \mathrm{d}\pi_{\mathrm{Gr}}\big((\,\mathrm{d}\pi_P)_H(v)\big) \in \tilde{H}$$

where the last implication is by the definition $\tilde{H}$.

Since the diagram commutes, we have

$$\mathrm{d}\rho \circ \mathrm{d}\Psi(v) = \mathrm{d}\pi_{\mathrm{Gr}}\big((\,\mathrm{d}\pi_P)_H(v)\big) \in \tilde{H}$$

hence $\mathrm{d}\Psi(v) \in \xi_{\mathrm{can}}(\tilde{H}) = (d\rho)^{-1}(\tilde{H})$ by the definition of the tautological (Liouville) contact distribution. Thus Φ is a contactomorphism.

Finally, because Φ is bijective and $\tilde{H}$ contains the complex hyperplane W, the map

$$\pi_{\mathbb{P}} \circ \Phi^{-1} : \mathbb{P}T^*X \longrightarrow \mathrm{Gr}(X, J), \qquad \tilde{H} \longmapsto \tilde{H} \cap J\tilde{H},$$

is the fiberwise geometric Hopf map by construction. The same argument applies to the co-oriented contactisation $\mathcal{C}(\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$. $\square$

4.4. **Standard fat structure on $\mathbb{C}^{2n+1}$.** The previous constructions can be specialised to define the standard fat distribution on $\mathbb{C}^{2n+1}$.

Equip $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ with an almost complex structure J. After choosing a complex framing, we trivialise the almost-complex Grassmannian as

$$\mathrm{Gr}(\mathbb{R}^{2n+2}, J) \simeq \mathbb{R}^{2n+2} \times \mathbb{CP}^n.$$

Fix a complex hyperplane $H \simeq \mathbb{CP}^{n-1}$ in $\mathbb{CP}^n$, and identify $\mathbb{CP}^n \setminus H \simeq \mathbb{C}^n$. Thus we obtain an open subset

$$\mathbb{R}^{2n+2} \times \mathbb{C}^n \subseteq \mathbb{R}^{2n+2} \times \mathbb{CP}^n \simeq \mathrm{Gr}(\mathbb{R}^{2n+2}, J).$$

The canonical fat structure $\mathcal{D}_{\mathrm{can}}$ on $\mathrm{Gr}(\mathbb{R}^{2n+2}, J)$ restricts to a fat distribution on $\mathbb{R}^{2n+2} \times \mathbb{C}^n \simeq \mathbb{R}^{4n+2}$, called the *standard fat structure twisted by* J and denoted $\mathcal{D}_{\mathrm{std},J}$.

Note that the space of hyperplanes in $\mathbb{CP}^n$ is naturally identified with the dual projective space $(\mathbb{CP}^n)^{\vee} \cong \mathbb{CP}^n$, which is homogeneous. Hence different choices of hyperplane and framing produce homotopic fat distributions on $\mathbb{R}^{4n+2}$, so $\mathcal{D}_{\mathrm{std},J}$ is well defined up to homotopy.

In the case of $J \equiv J_{\mathrm{std}}$ (the standard complex structure on $\mathbb{C}^{n+1}$), the construction yields a fat distribution on $\mathbb{C}^{2n+1}$, called the *standard fat structure* and denoted $\mathcal{D}_{\mathrm{std}}$. Moreover

Proposition 4.8. *The standard fat structure $\mathcal{D}_{\mathrm{std}}$ on $\mathbb{C}^{2n+1}$ coincides with standard holomorphic contact structure* $\ker(\Theta_{\mathrm{std}})$.

Proof. Write the complex coordinates on $\mathbb{C}^{n+1}$ as $(x_1,\dots,x_n,z)$ as in the proof of Lemma 4.4. Then the canonical fat distribution $\mathcal{D}_{\mathrm{can}}$ on the complex Grammarian $\mathrm{Gr}(\mathbb{C}^{n+1}) = \mathbb{C}^{n+1}\times\mathbb{CP}^n$ is given by

$$\mathcal{D}_{\mathrm{can}}\big((x_1,\cdot,x_n,z);[w_1:\cdots:w_{n+1}]\big) = \ker(w_1\,\mathrm{d}x_1+\cdots+w_{n+1}\,\mathrm{d}z)\big).$$

Choosing the hyperplane $H \coloneqq \{w_{n+1}=0\}\subseteq\mathbb{CP}^n$, and affine coordinates $y_j \coloneqq \frac{w_j}{w_{n+1}}$, the standard fat distribution is

$$\mathcal{D}_{\mathrm{std}}\big((x_1,\dots,x_n,z);(y_1,\dots,y_n)\big) = \ker(y_1\,\mathrm{d}x_1+\cdots+y_n\,\mathrm{d}x_n+\,\mathrm{d}z),$$

which is $\ker\Theta_{\mathrm{std}}$, as required. □

Remark 4.9. The space of almost complex structures on $\mathbb{R}^{2n+2}$ (inducing the standard orientation) is connected, hence all standard fat distributions $\mathcal{D}_{\mathrm{std},J}$ twisted by J, are homotopic to $\mathcal{D}_{\mathrm{std}}$. △

Corollary 4.10. *The contactisation $\mathcal{C}(\mathbb{C}^{2n+1},\mathcal{D}_{\mathrm{std}})$ is contactomorphic to $(J^1(\mathbb{R}^{2n+1}\times\mathbb{S}^1),\xi_{\mathrm{std}})$.*

Proof. By definition, $(\mathbb{C}^{2n+1},\mathcal{D}_{\mathrm{std}})\subseteq\mathrm{Gr}(\mathbb{C}^{n+1},J_{\mathrm{std}})$ is obtained from $\mathrm{Gr}(\mathbb{C}^{n+1},J_{\mathrm{std}})$ by taking an affine chart and, by Theorem 4.7, the contactisation is

$$\mathbb{C}^{n+1}\times\mathbb{S}^{2n+1} = \ \mathbb{S}T^*\mathbb{C}^{n+1}\simeq\mathcal{C}(\mathrm{Gr}(\mathbb{C}^{n+1},J_{\mathrm{std}}))\to\mathrm{Gr}(\mathbb{C}^{n+1},J_{\mathrm{std}})=\mathbb{C}^{n+1}\times\mathbb{CP}^n.$$

with the canonical contact structure. Writing the complex coordinates on $\mathbb{C}^{n+1}$ as $(x,\dots,x_n,z)$, and the induced homogeneous coordinates on $\mathbb{CP}^n$, we get

$$\mathcal{C}(\mathbb{C}^{2n+1},\mathcal{D}_{\mathrm{std}}) = \mathbb{C}^{n+1}\times\mathrm{Hopf}^{-1}(\{z\neq 0\}) = \mathbb{C}^{n+1}\times S,$$

where $\mathrm{Hopf}:\mathbb{S}^{2n+1}\subseteq\mathbb{C}^{n+1}\to\mathbb{CP}^n$ is the standard Hopf map and $S\coloneqq\mathbb{S}^{2n+1}\setminus\{z=0\}$.

On the other hand, there is a contactomorphism

$$\begin{aligned}\mathbb{S}T^*\mathbb{R}^{2n+2}=\mathbb{R}^{2n+2}\times\mathbb{S}^{2n+1}&\to J^1(\mathbb{S}^{2n+1})=T^*\mathbb{S}^{2n+1}\times\mathbb{R}\\ (\mathbf{p},\mathbf{q})&\mapsto(\mathbf{q},\mathbf{p}-\langle\mathbf{q},\mathbf{p}\rangle\mathbf{q},\langle\mathbf{q},\mathbf{p}\rangle),\end{aligned}$$

that induces a contactomorphism

$$(\mathbb{C}^{n+1}\times S,\xi_{\mathrm{can}})\simeq(J^1(S),\xi_{\mathrm{std}}).$$

Since $S\simeq\mathbb{R}^{2n}\times\mathbb{S}^1$, the result follows. □

4.5. **Affine charts in the Grassmannian.**

It is sometimes useful to work inside an affine chart of the abstract Grassmannian, as in the construction of $(\mathbb{C}^{2n+1},\mathcal{D}_{\mathrm{std}})$. In general, we cannot do this globally (there may not exist a global section of $\mathrm{Gr}(X,J)\to X$), so we restrict to the product case $X=Q\times\mathbb{R}$.

4.5.1. *Contact case.* Let $X=Q\times\mathbb{R}$ with coordinate z on the second factor and *vertical vector field* ∂_z. Define the horizontal Grassmannian of real hyperplanes by

$$\mathbb{P}(T^*_{\mathrm{hor}}X)\coloneqq\{H\subseteq T_pX:\partial_z\notin H\}\subseteq\mathbb{P}T^*X.$$

Being an open submanifold (a fiberwise affine chart) of $\mathbb{P}T^*X$, it inherits the canonical contact structure ξ_{can}.

Similarly, for oriented hyperplanes we write $\mathbb{S}(T^*_{\mathrm{hor}}X)\subseteq\mathbb{S}T^*X$. We have the following standard identification:

Lemma 4.11. *There exists a canonical contactomorphism between $(\mathbb{S}T^*_{\mathrm{hor}}(Q\times\mathbb{R}),\xi_{\mathrm{can}})$ and $(J^1Q,\xi_{\mathrm{can}})$.*

Proof. An element of $\mathbb{S}T^*_{\mathrm{hor}}(Q\times\mathbb{R})$ is a pair $((q,z);\alpha)$ with $\alpha : T_qQ \oplus T_z\mathbb{R} \to \mathbb{R}$ and $\alpha(\partial_z) \neq 0$, hence uniquely of the form

$$\alpha = \mathrm{d}z - \lambda$$

for some $\lambda \in T^*_qQ$.

Then we define the diffeomorphism:

$$\begin{gathered}\mathbb{S}T^*_{\mathrm{hor}}(Q\times\mathbb{R}) \to J^1Q = T^*Q\times\mathbb{R}\\ ((q,z);\alpha) \mapsto ((q,\lambda);z).\end{gathered}$$

This is straightforward to check that this is a contactomorphism. □

4.5.2. *Almost complex case.* Now equip $X = Q\times\mathbb{R}$ with an almost complex structure J. The *horizontal complex Grassmannian* is

$$\mathrm{Gr}_{\mathrm{hor}}(X,J) \coloneqq \{K \subseteq T_xX : \partial_z \notin K \text{ and } K \text{ is } J\text{-invariant}\} \subseteq \mathrm{Gr}(X,J),$$

i.e. the fiberwise open locus of complex hyperplanes avoiding the vertical line. Equivalently, K is horizontal if and only if it does not contain the complex line $\mathrm{span}\langle\partial_z, J\partial_z\rangle$. Thus, when $\dim_{\mathbb{R}}(Q\times\mathbb{R}) = 2n$, the fibre of $\mathrm{Gr}_{\mathrm{hor}}(X,J)\to X$ is diffeomorphic to $\mathbb{C}^{n-1}$. The fiberwise Hopf map is well defined on both horizontal Grassmannians:

$$\mathrm{Hopf} : \mathbb{S}(T^*_{\mathrm{hor}}X) \longrightarrow \mathrm{Gr}_{\mathrm{hor}}(X,J).$$

In particular, there a natural inclusion

$$(\mathbb{S}(T^*_{\mathrm{hor}}X),\xi_{\mathrm{can}}) \hookrightarrow \mathcal{C}(\mathrm{Gr}_{\mathrm{hor}}(X,J),\mathcal{D}_{\mathrm{can}}) \tag{4}$$

Hence any prelegendrian $\Lambda \subseteq \mathrm{Gr}(X,J)$ whose Legendrian lift $\mathcal{L}(\Lambda)$ lies in $\mathbb{S}T^*X$ is automatically contained in $\mathrm{Gr}_{\mathrm{hor}}(X,J)$ whenever $\mathcal{L}(\Lambda)\subseteq \mathbb{S}T^*_{\mathrm{hor}}X$.

In the model case $Q = \mathbb{R}^{2n+1}$ and $X = \mathbb{R}^{2n+1}\times\mathbb{R}$ with J_{std}, the horizontal complex Grassmannian reproduces the standard structure:

$$\big(\mathrm{Gr}_{\mathrm{hor}}(\mathbb{C}^{n+1},J_{\mathrm{std}}),\mathcal{D}_{\mathrm{can}}\big) \cong \big(\mathbb{C}^{2n+1},\mathcal{D}_{\mathrm{std}}\big),$$

and we obtain the inclusion

$$\big(J^1\mathbb{R}^{2n+1},\xi_{\mathrm{can}}\big) \hookrightarrow \mathcal{C}\big(\mathbb{C}^{2n+1},\mathcal{D}_{\mathrm{std}}\big). \tag{5}$$

5. Legendrian Fronts

In view of Theorem 4.7, contactisations of fat complex Grassmannians are spaces of contact elements. These are examples of *Legendrian fibrations*, and admit canonical front projections, which can be used to study the Legendrians therein. In this section we provide a basic review of Legendrian fibrations and their fronts. We will define a subclass of fronts, which we call *simple*; this class will be specially relevant in our study of *prelegendrian fronts* in Section 8. The primary references are [AN01], [AGV12], and [Arn90].

5.1. **Legendrian fibrations.** We begin by recalling the notion of Legendrian fibrations and front projections.

Definition 5.1. Let (Y^{2n+1},ξ) be a $(2n+1)$-dimensional contact manifold. A fibre bundle

$$\pi_{\mathrm{F}} : (Y^{2n+1},\xi) \to B^{n+1}$$

over a manifold B is called a *Legendrian fibration* if all its fibres are Legendrian submanifolds of Y, i.e. if $\ker(d\pi_{\mathrm{F}}) \subseteq \xi$. The bundle map π_{F} is referred as *front projection.* △

Example 5.2. The following are examples of Legendrian fibrations:

- $\pi_{\mathrm{F}} : (J^1Q,\xi_{\mathrm{std}}) \to J^0Q$ with Q a smooth manifold.
- The concrete case $\pi_{\mathrm{F}} : (J^1\mathbb{R}^n,\xi_{\mathrm{std}}) \to J^0\mathbb{R}^n$ with $Q=\mathbb{R}^n$ is the *standard front projection.* Locally, every Legendrian fibration is contactomorphic to this one; see for instance [AN01, p. 72].
- The space of contact elements over a manifold X^n is a $\mathbb{RP}^{n-1}$-Legendrian fibration $\pi_{\mathbb{P}} : (\mathbb{P}T^*X,\xi_{\mathrm{can}}) \to X$. Similarly, the space of co-oriented contact elements is an $\mathbb{S}^{n-1}$-Legendrian fibration $\pi : (\mathbb{S}T^*X,\xi_{\mathrm{can}}) \to X$. △

Definition 5.3. A front map is a diagram consisting of an embedding of a Legendrian submanifold $\iota : L^n \hookrightarrow (Y^{2n+1},\xi)$ and a Legendrian fibration $\pi_{\mathrm{F}} : (Y^{2n+1},\xi) \to B^{n+1}$:

$$f : L^n \xhookrightarrow{\iota} (Y^{2n+1},\xi) \xrightarrow{\pi_{\mathrm{F}}} B^{n+1}.$$

By abuse of notation, we often refer to f itself as the front map. The image $\mathcal{F} \coloneqq f(L) \subseteq B^{n+1}$ is called the *wave front* (or simply the *front*) of L or f. △

Front maps are smooth maps from an n-dimensional manifold to an $(n+1)$-dimensional ambient manifold, forming a distinguished class of smooth maps with some special singularities. Roughly speaking, at each point $p \in L$, whether regular or singular, you still have a well-defined tangent map d_pf of full rank. Those tangent images record the original Legendrian directions and let you recover the submanifold. Locally, every front map can be described by means of the standard front projection $\pi_{\mathrm{F}} : (\mathbb{R}^{2n+1},\xi_{\mathrm{std}}) \to \mathbb{R}^{n+1}$. Conversely, from a front $\mathcal{F} \subseteq \mathbb{R}^{n+1}$ one can recover the $\mathbf{y}$-coordinates by taking the tangencies of $T\mathcal{F}$ [4], at the very least in the smooth points of $\mathcal{F}$.

5.2. **Front singularities and simple fronts.** Let $f : L^n \to B^{n+1}$ be a front map. We now study the singularities of the map f at a point $p \in L$. By singularity, we mean an equivalence class of front map germs. Since any front map can be locally described by the standard front projection, we always present germs of front map as $f : (\mathbb{R}^n,0) \to (\mathbb{R}^{n+1},0)$.

The possible singularities of a germ of a front are studied by means of generating functions or generating families of hypersurfaces [Arn90]. Although in the lower dimensional cases ($\dim L < 6$), the possible front singularities are stable and completely classified [AN01, AGV12], in full generality front singularities are intractable. In this article, we will focus on very *simple* singularities:

Definition 5.4. Let $f : L \xhookrightarrow{\iota} (Y,\xi) \xrightarrow{\pi_{\mathrm{F}}} B^{n+1}$ be the front map of a Legendrian submanifold L, and $\mathcal{F} \coloneqq f(L) \subseteq B^{n+1}$ the corresponding front. We say that f has a *cusp singularity* or A_2-singularity

[4] Here we identify a non-vertical oriented hyperplane $H \subseteq T\mathbb{R}^{n+1}$ with the unique 1-form $\mathrm{d}z - \mathbf{y}\,\mathrm{d}\mathbf{x}$ on $\mathbb{R}^{n+1}$ that annihilates it.

at $q \in L$, and that $p := f(q) \in \mathcal{F}$ is a *cusp point* of the front, if the germ of the front map $f|_{\mathcal{O}p(q)} : \mathcal{O}p(q) \to B^{n+1}$ is equivalent to

$$\hat{f} : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^n(\mathbf{x}) \times \mathbb{R}(z)$$
$$(q_1, \ldots, q_{n-1}, t) \mapsto \Big((q_1, \ldots, q_{n-1}, -3t); 2t^3\Big) \qquad \triangle$$

The class of (non-generic) fronts that we will work with is defined as follows:

Definition 5.5. A front $\mathcal{F} \subseteq B^{n+1}$ is said to be a *simple front* if there exists a stratification

$$\mathcal{F} = \Sigma_{A_1}(\Sigma) \sqcup \Sigma_{A_1A_1}(\Sigma) \sqcup \Sigma_{A_2}(\Sigma)$$

such that

- for all $p \in \Sigma_{A_1}(\mathcal{F})$, p is a smooth point of $\mathcal{F}$, i.e., $\mathcal{F}$ is a smooth hypersurface near p;
- for $p \in \Sigma_{A_1A_1}(\Sigma)$, $\mathcal{F}$ has a transversal self-intersection at p;
- for $p \in \Sigma_{A_2}(\Sigma)$, p is a cusp point of $\mathcal{F}$.

In this case, the sets $\Sigma_{A_1}(\mathcal{F})$, $\Sigma_{A_1A_1}(\Sigma)$ and $\Sigma_{A_2}(\Sigma)$ are called *smooth strata*, *self-intersection strata* and *cusp strata* of $\mathcal{F}$, respectively. We define the subset of singular points by

$$\Sigma(\mathcal{F}) := \Sigma_{A_1A_1}(\Sigma) \sqcup \Sigma_{A_2}(\Sigma). \qquad \triangle$$

The following follows from standard transversality arguments:

Lemma 5.6. *Let $\mathcal{F} \subseteq B^{n+1}$ be a generic simple front, then the subset of singularities $\Sigma(\mathcal{F}) \subseteq B$ is a codimension 2 submanifold in B^{n+1}.*

6. Legendrian stabilization and loose Legendrians

In this section we review the notions of Legendrian stabilization and loose Legendrians in higher-dimensional contact manifolds, as introduced in [Eli90, EES05b, Mur19]. All the results presented here are well-known, although we give some new proofs that are better suited to the purposes of this article.

6.1. N-Pushing.

We introduce an operation on fronts, called N*-pushing*, which alters the Legendrian isotopy class of a submanifold while preserving its formal Legendrian class, provided certain conditions on N are satisfied. This operation is particularly useful when showing that two Legendrians belong to the same formal class.

This construction first appeared in the works of [Eli90, EES05b], where it was referred to as a "stabilization". However, the particular construction from [EES05b] does not, in general, produce a loose Legendrian submanifold [Mur19]. For this reason, we adopt the term *pushing*, and reserve the term of *stabilization* for the operation that produces a loose Legendrian, i.e. an N-pushing in a cusp chart.

Let Q be a manifold. Given a function $f \in C^\infty(Q)$ we denote by $j^1 f \subseteq J^1 Q = T^*Q \times \mathbb{R}$ the Legendrian submanifold of $(J^1Q, \xi_{\mathrm{std}})$ defined as the 1-jet of f. Given $c \in \mathbb{R}$, we will use the notation $c_Q : Q \to \mathbb{R}$ to denote the function with constant value c. Consider the disconnected Legendrian $L_0 = j^1 0_Q \sqcup j^1 1_Q$ and a compact submanifold $N \subseteq Q$ together with a smooth function $f : Q \to [0, 2]$ such that

(i) $f_{|Q \setminus \mathcal{O}p(N)} \equiv 0$,

(ii) $f(p) > 1$ for $p \in N$,
(iii) $f(p) > 0$ for $p \in \mathcal{O}p(N)$, and
(iv) The critical points of f in $\mathcal{O}p(N)$ have critical values greater than 1.

The N*-pushing* of L_0 is the disconnected Legendrian

$$L_N = j^1 f \sqcup j^1 1_Q$$

Definition 6.1. Let (Y^{2n+1}, ξ) be a contact manifold of dimension $2n+1 \geq 5$, $L \subseteq Y$ a Legendrian submanifold and $U \subseteq Y$ an open codimension 0 submanifold such that

$$(U, L \cap U, \xi) \simeq (J^1 Q, L_0, \xi_{\mathrm{std}}).$$

The N-pushing of the Legendrian L is the Legendrian $\mathrm{p}_N(L)$ obtained by replacing $(U, L \cap U, \xi)$ by $(J^1 Q, L_N, \xi_{\mathrm{std}})$. △

The following result is well-known. However, we will include a proof since we will need to adapt it to the prelegendrian setting later:

Proposition 6.2 (Ekholm-Etnyre-Sullivan, Murphy)**.** *Let $L \subseteq (Y, \xi)$ be a Legendrian submanifold, and suppose that the Euler characteristic $\chi(N) = 0$ is zero. Then, the N-pushing $\mathrm{p}_N(L)$ is formally Legendrian isotopic to L in (Y, ξ).*

Proof. We will to check that the Legendrians $L_0 = j^1 0_Q \sqcup j^1 1_X$ and $L_N = j^1 f \sqcup j^1 1_Q$ are formally Legendrian isotopic by a compactly supported formal isotopy. By simplicity we will assume that $Q = \mathbb{R}^n$ since the general case is totally analogous after fixing a Riemannian metric.

To do this, it is enough to find a formal Legendrian isotopy between the Legendrian embeddings $j^1 0_{\mathbb{R}^n} : \mathbb{R}^n \to J^1\mathbb{R}^n, \mathbf{x} \mapsto (\mathbf{x}, 0, 0)$ and $j^1 f : \mathbb{R}^n \to J^1\mathbb{R}^n, \mathbf{x} \mapsto (\mathbf{x}, \frac{\partial f}{\partial \mathbf{x}}, f(\mathbf{x}))$, not intersecting the Legendrian $j^1 1_{\mathbb{R}^n}$. This isotopy should be supported in a small neighbourhood of the support of f. To construct it we will first find an isotopy of smooth embeddings, homotopic through isotopies of embeddings to the obvious Legendrian isotopy between $j^1 0_{\mathbb{R}^n}$ and $j^1 f$ induced by the linear interpolation between the functions $0_{\mathbb{R}^n}$ and f.

The required isotopy of smooth embeddings will be defined by a homotopy of non-holonomic sections of the 1-jet bundle. To define it, we will rely on the fact that a section of the 1-jet bundle can be reinterpreted as a section of the 0-jet bundle equipped with a family of non-vertical planes. The geometric idea behind the proof then goes as follows: Since the Euler characteristic $\chi(\mathcal{O}p(N)) = \chi(N) = 0$ vanishes, we can find a nowhere zero vector field Z tangent to $j^0 f|_{\mathcal{O}p(N)}$. This is enough to change the 1-jet information of $j^1 f$ by a non-holonomic deformation. Indeed, at every point of $j^0 f$ we can tilt its non-vertical tangent planes towards the vertical, by rotating Z towards ∂_z. After doing this carefully, it will be possible to interpolate linearly between $j^0 f$ and $j^0 0_{\mathbb{R}^n}$, keeping the non-holonomic 1-jet data fixed. Having tilted towards the vertical, the non-holonomic 1-jets will not intersect $j^1 1_{\mathbb{R}^n}$. The argument will conclude by tilting the 1-jet until it becomes horizontal, while fixing the 0-jet. The detailed construction of this idea is explained below.

The vector field Z is defined as follows: for $x \in f^{-1}(-\infty, 1+\epsilon)$, we define $Z = \nabla h$, where $h : j^0 f \to \mathbb{R}$ is the height function respect to the standard Euclidean metric in $\mathbb{R}^{n+1}$, where $\varepsilon > 0$ is so small that there is not critical value below $1 + \varepsilon$. In particular, $f^{-1}([1,2]) \simeq \overline{\mathcal{O}p(N)}$ and $f^{-1}(\{1\}) \simeq \partial\overline{\mathcal{O}p(N)}$. Hence, $\chi(f^{-1}([1,2])) = \chi(\mathcal{O}p(N)) = \chi(N) = 0$, and we can extend Z to a nowhere-zero vector field, tangent to $j^0 f$. The key property of this vector field is that the angle between Z with ∂_z is less that $\pi/2$ in the region $f^{-1}[1, 1+\epsilon]$.

We will use this vector field Z to rotate the tangent space of 0-jet $j^0 f$, which at a point $(\mathbf{x}, f(\mathbf{x}))$ is the non-vertical plane

$$T_{(\mathbf{x},f(\mathbf{x}))}(j^0 f) = \ker(\,\mathrm{d}z - \frac{\partial f}{\partial \mathbf{x}}\,\mathrm{d}\mathbf{x}) \subseteq T_{(\mathbf{x},f(\mathbf{x}))}\mathbb{R}^{n+1},$$

where $\mathbf{y} = \frac{\partial f}{\partial \mathbf{x}}$ are precisely the $\mathbf{y}$-coordinates of the Legendrian $j^1 f$.

To define the non-holonomic deformation of the planes $T_{(\mathbf{x},h(\mathbf{x}))}(j^0 f)$ we will use a bundle isomorphism

$$R_\theta : (j^0 f)^* T\mathbb{R}^{n+1} \to (j^0 f)^* T\mathbb{R}^{n+1},$$

where $\theta : \mathbb{R}^n \to [0, \frac{\pi}{2})$ is a smooth function with compact support $\mathrm{supp}(\theta) \subseteq f^{-1}[1,2]$, and $R_\theta(\mathbf{x}) : T_{(\mathbf{x},f(\mathbf{x}))}\mathbb{R}^{n+1} \to T_{(\mathbf{x},f(\mathbf{x}))}\mathbb{R}^{n+1}$ is a rotation of angle $\theta(\mathbf{x})$ about the $(n-1)$-dimensional subspace orthogonal to the oriented plane $\mathrm{span}\langle Z(\mathbf{x}), \partial_z\rangle$.

We define the function $\theta : \mathbb{R}^n \to [0, \frac{\pi}{2})$ to be

$$\theta(\mathbf{x}) = \rho(f(\mathbf{x}))\frac{\beta(\mathbf{x})}{2},$$

where

(i) $\rho : \mathbb{R} \to [0,1]$ is an increasing smooth function so that $\rho_{|\{z\leq 1\}} \equiv 0$, and $\rho_{|\{z\geq 1+\varepsilon\}} \equiv 1$.
(ii) $\beta : \mathbb{R}^n \to (0,\pi)$ satisfies that $\beta_{|\mathbb{R}^n\setminus \mathcal{O}p(N)} \equiv 0$, and over $\mathcal{O}p(N)$ is defined to be the angle

$$\beta(\mathbf{x}) \coloneqq \angle(Z(\mathbf{x}), \partial_z), \mathbf{x} \in \mathcal{O}p(N).$$

Now we are ready to define the required homotopy of non-holonomic sections $g_t : \mathbb{R}^n \to J^1\mathbb{R}^n$, $t \in [0,3]$, between $g_0 = j^1 f$ and $g_3 = j^1 0_{\mathbb{R}^n}$.

- For $t \in [0,1]$, define
$$g_t(\mathbf{x}) = (\mathbf{x}, \mathbf{y}_t(\mathbf{x}), f(\mathbf{x})),$$
where $\mathbf{y}_t(\mathbf{x})$ is defined by the field of non-vertical hyperplanes along $j^0 f$ given by
$$H_t(\mathbf{x}) = R_{t\theta}(T_{(\mathbf{x},f(\mathbf{x}))} j^0 f) = \ker(\,\mathrm{d}z - \mathbf{y}_t(\mathbf{x})\,\mathrm{d}\mathbf{x})$$
- For $t \in [1,2]$, define g_t just by changing the 0-jet information via a linear interpolation between the functions $f(\mathbf{x})$ and $0_{\mathbb{R}^n}$ while fixing the 1-jet information. Observe that for $\mathbf{x} \in f^{-1}[1+\varepsilon, 2]$ the angle
$$\angle(R_{\theta(\mathbf{x})}(Z(\mathbf{x})), \partial_z) = \beta(\mathbf{x}) - \frac{\beta(\mathbf{x})}{2} = \frac{\beta(\mathbf{x})}{2} < \frac{\pi}{2},$$
and for $\mathbf{x} \in f^{-1}[1, 1+\varepsilon]$,
$$\angle(R_{\theta(\mathbf{x})}(Z(\mathbf{x})), \partial_z) = \beta(\mathbf{x}) - \rho(h(\mathbf{x}))\frac{\beta(\mathbf{x})}{2} \leq \beta(\mathbf{x}) < \frac{\pi}{2}.$$
Therefore, the non-vertical hyperplanes $H_1(\mathbf{x}) = R_\theta(\mathbf{x})(T_{(\mathbf{x},f(\mathbf{x}))} j^0 f)$ are non-horizontal for $\mathbf{x} \in f^{-1}[1,2]$ and this homotopy does not intersect $j^1 1_{\mathbb{R}^n}$.
- Finally, for $t \in [2,3]$, define the homotopy by fixing the 0-jet datum while interpolating the 1-jet information $\mathbf{y}_1(\mathbf{x})$ linearly to $0_{\mathbb{R}^n}$.

This homotopy of non-holonomic sections g_t is clearly homotopic to the homotopy of holonomic sections (Legendrian embeddings) between $j^1 f$ and $j^1 0_{\mathbb{R}^n}$ induced by a linear interpolation between the functions f and $0_{\mathbb{R}^n}$. This concludes the argument. □

6.2. **Legendrian Stabilization and Loose Legendrians.** The 1-dimensional Legendrian cusp $L^0_{\text{cusp}} \subseteq J^1\mathbb{R}$ is defined by $L^0_{\text{cusp}} \cap \{x \leq 0\} = j^1 0_{\mathbb{R}} \sqcup j^1 1_{\mathbb{R}} \cap \{x \leq 0\}$, while over $\{x \geq 0\}$ the front $\pi_{\mathrm{F}}(L^0_{\text{cusp}})$ is given by a single cusp connecting $j^0 0_{\mathbb{R}}$ and $j^0 1_{\mathbb{R}}$. See Fig. 1

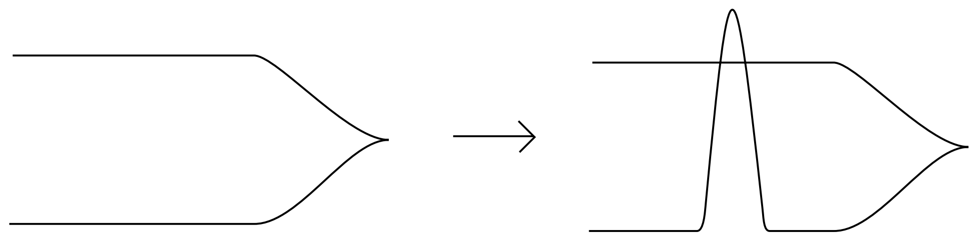

FIGURE 1. Left: the front of the cusp model L^0_{cusp}. Right: the stabilized front L_{loose}.

Let Q be a $(n-1)$-dimensional manifold, and define $X = \mathbb{R} \times Q$. Consider 1-jet bundle $J^1X = J^1\mathbb{R} \times T^*Q$, and a Legendrian submanifold $L_{\text{cusp}} \subseteq J^1\mathbb{R} \times T^*Q$ defined as:

$$L_{\text{cusp}} \coloneqq L^0_{\text{cusp}} \times Q \subseteq J^1\mathbb{R} \times T^*Q$$

where $Q \subseteq T^*Q$ is the 0-section.

Notice that over $\{x \leq 0\}$ we have that

$$L_{\text{cusp}} \cap \{x \leq 0\} = (j^1 0_{\mathbb{R}} \sqcup j^1 1_{\mathbb{R}} \cap \{x \leq 0\}) \times Q = j^1 0_{\{x \leq 0\} \times Q} \sqcup j^1 1_{\{x \leq 0\} \times Q},$$

which is the situation described in the previous section. Therefore, given a closed submanifold $N \subseteq \mathbb{R} \times Q \cap \{x < 0\}$ with $\chi(N) = 0$, we define the $L_{\text{loose}} \coloneqq \mathrm{p}_N(L_{\text{cusp}})$.

Definition 6.3. Let (Y^{2n+1}, ξ) be a contact manifold of dimension $2n+1 \geq 5$, $L \subseteq Y$ a Legendrian submanifold and $U \subseteq Y$ an open neighbourhood such that

$$(U, L \cap U, \xi) \simeq (J^1X, L_{\text{cusp}}, \xi_{\text{std}}),$$

for some manifold $X = \mathbb{R} \times Q$. The *stabilization* of L is the Legendrian $s(L)$ obtained by replacing $(U, L \cap U, \xi)$ by $(J^1X, L_{\text{loose}}, \xi_{\text{std}})$. △

Definition 6.4. Let (Y^{2n+1}, ξ) be a contact manifold of dimension $2n+1 \geq 5$. A Legendrian submanifold $L \subseteq Y$ is *loose* if $L = s(L')$ for some Legendrian submanifold L'. △

The relevance of loose Legendrians and the justification of dropping N from the notation in the definitions relies on the following h-principle of Murphy:

Theorem 6.5 (Murphy [Mur19])**.** *Two loose Legendrians are Legendrian isotopic if and only if are formally Legendrian isotopic. Moreover, every formal Legendrian is formally Legendrian isotopic to a genuine Legendrian by a C^0-small isotopy.*

7. Legendrian spinning

In low dimensions, generic fronts are simple. In contrast, a front in high dimensions may be extremely singular and thus complicated to represent. In this section we introduce a construction of higher-dimensional fronts that uses lower-dimensional fronts as input. This construction is called *spinning*; it was introduced first in [EES05b] to construct of Legendrian submanifolds in $\mathbb{R}^{2n+1}$. It has been further studied in [EES09, Gol14, LC13].

7.1. **Front spinning.**

Let us recall some basic facts about the 1-jet bundles, which is a crucial part of the spinning construction.

Lemma 7.1. *Let L_N and L_M be bundles over N and M, respectively, with 1-dimensional fibres. Consider a diagram of embeddings:*

$$\begin{tikzcd} L_N \arrow[r, "\varphi"] \arrow[d] & L_M \arrow[d] \\ N \arrow[r, "e"] & M \end{tikzcd}$$

Then, there is an induced 1-jet bundle map:

$$\hat{\varphi} : (J^1(L_N), \xi_{\mathrm{std}}) \to (J^1(L_M), \xi_{\mathrm{std}}),$$

which is an isocontact embedding.

In particular, $\hat{\varphi}$ sends isotropics to isotropics. In the case $\dim(N) = \dim(M)$, it defines a map between the corresponding spaces of Legendrians:

$$\hat{\varphi}_* : \mathfrak{Leg}(J^1(L_N), \xi_{\mathrm{std}}) \to \mathfrak{Leg}(J^1(L_M), \xi_{\mathrm{std}}).$$

Moreover, this construction works parametrically. If φ_s is a homotopy of fibred embeddings, then the induced isocontact embeddings $\hat{\varphi}_0$ and $\hat{\varphi}_1$ are contact isotopic. In particular, in the equidimensional case, the induced maps between Legendrian embedding spaces are homotopic.

Remark 7.2. In the case where $L_N = N \times \mathbb{R}$ and $L_M = M \times \mathbb{R}$ are trivial line bundles, the embedding $e \colon N \hookrightarrow M$ induces an isocontact embedding

$$T^*N \times \mathbb{R} \to T^*M \times \mathbb{R},$$

once a choice of Riemannian metrics has been made; note that in this standard construction the $\mathbb{R}$–coordinate is mapped identically.

However, for our purposes we need to modify the construction in order to impose an additional geometric condition on the resulting spinning, which requires us to shift the $\mathbb{R}$-coordinate. Concretely, we will only consider the case of trivial line bundles, and we will work with bundle maps of the form

$$\begin{aligned} \varphi : N \times \mathbb{R} &\to M \times \mathbb{R}, \\ (n, z) &\mapsto \big(e(n), z + h(n)\big); \end{aligned}$$

where $h \in C^\infty(N; \mathbb{R})$, and the lemma then provides the required isocontact embedding. △

The second easy observation that we use is the following:

Lemma 7.3. *For every smooth manifolds N^m and K, there exists a well-defined front-graph map:*

$$G_K : \mathrm{Map}(K, \mathfrak{Leg}(N, (J^1\mathbb{R}^m, \xi_{\mathrm{std}}))) \to \mathfrak{Leg}(K \times N, (J^1(K \times \mathbb{R}^m), \xi_{\mathrm{std}})).$$

Proof. Given a $L_k \in \mathrm{Map}(K, \mathfrak{Leg}(N, (J^1\mathbb{R}^m, \xi_{\mathrm{std}})))$, define the Legendrian embedding $G_K(L_k) \in \mathfrak{Leg}(K \times N, (J^1(K \times \mathbb{R}^m), \xi_{\mathrm{std}}))$ by the front map:

$$G_K(L_k) : K \times N \to J^0(K \times \mathbb{R}^m) = K \times J^0\mathbb{R}^m, \quad (k, n) \mapsto (k, \pi_{\mathrm{F}}(L_k(n))),$$

where $\pi_{\mathrm{F}} : J^1\mathbb{R}^m \to J^0\mathbb{R}^m$ is the front projection. $\square$

Now, we can define the spinning as the following.

Definition 7.4. Let $e : K \times \mathbb{R}^m \to \mathbb{R}^n$ be a smooth embedding of codimension 0, covered by a fibred embedding $\varphi : J^0(K \times \mathbb{R}^m) \to J^0\mathbb{R}^n$. Let N be a smooth manifold of dimension m. The *spinning map* of N along K via φ is defined as the composition:

$$\hat{\varphi}_* \circ G_K : \mathrm{Map}(K, \mathfrak{Leg}(N, (J^1\mathbb{R}^m, \xi_{\mathrm{std}}))) \to \mathfrak{Leg}(K \times N, (J^1\mathbb{R}^n), \xi_{\mathrm{std}}). \qquad \triangle$$

7.2. **A specific spinning.** We will only use the spinning map in a very concrete situation, that we describe below. Let $\hat{e} : K \hookrightarrow J^0\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}_z$ be a smooth embedding graphical over $\mathbb{R}^n$, that means

$$\hat{e} = (\hat{e}_{\mathbb{R}^n}, h) : K \to \mathbb{R}^n \times \mathbb{R}_z,$$

where $\hat{e}_{\mathbb{R}^n} : K \hookrightarrow \mathbb{R}^n$ is also an embedding and $h \in C^\infty(K)$ is a smooth function. We further assume that the embedding $\hat{e}_{\mathbb{R}^n}$ has trivialisable normal bundle. We fix a framing Fr of the normal bundle of $\hat{e}_{\mathbb{R}^n}(K)$, to define an embedding:

$$e : K \times \mathbb{R}^m \to \mathbb{R}^n.$$

This codimension-0 embedding is naturally covered by the fibrewise isomorphism

$$\varphi : J^0(K \times \mathbb{R}^m) \to J^0\mathbb{R}^n, \quad (k, v, z) \mapsto (e(k, v), z + h(k)).$$

Note that the map φ is an affine isomorphism, and is invariant along the normal directions.

In this situation, given a family of Legendrian embeddings $L_k \in \mathrm{Map}(K, \mathfrak{Leg}(N, (J^1\mathbb{R}^m, \xi_{\mathrm{std}})))$, we will denote by

$$\mathrm{spin}_{\hat{e},\mathrm{Fr}}[L_k] \coloneqq \hat{\varphi}_* \circ G_K(L_k) \in \mathfrak{Leg}(N \times K, (J^1\mathbb{R}^n, \xi_{\mathrm{std}}))$$

the spinning of the family $\{L_k\}_{k \in K}$. In the cases that we will treat later the framing will be unique so we will simply write $\mathrm{spin}_{\hat{e}}[L_k]$, and if $L_k \equiv L$ is a constant family we will write $\mathrm{spin}_{\hat{e}}[L]$.

The following result follows by combining the Ekholm-Etnyre-Sullivan computation of the formal classes of a spinning [EES05a], together with Murphy classification of formal Legendrians [Mur19]. However, we provide a direct proof since it will be fundamental for our study of prelegendrians.

Lemma 7.5. *Let $\hat{e} : K^{n-1} \to J^0\mathbb{R}^n$ be a smooth embedding, graphical over $\mathbb{R}^n$, with $\chi(K) = 0$. Fix moreover a framing* Fr.

If $L_0, L_1 \in \mathfrak{Leg}(J^1\mathbb{R}^1, \xi_{\mathrm{std}})$ are two Legendrian knots with $\mathrm{Rot}(L_0) = \mathrm{Rot}(L_1)$, *then* $\mathrm{spin}_{\hat{e},\mathrm{Fr}}[L_0]$ *and* $\mathrm{spin}_{\hat{e},\mathrm{Fr}}[L_1]$ *are formally Legendrian isotopic. Moreover, the formal Legendrian isotopy is induced by a sequence of Reidemeister moves of the 1-dimensional Legendrians L_0 and L_1, and K-pushings along parallel copies of $\hat{e}(K)$.*

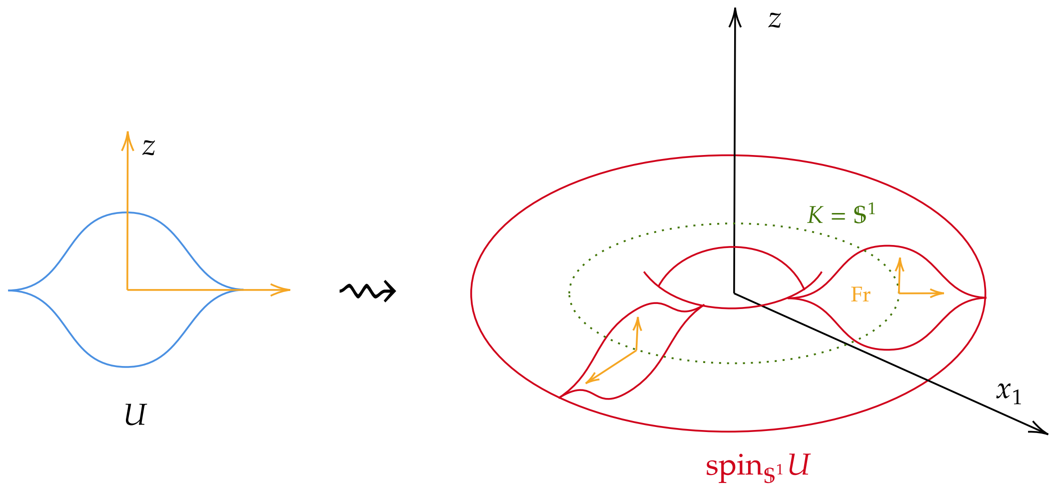

FIGURE 2. A schematic depiction of the front spinning of the Legendrian unknot $U \subseteq (J^1\mathbb{R}, \xi_{\text{std}})$ along the standard circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$.

Proof. Following the notations above we write $e : K \times \mathbb{R}^2 \to J^0\mathbb{R}^n$, to denote the tubular neighbourhood embedding of $\hat{e}$ induced by Fr, in which we identify $\{k\} \times \mathbb{R}^2 = J^0\mathbb{R}$.

By the h-principle for Legendrian immersions in $\mathbb{R}^3$, there is a generic homotopy of Legendrian immersions L_t, $t \in [0,1]$, between the given Legendrian knots. There is a finite number of times $0 < t_1 < \ldots < t_n < 1$ in which a unique transverse self-intersection of L_{t_i} appears at a point p_i. The spinning of L_t, $t \in [t_i - \varepsilon, t_i + \varepsilon]$, describes precisely a K-pushing along $e(K \times \{\pi_{\mathrm{F}}(p_i)\})$. The result follows from Proposition 6.2. ☐

The original definition of spinning was given in [EES05b] for $K = \mathbb{S}^1$ and later generalized by Golovko in [Gol14] to $K = \mathbb{S}^k$. They work with the standard inclusion; specifically, they consider

$$\hat{e} : \mathbb{S}^k \hookrightarrow \mathbb{R}^{k+1} \times \{0\} \times \mathbb{R}_z \subseteq \mathbb{R}^{k+1} \times \mathbb{R}^{m-1} \times \mathbb{R}_z,$$

equipped with the canonical normal framing of $\mathbb{S}^k \subseteq \mathbb{R}^{k+1} \times \mathbb{R}^{m-1}$. In this case, we denote the spinning construction simply by $\mathrm{spin}_{\mathbb{S}^k}[L]$ and refer to it as the *standard* $\mathbb{S}^k$*-spinning.* This yields a map

$$\mathrm{spin}_{\mathbb{S}^k} : \mathfrak{Leg}(N^m, (J^1\mathbb{R}^m)) \to \mathfrak{Leg}(\mathbb{S}^k \times N^m, J^1\mathbb{R}^{m+k}).$$

7.2.1. *Torus spinning.* For our purposes (constructing prelegendrians via spinning along a general embedding of the torus) we need to verify that torus spinning produces the same Legendrian isotopy class as iterated standard $\mathbb{S}^1$-spinning. The reason is that we understand the effect that $\mathbb{S}^1$-spinning has on Legendrian Contact Homology; see [EES05b, Example 4.20].

The *standard* k*-torus* $\mathbb{T}^k \subseteq \mathbb{R}^{k+1}$ is defined inductively as follows:

- $\mathbb{T}^1 = \mathbb{S}^1 \subseteq \mathbb{R}^2$ is the standard inclusion.
- Assume that $\mathbb{T}^{k-1} \subseteq \mathbb{R}^k$ is defined. Consider

$$\mathbb{S}^1 \subseteq \mathbb{R}^2(x_1, x_2) \times \{0\} \subseteq \mathbb{R}^2(x_1, x_2) \times \mathbb{R}^{k-1}(x_3, \ldots, x_{k+1})$$

> endowed with the canonical normal framing $\nu \simeq \mathrm{span}\langle \partial_r, \partial_{x_3}, \dots, \partial_{x_{k+1}} \rangle$. Identifying $\nu \cong \mathbb{R}^k$ and using the standard inclusion $\mathbb{T}^{k-1} \subseteq \mathbb{R}^k \simeq \nu$, we obtain the standard embedding $\mathbb{T}^k \subseteq \mathbb{R}^{k+1}$.

For the standard k-torus $\mathbb{T}^k \subseteq \mathbb{R}^{k+1}$ there is also a canonical normal framing; therefore we denote the corresponding spinning simply by $\mathrm{spin}_{\mathbb{T}^k}[L]$ for $L \in \mathfrak{Leg}(J^1\mathbb{R}^m, \xi_{\mathrm{std}})$, in complete analogy with the k-sphere case. The following lemma is immediate from the definition of the standard k-torus.

Lemma 7.6. *Let $L \subseteq J^1\mathbb{R}^m$ be a Legendrian submanifold. Consider the standard torus $\mathbb{T}^k \subseteq \mathbb{R}^{k+1}$, then*

$$\mathrm{spin}_{\mathbb{T}^k}[L] \simeq (\mathrm{spin}_{\mathbb{S}^1})^k[L].$$

8. Prelegendrian fronts

Consider an almost complex manifold (X, J), and the associated fat manifold $(\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$. By Theorem 4.7, the contactisation $\mathcal{C}(\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$ is contactomorphic to the unit cotangent bundle $\mathbb{S}T^*X$, this yields the commutative diagram:

$$\begin{tikzcd}
(\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}}) \arrow[dr, "\pi_{\mathrm{PF}}"'] & & (\mathbb{S}T^*X, \xi_{\mathrm{can}}) \arrow[ll, "\mathrm{Hopf}"'] \arrow[dl, "\pi_{\mathrm{F}}"] \\
& (X, J) &
\end{tikzcd}$$

For simplicity, we assume throughout that all prelegendrians are *co-oriented* (see Section 2.3).

Given an embedded prelegendrian $\Lambda \subseteq \mathrm{Gr}(X, J)$, we will refer to $\pi_{\mathrm{PF}}(\Lambda) \subseteq (X, J)$ as a *prelegendrian front*.

Every generic Legendrian submanifold of $\mathcal{L}(\Lambda) \subseteq \mathbb{S}T^*X$ can be represented by its front $\mathcal{F} = \pi_{\mathrm{F}}(\mathcal{L}(\Lambda))$ in X. However, not every front $\mathcal{F}$ in X is a prelegendrian front. Strong geometric constraints must be satisfied along the singularities of the front. The main result of this section, is a complete characterization of which *simple* fronts are prelegendrian. It is a rephrasing of Theorem 1.9, from the Introduction:

Theorem 8.1 (Simple Prelegendrian Fronts)**.** *A simple front $\mathcal{F} \subseteq (X, J)$ is a prelegendrian front if and only if its singular loci $\Sigma = \Sigma(\mathcal{F})$ are co-real. Moreover, the prelegendrian $\Lambda \subseteq (\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$ determined by $\mathcal{F}$ is unique.*

Remark 8.2. We point out that a simple front $\mathcal{F} \subseteq (X, J)$ whose cusp locus is co-real but whose self-intersection locus is not, also defines a unique *immersed* prelegendrian $\Lambda \subseteq (\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$. The self-intersection points of Λ are precisely the preimage by π_{PF} of the complex points of the self-intersection locus. △

8.1. **Proof of Theorem 8.1.**

Since the property of being a prelegendrian front is local, it suffices to work with the germ of $\mathcal{F}$. We therefore argue case by case according to the singularity type.

Let $p \in \mathcal{F} \subseteq (X, J)$. Since $\mathcal{F}$ is a simple front, there exists a diffeomorphism from an open neighbourhood of p in X onto an open neighbourhood of 0 in $\mathbb{R}^{2n+2}$:

$$\psi : (\mathcal{O}p(p), p) \to (\mathbb{R}^{2n+2}, 0),$$

such that, via ψ, the germ of the front $\mathcal{F}$ at $(\mathcal{O}p(p), p)$ is in the normal form of Definition 5.5, namely:

(i) In the case of A_1-singularity, $\mathcal{F} = \mathbb{R}^{2n+1} \times \{0\} \subseteq \mathbb{R}^{2n+2}$ is the horizontal hyperplane. These are smooth points of the front.

(ii) In the case of A_1A_1-singularity, the front $\mathcal{F}$ is the union of two hyperplanes intersecting transversely at 0. This corresponds to a transverse self–intersection of the front.

(iii) In the case of an A_2-singularity, $\mathcal{F}$ is parametrized by the front map $f : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+2}$ given by

$$\begin{aligned} f : (\mathbb{R}^{2n+1}, 0) &\to (\mathbb{R}^{2n+2}, 0) \\ (q_1, \dots, q_{2n}, t) &\mapsto (q_1, \cdots, q_{2n}, -3t^2, 2t^3) \end{aligned} \tag{6}$$

This corresponds to cusp points of the front.

For simplicity, we continue to denote by J the pushforward almost–complex structure $\psi_* J$ on $\mathbb{R}^{2n+2}$. Since the almost–complex Grassmannian construction is functorial, we may reduce to the case $X = \mathbb{R}^{2n+2}$, with $\mathcal{F}$ one of the three types above.

We treat each case separately. We begin with the smooth stratum (i). In this case, the prelegendrian is determined by the complex tangencies of the front (in the contact setting, by the real tangencies). This is the content of the following:

Proposition 8.3 (Smooth points)**.** *The (oriented) horizontal hyperplane $\mathcal{F} = \mathbb{R}^{2n+1} \times \{0\} \subseteq \mathbb{R}^{2n+2}$ admits a prelegendrian lift:*

$$\begin{aligned} g : \mathbb{R}^{2n+1} &\to \mathrm{Gr}(\mathbb{R}^{2n+2}, J) = (\mathbb{R}^{2n+2} \times \mathbb{CP}^n, \mathcal{D}_{\mathrm{std},J}); \\ \mathbf{q} &\mapsto \big((\mathbf{q}, 0); H \cap J_x H\big), \end{aligned}$$

such that the following diagram commutes:

$$\begin{tikzcd} & \mathbb{S}T^*\mathbb{R}^{2n+2} = \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1} \arrow[d, "\mathrm{Hopf}"] \\ \mathbb{R}^{2n+1} \arrow[ur, "\mathcal{L}(g)"] \arrow[r, "g"] & \mathrm{Gr}(\mathbb{R}^{2n+2}, J) = \mathbb{R}^{2n+2} \times \mathbb{CP}^n \end{tikzcd} \tag{7}$$

where $\mathcal{L}(g)(\mathbf{q}) = \big((\mathbf{q}, 0); H\big)$ is the Legendrian lift of g.

Proof. The fibrewise Hopf map Hopf : $\mathbb{S}T^*\mathbb{R}^{2n+2} \to \mathrm{Gr}(\mathbb{R}^{2n+2}, J)$ (see Section 3) sends an oriented hyperplane to its complex part. The Legendrian determined by the hyperplane front $\mathcal{F}$ is parametrised by

$$(q_1, \dots, q_{2n+1}) \to (q_1, \dots, q_{2n+1}, 0; H) \in \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1}$$

Thus the prelegendrian map is the composition

$$\begin{aligned} g : \mathbb{R}^{2n+1} &\to \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1} \to \mathbb{R}^{2n+2} \times \mathbb{CP}^n, \\ \mathbf{q} &\mapsto (\mathbf{q}, 0; H_0) \mapsto (\mathbf{q}, 0 : H_0 \cap JH_0). \end{aligned}$$

Since the first projection parametrises $\mathcal{F}$, g is an embedding; and it is prelegendrian by the definition of $\mathcal{D}_{\mathrm{can}}$. $\square$

Now, we deal with the self-intersection points (ii). In this case, the prelegendrian lift of each smooth branch is given by the complex tangencies of the front. The lift of both branches will be embedded if and only if the complex tangencies of each branch are transverse. Since they intersect in a $2n$-dimensional space, it is enough to require this intersection to not be complex:

Proposition 8.4 (Self-intersections)**.** *Let $\mathcal{F} = H_1 \cup H_2 \subseteq \mathbb{R}^{2n+2}$ be the front given by the union of two transversal hyperplanes. Then, it lifts to an embedded prelegendrian if and only if $\Sigma = H_1 \cap H_2$ is co-real.*

Proof. We have that

$$\Sigma^{2n} \cap J(\Sigma^{2n}) = (H_1 \cap JH_1)^{2n} \cap (H_2 \cap JH_2)^{2n} \subseteq \Sigma^{2n}.$$

By dimensional reasons, $H_1 \cap JH_1 = H_2 \cap JH_2$ if and only if Σ is J-complex. □

Finally, we consider the cusp stratum (iii). By the previous case, there is a genuine prelegendrian lift along the smooth stratum, which extends uniquely to the cusp points. We must check that this lift is an immersion. In the contact case this is true because the tangent space of the front rotates continuously when one crosses the cusp locus transversely. It follows that we must check that the complex tangencies also rotate across the cusp locus. The following geometric condition ensures this:

Proposition 8.5 (Cusps points)**.** *Let $f : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+2}$ be the front map given in (6), and $\mathcal{F} = f(\mathbb{R}^{2n+1})$. Then f lifts to a prelegendrian embedding if and only if $\Sigma = \Sigma(\mathcal{F})$ is co-real.*

Proof. First, write the coordinates of $\mathbb{R}^{2n+2}$ as $(x_1, \ldots, x_{2n+2})$. Define the family of 1-forms $\alpha_t := t\,\mathrm{d}x_{2n+1} + \mathrm{d}x_{2n+2}$. A simple computation shows that the tangent space of the front $\mathcal{F}$ is

$$T_{f(\mathbf{q},t)}\mathcal{F} = \ker \alpha_t.$$

The singular set of $\mathcal{F}$ is

$$\Sigma = f(\mathbb{R}^{2n} \times \{0\}) = \{x_{2n+1} = x_{2n+2} = 0\} \subseteq \mathcal{F},$$

with tangent space

$$T_{f(\mathbf{q},0)}\Sigma = \operatorname{span}(\partial_{x_1}, \ldots, \partial_{x_{2n}}) = \ker(\mathrm{d}x_{2n+1}) \cap \ker(\mathrm{d}x_{2n+2}),$$

Therefore, the Legendrian embedding associated to f is

$$\begin{aligned} \tilde{g} : \mathbb{R}^{2n+1} &\to \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1}; \\ (\mathbf{q}, t) &\mapsto (f(\mathbf{q}, t), \alpha_t), \end{aligned}$$

with dual identification. According to Proposition 8.3, the associated prelegendrian map must be

$$\begin{aligned} g : \mathbb{R}^{2n+1} &\to \mathbb{R}^{2n+2} \times \mathbb{CP}^n; \\ (\mathbf{q}, t) &\mapsto (f(\mathbf{q}, t); (\alpha_t)^J), \end{aligned}$$

where

$$(\alpha_t)^J = t(\mathrm{d}x_{2n+1})^J + (\mathrm{d}x_{2n+2})^J$$

is the complexification of α_t.

Observe that g is injective, and, moreover, is an embedding away from $t = 0$. At the points of $\{t = 0\}$, the vectors $\{Df_{(\mathbf{q},0)}(\partial_{q_i}), i = 1, \ldots, 2n\} \subseteq T_{f(\mathbf{q},t)}\mathbb{R}^{2n+2}$ span a $2n$-dimensional subspace, while $Df_{\mathbf{q},0}(\partial_t) = 0$. Therefore, g will be an immersion at a point $(\mathbf{q}, 0)$ if and only if $Dg_{(\mathbf{q},0)}(\partial_t) = (0, D_{(\mathbf{q},0)}(\alpha_t)^J(\partial_t)) \neq 0$.

Since $T_{f(\mathbf{q},0)}\Sigma = \ker(\mathrm{d}x_{2n+1}) \cap \ker(\mathrm{d}x_{2n+2})$, Lemma 3.4 applies, i.e. $T_{f(\mathbf{q},0)}\Sigma$ is J-invariant if and only if the complexifications $(\mathrm{d}x_{2n+1})^J$ and $(\mathrm{d}x_{2n+2})^J$ are $\mathbb{C}$–linearly dependent in $\mathrm{Hom}_{\mathbb{C}}(T_{f(\mathbf{q},0)}\mathbb{R}^{2n+2}, \mathbb{C})$.

Assume first that Σ is co-real. Then, $\{(\mathrm{d}x_{2n+1})^J, (\mathrm{d}x_{2n+2})^J\}$ are $\mathbb{C}$-linearly independent along Σ. Since being linearly independent is an open coition, we may further assume that they are linearly independent on $\mathbb{R}^{2n+2}$. Therefore, $\{(\mathrm{d}x_{2n+1})^J, (\mathrm{d}x_{2n+2})^J\}$ can be completed to a $\mathbb{C}$-basis of $\mathrm{Hom}_{\mathbb{C}}(T\mathbb{R}^{2n+2}, \mathbb{C})$. In that basis, the value of g in the $\mathbb{CP}^n$ component is of the form $[0 : \cdots : 0 : t : 1]$. In particular, the derivative with respect to t is non-zero, hence g is an immersion.

Conversely, if Σ is not co-real, then $T_{f(\mathbf{q}_0,0)}\Sigma$ is J-invariant for some $\mathbf{q}_0 \in \mathbb{R}^{n+1}$, hence $(\mathrm{d}x_{2n+1})^J$, $(\mathrm{d}x_{2n+2})^J$ are $\mathbb{C}$-linearly dependent at the point $f(\mathbf{q}_0, 0)$. As in the previous case, we complete $(\mathrm{d}x_{2n+2})^J$ to a basis $\{v_1, \dots, v_n, (\mathrm{d}x_{2n+2})^J\}$ over $\mathbb{R}^{2n+2}$. We express $(\mathrm{d}x_{2n+1})^J$ as a linear combination of this basis along $\mathcal{F}$:

$$(\mathrm{d}x_{2n+1})^J = a_1(\mathbf{q}, t)v_1 + \cdots + a_n(\mathbf{q}, t)v_n + a_{n+1}(\mathbf{q}, t)(\mathrm{d}x_{2n+2})^J,$$

where $a_i(\mathbf{q}, t) \in \mathbb{C}$, $i = 1, \dots, n+1$, are the coefficients. Moreover, by the assumption of $(\mathrm{d}x_{2n+1})^J$ and $(\mathrm{d}x_{2n+2})^J$ being $\mathbb{C}$-linearly dependent at the point $f(\mathbf{q}_0, 0)$ we have that:

- $a_{n+1}(\mathbf{q}_0, 0) \neq 0$
- $a_i(\mathbf{q}_0, t) = O(t)$ for $i = 1, \dots, n$.

Therefore, the value of $g(\mathbf{q}_0, t)$ with t sufficiently small, in the $\mathbb{CP}^n$ component is of the form

$$\begin{aligned}[t a_1(\mathbf{q}_0, t) : \cdots : t a_n(\mathbf{q}_0, t) : 1 + t a_{n+1}(\mathbf{q}_0, t)] &= [O(t^2) : \cdots : O(t^2) : 1 + O(t)] \\ &= [O(t^2) : \cdots : O(t^2) : 1].\end{aligned}$$

This implies that the derivative of g along the t-direction vanishes at the point $(\mathbf{q}, 0)$, and g fails to be an immersion. □

8.2. **Simple prelegendrian Reidemeister moves.** The study of prelegendrian simple fronts conducted above can be generalised to 1-parametric families. We will say that a one–parameter family of Legendrian fronts $(\mathcal{F}_s)_{s\in[0,1]}$ in (X, J) undergoes a Reidemeister I move if (1) there is a closed coorientable hypersurface $O \subseteq \mathcal{F}_0$ contained in the smooth locus such that (2) $\mathcal{F}_s$ is supported in $\mathcal{O}p(O) \cong O \times \mathbb{R}^2$ with $\mathcal{O}p(O) \cap \mathcal{F}_0 \cong O \times \mathbb{R}$, and (3) $\mathcal{F}_s$ is the product of O with the standard 1-dimensional move of $\mathbb{R}$ in $\mathbb{R}^2$ (Fig. 3). The Reidemeister moves II and III are defined similarly by considering O to be one of the components of the cusp and self-intersection loci, respectively. Then:

Lemma 8.6. *Suppose that $\mathcal{F}_0$ is a simple prelegendrian front. Introduce a Reidemeister move along a coreal submanifold $O \subseteq \mathcal{F}_0$. Then, the resulting family of fronts $(\mathcal{F}_s)_{s\in[0,1]}$ is prelegendrian.*

Proof. By construction, the singularities of $\mathcal{F}_{1/2}$ occur along an embedded co-real submanifold. The argument in the proof of Proposition 8.4 applies verbatim for the RII and RIII moves. For the RI move, the argument from Proposition 8.5 applies. □

The resulting prelegendrian homotopies are referred to as *prelegendrian Reidemeister* I, II and III moves.

Remark 8.7. In plain words, Lemma 8.6 says that Reidemeister moves for prelegendrian fronts can be performed along co-real submanifolds. In particular, let $\mathcal{F}$ be a prelegendrian front and $N \subseteq \Sigma_{A_1}(\mathcal{F})$ a codimension 1 co-real submanifold in the smooth strata. Then, there exists a

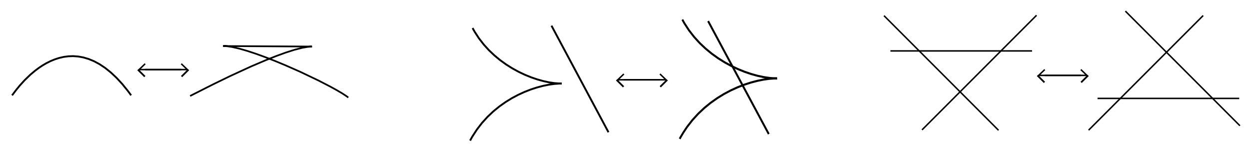

FIGURE 3. From left to right the fronts $\hat{\mathcal{F}}_s \subseteq J^0\mathbb{R}$, $s \in [0,1]$, of the 1-dimensional Reidemeister moves I, II and III. The higher dimensional moves are locally modelled on $\hat{\mathcal{F}}_s \times \mathbb{R}^n \subseteq J^0\mathbb{R} \times \mathbb{R}^n = J^0\mathbb{R}^{n+1}$, $s \in [0,1]$.

homotopy of prelegendrian fronts $\mathcal{F}_s$, $s \in [0,1]$, that describes an RI-move along N. Moreover, since being co-real is an open condition we may achieve that the region in the prelegendrian front $\mathcal{F}_1$ in which the RI appears is foliated by co-real submanifolds parallel to N. △

Remark 8.8. A generic 1-parametric family of Legendrians in higher-dimensions will undergo a wider class of bifurcations than those introduced above. We expect the same to be true for prelegendrians. Moreover, the three Reidemeister moves shown here are not generic and will generally appear as combinations of more elementary moves. Developing the theory further in this direction is left as an open question. △

8.3. **Prelegendrian spinning.** Theorem 8.1 provides a criterion ensuring that, under mild assumptions, the Legendrians in $(\mathbb{R}^{4n+3}, \xi_{\mathrm{std}}) = (J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$ produced by the Legendrian spinning from Section 7.2 admit a prelegendrian front and therefore define prelegendrian embeddings in $(\mathbb{R}^{4n+2}, \mathcal{D}_{\mathrm{std},J})$. This yields an effective method to build prelegendrian embeddings and leads us to the construction of exotic prelegendrian tori in Section 11.

The setup is as follows. Consider $\mathbb{R}^{2n+2} = J^0\mathbb{R}^{2n+1}$ equipped with an almost complex structure J. As explained in Section 7.2, given an embedding $\hat{e} = (\hat{e}_{\mathbb{R}^{2n+1}}, h) : K \hookrightarrow J^0\mathbb{R}^{2n+1}$ of codimension $m+1$ that is graphical over $\mathbb{R}^{2n+1}$, together with a framing of $\hat{e}_{\mathbb{R}^{2n+1}}$, there is a well-defined Legendrian spinning map. This operation takes as input a Legendrian $L \subseteq (J^1\mathbb{R}^m, \xi_{\mathrm{std}})$ and produces a higher-dimensional Legendrian

$$\mathrm{spin}_{\hat{e}}[L] \subseteq (J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$$

Our first observation concerns the singularities of the front of the Legendrian spinning:

Lemma 8.9. *The singular loci of the Legendrian spinning $\pi_{\mathrm{F}}(\mathrm{spin}_{\hat{e}}[L])$ satisfies:*

$$\Sigma\big(\pi_{\mathrm{F}}(\mathrm{spin}_{\hat{e}}[L])\big) = \varphi\big(K \times \pi_{\mathrm{F}}(L)\big),$$

In particular,

$$\Sigma\big(\pi_{\mathrm{F}}(\mathrm{spin}_{\hat{e}}[L])\big) = \bigsqcup_{p \in \Sigma(\pi_{\mathrm{F}}(L))} \varphi(K \times \{p\})$$

is foliated by submanifolds diffeomorphic to K. Moreover, the leaves of this foliation are C^∞-close to $\hat{e}(K) = \varphi(K \times \{0\})$ whenever $\pi_{\mathrm{F}}(L)$ is C^∞-small.

It is important to realize that every Legendrian in $(J^1\mathbb{R}^m, \xi_{\mathrm{std}})$ is Legendrian isotopic to a Legendrian whose front is C^∞-small by using the contact isotopy $\delta_t(\mathbf{x}, \mathbf{y}, z) = (t\mathbf{x}, t\mathbf{y}, t^2 z)$, for $t > 0$.

An important consequence is the following:

Corollary 8.10. *Assume that $\hat{e} : K \hookrightarrow (\mathbb{R}^{2n}, J)$ is a co-real embedding. Let $L \subseteq (J^1\mathbb{R}^m, \xi_{\mathrm{std}})$ be a Legendrian with sufficiently C^∞-small simple front. Then, the front $\pi_{\mathrm{F}}\big(\mathrm{spin}_{\hat{e}}[L]\big)$ is a prelegendrian front. In particular, it defines a prelegendrian embedding*

$$\mathrm{prespin}_{\hat{e}}[L] \subseteq (\mathbb{R}^{4n+2}, \mathcal{D}_{\mathrm{std},J})$$

that will be referred as prelegendrian spinning *of L along the (co-real embedding) $\hat{e}$.*

Proof. The previous lemma combined with Lemma 3.8 implies that the singularities of the front are co-real whenever $\pi_{\mathrm{F}}(L)$ is sufficiently C^∞-small. The result follows from Theorem 8.1. □

Remark 8.11. By definition, the Legendrian lift of the prelegendrian spinning is the Legendrian spinning:

$$\mathcal{L}(\mathrm{prespin}_{\hat{e}}[L]) = \mathrm{spin}_{\hat{e}}[L]. \qquad \triangle$$

Remark 8.12. We continue using the notation of the corollary, and work still under the same C^∞-small assumption on the fronts. Assume that $L_s \subseteq (J^1\mathbb{R}^m, \xi_{\mathrm{std}})$, $s \in [0,1]$, is a Legendrian isotopy whose front describes a Reidemeister I, II or III move between two simple fronts L_0 and L_1. Then, by Lemma 8.6, there is a prelegendrian isotopy

$$\mathrm{prespin}_{\hat{e}}[L_s] \subseteq (\mathbb{R}^{4n+2}, \mathcal{D}_{\mathrm{std},J}), s \in [0,1],$$

describing an RI, RII or RII prelegendrian move, respectively. In particular, assume that the co-real embedding $\hat{e}$ has codimension 2 and, therefore, L is a 1-dimensional Legendrian. Then, the prelegendrian spinning just depends, up to prelegendrian isotopy, on the Legendrian isotopy class of L. △

9. Prelegendrian stabilization

In this section we develop the prelegendrian N-pushing operation along a co-real submanifold N, describe its effect on the formal prelegendrian class, and explain how to stabilize prelegendrians. This will lead us to the proof of Theorem 1.12 in the next section.

9.1. **Prelegendrian N-pushing.**

In the contact case, the N-pushing construction is defined in the local model of the 1-jet bundle, where a natural front projection is available, and then transported via a Darboux chart to any contact manifold. For fat distributions of corank 2 (or higher), there is no Darboux-type theorem. For this reason, we restrict our definition to the almost complex Grassmannian, which admits a prelegendrian front projection.

9.1.1. *Model case.* First we work in a local model where $X := J^0Q = Q \times \mathbb{R}(z)$, which allows us to consider the horizontal Grassmannians:

$$\begin{tikzcd}
(\mathrm{Gr}_{\mathrm{hor}}(X,J), \mathcal{D}_{\mathrm{std},J}) \arrow[dr, "\pi_{\mathrm{PF}}"'] & & (\mathbb{S}T^*_{\mathrm{hor}}X, \xi_{\mathrm{can}}) \simeq (J^1Q, \xi_{\mathrm{std}}) \arrow[ll, "\mathrm{Hopf}"'] \arrow[dl, "\pi_{\mathrm{F}}"] \\
& (X, J) &
\end{tikzcd}$$

We use the notation of Section 6.1.

Proposition 9.1. *Let $\Lambda \subseteq \mathbb{C}^{2n+1}$ be the prelegendrian with $\pi_{\mathrm{PF}}(\Lambda) = j^0 0_Q \sqcup j^0 1_Q \subseteq J^0Q$. Suppose that $N \subseteq j^0 0$ is a co-real submanifold that admits a non-zero section in $\nu_{N\setminus Q}$. Then, there exists a prelegendrian $\mathrm{p}_N(\Lambda)$ such that*

- $\pi_{\mathrm{PF}}(\mathrm{p}_N(\Lambda)) = j^0 f \cup j^0 1_Q$,
- $\mathcal{L}(\mathrm{p}_N(\Lambda)) = \mathrm{p}_N(\mathcal{L}(\Lambda))$,
- *Λ and $\mathrm{p}_N(\Lambda)$ coincide outside of* $\mathrm{Gr}_{\mathrm{hor}}(\mathcal{O}p_X(N \times [0,1]), J)$,
- *$\mathrm{p}_N(\Lambda)$ is formally isotopic to Λ as prelegendrians.*

The prelegendrian $\mathrm{p}_N(\Lambda)$ is said to be an N-pushing of Λ.

Proof. By abuse of notation, given $A \subseteq \mathbb{R}$ we write $f^{-1}(A)$ to mean $j^0\big(f|_{f^{-1}(A)}\big) \subseteq j^0 f$.

Consider the front $\mathcal{F} := j^0 f \cup j^0 1 \subseteq \mathbb{C}^{n+1}$. This is a simple front: its singular set consists of the transverse self-intersection $j^0 f \cap j^0 1 = f^{-1}(\{1\})$. Set $M := f^{-1}([1,2]) \subseteq \mathbb{R}^{2n+1}$ and thus $\partial M = f^{-1}(\{1\})$. If we allow ourselves to homotope the section $j^0 1$ (relative boundary) and to shrink the model, we can assume, using the fact that co-reality is an open condition, that every shifted copy $N \times \{c\}$ is co-real and so is ∂M. Therefore, $\mathcal{F}$ is a prelegendrian front by Lemma 3.8 and thus defines a prelegendrian Λ'. We set $\mathrm{p}_N(\Lambda) := \Lambda'$ and the first three claims follow.

The last claim will follow by an argument similar to the one in Proposition 6.2. In this case, the extra coordinates of the prelegendrian, which are an element of $\mathbb{C}^n \subseteq \mathbb{CP}^n$, can be interpreted as the slope of the complex tangencies of the front. Suppose we now choose a non-vanishing vector field $Z \in T(j^0 f)^J$ defined over $\mathcal{O}p(M)$. By construction it is complex linearly independent from ∂_z and we can rotate it towards the latter. This provides a formal rotation of the complex tangencies, making them almost vertical. Once we do this, we can apply the reasoning in Proposition 6.2 to conclude.

It remains to produce such a $Z \in T(j^0 f)^J$. Since both ∂M and M are co-real, the same argument as in Lemma 3.10 shows that there is a splitting

$$TM|_{\partial M} = T(\partial M) \oplus \eta,$$

with $\eta \subseteq TM^J$ and $\eta \simeq \nu_{\partial M \setminus M}$. Choose a generator of the line bundle η that is inward-pointing along ∂M. This yields a vector field

$$V : \mathcal{O}p_M(\partial M) \longrightarrow TM^J,$$

that points transversely inward along ∂M.

Since the gradient ∇f is inward-pointing on ∂M, we can homotope it to V near the boundary and, by applying Proposition 3.9, obtain a vector field Z on $j^0 f$ such that

- $Z \in (T(j^0 f))^J$ on $f^{-1}([1,2])$,
- $Z = V$ on $f^{-1}([1-\varepsilon, 1+\varepsilon])$,
- $Z = \nabla f$ on $f^{-1}([0, 1-\varepsilon])$.

This concludes the proof. □

9.1.2. *General case.* We now describe the N–pushing in the general almost–complex Grassmannian setting:

Theorem 9.2 (Prelegendrian N-pushing). *Let $\Lambda \subseteq (\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ be a prelegendrian, $\mathcal{F} := \pi_{\mathrm{PF}}(\Lambda)$, N a closed manifold, and $\Phi : N \times [0,1] \hookrightarrow X$ an embedding such that:*

- *$N_0 := \Phi(N \times \{0\})$, $N_1 := \Phi(N \times \{1\})$ are contained in the smooth strata $\Sigma_{A_1}(\mathcal{F})$.*
- *N_1 is a co-real submanifold of (X, J).*
- *$\Phi(N \times [0,1]) \cap \mathcal{F} = \Phi(N \times \{0,1\})$,*
- *the normal bundle $\nu_{N_1 \setminus \mathcal{F}}$ admits a non-zero section.*

Then, there exists a prelegendrian $\mathrm{p}_N(\Lambda)$ *such that:*

- $\Lambda \cap \mathrm{Gr}(X \backslash \mathcal{O}p(\Phi), J) = \mathrm{p}_N(\Lambda) \cap \mathrm{Gr}(X \backslash \mathcal{O}p(\Phi), J)$.
- *The Legendrian lifts* $\mathcal{L}(\Lambda)$ *and* $\mathcal{L}(\mathrm{p}_N(\Lambda))$ *in* $(\mathbb{P}T^*X, \xi_{\mathrm{std}})$ *satisfy:*

$$\mathcal{L}(\mathrm{p}_N(\Lambda)) = \mathrm{p}_N(\mathcal{L}(\Lambda))$$

- $\mathrm{p}_N(\Lambda)$ *and* Λ *are formally prelegendrian isotopic.*

The prelegendrian $p_N(\Lambda)$ *is said to be an* N*-pushing of* Λ*.*

Proof. Let $Q := \mathcal{O}p_{\mathcal{F}}(N_0)$ be a neighbourhood of N_0 in $\mathcal{F}$. The embedding $\Phi : N \times [0,1] \hookrightarrow X$ extends to an embedding $\widetilde{\Phi} : Q \times [0,1] \hookrightarrow X$. Thus $\Phi(Q \times [0,1]) \subseteq (X, J)$ is identified with the model $(Q \times [0,1], \widetilde{\Phi}^* J)$ of Proposition 9.1, which yields the result. ☐

9.2. **Prelegendrian stabilization.** Following the contact geometric approach, we can use the prelegendrian N-pushing operation to define a prelegendrian stabilization.

Definition 9.3. Let $\Lambda \subseteq (\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$ be a prelegendrian, $p \in \mathcal{F} = \pi_{\mathrm{PF}}(\Lambda)$ be a smooth point, and $N \subseteq \mathcal{O}p(p) \cap \mathcal{F}$ a co-real hypersurface. We define a new prelegendrian as follows:

(i) Perform a prelegendrian RI move on Λ along N, see Remark 8.7. Denote the resulting prelegendrian by Λ_1.
(ii) Apply the N-pushing operation described in Theorem 9.2 to Λ_1 along a co-real $N' \cong N$ of $\mathcal{O}p(N) \cap \pi_{\mathrm{PF}}(\Lambda_1)$ very close to the cusp locus of the RI-move. Denote the resulting prelegendrian by $s(\Lambda)$

The prelegendrian $s(\Lambda)$ is said to be a *stabilization* of Λ. △

The following highlights the key property of the prelegendrian stabilization:

Proposition 9.4. *The prelegendrians* Λ *and* $s(\Lambda)$ *are formally prelegendrian isotopic. Moreover,*

$$s(\mathcal{L}(\Lambda)) = \mathcal{L}(s(\Lambda)).$$

In particular, $\mathcal{L}(s(\Lambda))$ *is loose.*

Remark 9.5. Even though the Legendrian lift of the prelegendrian stabilization is independent of the choice of co-real submanifold (by the h-principle of Theorem 6.5), it is unclear whether the stabilized prelegendrian satisfies an h-principle as well. It is an open question whether prelegendrian stabilization depends on the choice of co-real submanifold N. △

10. Existence of compact prelegendrians and prelegendrian stabilizations

In this section we establish the existence of compact prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ and prelegendrian stabilizations. To do this, we will give an explicit family of a co-real submanifolds in $\mathbb{C}^{n+1}$ which is graphical over $\mathbb{R}^{2n+1} \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} = \mathbb{R}^{2n+1} \times \mathbb{R}$. The existence of these submanifolds implies then the existence of prelegendrian stabilizations as discussed in the previous section, thus yielding the proof of Theorem 1.12.

10.1. **Co-real submanifolds in $\mathbb{C}^{n+1}$ graphical over $\mathbb{R}^{2n+1}$.** We begin by proving the existence of suitable co-real submanifolds to perform our prelegendrian spinning. After this, Corollary 8.10 trivially implies the existence of compact prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$.

Proposition 10.1. *Let $F \subseteq \mathbb{R}^n$ be a closed submanifold. Then, there exists a co-real embedding $\hat{e} : \mathbb{T}^{n+1} \times F \hookrightarrow \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} = \mathbb{R}^{2n+1} \times \mathbb{R}$ which is graphical over $\mathbb{R}^{2n+1}$. In particular, there exists a co-real embedding of $\mathbb{T}^{2n}$.*

Proof of Proposition 10.1. We use polar coordinates (r_i, θ_i), in each $\mathbb{C}$-factor of $\mathbb{C}^{n+1} = \mathbb{C} \times \cdots \times \mathbb{C}$. Our construction takes places close to the Clifford torus $\mathbb{T}^{n+1}_{\text{Cl}} \subseteq \mathbb{C}^{n+1}$, defined as the image of the embedding:

$$\begin{array}{rcl} \mathrm{Cl} : \mathbb{T}^{n+1} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 & \hookrightarrow & \mathbb{C}^{n+1}; \\ (\theta_1, \ldots, \theta_{n+1}) & \mapsto & \Big((1, \theta_1); \cdots, (1, \theta_n); (1, \theta_{n+1})\Big). \end{array}$$

The Clifford torus is clearly not graphical over $\mathbb{R}^{2n+1}$. However, the family of perturbations

$$\mathrm{Cl}^\delta(\theta_1, \ldots, \theta_{n+1}) = \Big((1 + \delta \sin(\theta_{n+1}), \theta_1); \cdots, (1, \theta_n); (1, \theta_{n+1})\Big),$$

for

$$0 < \delta < 1$$

, is easily seen to be graphical over $\mathbb{R}^{2n+1}$.

Since being totally real is a C^1-open condition we may fix $\delta > 0$ small enough such that Cl^δ is totally real and graphical over $\mathbb{R}^{2n+1}$. Write $\mathrm{Cl}^\delta = (g, \sin(\theta_{n+1}))$, where $g : \mathbb{T}^{n+1} \hookrightarrow \mathbb{R}^{2n+1}$ is the embedding obtained by projecting Cl^δ over $\mathbb{R}^{2n+1}$.

A global trivialisation of the normal bundle of Cl^δ can be given by

$$\langle \partial_{r_1} + \partial_{r_{n+1}}, \partial_{r_2}, \ldots, \partial_{r_n} \rangle \oplus \langle \partial_z \rangle,$$

where ∂_z spans the vertical direction in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+1} \times \mathbb{R}$. Moreover, the first n-vectors project to a trivialisation of the normal bundle of g in $\mathbb{R}^{2n+1}$.

Consider the inclusion $F \hookrightarrow \mathbb{R}^n$; without loss of generality we may assume that it has image in an ε-ball, for $\varepsilon > 0$ sufficiently small. After fixing a tubular neighbourhood embedding of g, this inclusion induces an embedding $e : \mathbb{T}^{n+1} \times F \hookrightarrow \mathbb{R}^{2n+1}$.

The embedding $\hat{e} = (e, \sin(\theta_{n+1})) : \mathbb{T}^{n+1} \to \mathbb{C}^{n+1} = \mathbb{R}^{2n+1} \times \mathbb{R}$, is co-real for $\varepsilon > 0$ sufficiently small, by an application of Lemma 3.8. This concludes the argument. $\square$

Note that the Clifford torus $\mathbb{T}^{n+1}_{\text{Cl}} \subseteq \mathbb{C}^{n+1}$ is isotopic to the standard torus (cf. Section 7.2). If we take $F = \mathbb{T}^{n-1} \subseteq \mathbb{R}^n$ to be the standard torus, then the resulting embedding $e : \mathbb{T}^{2n} \hookrightarrow \mathbb{R}^{2n+1}$ is also isotopic to the standard torus. Hence, by Lemma 7.6, we obtain:

Lemma 10.2. *In the situation of the above proposition, we can moreover assume that $\hat{e} : \mathbb{T}^{2n} \to \mathbb{C}^{n+1}$ satisfies:*

$$\mathrm{spin}_{\hat{e}}[L] \simeq (\mathrm{spin}_{\mathbb{S}^1})^{2n}[L],$$

for any Legendrian $L \subseteq J^1\mathbb{R}^m$.

10.2. **Existence of prelegendrian stabilizations.**

Proof of Theorem 1.12. Let Λ be a prelegendrian submanifold in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$. Consider a point $q \in \Lambda$ in the smooth stratum of the prelegendrian front $\pi_{\text{PF}}(\Lambda)$. It follows from Proposition 10.1 that, possibly after a prelegendrian front homotopy of $\mathcal{O}p(\pi_{\text{PF}}(p)) \cap \pi_{\text{PF}}(\Lambda)$, there exists a codimension 2 co-real torus $N \subseteq \mathcal{O}p(\pi_{\text{PF}}(p)) \cap \pi_{\text{PF}}(\Lambda) \subseteq \mathbb{C}^{n+1}$. This places us in the setup of Section 9.2. In particular, the stabilization construction defined there is not an empty operation.

The very same approach works in $(\text{Gr}(X, J), \mathcal{D}_{\text{can}})$. Indeed, the co-real torus from Proposition 10.1 can be assumed to be co-real for a general J if we work in a sufficiently small ball. This is because the property of being co-real is preserved under C^0-perturbations of the almost complex structure, and every almost complex structure can be assumed to coincide with the standard one at a given point. ☐

11. Exotic prelegendrians

In this section we will prove our main results about the failure of the h-principle for prelegendrian embeddings: Theorem 1.10, Corollary 1.14 and Corollary 1.15.

11.1. **Auxiliary Lemmas in Contact Topology.**

We recall two basic results in contact topology that we will need later. The first one is standard and well-known. It follows from the functioriality of jet spaces:

Lemma 11.1. *Let Q be a smooth manifold and $\widetilde{Q}$ its universal cover. Then, the contact universal cover of J^1Q is*

$$(\widetilde{J^1Q}, \widetilde{\xi_{\text{can}}}) \simeq (J^1\widetilde{Q}, \xi_{\text{can}}).$$

In particular, the contact universal cover of $(J^1(\mathbb{R}^m \times \mathbb{S}^1), \xi_{\text{can}})$ is contactomorphic to $(J^1\mathbb{R}^{m+1}, \xi_{\text{std}})$.

Proof. The differential of the covering map $\widetilde{Q} \to Q$ induces a symplectic covering map $(T\widetilde{Q}, d\lambda_{\text{can}}) \to (T^*Q, d\lambda_{\text{can}})$ which preserves the canonical 1-form. The result follows. ☐

The second auxiliary result that we need follows by an standard cut-off argument with contact Hamiltonians:

Lemma 11.2. *Let (Y^{2n+1}, ξ) be a contact manifold, such that its universal cover $(\widetilde{Y}, \tilde{\xi})$ is contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$, and $(B, \xi_{\text{std}}) \subseteq (Y, \xi)$ a Darboux ball. Let $L_0, L_1 \subseteq (B, \xi_{\text{std}}) \subseteq (Y, \xi)$ be two compact Legendrian submanifolds which are not Legendrian isotopic in the ball, then L_0 and L_1 are not Legendrian isotopic in (Y, ξ).*

Proof. We prove the contrapositive. Assume that there is a Legendrian isotopy $L_t \subseteq (Y, \xi)$, $t \in [0, 1]$, between L_0 and L_1. Consider a lift $(\tilde{B}, \xi_{\text{std}}) \subseteq (\mathbb{R}^{2n+1}, \xi_{\text{std}})$ of the Darboux ball (B, ξ_{std}), together with lifts of the initial Legendrians $\tilde{L}_0, \tilde{L}_1 \subseteq (\tilde{B}, \xi_{\text{std}})$.

We claim that, since there is a Legendrian isotopy $L_t \subseteq (Y, \xi)$, $t \in [0, 1]$, there exists a Legendrian isotopy $\tilde{L}_t \subseteq (\mathbb{R}^{2n+1}, \xi_{\text{std}})$, $t \in [0, 1]$, in the universal cover between the Legendrians $\tilde{L}_0$ and $\tilde{L}_1$. Indeed, the Legendrian isotopy can be lifted to a Legendrian isotopy $\hat{L}_t \subseteq (\mathbb{R}^{2n+1}, \xi_{\text{std}})$, $t \in [0, 1]$, starting at $\hat{L}_0 = \tilde{L}_0$ but possibly ending at Legendrian lift $\hat{L}_1$ of L_1 which is different to $\tilde{L}_1$. This lift would be in a different lift $(\hat{B}, \xi_{\text{std}})$ of the Darboux ball (B, ξ_{std}). Since the space of Darboux ball embeddings into a connected contact manifold is connected, the contact embeddings

$(\tilde{B}, \xi_{\text{std}}) \hookrightarrow (\mathbb{R}^{2n+1}, \xi_{\text{std}})$ and $(\hat{B}, \xi_{\text{std}}) \hookrightarrow (\mathbb{R}^{2n+1}, \xi_{\text{std}})$ are contact isotopic. This implies that $\hat{L}_1$ and $\tilde{L}_1$ are Legendrian isotopic. The concatenation of both Legendrian isotopies is the required Legendrian isotopy $\tilde{L}_t \subseteq (\mathbb{R}^{2n+1}, \xi_{\text{std}})$, $t \in [0,1]$.

Now we will show that the Legendrian isotopy $\tilde{L}_t$ is homotopic, relative to $t \in \{0,1\}$, to a Legendrian isotopy whose image is in $(\tilde{B}, \xi_{\text{std}})$. Since the covering map defines a contactomorphism $(\tilde{B}, \xi_{\text{std}}) \to (B, \xi_{\text{std}})$ this would imply the existence of a Legendrian isotopy in (B, ξ_{std}) between L_0 and L_1, thus concluding the proof.

Fix two auxiliary Darboux balls (B_0, ξ_{std}) and (B_1, ξ_{std}) in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$ such that

- $\tilde{L}_0, \tilde{L}_1 \subseteq B_0$
- $\tilde{L}_t \subseteq B_1$
- $B_0 \subseteq \tilde{B} \subseteq B_1$.

Since the space of Darboux balls is connected we may further assume that (B_0, ξ_{std}) is the unit ball in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$. Consider the contact vector field

$$X = 2z\partial z + \sum_{i=1}^{n} x_i \partial x_i + y_i \partial y_i,$$

in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$, whose contact flow is the contact dilation $\varphi_X^t(\mathbf{x}, \mathbf{y}, z) = (e^t\mathbf{x}, e^t\mathbf{y}, e^{2t}z)$, $t \in \mathbb{R}$. With respect to the standard contact form, X has contact Hamiltonian

$$H = \alpha_{\text{std}}(X) = 2z - \mathbf{y} \cdot \mathbf{x}$$

Let $\rho : \mathbb{R}^{2n+1} \to [0,1]$ be a smooth function such that

- $\rho_{|B_0} \equiv 0$ and
- $\rho_{|\mathbb{R}^{2n+1} \setminus \tilde{B}} \equiv 1$.

Denote by $\tilde{X}$ the contact vector field associated to the Hamiltonian $\tilde{H} \coloneqq \rho \cdot H$. The required Legendrian isotopy is then

$$\varphi_{\tilde{X}}^{-T}(\tilde{L}_t), t \in [0,1],$$

where $T >> 0$ is big enough. This concludes the argument. $\square$

The previous lemma yields the following consequence, which, although not used later in the manuscript, may be of independent interest.

Corollary 11.3. *Let (Y^{2n+1}, ξ) be a contact manifold, such that its universal cover $(\widetilde{Y}, \tilde{\xi})$ is contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$, and $L \subseteq (B, \xi_{\text{std}})$ a Legendrian submanifold. Then, if L is non-loose in (B, ξ_{std}) then it is non-loose in (Y, ξ).*

Proof. Apply the previous result to $L_0 \coloneqq L$ and $L_1 \coloneqq s(L) \subseteq B$. Indeed, if $L_0 = L$ was loose in (Y, ξ), then it will be Legendrian isotopic to $L_1 = s(L)$ by Murphy's h-principle Theorem 6.5. $\square$

Remark 11.4. A direct proof of the corollary can also be given, without invoking the h-principle, by using contact Hamiltonians—as in Lemma 11.2—to push any potential loose chart into (B, ξ). $\triangle$

11.2. **Exotic prelegendrian tori.**

Proof of Theorem 1.10. Consider the family of Legendrian Whitehead doubles of stabilized Legendrian unknots $W_s \subseteq (J^1\mathbb{R}, \xi_{\mathrm{std}})$ with $s = 3, 5, \cdots$, depicted in Fig. 4. By Proposition 10.1, there exists a co-real embedding $\hat{e} = (\hat{e}_{\mathbb{R}^{2n+1}}, h) : \mathbb{T}^{2n} \to \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} = \mathbb{R}^{2n+1} \times \mathbb{R}$, which is graphical over $\mathbb{R}^{2n+1}$.

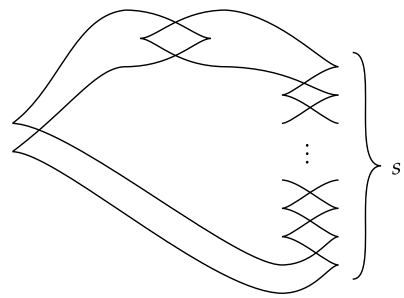

FIGURE 4. The Legendrian W_s, where s is the number of crossings in the right part.

By Corollary 8.10 there is well-defined infinite family of prelegendrian $(2n+1)$-tori:

$$\Lambda_s = \mathrm{prespin}_{\hat{e}}[W_s] \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$$

obtained by prelegendrian spinning.

Observe that $\mathrm{Rot}(W_s) = 0$ for all s. Therefore, by Lemma 7.5 and Remark 8.12 the prelegendrian fronts of Λ_s are related by a sequence of prelegendrian Reidemeister moves and N-pushings along parallel copies of $\hat{e}(\mathbb{T}^{2n})$. Theorem 9.2 then implies that all these prelegendrians are formally isotopic. A formal prelegendrian isotopy is depicted in Fig. 5.

Now we will show that the Legendrian lifts $\mathcal{L}(\Lambda_s)$ are pairwise not Legendrian isotopic, which by Corollary 2.15 will imply that the prelegendrians Λ_s are pairwise not prelegendrian isotopic and hence the result.

By construction,

$$\mathcal{L}(\Lambda_s) = \mathrm{spin}_{\hat{e}}[W_s] \subseteq (J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}}).$$

Moreover, as Legendrians in $(J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$, $\mathcal{L}(\Lambda_s)$ and $(\mathrm{spin}_{\mathbb{S}^1})^{2n}[W_s]$ are Legendrian isotopic by Lemma 10.2. It was shown in [EES05b, Example 4.20] that the Legendrian contact homologies of $\mathcal{L}(\Lambda_s)$ are all non quasi-isomorphic, so the $\mathcal{L}(\Lambda_s)$ are pairwise not Legendrian isotopic in $(J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$. To conclude we need to tell them apart in the contactisation $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$.

Notice that there is a natural inclusion

$$(J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}}) \cong (\mathbb{S}T^*_{\mathrm{hor}}(\mathbb{R}^{2n+2}), \xi_{\mathrm{can}}) \subseteq \mathcal{C}(\mathrm{Gr}_{\mathrm{hor}}(\mathbb{C}^{n+1}, J_{\mathrm{std}}), \mathcal{D}_{\mathrm{can}}) = \mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}}),$$

where the first contactomorphism follows from Lemma 4.11 and the inclusion is the one given in (5). Moreover, by Corollary 4.10 we have $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}}) \cong (J^1(\mathbb{R}^{2n} \times \mathbb{S}^1), \xi_{\mathrm{std}})$. Hence, Lemma 11.1 tell us that the contact universal cover of $\mathcal{C}(\mathbb{C}^{2n+1}, \xi_{\mathrm{std}})$ is $(J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$. Therefore, Lemma 11.2 can be applied to conclude that the Legendrian $\mathcal{L}(\Lambda_s)$ are pairwise not Legendrian non isotopic in the contactisation $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$. $\square$

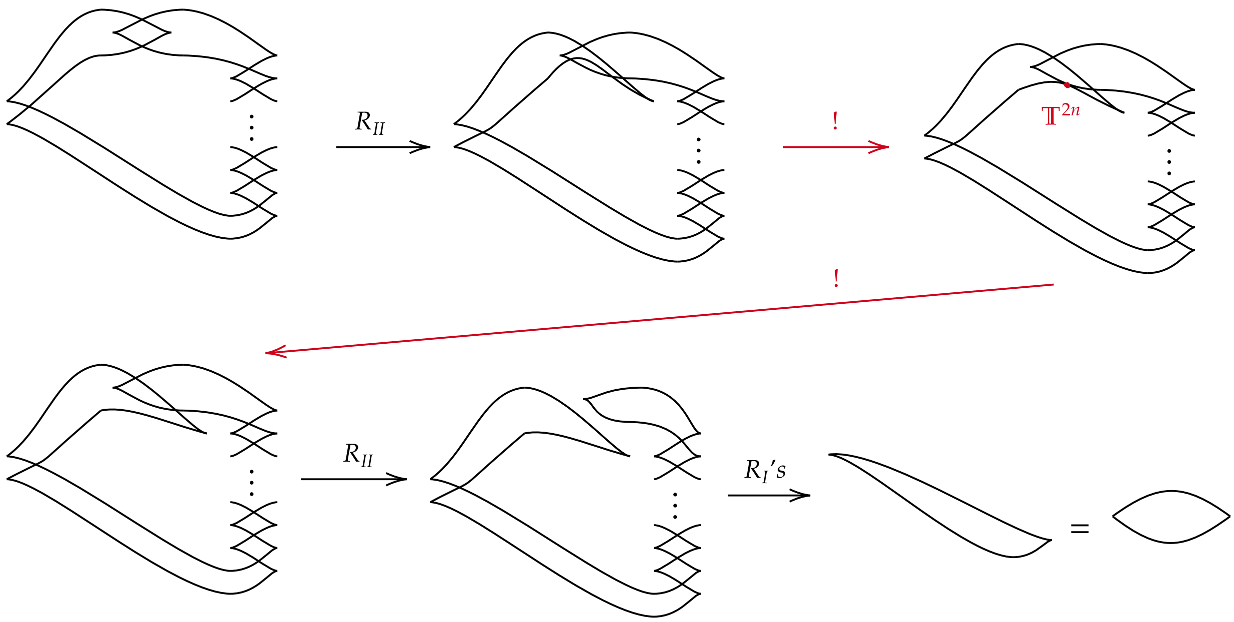

FIGURE 5. A formal prelegendrian isotopy between of $\Lambda_s = \text{prespin}_{\hat{e}}[W_s]$ and $\text{prespin}_{\hat{e}}[U]$, where $U \subseteq (J^1\mathbb{R}, \xi_{\text{std}})$ is the standard Legendrian unknot. The first and fourth arrow indicate prelegendrian RII moves, and the last one several prelegendrian RI moves along parallel copies of the co-real $\mathbb{T}^{2n}$. The second and third arrow describe a prelegendrian N-pushing along the co-real $\mathbb{T}^{2n}$.

Remark 11.5. Let $(Q^{2n+1} \times \mathbb{R}, J)$ be an almost complex manifold, with universal cover $\mathbb{R}^{2n+1} \times \mathbb{R}$. Then, it follows from the same argument that $(\text{Gr}_{\text{hor}}(Q \times \mathbb{R}), \mathcal{D}_{\text{can}})$ has infinitely many exotic prelegendrian $(2n+1)$-tori as well. Indeed, we consider the prelegendrian fronts appearing in the proof and we insert them in a very small ball $\mathcal{O}p(q, z)$, where $(q, z) \in Q \times \mathbb{R}$. It will then follow that the front singularities are co-real for J, and hence define prelegendrians in $(\text{Gr}_{\text{hor}}(Q \times \mathbb{R}), \mathcal{D}_{\text{can}})$. Our assumption on Q then implies that the contact universal cover of $\mathcal{C}(\text{Gr}_{\text{hor}}(Q \times \mathbb{R}), \mathcal{D}_{\text{can}})$ is contactomorphic to $(J^1\mathbb{R}^{2n+1}, \xi_{\text{std}})$, so the same proof can conclude in the same manner. This yields infinitely many examples of fat manifolds of corank 2 with infinitely many exotic prelegendrians. $\triangle$

11.3. **Proof of Corollary 1.14.** Consider the perturbation of the Clifford torus $\hat{e} : \mathbb{T}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ from Proposition 10.1. The prelegendrian embedding $\Lambda : \mathbb{T}^{n+1} \times \mathbb{S}^{j_0} \times \cdots \times \mathbb{S}^{j_k} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\text{can}})$ with non-loose Legendrian lift $\mathcal{L}(\Lambda)$ is going to be defined as

$$\Lambda := \text{prespin}_{\hat{e}}[L] \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}}),$$

for certain Legendrian $L \cong \mathbb{S}^{j_0} \times \cdots \times \mathbb{S}^{j_k} \subseteq (J^1\mathbb{R}^n, \xi_{\text{std}})$. To conclude that $\mathcal{L}(\Lambda) = \text{spin}_{\hat{e}}[L] \subseteq (J^1\mathbb{R}^{2n+1}, \xi_{\text{std}})$ is non-loose, we will show that $\mathcal{L}(\Lambda)$ is fillable [Ekh08, Mur19]. The second part of the result will then follow from Theorem 1.12.

By Lemma 10.2, the Legendrians $\mathrm{spin}_{\hat{e}}[L]$ and $(\mathrm{spin}_{\mathbb{S}^1})^{n+1}[L]$ are Legendrian isotopic. Therefore, the proof reduces to showing the existence of a Legendrian $L \cong \mathbb{S}^{j_0} \times \cdots \times \mathbb{S}^{j_k} \subseteq (J^1\mathbb{R}^n, \xi_{\mathrm{std}})$ such that

- L has a simple front, so $\Lambda = \mathrm{prespin}_{\hat{e}}[L] \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ is well-defined by Corollary 8.10.
- $(\mathrm{spin}_{\mathbb{S}^1})^{n+1}[L] \subseteq (J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$ is Lagrangian fillable.

Let $U \cong \mathbb{S}^{j_0} \subseteq (J^1\mathbb{R}^{j_0}, \xi_{\mathrm{std}})$ be the standard Legendrian unknot. Then, the Legendrian

$$L \coloneqq \mathrm{spin}_{\mathbb{S}^{j_k}} \circ \mathrm{spin}_{\mathbb{S}^{j_{k-1}}} \circ \cdots \circ \mathrm{spin}_{\mathbb{S}^{j_1}}[U] \subseteq (J^1\mathbb{R}^n, \xi_{\mathrm{std}}),$$

satisfies the required properties: it has a simple front and, moreover, $(\mathrm{spin}_{\mathbb{S}^1})^{n+1}[L]$ is Lagrangian fillable because the $\mathbb{S}^k$-spinning of a fillable Legendrian is fillable [Gol14, Remark 2]. This concludes the argument. □

Remark 11.6. There is an alternative argument to conclude non-looseness in the above construction which avoids appealing to Lagrangian fillability. Dimitroglou Rizell and Golovko prove in [DRG15, Theorem 3.1] that $\mathbb{S}^m$–spinning preserves augmentations: if the Chekanov–Eliashberg DGA of a Legendrian submanifold admits an augmentation, then the DGA of its $\mathbb{S}^m$–spinning also admits one. In particular, non-acyclicity of the DGA (equivalently, nonvanishing of linearized LCH) is preserved under spinning. Thus, whenever L has nonzero Legendrian contact homology, the same holds for $\mathrm{spin}_{\mathbb{S}^m}[L]$, which already implies that the latter is non-loose. △

Remark 11.7. It follows from [Gol14, Theorem 1.2] that we can actually find infinitely many prelegendrians as in Corollary 1.14 whose Legendrian lifts are not Legendrian isotopic. We expect these prelegendrians to be formally prelegendrian isotopic, yielding new infinite families of exotic prelegendrians with topology not restricted to be tori. However, the formal computation is more involved than the case of tori, since we cannot reduce it to 1-dimensional Legendrian knot theory, as in Lemma 7.5. This remains an open problem. △

11.4. **Proof of Corollary 1.15.** The Legendrian lift of the prelegendrian $\Lambda_M = (M, TM^J) \subseteq (\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ is

$$\mathcal{L}(\Lambda_M) = (M, TM) \subseteq (\mathbb{S}T^*X, \xi_{\mathrm{can}}) = \mathcal{C}(\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}}),$$

here the prelegendrian co-orientation is defined by the outward normal at $M = \partial Y$ defined by Y.

We claim that $\mathcal{L}(\Lambda_M)$ admits an exact Lagrangian filling in $(\mathbb{D}T^*X, \mathrm{d}\lambda_{\mathrm{can}})$, hence is non-loose [Ekh08, Mur19] and the result follows from Theorem 1.12. This must be well-known, however we include the proof here for completeness. Fix a Riemannian metric in M, and consider the associated metric in T^*M. Then, if ν_p, $p \in M = \partial Y$, is the outward unit normal to M, we can identify the Legendrian lift as the unit co-normal lift associated to ν, i.e. $\mathcal{L}(\Lambda_M) = \{(p, \nu_p^*) : p \in M\} \subseteq (\mathbb{S}T^*X, \xi_{\mathrm{can}})$.

Consider a tubular neighbourhood of M

$$\Phi : M \times (-\varepsilon, \varepsilon) \to U \subseteq X$$
$$(p,t) \mapsto \exp_p(t\nu_p),$$

Define $\psi : U \to \mathbb{R}$ by $\psi(\Phi(p,t)) \coloneqq t$. Then, $\mathrm{d}_p\psi = \nu_p^*$ for $p \in M$, $\|\mathrm{d}\psi\| = 1$, $\psi_{|Y \cap U} < 0$ and $\psi_{|M} = 0$. Since M is separating we may extend ψ to a smooth function in X, still denoted by ψ such that $\phi_{|Y \setminus U} \leq -\varepsilon$ and $\psi_{|X \setminus Y} > 0$.

Fix a step function $\chi : (-\infty, 0] \to [-1, 0]$ such that

- $\chi(0) = 0$,

- $\chi'(0) = 1$,
- $0 \leq \chi'(t) < 1$ for all $t < 0$,
- $\mathrm{supp}(\chi') \subseteq (-\varepsilon, 0]$.

Set $f : Y \to \mathbb{R}$ by $f(p) \coloneqq \chi(\psi(p))$. Then $L \coloneqq \mathrm{graph}(\mathrm{d}f) \subseteq T^*X$ is an exact Lagrangian. By definition,

$$\mathrm{d}_p f = \chi'(\psi(p))\, \mathrm{d}_p \psi.$$

From here it follows that $\|(\mathrm{d}f)_p\| < 1$ for $p \in \mathrm{Int}(Y)$ and $\mathrm{d}_p f = \nu_p^*$ for $p \in M = \partial Y$. That is, $L \subseteq (\mathbb{D}T^*X, \mathrm{d}\lambda)$ is an exact Lagrangian filling of $\Lambda_M \subseteq (\mathbb{S}T^*X, \xi_{\mathrm{can}})$ and the result follows. □

12. Constructing prelegendrians in non-standard fat distributions

In this section we translate our previous results to more general fat distributions. To do this we use two main tools: (1) the nilpotentisation, to be introduced next, and (2) Gray stability for contact structures. These tools partially remedy the lack of Darboux and Gray for fat structures of higher corank.

12.1. The nilpotentisation.

According to Cartan's study of normal forms of distributions, corank-2 fat distributions do not admit a Darboux type theorem. However, an approximate local form does exist [Ge92, Mon02], as we now explain.

Consider $(M^{4n+2}, \mathcal{D}^{4n})$ fat and fix a point $p \in M$. Then, there are local coordinates $(x_1, \ldots, x_{4n}, z_1, z_2)$ and 1-forms

$$\lambda^i = \mathrm{d}z_i + \sum_{j,k} \Gamma^i_{jk} x_j \, \mathrm{d}x_k + R_i, \quad i = 1, 2,$$

such that $\mathcal{D} \overset{loc.}{=} \ker \lambda^1 \cap \ker \lambda^2$. Here R_i is a remainder/defect term

$$R_i = \sum_j f_{ij} \, \mathrm{d}z_j + \sum_j g_{ij} \, \mathrm{d}x_j,$$

with $f, g \in O(|x|^2 + |z|^2)$.

The coefficients $\{\Gamma^i_{jk}\}$ depend on p and are the structure constants of the Lie algebra structure on $\mathcal{D}_p^{2n} \oplus TM_p/\mathcal{D}_p$ induced by the curvature of $\mathcal{D}$. As p varies, this yields a weak Lie algebra bundle over M, meaning that the isomorphism type of the Lie algebra may vary from point to point. This bundle is called the *nilpotentisation* of $\mathcal{D}$. The Lie algebra at p defines a left-invariant fat distribution on the corresponding Lie group; it is given precisely by the 1-forms

$$\lambda^i = \mathrm{d}z_i + \sum_{j,k} \Gamma^i_{jk} x_j \, \mathrm{d}x_k,$$

i.e. by removing the remainder. The following standard result tells us that this provides an approximate model for $\mathcal{D}$:

Proposition 12.1. *Fix a fat distribution $(M^{4n+2}, \mathcal{D}^{4n})$, a point $p \in M$, and local coordinates $(x_1, \ldots, x_{4n}, z_1, z_2)$, as above. Let $\mathcal{D}_{model}$ be the model fat distribution associated to the nilpotentisation at p.*

Then, there is a path of fat distributions $(\mathcal{D}_s)_{s \in [0,1]}$ on $\mathbb{R}^{4n+2}$ such that:

- *If $s > 0$, $\mathcal{D}_s$ is diffeomorphic to the restriction of $\mathcal{D}$ to the chart.*
- *$\mathcal{D}_0 = \mathcal{D}_{model}$.*

The proof amounts to performing a suitable weighted dilation in order to get rid of the remainder. As we take the limit as the dilation factor goes to zero, we obtain the model structure.

For the standard fat space $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$, this space is itself the model, and its Lie algebra is known as the *complex Heisenberg algebra*. A particularly interesting case is $n = 1$. By the classification of nilpotent 6-dimensional Lie algebras [CFS05, Mag86], there is a unique Lie algebra structure that is fat. Hence:

Corollary 12.2. *Fix a fat distribution $(M^6, \mathcal{D}^4)$, a point $p \in M$, and local coordinates $(x_1, \dots, x_4, z_1, z_2)$, as above. Then, there is a path of fat distributions $(\mathcal{D}_s)_{s\in[0,1]}$ on $\mathbb{C}^3$ such that:*

- *If $s > 0$, $\mathcal{D}_s$ is diffeomorphic to the restriction of $\mathcal{D}$ to the chart.*
- *$\mathcal{D}_0 = \mathcal{D}_{\mathrm{std}}$.*

12.2. **Prelegendrians in general fat manifolds.** This language of approximate models allows us to prove:

Proof of Theorem 1.17. Use Corollary 12.2 to produce a path of fat distributions $\mathcal{D}_s$ starting with $\mathcal{D}_0 = \mathcal{D}_{\mathrm{std}}$ such that $\mathcal{D}_s$ is diffeomorphic to the restriction of $\mathcal{D}$ to some small ball whose radius depends on $s > 0$. Now, consider the contactisation of the family $(M, \mathcal{D}_s)$, these define a family of (isotropic) bundle projections:

$$\mathrm{pr}_s : (U, \xi_s) \to (B, \mathcal{D}_s),$$

where U is an open manifold diffeomorphic to $B \times \mathbb{S}^1$ and B is some open 6-ball around the origin. Consider now the prelegendrians P of $(\mathbb{C}^3, \mathcal{D}_{\mathrm{std}}) = (B, \mathcal{D}_0)$ constructed in Theorem 1.10, they define corresponding Legendrian lifts L on $\mathcal{C}(B, \mathcal{D}_0) = (U, \xi_0)$.

These Legendrian lifts can be assumed to be contained in some compact subset $K \subseteq U$. Gray stability tells us that there is a small time $s_0 > 0$ such that a family of isocontact embeddings $\phi_s : (K, \xi_0) \to (U, \xi_s)$, $s \in [0, s_0]$, exists. Consider then the images $\phi_s(L)$, these are Legendrians in (U, ξ_s). Since $\phi_0(L)$ is transverse to the projection pr_0 and transversality is an open condition, then $\phi_s(L)$ is transverse to pr_s for sufficiently small $s > 0$, hence it projects down to a prelegendrian in $(B, \mathcal{D}_s)$, as desired. This proves the first claim.

For the second claim, we consider any prelegendrian Λ in $(M, \mathcal{D})$ and zoom in at a point $p \in \Lambda$. The zooming argument above, in a chart, allows us to produce a path of prelegendrians $\Lambda_s \subseteq (B, \mathcal{D}_s)$ such that $\Lambda_0 \simeq T_p\Lambda$ is a prelegendrian linear subspace in $(B, \mathcal{D}_0) = (\mathbb{C}^3, \mathcal{D}_{\mathrm{std}})$. By the symmetry of the complex Heisenberg algebra, we can assume that Λ_0 projects to a prelegendrian front; we can then stabilise and denote $\tilde{\Lambda}_0 := s(\Lambda_0)$, and we denote its Legendrian lift by $\tilde{L}_0$.

In the contactisations $\mathrm{pr}_s\colon (U, \xi_s) \to (B, \mathcal{D}_s)$, the same construction as in the first claim produces a path of Legendrians $\tilde{L}_s := \phi_s(\tilde{L}_0)$ contained in a compact set $K \subseteq U$, for $s \in [0, s_0]$ with $s_0 > 0$ small. Furthermore, we can assume that the loose chart of each $\tilde{L}_s$ is contained in a compact set $W \subseteq K$.

Note that $\tilde{L}_s$ is C^∞-close to the Legendrian lift $L_s = \mathcal{L}(\Lambda_s)$. Then $\tilde{L}_s$ is a graph in a Weinstein neighbourhood chart of L_s, and by cutting off with a suitable function we produce a new path of Legendrians L'_s such that

- L'_s coincides with $\tilde{L}_s$ in W, which contains a loose chart;
- L'_s coincides with L_s away from $\mathcal{O}p(W)$.

Observe that $L'_0 = \tilde{L}_0$. By openness of transversality, we know that L'_s is transverse to pr_s for sufficiently small $s > 0$; this defines the stabilised prelegendrian of Λ.

□

The exact same reasoning shows:

Theorem 12.3. *Fix a fat distribution $(M^{4n+2}, \mathcal{D}^{4n})$. Suppose it contains a point whose nilpotentisation is the complex Heisenberg algebra. Then, it contains compact prelegendrians.*

12.3. **Stability phenomena.** The absence of a Darboux-type theorem for fat distributions of corank greater than 1 implies that Gray stability does not hold for these classes. Nonetheless:

Proof of Lemma 1.18. For the first claim we reason as in the proof of Theorem 1.17. We apply Gray stability in the contactisation to the Legendrian lift. This produces a family of Legendrians that, at least for small time, remain transverse to the projection to the fat manifold.

For the second claim we apply the reasoning appearing in the proof of Theorem 1.17 parametrically. That is: we apply the nilpotentisation parametrically on s. Then, given any prelegendrian for $\mathcal{D}_{\mathrm{std}}$ we may assume that it is contained in an arbitrary small ball, thanks to a dilation. Since $\mathcal{D}_{\mathrm{std}}$ is the model structure on the nilpotentisation, we see a copy of the prelegendrian in each $\mathcal{D}_s$, parametrically. □

Proof of Theorem 1.19. Suppose that such a path does exist. Since the homotopy $\mathcal{D}_s$ is compactly supported, the contactisations define a path of contact structures that is also compactly supported. As such, Gray stability applies, telling us that the Legendrian lifts are contactomorphic. Since the Legendrian lifts are distinguished by LCH, this is a contradiction. □

References

[Ada03] Jiro Adachi. Global stability of distributions of higher corank of derived length one. *Int. Math. Res. Not.*, (49):2621–2638, 2003. 1

[AFL17] Antonio Alarcón, Franc Forstnerič, and Francisco J. López. Holomorphic Legendrian curves. *Compositio Mathematica*, 153(9):1945–1986, 2017. 20

[AG18] Daniel Álvarez-Gavela. The simplification of singularities of Lagrangian and Legendrian fronts. *Invent. Math.*, 214(2):641–737, 2018. 8

[AGV12] V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko. *Singularities of Differentiable Maps, Volume 1: Classification of Critical Points, Caustics and Wave Fronts*. Birkhäuser, Boston, 2012. 25, 26

[AN01] V. I. Arnold and S. P. Novikov, editors. *Dynamical systems. IV*, volume 4 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, revised edition, 2001. Symplectic geometry and its applications, A translation of *Current problems in mathematics. Fundamental directions, Vol. 4 (Russian)*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1985 [MR0842907 (87j:58032)], Translated by G. Wasserman. 25, 26

[Arn90] V. I. Arnold. *Singularities of caustics and wave fronts*, volume 62 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990. 25, 26

[BD23] Aritra Bhowmick and Mahuya Datta. Existence of horizontal immersions in fat distributions. *Internat. J. Math.*, 34(10):Paper No. 2350056, 33, 2023. 2, 10, 20, 22

[BEM15] Matthew Strom Borman, Yakov Eliashberg, and Emmy Murphy. Existence and classification of overtwisted contact structures in all dimensions. *Acta Math.*, 215(2):281–361, 2015. 1

[Ben83] Daniel Bennequin. Entrelacements et équations de Pfaff. In *Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982)*, volume 107-108 of *Astérisque*, pages 87–161. Soc. Math. France, Paris, 1983. 1, 2

[Bho22] Aritra Bhowmick. *Horizontal, Contact and Partially Horizontal Immersions in Fat Distributions*. PhD thesis, Indian Statistical Institute, 2022. `https://digitalcommons.isical.ac.in/doctoral-theses/569/`. 2, 7, 15

[Bho24] Aritra Bhowmick. On horizontal immersions of discs in fat distributions of type (4,6). *J. Topol. Anal.*, 16(1):125–153, 2024. 2, 19

[CdPP20] Roger Casals, Álvaro del Pino, and Francisco Presas. Loose Engel structures. *Compos. Math.*, 156(2):412–434, 2020. 1

[CEM24] Kai Cieliebak, Yakov Eliashberg, and Nikolai Mishachev. *Introduction to the h -Principle*, volume 239. American Mathematical Society, 2024. 1, 8, 11, 15, 17

[CFS05] Sergio Console, Anna Fino, and Evangelia Samiou. The moduli space of six-dimensional two-step nilpotent Lie algebras. *Ann. Global Anal. Geom.*, 27(1):17–32, 2005. 49

[Che02] Yuri Chekanov. Differential algebra of Legendrian links. *Invent. Math.*, 150(3):441–483, 2002. 5

[CMP19] Roger Casals, Emmy Murphy, and Francisco Presas. Geometric criteria for overtwistedness. *J. Amer. Math. Soc.*, 32(2):563–604, 2019. 2

[CPdPP17] Roger Casals, José Luis Pérez, Álvaro del Pino, and Francisco Presas. Existence h-principle for Engel structures. *Invent. Math.*, 210(2):417–451, 2017. 1

[dPV20] Álvaro del Pino and Thomas Vogel. The Engel-Lutz twist and overtwisted Engel structures. *Geom. Topol.*, 24(5):2471–2546, 2020. 1

[DRG15] Georgios Dimitroglou Rizell and Roman Golovko. Estimating the number of Reeb chords using a linear representation of the characteristic algebra. *Algebr. Geom. Topol.*, 15(5):2887–2920, 2015. 47

[EES05a] Tobias Ekholm, John Etnyre, and Michael Sullivan. The contact homology of Legendrian submanifolds in $\mathbb{R}^{2n+1}$. *J. Differential Geom.*, 71(2):177–305, 2005. 5, 32

[EES05b] Tobias Ekholm, John Etnyre, and Michael Sullivan. Non-isotopic Legendrian submanifolds in $\mathbb{R}^{2n+1}$. *J. Differential Geom.*, 71(1):85–128, 2005. 5, 27, 31, 33, 45

[EES07] Tobias Ekholm, John Etnyre, and Michael Sullivan. Legendrian contact homology in $P \times \mathbb{R}$. *Trans. Amer. Math. Soc.*, 359(7):3301–3335, 2007. 5

[EES09] Tobias Ekholm, John B. Etnyre, and Joshua M. Sabloff. A duality exact sequence for Legendrian contact homology. *Duke Math. J.*, 150(1):1–75, 2009. 5, 31

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. pages 560–673. 2000. GAFA 2000 (Tel Aviv, 1999). 5

[Ekh08] Tobias Ekholm. Rational symplectic field theory over $\mathbb{Z}_2$ for exact Lagrangian cobordisms. *J. Eur. Math. Soc. (JEMS)*, 10(3):641–704, 2008. 46, 47

[Eli89] Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98(3):623–637, 1989. 1

[Eli90] Yakov Eliashberg. Topological characterization of Stein manifolds of dimension > 2. *Internat. J. Math.*, 1(1):29–46, 1990. 5, 27

[Eli92] Yakov Eliashberg. Contact 3-manifolds twenty years since J. Martinet's work. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):165–192, 1992. 2

[Eli98] Yakov Eliashberg. Invariants in contact topology. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, pages 327–338, 1998. 5

[ENS18] Tobias Ekholm, Lenhard Ng, and Vivek Shende. A complete knot invariant from contact homology. *Invent. Math.*, 211(3):1149–1200, 2018. 8

[FL22] Franc Forstnerič and Finnur Lárusson. Holomorphic Legendrian curves in projectivised cotangent bundles. *Indiana Univ. Math. J.*, 71(1):93–124, 2022. 21

[Ge92] Zhong Ge. Betti numbers, characteristic classes and sub-Riemannian geometry. *Illinois J. Math.*, 36(3):372–403, 1992. 48

[Gei08] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2008. 11

[Gol14] R. Golovko. A note on the front spinning construction. *Bull. Lond. Math. Soc.*, 46(2):258–268, 2014. 5, 31, 33, 47

[Gro86] Mikhael Gromov. *Partial Differential Relations*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1986. 1, 8, 17

[Gro96] Mikhael Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian Geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996. 7, 15

[LC13] Peter Lambert-Cole. Legendrian products, 2013. https://arxiv.org/abs/1301.3700. 5, 31

[MA23] Javier Martínez-Aguinaga. Existence and classification of maximal growth distributions, 2023. https://arxiv.org/abs/2308.10762. 1

[MAdP21] Javier Martínez-Aguinaga and Álvaro del Pino. Convex integration with avoidance and hyperbolic (4,6) distributions, 2021. https://arxiv.org/abs/2112.14632. 1, 2, 8

[Mag86] L. Magnin. Sur les algèbres de Lie nilpotentes de dimension ≤ 7. *J. Geom. Phys.*, 3(1):119–144, 1986. 49

[McD87] Dusa McDuff. Applications of convex integration to symplectic and contact geometry. *Ann. Inst. Fourier (Grenoble)*, 37(1):107–133, 1987. 1

[Mon02] Richard Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. 1, 2, 48

[MS74] John Willard Milnor and James D. Stasheff. *Characteristic Classes*. Number 76. Princeton university press, 1974. 17

[Mur19] Emmy Murphy. Loose legendrian embeddings in high dimensional contact manifolds, 2019. https://arxiv.org/abs/1201.2245. 5, 27, 30, 32, 46, 47

[Ng05] Lenhard Ng. Knot and braid invariants from contact homology. I. *Geom. Topol.*, 9:247–297, 2005. 4

[Ng14] Lenhard Ng. A topological introduction to knot contact homology. In *Contact and symplectic topology*, volume 26 of *Bolyai Soc. Math. Stud.*, pages 485–530. János Bolyai Math. Soc., Budapest, 2014. 4

[OV00] Hirosi Ooguri and Cumrun Vafa. Knot invariants and topological strings. *Nuclear Phys. B*, 577(3):419–438, 2000. 4

[Ray68] C. B. Rayner. The exponential map for the Lagrange problem on differentiable manifolds. *Philos. Trans. Roy. Soc. London Ser. A*, 262:299–344, 1967/68. 11

[Rut23] Nina Rutten. Fat prolonged distributions of type (4,6). Master's thesis, Utrecht University, 2023. https://studenttheses.uu.nl/handle/20.500.12932/44715. 19, 20, 21

[She19] Vivek Shende. The conormal torus is a complete knot invariant. *Forum Math. Pi*, 7:e6, 16, 2019. 4, 8

[Vog09] Thomas Vogel. Existence of Engel structures. *Ann. of Math. (2)*, 169(1):79–137, 2009. 1

[Wei80] Alan Weinstein. Fat bundles and symplectic manifolds. *Adv. in Math.*, 37(3):239–250, 1980. 1

[Wel69] R. O. Wells, Jr. Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles. *Math. Ann.*, 179:123–129, 1969. 17

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