

# Decay matrix of $B$ - $\bar{B}$ mixing: Mixing of dimension-seven operators into dimension-six operators under renormalization

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**ABSTRACT:** The precise measurement of the width difference  $\Delta\Gamma_s$  among the mass eigenstates of the  $B_s$ - $\bar{B}_s$  system requires the calculation of the corresponding decay matrix to order  $\alpha_s/m_b$ . QCD corrections to power-suppressed terms in the Heavy Quark Expansion involve the renormalization of dimension-7 four-quark operators for which no general methodology is available yet. In the  $\overline{\text{MS}}$  scheme one-loop corrections to matrix elements of dimension-7 operators violate the power counting, but we find the responsible terms to be infrared-finite and show that they can be absorbed into finite counterterms proportional to dimension-6 operators. We calculate all these counterterms and subsequently verify the consistency of our results with hadronic matrix elements calculated in the limit of a large number  $N_c$  of colours. The condition of correct power counting implies constraints on the possible definitions of evanescent operators.

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## 1 Introduction

A neutral  $B_q$  meson mixes with its antimeson  $\bar{B}_q$  leading to a time evolution formula featuring damped oscillations between these two states:

$$i \frac{d}{dt} \begin{pmatrix} |B_q(t)\rangle \\ |\bar{B}_q(t)\rangle \end{pmatrix} = \left( M^q - \frac{i}{2} \Gamma^q \right) \begin{pmatrix} |B_q(t)\rangle \\ |\bar{B}_q(t)\rangle \end{pmatrix}, \quad q = d \text{ or } s, \quad (1.1)$$

with the hermitian  $2 \times 2$  mass and decay matrices  $M^q$  and  $\Gamma^q$ , respectively. Relevant for  $B_q - \bar{B}_q$  mixing are the off-diagonal elements  $M_{12}^q = M_{21}^{q*}$  and  $\Gamma_{12}^q = \Gamma_{21}^{q*}$ . The diagonalization of  $M^q - i\Gamma^q/2$  yields the mass eigenstates  $B_L^q$  and  $B_H^q$  with “L” and “H” denoting “light” and “heavy”.  $B_L^q$  and  $B_H^q$  have different masses  $M_{H,L}^q$  and decay widths  $\Gamma_{H,L}^q$ .  $|M_{12}^q| \simeq \Delta M_q/2$  determines the mass difference  $\Delta M_q = M_H^q - M_L^q$ , which coincides with the well-measured  $B_q - \bar{B}_q$  oscillation frequency.  $\Gamma_{12}^q$  is much smaller in magnitude than  $M_{12}^q$  and enters the width difference  $\Delta\Gamma_q = \Gamma_L^q - \Gamma_H^q$  and the CP asymmetry in flavour-specific decays,  $a_{\text{fs}}^q$  as

$$\frac{\Delta\Gamma_q}{\Delta M_q} = -\text{Re} \frac{\Gamma_{12}^q}{M_{12}^q}, \quad a_{\text{fs}}^q = \text{Im} \frac{\Gamma_{12}^q}{M_{12}^q}. \quad (1.2)$$

We know much less about  $\Gamma_{12}^q$  than  $M_{12}^q$ : While  $\Delta\Gamma_s$  is measured precisely,

$$\Delta\Gamma_s = (0.0781 \pm 0.0035) \text{ ps}^{-1}, \quad (1.3)$$

experimental results on  $\Delta\Gamma_d$ ,  $a_{\text{fs}}^d$ , and  $a_{\text{fs}}^s$  are all consistent with zero. The 2025 HFLAV average [1] in Eq. (1.3) uses data from the DØ, CDF, ATLAS, LHCb, and CMS experiments [2–15]. On the theory side,  $\Gamma_{12}^q$  is much more difficult to calculate than  $M_{12}^q$ : Firstly, the scale governing the perturbative QCD corrections is  $\mu \sim m_b$  in  $\Gamma_{12}^q$ , opposed to  $\mu \sim m_t$  in  $M_{12}^q$ . Secondly, unlike  $M_{12}^q$ , which only involves a single dimension-6 operator, the Heavy Quark Expansion (HQE) [16–19] of  $\Gamma_{12}^q$  involves corrections of order  $2\Lambda_{QCD}/m_b = \mathcal{O}(20\%)$  containing five matrix elements of dimension-7 operators [20]. The calculation of these matrix elements with lattice QCD is difficult, because they mix with dimension-6 operators requiring the control of effects which diverge with one power of the inverse lattice spacing. Currently errors of order 40% are quoted for these matrix elements [21].

For the discussion of  $1/m_b$  corrections it is convenient to work with  $\Gamma_{21}^q$  rather than  $\Gamma_{12}^q$ . We further specify to  $q = s$ , for which the envisaged corrections are most urgently needed; the generalization to  $q = d$  is straightforward. The  $(\Lambda_{QCD}/m_b)^0$  leading-power term of the HQE for  $\Gamma_{21}^s$  involves two dimension-six  $\Delta B = -2$  operators ( $B$  denotes the beauty quantum number)

$$Q = \bar{b}_i \gamma_\mu L s_i \bar{b}_j \gamma^\mu L s_j, \quad \tilde{Q}_S = \bar{b}_i L s_j \bar{b}_j L s_i \quad (1.4)$$

Here the  $i, j$  are color indices and  $L = (1 - \gamma_5)$ . The Wilson coefficients of these operators have been calculated at next-to-leading order (NLO) of QCD perturbation theory [22–27] and NNLO results have been first obtained in the limit of a large number  $N_f$  of flavors [28] and subsequently for contributions of the chromomagnetic operator [27, 28]. Finally the complete NNLO contributions involving the dominant current-current operators have been calculated [29, 30].

The hadronic matrix elements of the operators in Eq. (1.4), which are calculated with non-perturbative methods such as lattice QCD [31] or QCD sum rules [32], are commonly parametrized as

$$\langle B_s | Q(\mu) | \bar{B}_s \rangle = M_{B_s}^2 f_{B_s}^2 \left( 2 + \frac{2}{N_c} \right) B(\mu), \quad \langle B_s | \tilde{Q}_S(\mu) | \bar{B}_s \rangle = M_{B_s}^2 f_{B_s}^2 \left( 1 - \frac{2}{N_c} \right) \tilde{B}'_S(\mu). \quad (1.5)$$

Here  $M_{B_s}$  and  $f_{B_s}$  are the mass and the decay constant of the  $B_s$  meson,  $N_c$  is the number of colors and  $\mu = \mathcal{O}(m_b)$  is the renormalization scale at which the matrix elements are calculated. In lattice-gauge theory  $\mu$  is the scale at which the lattice continuum matching is performed.

The prediction of Ref.[30] for  $\Delta\Gamma_s$  reads

$$\Delta\Gamma_s = \left[ 0.077 \begin{smallmatrix} +0.005 \\ -0.007 \end{smallmatrix} \Big|_{\text{scale}} \pm 0.014 \Big|_{1/m_b} \pm 0.002 \Big|_{B, \tilde{B}'_S} \pm 0.001 \Big|_{\text{input}} \right] \text{ps}^{-1}. \quad (1.6)$$

Here the first uncertainty shows the dependence on the renormalization scale of the NNLO result for the leading-power term in  $\Gamma_{12}^s$ . The second uncertainty stems from the  $1/m_b$  term. The third and last uncertainties refer to the hadronic matrix elements in Eq. (1.5) and the parametric input, namely quark masses and CKM elements, respectively. The quoted number  $0.014 \text{ps}^{-1}$  reflects the uncertainties from both the lattice calculation [21] and the perturbative LO calculation of the coefficients [20]. In fact both uncertainties cannot be separated, because a meaningful lattice-continuum matching requires coefficients calculated at least to NLO.

The large uncertainty in Eq. (1.6) calls for an NLO calculation of the coefficients of the dimension-7 operators occurring in the  $1/m_b$  corrections. In this paper we present a first step in that direction by calculating the mixing of these operators with the dimension-6 operators  $Q$  and  $\tilde{Q}_S$  and the associated

renormalization prescription. The renormalization constants presented below are needed for both the lattice-continuum matching and the calculation of the NLO coefficients. Our paper is organized as follows: In Sec. 2 we list the operators and summarize preliminary work. Secs. 3 and 4 are devoted to the derivation of the renormalization constants and the calculation of matrix elements in the limit of a large number of colors,  $N_c \rightarrow \infty$ , respectively. In Sec. 5 we conclude.

## 2 Preliminaries

$\Gamma_{21}^s = \Gamma_{12}^{s*}$  is decomposed as [22]

$$\Gamma_{21}^s = -(V_{cb}^* V_{cs})^2 \Gamma_{21}^{cc} - 2V_{cb}^* V_{cs} V_{ub}^* V_{us} \Gamma_{21}^{uc} - (V_{ub}^* V_{us})^2 \Gamma_{21}^{uu}. \quad (2.1)$$

The individual terms read

$$\Gamma_{21}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[ H^{ab}(z) \langle \bar{B}_s | Q | B_s \rangle + \tilde{H}_S^{ab}(z) \langle \bar{B}_s | \tilde{Q}_S | B_s \rangle \right] + \tilde{\Gamma}_{21,1/m_b}^{ab} + \dots \quad (2.2)$$

Here the first term is the leading-power term with various contributions to the coefficients  $H^{ab}$  and  $\tilde{H}_S^{ab}$  calculated to NLO and NNLO as described in the Introduction.  $z = m_c^2/m_b^2$  encodes the heavy-quark mass ratio.  $\tilde{\Gamma}_{21,1/m_b}^{ab}$  contains the  $1/m_b$  corrections. We use the definition of Ref. [25] which differs from the original one in Ref. [20], because the original calculation was not done in the basis  $(Q, \tilde{Q}_S)$  for the dimension-6 operators used since 2006. The dots denote terms of order  $1/m_b^2$  and higher. One has

$$\tilde{\Gamma}_{12,1/m_b}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[ g_0^{ab}(z) \langle \bar{B}_s | R_0 | B_s \rangle + \sum_{j=1}^3 \left[ g_j^{ab}(z) \langle \bar{B}_s | R_j | B_s \rangle + \tilde{g}_j^{ab}(z) \langle \bar{B}_s | \tilde{R}_j | B_s \rangle \right] \right] \quad (2.3)$$

the coefficients  $g_j^{ab}(z)$ , derived from the results in Refs. [20] and [33], are known to LO in  $\alpha_s$  and can be found in Eqs. (25) and (26) of Ref. [25].

The dimension-7 operators entering Eq. (2.3) are [20]

$$R_0 = Q_S + \tilde{Q}_S + \frac{1}{2}Q \quad \text{with} \quad Q_S = \bar{b}_i L s_i \bar{b}_j L s_j, \quad (2.4)$$

$$R_1 = \frac{m_s}{m_b} \bar{b}_i L s_i \bar{b}_i R s_i \quad (2.5)$$

$$R_2 = \frac{1}{m_b^2} \left( \bar{b} \overleftarrow{D}_\rho \right)_i \gamma^\mu L (D^\rho s)_i \bar{b}_j \gamma_\mu L s_j, \quad (2.6)$$

$$R_3 = \frac{1}{m_b^2} \left( \bar{b} \overleftarrow{D}_\rho \right)_i L (D^\rho s)_i \bar{b}_j L s_j, \quad (2.7)$$

with  $L, R = (1 \mp \gamma_5)$  and  $D_\rho = \partial_\rho - ig_s A_\rho^a T^a$ . There are further color-rearranged operators  $\tilde{R}_i$  for  $i = 1, 2, 3$ , obtained from the expressions above by interchanging  $s_i$  and  $s_j$ . At order  $1/m_b$  only a subset of these operators is needed, as some of their linear combinations have  $1/m_b^2$  suppressed matrix elements; in particular one has  $\langle R_2 \rangle = -\langle \tilde{R}_2 \rangle + \mathcal{O}(1/m_b^2)$ .

In the power counting factors of  $m_s$  and derivatives  $\partial^\mu$  on the  $s$  field count as  $\Lambda_{QCD}$  as they lead to factors of  $m_s$  or  $p_s^\mu$  in the tree-level matrix elements between quark states, accompanied by factors

of  $1/m_b$  in the coefficients. The bare loop matrix elements do not have this feature, because  $\partial^\mu s$  can give a power of the loop momentum with the subsequent loop integration giving a power of  $m_b$  instead of  $m_s$  or  $p_s^\mu$ . Clearly, this is an ultraviolet (UV) effect stemming from the region of large loop momenta. We therefore expect to be able to absorb these terms into finite counterterms to restore the correct power counting of the loop-corrected matrix elements. To this end we employ dimensional regularization in  $D = 4 - 2\epsilon$  dimensions and define

$$R_j^{\text{ren}} = Z_{jQ} Q^{\text{bare}} + Z_{j\tilde{Q}_S} \tilde{Q}_S^{\text{bare}} + \sum_{k=0}^3 \left[ Z_{jk} R_k^{\text{bare}} + \tilde{Z}_{jk+3} \tilde{R}_k^{\text{bare}} \right], \quad (2.8)$$

$$\tilde{R}_j^{\text{ren}} = \tilde{Z}_{jQ} Q^{\text{bare}} + \tilde{Z}_{j\tilde{Q}_S} \tilde{Q}_S^{\text{bare}} + \sum_{k=0}^3 \left[ \tilde{Z}_{jk} R_k^{\text{bare}} + \tilde{Z}_{jk+3} \tilde{R}_k^{\text{bare}} \right], \quad (2.9)$$

and

$$Z_{jQ} = \frac{\alpha_s}{4\pi} \left[ \frac{Z_{jQ}^{(11)}}{\epsilon} + Z_{jQ}^{(10)} \right] + \mathcal{O}(\alpha_s^2), \quad Z_{j\tilde{Q}_S} = \frac{\alpha_s}{4\pi} \left[ \frac{\tilde{Z}_{j\tilde{Q}_S}^{(11)}}{\epsilon} + Z_{j\tilde{Q}_S}^{(10)} \right] + \mathcal{O}(\alpha_s^2) \quad (2.10)$$

with analogous definitions of  $\tilde{Z}_{jQ}$  and  $\tilde{Z}_{j\tilde{Q}_S}$ . We replace  $\mu^\epsilon g$  by  $(\mu e^{-\gamma_E}/(4\pi))^\epsilon g$  in the Feynman rules for the gauge coupling to remove the Euler constant  $\gamma_E$  and the term  $\log(4\pi)$  from our matrix elements.  $\mu$  is the renormalization scale at which the matrix elements are calculated and  $\alpha_s = g^2/(4\pi)$  is to be read as  $\alpha_s(\mu)$  everywhere. We further write

$$\langle R_j \rangle^{\text{ren}} = \langle R_j \rangle^{(0)} + \frac{\alpha_s}{4\pi} \langle R_j^{\text{ren}} \rangle^{(1)} + \mathcal{O}(\alpha_s^2) \quad (2.11)$$

and similarly for  $\langle \tilde{R}_j \rangle^{\text{ren}}$  and the bare matrix elements. We will expand the one-loop matrix elements in terms of tree-level matrix elements counting  $\langle R_j \rangle^{(0)} = \mathcal{O}(1/m_b)$  and  $\langle Q \rangle^{(0)}, \langle \tilde{Q}_S \rangle^{(0)} = \mathcal{O}(m_b^0)$ .

In this paper only terms of  $\langle R_j \rangle^{(1)}$  proportional to  $\langle Q \rangle^{(0)}$  and  $\langle \tilde{Q}_S \rangle^{(0)}$  are to be considered. In the  $\overline{\text{MS}}$  scheme the finite pieces in Eq. (2.10) all vanish but  $\langle R_j \rangle^{(1)}$  contains these unwanted matrix elements of dimension-6 operators. The purpose of this paper is the determination of  $Z_{jQ}^{(10)}, Z_{j\tilde{Q}_S}^{(10)}, \tilde{Z}_{jQ}^{(10)}$ , and  $\tilde{Z}_{j\tilde{Q}_S}^{(10)}$  in

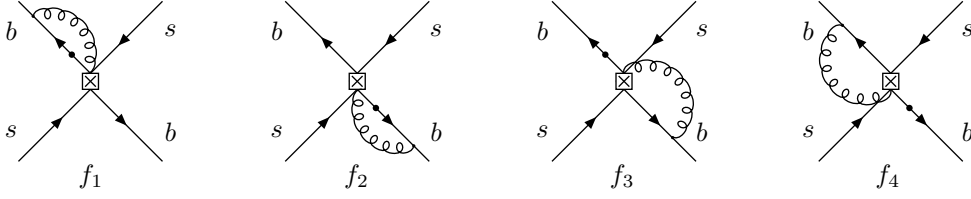
$$\delta R_j \equiv \frac{\alpha_s}{4\pi} \delta R_j^{(1)}, \quad \delta R_j^{(1)} \equiv \left[ \frac{Z_{jQ}^{(11)}}{\epsilon} + Z_{jQ}^{(10)} \right] Q + \left[ \frac{Z_{j\tilde{Q}_S}^{(11)}}{\epsilon} + Z_{j\tilde{Q}_S}^{(10)} \right] \tilde{Q}_S \quad (2.12)$$

$$\delta \tilde{R}_j \equiv \frac{\alpha_s}{4\pi} \delta \tilde{R}_j^{(1)}, \quad \delta \tilde{R}_j^{(1)} \equiv \left[ \frac{\tilde{Z}_{jQ}^{(11)}}{\epsilon} + \tilde{Z}_{jQ}^{(10)} \right] Q + \left[ \frac{\tilde{Z}_{j\tilde{Q}_S}^{(11)}}{\epsilon} + \tilde{Z}_{j\tilde{Q}_S}^{(10)} \right] \tilde{Q}_S \quad (2.13)$$

such that

$$\langle R_j^{\text{ren}} \rangle^{(1)} = \langle R_j^{\text{bare}} \rangle^{(1)} + \langle \delta R_j \rangle^{(1)} = \mathcal{O}\left(\frac{1}{m_b}\right) \quad (2.14)$$

and analogously for  $\tilde{R}_j^{\text{ren}}$ .



**Figure 1.** Infrared-finite diagrams originating from the couplings of the operators  $\tilde{R}_2$  and  $\tilde{R}_3$  with a gluon field and a derivative acting on a  $b$  field. The  $b$  line with derivative is marked with a dot.

For  $R_0$  this task has already been completed in Ref. [20]. For the dimension-6 basis  $(Q, \tilde{Q}_S)$  the counterterm reads

$$\begin{aligned} \delta R_0^{(1)} = & \left( (N_c + 1) \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) + \frac{N_c^2 + 2N_c - 2}{N_c} \right) Q \\ & + 2(N_c + 1) \left( \frac{1}{\epsilon} + 2 \log \frac{\mu}{m_b} + 1 \right) \tilde{Q}_S. \end{aligned} \quad (2.15)$$

$C_F = (N_c^2 - 1)/(2N_c)$  is a color factor. The result for  $N_c = 3$  was given in Ref. [25].  $\delta R_0$  has been calculated to order  $\alpha_s^2 N_f$  in Ref. [28]; the full result to order  $\alpha_s^2$  is presented in Ref. [30].

$\delta R_1^{(1)} = \delta \tilde{R}_1^{(1)} = 0$ , because the factor of  $m_s$  in the definition of these operators (see Eq. (2.5)) prevents any mixings with  $Q$  or  $\tilde{Q}_S$ .

### 3 Renormalization of operators $R_2$ , $\tilde{R}_2$ , $R_3$ , and $\tilde{R}_3$

In this section we calculate  $\delta R_{2,3}^{(1)}$  and  $\delta \tilde{R}_{2,3}^{(1)}$ . We encounter two types of contributions to the desired finite renormalization. Fig. 1 shows those Feynman diagrams which have a single derivative acting only on a  $b$  field and a gluon field from  $D^\rho s$ . Fig. 2 depicts the diagrams which involve both  $\partial_\rho b$  and  $\partial^\rho s$ . In the integrals corresponding to these Feynman diagrams, we set the small strange quark momentum  $p_s$  and the strange quark mass  $m_s$ , which we count as  $\Lambda_{QCD}$ , equal to zero. Terms proportional to  $p_s$  or  $m_s$  will only be needed for the calculation of  $Z_{jk}$ ,  $\tilde{Z}_{jk}$  in Eqs. (2.8) and (2.9), but not for the mixings into  $Q$ ,  $\tilde{Q}_S$ . Now  $\partial^\rho s$  can only give a non-zero contribution when the derivative gives a power of the loop momentum, reducing the number of diagrams to the four of Fig. 1.

#### 3.1 Evanescent operators

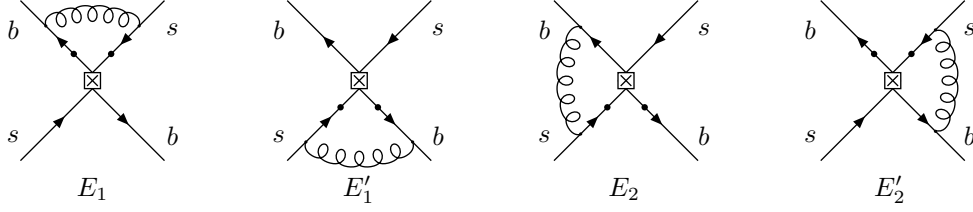
Our results depend on the chosen renormalization scheme, characterized by the use of an anticommuting  $\gamma_5$  (NDR scheme) and the following definitions of the evanescent operators [22, 34]

$$E_1[Q] \equiv \bar{b}_i \gamma_\mu \gamma_\nu \gamma_\rho L s_i \bar{b}_j \gamma^\rho \gamma^\nu \gamma^\mu L s_j - (4 - 8\epsilon) Q + \mathcal{O}(\epsilon^2), \quad (3.1)$$

$$E_2[Q_S] \equiv \bar{b}_i \gamma_\mu \gamma_\nu L s_i \bar{b}_j \gamma_\nu \gamma_\mu L s_j + (8 - 8\epsilon) \tilde{Q}_S + \mathcal{O}(\epsilon^2), \quad (3.2)$$

$$E_2[\tilde{Q}_S] \equiv \bar{b}_i \gamma_\mu \gamma_\nu L s_j \bar{b}_j \gamma_\nu \gamma_\mu L s_i + (8 - 8\epsilon) Q_S + \mathcal{O}(\epsilon^2). \quad (3.3)$$

Evanescent operators receive finite counterterm such that their renormalized matrix elements vanish for  $\epsilon = 0$ . The coefficients multiplying  $\epsilon$  in the definitions in Eqs. (3.1) to (3.3), however, combine with



**Figure 2.** Diagrams  $E_1$ ,  $E'_1$ ,  $E_2$  and  $E'_2$  which contribute to the finite renormalization of operators  $R_2$ ,  $R_3$ ,  $\tilde{R}_2$  and  $\tilde{R}_3$ . The derivatives, acting on the  $b$  and  $s$  fields, are denoted as dots on the corresponding lines.

$1/\epsilon$  poles to finite contributions affecting our  $Z_{jQ}^{(10)}, \dots, \tilde{Z}_{j\tilde{Q}_S}^{(10)}$ . In the practical calculation, for instance, Eqs. (3.2) and (3.3) mean that one replaces structures with two Dirac matrices on each fermion line as follows:

$$\begin{aligned} [\gamma_\mu \gamma_\nu L]_{lm} \otimes [\gamma_\nu \gamma_\mu L]_{np} &\rightarrow (8 - 8\epsilon) L_{lp} \otimes L_{nm}, \\ [\gamma_\mu \gamma_\nu L]_{lm} \otimes [\gamma_\mu \gamma_\nu L]_{np} &\rightarrow -(8 - 8\epsilon) L_{lp} \otimes L_{nm} + (8 - 4\epsilon) L_{lm} \otimes L_{np} \end{aligned} \quad (3.4)$$

with the second prescription obtained from the first one by a simple anti-commutation of the two Dirac matrices.  $l, \dots, p$  are Dirac indices. Note that these relations involve Fierz transformations (interchange of indices  $m$  and  $p$  on the fermion lines), which a priori do not hold in  $D \neq 4$  dimensions. The choice of the coefficients of  $\epsilon$  in Eqs. (3.2) and (3.3) insures that the renormalized one-loop matrix elements obey Fierz symmetry, *i.e.* these matrix elements are equal to those of the operators  $Q^F = \tilde{Q}$ ,  $Q_S^F, \tilde{Q}_S^F$ , found from  $Q, Q_S, \tilde{Q}_S$  by Fierz transformations.

We define the evanescent operators for the dimension-7 operators as

$$E_1[R_2] \equiv \frac{1}{m_b^2} \left( \bar{b} \overleftarrow{D}_\rho \right)_i \gamma_\mu \gamma_\nu \gamma_\alpha L (D^\rho s)_i \bar{b}_j \gamma^\alpha \gamma^\nu \gamma^\mu L s_j - (4 - a_2 \epsilon) R_2 + \mathcal{O}(\epsilon^2), \quad (3.5)$$

$$E_2[R_2] \equiv \frac{1}{m_b^2} \left( \bar{b} \overleftarrow{D}_\rho \right)_i \gamma_\mu \gamma_\nu L (D^\rho s)_i \bar{b}_j \gamma^\nu \gamma^\mu L s_j - (8 - b_2 \epsilon) \tilde{R}_3 + \mathcal{O}(\epsilon^2), \quad (3.6)$$

$$E_2[R_3] \equiv \frac{1}{m_b^2} \left( \bar{b} \overleftarrow{D}_\rho \right)_i \gamma_\mu \gamma_\nu L (D^\rho s)_i \bar{b}_j \gamma^\nu \gamma^\mu L s_j - (8 - b_3 \epsilon) \tilde{R}_3 + \mathcal{O}(\epsilon^2), \quad (3.7)$$

and similar for color-rearranged operators  $\tilde{R}_n$  ( $n = 2, 3$ ), obtained from the expressions above by interchanging  $s_i$  and  $s_j$ , and replacing constants  $a_n$  and  $b_n$  by  $\tilde{a}_n$  and  $\tilde{b}_n$  correspondingly. The numerical values of these constants are to be fixed later. If we impose the requirement of Fierz invariance, in analogy to  $Q$  and  $\tilde{Q}_S$  above, on  $R_j$  and  $\tilde{R}_j$ , we obtain  $a_2 = \tilde{a}_2 = 12$  and  $b_2 = \tilde{b}_2 = 4$ , while the Fierz invariance of operators  $R_3$  and  $\tilde{R}_3$  yields  $b_3 = \tilde{b}_3 = 10$ . Thus, since  $b_2 = \tilde{b}_2$  and  $b_3 = \tilde{b}_3$  differ from each other and from the corresponding coefficient in Eq. (3.3), the replacement rule in Eq. (3.4) is different for  $Q_S, \tilde{Q}_S, R_2, \tilde{R}_2, R_3$ , and  $\tilde{R}_3$ , if one applies the criterion that  $\overline{\text{MS}}$ -renormalized matrix elements of these operators should be equal to those of the operators obtained from them by Fierz transformations. In Sec. 4 we will see that Fierz symmetry matters for correct results for  $\tilde{Z}_{jQ}^{(10)}$  and  $\tilde{Z}_{j\tilde{Q}_S}^{(10)}$ .

### 3.2 Results for the counterterms

Collecting all the results, which contribute to the finite renormalization of the operator  $R_2$  in the  $\overline{\text{MS}}$  scheme we obtain

$$\begin{aligned}\delta\langle R_2^{\overline{\text{MS}}} \rangle^{(1)} = & \left( \left( 2 - \frac{2}{N_c} - \frac{8C_F}{3} \right) \log \frac{\mu}{m_b} + 2 - \frac{a_2}{24} - \left( 2 - \frac{a_2}{24} \right) \frac{1}{N_c} - \frac{38C_F}{9} \right) \langle Q \rangle^{(0)} \\ & + \left( \left( \frac{2}{3} - \frac{4}{3N_c} \right) \log \frac{\mu}{m_b} + \frac{2}{9} - \left( \frac{4}{9} - \frac{b_2}{12} \right) \frac{1}{N_c} \right) \langle \tilde{Q}_S \rangle^{(0)} \\ & + \left( \left( \frac{4}{3} - \frac{2}{3N_c} + \frac{4C_F}{3} \right) \log \frac{\mu}{m_b} + \left( \frac{4}{9} - \frac{b_2}{12} \right) - \frac{2}{9N_c} - \frac{2C_F}{9} \right) \langle Q_S \rangle^{(0)}\end{aligned}\quad (3.8)$$

$$\begin{aligned}= & \left( \left( \frac{4}{3} - \frac{5}{3N_c} - \frac{10C_F}{3} \right) \log \frac{\mu}{m_b} + \frac{16}{9} + \frac{b_2 - a_2}{24} - \left( \frac{17}{9} - \frac{a_2}{24} \right) \frac{1}{N_c} - \frac{37C_F}{9} \right) \langle Q \rangle^{(0)} \\ & + \left( \left( -\frac{2}{3} - \frac{2}{3N_c} - \frac{4C_F}{3} \right) \log \frac{\mu}{m_b} - \frac{2}{9} + \frac{b_2}{12} - \left( \frac{2}{9} - \frac{b_2}{12} \right) \frac{1}{N_c} + \frac{2C_F}{9} \right) \langle \tilde{Q}_S \rangle^{(0)}.\end{aligned}\quad (3.9)$$

In Eq. (3.9) we have used Eq. (2.4) to eliminate  $Q_S = R_0 - \tilde{Q}_S - Q/2$  and used that  $\langle R_0 \rangle^{(0)} = 0$  for  $p_s = 0$ . Thus

$$\begin{aligned}\delta R_2^{(1)} = & \left( \left( -\frac{4}{3} + \frac{5}{3N_c} + \frac{10C_F}{3} \right) \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) - \frac{16}{9} - \frac{b_2 - a_2}{24} + \left( \frac{17}{9} - \frac{a_2}{24} \right) \frac{1}{N_c} + \frac{37C_F}{9} \right) Q \\ & + \left( \left( \frac{2}{3} + \frac{2}{3N_c} + \frac{4C_F}{3} \right) \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) + \frac{2}{9} - \frac{b_2}{12} + \left( \frac{2}{9} - \frac{b_2}{12} \right) \frac{1}{N_c} - \frac{2C_F}{9} \right) \tilde{Q}_S\end{aligned}\quad (3.10)$$

and one arrives at  $\langle R_2^{\text{bare}} + \delta R_2 \rangle^{(1)} = \mathcal{O}(1/m_b)$  as desired.

Similarly for operator  $R_3$  we obtain

$$\begin{aligned}\langle R_3^{\overline{\text{MS}}} \rangle^{(1)} = & \left( \left( \frac{1}{6} - \frac{1}{6N_c} \right) \log \frac{\mu}{m_b} + \frac{1}{18} - \frac{1}{18N_c} \right) \langle Q \rangle^{(0)} \\ & + \left( \left( \frac{4}{3} + \frac{2}{3N_c} \right) \log \frac{\mu}{m_b} + \frac{29}{18} + \left( \frac{5}{9} - \frac{b_3}{24} \right) \frac{1}{N_c} \right) \langle \tilde{Q}_S \rangle^{(0)} \\ & + \left( -\left( \frac{2}{3} + \frac{4}{3N_c} \right) \log \frac{\mu}{m_b} - \frac{5}{9} + \frac{b_3}{24} - \frac{29}{18N_c} - 3C_F \right) \langle Q_S \rangle^{(0)},\end{aligned}\quad (3.11)$$

entailing

$$\begin{aligned}\delta R_3^{(1)} = & - \left( \left( \frac{1}{2} + \frac{1}{2N_c} \right) \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) + \frac{1}{3} - \frac{b_3}{48} + \frac{3}{4N_c} + \frac{3C_F}{2} \right) Q \\ & - \left( \left( 2 + \frac{2}{N_c} \right) \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) + \frac{13}{6} - \frac{b_3}{24} + \left( \frac{13}{6} - \frac{b_3}{24} \right) \frac{1}{N_c} + 3C_F \right) \tilde{Q}_S.\end{aligned}\quad (3.12)$$

For  $\tilde{R}_2$  we find

$$\begin{aligned}\langle \tilde{R}_2^{\overline{\text{MS}}} \rangle^{(1)} = & \left( \left( -\frac{4}{3} + \frac{4}{3N_c} + 4C_F \right) \log \frac{\mu}{m_b} - \frac{19}{9} + \frac{19}{9N_c} + \left( 4 - \frac{\tilde{a}_2}{12} \right) C_F \right) \langle Q \rangle^{(0)} \\ & + \left( \left( -\frac{2}{3N_c} + \frac{4C_F}{3} \right) \log \frac{\mu}{m_b} + \frac{1}{9N_c} + \frac{4C_F}{9} \right) \langle \tilde{Q}_S \rangle^{(0)} \\ & + \left( \left( \frac{2}{3} + \frac{8C_F}{3} \right) \log \frac{\mu}{m_b} - \frac{1}{9} + \left( \frac{8}{9} - \frac{\tilde{b}_2}{6} \right) C_F \right) \langle Q_S \rangle^{(0)},\end{aligned}\quad (3.13)$$

so that

$$\begin{aligned}\delta\tilde{R}_2^{(1)} = & \left( \left( \frac{5}{3} - \frac{4}{3N_c} - \frac{8C_F}{3} \right) \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) + \frac{37}{18} - \frac{19}{9N_c} - \left( \frac{32}{9} + \frac{\tilde{b}_2 - \tilde{a}_2}{12} \right) C_F \right) Q \\ & + \left( \left( \frac{2}{3} + \frac{2}{3N_c} + \frac{4C_F}{3} \right) \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) - \frac{1}{9} - \frac{1}{9N_c} + \left( \frac{4}{9} - \frac{\tilde{b}_2}{6} \right) C_F \right) \tilde{Q}_S.\end{aligned}\quad (3.14)$$

Finally for operator  $\tilde{R}_3$  we obtain

$$\begin{aligned}\langle \tilde{R}_3^{\overline{\text{MS}}} \rangle^{(1)} = & \left( \frac{1}{3} C_F \log \frac{\mu}{m_b} + \frac{C_F}{9} \right) \langle Q \rangle^{(0)} + \left( \frac{8C_F}{3} \log \frac{\mu}{m_b} + \frac{3}{2N_c} + \frac{29C_F}{9} \right) \langle \tilde{Q} \rangle_S^{(0)} \\ & + \left( -\frac{4C_F}{3} \log \frac{\mu}{m_b} - \frac{3}{2} - \left( \frac{10}{9} - \frac{\tilde{b}_3}{12} \right) C_F \right) \langle Q_S \rangle^{(0)},\end{aligned}\quad (3.15)$$

leading to the counterterm

$$\begin{aligned}\delta\tilde{R}_3^{(1)} = & - \left( C_F \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) + \frac{3}{4} + \left( \frac{2}{3} - \frac{\tilde{b}_3}{24} \right) C_F \right) Q \\ & - \left( 4C_F \left( \frac{1}{2\epsilon} + \log \frac{\mu}{m_b} \right) + \frac{3}{2} + \frac{3}{2N_c} + \left( \frac{13}{3} - \frac{\tilde{b}_3}{12} \right) C_F \right) \tilde{Q}_S.\end{aligned}\quad (3.16)$$

The results for the renormalization constants can be simply read off from Eqs. (3.10), (3.12), (3.14), and (3.16); we summarize  $Z_{jQ}^{(10)}$ ,  $Z_{j\tilde{Q}_S}^{(10)}$ ,  $\tilde{Z}_{jQ}^{(10)}$ , and  $\tilde{Z}_{j\tilde{Q}_S}^{(10)}$  in Tab. 1.

#### 4 Matrix elements for $N_c \rightarrow \infty$

In the limit  $N_c \rightarrow \infty$  one can calculate the hadronic matrix elements exactly, because they factorize into the product of two current operators which can be expressed in term of the  $B_s$  meson decay constant  $f_{B_s}$ . In this section we do this calculation for the  $\overline{\text{MS}}$ -renormalized one-loop matrix elements of the studied dimension-7 operators. These calculations provide sophisticated checks of the conceptual and calculational correctness of our results and shed light on the issue of allowed choices for the parameters  $\tilde{a}_2$  and  $\tilde{b}_{2,3}$  in the definitions of the evanescent operators in Eq. (3.7). The latter feature has to do with the fact that the large- $N_c$  calculation of the matrix elements of the colour-flipped operators  $\tilde{R}_{2,3}$  involves a four-dimensional Fierz transform.

In Ref. [22] the limit  $N_c \rightarrow \infty$  has lead to a non-trivial result of the calculation of  $\delta R_0$ : In the  $\overline{\text{MS}}$  scheme the factorized matrix element  $\langle \bar{B}_s | R_0^{\overline{\text{MS}}} | B_s \rangle^{\text{fac}}$  is proportional to  $(\bar{m}_b + \bar{m}_s)^2 - M_{B_s}^2$  with the  $\overline{\text{MS}}$  quark masses  $\bar{m}_{b,s}$ . This factor does not scale like  $m_b \Lambda_{QCD}$  as required for the proper power counting, one needs instead a (properly renormalon-subtracted) pole mass  $m_b^{\text{pole}}$ . The contribution from  $\delta R_0$  was found to fix this problem by providing the  $\alpha_s$  correction which changes  $\bar{m}_b^2 - M_{B_s}^2$  to  $m_b^{\text{pole}2} - M_{B_s}^2$ . So the large- $N_c$  check illustrates that  $\langle \bar{B}_s | R_0^{\overline{\text{MS}}} | B_s \rangle$  does not scale like  $\Lambda_{QCD}/m_b$ , while  $\langle \bar{B}_s | R_0^{\text{ren}} | B_s \rangle = \langle \bar{B}_s | R_0^{\overline{\text{MS}}} + \delta R_0 | B_s \rangle$  does.

To calculate the factorized matrix elements one uses

$$\langle 0 | \bar{b} \gamma_\mu \gamma_5 s(x) | B_s \rangle = i f_{B_s} P_\mu e^{-iPx}, \quad \langle \bar{B}_s | \bar{b} \gamma_\mu \gamma_5 s(x) | 0 \rangle = -i f_{B_s} P_\mu e^{iPx}, \quad (4.1)$$

$$\langle 0|\bar{b}\gamma_5 s(x)|B_s\rangle = -if_{B_s}\frac{M_{B_s}^2}{m_b+m_s}e^{-iPx}, \quad \langle \bar{B}_s|\bar{b}\gamma_5 s(x)|0\rangle = -if_{B_s}\frac{M_{B_s}^2}{m_b+m_s}e^{iPx}, \quad (4.2)$$

where  $|B_s\rangle = |B_s(P)\rangle$ ,  $|\bar{B}_s\rangle = |\bar{B}_s(P)\rangle$  and the results in Eq. (4.2) are obtained by applying  $\partial^\mu$  to Eq. (4.1) and using  $\partial^\mu \bar{b}\gamma_\mu \gamma_5 s = \bar{b}(\overleftarrow{D}^\mu + D^\mu)\gamma_\mu \gamma_5 s$ . With these results one finds the factorized matrix elements [20, 25]

$$\langle \bar{B}_s|Q|B_s\rangle^{\text{fac}} = f_{B_s}^2 M_{B_s}^2 \left(2 + \frac{2}{N_c}\right), \quad \langle \bar{B}_s|\tilde{Q}_S^{\overline{\text{MS}}}(\mu)|B_s\rangle^{\text{fac}} = f_{B_s}^2 M_{B_s}^2 \frac{M_{B_s}^2}{(\overline{m}_b + \overline{m}_s)^2} \left(1 - \frac{2}{N_c}\right). \quad (4.3)$$

#### 4.1 Operator $R_2$

In the large- $N_c$  limit we obtain from Eq. (3.9) with Eq. (4.3):

$$\langle \bar{B}_s|R_2^{\overline{\text{MS}}}|B_s\rangle^{\text{large } N_c} = f_{B_s}^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} \cdot \left[4 \log \frac{m_b}{\mu} - 4\right] + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.4)$$

The large- $N_c$  matrix element of  $m_b^2 R_2$  reads

$$m_b^2 \langle \bar{B}_s|R_2|B_s\rangle^{\text{large } N_c} = \langle \bar{B}_s|\bar{b}\gamma_\mu \gamma_5 s|0\rangle \langle 0|\bar{b}\overleftarrow{D}_\rho \gamma_\mu \gamma_5 D^\rho s|B_s\rangle + \langle \bar{B}_s|\bar{b}\overleftarrow{D}_\rho \gamma_\mu \gamma_5 D^\rho s|0\rangle \langle 0|\bar{b}\gamma_\mu \gamma_5 s|B_s\rangle. \quad (4.5)$$

In order to calculate the current matrix element with derivatives, one applies  $\partial_\rho \partial^\rho$  to Eq. (4.1) giving the desired term twice and terms with two covariant derivatives acting on either  $\bar{b}$  or  $s$ :

$$2\langle 0|\bar{b}\overleftarrow{D}_\rho \gamma_\mu \gamma_5 D^\rho s|B_s\rangle + \langle 0|\bar{b}\overleftarrow{D}_\rho \overleftarrow{D}^\rho \gamma_\mu \gamma_5 s|B_s\rangle + \langle 0|\bar{b}\gamma_\mu \gamma_5 D_\rho D^\rho s|B_s\rangle = -if_{B_s} P_\mu M_{B_s}^2 e^{-iPx}. \quad (4.6)$$

For the latter one needs

$$D^2 = \not{D} \not{D} + \frac{g_s}{2} \sigma_{\rho\nu} F^{\rho\nu, a} T^a, \quad (4.7)$$

with the gluon field strength tensor  $F^{\rho\nu, a}$  in order to apply  $i\bar{b}\overleftarrow{\not{D}} = -m_b \bar{b}$  and  $i\not{D}s = m_s s$ , and  $\sigma_{\rho\nu} = i(\gamma_\rho \gamma_\nu - \gamma_\nu \gamma_\rho)/2$ . Using this relation we can write

$$\langle 0|\bar{b}\gamma_\mu \gamma_5 D_\rho D^\rho s|B_s\rangle = -m_s^2 \langle 0|\bar{b}\gamma_\mu \gamma_5 s|B_s\rangle + \frac{1}{2} \langle 0|\bar{b}\gamma_\mu \sigma_{\rho\nu} g_s F^{\rho\nu, a} T^a \gamma_5 s|B_s\rangle, \quad (4.8)$$

$$\langle 0|\bar{b}\overleftarrow{D}_\rho \overleftarrow{D}^\rho \gamma_\mu \gamma_5 s|B_s\rangle = -m_b^2 \langle 0|\bar{b}\gamma_\mu \gamma_5 s|B_s\rangle + \frac{1}{2} \langle 0|\bar{b}\sigma_{\rho\nu} \gamma_\mu g_s F^{\rho\nu, a} T^a \gamma_5 s|B_s\rangle. \quad (4.9)$$

Combining these results one arrives at

$$\langle 0|\bar{b}\overleftarrow{D}_\rho \gamma_\mu \gamma_5 D^\rho s|B_s\rangle = \frac{i}{2} f_{B_s} P_\mu e^{-iPx} (m_b^2 + m_s^2 - M_{B_s}^2) - \frac{1}{2} \langle 0|S_\mu|B_s\rangle. \quad (4.10)$$

where we have introduced the operator

$$S_\mu \equiv \frac{g_s}{2} \bar{b} F^{\rho\nu, a} T^a \{\gamma_\mu, \sigma_{\rho\nu}\} \gamma_5 s. \quad (4.11)$$

In Eq. (4.10) one can drop the term  $m_s^2$  because it is of higher order in our  $1/m_b$  expansion. Eq. (4.10) holds in any renormalization scheme. The terms with  $\langle 0|S_\mu|B_s\rangle$  are formally of higher order in  $1/m_b$ , but in the  $\overline{\text{MS}}$  scheme the power counting is not manifest. Just as in the case of  $R_2, \dots, \tilde{R}_3$  in Sec. 3 the matrix elements of the current operator  $S_\mu^{\overline{\text{MS}}}$  contains unsuppressed pieces, this time

$\propto \alpha_s \langle 0 | \bar{b} \gamma_\mu \gamma_5 s(x) | B_s \rangle$ . Again, these terms result from hard loop momenta, are IR finite and perturbatively calculable. Putting everything together yields

$$m_b^2 \langle \bar{B}_s | R_2^{\overline{\text{MS}}} | B_s \rangle^{\text{large } N_c} = -f_{B_s}^2 M_{B_s}^2 (M_{B_s}^2 - \bar{m}_b^2 - \bar{m}_s^2) - \frac{1}{2} \langle \bar{B}_s | \bar{b} \gamma_\mu \gamma_5 s | 0 \rangle \langle 0 | S_\mu^{\overline{\text{MS}}} | B_s \rangle - \frac{1}{2} \langle \bar{B}_s | S_\mu^{\overline{\text{MS}}} | 0 \rangle \langle 0 | \bar{b} \gamma_\mu \gamma_5 s | B_s \rangle. \quad (4.12)$$

Dropping the  $S_\mu$  terms, which are NLO in  $\alpha_s$ , reproduces the result of the LO calculation in Ref. [20].

It remains to calculate the matrix element of  $S_\mu$ . The operator  $S_\mu$  defined in Eq. (4.11) has the following Feynman rule

$$-ig_s k^\rho \{ \gamma_\mu, \sigma_{\rho\nu} \} T_{ij}^a \gamma_5, \quad (4.13)$$

where  $k$  is the incoming momentum of the gluon. A second coupling involves two gluons and does not contribute at order  $\alpha_s$ . Using this Feynman rule we find the one-loop matrix element  $S_\mu$  as

$$\langle 0 | S_\mu^{\overline{\text{MS}}} | B_s \rangle^{(1)} = m_b^2 \frac{\alpha_s C_F}{4\pi} 4 \log \frac{m_b}{\mu} i f_{B_s} P_\mu e^{-iPx} + \mathcal{O}(\alpha_s^2, m_b). \quad (4.14)$$

This result solely stems from hard loop momenta, is insensitive to the external states and could therefore be calculated in perturbative QCD with external quark states. At the calculated order the renormalization scheme for  $m_b$  in Eq. (4.14) is not defined, this will matter only in an NNLO calculation.

In calculation of Eq. (4.14) it was necessary to calculate  $\langle 0 | \bar{b} \gamma_5 D_\rho s | B_s \rangle$ . We use  $D_\rho = \{ \gamma_\rho, \not{D} \} / 2$ :

$$\begin{aligned} \langle 0 | \bar{b} \gamma_5 D_\rho s | B_s \rangle &= \frac{1}{2} \langle 0 | \bar{b} \gamma_5 \not{D} \gamma_\rho s | B_s \rangle - \frac{1}{2} i m_s \langle 0 | \bar{b} \gamma_5 \gamma_\rho s | B_s \rangle \\ &= \frac{1}{2} \partial_\mu \langle 0 | \bar{b} \gamma_5 \gamma_\mu \gamma_\rho s | B_s \rangle - \frac{1}{2} \langle 0 | \bar{b} \gamma_5 \not{D} \gamma_\rho s | B_s \rangle + \frac{1}{2} i m_s \langle 0 | \bar{b} \gamma_\rho \gamma_5 s | B_s \rangle \\ &= \frac{1}{2} \partial_\mu \langle 0 | \bar{b} \gamma_5 \gamma_\mu \gamma_\rho s | B_s \rangle - \frac{1}{2} i (m_b - m_s) \langle 0 | \bar{b} \gamma_\rho \gamma_5 s | B_s \rangle. \end{aligned} \quad (4.15)$$

Using  $\gamma_\mu \gamma_\rho = g_{\mu\rho} - i \sigma_{\mu\rho}$  and  $\langle 0 | \bar{b} \gamma_5 \sigma_{\mu\rho} s | B_s \rangle = 0$  one arrives at

$$\begin{aligned} \langle 0 | \bar{b} \gamma_5 D_\rho s | B_s \rangle &= -\frac{1}{2} f_{B_s} P_\rho \frac{M_{B_s}^2}{m_b + m_s} e^{-iPx} + \frac{1}{2} (m_b - m_s) P_\rho f_{B_s} e^{-iPx} \\ &= -\frac{1}{2} f_{B_s} P_\rho \frac{M_{B_s}^2 - m_b^2 - m_s^2}{m_b + m_s} e^{-iPx}. \end{aligned} \quad (4.16)$$

To specify to the  $\overline{\text{MS}}$  scheme we finally need

$$\bar{m}_b(\mu) = m_b^{\text{pole}} \left[ 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left( -4 + 6 \log \frac{m_b}{\mu} \right) \right] \quad (4.17)$$

which implies

$$\begin{aligned} \frac{M_{B_s}^2}{\bar{m}_b^2} - 1 &= \left[ \frac{M_{B_s}^2}{m_b^{\text{pole}2}} - 1 \right] + \frac{M_{B_s}^2}{m_b^2} \frac{\alpha_s(\mu)}{4\pi} 2C_F \left( 4 - 6 \log \frac{m_b}{\mu} \right) + \mathcal{O}(\alpha_s^2) \\ &= \frac{\alpha_s(\mu)}{4\pi} 2C_F \left( 4 - 6 \log \frac{m_b}{\mu} \right) + \mathcal{O}(\alpha_s^2, 1/m_b). \end{aligned} \quad (4.18)$$

Using this in Eq. (4.16) we obtain the final result for the current matrix element

$$\langle 0 | [\bar{b}\gamma_5 D_\rho s]^{\overline{\text{MS}}} | B_s \rangle = -f_{B_s} m_b P_\rho e^{-iPx} \frac{\alpha_s(\mu)}{4\pi} C_F \left( 4 - 6 \log \frac{m_b}{\mu} \right) + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.19)$$

Here the factor in brackets is specific to the  $\overline{\text{MS}}$  scheme, while the the scheme of  $m_b$  is still arbitrary, because a scheme transformation will affect the result at order  $\alpha_s^2$ .  $m_s$  has been neglected, since terms of order  $m_s/m_b$  are power-suppressed.

Recalling  $C_F \simeq N_c/2$  and plugging Eqs. (4.14) and (4.19) into Eq.(4.12) we find

$$\langle \bar{B}_s | R_2^{\overline{\text{MS}}} | B_s \rangle^{\text{large } N_c} = f_{B_s}^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} \left( 4 \log \frac{m_b}{\mu} - 4 \right) + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.20)$$

Comparing this result with Eq. (4.4) we see that the two results are in agreement with each other. Thus the properly renormalized matrix elements satisfy  $\langle \bar{B}_s | R_2^{\text{ren}} | B_s \rangle^{\text{large } N_c} = \mathcal{O}(1/m_b)$  at NLO, because the  $\alpha_s$  term in Eq. (4.20) is canceled by  $\delta R_2$ .

## 4.2 Operator $R_3$

From Eq. (3.11) one finds with Eq. (4.3):

$$\langle \bar{B}_s | R_3^{\overline{\text{MS}}} | B_s \rangle^{\text{large } N_c} = 3f_{B_s}^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.21)$$

The large- $N_c$  matrix element of  $R_3$  reads

$$m_b^2 \langle R_3 \rangle^{\text{large } N_c} = \langle \bar{B}_s | \bar{b}\gamma_5 s | 0 \rangle \langle 0 | \bar{b} \overleftarrow{D}_\rho \gamma_5 D^\rho s | B_s \rangle + \langle \bar{B}_s | \bar{b} \overleftarrow{D}_\rho \gamma_5 D^\rho s | 0 \rangle \langle 0 | \bar{b}\gamma_5 s | B_s \rangle. \quad (4.22)$$

Repeating the steps shown in the previous section associated operator  $R_2$ , we get the following factorized formula (similar to Eq. (4.12)) for the color enhanced part of  $R_3$

$$\begin{aligned} m_b^2 \langle \bar{B}_s | R_3^{\overline{\text{MS}}} | B_s \rangle^{\text{large } N_c} = & f_{B_s}^2 \frac{M_{B_s}^2}{(\bar{m}_b + \bar{m}_s)^2} (M_{B_s}^2 - \bar{m}_b^2 - \bar{m}_s^2) \\ & - \frac{1}{2} \langle \bar{B}_s | \bar{b}\gamma_5 s | 0 \rangle \langle 0 | G_S^{\overline{\text{MS}}} | B_s \rangle - \frac{1}{2} \langle \bar{B}_s | G_S^{\overline{\text{MS}}} | 0 \rangle \langle 0 | \bar{b}\gamma_5 s | B_s \rangle, \end{aligned} \quad (4.23)$$

where we have introduced the operator

$$G_S = g_s \bar{b}_i F^{\rho\nu, a} T_{ij}^a \sigma_{\rho\nu} \gamma_5 s_j, \quad (4.24)$$

with the following Feynman rule

$$-2ig_s k^\rho \sigma_{\rho\nu} T_{ij}^a \gamma_5, \quad (4.25)$$

where  $k$  is the incoming gluon momentum. At one-loop level the matrix element of  $G_S$  is found as

$$\langle 0 | G_S^{\overline{\text{MS}}} | B_s \rangle^{(1)} = -m_b^2 \frac{\alpha_s C_F}{4\pi} \cdot \left[ -2 + 12 \log \frac{m_b}{\mu} \right] i f_{B_s} \frac{M_{B_s}^2}{m_b + m_s} e^{-iPx} + \mathcal{O}(\alpha_s^2, m_b). \quad (4.26)$$

Also here the renormalization scheme for  $m_{b,s}$  will only be fixed if one extends the calculation to NNLO. Inserting Eqs. (4.18) and (4.26) into Eq. (4.23) gives

$$\langle \bar{B}_s | R_3 | B_s \rangle^{\text{large } N_c} = -f_{B_s}^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} \left( 6 \log \frac{m_b}{\mu} - 4 \right)$$

$$\begin{aligned}
& + f_B^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} \left( 6 \log \frac{m_b}{\mu} - 1 \right) + \mathcal{O}(\alpha_s^2, 1/m_b) \\
& = 3 f_B^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} + \mathcal{O}(\alpha_s^2, 1/m_b).
\end{aligned} \tag{4.27}$$

Comparing this result with (4.21) we verify that they are in agreement with each other and thereby confirm  $\langle \bar{B}_s | R_3^{\text{ren}} | B_s \rangle^{\text{large } N_c} = \mathcal{O}(1/m_b)$  at NLO.

### 4.3 Operator $\tilde{R}_2$

From Eqs. (3.13) and (4.3) we get

$$\langle \bar{B}_s | \tilde{R}_2^{\text{MS}} | B_s \rangle^{\text{large } N_c} = f_{B_s}^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} \cdot \left[ -2 \log \frac{m_b}{\mu} + \frac{10}{3} - \frac{\tilde{a}_2}{12} + \frac{\tilde{b}_2}{6} \right] + \mathcal{O}(\alpha_s^2, 1/m_b). \tag{4.28}$$

The large- $N_c$  limit of  $\langle \bar{B}_s | \tilde{R}_2^{\text{MS}} | B_s \rangle^{\text{large } N_c}$  involves the Fierz transform of  $\tilde{R}_2$ :

$$\tilde{R}_2^F \equiv \frac{1}{m_b^2} \left( \bar{b} \overleftarrow{D}_\rho \right)_\alpha \gamma_\mu L s_\alpha \otimes \bar{b}_\beta \gamma^\mu L (D^\rho s)_\beta. \tag{4.29}$$

$$m_b^2 \langle \bar{B}_s | \tilde{R}_2 | B_s \rangle^{\text{large } N_c} = \langle \bar{B}_s | \bar{b} \overleftarrow{D}_\rho \gamma_\mu \gamma_5 s | 0 \rangle \langle 0 | \bar{b} \gamma_\mu \gamma_5 D^\rho s | B_s \rangle + \langle \bar{B}_s | \bar{b} \gamma_\mu \gamma_5 D^\rho s | 0 \rangle \langle 0 | \bar{b} \overleftarrow{D}_\rho \gamma_\mu \gamma_5 s | B_s \rangle. \tag{4.30}$$

Owing to

$$\begin{aligned}
\langle \bar{B}_s | \bar{b} \overleftarrow{D}_\rho \gamma_\mu \gamma_5 s | 0 \rangle &= \partial_\rho \langle \bar{B}_s | \bar{b} \gamma_\mu \gamma_5 s | 0 \rangle - \langle \bar{B}_s | \bar{b} \gamma_\mu \gamma_5 D_\rho s | 0 \rangle \\
&= f_{B_s} P_\rho P_\mu e^{iPx} - \langle \bar{B}_s | \bar{b} \gamma_\mu \gamma_5 D_\rho s | 0 \rangle,
\end{aligned} \tag{4.31}$$

we only need to calculate

$$\langle 0 | \bar{b} \gamma_\mu \gamma_5 D_\rho s | B_s \rangle = [a P_\mu P_\rho + b g_{\mu\rho} M_{B_s}^2] f_{B_s} e^{-iPx}. \tag{4.32}$$

The desired matrix element in Eq. (4.30) reads

$$m_b^2 \langle \bar{B}_s | \tilde{R}_2 | B_s \rangle^{\text{large } N_c} = 2 f_{B_s}^2 M_{B_s}^4 (1 - a - b)(a + b) \tag{4.33}$$

when expressed in terms of the real parameters  $a$  and  $b$ . To the order considered in this paper we can neglect  $-a - b = \mathcal{O}(\alpha_s, 1/m_b)$  in the first bracket.

To determine the coefficients  $a, b$  we first contract with  $g^{\mu\rho}$  to find:

$$a + 4b = -\frac{m_s}{m_b + m_s}. \tag{4.34}$$

The matrix element in Eq. (4.32) depends on the renormalization scheme employed for the operator and thus  $a$  and  $b$  are scheme-dependent. Now we apply  $\partial_\rho$  to Eq. (4.32), it gives

$$\langle 0 | \bar{b} \overleftarrow{D}_\rho \gamma_\mu \gamma_5 D^\rho s | B_s \rangle + \langle 0 | \bar{b} \gamma_\mu \gamma_5 D_\rho D^\rho s | B_s \rangle = -P_\rho [a P_\mu P_\rho + b g_{\mu\rho} M_{B_s}^2] i f_{B_s} e^{-iPx}. \tag{4.35}$$

To solve this equation we use Eq. (4.6) and find

$$\langle 0 | \bar{b} \overleftarrow{D}_\rho \gamma_\mu \gamma_5 D^\rho s | B_s \rangle = \frac{1}{2} \left( -i f_{B_s} P_\mu M_{B_s}^2 e^{-iPx} - \langle 0 | \bar{b} \overleftarrow{D}_\rho \overleftarrow{D}^\rho \gamma_\mu \gamma_5 s | B_s \rangle - \langle 0 | \bar{b} \gamma_\mu \gamma_5 D_\rho D^\rho s | B_s \rangle \right). \tag{4.36}$$

Using this result we make substitution in Eq. (4.35) and get

$$\langle 0 | \bar{b} \overleftarrow{D}_\rho \overleftarrow{D}^\rho \gamma_\mu \gamma_5 s | B_s \rangle - \langle 0 | \bar{b} \gamma_\mu \gamma_5 D_\rho D^\rho s | B_s \rangle = [2a + 2b - 1] i f_{B_s} P_\mu M_{B_s}^2 e^{-iPx}. \quad (4.37)$$

Also from Eqs. (4.8) and (4.9) we have

$$\begin{aligned} \langle 0 | \bar{b} \overleftarrow{D}_\rho \overleftarrow{D}^\rho \gamma_\mu \gamma_5 s | B_s \rangle - \langle 0 | \bar{b} \gamma_\mu \gamma_5 D_\rho D^\rho s | B_s \rangle \\ = -(m_b^2 - m_s^2) \langle 0 | \bar{b} \gamma_\mu \gamma_5 s | B_s \rangle + \frac{1}{2} \langle 0 | \bar{b} [\sigma_{\rho\nu}, \gamma_\mu] g_s F^{\rho\nu, a} T^a \gamma_5 s | B_s \rangle \\ = -(m_b^2 - m_s^2) i f_{B_s} P_\mu e^{-iPx} + \langle 0 | \tilde{S}_\mu | B_s \rangle, \end{aligned} \quad (4.38)$$

where we have introduced

$$\tilde{S}_\mu = \frac{g_s}{2} \bar{b} F^{\rho\nu, a} T^a [\sigma_{\rho\nu}, \gamma_\mu] \gamma_5 s. \quad (4.39)$$

Inserting this result into Eq. (4.37) we get

$$-(m_b^2 - m_s^2) i f_{B_s} P_\mu e^{-iPx} + \langle 0 | \tilde{S}_\mu | B_s \rangle = [2a + 2b - 1] i f_{B_s} P_\mu M_{B_s}^2 e^{-iPx}. \quad (4.40)$$

From  $\tilde{S}_\mu$  we have the following Feynman rule for its one-gluon coupling

$$-i g_s k^\rho [\sigma_{\rho\nu}, \gamma_\mu] T_{ij}^a \gamma_5, \quad (4.41)$$

where  $k$  is an incoming momentum of a gluon. At one loop, using for outgoing anti- $b$  quark  $\bar{b} \not{p}_b = -m_b \bar{b}$ , we have

$$\langle 0 | \tilde{S}_\mu^{\overline{\text{MS}}} | B_s \rangle^{(1)} = \frac{\alpha_s C_F}{4\pi} m_b^2 \left( -2 + 8 \log \frac{m_b}{\mu} \right) i f_{B_s} P_\mu e^{-iPx} + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.42)$$

Inserting this result into Eq. (4.40) we obtain

$$\begin{aligned} -(\bar{m}_b^2 - \bar{m}_s^2) i f_{B_s} P_\mu e^{-iPx} + \frac{\alpha_s C_F}{4\pi} m_b^2 \left( -2 + 8 \log \frac{m_b}{\mu} \right) i f_{B_s} P_\mu e^{-iPx} + \mathcal{O}(\alpha_s^2, 1/m_b) \\ = \left[ 2a^{\overline{\text{MS}}} + 2b^{\overline{\text{MS}}} - 1 \right] i f_{B_s} P_\mu M_{B_s}^2 e^{-iPx}, \end{aligned} \quad (4.43)$$

where we have added superscripts to  $a$  and  $b$  as well as bars to the masses to clarify that the result is specific to the  $\overline{\text{MS}}$  scheme. Next we find

$$a^{\overline{\text{MS}}} + b^{\overline{\text{MS}}} = \frac{1}{2} \left( 1 - \frac{\bar{m}_b^2 - \bar{m}_s^2}{M_{B_s}^2} + \frac{\alpha_s C_F}{4\pi} \frac{m_b^2}{M_{B_s}^2} \left( -2 + 8 \log \frac{m_b}{\mu} \right) \right) + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.44)$$

Dropping  $\bar{m}_s^2$  and using Eq. (4.17) this leads to

$$a^{\overline{\text{MS}}} + b^{\overline{\text{MS}}} = \frac{1}{2} \left( 1 - \frac{m_b^{\text{pole}2}}{M_{B_s}^2} + \frac{\alpha_s C_F}{4\pi} \frac{m_b^2}{M_{B_s}^2} \left( 6 - 4 \log \frac{m_b}{\mu} \right) \right) + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.45)$$

Here the first two terms in the brackets combine to a term of order  $\Lambda_{QCD}/m_b$  and can be dropped, while the  $\alpha_s$  piece is the desired power-unsuppressed radiative correction. By inserting Eq. (4.45) into Eq. (4.33) we obtain

$$\langle \bar{B}_s | \tilde{R}_2^{\overline{\text{MS}}} | B_s \rangle^{\text{large } N_c} = 2f_{B_s}^2 M_{B_s}^2 \left( a^{\overline{\text{MS}}} + b^{\overline{\text{MS}}} \right) + \mathcal{O}(1/m_b)$$

**Table 1.** Finite renormalization constants (see Eqs. (2.8) and (2.9)). The UV divergent constants  $Z_{jQ}^{(11)}$ ,  $Z_{j\bar{Q}s}^{(11)}$ ,  $\tilde{Z}_{jQ}^{(11)}$ , and  $\tilde{Z}_{j\bar{Q}s}^{(11)}$  are 1/2 of the coefficients of  $\log \mu$ . The large- $N_c$  analysis of Sec. 4 has shown that a consistent result requires a Fierz-symmetric definition of the colour-flipped operators, namely  $\tilde{a}_2 = 12$ ,  $\tilde{b}_2 = 4$ , and  $\tilde{b}_3 = 10$ .

| $j$                            | 0                                                                   | 2                                                                                                                                                                                                          | 3                                                                                                                                                   |
|--------------------------------|---------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------|
| $Z_{jQ}^{(10)}$                | $(N_c + 1) \log \frac{\mu}{m_b}$<br>$+\frac{N_c^2 + 2N_c - 2}{N_c}$ | $\left(-\frac{4}{3} + \frac{5}{3N_c} + \frac{10C_F}{3}\right) \log \frac{\mu}{m_b} - \frac{16}{9}$<br>$-\frac{b_2 - a_2}{24} + \left(\frac{17}{9} - \frac{a_2}{24}\right) \frac{1}{N_c} + \frac{37C_F}{9}$ | $-\left(\frac{1}{2} + \frac{1}{2N_c}\right) \log \frac{\mu}{m_b}$<br>$-\left(\frac{1}{3} - \frac{b_3}{48} + \frac{3}{4N_c} + \frac{3C_F}{2}\right)$ |
| $Z_{j\bar{Q}s}^{(10)}$         | $4(N_c + 1) \log \frac{\mu}{m_b}$<br>$+2(N_c + 1)$                  | $\left(\frac{2}{3} + \frac{2}{3N_c} + \frac{4C_F}{3}\right) \log \frac{\mu}{m_b}$<br>$+\frac{2}{9} - \frac{b_2}{12} + \left(\frac{2}{9} - \frac{b_2}{12}\right) \frac{1}{N_c} - \frac{2C_F}{9}$            | $-\left(2 + \frac{2}{N_c}\right) \log \frac{\mu}{m_b}$<br>$-\left(\frac{13}{6} - \frac{b_3}{24}\right) \left(1 + \frac{1}{N_c}\right) - 3C_F$       |
| $\tilde{Z}_{jQ}^{(10)}$        |                                                                     | $\left(\frac{5}{3} - \frac{4}{3N_c} - \frac{8C_F}{3}\right) \log \frac{\mu}{m_b}$<br>$+\frac{37}{18} - \frac{19}{9N_c} - \left(\frac{32}{9} + \frac{\tilde{b}_2 - \tilde{a}_2}{12}\right) C_F$             | $-C_F \log \frac{\mu}{m_b}$<br>$-\frac{3}{4} - \left(\frac{2}{3} - \frac{\tilde{b}_3}{24}\right) C_F$                                               |
| $\tilde{Z}_{j\bar{Q}s}^{(10)}$ |                                                                     | $\left(\frac{2}{3} + \frac{2}{3N_c} + \frac{4C_F}{3}\right) \log \frac{\mu}{m_b}$<br>$-\frac{1}{9} - \frac{1}{9N_c} + \left(\frac{4}{9} - \frac{\tilde{b}_2}{6}\right) C_F$                                | $-4C_F \log \frac{\mu}{m_b} - \frac{3}{2}$<br>$-\frac{3}{2N_c} - \left(\frac{13}{3} - \frac{\tilde{b}_3}{12}\right) C_F$                            |

$$= f_{B_s}^2 M_{B_s}^2 \frac{\alpha_s}{4\pi} N_c \left(3 - 2 \ln \frac{m_b}{\mu}\right) + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.46)$$

Comparing this result with Eq. (4.28) we verify that they are in agreement with each other and thereby confirm  $\langle \bar{B}_s | \tilde{R}_2^{\text{ren}} | B_s \rangle^{\text{large } N_c} = \mathcal{O}(1/m_b)$  at NLO for the choices  $\tilde{a}_2 = 12$  and  $\tilde{b}_2 = 4$  found in Sec. 3.1 by imposing the requirement that Fierz symmetry is respected at the one-loop level. We remark that Fierz symmetry is a stronger constraint than consistency of Eqs. (4.28) and (4.46), which holds for any values of  $\tilde{a}_2$  and  $\tilde{b}_2$  satisfying  $\tilde{a}_2 - 2\tilde{b}_2 = 4$ .

#### 4.4 Operator $\tilde{R}_3$

From Eqs. (3.15) and (4.3) in the large- $N_c$  limit we have

$$\langle \bar{B}_s | \tilde{R}_3^{\text{MS}} | B_s \rangle^{\text{large } N_c} = f_{B_s}^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} \cdot \left[ -3 \log \frac{m_b}{\mu} + \frac{17}{6} - \frac{\tilde{b}_3}{12} \right] + \mathcal{O}(\alpha_s^2, 1/m_b). \quad (4.47)$$

The large- $N_c$  matrix element of operator  $\tilde{R}_3$  involves the Fierz transform of  $\tilde{R}_3$ :

$$m_b^2 \tilde{R}_3^F = -\frac{1}{2} \left( \bar{b} \overleftarrow{D}_\rho \right)_\alpha L s_\alpha \otimes \bar{b}_\beta L (D^\rho s)_\beta - \frac{1}{8} \left( \bar{b} \overleftarrow{D}_\rho \right)_\alpha \sigma_{\mu\nu} L s_\alpha \otimes \bar{b}_\beta \sigma^{\mu\nu} L (D^\rho s)_\beta. \quad (4.48)$$

For the color enhanced part of factorized  $\tilde{R}_3$  we get

$$\begin{aligned} m_b^2 \langle \bar{B}_s | \tilde{R}_3 | B_s \rangle^{\text{large } N_c} = & \\ & -\frac{1}{2} \langle \bar{B}_s | \bar{b} \overleftarrow{D}_\rho \gamma_5 s | 0 \rangle \langle 0 | \bar{b} \gamma_5 D^\rho s | B_s \rangle - \frac{1}{2} \langle \bar{B}_s | \bar{b} \gamma_5 D^\rho s | 0 \rangle \langle 0 | \bar{b} \overleftarrow{D}_\rho \gamma_5 s | B_s \rangle \\ & -\frac{1}{8} \langle \bar{B}_s | \bar{b} \overleftarrow{D}_\rho \sigma_{\mu\nu} \gamma_5 s | 0 \rangle \langle 0 | \bar{b} \sigma^{\mu\nu} \gamma_5 D^\rho s | B_s \rangle - \frac{1}{8} \langle \bar{B}_s | \bar{b} \sigma^{\mu\nu} \gamma_5 D^\rho s | 0 \rangle \langle 0 | \bar{b} \overleftarrow{D}_\rho \sigma_{\mu\nu} \gamma_5 s | B_s \rangle, \end{aligned} \quad (4.49)$$

which we further modify to the following form:

$$\begin{aligned} m_b^2 \langle \bar{B}_s | \tilde{R}_3 | B_s \rangle^{\text{large } N_c} = & -\frac{1}{2} \left( \partial_\rho \langle \bar{B}_s | \bar{b} \gamma_5 s | 0 \rangle - \langle \bar{B}_s | \bar{b} \gamma_5 D_\rho s | 0 \rangle \right) \langle 0 | \bar{b} \gamma_5 D^\rho s | B_s \rangle + c.c. \\ & -\frac{1}{8} \left( \partial_\rho \langle \bar{B}_s | \bar{b} \sigma_{\mu\nu} \gamma_5 s | 0 \rangle - \langle \bar{B}_s | \bar{b} \sigma_{\mu\nu} \gamma_5 D_\rho s | 0 \rangle \right) \langle 0 | \bar{b} \sigma^{\mu\nu} \gamma_5 D^\rho s | B_s \rangle + c.c.. \end{aligned} \quad (4.50)$$

We have

$$\langle \bar{B}_s | \bar{b} \sigma_{\mu\nu} \gamma_5 D_\rho s | 0 \rangle = 0, \quad (4.51)$$

thus we obtain

$$\begin{aligned} m_b^2 \langle \bar{B}_s | \tilde{R}_3 | B_s \rangle^{\text{large } N_c} = & -\frac{1}{2} \left( \partial_\rho \langle \bar{B}_s | \bar{b} \gamma_5 s | 0 \rangle \right) \langle 0 | \bar{b} \gamma_5 D^\rho s | B_s \rangle + c.c. \\ & -\frac{1}{2} \langle \bar{B}_s | \bar{b} \gamma_5 D_\rho s | 0 \rangle \langle 0 | \bar{b} \gamma_5 D^\rho s | B_s \rangle + c.c. \\ & -\frac{1}{8} \langle \bar{B}_s | \bar{b} \sigma^{\mu\nu} \gamma_5 D^\rho s | 0 \rangle \langle 0 | \bar{b} \sigma_{\mu\nu} \gamma_5 D_\rho s | B_s \rangle + c.c.. \end{aligned} \quad (4.52)$$

In this expression the terms in the last two lines are suppressed by powers of  $\alpha_s$  or  $\Lambda_{QCD}/m_b$  w.r.t. the first term and can be neglected. So using Eqs. (4.2) and (4.19) we get

$$\begin{aligned} \langle \bar{B}_s | \tilde{R}_3^{\overline{\text{MS}}} | B_s \rangle^{\text{large } N_c} = & -\frac{1}{2m_b^2} P_\rho f_{B_s} \frac{M_{B_s}^2}{\bar{m}_b + \bar{m}_s} e^{iPx} \langle 0 | \bar{b} \gamma_5 D^\rho s | B_s \rangle + c.c. \\ = & f_{B_s}^2 M_{B_s}^2 \frac{\alpha_s N_c}{4\pi} \left( 2 - 3 \ln \frac{m_b}{\mu} \right) + \mathcal{O}(\alpha_s^2, 1/m_b), \end{aligned} \quad (4.53)$$

which is in agreement with Eq. (4.47) for  $\tilde{b}_3 = 10$ , and Fierz symmetry is necessary and sufficient for the correct result for  $\langle \tilde{R}_3 \rangle^{\text{large } N_c}$ .

## 5 Discussion and conclusions

Current data on the width difference  $\Delta\Gamma_s$  in the  $B_s$ - $\bar{B}_s$  system are more precise than the theory prediction [29, 30] and the uncertainty of the latter is dominated by the size of the power-suppressed contribution  $\tilde{\Gamma}_{21,1/m_b}^{cc}$  in Eq. (2.2) to the decay matrix. This calls for a calculation of NLO QCD corrections to this quantity, which is novel territory for the renormalization problem of four-quark operators. It is known that, generally, in the  $\overline{\text{MS}}$  scheme the decoupling of heavy scales is not manifest

and higher-dimensional operators can mix into lower-dimensional ones under renormalization. At present, it is not known whether the commonly adopted HQE [16–20, 22–30] for  $\Gamma_{21}^q$  can be consistently defined and, if yes, how the finite counterterms to the dimension-7 operators restoring the power counting must be defined.

In this work we have done a first step in this direction and calculated those parts of the  $\overline{\text{MS}}$ -renormalized matrix elements of the dimension-7 operators  $R_{2,3}$  and  $\tilde{R}_{2,3}$  at order  $\alpha_s$ , which are associated with the finite renormalization of these operators by terms proportional to dimension-6 operators. We have found that the loop contributions to  $\langle R_{2,3} \rangle$  and  $\langle \tilde{R}_{2,3} \rangle$  scaling like  $m_b^0$  are indeed infrared-finite, so that they can be cancelled by counterterms proportional to the dimension-6 operators  $Q$  and  $\tilde{Q}_S$  to restore the correct power counting,  $\langle R_{2,3}^{\text{ren}} \rangle, \langle \tilde{R}_{2,3}^{\text{ren}} \rangle = \mathcal{O}(1/m_b)$ . However, we have also found that these counterterms depend on the definitions of evanescent operators and have shown that in the case of the colour-flipped operators  $\tilde{R}_{2,3}$  the  $\epsilon$  terms of these operators must be chosen to obey Fierz symmetry. This feature has been observed from the hadronic matrix elements calculated in the large- $N_c$  limit, which turned out to be very subtle, with power-unsuppressed contributions from hard loop corrections to operators involving the gluon field strength tensor. The large- $N_c$  criterion does not apply to the colour-straight operators  $R_{2,3}$  and it is still an open question, whether arbitrary choices of the  $\epsilon$  terms in the evanescent operators are permissible or whether other conditions will fix them, for example  $\langle R_2 \rangle = -\langle \tilde{R}_2 \rangle + \mathcal{O}(1/m_b^2)$ , which automatically holds at tree-level [20] but needs a finite counterterm at NLO. Since  $Z_{1Q} = Z_{1\tilde{Q}_S} = 0$  and  $Z_{0Q}, Z_{0\tilde{Q}_S}$  could be extracted from Ref. [22], the calculations presented in this paper complete the set of renormalisation constants related to the mixing of  $R_j, \tilde{R}_j$  into dimension-6 operators.

In conclusion, we have made a first step towards the calculation of the  $B\text{-}\bar{B}$  decay matrix at order  $\alpha_s/m_b$ , *i.e.* to extend the 29-year-old LO result of Ref. [20] to NLO, by calculating the counterterms listed in Tab. 1. As a by-product, we have calculated matrix elements of several  $\bar{b}$ - $s$  current operators. Our results for the hard, infrared-finite loop contributions are needed to restore the correct power counting in any application involving these currents.

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