

Frobenius Revivals in Laplacian Cellular Automata: Chaos, Replication, and Reversible Encoding

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Abstract

We investigate Frobenius-driven revivals in prime-modulus Laplacian cellular automata, a phenomenon in which long chaotic transients collapse into exact, multi-tile replicas of an initial seed at algebraically prescribed times $t = p^m$. The mechanism follows directly from the Frobenius identity $(I+B)^{p^m} = I + B^{p^m}$, which eliminates all mixed binomial terms and enforces deterministic reappearance of the seed after dispersion. We provide a detailed numerical and analytical characterisation of these revivals across several moduli, examining entropy dynamics, spatial organisation, and local stability under perturbations.

The revival structure yields several useful features: predictable transitions between chaotic and ordered phases, intrinsic spatial redundancy, and robust reconstruction via replica consensus in the presence of weak additive noise. We further show that composing Laplacian operators modulo multiple primes generates significantly extended periodic orbits while preserving exact reversibility.

Building on these observations, we propose an explicit reversible encoding scheme based on chaotic transients and Frobenius returns, together with practical separation conditions and noise-tolerance estimates. Potential applications include reversible steganography, structured pseudorandomness, error-tolerant information representation, and procedural pattern synthesis. The results highlight an interplay between algebraic combinatorics and cellular-automaton dynamics, suggesting further avenues for theoretical and applied development.

Keywords: Cellular automata; Frobenius endomorphism; Laplacian dynamics; Prime-modulus evolution; Chaotic transients; Reversible encoding; Pattern replication; Error-tolerant information processing.

Highlights

- Prime-modulus Laplacian cellular automata display abrupt seed revivals after long dispersive transients.
- Replica formation at prescribed Frobenius times $t = p^m$ follows directly from the algebraic collapse $(I+B)^{p^m} = I + B^{p^m}$.
- Multi-prime compositions yield extended reversible cycles with multiple high-entropy intervals.
- Intermediate states appear statistically featureless while remaining exactly recoverable at the next revival.
- Localised perturbations remain confined to individual replica tiles, and tile consensus provides strong inherent robustness.

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1 Introduction

Cellular automata (CAs) are among the simplest discrete dynamical systems capable of producing remarkably rich spatiotemporal behaviour. Since the pioneering work of Wolfram [Wolfram, 1983] and subsequent developments in symbolic and nonlinear dynamics [Fuk s, 2004, Strogatz, 2018, Ott, 2002], CAs have become canonical models for studying the emergence of complexity from simple local interactions. Depending on the rule and underlying algebraic structure, their orbits may exhibit rapid dispersion, high-entropy chaotic transients, intermittent phases, or the spontaneous formation of coherent global patterns. Such transitions between disorder and order—well known in nonlinear dynamical systems through bifurcation cascades and periodic windows [Cvitanovi c, 2019, Politi and Torcini, 1992]—remain especially intriguing in linear CA settings, where the global behaviour is often expected to be more rigid.

In this work we investigate a class of Laplacian cellular automata defined over finite prime fields. Despite their algebraic linearity, these systems display long chaotic transients followed by abrupt large-scale reorganization: at characteristic prime-power times $t = p^m$, the configuration suddenly collapses into multiple spatially disjoint replicas of the initial seed. We demonstrate that this revival phenomenon follows directly from the Frobenius endomorphism in characteristic p , which eliminates all mixed binomial terms in the operator $(I + B)^{p^m}$. As a consequence, a configuration that appears fully chaotic can reassemble into an ordered tiling of exact copies of the original pattern.

This mechanism may be viewed as a discrete analogue of chaos-order transitions in nonlinear systems, but here it arises purely from algebraic constraints. We analyze the dynamical structure of the phenomenon, including the growth of spatiotemporal entropy, the geometry of the emerging replicas, and the confinement of local perturbations to limited light-cone regions. We also show that compositions of prime-modulus Laplacian stages produce long composite orbits with multiple chaotic windows and delayed revivals, enabling precise control over the temporal complexity of the evolution.

Our results place Frobenius-driven revivals within the broader framework of nonlinear and symbolic dynamics. They provide a dynamical interpretation of replica formation in Laplacian CAs and demonstrate how algebraically induced transitions from chaos to order may be exploited for reversible encoding, redundant information representation, and controlled spatiotemporal pattern generation in discrete systems.

2 Background and Previous Work

2.1 Related work

The study of complex behaviour in discrete dynamical systems has a long history spanning nonlinear dynamics, symbolic dynamics, and the theory of cellular automata (CAs). Classical works in chaos theory [Ott, 2002, Strogatz, 2018] have established a rich vocabulary for describing transitions between ordered and disordered regimes, ranging from bifurcation cascades to intermittent chaotic windows and sudden returns to regular behaviour. These notions have been extended to spatially extended systems, where spatiotemporal chaos and emergent pattern formation play a central role [Politi and Torcini, 1992, Crutchfield and Young, 1989].

In the context of CAs, Wolfram’s foundational programme [Wolfram, 1983] demonstrated that simple local rules can produce behaviour characteristic of nonlinear dynamical systems, including long chaotic transients, spatial disorder, and spontaneous emergence of coherent structures. Subsequent advances in symbolic dynamics have clarified how CA evolution can be interpreted through the lens of shift spaces, subshifts of finite type, and algebraic constraints [Fuk s, 2004]. These perspectives provide a natural framework for analysing the global structure of CA trajectories, their periodic orbits, and the mechanisms by which low-entropy motifs arise within seemingly chaotic evolution.

Spatiotemporal organisation in discrete dynamical systems has also been studied extensively in the setting of coupled map lattices [Kaneko, 1993], where coherent structures, domain formation, and intermittency emerge from interactions between local update rules and global phase-space geometry. These systems exhibit transitions reminiscent of pattern formation in continuous media, yet generated by purely discrete dynamics, mirroring some of the behaviours observed in Laplacian CAs.

Recent work has further explored the dynamical properties of linear and algebraic CAs, including the role of periodic orbits, invertibility, and spectral characteristics in determining long-term behaviour. However, the mechanisms by which chaotic dispersive dynamics give rise to abrupt large-scale order remain insufficiently understood, particularly in the setting of modular Laplacian operators. While algebraic identities governing such operators are well known, their dynamical consequences—especially the emergence of multi-tile replications at prime-power times—have not been systematically analysed.

The present work contributes to this line of research by placing Frobenius-driven revivals within the broader landscape of nonlinear and symbolic dynamics. In contrast to previous studies that emphasise pseudorandomness, diffusion, or local complexity, we focus on the global dynamical mechanism enabling transitions from high-entropy chaotic evolution to sharply ordered, algebraically constrained replica configurations. This perspective connects algebraic CA theory with dynamical phenomena traditionally associated with nonlinear systems, offering a unified interpretation of chaos-order transitions in discrete settings.

2.2 Mathematical Background

Linear cellular automata over finite fields exhibit a wide range of algebraically induced periodicities. Despite their linearity, the iterates of such operators may generate highly nontrivial spatiotemporal behaviour, including long chaotic transients followed by abrupt returns to ordered configurations. When the update rule is a discrete Laplacian modulo a prime p , the system may display *exact seed revivals* at the prime-power times $t = p^m$.

This phenomenon follows directly from the Frobenius endomorphism in characteristic p , which guarantees

$$(a + b)^{p^m} = a^{p^m} + b^{p^m} \quad \text{in } \mathbb{F}_p.$$

For an evolution operator of the form $T = I + B$, where B is the neighbourhood convolution, one obtains

$$T^{p^m} = (I + B)^{p^m} = I + B^{p^m}.$$

All mixed binomial terms vanish, and B^{p^m} acts as a large spatial shift. Thus the central region reproduces the initial seed exactly, while additional shifted copies appear near the boundary. This algebraic mechanism accounts for the sudden chaos-order transitions observed in Laplacian cellular automata and links the replication phenomenon directly to the Frobenius endomorphism [Wikipedia contributors, 2025, Byczewski and Cornelissen, 2022].

2.3 Periods and Replication

Let L denote the Laplacian operator. Every finite pattern F is contained in a minimal axis-aligned bounding rectangle $r(F)$. A positive integer τ is called a *period* of F if

$$L^\tau(F) = \bigcup_{i=1}^s T_i F, \quad s \geq 2,$$

for some lattice shifts T_i . This formalises the empirically observed replication events occurring at regular times within an otherwise chaotic orbit.

- A *small period* occurs when the rectangles $r(T_i F)$ overlap.

- A *large period* occurs when the rectangles $r(T_i F)$ are disjoint.
- A *shifted period* occurs when $F = L^{t_0}(S)$ for $t_0 > 0$ and

$$L^{t_0+\tau}(S) = \bigcup_{i=1}^s T_i L^{t_0}(S).$$

These notions allow one to distinguish early overlapping returns from the fully separated tilings characteristic of Frobenius revivals.

2.4 Composition of Laplacian Operators

Let L_2 and L_p denote Laplacian evolutions modulo 2 and a prime $p \geq 3$, acting on a common seed. Let T be a period of L_2 and T' a period of L_p along the L_2 -orbit. The composite operator

$$L_2^{T-x} L_p^{T'} L_2^x, \quad 0 \leq x < T,$$

produces orbits containing long chaotic segments interspersed with structured replication phases. Such mixed-modulus compositions act as discrete multi-scale forcing and naturally generate transitions reminiscent of intermittency phenomena in nonlinear dynamics.

2.5 Multi-Prime Composition

Let L_{p_i} denote Laplacian updates modulo distinct primes p_i , with intrinsic periods T_{p_i} . For offsets x_i with $0 \leq x_i < T_{p_i}$, consider the composite cycle

$$L_{p_1}^{T_{p_1}-x_1} \dots L_{p_m}^{T_{p_m}-x_m} L_{p_m}^{x_m} \dots L_{p_1}^{x_1}. \quad (1)$$

The first half of the cycle aligns each modulus with its own revival, while the second half reverses the offsets and restores the initial seed exactly. Because Laplacian operators generally do not commute, the ordering $p_1 < p_2 < \dots < p_m$ is essential for maintaining phase coherence.

The resulting global period

$$T_{\text{global}} = \text{lcm}(T_{p_1}, \dots, T_{p_m})$$

is typically several orders of magnitude larger than any individual T_{p_i} , producing extended chaotic windows and delayed revivals. This hierarchical structure provides a controllable mechanism for generating long, fully reversible orbits with rich temporal complexity.

2.6 Discrete Laplacian Cellular Automata

We consider two-dimensional cellular automata whose local update rule is given by the discrete Laplacian

$$\Delta u(p) = \sum_{g \in N(p)} (u(g) - u(p)) \pmod{k_i}, \quad (2)$$

where $N(p)$ is the neighbourhood of p determined by a prescribed mask. The modulus k_i at iteration i is taken from a fixed sequence

$$k_1, k_2, k_3, \dots, \quad k_i \in \mathbb{Z}_{\geq 2}. \quad (3)$$

At iteration t , the configuration evolves as

$$u_{t+1} = L u_t \pmod{k_t},$$

where L is the Laplacian associated with the chosen mask. Although the rule is linear, the induced dynamics are often strongly dispersive and may enter high-entropy spatiotemporal regimes before any revival occurs.

While simulations are frequently performed on a finite torus $\mathbb{Z}_N^2$, the revival phenomenon is most naturally analysed on the infinite lattice, where no wrap-around effects interfere with replica formation.

3 Methods

This section summarises the mathematical model, seed geometry, observables, noise mechanisms and reconstruction rules used throughout the numerical study. The algebraic theory of Frobenius revivals and period structure was presented in Section 2; here we describe only the computational and experimental procedures.

3.1 Model

We consider two-dimensional linear cellular automata over a finite field $\mathbb{F}_p$, where p is a prime modulus. Let $u_t : \mathbb{Z}^2 \rightarrow \mathbb{F}_p$ denote the configuration at iteration t . The update rule is the Laplacian map

$$u_{t+1} = (I + B) u_t \pmod{p},$$

where B is the discrete convolution induced by the 3×3 Moore neighbourhood. The presence of the self-term I is essential for Frobenius revivals.

All simulations are performed on an *expanding window*: the canvas grows by one pixel per iteration in each direction, preventing wrap-around interactions and ensuring that revived replicas do not intersect the central crop. For multi-prime constructions, different moduli act sequentially but always via the same operator $T = I + B$.

3.2 Seeds and initial geometry

A *seed* is a compact nonzero pattern supported in a finite $N \times N$ window. We use three representative seed types:

- small seeds (up to 3×3 support),
- medium seeds (natural-image silhouettes, $\sim 18 \times 18$),
- large seeds (high-resolution shapes, $\sim 80 \times 80$).

Seeds are placed at the centre of the expanding canvas. Silhouettes are particularly convenient because geometric distortions and replica misalignments are easily detectable.

3.3 Observables

We employ three complementary observables to quantify evolution, revival quality and reconstruction fidelity.

Entropy. For a configuration u_t evolving modulo p , define the symbol frequencies

$$\rho_t^{(c)} = \frac{|\{x : u_t(x) = c\}|}{|r_t|}, \quad c = 0, \dots, p-1,$$

where r_t is the minimal bounding box of nonzero values. The Shannon entropy is

$$H_t = - \sum_{c=0}^{p-1} \rho_t^{(c)} \log \rho_t^{(c)}, \quad (0 \log 0 := 0).$$

Sharp minima of H_t coincide with revival windows, while high values correspond to chaotic dispersive phases.

Hamming distance. For reconstructed $\hat{u}_0$ and ground truth u_0 ,

$$\text{Ham}(\hat{u}_0, u_0) = \frac{1}{N^2} \sum_x \mathbf{1}\{\hat{u}_0(x) \neq u_0(x)\}.$$

Perceptual metrics. For binary and ternary silhouettes we additionally report SSIM and PSNR to capture structural and pixelwise fidelity.

3.4 Noise model

Noise is applied independently at each iteration. During the update $t \mapsto t + 1$, each pixel is perturbed with probability p_{noise} :

- for $p = 2$: the bit is flipped,
- for $p > 2$: the value is reassigned uniformly in $\{0, \dots, p - 1\}$.

If noise acts for N steps with per-step rate p , the accumulated corruption satisfies approximately

$$q \approx \frac{1}{2}(1 - e^{-2pN}),$$

consistent with the observed saturation in long chaotic transients.

All robustness experiments use this independent-noise model with multiple Monte–Carlo trials.

3.5 Spatial redundancy and decoding rule

At revival times $t^* = p^m$, the configuration contains several disjoint replicas of the original seed due to the identity

$$(I + B)^{p^m} = I + B^{p^m}.$$

When $p^m \geq N$, these shifted copies do not overlap the central region, providing inherent spatial redundancy.

Let M denote the number of full-size replica tiles extracted at t^* . Reconstruction uses pixelwise majority (or mode) voting:

$$\hat{u}_0(x) = \text{mode}_{1 \leq i \leq M} u^{(i)}(x).$$

If each replica suffers independent per-pixel corruption q , the probability of an erroneous majority decision is

$$q_{\text{maj}}(M) = \sum_{i=\lceil (M+1)/2 \rceil}^M \binom{M}{i} q^i (1-q)^{M-i},$$

which decays rapidly with M . This scheme forms the reconstruction mechanism used in all noise-tolerance experiments.

3.6 Monte–Carlo protocol for noise tolerance

Because error propagation depends sensitively on replica geometry, closed–form noise thresholds are difficult to obtain. We therefore estimate effective stability bounds using a standard Monte–Carlo procedure applied to the redundant (no–crop) evolution at revival times $t^* = p^m$.

1. **Parameters.** Fix a seed u_0 of size $s \times s$, a neighbourhood mask, and a target revival time $t^* = p^m$. Choose an admissible reconstruction error ε (e.g. $\varepsilon = 10^{-3}$ for 0.1% Hamming error).
2. **Noise grid.** Select a set of per-step noise rates

$$p_{\text{noise}} \in \{10^{-5}, 5 \cdot 10^{-5}, 10^{-4}, \dots\}.$$

3. **Simulation.** For each p_{noise} :

- (a) Initialise $u^{(0)} = u_0$ on a canvas large enough to contain all replicas at time t^* .
- (b) For $t = 1, \dots, t^*$:
 - i. apply one Laplacian update modulo p ,
 - ii. apply independent noise: each pixel is reassigned uniformly in $\{0, \dots, p-1\}$ with probability p_{noise} .
- (c) At $t = t^*$, extract all disjoint replica tiles, reconstruct $\hat{u}_0$ via pixelwise majority (or mode) voting, and compute the Hamming error

$$E(p_{\text{noise}}) = \text{Ham}(\hat{u}_0, u_0).$$

4. **Averaging.** Repeat the simulation for R trials (typically $R = 20$) and record the mean error $\bar{E}(p_{\text{noise}})$.
5. **Tolerance bound.** Define the empirical stability threshold as

$$p_{\max}(p, t^*; \varepsilon) = \max \{p_{\text{noise}} : \bar{E}(p_{\text{noise}}) \leq \varepsilon\}.$$

This protocol yields effective per-step noise tolerances and quantifies how longer revival times and increased replica multiplicity compensate for persistent perturbations.

Temporal redundancy. In addition to spatial redundancy at t^* , a second, independent layer of robustness may be introduced. Instead of transmitting a single chaotic state u_s with $s < t^*$, choose a set of encoding times

$$S = \{s_1, s_2, \dots, s_L\} \subset [0, t^*),$$

each producing a distinct noisy snapshot u_{s_ℓ} . For each s_ℓ , the receiver performs the top-up to t^* , obtaining independent reconstructions $\hat{u}_0^{(\ell)}$. A final estimate is obtained by pixelwise majority (or mode) voting:

$$\hat{u}_0(x) = \text{mode}_{1 \leq \ell \leq L} \hat{u}_0^{(\ell)}(x).$$

This two-layer scheme—spatial replication within a single revival frame and temporal repetition across multiple encoding times—significantly enhances robustness in the noisy-evolution regime and suppresses sporadic local defects.

4 Results

We first examine the revival structure of prime-modulus Laplacian dynamics, then quantify their entropy signatures, extend them to multi-prime compositions, and finally assess robustness under additive noise.

4.1 Illustrative chaos–order transitions

Before presenting the algebraic framework, we show a representative example of the phenomenon studied in this work. This serves as the entry point into our results. Figure 1 illustrates the evolution of a small binary seed under the Moore–Laplacian rule. The pattern alternates between chaotic high-entropy states and ordered replication phases, culminating in a fully separated revival at $t = 128$.

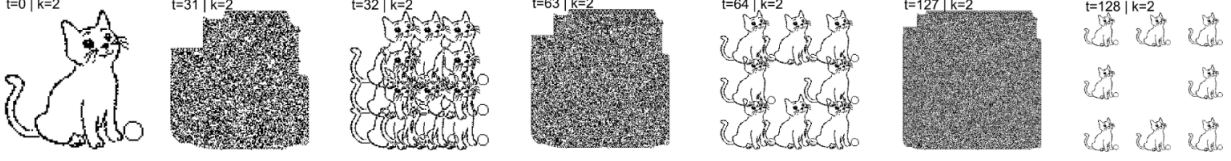


Figure 1: Binary Moore–Laplacian evolution of a “cat” seed showing alternating chaotic and ordered phases. Iterations $t = 31, 63, 127$ exhibit high-entropy, visually chaotic patterns, while $t = 32, 64, 128$ display sudden transitions to ordered replication. At $t = 128$ the replicas become fully separated, marking a large-period revival analogous to a chaos–order window in nonlinear dynamical systems.

4.2 Constant-modulus baseline

The characteristic periodic behaviour of Laplacian dynamics under a fixed modulus has been analysed in detail in our earlier work [Nowak-Kępczyk, 2025]. Here we summarise only the aspects most relevant for the multi-prime framework, emphasising how the principal revival times scale with the modulus.

For a seed of width s evolved under modulus k , the corresponding “big period” $T_{\text{big}}(s; k)$ is the smallest replication time $t \in \mathcal{R}_k$ satisfying

$$2t \geq ws,$$

where w denotes the number of horizontally replicated copies (or vertically, for symmetric neighbourhoods). This condition ensures spatial separation of replica tiles and marks the onset of the fully ordered regime. Table 1 lists representative small periods and the resulting big-period rule for several moduli, illustrating the simple modular structure of these replication ladders.

Starting from a natural-image silhouette (the “cat” seed), the system undergoes a high-entropy dispersive transient before entering a replication window in which clean copies of the seed re-emerge. For the binary and quinary Laplacian (Fig. 2a,b), the first such structured return occurs at the modulus-dependent revival times $t^* = 127$ and $t^* = 125$, respectively. These constant-modulus revivals form the baseline against which the multi-prime constructions in the next section will be evaluated.

Injected noise can delay or partially suppress replica formation, but the presence of multiple nonoverlapping copies enables reliable reconstruction via replica consensus. These transitions from chaotic dispersion to ordered replication provide the qualitative foundation for the cryptographic encoding mechanism, although we make no hardness claims here.

4.3 Entropy dynamics

Figure 3 shows the evolution of the Shannon entropy H_t for the binary, ternary and quinary Laplacian dynamics of the cat seed over 128 iterations. All three systems follow the same qualitative pattern: an initially low-entropy configuration generated by the compact seed, a rapid rise into a high-entropy plateau associated with the dispersive transient, and a sequence of sharp entropy drops marking the replication windows.

The height of the plateau increases with the modulus, reflecting the larger alphabet size ($H \approx \log 2$ for $p = 2$, $\log 3$ for $p = 3$, and $\log 5$ for $p = 5$). Superimposed on these plateaus are well-localised

Table 1: Constant-modulus dynamics: revival ladders and big-period rule.

Modulus	Small periods	Big period rule
2, 4, 8 (powers of two)	16, 32, 64, ... (binary ladder)	$T_{\text{big}} = \min\{t \in \mathcal{R}_2 : 2t \geq ws\}$
3	27, 54, 81, ... (ternary ladder)	$T_{\text{big}} = \min\{t \in \{27m\} : 2t \geq ws\}$
9, 27 (powers of three)	81, 162, ... (scaled ladder)	$T_{\text{big}} = \min\{t \in \{81m\} : 2t \geq ws\}$
5	25, 50, 75, ... (quinary ladder)	$T_{\text{big}} = \min\{t \in \{25m\} : 2t \geq ws\}$
$6 = 2 \cdot 3$ (composite)	mixed: 32, 64, ... and 27, 54, ...	$T_{\text{big}} = \min\{t \in \mathcal{R}_2 \cap \mathcal{R}_3 : 2t \geq ws\}$

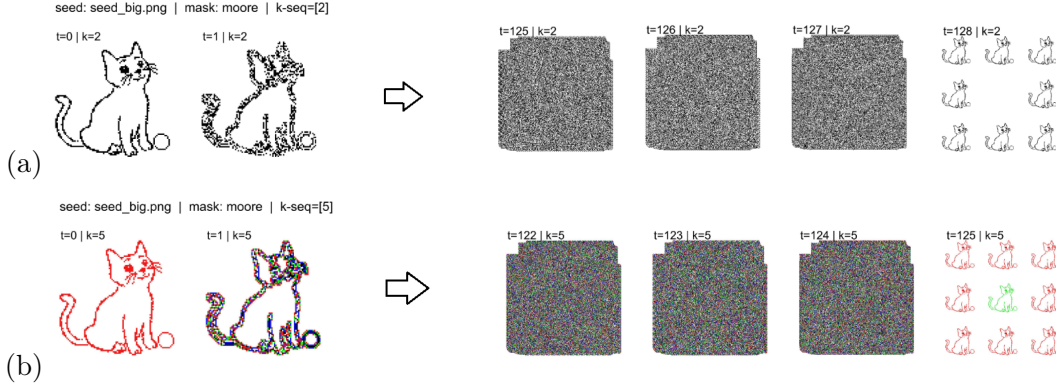


Figure 2: Characteristic-period returns for a natural-image silhouette under Moore–Laplacian dynamics. In both binary (a) and quinary (b) rules, the seed disperses through a chaotic high-entropy transient before reappearing as structured replicas at the revival time t^* . These constant-modulus revivals form the baseline for the multi-prime constructions discussed in the next section.

minima corresponding to the principal revival times of each modulus. For $p = 2$ the deepest drop appears at $t = 128$, while the ternary and quinary rules exhibit their primary minima at $t = 81$ and $t = 125$, respectively. These minima coincide exactly with the emergence of clean replica tilings in Fig. 2, demonstrating that H_t captures the onset of geometric order quantitatively.

Between the minima the entropy returns to its plateau, indicating that the system has re-entered a statistically chaotic regime. Thus the entropy trace serves as a precise observable for detecting replication times across different moduli. It distinguishes chaotic transients from order-restoring phases and mirrors the modulus-dependent structure of the Laplacian revival ladder. The close alignment of entropy minima with the visual events in Fig. 2 confirms that H_t provides a robust quantitative signature for the chaos–order alternation characteristic of these dynamics.

4.4 Multi-prime composition

When several Laplacian operators with distinct prime moduli are combined, the resulting dynamics exhibit substantially extended periodic orbits and delayed revivals. The interaction of the individual prime periods produces long composite cycles of total length

$$T_{\text{global}} = \text{lcm}(T_{p_1}, \dots, T_{p_m}),$$

often several orders of magnitude larger than any single-prime period.

Figure 2 illustrates the characteristic structure: intermediate stages of the orbit appear visually chaotic, while the later phases recover clean, well-separated replicas of the original seed. Within this transient interval, configurations are statistically indistinguishable from noise—a property central to the encoding mechanism. An encoding instance is obtained by selecting any state within this high-entropy

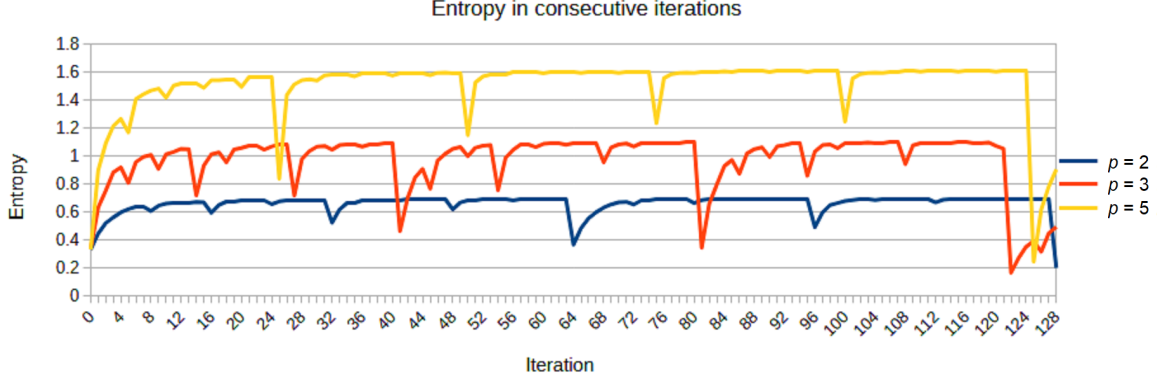


Figure 3: Entropy H_t for the Moore–Laplacian evolution of the cat seed under moduli $p = 2$, $p = 3$, and $p = 5$ over 128 iterations. Each modulus exhibits a high-entropy chaotic plateau interspersed with sharp dips corresponding to revival windows. The depth and timing of the minima depend on the modulus ($t = 128$ for $p = 2$, $t = 81$ for $p = 3$, $t = 125$ for $p = 5$), illustrating the prime-specific replication ladders.

segment. Decoding then completes the remaining portion of the composite cycle, which automatically restores the replication pattern and ultimately the seed.

The multi-prime construction therefore creates a robust dichotomy between the entropy-maximising transient and the order-restoring revival phase. Because the underlying Laplacian operators are non-commuting, the order of the prime moduli forms an essential part of the key: reversing the order typically destabilises the composite cycle and suppresses the revival. These properties naturally extend the constant-modulus behaviour and form the basis for the robustness analysis presented below.

4.5 Noise robustness and reconstruction

Under weak additive noise, distortions remain spatially localised rather than propagating throughout the pattern. Typical experiments show that a few corrupted tiles appear among many pristine replicas, while undamaged tiles retain full fidelity. This reveals a form of *local stability*: errors introduced in one region do not spread to neighbouring replicas. The chaotic transient between revivals acts as a natural separator, preventing cross-tile contamination.

This intrinsic isolation provides a baseline robustness mechanism even in the absence of explicit error-correction layers. Analytical bounds and direct experiments further support this behaviour, as demonstrated below.

Local patch experiment. Persistent per-step noise (“snowing”) accumulates gradually along an orbit, but realistic transmission errors often manifest as a few localised block corruptions. To model such events, let $T = I + B$ and $u_t = T^t u_0$. At an intermediate time s we overwrite a small patch by η , forming $y_s = u_s + \eta$. After the remaining $\Delta = 2^m - s$ steps up to the Frobenius time $t = 2^m$,

$$T^\Delta y_s = T^{2^m} u_0 + T^\Delta \eta = (I + B^{2^m}) u_0 + T^\Delta \eta.$$

The first term produces the standard 3×3 replica tiling. The perturbation term remains confined to a light cone of radius $r\Delta$ with $r = 1$ for the Moore mask. Hence a patch affects only the replica intersecting this cone provided

$$r\Delta < \frac{1}{2} 2^m.$$

Concrete instance. In Fig. 4, four block perturbations are inserted at $s = 127$ for a binary Frobenius time $t = 128$ ($p = 2$, $m = 7$). Here $\Delta = 1$, so $r\Delta = 1 \ll 64$, and each perturbation remains fully local. After the top-up to $t = 128$, every error appears only within its own replica tile; all remaining tiles, and the reconstructed seed, are unaffected.

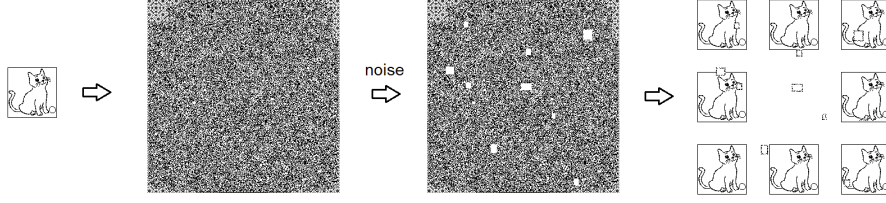


Figure 4: Localised block perturbations inserted at $t = 127$, just before the Frobenius revival at $t = 128$ ($p = 2$). With only $\Delta = 1$ remaining step, each perturbation remains confined to its own replica tile while the others remain pristine. This demonstrates inherent spatial isolation of one-shot errors.

This shows that block-type errors—common in packet loss and file transfer—do not interfere with seed recovery. Only persistent additive noise affects multiple tiles simultaneously.

Accumulated noise and replica repair. Figure 5 illustrates the effect of weak additive noise on a ternary Laplacian evolution over $t = 81$ steps. In the noiseless case (a), the seed disperses chaotically and then reappears as nine non-overlapping replicas at $t^* = 81$, allowing exact reconstruction.

Panel (b) repeats the experiment with per-step noise $p_{\text{noise}} = 0.05\%$. Although the final state appears random, the replica tiling persists: dividing the frame into nine tiles and applying majority (mode) voting recovers the seed with negligible error. Spatial replication thus provides an intrinsic error-correction mechanism.

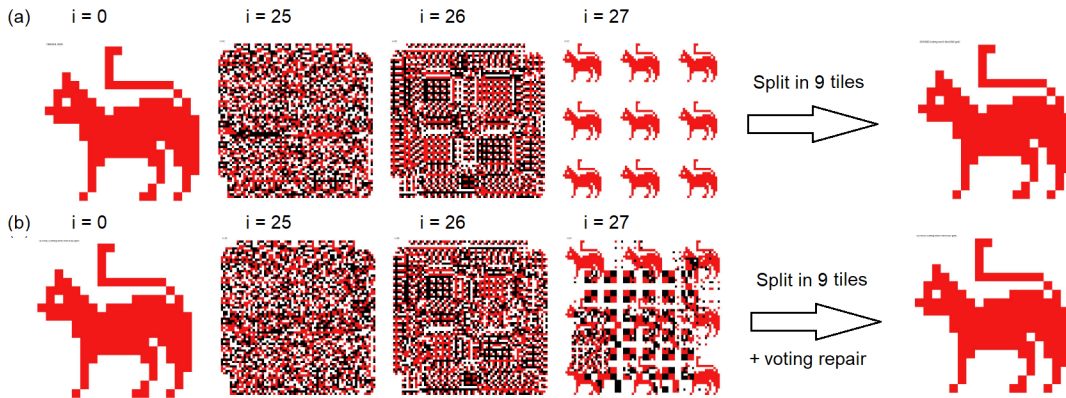


Figure 5: Effect of additive noise on ternary Laplacian evolution. (a) Noiseless case: perfect recovery after $t^* = 81$. (b) With per-step noise $p_{\text{noise}} = 0.05\%$: although the final state appears random, replica voting restores the seed with negligible error.

Quantitative measurements in Table 2 support these observations. For each modulus p , Laplacian revivals tolerate a nonzero but bounded per-step noise rate while maintaining accurate reconstruction via tile voting. Longer revival times correspond to slightly smaller noise budgets, consistent with cumulative error growth across more iterations.

Monte–Carlo tolerance. Table 2 lists the empirical per-step threshold p_{max} obtained using the Monte–Carlo protocol of Section 3.6, defined as the largest tested noise rate such that the mean post-

repair Hamming error satisfies $\bar{E} \leq 5.0 \times 10^{-2}$. Even for modest revival times, replica voting significantly enhances robustness.

Table 2: Empirical per-step noise tolerance $p_{\max}$ for the ternary Laplacian revival ($p = 3, t^* = 27$) with replica voting. Note: tolerance levels reflect empirical robustness and not theoretical bounds.

p	t^*	$p_{\max}$	mean Hamming at $p_{\max}$
3	27	5×10^{-5}	2.0×10^{-2}

4.6 Temporal redundancy.

Beyond the spatial redundancy provided by the replica tiling, one may introduce a second, independent layer of *temporal redundancy*. Instead of transmitting a single chaotic state u_s with $s < t^*$, we choose a set of encoding times

$$S = \{s_1, s_2, \dots, s_L\} \subset [0, t^*),$$

each yielding a distinct noisy snapshot. For each s_ℓ , the receiver performs the top-up to t^* , obtaining an independent reconstruction $\hat{u}_0^{(\ell)}$. The final estimate of the seed is obtained by majority (or mode) vote over all L reconstructions.

The combined spatial-temporal redundancy substantially improves robustness. Even when individual snapshots are heavily corrupted, consensus over independent reconstructions suppresses sporadic defects and lowers the effective error rate. This is particularly advantageous in long trajectories where noise accumulates.

Discussion. For the ternary case ($p = 3$), replica voting remains effective up to a per-step noise rate of roughly 5×10^{-5} , yielding a mean post-repair Hamming error of about 2×10^{-2} . For higher primes ($p = 5, 7$), the tested grid of noise rates exceeded the reconstruction threshold, and none of the tested values achieved post-repair errors below 5%. These thresholds should be viewed as conservative: robustness can be increased by enlarging the number of replicas, refining crop geometry, or incorporating multi-cycle temporal consensus. The reported $p_{\max}$ values thus represent empirical performance, not the fundamental stability limits of Frobenius-driven Laplacian revivals.

5 Applications

The Frobenius-based revival mechanism combines three essential features: reversibility, structured chaotic phases, and local stability under perturbations. Together, these properties enable several practical applications ranging from procedural pattern synthesis to reversible encoding and error-tolerant reconstruction.

5.1 Decoding and security

Encoding takes place after

$$(T_{p_1} - x_1) + (T_{p_2} - x_2) + \dots + (T_{p_m} - x_m)$$

iterations of the multi-prime cycle (1), when the configuration lies in a visually chaotic, high-entropy state. Decoding then applies the remaining offsets in the reverse order,

$$x_m + x_{m-1} + \dots + x_1,$$

thereby undoing the composition and recovering the original seed exactly. The reversal is essential because the Laplacian operators L_{p_i} generally do not commute.

For authorised users the procedure runs in linear time. In contrast, an adversary lacking the moduli, their order, and the offsets must solve a combinatorial phase-recovery problem. Since noncommutativity makes the ordered prime tuple $(p_1, \dots, p_m)$ part of the key itself, the multi-prime construction both lengthens the global orbit and increases resistance to brute-force analysis.

Pattern synthesis and self-replication. Laplacian dynamics naturally generate multiple non-overlapping replicas of the seed at revival times $t = p^m$. This makes the system suitable for procedural texture generation, deterministic tiling, and self-similar image synthesis. Because the replicated tiles are algebraically determined, their spatial alignment and relative phase are exact, yielding a deterministic alternative to stochastic or noise-based texture methods.

Steganography and reversible encoding. During the chaotic phase preceding the revival window, intermediate configurations appear visually random and statistically featureless. A message (seed) can therefore be encoded by releasing a state u_s with $s < t^*$, after which an authorised recipient decodes it by completing the remaining $\Delta = t^* - s$ iterations. The key consists of the tuple (p, mask, m, s) ; incorrect parameters do not restore the seed. This yields a reversible, dynamical steganographic mechanism based on emergent structure rather than additive encryption.

Error-tolerant reconstruction. The multi-replica structure at t^* provides inherent redundancy. As demonstrated in Section 4.5, majority (or mode) voting across replicated tiles suppresses local corruption even under weak additive noise. This suggests hybrid designs that combine Laplacian evolution with conventional error-control techniques, using replica consensus as a lightweight error-repair primitive.

Outlook. These examples show how a purely algebraic phenomenon in cellular dynamics can be repurposed into practical tools for reversible encoding, structured pseudorandomness, and robust information representation. To make these ideas concrete, we now present an explicit algorithmic framework for Frobenius-driven encoding and decoding.

6 Discussion

The analytical and numerical results presented above reveal a consistent structure in Frobenius-driven Laplacian dynamics across prime moduli. Three features appear robust.

(i) **Deterministic emergence from chaotic transients.** Although intermediate states are visually disordered, their evolution is constrained by the identity

$$(I + B)^{p^m} = I + B^{p^m}.$$

At the Frobenius times $t = p^m$ all mixed terms vanish, causing the chaotic transient to collapse into a regular multi-tile revival. This mechanism is algebraic rather than statistical and contrasts with CA designs aimed at maximising unpredictability.

(ii) **Local propagation and replica-level stability.** Perturbations propagate within a bounded light cone and therefore remain confined to a single replica at revival times. Even under weak additive noise, the final configuration retains a valid tiling, and majority or mode voting across tiles yields reliable reconstruction. This form of error localisation is intrinsic to the Laplacian operator and requires no auxiliary correction layer.

(iii) **Extended cycles through multi-prime composition.** Compositions of Laplacian operators with distinct prime moduli produce periodic orbits of length

$$T_{\text{global}} = \text{lcm}(T_{p_1}, \dots, T_{p_m}),$$

substantially exceeding the individual prime periods. These orbits contain long chaotic segments followed by deterministic revivals, yielding controlled alternation between dispersive and reconstructive phases.

Limitations. Two structural constraints are inherent to the mechanism. First, Frobenius revivals rely on the presence of the self-term in $T = I + B$; pure Laplacians do not generally exhibit this behaviour. Second, exact decoding requires zero boundary conditions on an expanding domain; periodic boundaries may induce tile overlap at revival times.

Relation to existing CA frameworks. Classical CA schemes for encryption or pseudorandomness focus on diffusion and confusion properties. The present setting differs in that long-term behaviour is algebraically rigid and fully reversible. The dynamics therefore occupy an intermediate position between symbolic dynamics, linear CA theory, and reversible encoding.

Perspectives. Several extensions merit further investigation:

- multi-scale redundancy through combined spatial and temporal voting,
- anisotropic neighbourhoods and their effect on tiling geometry,
- adaptation of the mechanism to sparse or symbolic data,
- characterisation of composite-period growth rates,
- analytic estimates of noise tolerance via operator norms on $\mathbb{F}_p$.

These directions suggest that Frobenius-based dynamics may provide a useful testbed for controlled reversibility in linear CA.

7 Conclusion

We studied Frobenius-driven revivals in prime-modulus Laplacian cellular automata and analysed their use as a reversible encoding mechanism. A compact seed undergoes a dispersive, high-entropy transient and subsequently reappears as multiple disjoint replicas at times $t = p^m$. This phenomenon follows directly from the identity $(I + B)^{p^m} = I + B^{p^m}$ and is robust across symmetric neighbourhoods.

The mechanism offers:

- predictable transitions between chaotic and ordered regimes,
- exact reversibility in the noiseless case,
- intrinsic spatial redundancy enabling replica-based reconstruction,
- tunable orbit lengths via multi-prime composition.

An explicit encoding–decoding scheme based on these revivals was formulated, together with separation conditions for exact recovery. Monte–Carlo tests show that replica voting significantly increases tolerance to weak additive noise, even when the chaotic transient appears statistically random.

Beyond its algebraic interest, the mechanism suggests applications in reversible encoding, structured pseudorandomness, texture synthesis, and robust information representation. Future work will address sharper analytic bounds on noise propagation, extensions to CRT-driven multi-prime systems, and revival phenomena in higher-dimensional or nonlinear CA settings.

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