EXPONENTIAL MATRICES, $\mathbb{G}_a$-ACTIONS ON PROJECTIVE SPACES AND MODULAR REPRESENTATIONS OF ELEMENTARY ABELIAN p-GROUPS

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Abstract. Let k be an algebraically closed field of characteristic $p\,(\geq 0)$ and let $\mathbb{G}_a$ denote the additive group of k. In this article, we reveal the following:

In the first, we show that there exist one-to-one correspondences among the following sets (a), (b), (c), (d), (e), (f):

(a) the set $\mathrm{Mat}(n, k[T])^E$ of all exponential matrices of $\mathrm{Mat}(n, k[T])$;
(b) the set of all homomorphisms $h : k[\,\mathrm{GL}(n, k)\,] \to k[T]$ of Hopf algebras;
(c) the set of all homomorphisms $\varphi : \mathbb{G}_a \to \mathrm{GL}(n, k)$ of algebraic groups;
(d) the set of all homomorphisms $\theta : \mathbb{G}_a \to \mathrm{PGL}(n, k)$ of algebraic groups;
(e) the set of all homomorphisms $\vartheta : \mathbb{G}_a \to \mathrm{Aut}(\mathbb{P}^{n-1})$ of algebraic groups;
(f) the set of all $\mathbb{G}_a$-actions μ on $\mathbb{P}^{n-1}$.

These correspondences are given up to equivalences.

In the second, assuming $p > 0$, we build a relationship between modular representations of elementary abelian p-groups and exponential matrices. We show that there exists a one-to-one correspondence between the following sets (α) and (β):

(α) the set of all group homomorphisms from $(\mathbb{Z}/p\mathbb{Z})^r$ to $\mathrm{GL}(n, k)$, where r ranges over all non-negative integers.
(β) the set $\mathrm{Mat}(n, k[T])^E \times \mathbb{Z}_{\geq 0}$.

0. Introduction

In this article, we investigate relationships among exponential matrices, $\mathbb{G}_a$-actions on projective spaces and modular representations of elementary abelian p-groups. These three objects belong to different reseach areas. At present, there seems no interactive result among these objects. We are in the following research situations:

In characteristi zero, exponential matrices are classified (see [5]). In positive characteristic, exponential matrices of size n-by-n for $1 \leq n \leq 5$ are described (see [5, 6]).

In characteristic zero, $\mathbb{G}_a$-actions on the n-dimensional projective space are classified (see [3]). Perhaps, in positive characteristic, we need to discover p-techniques for exploring $\mathbb{G}_a$-actions on projective spaces.

Modular representations of elemetary abelian p-groups are wild except for few cases (see [2]). We refer to [1] about various results of modular representations of elementary abelian p-groups.

Toward collaborative developments of exponential matrices, $\mathbb{G}_a$-actions on projective spaces and modular representations of elementary abelian p-groups, we build relationships among these objects. Our main results are the following Theorems 0.1 and 0.2, where k is an algebraically closed field of characteristic $p\,(\geq 0)$ and $\mathbb{G}_a$ denotes the additive group of k:

Theorem 0.1. *There exist one-to-one correspondences among the following sets* (a)*,* (b)*,* (c)*,* (d)*,* (e)*,* (f)*:*

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(a) *the set* $\mathrm{Mat}(n, k[T])^E$ *of all exponential matrices of* $\mathrm{Mat}(n, k[T])$*;*
(b) *the set of all homomorphisms* $h : k[\,\mathrm{GL}(n, k)\,] \to k[T]$ *of Hopf algebras;*
(c) *the set of all homomorphisms* $\varphi : \mathbb{G}_a \to \mathrm{GL}(n, k)$ *of algebraic groups;*
(d) *the set of all homomorphisms* $\theta : \mathbb{G}_a \to \mathrm{PGL}(n, k)$ *of algebraic groups;*
(e) *the set of all homomorphisms* $\vartheta : \mathbb{G}_a \to \mathrm{Aut}(\mathbb{P}^{n-1})$ *of algebraic groups;*
(f) *the set of all* $\mathbb{G}_a$*-actions* μ *on* $\mathbb{P}^{n-1}$*.*

These correspondences are given up to equivalences.

The following theorem works provided that $p > 0$.

Theorem 0.2. *Assume* $p > 0$*. There exists a one-to-one correspondence between the following sets* (α) *and* (β)*:*

(α) *the set of all group homomorphisms from* $(\mathbb{Z}/p\mathbb{Z})^r$ *to* $\mathrm{GL}(n, k)$*, where* r *ranges over all non-negative integers;*
(β) *the set* $\mathrm{Mat}(n, k[T])^E \times \mathbb{Z}_{\geq 0}$*.*

We employ the following notation and a definition:

For any commutative ring R, we denote by $\mathrm{Mat}(n, R)$ the set of all $n \times n$ matrices whose entries belong to R and denote by $R[T]$ the polynomial ring in one variable over R.

For any field k, a matrix $A(T)$ of $\mathrm{Mat}(n, k[T])$ is said to be an *exponential matrix* of $\mathrm{Mat}(n, k[T])$ if $A(T)$ satisfies the following conditions (1) and (2):

(1) $A(T)\,A(T') = A(T + T')$ in $\mathrm{Mat}(n, k[T, T'])$, where $k[T, T']$ denotes the polynomial ring in two variables over k.
(2) $A(0) = I_n$.

We denote by $\mathrm{Mat}(n, k[T])^E$ the set of all exponential matrices $A(T)$ of $\mathrm{Mat}(n, k[T])$.

1. Correspondences

1.1. Equivalences

(a) **Equivalence of exponential matrices of** $\mathrm{Mat}(n, k[T])$**.**

For any regular matrix P of $\mathrm{GL}(n, k)$, we can define a map $\mathrm{inn}_P^{(\mathrm{a})} : \mathrm{Mat}(n, k[T])^E \to \mathrm{Mat}(n, k[T])^E$ as

$$\mathrm{inn}_P^{(\mathrm{a})}(\,A(T)\,) := P\,A(T)\,P^{-1}.$$

Let $A_1(T)$ and $A_2(T)$ be exponential matrices of $\mathrm{Mat}(n, k[T])$. We say that $A_1(T)$ and $A_2(T)$ are *equivalent* if there exists a regular matrix P of $\mathrm{GL}(n, k)$ such that

$$A_2(T) = \mathrm{inn}_P^{(\mathrm{a})}(\,A_1(T)\,).$$

(b) **Equivalence of homomorphisms** $k[\,\mathrm{GL}(n, k)\,] \to k[T]$ **of Hopf algebras.**

Let

$$\big(\,k[\,\mathrm{GL}(n, k)\,],\ \Delta_{k[\,\mathrm{GL}(n,k)\,]},\ \epsilon_{k[\,\mathrm{GL}(n,k)\,]},\ S_{k[\,\mathrm{GL}(n,k)\,]}\,\big)$$

be the commutative Hopf algebra corresponding to the linear algebraic group $\mathrm{GL}(n, k)$, where $k[\,\mathrm{GL}(n, k)\,]$ denotes the coordinate ring of $\mathrm{GL}(n, k)$, i.e.,

$$k[\,\mathrm{GL}(n, k)\,] = k\big[\,x_{i,j}\ (\,1 \leq i, j \leq n\,),\ 1/d\,\big], \quad d := \det\big(\,(x_{i,j})_{1 \leq i,j \leq n}\,\big),$$

and where $\Delta_{k[\,\mathrm{GL}(n,k)\,]}$ is a comultiplication, $\epsilon_{k[\,\mathrm{GL}(n,k)\,]}$ is a counit and $S_{k[\,\mathrm{GL}(n,k)\,]}$ is an antipode, i.e.,

$$\Delta_{k[\,\mathrm{GL}(n,k)\,]}(x_{i,j}) = \sum_{\ell=1}^{n} x_{i,\ell} \otimes x_{\ell,j} \qquad \text{for all} \quad 1 \le i, j \le n,$$

$$\epsilon_{k[\,\mathrm{GL}(n,k)\,]}(x_{i,j}) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

$$S_{k[\,\mathrm{GL}(n,k)\,]}(x_{i,j}) = (-1)^{i+j}\, d^{-1} \det\big(\, (x_{r,s})_{r\neq j,\, s\neq i}\,\big) \qquad \text{for all} \quad 1 \le i, j \le n.$$

Let $(k[T], \Delta_{k[T]}, \epsilon_{k[T]}, S_{k[T]})$ be the Hopf algebra corresponding to the additive group $\mathbb{G}_a$, where $\Delta_{k[T]}$ is a comultiplication, $\epsilon_{k[T]}$ is a counit and $S_{k[T]}$ is an antipode. So, $\Delta_{k[T]}(T) = T \otimes 1 + 1 \otimes T$, $\epsilon_{k[T]}(T) = 0$ and $S_{k[T]}(T) = -T$.

For any regular matrix P of $\mathrm{GL}(n,k)$, we denote by $\widehat{p}_{i,j}$ the (i,j)-th entry of P^{-1} ($1 \le i, j \le n$), i.e., $P^{-1} = (\,\widehat{p}_{i,j}\,)_{1\le i,j\le n}$. We can define a homomorphism $\mathrm{inn}_P^{(\mathrm{b})} : k[\,\mathrm{GL}(n,k)\,] \to k[\,\mathrm{GL}(n,k)\,]$ as

$$\mathrm{inn}_P^{(\mathrm{b})}(x_{i,j}) = \sum_{1\le \lambda,\mu \le n} p_{i,\lambda} \cdot x_{\lambda,\mu} \cdot \widehat{p}_{\mu,j} \qquad \text{for all} \quad 1 \le i, j \le n.$$

Let $h_1 : k[\,\mathrm{GL}(n,k)\,] \to k[T]$ and $h_2 : k[\,\mathrm{GL}(n,k)\,] \to k[T]$ be homomorphisms of Hopf algebras. We say that h_1 and h_2 are *equivalent* if there exists a regular matrix P of $\mathrm{GL}(n,k)$ such that

$$h_2 = h_1 \circ \mathrm{inn}_P^{(\mathrm{b})}.$$

(c) **Equivalence of homomorphisms $\mathbb{G}_a \to \mathrm{GL}(n,k)$ of algebraic groups.**

For any regular matrix P of $\mathrm{GL}(n,k)$, we can define a homomorphism $\mathrm{inn}_P^{(\mathrm{c})} : \mathrm{GL}(n,k) \to \mathrm{GL}(n,k)$ as

$$\mathrm{inn}_P^{(\mathrm{c})}(Q) := P\,Q\,P^{-1}.$$

Let $\varphi_1 : \mathbb{G}_a \to \mathrm{GL}(n,k)$ and $\varphi_2 : \mathbb{G}_a \to \mathrm{GL}(n,k)$ be homomorphisms of algebraic groups. We say that φ_1 and φ_2 are *equivalent* if there exists a regular matrix P of $\mathrm{GL}(n,k)$ such that

$$\varphi_2 = \mathrm{inn}_P^{(\mathrm{c})} \circ \varphi_1.$$

(d) **Equivalence of homomorphisms $\mathbb{G}_a \to \mathrm{PGL}(n,k)$ of algebraic groups.**

For any regular matrix P of $\mathrm{GL}(n,k)$, we can define a homomorphism $\mathrm{inn}_P^{(\mathrm{d})} : \mathrm{PGL}(n,k) \to \mathrm{PGL}(n,k)$ as

$$\mathrm{inn}_P^{(\mathrm{d})}([Q]) := [\,P\,Q\,P^{-1}\,].$$

Let $\theta_1 : \mathbb{G}_a \to \mathrm{PGL}(n,k)$ and $\theta_2 : \mathbb{G}_a \to \mathrm{PGL}(n,k)$ be homomorphisms of algebraic groups. We say that θ_1 and θ_2 are *equivalent* if there exists a regular matrix P of $\mathrm{GL}(n,k)$ such that

$$\theta_2 = \mathrm{inn}_P^{(\mathrm{d})} \circ \theta_1.$$

(e) **Equivalence of homomorphisms $\mathbb{G}_a \to \mathrm{Aut}(\mathbb{P}^{n-1})$ of algebraic groups.**

For any $\sigma \in \mathrm{Aut}(\mathbb{P}^{n-1})$, we can define a homomorphism $\mathrm{inn}_\sigma^{(\mathrm{e})} : \mathrm{Aut}(\mathbb{P}^{n-1}) \to \mathrm{Aut}(\mathbb{P}^{n-1})$ as

$$\mathrm{inn}_\sigma^{(\mathrm{e})}(f) := \sigma \circ f \circ \sigma^{-1}.$$

Let $\vartheta_1 : \mathbb{G}_a \to \mathrm{Aut}(\mathbb{P}^{n-1})$ and $\vartheta_2 : \mathbb{G}_a \to \mathrm{Aut}(\mathbb{P}^{n-1})$ be homomorphisms of algebraic groups. We say that ϑ_1 and ϑ_2 are *equivalent* if there exists an element σ of $\mathrm{Aut}(\mathbb{P}^{n-1})$ such that $\vartheta_2 = \mathrm{inn}_\sigma^{(\mathrm{e})} \circ \vartheta_1$.

(f) **Equivalence of $\mathbb{G}_a$-actions on $\mathbb{P}^{n-1}$.**

Two $\mathbb{G}_a$-actions $\mu_1 : \mathbb{G}_a \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ and $\mu_2 : \mathbb{G}_a \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ are said to be *equivalent* if there exists an automorphism $\sigma : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ such that $\mu_2 \circ (\mathrm{id}_{\mathbb{G}_a} \times \sigma) = \sigma \circ \mu_1$, i.e., the following diagram is commutative, where $\mathrm{id}_{\mathbb{G}_a} : \mathbb{G}_a \to \mathbb{G}_a$ denotes the identity map:

$$\begin{tikzcd}
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\mu_1"] \arrow[d, "\mathrm{id}_{\mathbb{G}_a} \times \sigma"'] & \mathbb{P}^{n-1} \arrow[d, "\sigma"] \\
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\mu_2"] & \mathbb{P}^{n-1}
\end{tikzcd}$$

1.2. Proof of Theorem 0.1

1.2.1. (a) $\cong$ (b) $\cong$ (c)

We shall give one-to-one correspondences among sets (a), (b), (c).

We denote by $\mathcal{A}^+$ the set of all matrices $A(T)$ of $\mathrm{Mat}(n, k[T])$ such that $\det A(T) \in k\backslash\{0\}$, i.e.,

$$\mathcal{A}^+ := \{ A(T) \in \mathrm{Mat}(n, k[T]) \mid \det A(T) \in k\backslash\{0\} \}.$$

Clearly, $\mathrm{Mat}(n, k[T])^E \subset \mathcal{A}^+$ (see [5, Lemmas 1.8 and 1.9]).

We denote by $\mathcal{B}^+$ the set of all homomorphisms $h : k[\,\mathrm{GL}(n,k)\,] \to k[T]$ of k-algebras, i.e.,

$$\mathcal{B}^+ := \mathrm{Hom}_{k\text{-alg}}(k[\,\mathrm{GL}(n,k)\,], k[T]).$$

Clearly, $\mathrm{Hom}_{\mathrm{Hopf}}(k[\,\mathrm{GL}(n,k)\,], k[T]) \subset \mathcal{B}^+$, where $\mathrm{Hom}_{\mathrm{Hopf}}$ denotes homomorphisms of Hopf algebras.

We denote by $\mathcal{C}^+$ the set of all homomorphisms $\varphi : \mathbb{G}_a \to \mathrm{GL}(n, k[T])$ of affine varieties, i.e.,

$$\mathcal{C}^+ := \mathrm{Hom}_{\mathrm{aff.var}}(\mathbb{G}_a, \mathrm{GL}(n,k)).$$

Clearly, $\mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{GL}(n,k)) \subset \mathcal{C}^+$, where $\mathrm{Hom}_{\mathrm{alg.gr}}$ denotes homomorphisms of algebraic groups.

Let $F_{\mathcal{B}^+,\mathcal{A}^+} : \mathcal{A}^+ \to \mathcal{B}^+$ be the map defined, as follows: To a matrix $A(T) = (\,a_{i,j}(T)\,)_{1\le i,j\le n}$ of $\mathcal{A}^+$, we assign an element $h : k[\,\mathrm{GL}(n,k)\,] \to k[T]$ of $\mathcal{B}^+$ as

$$h(x_{i,j}) := a_{i,j}(T) \qquad \text{for all} \quad 1 \le i,j \le n.$$

Let $F_{\mathcal{C}^+,\mathcal{B}^+} : \mathcal{B}^+ \to \mathcal{C}^+$ be the map defined, as follows: To an element $h : k[\,\mathrm{GL}(n,k)\,] \to k[T]$ of $\mathcal{B}^+$, we naturally assign an element $\varphi : \mathbb{G}_a \to \mathrm{GL}(n,k)$ of $\mathcal{C}^+$.

We have the following diagram:

$$\begin{tikzcd}
\mathcal{A}^+ \arrow[r, "F_{\mathcal{B}^+,\mathcal{A}^+}"] & \mathcal{B}^+ \arrow[r, "F_{\mathcal{C}^+,\mathcal{B}^+}"] & \mathcal{C}^+ \\
\mathrm{Mat}(n, k[T])^E \arrow[u, hook] & \mathrm{Hom}_{\mathrm{Hopf}}(k[\,\mathrm{GL}(n,k)\,], k[T]) \arrow[u, hook] & \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{GL}(n,k)) \arrow[u, hook]
\end{tikzcd}$$

Lemma 1.1. *The following assertions* (1) *and* (2) *hold true:*

(1) *The map $F_{\mathcal{B}^+, \mathcal{A}^+}$ gives a one-to-one coresspondence between the sets $\mathcal{A}^+$ and $\mathcal{B}^+$.*
(2) *The map $F_{\mathcal{C}^+, \mathcal{B}^+}$ gives a one-to-one correspondence between the sets $\mathcal{B}^+$ and $\mathcal{C}^+$.*

Proof. The proofs of assertions (1) and (2) are straightforward. $\square$

Lemma 1.2. *Let*

$$A(T) = (\, a_{i,j}(T) \,)_{1 \leq i,j \leq n} \in \mathcal{A}^+, \qquad h := F_{\mathcal{B}^+, \mathcal{A}^+}(A(T)) \in \mathcal{B}^+, \qquad \varphi := F_{\mathcal{C}^+, \mathcal{B}^+}(h) \in \mathcal{C}^+.$$

Then the following conditions (1), (2), (3), (4) *are equivalent:*

(1) $A(T) \in \mathrm{Mat}(n, k[T])^E$.
(2) *We have*

$$a_{i,j}(T \otimes 1 + 1 \otimes T) = \sum_{\ell=1}^{n} a_{i,\ell}(T) \otimes a_{\ell,j}(T) \qquad \text{for all} \quad 1 \leq i, j \leq n$$

and

$$a_{i,j}(0) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

(3) $h \in \mathrm{Hom}_{\mathrm{Hopf}}(k[\,\mathrm{GL}(n,k)\,], k[T])$. *That is to say, the following two diagrams are commutative:*

$$\begin{tikzcd} k[\,\mathrm{GL}(n,k)\,] \otimes_k k[\,\mathrm{GL}(n,k)\,] \arrow[r, "h \otimes h"] & k[T] \otimes_k k[T] \\ k[\,\mathrm{GL}(n,k)\,] \arrow[u, "\Delta_{k[\,\mathrm{GL}(n,k)\,]}"] \arrow[r, "h"'] & k[T] \arrow[u, "\Delta_{k[T]}"'] \end{tikzcd}$$

$$\begin{tikzcd} k[\,\mathrm{GL}(n,k)\,] \arrow[rr, "h"] \arrow[dr, "\epsilon_{k[\,\mathrm{GL}(n,k)\,]}"'] & & k[T] \arrow[dl, "\epsilon_{k[T]}"] \\ & k & \end{tikzcd}$$

(4) $\varphi \in \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{GL}(n,k))$.

Proof. We first prove (1) $\iff$ (2). Condition (1) holds true if and only if

$$a_{i,j}(T + T') = \sum_{\ell=1}^{n} a_{i,\ell}(T) \cdot a_{\ell,j}(T') \qquad \text{for all} \quad 1 \leq i, j \leq n$$

and

$$a_{i,j}(0) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Consider the k-algebra isomorphsim $\xi : k[T, T'] \to k[T] \otimes_k k[T]$ defined by

$$\xi(T) := T \otimes 1, \qquad \xi(T') := 1 \otimes T.$$

The above two conditions hold true if and only if

$$a_{i,j}(T \otimes 1 + 1 \otimes T) = \sum_{\ell=1}^{n} a_{i,\ell}(T) \otimes a_{\ell,j}(T) \qquad \text{for all} \quad 1 \leq i, j \leq n$$

and

$$a_{i,j}(0) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

We next prove (2) $\iff$ (3). For all $1 \leq i, j \leq n$, we have

$$\big((h \otimes h) \circ \Delta_{k[\,\mathrm{GL}(n,k)\,]} \big)(x_{i,j}) = (h \otimes h) \left(\sum_{\ell=1}^{n} x_{i,\ell} \otimes x_{\ell,j} \right) = \sum_{\ell=1}^{n} a_{i,\ell}(T) \otimes a_{\ell,j}(T),$$

$$(\Delta_{k[T]} \circ h)(x_{i,j}) = \Delta_{k[T]}(a_{i,j}(T)) = a_{i,j}(T \otimes 1 + 1 \otimes T),$$

$$\big(\epsilon_{k[T]} \circ h \big)(x_{i,j}) = \epsilon_{k[T]}(\, a_{i,j}(T)\,) = a_{i,j}(0),$$

$$\epsilon_{k[\,\mathrm{GL}(n,k)\,]}(x_{i,j}) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

So, the implication (3) $\Longrightarrow$ (2) is clear. Conversely, if condition (2) holds true, then we have $((h \otimes h) \circ \Delta_{k[\,\mathrm{GL}(n,k)\,]})(x_{i,j}) = (\Delta_{k[T]} \circ h)(x_{i,j})$ and $(\epsilon_{k[T]} \circ h)(x_{i,j}) = \epsilon_{k[\,\mathrm{GL}(n,k)\,]}(x_{i,j})$ for all $1 \leq i, j \leq n+1$, and thereby have

$$\big((h \otimes h) \circ \Delta_{k[\,\mathrm{GL}(n,k)\,]} \big) \left(\frac{1}{d} \right) = \frac{1}{\big((h \otimes h) \circ \Delta_{k[\,\mathrm{GL}(n,k)\,]} \big)(d)} = \frac{1}{(\Delta_{k[T]} \circ h)(d)} = (\Delta_{k[T]} \circ h) \left(\frac{1}{d} \right),$$

$$(\epsilon_{k[T]} \circ h) \left(\frac{1}{d} \right) = \frac{1}{(\epsilon_{k[T]} \circ h)(d)} = \frac{1}{\epsilon_{k[\,\mathrm{GL}(n,k)\,]}(d)} = \epsilon_{k[\,\mathrm{GL}(n,k)\,]} \left(\frac{1}{d} \right).$$

Thus condition (3) holds true.

The equivalence (3) $\iff$ (4) is clear. □

Lemma 1.3. *The following assertions* (1) *and* (2) *hold true:*

(1) *The map* $F_{\mathcal{B}^+,\mathcal{A}^+}$ *gives a one-to-one correspondence between the sets* $\mathrm{Mat}(n, k[T])^E$ *and* $\mathrm{Hom}_{\mathrm{Hopf}}(k[\,\mathrm{GL}(n,k)\,], k[T])$.

(2) *The map* $F_{\mathcal{C}^+,\mathcal{B}^+}$ *gives a one-to-one correspondence between the sets* $\mathrm{Hom}_{\mathrm{Hopf}}(k[\,\mathrm{GL}(n,k)\,], k[T])$ *and* $\mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{GL}(n,k))$.

Proof. See Lemmas 1.1 and 1.2. □

Lemma 1.4. *For each* $i = 1, 2$, *let* $A_i(T)$ *be an exponential matrix of* $\mathrm{Mat}(n, k[T])^E$, *let* $h_i : k[\,\mathrm{GL}(n,k)\,] \to k[T]$ *be a homomorphism of Hopf algebras such that*

$$h_i := F_{\mathcal{B}^+,\mathcal{A}^+}(A_i(T)),$$

and let $\varphi_i : \mathbb{G}_a \to \mathrm{GL}(n,k)$ *be a homomorphism of algebraic groups over* k *such that*

$$\varphi_i := F_{\mathcal{C}^+,\mathcal{B}^+}(h_i(T)).$$

Then the following conditions (1), (2), (3) *are equivalent:*

(1) $A_1(T)$ *and* $A_2(T)$ *are equivalent.*

(2) h_1 *and* h_2 *are equivalent.*

(3) φ_1 *and* φ_2 *are equivalent.*

Proof. We first prove (1) $\iff$ (2). Let $P = (\, p_{i,j}\,)_{1 \leq i,j \leq n}$ be a regular matrix of $\mathrm{GL}(n,k)$. Write $P^{-1} = (\, \widehat{p}_{i,j}\,)_{1 \leq i,j \leq n}$, $A_1(T) = (\, a^{(1)}_{i,j}\,)_{1 \leq i,j \leq n}$ and $A_2(T) = (\, a^{(2)}_{i,j}\,)_{1 \leq i,j \leq n}$. So, $A_2(T) = \mathrm{inn}^{(\mathrm{a})}_P \circ A_1(T)$ if and only if

$$a^{(2)}_{i,j} = \sum_{1 \leq \lambda,\mu \leq n} p_{i,\lambda} \cdot a^{(1)}_{\lambda,\mu} \cdot \widehat{p}_{\mu,j} \qquad \text{for all} \quad 1 \leq i, j \leq n.$$

We have

$$h_2(x_{i,j}) = a^{(2)}_{i,j}$$

and

$$h_1(\,\mathrm{inn}^{(\mathrm{b})}_P(x_{i,j})\,) = h_1\left(\sum_{1\le\lambda,\mu\le n} p_{i,\lambda}\cdot x_{\lambda,\mu}\cdot \widehat{p}_{\mu,j}\right) = \sum_{1\le\lambda,\mu\le n} p_{i,\lambda}\cdot a^{(1)}_{\lambda,\mu}\cdot \widehat{p}_{\mu,j}$$
$$\text{for all}\quad 1\le i,j\le n.$$

Thus $A_2(T) = \mathrm{inn}^{(\mathrm{a})}_P(A_1(T))$ if and only if $h_2(x_{i,j}) = (h_1\circ \mathrm{inn}^{(\mathrm{b})}_P)(x_{i,j})$ for all $1\le i,j\le n$. This equivalence implies the equivalence (1) $\Longleftrightarrow$ (2).

The proof of equivalence (2) $\Longleftrightarrow$ (3) is straightforward. □

1.2.2. **(c) ≅ (d)**

We shall give a one-to-one correspondence between sets (c) and (d).

Let $\pi : \mathrm{GL}(n,k) \to \mathrm{PGL}(n,k)$ be the natural surjection. Clearly, π is a homomorphism of algebraic groups.

Lemma 1.5. *For any homomorphism $\theta : \mathbb{G}_a \to \mathrm{PGL}(n,k)$, there exists a unique homomorphism $\varphi : \mathbb{G}_a \to \mathrm{GL}(n,k)$ such that $\pi\circ\varphi = \theta$, i.e., the following diagram is commutative:*

$$\begin{tikzcd} & \mathrm{GL}(n,k) \arrow[d, "\pi"] \\ \mathbb{G}_a \arrow[ur, "\varphi"] \arrow[r, "\theta"'] & \mathrm{PGL}(n,k) \end{tikzcd}$$

Proof. If θ is trivial, i.e., $\theta(\mathbb{G}_a) = \{\,e_{\mathrm{PGL}(n,k)}\,\}$, we are done. So, assume that θ is nontrivial. The homomorphism θ factors into a composite of homomorphisms

$$\mathbb{G}_a \xrightarrow{\ q\ } X \xrightarrow{\ i\ } \mathrm{PGL}(n,k)$$

with q a faithfully flat morphism and i a closed embedding (see [4, Theorem 5.39]). The image $X = \theta(\mathbb{G}_a)_{\mathrm{red}}$ is smooth (see [4, Proposition 1.28 and Corollary 5.26]). Thus X is isomorphic to the affine line, i.e., $X\cong\mathbb{A}^1$. We consider the inverse image $E := \pi^{-1}(X)_{\mathrm{red}}$ of X under π. So, $\pi|_E : E\to X$ is a $\mathbb{G}_m$-bundle over X, where $\pi|_E$ is the restriction of π to E. Note that $\pi|_E$ is a trivial $\mathbb{G}_m$-bundle over X. In fact, there exist a line bundle $\pi' : L\to X$ and an open embedding $\iota : E\to L$ so that $\pi'\circ\iota = \pi|_E$. Since $\mathrm{Pic}\,(X) = 0$, the line bundle $\pi' : L\to X$ is trivial, and thereby $\pi|_E$ is a trivial $\mathbb{G}_m$-bundle over X. We have the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, hook, "\iota"] \arrow[d, "\pi|_E"'] & L \arrow[r, "\cong"] \arrow[dl, "\pi'"'] & X\times\mathbb{A}^1 \arrow[dll, "(x,a)\mapsto x"] \\ X & & \end{tikzcd}$$

Since $E\cong X\times\mathbb{G}_m$, there exists a section s of $\pi|_E$. We can define a morphism $f : \mathbb{G}_a\to\mathrm{GL}(n,k)$ as

$$f := \iota_E\circ s\circ q,$$

where $\iota_E : E \to \mathrm{GL}(n,k)$ is the natural closed immersion. Clearly,

$$\pi \circ f = \theta. \tag{1.5.1}$$

Let $\nu : \mathrm{GL}(n,k) \to \mathrm{GL}(n,k)$ be the homomorphism defined by

$$\nu(A) := \frac{1}{\det A} A.$$

Clearly,

$$\pi \circ \nu = \pi. \tag{1.5.2}$$

Since θ is a homomorphism, we have

$$m_{\mathrm{PGL}(n,k)} \circ (\theta \times \theta) = \theta \circ m_{\mathbb{G}_a}, \tag{1.5.3}$$

i.e., the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{G}_a \times \mathbb{G}_a \arrow[r, "\theta\times\theta"] \arrow[d, "m_{\mathbb{G}_a}"'] & \mathrm{PGL}(n,k) \times \mathrm{PGL}(n,k) \arrow[d, "m_{\mathrm{PGL}(n,k)}"] \\
\mathbb{G}_a \arrow[r, "\theta"] & \mathrm{PGL}(n,k)
\end{tikzcd}$$

We have

$$\pi \circ m_{\mathrm{GL}(n,k)} \circ (\nu \times \nu) \circ (f \times f) = \pi \circ \nu \circ f \circ m_{\mathbb{G}_a}$$

since

$$\begin{aligned}
&\pi \circ m_{\mathrm{GL}(n,k)} \circ (\nu \times \nu) \circ (f \times f) \\
&\quad = \quad m_{\mathrm{PGL}(n,k)} \circ (\pi \times \pi) \circ (\nu \times \nu) \circ (f \times f) \\
&\quad \overset{(1.5.2)}{=} \quad m_{\mathrm{PGL}(n,k)} \circ (\pi \times \pi) \circ (f \times f) \\
&\quad \overset{(1.5.1)}{=} \quad m_{\mathrm{PGL}(n,k)} \circ (\theta \times \theta) \\
&\quad \overset{(1.5.3)}{=} \quad \theta \circ m_{\mathbb{G}_a} \\
&\quad \overset{(1.5.1)}{=} \quad \pi \circ f \circ m_{\mathbb{G}_a} \\
&\quad \overset{(1.5.2)}{=} \quad \pi \circ \nu \circ f \circ m_{\mathbb{G}_a}.
\end{aligned}$$

Now, we have

$$m_{\mathrm{GL}(n,k)} \circ (\nu \times \nu) \circ (f \times f) = \nu \circ f \circ m_{\mathbb{G}_a}.$$

Letting $\varphi := \nu \circ f$, we have a homomorphism $\varphi : \mathbb{G}_a \to \mathrm{GL}(n,k)$ so that $\pi \circ \varphi = \theta$.

Let $\varphi : \mathbb{G}_a \to \mathrm{GL}(n,k)$ and let $\phi : \mathbb{G}_a \to \mathrm{GL}(n,k)$ be homomorphisms so that $\pi \circ \varphi = \theta$ and $\pi \circ \phi = \theta$. So, we have $\pi \circ \nu \circ \varphi = \pi \circ \nu \circ \phi$, which implies $\nu \circ \varphi = \nu \circ \phi$. We can show $\nu \circ \varphi = \varphi$ and $\nu \circ \phi = \phi$ since $\det \varphi(t) = \det \phi(t) = 1$ for all $t \in \mathbb{G}_a$ (see [5, Lemmas 1.8 and 1.9]). Thus $\varphi = \phi$. □

We can define a map $\varpi : \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{GL}(n,k)) \to \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{PGL}(n,k))$ as

$$\varpi(\varphi) := \pi \circ \varphi.$$

Lemma 1.6. *The map* $\varpi : \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{GL}(n,k)) \to \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{PGL}(n,k))$ *is bijective.*

Proof. See Lemma 1.5. □

Lemma 1.7. *Let $\theta_i : \mathbb{G}_a \to \mathrm{PGL}(n,k)$ $(i = 1,2)$ be homomorphisms and let $\varphi_i : \mathbb{G}_a \to \mathrm{GL}(n,k)$ $(i = 1,2)$ be homomorphisms such that $\theta_i = \pi \circ \varphi_i$ for $i = 1,2$. Then the following conditions* (1) *and* (2) *are equivalent:*

(1) *θ_1 and θ_2 are equivalent.*
(2) *φ_1 and φ_2 are equivalent.*

Proof. Let $P \in \mathrm{PGL}(n,k)$ and $Q \in \mathrm{GL}(n,k)$ satisfy $P = \pi(Q)$. Clearly, $\mathrm{inn}_P^{(\mathrm{d})} \circ \pi = \pi \circ \mathrm{inn}_Q^{(\mathrm{c})}$, i.e., the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{GL}(n,k) \arrow[r, "\mathrm{inn}_Q^{(\mathrm{c})}"] \arrow[d, "\pi"'] & \mathrm{GL}(n,k) \arrow[d, "\pi"] \\
\mathrm{PGL}(n,k) \arrow[r, "\mathrm{inn}_P^{(\mathrm{d})}"'] & \mathrm{PGL}(n,k)
\end{tikzcd}$$

If $\theta_2 = \mathrm{inn}_P^{(\mathrm{d})} \circ \theta_1$, then

$$\pi \circ \varphi_2 = \theta_2 = \mathrm{inn}_P^{(\mathrm{d})} \circ \theta_1 = \mathrm{inn}_P^{(\mathrm{d})} \circ \pi \circ \varphi_1 = \pi \circ \mathrm{inn}_Q^{(\mathrm{c})} \circ \varphi_1,$$

which implies $\varphi_2 = \mathrm{inn}_Q^{(\mathrm{c})} \circ \varphi_1$ (see Lemma 1.6). Conversely, if $\varphi_2 = \mathrm{inn}_Q^{(\mathrm{c})} \circ \varphi_1$, it is clear that $\theta_2 = \mathrm{inn}_P^{(\mathrm{d})} \circ \theta_1$.

□

1.2.3. **(d) $\cong$ (e)**

We shall give a one-to-one correspondence between sets (d) and (e).

We can define an isomorphism $\jmath : \mathrm{PGL}(n,k) \to \mathrm{Aut}(\mathbb{P}^{n-1})$ of algebraic groups as

$$P \quad \mapsto \quad \big(\, (x_0 : x_1 : \cdots : x_{n-1}) \ \mapsto \ (x_0 : x_1 : \cdots : x_{n-1})\, {}^tQ \,\big),$$

where $P = \pi(Q)$ and $\pi : \mathrm{GL}(n,k) \to \mathrm{PGL}(n,k)$ is the natural surjection.

Lemma 1.8. *The map $\mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{PGL}(n,k)) \to \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{Aut}(\mathbb{P}^{n-1}))$ defined by $\theta \mapsto \jmath \circ \theta$ is a bijection.*

Proof. The proof is straightforward. □

Lemma 1.9. *Let θ_i $(i = 1,2)$ be elements of $\mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{PGL}(n,k))$ and let $\vartheta_i := \jmath \circ \theta_i$ $(i = 1,2)$ be elements of $\mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{Aut}(\mathbb{P}^{n-1}))$. Then the following conditions* (1) *and* (2) *are equivalent:*

(1) *θ_1 and θ_2 are equivalent.*
(2) *ϑ_1 and ϑ_2 are equivalent.*

Proof. For any element $P \in \mathrm{PGL}(n,k)$, we have $\mathrm{inn}_{\jmath(P)}^{(\mathrm{e})} = \jmath \circ \mathrm{inn}_P^{(\mathrm{d})} \circ \jmath^{-1}$. So, the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{PGL}(n,k) \arrow[r, "\jmath"] \arrow[d, "\mathrm{inn}_P^{(\mathrm{d})}"'] & \mathrm{Aut}(\mathbb{P}^{n-1}) \arrow[d, "\mathrm{inn}_{\jmath(P)}^{(\mathrm{e})}"] \\
\mathrm{PGL}(n,k) \arrow[r, "\jmath"] & \mathrm{Aut}(\mathbb{P}^{n-1})
\end{tikzcd}$$

This diagram implies the equivalence (1) $\Longleftrightarrow$ (2). □

1.2.4. **(e) $\cong$ (f)**

We shall give a one-to-one correspondence between sets (e) and (f).

We can define a map $\Phi : \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{Aut}(\mathbb{P}^{n-1})) \to \{\, \mathbb{G}_a\text{-actions on } \mathbb{P}^{n-1} \,\}$ as

$$\Phi(\vartheta) := \mu_\vartheta, \qquad \mu_\vartheta(t,x) := \vartheta(t)(x).$$

So, the following diagram is commutative, where $\mu_{\mathbb{P}^{n-1}}(f,x) := f(x)$, where $\mathrm{id}_{\mathbb{P}^{n-1}} : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ denotes the identity map:

$$\begin{tikzcd}
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\mu_\theta"] \arrow[d, "\vartheta \times \mathrm{id}_{\mathbb{P}^{n-1}}"'] & \mathbb{P}^{n-1} \\
\mathrm{Aut}(\mathbb{P}^{n-1}) \times \mathbb{P}^{n-1} \arrow[ur, "\mu_{\mathbb{P}^{n-1}}"'] &
\end{tikzcd}$$

We can define a map $\Psi : \{\, \mathbb{G}_a\text{-actions on } \mathbb{P}^{n-1} \,\} \to \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{Aut}(\mathbb{P}^{n-1}))$ as

$$\Psi(\mu) := \vartheta_\mu, \qquad \vartheta_\mu(t) := (\, x \mapsto \mu(t,x) \,) \in \mathrm{Aut}(\mathbb{P}^{n-1}).$$

Lemma 1.10. $\Psi \circ \Phi = \mathrm{id}$ *and* $\Phi \circ \Psi = \mathrm{id}$.

Proof. The proof is straightforward. □

Lemma 1.11. *Let* ϑ_i $(i = 1, 2)$ *be elements of* $\mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{Aut}(\mathbb{P}^{n-1}))$. *For any* $\sigma \in \mathrm{Aut}(\mathbb{P}^{n-1})$, *the following conditions* (1) *and* (2) *are equivalent:*

(1) $\vartheta_2 = \mathrm{inn}^{(\mathrm{e})}_\sigma \circ \vartheta_1$.

(2) *The following diagram is commutative:*

$$\begin{tikzcd}
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\vartheta_1 \times \mathrm{id}_{\mathbb{P}^{n-1}}"] \arrow[d, "\mathrm{id}_{\mathbb{G}_a} \times \sigma"'] & \mathrm{Aut}(\mathbb{P}^{n-1}) \times \mathbb{P}^{n-1} \arrow[r, "\mu_{\mathbb{P}^{n-1}}"] & \mathbb{P}^{n-1} \arrow[d, "\sigma"] \\
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\vartheta_2 \times \mathrm{id}_{\mathbb{P}^{n-1}}"] & \mathrm{Aut}(\mathbb{P}^{n-1}) \times \mathbb{P}^{n-1} \arrow[r, "\mu_{\mathbb{P}^{n-1}}"] & \mathbb{P}^{n-1}
\end{tikzcd}$$

Proof. The proof is straightforward. □

Lemma 1.12. *Let* ϑ_i $(i = 1, 2)$ *be elements of* $\mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_a, \mathrm{Aut}(\mathbb{P}^{n-1}))$. *Let* $\mu_i := \Phi(\vartheta_i)$ $(i = 1, 2)$ *be* $\mathbb{G}_a$*-actions on* $\mathbb{P}^{n-1}$. *Then the following conditions* (1) *and* (2) *are equivalent:*

(1) μ_1 *and* μ_2 *are equivalent.*

(2) ϑ_1 *and* ϑ_2 *are equivalent.*

Proof. The implication (1) $\Longrightarrow$ (2) follows from the following diagram:

$$\begin{tikzcd}
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\vartheta_1 \times \mathrm{id}_{\mathbb{P}^{n-1}}"] \arrow[d, "\mathrm{id}_{\mathbb{G}_a} \times \sigma"'] \arrow[rr, bend right, "\mu_1"'] & \mathrm{Aut}(\mathbb{P}^{n-1}) \times \mathbb{P}^{n-1} \arrow[r, "\mu_{\mathbb{P}^{n-1}}"] & \mathbb{P}^{n-1} \arrow[d, "\sigma"] \\
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\vartheta_2 \times \mathrm{id}_{\mathbb{P}^{n-1}}"'] \arrow[rr, bend left, "\mu_2"] & \mathrm{Aut}(\mathbb{P}^{n-1}) \times \mathbb{P}^{n-1} \arrow[r, "\mu_{\mathbb{P}^{n-1}}"'] & \mathbb{P}^{n-1}
\end{tikzcd}$$

The implication (2) $\Longrightarrow$ (1) follows from the following diagram:

$$\begin{tikzcd}
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\vartheta_1 \times \mathrm{id}_{\mathbb{P}^{n-1}}"] \arrow[d, "\mathrm{id}_{\mathbb{G}_a} \times \sigma"'] \arrow[rr, bend left, "\mu_1"] & \mathrm{Aut}(\mathbb{P}^{n-1}) \times \mathbb{P}^{n-1} \arrow[r, "\mu_{\mathbb{P}^{n-1}}"] & \mathbb{P}^{n-1} \arrow[d, "\sigma"] \\
\mathbb{G}_a \times \mathbb{P}^{n-1} \arrow[r, "\vartheta_2 \times \mathrm{id}_{\mathbb{P}^{n-1}}"'] \arrow[rr, bend right, "\mu_2"'] & \mathrm{Aut}(\mathbb{P}^{n-1}) \times \mathbb{P}^{n-1} \arrow[r, "\mu_{\mathbb{P}^{n-1}}"'] & \mathbb{P}^{n-1}
\end{tikzcd}$$

□

2. Modular representations of elementary abelian p-groups and exponential matrices

From now on, we assume that the characteristic p of k is positive.

2.1. Preparations for proving Theorem 0.2

2.1.1. $\mathbb{E}_r(n,k)$, $\mathbb{E}_{\geq 0}(n,k)$

For any $r \geq 0$ and $n \geq 1$, we denote by $\mathbb{E}_r(n,k)$ the set of all group homomorphisms from $(\mathbb{Z}/p\mathbb{Z})^r$ to $\mathrm{GL}(n,k)$ and denote by $\mathbb{E}_{\geq 0}(n,k)$ the direct sum of $\mathbb{E}_r(n,k)$, where r ranges over all non-negative integers, i.e.,

$$\begin{cases} \mathbb{E}_r(n,k) := \mathrm{Hom}_{\mathrm{gr}}((\mathbb{Z}/p\mathbb{Z})^r, \mathrm{GL}(n,k)) \qquad \text{for all} \quad r \geq 0, \\ \mathbb{E}_{\geq 0}(n,k) := \coprod_{r \geq 0} \mathbb{E}_r(n,k). \end{cases}$$

Clearly, $\mathbb{E}_0(n,k)$ consists of one element, which we denote by ρ_0.

2.1.2. $\mathcal{N}_r(n,k)$, $\rho_{N_1,\ldots,N_r}$

For any $r \geq 1$ and $n \geq 1$, we let

$$\mathcal{N}_r(n,k) := \left\{ (N_1, \ldots, N_r) \in \mathrm{Mat}(n,k)^r \;\middle|\; \begin{array}{l} N_i^p = O_n \text{ for all } 1 \leq i \leq r, \\ N_i\, N_j = N_j\, N_i \text{ for all } 1 \leq i,j \leq r \end{array} \right\}.$$

For any element $(N_1, \ldots, N_r)$ of $\mathcal{N}_r(n,k)$, we can define an element $\rho_{N_1,\ldots,N_r}$ of $\mathbb{E}_r(n,k)$ as

$$\rho_{N_1,\ldots,N_r}(a_1, \ldots, a_r) := \prod_{i=1}^{r} (I_n + N_i)^{a_i}.$$

Lemma 2.1. *Assume $r \geq 1$. The map $\mathcal{N}_r(n,k) \to \mathbb{E}_r(n,k)$ defined by $(N_1, \ldots, N_r) \mapsto \rho_{N_1,\ldots,N_r}$ is a bijection.*

Proof. The proof is straightforward. □

2.1.3. π_r, π

For any $r \geq 0$, we can define a map

$$\pi_r : \mathbb{E}_r(n,k) \to \mathrm{Mat}(n, k[T])^E$$

by separating the cases $r = 0$ and $r \geq 1$. In the case where $r = 0$, we let $\pi_0(\rho_0) := I_n$. In the case where $r \geq 1$, for any element ρ of $\mathbb{E}_r(n,k)$, there exists a unique element $(N_1, \ldots, N_r)$ of $\mathcal{N}_r(n,k)$ such that $\rho = \rho_{N_1,\ldots,N_r}$. Let $A_\rho(T)$ be the matrix of $\mathrm{Mat}(n, k[T])^E$ defined by

$$A_\rho(T) := \prod_{i=1}^{r} \mathrm{Exp}(T^{p^{i-1}} N_i).$$

Clearly, $A_\rho(T)$ is an exponential matrix. Thus we can define $\pi_r(\rho)$ as

$$\pi_r(\rho) := A_\rho(T).$$

Now, let $\pi : \mathbb{E}_{\geq 0}(n,k) \to \mathrm{Mat}(n, k[T])^E$ be the map defined by

$$\pi(\rho) := \pi_r(\rho) \qquad \text{if} \quad \rho \in \mathbb{E}_r(n,k).$$

Lemma 2.2. *π is surjective.*

Proof. Let $A(T)$ be an exponential matrix of $\mathrm{Mat}(n, k[T])^E$. We can express $A(T)$ as

$$A(T) = \prod_{i=1}^{r} \mathrm{Exp}(T^{p^{i-1}} N_i)$$

for some $r \geq 1$ and $(N_1, \ldots, N_r) \in \mathcal{N}_r(n,k)$ (see [5, Lemma 1.5]). So, $\rho_{N_1,\ldots,N_r} \in \mathbb{E}_r(n,k)$ and

$$\pi(\rho_{N_1,\ldots,N_r}) = \pi_r(\rho_{N_1,\ldots,N_r}) = A_{\rho_{N_1,\ldots,N_r}}(T) = A(T).$$

□

2.1.4. $l(\rho)$, $\rho_{\min}$

Let $\rho \in \mathbb{E}_{\geq 0}(n,k)$. If $\rho \in \mathbb{E}_r(n,k)$ for some $r \geq 1$, there exists a unique element $(N_1, \ldots, N_r)$ of $\mathcal{N}_r(n,k)$ such that $\rho = \rho_{N_1,\ldots,N_r}$. Let

$$l(\rho) := \begin{cases} \min\{\, i \in \mathbb{Z}_{\geq 0} \mid N_j = O_n \text{ for all } j \geq i+1 \,\} & \text{if} \quad r \geq 1, \\ 0 & \text{if} \quad r = 0. \end{cases}$$

We can define a homomorphism $\rho_{\min} : (\mathbb{Z}/p\mathbb{Z})^{l(\rho)} \to \mathrm{GL}(n,k)$ as

$$\rho_{\min} := \begin{cases} \rho_{N_1,\ldots,N_{l(\rho)}} & \text{if} \quad l(\rho) \geq 1, \\ \rho_0 & \text{if} \quad l(\rho) = 0. \end{cases}$$

Clearly, the equality $l(\rho) = 0$ holds true if and only if ρ is trivial.

Lemma 2.3. *For any $\rho \in \mathbb{E}_{\geq 0}(n,k)$, we have*

$$\pi(\rho) = \pi(\rho_{\min}).$$

Proof. If $l(\rho) \geq 1$, we can express ρ as $\rho = \rho_{N_1,\ldots,N_r}$ for some $r \geq 1$ and $(N_1, \ldots, N_r) \in \mathcal{N}_r(n,k)$. We have $(N_1, \ldots, N_r) = (N_1, \ldots, N_{l(\rho)}, O_n, \ldots, O_n)$, which implies

$$\pi(\rho) = A_{\rho_{N_1,\ldots,N_r}}(T) = \prod_{i=1}^{r} \mathrm{Exp}(T^{p^{i-1}} N_i) = \prod_{i=1}^{l(\rho)} \mathrm{Exp}(T^{p^{i-1}} N_i) = A_{\rho_{N_1,\ldots,N_{l(\rho)}}}(T) = \pi(\rho_{\min}).$$

If $l(\rho) = 0$, then ρ is trivial. So, $\pi(\rho) = I_n$. Since $\rho_{\min} = \rho_0$, we have $\pi(\rho_{\min}) = I_n$. □

2.2. **Proof of Theorem 0.2**

Let

$$\mathbb{E}_{\min}(n,k) := \{\, \rho \in \mathbb{E}_{\geq 0}(n,k) \mid \rho = \rho_{\min} \,\}.$$

Clearly, $\rho_0 \in \mathbb{E}_{\min}(n,k)$. Let $\pi|_{\mathbb{E}_{\min}(n,k)} : \mathbb{E}_{\min}(n,k) \to \mathrm{Mat}(n,k[T])^E$ be the restriction of $\pi : \mathbb{E}_{\geq 0}(n,k) \to \mathrm{Mat}(n,k[T])^E$ to the subset $\mathbb{E}_{\min}(n,k)$ of $\mathbb{E}_{\geq 0}(n,k)$.

Claim 2.4. *$\pi|_{\mathbb{E}_{\min}(n,k)}$ is a bijection.*

Proof. The map $\pi|_{\mathbb{E}_{\min}(n,k)}$ is surjective (see Lemmas 2.2 and 2.3).

We shall show that $\pi|_{\mathbb{E}_{\min}(n,k)}$ is injective by separating two cases. Let $\rho_1, \rho_2 \in \mathbb{E}_{\min}(n,k)$ and assume $\pi(\rho_1) = \pi(\rho_2)$.

Case $l(\rho_1) = 0$. In this case, ρ_1 is trivial and thereby $\pi(\rho_1) = I_n$. So, $\pi(\rho_2) = I_n$, and thereby ρ_2 is trivial. Thus $\rho_1 = \rho_2$.

Case $l(\rho_1) \geq 1$. From the above case, we know $l(\rho_2) \geq 1$. Let $r := l(\rho_1)$ and $s := l(\rho_2)$. So, we can express ρ_1 and ρ_2 as

$$\begin{cases} \rho_1 = \rho_{N_1,\ldots,N_r} & \text{for some} \quad (N_1,\ldots,N_r) \in \mathcal{N}_r(n,k), \\ \rho_2 = \rho_{M_1,\ldots,M_s} & \text{for some} \quad (M_1,\ldots,M_s) \in \mathcal{N}_s(n,k). \end{cases}$$

We may assume $r \geq s$. Since $A_{\rho_1}(T) = \pi(\rho_1) = \pi(\rho_2) = A_{\rho_2}(T)$, we have

$$(N_1,\ldots,N_r) = (M_1,\ldots,M_s,O_n,\ldots,O_n).$$

Since $N_r \neq O_n$, we must have $r = s$. Thus $\rho_1 = \rho_2$. □

For $i \geq 1$, we denote by $\mathbb{E}^{+i}_{\min}(n,k)$ the set of all homomorphisms

$$\rho_{N_1,\ldots,N_{r+i}} : (\mathbb{Z}/p\mathbb{Z})^{r+i} \to \mathrm{GL}(n,k), \qquad (N_1,\ldots,N_{r+i}) \in \mathcal{N}_{r+i}(n,k)$$

such that the following conditions (1) and (2) hold true:

(1) $\rho_{N_1,\ldots,N_r} \in \mathbb{E}_{\min}(n,k)$.
(2) $N_{r+j} = O_n$ for all $1 \leq j \leq i$.

Claim 2.5. *We have*

$$\mathbb{E}_{\geq 0}(n,k) = \coprod_{i\geq 0} \mathbb{E}^{+i}_{\min}(n,k),$$

where $\mathbb{E}^{+0}_{\min}(n,k) := \mathbb{E}_{\min}(n,k)$.

Proof. Choose any element ρ of $\mathbb{E}_{\geq 0}(n,k)$. If $\rho \in \mathbb{E}_r(n,k)$ for some $r \geq 1$, we can express ρ as $\rho = \rho_{N_1,\ldots,N_r}$ for some $(N_1,\ldots,N_r) \in \mathcal{N}_r(n,k)$. Let $i := r - l(\rho)$. We have $\rho \in \mathbb{E}^{+i}_{\min}(n,k)$. Thus $\mathbb{E}_{\geq 0}(n,k) \subset \bigcup_{i\geq 0} \mathbb{E}^{+i}_{\min}(n,k)$. Clearly, $\mathbb{E}^{+i}_{\min}(n,k) \cap \mathbb{E}^{+j}_{\min}(n,k) = \emptyset$ for all integers $i,j \geq 0$ with $i \neq j$. □

Claim 2.6. *There exists a natural one-to-one correspondence between the sets $\mathbb{E}_{\geq 0}(n,k)$ and $\mathbb{E}_{\min}(n,k) \times \mathbb{Z}_{\geq 0}$, i.e.,*

$$\mathbb{E}_{\geq 0}(n,k) \cong \mathbb{E}_{\min}(n,k) \times \mathbb{Z}_{\geq 0}.$$

Proof. For any $i \geq 1$, there exists a natural one-to-one correspondence between the sets $\mathbb{E}^{+i}_{\min}(n,k)$ and $\mathbb{E}_{\min}(n,k)$. Thus we have the desired bijection (see Claim 2.5).

□

We can obtain from Claims 2.6 and 2.4 the natural bijection

$$\mathbb{E}_{\geq 0}(n, k) \cong \mathrm{Mat}(n, k[T])^E \times \mathbb{Z}_{\geq 0}.$$

Now, we complete the proof of Theorem 0.2.

From the above proof, we have the following commutative diagram, where $p_1 : \mathrm{Mat}(n, k[T])^E \times \mathbb{Z}_{\geq 0} \to \mathrm{Mat}(n, k[T])^E$ is the projection onto the first component:

$$\begin{tikzcd}
\mathbb{E}_{\geq 0}(n, k) \arrow[r, "\cong"] \arrow[d, "\pi"'] & \mathrm{Mat}(n, k[T])^E \times \mathbb{Z}_{\geq 0} \arrow[dl, "p_1"] \\
\mathrm{Mat}(n, k[T])^E &
\end{tikzcd}$$

We can conclude that π is a trivial $\mathbb{Z}_{\geq 0}$-fibration.

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