A mixed fractional CIR model: positivity and an implicit Euler scheme

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Abstract

We consider a Cox–Ingersoll–Ross (CIR) type short rate model driven by a mixed fractional Brownian motion. Let $M = B + B^H$ be a one-dimensional mixed fractional Brownian motion with Hurst index $H > 1/2$, and let $\mathbf{M} = (M, \mathbb{M}^{\text{Itô}})$ denote its canonical Itô rough path lift. We study the rough differential equation

$$\mathrm{d}r_t = k(\theta - r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}\mathbf{M}_t, \qquad r_0 > 0, \tag{1}$$

and prove that, under the Feller condition $2k\theta > \sigma^2$, the unique rough path solution is almost surely strictly positive for all times. The proof relies on an Itô type formula for rough paths, together with refined pathwise estimates for the mixed fractional Brownian motion, including Lévy's modulus of continuity for the Brownian part and a law of the iterated logarithm for the fractional component. As a consequence, the positivity property of the classical CIR model extends to this non-Markovian rough path setting. We also establish the convergence of an implicit Euler scheme for the associated singular equation obtained by a square-root transformation.

1 Introduction

We consider the one-dimensional rough differential equation

$$\mathrm{d}r_t = k(\theta - r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}\mathbf{M}_t, \qquad r_0 > 0, \tag{2}$$

where $k, \theta, \sigma > 0$, and $\mathbf{M} = (M, \mathbb{M}^{\text{Itô}})$ is the Itô rough path lift of the mixed fractional Brownian motion $M = B + B^H$. Here B is a standard Brownian motion, B^H is an independent fractional Brownian motion with Hurst parameter $H > 1/2$, and

$$\mathbb{M}^{\text{Itô}}_{s,t} = \int_s^t M_{s,r} \otimes \mathrm{d}M_r$$

is the Itô iterated integral. The process $\mathbf{M}$ is a canonical example of a Gaussian rough path.

For the classical CIR model driven by a Brownian motion W,

$$\mathrm{d}r_t = k(\theta - r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t, \qquad r_0 > 0,$$

it is well known that the solution remains nonnegative, and even strictly positive, under the Feller condition $2k\theta > \sigma^2$ (see e.g. Feller's classification of boundaries and the monograph of Karatzas

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and Shreve). Our goal is to show that a similar positivity property holds for the mixed fractional model (1), in a rough path framework.

The main steps of the analysis are as follows. First, we recall the notion of bracket for a rough path and prove an Itô type formula for the mixed fractional rough path $\mathbf{M}$. This allows us to apply the square-root transformation $z_t = \frac{2}{\sigma}\sqrt{r_t}$ up to the (hypothetical) hitting time of zero, and we obtain a singular mean-reverting equation for z. Second, we use pathwise estimates for $M = B + B^H$, namely Lévy's modulus for B and a law of the iterated logarithm for B^H, to show that the singular drift in the z-equation prevents z from hitting zero in finite time. This implies that $r_t > 0$ almost surely for all $t \geq 0$. Finally, we study an implicit Euler scheme for the singular equation and prove its convergence in the uniform norm, following ideas of Marie [4].

2 An Itô formula for the mixed fractional rough path

In this section, we consider rough paths $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ with $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ unless otherwise noted. Let V be a Banach space. For $x \in V \otimes V$, we introduce the symmetric part of x:

$$\mathrm{Sym}(x) = \frac{1}{2}\left(x + x^T\right).$$

Definition 2.1 (Bracket of a rough path). *Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ be a rough path. The bracket of $\mathbf{X}$ is the path $[\mathbf{X}] : [0, T] \to \mathbb{R}^{d\times d}$ given by*

$$[\mathbf{X}]_t := (X_{0,t} \otimes X_{0,t}) - 2\,\mathrm{Sym}(\mathbb{X}_{0,t}). \tag{3}$$

Remark 2.2. *By Chen's relation for $\mathbb{X}$, one checks that, for all $(s,t) \in \Delta_{[0,T]}$,*

$$[\mathbf{X}]_{s,t} = (X_{s,t} \otimes X_{s,t}) - 2\,\mathrm{Sym}(\mathbb{X}_{s,t}). \tag{4}$$

We now specialize to the mixed fractional Brownian motion.

Lemma 2.3. *Let $M = B + B^H$ be a one-dimensional mixed fractional Brownian motion, given as the sum of a standard Brownian motion B and an independent fractional Brownian motion B^H with Hurst index $H > \frac{1}{2}$. Consider $\mathbf{M} = (M, \mathbb{M}^{\text{Itô}})$ where*

$$\mathbb{M}^{\text{Itô}}_{s,t} := \int_s^t M_{s,r} \otimes \mathrm{d}M_r. \tag{5}$$

Then for any finite $t > 0$,

$$[\mathbf{M}]_t = t. \tag{6}$$

Proof. In the one-dimensional case we have $\mathrm{Sym}(\mathbb{M}_{s,t}) = \mathbb{M}_{s,t}$.

For brevity, write $\mathbb{M}$ for $\mathbb{M}^{\text{Itô}}$. Fix $t > 0$, and let $\pi : 0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[0,t]$ with mesh size $|\pi| := \max_i |t_{i+1} - t_i|$. By Remark 2.2,

$$[\mathbf{M}]_t = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} [\mathbf{M}]_{t_i,t_{i+1}},$$

where

$$[\mathbf{M}]_{t_i,t_{i+1}} = (M_{t_i,t_{i+1}})^2 - 2\mathbb{M}_{t_i,t_{i+1}}.$$

Since $M = B + B^H$, we obtain

$$\begin{aligned}[\mathbf{M}]_{t_i,t_{i+1}} &= B^2_{t_i,t_{i+1}} + 2B_{t_i,t_{i+1}} B^H_{t_i,t_{i+1}} + (B^H_{t_i,t_{i+1}})^2 \\ &\quad - 2\mathbb{B}_{t_i,t_{i+1}} - 2\mathbb{B}^H_{t_i,t_{i+1}} - 2\int_{t_i}^{t_{i+1}} B^H_{t_i,r}\,\mathrm{d}B_r - 2\int_{t_i}^{t_{i+1}} B_{t_i,r}\,\mathrm{d}B^H_r,\end{aligned} \tag{7}$$

where

$$\mathbb{B}_{s,t} := \int_s^t B_{s,r}\,\mathrm{d}B_r, \qquad \mathbb{B}^H_{s,t} := \int_s^t B^H_{s,r}\,\mathrm{d}B^H_r.$$

We group the non-Brownian terms as

$$\mathrm{I}_i := 2B_{t_i,t_{i+1}} B^H_{t_i,t_{i+1}} + (B^H_{t_i,t_{i+1}})^2,$$

and

$$\mathrm{II}_i := \mathbb{B}^H_{t_i,t_{i+1}} + \int_{t_i}^{t_{i+1}} B^H_{t_i,r}\,\mathrm{d}B_r + \int_{t_i}^{t_{i+1}} B_{t_i,r}\,\mathrm{d}B^H_r.$$

Then (7) can be rewritten as

$$[\mathbf{M}]_{t_i,t_{i+1}} = B^2_{t_i,t_{i+1}} - 2\mathbb{B}_{t_i,t_{i+1}} + \mathrm{I}_i - 2\mathrm{II}_i.$$

Since B and B^H are almost surely Hölder continuous of any order strictly less than $\frac{1}{2}$ and H, respectively, there exists $\varepsilon \in (0, \frac{H}{2} - \frac{1}{4})$ such that B is $(\frac{1}{2} - \varepsilon)$-Hölder and B^H is $(H - \varepsilon)$-Hölder. Hence

$$|B_{t_i,t_{i+1}}| \lesssim |t_{i+1} - t_i|^{\frac{1}{2}-\varepsilon}, \qquad |B^H_{t_i,t_{i+1}}| \lesssim |t_{i+1} - t_i|^{H-\varepsilon},$$

and therefore

$$\mathrm{I}_i = O\big((t_{i+1} - t_i)^{H+\frac{1}{2}-2\varepsilon}\big). \tag{8}$$

For the integrals in II_i we use Young integration: since $(\frac{1}{2}-\varepsilon)+(H-\varepsilon) > 1$, the Young integrals $\int B^H\,\mathrm{d}B$ and $\int B\,\mathrm{d}B^H$ exist and satisfy

$$\left|\int_{t_i}^{t_{i+1}} B^H_{t_i,r}\,\mathrm{d}B_r\right| \lesssim \|B^H\|_{H-\varepsilon,[t_i,t_{i+1}]}\,\|B\|_{\frac{1}{2}-\varepsilon,[t_i,t_{i+1}]}\,|t_{i+1} - t_i|^{H+\frac{1}{2}-2\varepsilon},$$

and similarly for $\int B_{t_i,r}\,\mathrm{d}B^H_r$. Moreover,

$$\mathbb{B}^H_{t_i,t_{i+1}} = O\big((t_{i+1} - t_i)^{2H-2\varepsilon}\big),$$

with $2H - 2\varepsilon > H + \frac{1}{2} - 2\varepsilon$. Altogether we obtain

$$\mathrm{II}_i = O\big((t_{i+1} - t_i)^{H+\frac{1}{2}-2\varepsilon}\big). \tag{9}$$

Combining (8) and (9), we arrive at

$$\sum_{i=0}^{n-1} (\mathrm{I}_i - 2\mathrm{II}_i) = O\Big(\sum_{i=0}^{n-1} (t_{i+1} - t_i)^{H+\frac{1}{2}-2\varepsilon}\Big) \longrightarrow 0 \quad \text{as } |\pi| \to 0, \tag{10}$$

since $H + \frac{1}{2} - 2\varepsilon > 1$.

On the other hand, by the usual Itô calculus for B we know that

$$\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} \Big(B^2_{t_i,t_{i+1}} - 2\mathbb{B}_{t_i,t_{i+1}}\Big) = t$$

almost surely. Together with (10) this yields

$$[\mathbf{M}]_t = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} [\mathbf{M}]_{t_i,t_{i+1}} = t,$$

which completes the proof. □

Proposition 2.4 (Itô formula for rough paths)**.** *Let $\mathbf{M} = (M, \mathbb{M}) \in \mathscr{C}^\alpha$ be a rough path, and let $f \in C^3(\mathbb{R})$. Then*

$$f(M_T) = f(M_0) + \int_0^T Df(M_u)\,\mathrm{d}\mathbf{M}_u + \frac{1}{2}\int_0^T D^2 f(M_u)\,\mathrm{d}[\mathbf{M}]_u. \tag{11}$$

Proof. Assume without loss of generality that $f \in C_b^3$. Since $Df \in C_b^2$, the pair $(Df(M), D^2 f(M))$ is a controlled path with respect to M. For $0 \le s < t \le T$ we have the Taylor expansion

$$f(M_t) - f(M_s) = Df(M_s)M_{s,t} + D^2 f(M_s)\mathbb{M}_{s,t} + R_{s,t}, \tag{12}$$

where

$$R_{s,t} := \int_0^1 \int_0^1 D^3 f\big(M_s + r_1 r_2 M_{s,t}\big)(M_{s,t} \otimes M_{s,t})\, r_1\,\mathrm{d}r_2\,\mathrm{d}r_1. \tag{13}$$

Thus

$$|R_{s,t}| \le \|f\|_{C_b^3} |M_{s,t}|^3 \le \|f\|_{C_b^3} \|M\|_\alpha^3 |t-s|^{3\alpha},$$

so that, for any sequence of partitions π with mesh $|\pi| \to 0$,

$$\sum_{[s,t]\in\pi} |R_{s,t}| \longrightarrow 0.$$

Since the Hessian $D^2 f(M_s)$ is symmetric, it kills the antisymmetric part of $\mathbb{M}_{s,t}$, that is,

$$D^2 f(M_s)\mathbb{M}_{s,t} = D^2 f(M_s)\,\mathrm{Sym}(\mathbb{M}_{s,t}).$$

Using Definition 2.2 we have

$$\mathrm{Sym}(\mathbb{M}_{s,t}) = \tfrac{1}{2}\big(M_{s,t} \otimes M_{s,t} - [\mathbf{M}]_{s,t}\big),$$

and therefore

$$D^2 f(M_s)\mathbb{M}_{s,t} = \frac{1}{2} D^2 f(M_s)(M_{s,t} \otimes M_{s,t}) - \frac{1}{2} D^2 f(M_s)[\mathbf{M}]_{s,t}.$$

Summing over a partition and passing to the limit $|\pi| \to 0$ gives (11). □

Now return to the mixed fractional CIR equation (1). Fix $\varepsilon_0 > 0$ and define the stopping time

$$\tau^r_{\varepsilon_0} := \inf\{s > 0 : r_s = \varepsilon_0\}.$$

For any $0 \le s \le t \le \tau^r_{\varepsilon_0}$, define

$$z_t := \frac{2}{\sigma}\sqrt{r_t}.$$

On $[\varepsilon_0, \infty)$ the function $f(x) = \frac{2}{\sigma}\sqrt{x}$ is C_b^3, so by Proposition 2.4 and Lemma 2.3 we obtain

$$\begin{aligned} z_t =& z_s + \int_s^t \frac{1}{\sigma\sqrt{r_u}} k(\theta - r_u)\,\mathrm{d}u + \int_s^t \frac{1}{\sigma\sqrt{r_u}} \sigma\sqrt{r_u}\,\mathrm{d}\mathbf{M}_u \\ &+ \frac{1}{2}\int_s^t -\frac{1}{2}\frac{1}{\sigma} r_u^{-\frac{3}{2}} \sigma^2 r_u\,\mathrm{d}[\mathbf{M}]_u \\ =& z_s + \int_s^t \left(\frac{2k\theta}{\sigma^2 z_u} - \frac{k z_u}{2}\right)\mathrm{d}u + M_{s,t} - \int_s^t \frac{1}{2z_u}\,\mathrm{d}u \\ =& z_s + \int_s^t \left(\frac{4k\theta - \sigma^2}{2\sigma^2 z_u} - \frac{k z_u}{2}\right)\mathrm{d}u + M_{s,t}. \end{aligned} \tag{14}$$

Define

$$m := \frac{2k\theta - \sigma^2}{\sigma^2}, \tag{15}$$

we obtain the singular equation

$$\mathrm{d}z_t = \left[(m + \frac{1}{2}) z_t^{-1} - \frac{k}{2} z_t\right]\mathrm{d}t + \mathrm{d}M_t. \tag{16}$$

A standard localization argument (using the stopping times $\tau_{\varepsilon_0}^r$ or a smooth approximation of $\sqrt{\cdot}$ near zero) shows that (16) actually holds for all t strictly smaller than the first hitting time of 0 by z.

3 Positivity of the mixed fractional CIR process

Before proving the positivity of the solution $(r_t)_{t\geq 0}$ to (1), we recall two classical pathwise results for Brownian motion and fractional Brownian motion.

Lemma 3.1 (Lévy's modulus of continuity, see e.g. [1])**.** *Let B be a one-dimensional standard Brownian motion. Put $g(t) := \sqrt{2t\log(1/t)}$. Then*

$$\mathbb{P}\left[\limsup_{\delta\to 0^+} \frac{1}{g(\delta)} \sup_{\substack{0\ \leq s\ <\ t\ <\ 1 \\ t\ -\ s\ \leq\ \delta}} |B_t - B_s| = 1\right] = 1. \tag{17}$$

Lemma 3.2 (Law of the iterated logarithm for fBm, see [2, 3])**.** *Let B^H be a one-dimensional fractional Brownian motion with Hurst index $H \in (0, 1)$. Then*

$$\limsup_{t\to 0^+} \frac{|B_t^H|}{t^H\sqrt{\log\log t^{-1}}} = c_H \tag{18}$$

with probability one, where $c_H \in (0, \infty)$ is a suitable constant depending only on H.

We can now state the main positivity result.

Theorem 3.3. *Consider* (1) *with $r_0 > 0$. Let $k, \theta > 0$, $H > \frac{1}{2}$, and assume that*

$$2k\theta > \sigma^2, \tag{19}$$

which is the classical Feller condition. Then the process $(r_t)_{t\geq 0}$ defined by (1) *is strictly positive almost surely.*

Proof. We work with the transformed process $z_t = \frac{2}{\sigma}\sqrt{r_t}$, which satisfies (16) up to its first hitting time of 0. Note that $m > 0$ by (19). Define

$$\tau := \inf\{s > 0 : z_s = 0\}.$$

We will show that $\mathbb{P}(\tau < \infty) = 0$. Suppose τ is finite, by Lemma 3.1,for any $\varepsilon > 0$ such that $2\alpha_0 m + 1 > (1+\varepsilon)^2$ for some $\alpha_0 \in (0,1)$, there always exists $\delta(\omega, \varepsilon)$ such that

$$|B_\tau - B_t| \leq (1+\varepsilon)\sqrt{2\delta_1(t)\log\frac{1}{\delta_1(t)}} \quad a.s. \tag{20}$$

for all $\delta_1(t) := \tau - t \in (0, \delta)$.

Assume by contradiction that $\tau < \infty$ with positive probability. Fix ω in a full-measure event on which all the pathwise estimates below hold, and work pathwise. For $t < \tau$, integrating (16) from t to τ yields

$$z_t + \int_t^\tau \left((m + \frac{1}{2})z_s^{-1} - \frac{k}{2}z_s\right) \mathrm{d}s = M_t - M_\tau, \tag{21}$$

where $M = B + B^H$.

Fix $\gamma^* \in (0, H - \frac{1}{2})$. Consider $\tau - \delta_2 < t < \tau$ where τ_2 is small enough so that we can always ensure $(m+\frac{1}{2})z_s^{-1} - \frac{k}{2}z_s > 0$. For

$$z_t + \int_t^\tau \left((m + \frac{1}{2})z_s^{-1} - \frac{k}{2}z_s\right) ds = M_t - M_\tau, \tag{22}$$

since M has Hölder continuous paths of any order $\alpha < 1/2$, there exist random constants $C(\omega, \gamma^*) > 0$ and $\delta_2(\omega) > 0$ such that

$$z_t \leq C(\omega, \gamma^*)|\tau - t|^{\frac{1}{2} - \gamma^*}, \qquad \text{for all } t \in (\tau - \delta_2, \tau). \tag{23}$$

In particular, $z_t \to 0$ as $t \uparrow \tau$.

Fix $\xi > 0$ small and define $\tau_\xi := \sup\{s \in (\tau - \min\{\delta, \delta_2\}, \tau) : z_s = \xi\}$, so that we have

$$-\xi = \int_{\tau_\xi}^\tau \left((m + \frac{1}{2})z_s^{-1} - \frac{k}{2}z_s\right) ds + M_\tau - M_{\tau_\xi}. \tag{24}$$

Since we Suppose

$$M_t = B_t + B_t^H, \tag{25}$$

we have

$$-\xi = \int_{\tau_\xi}^\tau \left((m + \frac{1}{2})z_s^{-1} - \frac{k}{2}z_s\right) ds + B_\tau - B_{\tau_\xi} + B_\tau^H - B_{\tau_\xi}^H. \tag{26}$$

Then there $\exists \alpha \in (\alpha_0, 1)$ that

$$\begin{aligned}\left|B_\tau - B_{\tau_\xi}\right| + \left|B_\tau^H - B_{\tau_\xi}^H\right| \geq &\int_{\tau_\xi}^\tau (1-\alpha)m z_s^{-1}\, ds \\ &+ \left((\alpha m + \frac{1}{2})\xi^{-1} - \frac{k}{2}\xi\right)(\tau - \tau_\xi) + \xi,\end{aligned} \tag{27}$$

notice $\tau - \tau_\xi < \delta$, almost surely, we have the following

$$(1+\varepsilon)\sqrt{2\delta_1(\tau_\xi)\log\frac{1}{\delta_1(\tau_\xi)}} + \left|B^H_\tau - B^H_{\tau_\xi}\right| \ge \int_{\tau_\xi}^{\tau}(1-\alpha)mz_s^{-1}\,ds + \left((\alpha m + \frac{1}{2})\xi^{-1} - \frac{k}{2}\xi\right)(\tau-\tau_\xi) + \xi,$$

$$\left|B^H_\tau - B^H_{\tau_\xi}\right| \ge \int_{\tau_\xi}^{\tau}(1-\alpha)mz_s^{-1}\,ds + \left((\alpha m + \frac{1}{2})\xi^{-1} - \frac{k}{2}\xi\right)(\tau-\tau_\xi) - (1+\varepsilon)\sqrt{2\delta_1(\tau_\xi)\log\frac{1}{\delta_1(\tau_\xi)}} + \xi. \tag{28}$$

From now to discuss $\left((\alpha m + \frac{1}{2})\xi^{-1} - \frac{k}{2}\xi\right)(\tau-\tau_\xi) - (1+\varepsilon)\sqrt{2\delta_1(\tau_\xi)\log\frac{1}{\delta_1(\tau_\xi)}} + \xi$, notice $\exists\tilde{\varepsilon} > 0$ such that $\forall\varepsilon < \tilde{\varepsilon}$, $\frac{(\alpha m+\frac{1}{2})}{\varepsilon} - \frac{k}{2}\varepsilon > \frac{\alpha_0 m+\frac{1}{2}}{\varepsilon}$ (Note that α_0 has been fixed before (20) and $\alpha \in (\alpha_0, 1)$). Consider

$$\begin{aligned} f(x) &= \frac{2\alpha_0 m+1}{2\xi}x - (1+\varepsilon)\sqrt{2x\log\frac{1}{x}} + \xi, \\ f_\gamma(x) &= \frac{2\alpha_0 m+1}{2\xi}x - (1+\varepsilon)\sqrt{2}x^\gamma + \xi, \end{aligned} \tag{29}$$

where $\gamma \in (0, \frac{1}{2})$. We know

$$f'_\gamma(x) = \frac{2\alpha_0 m+1}{2\xi} - (1+\varepsilon)\sqrt{2}\gamma x^{\gamma-1},$$

let $f'_\gamma(x_0) = 0$, then $x_0 = \left(\frac{2\alpha_0 m+1}{2\sqrt{2}(1+\varepsilon)\xi\gamma}\right)^{\frac{1}{\gamma-1}}$,

$$\begin{aligned} f_\gamma(x_0) =& (1+\varepsilon)\sqrt{2}(\gamma-1)x_0^\gamma + \xi \\ =& \left(\sqrt{2}(1+\varepsilon)\right)^{-\frac{1}{\gamma-1}}\left(\frac{2\gamma}{2\alpha_0 m+1}\right)^{-\frac{\gamma}{\gamma-1}}(\gamma-1)\xi^{-1-\frac{1}{\gamma-1}} + \xi \\ =& g(\gamma). \end{aligned}$$

Note when $\gamma \to \frac{1}{2}^-$,

$$g(\gamma) \to \left(1 - \frac{(1+\varepsilon)^2}{2\alpha_0 m+1}\right)\xi > 0,$$

since $g(\gamma)$ is continuous in $(0, \frac{1}{2})$, so $\exists\gamma_0 > 0$, $\forall\gamma \in (\gamma_0, \frac{1}{2})$, $g(\gamma) > 0$. Fix any $\gamma_1 \in (\gamma_0, \frac{1}{2})$. Then we have

$$\begin{aligned} \left|B^H_\tau - B^H_{\tau_\xi}\right| &\ge \int_{\tau_\xi}^{\tau}(1-\alpha)mz_s^{-1}\,ds + f(\tau-\tau_\xi) - f_{\gamma_1}(\tau-\tau_\xi) + f_{\gamma_1}(\tau-\tau_\xi) \\ &\ge \int_{\tau_\xi}^{\tau}(1-\alpha)mz_s^{-1}\,ds + f(\tau-\tau_\xi) - f_{\gamma_1}(\tau-\tau_\xi), \quad a.s. \end{aligned} \tag{30}$$

by Lemma 3.2, for ξ small enough, there exists a constant $C_H^* > 0$ such that we have

$$|B^H_\tau - B^H_{\tau_\xi}| \le C_H^*(\tau-\tau_\xi)^H\sqrt{\log\log\frac{1}{\tau-\tau_\xi}}. \tag{31}$$

It follows that

$$
\begin{aligned}
C_H^* (\tau-\tau_\xi)^H \sqrt{\log\log(\tau-\tau_\xi)^{-1}} &\geq C(1-\alpha)m \int_{\tau_\xi}^{\tau} (\tau-s)^{-\left(\frac{1}{2}-\gamma^*\right)} ds + f(\tau-\tau_\xi) - f_{\gamma_1}(\tau-\tau_\xi) \\
&\geq C(1-\alpha)m\left(\frac{1}{2}+\gamma^*\right)^{-1} (\tau-\tau_\xi)^{\frac{1}{2}+\gamma^*} + f(\tau-\tau_\xi) - f_{\gamma_1}(\tau-\tau_\xi), \quad a.s.
\end{aligned} \tag{32}
$$

and notice for any $\gamma > 0$, $f(x) - f_\gamma(x) = (1+\varepsilon)\sqrt{2}\left(x^\gamma - \sqrt{x\log\frac{1}{x}}\right) \geq 0$ for any x sufficiently small, so let $\xi \to 0$, then $\tau_\xi \to \tau$, we only need to consider

$$
(\tau-\tau_\xi)^{H-\frac{1}{2}-\gamma^*} \sqrt{\log\log(\tau-\tau_\xi)^{-1}} \geq C_f(1-\alpha)m\left(\frac{1}{2}+\gamma^*\right)^{-1}, \quad a.s. \tag{33}
$$

as $\xi \to 0$, then it's clear that as $\tau_\xi \to \tau$,

$$
(\tau-\tau_\xi)^{H-\frac{1}{2}-\gamma^*} \sqrt{\log\log(\tau-\tau_\xi)^{-1}} \to 0, \quad \text{a.s.} \tag{34}
$$

since $H - \frac{1}{2} > \gamma^*$, then we have

$$
0 \geq C_f(1-\alpha)m\left(\frac{1}{2}+\gamma^*\right)^{-1} > 0. \tag{35}
$$

It follows that there is a contradiction from the limits of eq.(33). □

4 Convergence of the implicit Euler scheme for the singular equation

Recall the singular equation

$$
\mathrm{d}z_t = \left[(m+\tfrac{1}{2})z_t^{-1} - \tfrac{k}{2}z_t\right]\mathrm{d}t + \mathrm{d}M_t, \tag{36}
$$

where $M = B + B^H$ is the mixed fractional Brownian motion and

$$
m = \frac{2k\theta - \sigma^2}{\sigma^2} > 0.
$$

Fix a time horizon $T > 0$, and let $(t_k^n)_{k=0,\dots,n}$ be the uniform partition of $[0, T]$ given by

$$
t_k^n := \frac{kT}{n}, \qquad \Delta t := t_{k+1}^n - t_k^n = \frac{T}{n}.
$$

The implicit Euler scheme associated to (36) and to $(t_0^n, t_1^n, \dots, t_n^n)$ is

$$
z_{k+1}^n = z_k^n + b(z_{k+1}^n)\,\Delta t + M_{t_{k+1}^n} - M_{t_k^n}, \qquad k = 0, \dots, n-1, \tag{37}
$$

with

$$
b(x) = \left(m + \frac{1}{2}\right)x^{-1} - \frac{k}{2}x, \qquad z_0^n := z_0 > 0.
$$

The proof of convergence is inspired by Marie [4].

Remark 4.1. *Although* (36) *does not satisfy Assumption 1.1.(4) of [4], the fact that it satisfies Assumptions 1.1.(1)–1.1.(3) makes the convergence argument in Marie [4] applicable in our setting.*

We define the continuous-time interpolation $(z_t^n)_{t\in[0,T]}$ by

$$z_t^n = \sum_{k=0}^{n-1} \left[z_k^n + \frac{z_{k+1}^n - z_k^n}{t_{k+1}^n - t_k^n}(t - t_k^n) \right] \mathbb{1}_{(t_k^n, t_{k+1}^n]}(t), \qquad t \in [0,T]. \tag{38}$$

We also use the notation

$$\|f\|_{\infty,T} := \sup_{t\in[0,T]} |f_t|, \qquad \|f\|_{\alpha,T} := \sup_{0\le s<t\le T} \frac{|f_t - f_s|}{|t-s|^\alpha}$$

for the uniform and Hölder norms on $[0,T]$.

Theorem 4.2. *Assume that $2k\theta > \sigma^2$ and $k > 0$. Then the implicit Euler scheme* (37) *is well-defined and has a unique positive solution. Moreover, let $\alpha := \frac{1}{2} - \varepsilon$ with any sufficiently small $\varepsilon > 0$. There exists a constant $C > 0$ depending on T, m, k, and the sample path of M such that*

$$\|z^n - z\|_{\infty,T} \le C\, n^{-\alpha} = C\, n^{-\left(\frac{1}{2}-\varepsilon\right)}. \tag{39}$$

Proof. The proof follows the approach of Marie [4] and consists of three main steps.

Step 1: Well-posedness of the scheme.

For each k, equation (37) is an implicit equation for z_{k+1}^n. Define the function

$$\varphi(x) = x - b(x)\,\Delta t - z_k^n - \Delta M_k, \tag{40}$$

where $\Delta t = T/n$ and $\Delta M_k = M_{t_{k+1}^n} - M_{t_k^n}$.

We compute the derivative:

$$\varphi'(x) = 1 - b'(x)\,\Delta t = 1 + \left[\left(m + \frac{1}{2}\right) x^{-2} + \frac{k}{2} \right] \Delta t > 0 \quad \text{for all } x > 0, \tag{41}$$

so φ is strictly increasing on $(0,\infty)$. Moreover, $\varphi(x) \to -\infty$ as $x \to 0^+$ and $\varphi(x) \to +\infty$ as $x \to +\infty$. Therefore, by the intermediate value theorem, there exists a unique positive solution $z_{k+1}^n > 0$. Inductively, the scheme is well-defined and remains strictly positive.

Step 2: Uniform boundedness.

We first show that the scheme is uniformly bounded on $[0,T]$.

Fix $k \in \{1,\dots,n\}$ and define

$$n(z_0,k) := \max\{i \in \{0,\dots,k\} : z_i^n \le z_0\}. \tag{42}$$

If $n(z_0,k) = k$, then $0 < z_k^n \le z_0$. If $n(z_0,k) < k$, then

$$\begin{aligned} z_k^n - z_{n(z_0,k)}^n &= \sum_{i=n(z_0,k)}^{k-1} (z_{i+1}^n - z_i^n) \\ &= \sum_{i=n(z_0,k)}^{k-1} \Big(b(z_{i+1}^n)\Delta t + \Delta M_i \Big) \\ &= \Delta M_{n(z_0,k),k} + \sum_{i=n(z_0,k)}^{k-1} b(z_{i+1}^n)\Delta t, \end{aligned} \tag{43}$$

where $\Delta M_{n(z_0,k),k} = M_{t_k^n} - M_{t^n_{n(z_0,k)}}$.

Since b is strictly decreasing (because $b'(x) < 0$ for all $x > 0$) and $z_{i+1}^n > z_0$ for $i \geq n(z_0,k)$, we have

$$b(z_{i+1}^n) \leq b(z_0) \quad \text{for all } i \geq n(z_0,k). \tag{44}$$

Therefore,

$$\sum_{i=n(z_0,k)}^{k-1} b(z_{i+1}^n)\Delta t \leq b(z_0)(t_k^n - t^n_{n(z_0,k)}) \leq |b(z_0)|T. \tag{45}$$

Moreover,

$$|\Delta M_{n(z_0,k),k}| \leq 2\|M\|_{\infty,T}. \tag{46}$$

Hence,

$$0 < z_k^n \leq z_0 + |b(z_0)|T + 2\|M\|_{\infty,T}, \tag{47}$$

which shows that $(z_k^n)_{k=0,\dots,n}$ is uniformly bounded in k (for fixed n and T). In particular, there exist random constants $0 < z_* \leq z^* < \infty$ such that

$$z_* \leq z_k^n \leq z^* \qquad \text{for all } k = 0,\dots,n. \tag{48}$$

Step 3: Convergence estimate.

Let $\xi_k^n := z_{t_k^n}$ be the exact solution evaluated at the grid points. The exact solution satisfies

$$\xi_{k+1}^n = \xi_k^n + \int_{t_k^n}^{t_{k+1}^n} b(z_s)\,\mathrm{d}s + \big(M_{t_{k+1}^n} - M_{t_k^n}\big). \tag{49}$$

Subtracting this from the scheme (37), we obtain

$$z_{k+1}^n - \xi_{k+1}^n = z_k^n - \xi_k^n + \big[b(z_{k+1}^n) - b(\xi_{k+1}^n)\big]\Delta t + \varepsilon_k^n, \tag{50}$$

where

$$\varepsilon_k^n := b(\xi_{k+1}^n)\Delta t - \int_{t_k^n}^{t_{k+1}^n} b(z_s)\,\mathrm{d}s. \tag{51}$$

Since b is strictly decreasing, we distinguish two cases.

Case 1: If $z_{k+1}^n > \xi_{k+1}^n$, then $b(z_{k+1}^n) - b(\xi_{k+1}^n) \leq 0$, and thus

$$|z_{k+1}^n - \xi_{k+1}^n| = z_{k+1}^n - \xi_{k+1}^n \leq |z_k^n - \xi_k^n| + |\varepsilon_k^n|. \tag{52}$$

Case 2: If $z_{k+1}^n \leq \xi_{k+1}^n$, then $b(\xi_{k+1}^n) - b(z_{k+1}^n) \leq 0$, and

$$|z_{k+1}^n - \xi_{k+1}^n| = \xi_{k+1}^n - z_{k+1}^n \leq |z_k^n - \xi_k^n| + |\varepsilon_k^n|. \tag{53}$$

In both cases we have

$$|z_{k+1}^n - \xi_{k+1}^n| \leq |z_k^n - \xi_k^n| + |\varepsilon_k^n|. \tag{54}$$

By induction it follows that

$$|z_k^n - \xi_k^n| \leq \sum_{i=0}^{k-1} |\varepsilon_i^n|, \qquad k = 1,\dots,n. \tag{55}$$

We now estimate the error term ε_i^n:

$$|\varepsilon_i^n| = \left| \int_{t_i^n}^{t_{i+1}^n} \left[b(\xi_{i+1}^n) - b(z_s) \right] \mathrm{d}s \right|. \tag{56}$$

Let $z_* := \inf_{t\in[0,T]} z_t$ and $z^* := \sup_{t\in[0,T]} z_t$. Since b is continuously differentiable on $[z_*, z^*]$, it is Lipschitz continuous on this interval with constant $L_b > 0$. Therefore,

$$|\varepsilon_i^n| \le L_b \int_{t_i^n}^{t_{i+1}^n} |\xi_{i+1}^n - z_s| \,\mathrm{d}s. \tag{57}$$

Since z is α-Hölder continuous on $[0,T]$ with $\alpha = \frac{1}{2} - \varepsilon$ (this follows from the fact that M has Hölder regularity α and the rough differential equation theory), we have

$$|\xi_{i+1}^n - z_s| = |z_{t_{i+1}^n} - z_s| \le \|z\|_{\alpha,T} (t_{i+1}^n - s)^\alpha.$$

Hence,

$$|\varepsilon_i^n| \le L_b \|z\|_{\alpha,T} \int_{t_i^n}^{t_{i+1}^n} (t_{i+1}^n - s)^\alpha \mathrm{d}s = L_b \|z\|_{\alpha,T} \frac{(\Delta t)^{\alpha+1}}{\alpha+1}. \tag{58}$$

Thus,

$$\sum_{i=0}^{k-1} |\varepsilon_i^n| \le k \cdot L_b \|z\|_{\alpha,T} \frac{(\Delta t)^{\alpha+1}}{\alpha+1} \le L_b \|z\|_{\alpha,T} \frac{T^{\alpha+1}}{(\alpha+1)n^\alpha}, \tag{59}$$

and therefore, at grid points,

$$\max_{0\le k\le n} |z_k^n - \xi_k^n| \le L_b \|z\|_{\alpha,T} \frac{T^{\alpha+1}}{(\alpha+1)n^\alpha}. \tag{60}$$

Next, consider the continuous-time interpolation. For $t \in (t_k^n, t_{k+1}^n]$ we have

$$|z_t^n - z_t| \le |z_t^n - z_k^n| + |z_k^n - \xi_k^n| + |\xi_k^n - z_t|. \tag{61}$$

For the first term, using (37),

$$\begin{aligned} |z_t^n - z_k^n| &\le |z_{k+1}^n - z_k^n| \\ &= |b(z_{k+1}^n)|\Delta t + |\Delta M_k| \\ &\le \left(\|b\|_{\infty,[z_*,z^*]} + \|M\|_{\alpha,T} \Delta t^\alpha \right) \Delta t. \end{aligned} \tag{62}$$

For the third term, using the Hölder regularity of z,

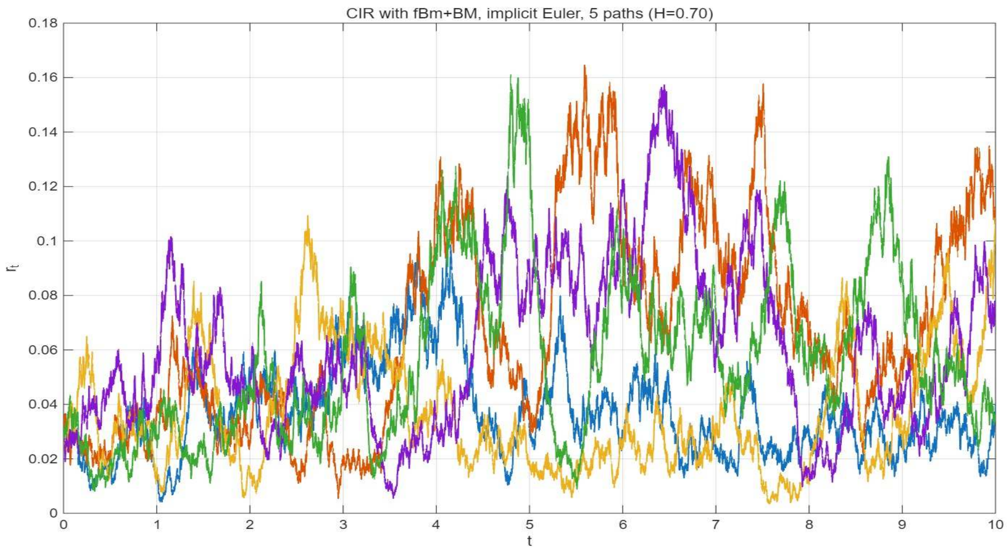
$$|\xi_k^n - z_t| = |z_{t_k^n} - z_t| \le \|z\|_{\alpha,T} \Delta t^\alpha. \tag{63}$$

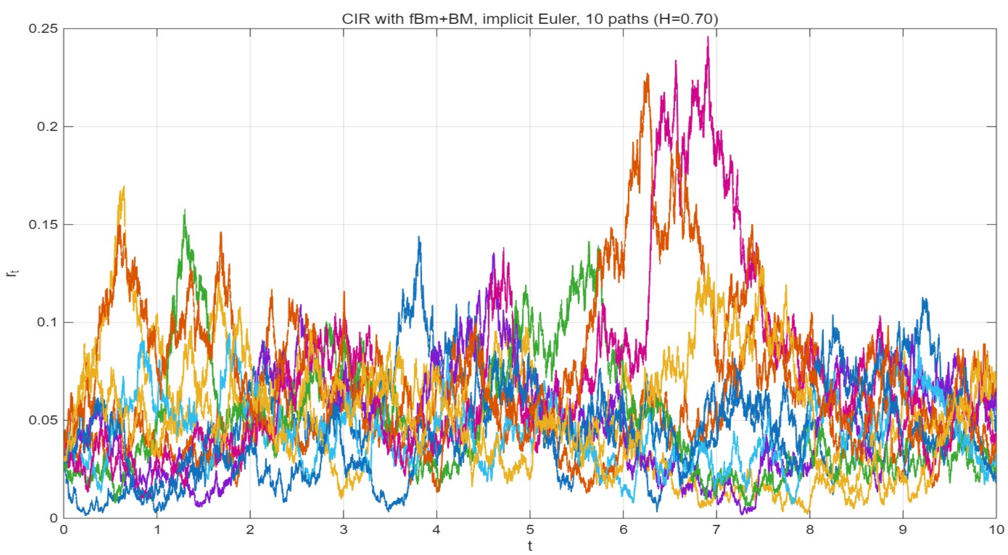
Combining all estimates and using the bound for $|z_k^n - \xi_k^n|$, we obtain

$$\|z^n - z\|_{\infty,T} \le C\, n^{-\alpha}, \tag{64}$$

for some (random) constant $C > 0$ depending on T, b, $\|z\|_{\alpha,T}$, and $\|M\|_{\alpha,T}$. This completes the proof. □

In the following, we illustrate the behavior of the mixed fractional CIR model and the implicit Euler scheme. We simulate the short rate process r_t on the interval $[0, 10]$ with different numbers of sample paths.


CIR with fBm+BM, implicit Euler, 5 paths (H=0.70)
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
r_t
0
1
2
3
4
5
6
7
8
9
10
t


CIR with fBm+BM, implicit Euler, 10 paths (H=0.70)
0.25
0.2
0.15
0.1
0.05
0
r_t
0
1
2
3
4
5
6
7
8
9
10
t

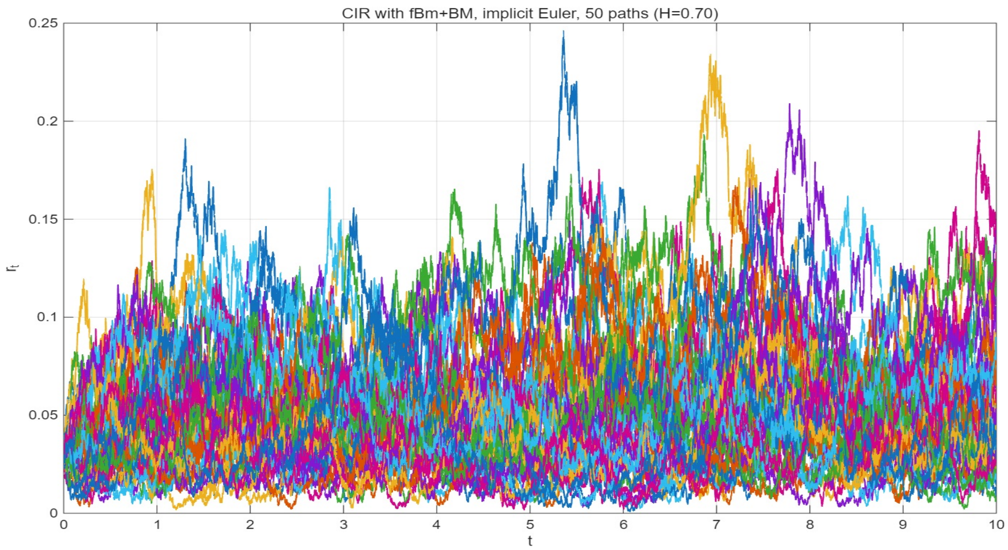


Fig.1.Implicit Euler simulations of the mixed fractional CIR short rate on the interval $[0, 10]$. The figure shows trajectories of r_t obtained from the implicit Euler scheme for 5, 10 and 50 independent sample paths.

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