All steerable quantum correlations can provide thermodynamic advantages in cooling

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The removal of heat generated during computation poses a major challenge for both classical and quantum computation and information processing. In particular, removal of this heat is intrinsically tied to one of the fundamental requirements of quantum computation—the need to reset the system to a pure state before computation. Therefore, efficient cooling is of paramount importance, not only for deepening our understanding of thermodynamics in the quantum regime but also for advancing the development of modern quantum technologies. In this article, we devise a cooling task that exploits steerability—a fundamental feature of quantum correlations—to demonstrate a provable quantum advantage over classically correlated scenarios where steerability is absent. We quantify the advantage by the ratio of the amount of heat removed using steerable (quantum) correlations to that obtained with unsteerable (classical) correlations. Specifically, we show that steerable quantum correlations always yield a thermodynamic advantage in a cooling task over their classical counterparts. Remarkably, we further establish that the maximum achievable advantage is directly related to a geometric measure of steerability called *steerability robustness*. Our results suggest that this thermodynamic advantage can be interpreted as a witness of steerability. Finally, we present examples in which the advantage grows with the dimension of the underlying system.

I. INTRODUCTION

Cooling quantum systems is a fundamental thermodynamic task, essential both for practical applications in modern quantum technologies and for foundational investigations [1–7]. In particular, quantum computers inevitably generate heat during computation. Since qubits operate at the quantum scale, even small amounts of thermal noise can disrupt superposition and entanglement—the essential resources for quantum computation. Beyond this fundamental limitation, additional heat is generated due to experimental constraints and the characteristics of the specific physical platform used to implement the computation. Therefore, the quantum processor executing the computation must be maintained at extremely low temperatures by removing the generated heat. Efficiently removing this heat remains a key challenge for the realization of scalable quantum computers.

Inspired by the operation of a microscopic Otto engine [8–14] and refrigerator [15–18], we propose a cooling task that utilizes quantum correlations shared between two spatially separated parties as a fundamental resource, enabling the withdrawal of a substantially larger amount of heat from a thermal bath compared to what is achievable using only classical correlations. We show that the steerability of quantum correlations is the key feature underlying these advantages, where the advantage is quantified by the ratio of the heat removed in the presence and absence of steerability.

The concept of steerability was first introduced by Schrodinger in response to the Einstein-Podolsky-Rosen (EPR) paradox [19–21]. Steerability of quantum correlations, also known as *quantum steering*, is a feature of quantum correlations whereby a spatially separated observer can influence the set of conditional quantum states, referred to as *assemblages*, accessible to a distant party as a result of performing uncharacterized measurements, in a manner that cannot be explained by local causal models. [22–24]. Here, *uncharacterized measurements* refer to measurements whose specific measurement operators are unknown to the party, so that the corresponding conditional states are not accessible. When both parties perform uncharacterized measurements, the scenario corresponds to the device-independent framework of Bell nonlocality [25–29]. Quantum correlations in entangled states that violate Bell inequalities are always steerable, but the converse does not necessarily hold.

Quantum steering encapsulates intrinsically quantum features that have no classical counterpart. Previous studies have demonstrated that steerability of quantum correlations offers significant advantages across a wide range of information-theoretic tasks, including quantum key distribution [30, 31], quantum key authentication [32], randomness certification [33, 34], sub-channel discrimination [35, 36], distinguishing quantum measurements [37], characterization of measurement incompatibility [38, 39], secret sharing [40, 41], quantum metrology [42], quantum teleportation [43–46] and work extraction [47–53].

In this article, we demonstrate that there always exists a cooling task wherein the steerability of quantum correlations serves as a key resource, enabling enhanced cooling characterized by the removal of a greater amount of heat compared to scenarios where steerability is absent. We demonstrate that for every steerable assemblage, one can design a cooling protocol in which steerability acts as a genuine resource, enabling the extraction of more heat than is achievable in any unsteerable scenario. Remarkably, the maximum advantage achievable using steerable quantum correlations is lower bounded by the steerability robustness—a geometric quantifier of steerability. We further show that this advantage can itself serve as a witness of steerability.

In previous studies, robustness has been extensively investigated in several resource theories—such as entanglement [54–

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56], coherence [57, 58], asymmetry [59], non-stabilizerness [60, 61] and measurement informativeness [62] —and relate it with state and subchannel discrimination tasks [63–69]. To the best of our knowledge, our work provides the first operational interpretation of robustness in a thermodynamic setting. As a concrete example, motivated by the connection between steerability and measurement incompatibility as described in Ref. [70], we consider assemblages generated by measurements in mutually unbiased bases (MUBs) on one subsystem of a maximally entangled state [71, 72]. These MUBs exhibit maximal measurement incompatibility, which leads to maximal steerability robustness. The robustness of these assemblages scales as $O(\sqrt{d})$ with the dimension d of the underlying Hilbert space. Consequently, when such quantum assemblages are used as a resource for cooling, the resulting quantum advantage increases with the underlying system's dimension, sometimes referred to as an unbounded quantum advantage in the quantum information literature.

II. MAIN RESULTS

A. Cooling task using classical and quantum resources

We now describe the cooling task, where the objective is to withdraw heat from a thermal reservoir. This cooling task resembles the standard operation of an Otto refrigerator, which withdraws heat from a cold reservoir and transfers it to a hotter one at the expense of external work. Since our goal is to explore potential advantages in cooling arising from quantum resources over classical ones, we will focus on a single heat reservoir and aim to withdraw heat from it using such non-thermal quantum and classical resources.

Assume that Bob has access to a heat reservoir at inverse temperature β, from which he seeks to withdraw heat with the assistance of Alice, who is spatially separated. Alice and Bob share a joint quantum state $\hat{\rho}_{AB}$. We consider Bob's system to have finite dimension $d < \infty$, with its initial Hamiltonian specified as $H = 0$.

To prepare resources that Bob can exploit for cooling, Alice samples an input x from the set $\mathcal{X} = \{0, \ldots, n-1\}$ and performs the measurement M_x on the bipartite state $\hat{\rho}_{AB}$. Each measurement M_x is described by a positive operator-valued measure (POVM) $M_x = \{M_{a|x}\}_{a=0}^{o-1}$, where o is the total number of outcomes. Upon obtaining the outcome a, she communicates the pair (a, x) to Bob via a classical communication channel. As a result of Alice's measurement, Bob's subsystem collapses to the conditional state

$$\hat{\rho}_{a|x} = \frac{\mathrm{Tr}_A[(M_{a|x} \otimes \mathbb{I})\,\hat{\rho}_{AB}]}{\mathrm{Tr}[(M_{a|x} \otimes \mathbb{I})\,\hat{\rho}_{AB}]}, \tag{1}$$

with the corresponding probability

$$p(a|x) = \mathrm{Tr}[(M_{a|x} \otimes \mathbb{I})\,\hat{\rho}_{AB}]\,. \tag{2}$$

To withdraw heat from the reservoir, Bob carries out the following procedure:

1) After receiving a and x, Bob quenches his initial Hamiltonian $H = 0$ to $H_{a|x}$.

2) Next, Bob thermalizes his system $\hat{\rho}_{a|x}$ to the thermal state $\hat{\gamma}_{a|x}$ with respect to the Hamiltonian $H_{a|x}$ using a heat reservoir at temperature T. This process withdraws an amount of heat from the reservoir given by

$$Q_c(\hat{\rho}_{a|x}, H_{a|x}) = \mathrm{Tr}(H_{a|x}\hat{\rho}_{a|x}) - \mathrm{Tr}(H_{a|x}\hat{\gamma}_{a|x}), \tag{3}$$

$$\text{where} \quad \hat{\gamma}_{a|x} := \frac{e^{-\beta H_{a|x}}}{\mathrm{Tr}\left(e^{-\beta H_{a|x}}\right)} \quad \text{with} \quad \beta = (k_B T)^{-1}, \tag{4}$$

and k_B denotes the Boltzmann constant.

3) In the final step, Bob restores the system to its initial Hamiltonian $H = 0$ through another quench performed without any interaction with the heat reservoir.

The steps performed by Bob to withdraw heat from the bath correspond precisely to the operation of a four-stroke Otto refrigerator. Unlike a conventional Otto refrigerator, where it is essential to restore the system to its initial state by coupling it to another bath in order to operate in a cyclic manner, our focus here is on cooling that exploits the resources arising from the shared correlations between Alice and Bob. Consequently, we do not aim to return the refrigerator to its initial state; instead, after withdrawing heat from the reservoir, we simply discard it and initiate the process again from Alice's side.

The amount of heat withdrawn from the heat reservoir, averaged over all possible inputs x and outcomes a, is given by

$$\begin{aligned} Q_c\left(\{\rho_{a|x}\}, \{H_{a|x}\}\right) &= \frac{1}{n}\sum_{a,x} p(a|x) Q_c(\hat{\rho}_{a|x}, H_{a|x}) \\ &= \frac{1}{n}\sum_{a,x} p(a|x)\left[\mathrm{Tr}(H_{a|x}\hat{\rho}_{a|x}) - \mathrm{Tr}(H_{a|x}\hat{\gamma}_{a|x})\right] \\ &= \frac{1}{n}\sum_{a,x} \left[\mathrm{Tr}(H_{a|x}\rho_{a|x}) - \mathrm{Tr}(H_{a|x}\gamma_{a|x})\right], \end{aligned} \tag{5}$$

where we define

$$\rho_{a|x} := p(a|x)\hat{\rho}_{a|x}, \qquad \gamma_{a|x} := p(a|x)\hat{\gamma}_{a|x}. \tag{6}$$

The prefactor $1/n$ in Eq. (5) arises from the uniform sampling of x over the set $\mathcal{X} = \{0, \ldots, n-1\}$. We refer to collections of such unnormalized states (i.e., $\{\rho_{a|x}\}_{a,x}$, $\{\gamma_{a|x}\}_{a,x}$) as *assemblages* [23]. Their traces yield the probabilities of obtaining outcome a for the measurement setting x sampled by Alice, i.e., $\mathrm{Tr}(\rho_{a|x}) = \mathrm{Tr}(\gamma_{a|x}) = p(a|x)$. Consequently, an assemblage $\{\rho_{a|x}\}_{a,x}$ can be regarded as a source, indexed by x, producing the normalized quantum state $\hat{\rho}_{a|x}$ with probability $\mathrm{Tr}(\rho_{a|x})$.

To identify signatures of *quantumness*, we introduce the notion of steerability. If the postmeasurement assemblages $\{\rho_{a|x}\}_{a,x}$ can be written as

$$\rho_{a|x} = \int d\lambda\, p(\lambda)\, p(a|x,\lambda)\, \sigma_\lambda \qquad \forall\, a, x, \tag{7}$$

then $\{\rho_{a|x}\}_{a,x}$ is called *unsteerable* assemblage. This decomposition in Eq. (7) is referred to as the *local hidden state* (LHS)

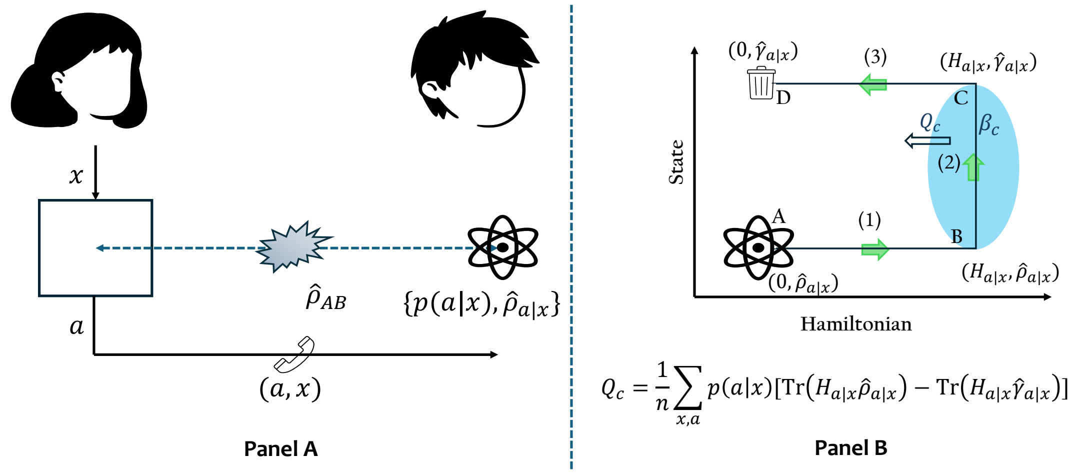

FIG. 1. In Panel A, we illustrate the preparation of resources for heat removal (cooling) from the bath. Alice and Bob share a quantum state $\hat{\rho}_{AB}$. Alice samples an input x uniformly from the set $\mathcal{X} = \{0, \ldots, n-1\}$ and performs the measurement M_x on her subsystem of the joint state $\hat{\rho}_{AB}$. After the measurement, Alice communicates the outcome a and the input x to Bob. As a result of Alice's measurement, Bob obtains the conditional state $\hat{\rho}_{a|x}$, given in Eq. (1), with probability $p(a|x)$, as given in Eq. (2). This state then serves as a resource for withdrawing heat from the bath. Panel B illustrates the procedure for heat removal from a bath at inverse temperature β. Upon receiving the pair (a, x) from Alice, Bob quenches his Hamiltonian from the initial trivial Hamiltonian $H = 0$ to a final Hamiltonian $H_{a|x}$. This step involves no contact with the bath, so the entropy remains unchanged, as indicated by arrow (1) in the average energy–entropy diagram. In the next step, represented by arrow (2), Bob thermalizes his subsystem by placing it in contact with the thermal bath. During this process, heat is withdrawn from the bath and the entropy of the state changes. Finally, in the step indicated by arrow (3), Bob returns the Hamiltonian to its initial trivial form via another quench, again without contacting the bath. On average, the heat withdrawn from the bath is given by $Q_c = 1/n \sum_{a,x} p(a|x)\,[\mathrm{Tr}(\hat{\rho}_{a|x}H_{a|x}) - \mathrm{Tr}(\hat{\gamma}_{a|x}H_{a|x})]$.

decomposition. An assemblage that is not unsteerable is *steerable*. We denote the set of all assemblages by $\mathcal{L}$.

Unsteerable assemblages are regarded as *classical* because they admit an LHS decomposition, meaning all measurement statistics can be simulated using classical correlations (e.g., correlations that arise from a separable state). Conversely, if the assemblages $\rho_{a|x}$ do not admit an LHS decomposition as in Eq. (7) then they cannot be generated by local measurements on a separable state. In this case, an entangled state $\hat{\rho}_{AB}$ must necessarily be shared between Alice and Bob to reproduce the such an assemblage.

In this work, we investigate the amount of heat withdrawn from the reservoir using both unsteerable (classical) and steerable (quantum) assemblages. To perform a fair comparison between the classical and quantum scenarios, we impose that the probability associated with the classical and quantum assemblages is identical i.e.,

$$\forall a, x \quad \mathrm{Tr}(\rho_{a|x}) = \mathrm{Tr}(\sigma_{a|x}) = p(a|x). \tag{8}$$

Here, $\sigma_{a|x}$ is unsteerable, i.e., it admits a local hidden state (LHS) decomposition as defined in Eq. (7), whereas $\rho_{a|x}$ is steerable [51]. This means that Bob has access to sources producing quantum resources (i.e., collections of normalized states $\hat{\rho}_{a|x}$ exhibiting quantum steerability) with probability $p(a|x)$, which he then utilizes to withdraw heat from the reservoir. We compare this removal amount of heat in the classical scenario, where Bob instead has access to sources producing classical resources (i.e., normalized states $\hat{\sigma}_{a|x}$ that cannot exhibit steerability) with the same probability $p(a|x)$. For a given assemblage $\rho_{a|x}$, we define the set of unsteerable assemblages obeying Eq. (8) as $\mathcal{L}_\rho$

$$\mathcal{L}_\rho := \Big\{\{\sigma_{a|x}\}_{a,x} \in \mathcal{L} \;\Big|\; \mathrm{Tr}(\rho_{a|x}) = \mathrm{Tr}(\sigma_{a|x}) = p(a|x) \quad \forall\, a, x\Big\}. \tag{9}$$

B. Cooling task using classical and quantum resources

To quantify the degree of steerability of states, we consider the *steering robustness of an assemblage* [35, 73]. Analogous to entanglement robustness, the steering robustness captures the minimal amount of noise that must be mixed with a given assemblage to make it unsteerable. The definition of the steerability robustness of an assemblage is

$$\mathcal{R}(\{\rho_{a|x}\}_{a,x}) = \min_s \left\{ \left\{\frac{\rho_{a|x} + s\tau_{a|x}}{1+s}\right\}_{a,x} \in \mathcal{L}, \right.$$
$$\left. \text{where } \{\tau_{a|x}\}_{a,x} \text{ is an arbitrary assemblage} \right\}. \tag{10}$$

Here, $\mathcal{L}$ denotes the set of all unsteerable assemblages. It straightforwardly follows that $\mathcal{R}(\{\rho_{a|x}\}_{a,x}) = 0$ if and only if $\{\rho_{a|x}\}_{a,x} \in \mathcal{L}$, and it is strictly greater than 0, otherwise.

We are now ready to present the main result of this article, which establishes a thermodynamic advantage in cooling using steerable versus unsteerable assemblages satisfying Eq. (8). To quantify the advantage, we consider the ratio between the heat removal from the bath using a steerable assemblage and the maximum heat removal achievable using any unsteerable assemblage. We then relate this ratio to the

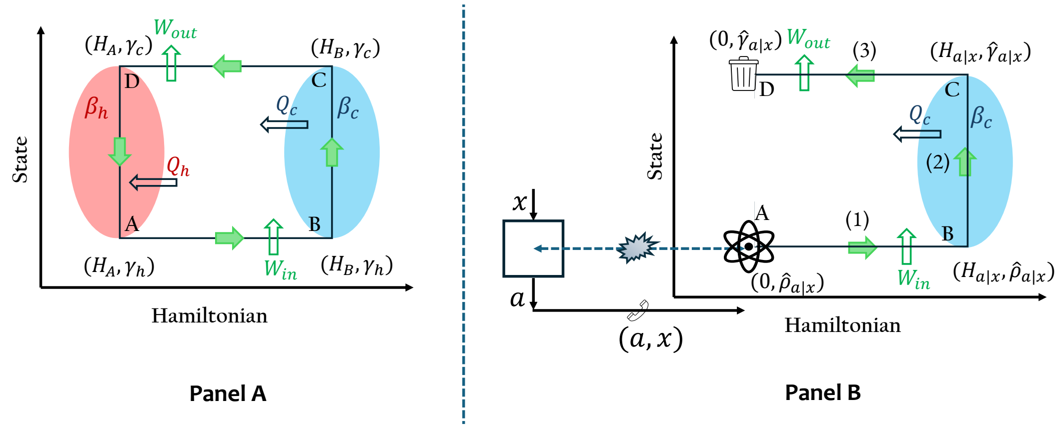

FIG. 2. In this diagram, we present a parallel comparison between the Otto refrigerator and the cooling protocol assisted by correlations introduced in this article. Panel (A) illustrates a bidimensional representation of the Otto refrigeration cycle in the "phase space" defined by the system's state and Hamiltonian. Driven by the temperature difference between the two baths, the refrigerator withdraws heat Q_c from the cold reservoir and releases heat Q_h into the hot reservoir. To complete the cycle, an input work of total magnitude $Q_h + Q_c = W_{\text{in}} + W_{\text{out}}$ is required, where $W_{\text{in}} = \text{Tr}(H_B\gamma_h) - \text{Tr}(H_A\gamma_h)$ and $W_{\text{out}} = \text{Tr}(H_A\gamma_c) - \text{Tr}(H_B\gamma_c)$. In panel (B), we represent the cooling mechanism based on assemblages arising from unsteerable and steerable correlations. We compare the amount of heat Q_c withdrawn when the refrigerator operates with assemblages generated by unsteerable correlations versus those produced by steerable correlations. The latter cool the bath more, providing a quantum thermodynamic advantage.

steerability robustness defined in Eq. (10). Our main result is stated as follows:

Theorem 1. *For any steerable quantum assemblage* $\{\rho_{a|x}\}_{a,x} \notin \mathcal{L}$*, we have*

$$\xi_{\max}(\{\rho_{a|x}\}_{a,x}) := \max_{\{H_{a|x}\}_{a,x}} \frac{Q_c(\{\rho_{a|x}\}, \{H_{a|x}\})}{\max_{\{\sigma_{a|x}\}\in\mathcal{L}_\rho} Q_c(\{\sigma_{a|x}\}, \{H_{a|x}\})} \geq 1 + \mathcal{R}(\{\rho_{a|x}\}_{a,x}), \tag{11}$$

where $\mathcal{R}(\{\rho_{a|x}\}_{a,x})$ *denotes the* steerability robustness *of the assemblage* $\{\rho_{a|x}\}_{a,x}$*, as defined in Eq.* (10)*, and* $\mathcal{L}_\rho$ *is given in Eq.* (9)*.*

As established earlier, the steerability robustness $\mathcal{R}(\{\rho_{a|x}\}_{a,x})$ of any steerable assemblage $\{\rho_{a|x}\}_{a,x}$ is strictly positive. Consequently, $\xi_{\max}(\{\rho_{a|x}\}_{a,x}) > 1$. Therefore, Theorem 1 guarantees that there always exists a cooling task (i.e., an appropriate choice of Hamiltonians) in which using a steerable assemblage as a resource outperforms any unsteerable assemblage contained in the set $\mathcal{L}_\rho$. Remarkably, the maximal advantage in such a task is lower bounded by the steerability robustness. This establishes an operational interpretation of steerability robustness—a geometric quantity originally introduced to quantify steerability—in terms of a thermodynamic task.

Moreover, an advantage in withdrawing heat from the bath using an assemblage $\{\rho_{a|x}\}$, i.e., when $\xi_{\max}(\{\rho_{a|x}\}_{a,x}) > 1$, can be interpreted as a *thermodynamic witness of steerability*. In particular, if the assemblage $\{\rho_{a|x}\}$ is unsteerable, then by definition $\{\rho_{a|x}\} \in \mathcal{L}_\rho$ [see Eq. (8)]. Since the denominator in Eq. (11) involves a maximization taken over all assemblages in $\mathcal{L}_\rho$, any unsteerable assemblage must satisfy $\xi_{\max}(\rho_{a|x}) \leq 1$. Therefore, observing $\xi_{\max}(\rho_{a|x}) > 1$ is both necessary and sufficient to certify that the assemblage $\{\rho_{a|x}\}$ is steerable.

Next, we focus on a particular steerable assemblage motivated by the connection between steerability and measurement incompatibility. As shown in Ref. [70], the steerability of any assemblage can be equivalently formulated as a joint measurability problem, and vice versa. We focus on the steerable assemblage constructed from MUBs, since MUBs exhibit maximal measurement incompatibility, which in turn leads to an assemblage with maximum steerability robustness. Therefore, using inequality from Eq. (11) we expect to see a significant quantum advantage in cooling.

To establish this, we consider the following assemblage:

$$\rho^{\text{isotropic}}_{a|x}(\eta) := \eta\,\frac{1}{d}\left|\phi^a_x\middle\rangle\!\middle\langle\phi^a_x\right| + (1-\eta)\,\frac{\mathbb{I}}{d^2}, \qquad 0 \leq \eta \leq 1, \tag{12}$$

where $a \in \{0, \ldots, d-1\}$ and $x \in \{0, \ldots, n\}$. An assemblage of the form $\{\rho^{\text{isotropic}}_{a|x}(\eta)\}_{a,x}$ can be prepared if Alice and Bob share the isotropic state

$$\rho^{\text{isotropic}}_{AB}(\eta) = \eta\left(\frac{1}{d}\sum_{i,j=0}^{d-1} |i\rangle\!\langle j|_A \otimes |i\rangle\!\langle j|_B\right) + (1-\eta)\frac{\mathbb{I}_{AB}}{d^2}, \tag{13}$$

and Alice performs a measurement described by the projectors

$$\left\{\left|\phi^{*a}_x\middle\rangle\!\middle\langle\phi^{*a}_x\right|\right\}_{a=0}^{d-1}, \qquad \left|\phi^{*a}_x\right\rangle = \sum_{i=0}^{d-1} \langle\phi^a_x|i\rangle\,|i\rangle\,. \tag{14}$$

Here, $\left|\phi^a_x\right\rangle$ denotes the a-th vector of the mutually unbiased basis (MUB) indexed by x, satisfying

$$\left|\left\langle\phi^a_x\middle|\phi^b_y\right\rangle\right| = \begin{cases} \delta_{ab}, & x = y, \\ 1/\sqrt{d}, & x \neq y. \end{cases}$$

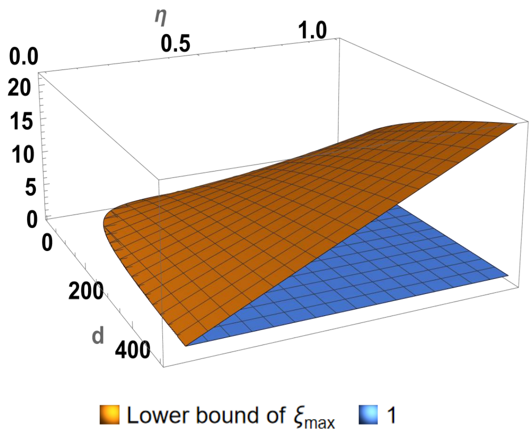

FIG. 3. We plot the lower bound of $\xi_{\max}\left(\left\{\rho^{\text{isotropic}}_{a|x}(\eta)\right\}_{a,x}\right)$, given in Eq. (17), as a function of d and η (orange surface) for $2 \leq d \leq 500$ and $\left(\sqrt{d} + \frac{1}{d+\sqrt{d}+1}\right)^{-1} < \eta \leq 1$, as specified in Eq. (19). The plot shows that this surface lies strictly above the plane $z = 1$ (shown in blue), demonstrating that $\xi_{\max}\left(\left\{\rho^{\text{isotropic}}_{a|x}(\eta)\right\}_{a,x}\right) > 1$. This confirms the presence of quantum advantage, which increases with the dimension d throughout the parameter regime $\left(\sqrt{d} + \frac{1}{d+\sqrt{d}+1}\right)^{-1} < \eta \leq 1$.

Note that for $\eta = 1$, the state $\rho^{\text{isotropic}}_{AB}(\eta)$ becomes maximally entangled, and the assemblage $\rho^{\text{isotropic}}_{a|x}(\eta)$ in Eq. (12) reduces to

$$\rho^{\text{isotropic}}_{a|x}(\eta = 1) = \frac{1}{d}\left|\phi^a_x\middle\rangle\middle\langle\phi^a_x\right| =: \rho^{\text{max. ent.}}_{a|x}. \tag{15}$$

We emphasize that in any Hilbert space of dimension $d = p^n$, where p is a prime and n is a positive integer, a complete set of $d+1$ MUBs always exists. In such dimensions, we can always take the number of MUBs to be $n = d+1$ [72, 74]. As shown in Appendix D, in such dimensions taking $n = d+1$ gives

$$\mathcal{R}\left(\left\{\rho^{\text{isotropic}}_{a|x}(\eta)\right\}_{a,x}\right) \geq \sqrt{d}\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)\eta - \left(\frac{d\sqrt{d}-1}{d(\sqrt{d}+1)}\right)(1-\eta), \tag{16}$$

which further gives using the inequality in Eq. (11),

$$\xi_{\max}\left(\left\{\rho^{\text{isotropic}}_{a|x}(\eta)\right\}_{a,x}\right) \geq \frac{(d+1)(1+\eta(d-1))}{d\,(1+\sqrt{d})}. \tag{17}$$

For $\eta = 1$, we obtain

$$\xi_{\max}\left(\left\{\rho^{\text{max. ent.}}_{a|x}\right\}_{a,x}\right) \geq \frac{d+1}{\sqrt{d}+1} = O(\sqrt{d}). \tag{18}$$

This demonstrates that the quantum advantage in the cooling task—when using steerable assemblages $\left\{\rho^{\text{max. ent.}}_{a|x}\right\}_{a,x}$ as a resource—increases with the system dimension. This scaling of advantage growing with dimension is sometimes referred to as an unbounded quantum advantage [75–77].

For the assemblage $\left\{\rho^{\text{isotropic}}_{a|x}(\eta)\right\}_{a,x}$, whether the advantage grows with the dimension depends explicitly on how the parameter η scales with d. In particular, achieving an advantage that increases with the dimension requires

$$\eta > \frac{1}{\sqrt{d} + \frac{1}{d+\sqrt{d}+1}}, \tag{19}$$

as illustrated in Fig. 3 . The derivation of this condition is provided in Appendix D.

III. DISCUSSION

We demonstrate that quantum steerable states constitute a valuable resource for cooling a heat bath, outperforming all unsteerable states. In particular, we show that for any steerable assemblage, one can construct a cooling scenario in which the assemblage achieves strictly better performance than any unsteerable counterpart. As a figure of merit, we consider the ratio between the heat withdrawn in the quantum (steerable) and classical (unsteerable) settings. We prove that this quantum-to-classical heat withdrawal ratio is lower bounded by the steering robustness, thereby establishing a quantitative link between operational advantages in thermodynamic tasks and a geometric measure of quantum steerability.

Our result carries several noteworthy implications. First, the observed thermodynamic advantage can serve as an operational witness of quantum steering: if a given assemblage enables enhanced cooling beyond the classical limit, it must necessarily be steerable. Next, in the case of prime dimensions, we show that the quantum-to-classical heat withdrawal ratio scales with the dimension, thereby demonstrating an unbounded advantage. Furthermore, we analyze the lower bound for specific classes of quantum states, such as isotropic states, illustrating how the steering robustness—and consequently the thermodynamic advantage—behaves for these familiar states.

These findings open up several promising directions within the domain of quantum thermodynamics. As efficient cooling remains a fundamental challenge for the practical implementation of quantum computing, the demonstrated advantage offered by quantum correlations in enhancing cooling efficiency could play a pivotal role in the development of future quantum technologies. Furthermore, since the proposed protocol relies solely on quenching and thermalization—standard operations in the context of microscopic thermal machines—its implementation appears to be experimentally accessible within current technological capabilities. Another natural direction for future research is to extend the current framework to multipartite systems. This could lead to the discovery of richer thermodynamic behavior and new types of quantum advantages. In particular, studying how multipartite entanglement and steering affect thermodynamic tasks such as cooling or work withdrawal may provide a deeper understanding of the role of quantum correlations in thermodynamics.

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Appendix A: Proof of Theorem 1

For a given assemblage $\{\rho_{a|x}\}_{a,x}$, one can calculate the steerability robustness (see Refs. [23, 36]) using the following semidefinite program (SDP):

$$\begin{aligned} 1+\mathcal{R}(\{\rho_{a|x}\}_{a,x}) = \text{maximize} \quad & \sum_{a,x} \mathrm{Tr}\left[F_{a|x}\rho_{a|x}\right] \\ \text{subject to} \quad & \sum_{a,x} D(a|x,\lambda) F_{a|x} \leq \mathbb{I}, \quad \forall \lambda, \\ & F_{a|x} \geq 0, \quad \forall a, x, \end{aligned} \tag{A1}$$

where $D(a|x,\lambda)$ is the deterministic response function i.e., $D(a|x,\lambda) = \delta_{a,\lambda(x)}$. We denote $\{F^*_{a|x}\}_{a,x}$ as the solution of the SDP given in Eq. (A1). Recall that

$$\mathrm{Tr}(\rho_{a|x}) = p(a|x). \tag{A2}$$

Consider a Hamiltonian of the form

$$H^*_{a|x} = \epsilon\, F^*_{a|x}, \tag{A3}$$

which Bob quenches to after receiving the pair (a, x) from Alice, where $\epsilon > 0$. We define the assemblage $\{\sigma^*_{a|x}\}_{a,x}$ via the following relation:

$$\max_{\{\sigma_{a|x}\}\in\mathcal{L}_\rho} \sum_{a,x} \mathrm{Tr}(F^*_{a|x}\sigma_{a|x}) := \sum_{a,x} \mathrm{Tr}(F^*_{a|x}\sigma^*_{a|x}). \tag{A4}$$

Since $\{\sigma^*_{a|x}\}_{a,x}$ is an unsteerable assemblage, it holds that [23, 24]:

$$\sum_{a,x} \mathrm{Tr}(F^*_{a|x}\sigma^*_{a|x}) \leq 1, \qquad \forall\, a, x. \tag{A5}$$

We also denote

$$\begin{aligned} \gamma^*_{a|x}(\epsilon) &:= p(a|x)\hat{\gamma}^*_{a|x}(\epsilon), \quad \text{where} \\ \hat{\gamma}^*_{a|x}(\epsilon) &:= \frac{e^{-\beta H^*_{a|x}}}{\mathrm{Tr}\left(e^{-\epsilon\beta H^*_{a|x}}\right)} = \frac{e^{-\epsilon\beta F^*_{a|x}}}{\mathrm{Tr}\left(e^{-\epsilon\beta F^*_{a|x}}\right)}. \end{aligned} \tag{A6}$$

Then, the following inequality holds:

$$\begin{aligned} \xi_{\max}\left(\{\rho_{a|x}\}_{a,x}\right) &:= \max_{\{H_{a|x}\}_{a,x}} \frac{Q_c(\{\rho_{a|x}\},\{H_{a|x}\})}{\max_{\{\sigma_{a|x}\}\in\mathcal{L}_\rho} Q_c(\{\sigma_{a|x}\},\{H_{a|x}\})} \\ &\geq \frac{Q_c\left(\{\rho_{a|x}\},\{H^*_{a|x}\}\right)}{Q_c\left(\{\sigma^*_{a|x}\},\{H^*_{a|x}\}\right)} \\ &= \frac{\frac{1}{n}\sum_{a,x}\left[\mathrm{Tr}(H^*_{a|x}\rho_{a|x}) - \mathrm{Tr}(H^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]}{\frac{1}{n}\sum_{a,x}\left[\mathrm{Tr}(H^*_{a|x}\sigma^*_{a|x}) - \mathrm{Tr}(H^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]} \\ &= \frac{\sum_{a,x}\left[\mathrm{Tr}(F^*_{a|x}\rho_{a|x}) - \mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]}{\sum_{a,x}\left[\mathrm{Tr}(F^*_{a|x}\sigma^*_{a|x}) - \mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]} \\ &= \frac{\left[1+\mathcal{R}(\{\rho_{a|x}\}_{a,x}) - \sum_{a,x}\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]}{\sum_{a,x}\left[\mathrm{Tr}(F^*_{a|x}\sigma^*_{a|x}) - \mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]}. \end{aligned} \tag{A7}$$

For any $\epsilon > 0$, we have

$$\sum_{a,x}\left[\mathrm{Tr}(F^*_{a|x}\sigma^*_{a|x}) - \mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right] \geq 0. \tag{A8}$$

To show this, we use Eq. (A6) to observe that

$$\gamma^*_{a|x}(0) = p(a|x)\frac{\mathbb{I}}{d} := I_{a|x} \quad \forall\, a, x, \tag{A9}$$

for $\epsilon = 0$. $\gamma^*_{a|x}(0)$ thus constitutes an unsteerable assemblage in the set $\mathcal{L}_\rho$. [One can simply choose $p(\lambda) = 1$, $p(a|x,\lambda) = p(a|x)$ for all a, x, and $\rho_\lambda = \mathbb{I}/d$ in Eq. (7) to verify that the assemblage in Eq. (A9) is unsteerable. Moreover $\forall a, x$; $\mathrm{Tr}\left(\gamma^*_{a|x}(0)\right) = p(a|x) = \mathrm{Tr}(\rho_{a|x})$ implies $\gamma^*_{a|x}(0) \in \mathcal{L}_\rho$.] Then, employing Eq. (A4), we can infer that inequality in Eq. (A8) holds when $\epsilon = 0$. Satisfying the inequality in Eq. (A8) at $\epsilon = 0$ is enough to show that the inequality holds for any $\epsilon > 0$. To see this, we observe that $\hat{\gamma}_{a|x}$ in Eq. (A6) can be interpreted as a Gibbs state with respect to the effective Hamiltonian $F^*_{a|x}$ at an effective temperature $\beta_{\text{eff}}(\epsilon) := \epsilon\beta$, i.e.,

$$\hat{\gamma}_{a|x} = \frac{e^{-\beta_{\text{eff}}(\epsilon)F^*_{a|x}}}{\mathrm{Tr}\left(e^{-\beta_{\text{eff}}(\epsilon)F^*_{a|x}}\right)} \quad . \tag{A10}$$

Consequently, $\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))$ can be interpreted as the average energy at the Gibbs state $\hat{\gamma}_{a|x}$ [as given in Eq. (A10)] multiplied by $p(a|x)$ i.e.,

$$\begin{aligned} \mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon)) &= p(a|x)\,\mathrm{Tr}\left(F^*_{a|x}\frac{e^{-\epsilon\beta F^*_{a|x}}}{\mathrm{Tr}\left(e^{-\epsilon\beta F^*_{a|x}}\right)}\right) \\ &= p(a|x)\,\mathrm{Tr}\left(F^*_{a|x}\frac{e^{-\beta_{\text{eff}}(\epsilon)F^*_{a|x}}}{\mathrm{Tr}\left(e^{-\beta_{\text{eff}}(\epsilon)F^*_{a|x}}\right)}\right). \end{aligned} \tag{A11}$$

Since the average energy of a Gibbs state is a monotonically decreasing function of the inverse temperature, we have

$$\begin{aligned}\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(0)) &= p(a|x)\,\mathrm{Tr}\left(F^*_{a|x}\frac{e^{-\beta_{\rm eff}(0)F^*_{a|x}}}{\mathrm{Tr}\left(e^{-\beta_{\rm eff}(0)F^*_{a|x}}\right)}\right)\\ &\geq p(a|x)\,\mathrm{Tr}\left(F^*_{a|x}\frac{e^{-\beta_{\rm eff}(\epsilon)F^*_{a|x}}}{\mathrm{Tr}\left(e^{-\beta_{\rm eff}(\epsilon)F^*_{a|x}}\right)}\right)\\ &= \mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon)),\end{aligned} \tag{A12}$$

as

$$\forall\,\epsilon>0,\qquad \beta_{\rm eff}(\epsilon)=\epsilon\beta>0=\beta_{\rm eff}(0). \tag{A13}$$

Using Eqs. (A4) and (A12), we obtain:

$$\sum_{a,x}\left[\mathrm{Tr}(F^*_{a|x}\sigma^*_{a|x})\right]\geq\sum_{a,x}\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(0))\geq\sum_{a,x}\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon)). \tag{A14}$$

Then, from Eq. (A7) we can find that

$$\begin{aligned}\xi_{\max}(\{\rho_{a|x}\}) &\geq \frac{\left[1+\mathcal{R}(\{\rho_{a|x}\}_{a,x})-\sum_{a,x}\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]}{\sum_{a,x}\left[\mathrm{Tr}(F^*_{a|x}\sigma^*_{a|x})-\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]}\\ &\geq \frac{\left[1+\mathcal{R}(\{\rho_{a|x}\}_{a,x})-\sum_{a,x}\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]}{\left[1-\sum_{a,x}\mathrm{Tr}(F^*_{a|x}\gamma^*_{a|x}(\epsilon))\right]}\\ &\geq 1+\mathcal{R}(\{\rho_{a|x}\}_{a,x}),\end{aligned} \tag{A15}$$

where in the second inequality we use Eq. (A5). To derive the final inequality in Eq. (A15), we use the fact that

$$\frac{x-z}{y-z}\geq\frac{x}{y}\quad\Longleftrightarrow\quad y-z\geq0\quad\text{and}\quad x\geq y, \tag{A16}$$

with the identifications

$$x=1+\mathcal{R}(\{\rho_{a|x}\}_{a,x}),\quad y=1,\quad z=\sum_{a,x}\mathrm{Tr}(F^*_{a|x}\,\gamma^*_{a|x}(\epsilon)).$$

Since $\mathcal{R}(\{\rho_{a|x}\}_{a,x})>0$ for any steerable assemblage $\{\rho_{a|x}\}_{a,x}$, it follows that $x\geq y$, while $y-z\geq0$ follows directly from Eq. (A14).

Appendix B: Satisfiability of the constraint under the chosen suboptimal operators $F_{a|x}$

Here, we prove that

$$F_{a|x}=\frac{1}{1+\sqrt{d}}\left|\phi^a_x\middle\rangle\middle\langle\phi^a_x\right|\qquad\forall a,x, \tag{B1}$$

satisfies the constraint of the SDP given in Eq. (A1), where $a\in\{0,\ldots,d-1\}$ and $x\in\{0,\ldots,d\}$. The operators $F_{a|x}$ defined in Eq. (B1) are positive semidefinite for all a and x. Next, we show that

$$\sum_{a,x}D(a|x,\lambda)F_{a|x}\leq\mathbb{I}\qquad\forall\lambda, \tag{B2}$$

which is equivalent to showing that

$$\left\|\sum_{a,x}D(a|x,\lambda)F_{a|x}\right\|_\infty\leq1\qquad\forall\lambda, \tag{B3}$$

where $D(a|x,\lambda)=\delta_{a,\lambda(x)}$ denotes the deterministic response function.

Before proving Eq. (B3), we establish a relation that will be used in the proof. Consider a bipartite pure state on systems C and D of the form

$$|\zeta\rangle_{CD}=\sum_{x=1}^{d+1}\left|\phi^{\lambda(x)}_x\right\rangle_C|x\rangle_D\quad\text{where the }|x\rangle\text{'s form an orthonormal basis.} \tag{B4}$$

Using the Schmidt decomposition of $|\zeta\rangle_{CD}$, we can infer that

$$\begin{aligned}&\mathrm{Spec}\left[\mathrm{Tr_C}\left(|\zeta\rangle_{\rm CD}\right)\right]=\mathrm{Spec}\left[\mathrm{Tr_D}\left(|\zeta\rangle_{\rm CD}\right)\right]\\ \Longleftrightarrow\ &\mathrm{Spec}\left[\sum_{\rm x}\left|\phi^{\lambda(\rm x)}_{\rm x}\middle\rangle\middle\langle\phi^{\lambda(\rm x)}_{\rm x}\right|\right]=\mathrm{Spec}\left[\sum_{\rm x,y}\left\langle\phi^{\lambda(\rm y)}_{\rm y}\middle|\phi^{\lambda(\rm x)}_{\rm x}\right\rangle|{\rm y}\rangle\langle{\rm x}|\right]\\ \Longleftrightarrow\ &\left\|\sum_x\left|\phi^{\lambda(x)}_x\middle\rangle\middle\langle\phi^{\lambda(x)}_x\right|\right\|_\infty=\left\|\sum_{x,y}\left\langle\phi^{\lambda(y)}_y\middle|\phi^{\lambda(x)}_x\right\rangle|y\rangle\langle x|\right\|_\infty,\end{aligned} \tag{B5}$$

where $\mathrm{Spec}[\cdot]$ denotes the spectrum and $\|\cdot\|_\infty$ the operator norm. Then, the proof of Eq. (B3) follows:

$$\begin{aligned}&\left\|\sum_{a,x}D(a|x,\lambda)F_{a|x}\right\|_\infty\\ &=\frac{1}{1+\sqrt{d}}\left\|\sum_{a,x}D(a|x,\lambda)\left|\phi^a_x\middle\rangle\middle\langle\phi^a_x\right|\right\|_\infty\\ &=\frac{1}{1+\sqrt{d}}\left\|\sum_{a,x}\delta_{a,\lambda(x)}\left|\phi^a_x\middle\rangle\middle\langle\phi^a_x\right|\right\|_\infty\\ &=\frac{1}{1+\sqrt{d}}\left\|\sum_x\left|\phi^{\lambda(x)}_x\middle\rangle\middle\langle\phi^{\lambda(x)}_x\right|\right\|_\infty\\ &=\frac{1}{1+\sqrt{d}}\left\|\sum_{x,y}\left\langle\phi^{\lambda(y)}_y\middle|\phi^{\lambda(x)}_x\right\rangle|y\rangle\langle x|\right\|_\infty\\ &=\frac{1}{1+\sqrt{d}}\left\|\sum_x\left\langle\phi^{\lambda(x)}_x\middle|\phi^{\lambda(x)}_x\right\rangle|x\rangle\langle x|+\sum_{x\neq y}\left\langle\phi^{\lambda(y)}_y\middle|\phi^{\lambda(x)}_x\right\rangle|y\rangle\langle x|\right\|_\infty\\ &=\frac{1}{1+\sqrt{d}}\left\|\sum_x|x\rangle\langle x|+\sum_{x\neq y}\frac{1}{\sqrt{d}}e^{i\phi_{x,y}}|y\rangle\langle x|\right\|_\infty\\ &=\frac{1}{1+\sqrt{d}}\left\|\sum_x\left(1-\frac{1}{\sqrt{d}}\right)|x\rangle\langle x|+\sum_{x,y}\frac{1}{\sqrt{d}}e^{i\phi_{x,y}}|y\rangle\langle x|\right\|_\infty\\ &\leq\frac{1}{1+\sqrt{d}}\left\|\sum_x\left(1-\frac{1}{\sqrt{d}}\right)|x\rangle\langle x|\right\|_\infty+\left\|\sum_{x,y}\frac{1}{\sqrt{d}}e^{i\phi_{x,y}}|y\rangle\langle x|\right\|_\infty\\ &\leq\frac{1}{1+\sqrt{d}}\left\|\sum_x\left(1-\frac{1}{\sqrt{d}}\right)|x\rangle\langle x|\right\|_\infty+\left\|\sum_{x,y}\frac{1}{\sqrt{d}}e^{i\phi_{x,y}}|y\rangle\langle x|\right\|_F\end{aligned} \tag{B6}$$

$$=\frac{1}{1+\sqrt{d}}\left[\left(1-\frac{1}{\sqrt{d}}\right)+\frac{1}{\sqrt{d}}(d+1)\right]=1, \tag{B7}$$

where the fifth equality follows from Eq. (B5), and in the sixth inequality we use the property of mutually unbiased bases (MUBs),

$$\left\langle \phi_y^{\lambda(y)} \middle| \phi_x^{\lambda(x)} \right\rangle = \frac{1}{\sqrt{d}} e^{i\phi_{x,y}} \qquad \phi_{x,y} \in \mathbb{R}, \tag{B8}$$

while the final inequality follows from the relation $\| \cdot \|_\infty \leq \| \cdot \|_F$, where the Frobenius norm is defined as

$$\|A\|_F = \sqrt{\mathrm{Tr}(A^\dagger A)} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2}. \tag{B9}$$

Appendix C: Lower bound on the steerability robustness for an assemblage generated from a maximally entangled state

In this section, we evaluate the steerability robustness of assemblages derived from the maximally entangled state

$$\rho_{AB}^{\mathrm{max.\,ent}} = \frac{1}{d} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B\,, \tag{C1}$$

by performing the measurements defined in Eq. (14) on one subsystem. This procedure yields the following assemblage on Bob's side:

$$\rho_{a|x}^{\mathrm{max.\,ent}} := \frac{1}{d} \left|\phi_x^a\right\rangle\left\langle\phi_x^a\right|, \qquad a \in \{0, \ldots, d-1\},\; x \in \{0, \ldots, n\}. \tag{C2}$$

In what follows, we assume $d = p^k$, where p is a prime number and k is a positive integer, allowing for $n = d + 1$. To compute a lower bound on the steerability robustness, we employ the semidefinite program (SDP) introduced in Eq. (A1). We take a suboptimal but feasible choice of $F_{a|x}$ that satisfies the SDP constraint (as shown in Appendix B):

$$F_{a|x} = \frac{1}{1 + \sqrt{d}} \left|\phi_x^a\right\rangle\left\langle\phi_x^a\right|. \tag{C3}$$

Substituting this into Eq. (A1), we obtain the following lower bound:

$$\begin{aligned} \mathcal{R}(\rho_{a|x}^{\mathrm{max.\,ent.}}) &\geq \sum_{a,x} \mathrm{Tr}\left(F_{a|x}\rho_{a|x}^{\mathrm{max.\,ent.}}\right) - 1 \\ &= \sqrt{d}\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right) = O(\sqrt{d}). \end{aligned} \tag{C4}$$

Hence, for such an assemblage, the lower bound on the steerability robustness scales as $O(\sqrt{d})$. Consequently, when this assemblage is employed, the advantage in cooling given by Eq. (11) also scales as $O(\sqrt{d})$.

Appendix D: Lower bound on the steerability robustness for an assemblage generated from an isotropic state

In this section, we evaluate the robustness of the assemblages obtained from the isotropic state by performing measurements given in Eq. (14) on one subsystem. The isotropic state is defined as a mixture of a maximally entangled state and the maximally mixed state i.e.,

$$\rho_{AB}^{\mathrm{isotropic}}(\eta) = \eta\left(\frac{1}{d}\sum_{i=0}^{d-1}\sum_{j=0}^{d-1}|i\rangle\langle j|_A \otimes |i\rangle\langle j|_B\right) + (1-\eta)\frac{\mathbb{I}_{\mathrm{AB}}}{d^2}. \tag{D1}$$

We see from Eq. (D1) that when $\eta = 1$, $\rho_{\mathrm{isotropic}}(\eta)$ reduces to the maximally entangled state, and when $\eta = 0$, $\rho_{\mathrm{isotropic}}(\eta)$ reduces to the maximally mixed state. We can calculate the assemblage obtained by measuring subsystem A in the state $\left|\phi_x^a\right\rangle$, which yields for all a and x:

$$\begin{aligned} \mathrm{Tr}_A\left[\left(\left|\phi_x^{*a}\right\rangle\left\langle\phi_x^{*a}\right|\right)\rho_{AB}^{\mathrm{isotropic}}(\eta)\right] &= \eta\,\frac{1}{d}\left|\phi_x^a\right\rangle\left\langle\phi_x^a\right|_B + (1-\eta)\,\frac{\mathbb{I}_B}{d^2} \\ &= \eta\,\rho_{a|x}^{\mathrm{max.\,ent.}} + (1-\eta)\,\frac{\mathbb{I}}{d^2} \\ &:= \rho_{a|x}^{\mathrm{isotropic}}(\eta) \end{aligned} \tag{D2}$$

where we have dropped the subscript for brevity to write the second equality. In Refs. [22, 24], it has been shown that when

$$\eta \leq \frac{H_d - 1}{d - 1}, \tag{D3}$$

the assemblage $\rho_{a|x}^{\mathrm{isotropic}}(\eta)$ given in Eq. (D2) is unsteerable, whereas it becomes steerable for

$$\eta > \frac{H_d - 1}{d - 1}, \tag{D4}$$

where $H_d := \sum_{i=1}^{d} i^{-1}$ denotes the d-th harmonic number. In the following, we calculate a lower bound on the steerability robustness for the assemblage $\rho_{a|x}^{\mathrm{isotropic}}(\eta)$.

To lower bound the steerability robustness, we employ the SDP given in Eq. (A1). We consider the same sub-optimal choice for $F_{a|x}$ as in Eq. (C3). Substituting such a suboptimal choice for $F_{a|x}$ from Eq. (C3), we can calculate the lower bound of steerability robustness as follows:

$$\begin{aligned} \mathcal{R}(\rho_{a|x}^{\mathrm{isotropic}}(\eta)) &\geq \sum_{a,x} \mathrm{Tr}\left(F_{a|x}\rho_{a|x}^{\mathrm{isotropic}}(\eta)\right) - 1 \\ &= \sqrt{d}\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)\eta - \left(\frac{d\sqrt{d}-1}{d\left(\sqrt{d}+1\right)}\right)(1-\eta). \end{aligned} \tag{D5}$$

The prefactor of η in Eq. (D5) scales as

$$\sqrt{d}\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right) = O(\sqrt{d}), \tag{D6}$$

while the prefactor of $(1-\eta)$, satisfies

$$\frac{d\sqrt{d}-1}{d(\sqrt{d}+1)} < 1 = O(1). \tag{D7}$$

The quantum advantage in cooling is observed when the lower bound in Eq. (D5) is strictly positive as given by

Eq. (11), i.e.,

$$\sqrt{d}\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)\eta-\left(\frac{d\sqrt{d}-1}{d(\sqrt{d}+1)}\right)(1-\eta)>0 \tag{D8}$$

$$\iff \quad \eta>\frac{1}{\sqrt{d}+\frac{1}{d+\sqrt{d}+1}}=O\left(\frac{1}{\sqrt{d}}\right). \tag{D9}$$

Hence, for any η satisfying Eq. (D9), the lower bound on the steerability robustness of the assemblage $\rho_{a|x}^{\text{isotropic}}(\eta)$ in Eq. (D5) remains positive, thereby ensuring a quantum advantage in cooling. Moreover, we observe that this lower bound decreases with increasing system dimension, indicating that a smaller admixture of the maximally entangled state is sufficient to achieve a quantum advantage in higher dimensions.

In particular, for $\eta = 1$, the isotropic state reduces to the maximally entangled state, yielding

$$\rho_{a|x}^{\text{isotropic}}(\eta=1)=\frac{1}{d}\left|\phi_x^a\middle\rangle\middle\langle\phi_x^a\right|=\rho_{a|x}^{\text{max. ent.}}, \qquad \forall a,x. \tag{D10}$$

In this case, the lower bound in Eq. (D5) simplifies to

$$\sqrt{d}\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)=O(\sqrt{d}). \tag{D11}$$

Consequently, the quantum advantage in cooling, as given by Eq. (11), scales as $O(\sqrt{d})$. Such an enhancement, which grows with the dimension of the underlying system, is sometimes referred to as an "unbounded quantum advantage".

To establish a general condition for obtaining a dimension-dependent advantage, we seek the range of values of η for which the lower bound is not only positive but also increases with the system dimension. Earlier, in Eq. (D9), we identified the region of η that ensures the positivity of the lower bound in Eq. (D5). We now determine the range of η for which this lower bound grows with dimension.

To this end, we impose that the first-order derivative of the lower bound with respect to d be positive:

$$\frac{\partial}{\partial d}\left[\sqrt{d}\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)\eta-\left(\frac{d\sqrt{d}-1}{d(\sqrt{d}+1)}\right)(1-\eta)\right]>0. \tag{D12}$$

Solving this inequality yields

$$\eta>\frac{2+\sqrt{d}(3+d)}{2+3\sqrt{d}+2d^2+d^{5/2}}=O\left(\frac{1}{d}\right). \tag{D13}$$

It is straightforward to verify that

$$\frac{1}{\sqrt{d}+\frac{1}{d+\sqrt{d}+1}}>\frac{2+\sqrt{d}(3+d)}{2+3\sqrt{d}+2d^2+d^{5/2}}. \tag{D14}$$

This implies that whenever η scales as

$$\eta>\frac{1}{\sqrt{d}+\frac{1}{d+\sqrt{d}+1}}, \tag{D15}$$

one not only achieves a quantum advantage in cooling but also an advantage that scales with the dimension of the underlying system.

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