Quantum speed limit for observables from quantum asymmetry

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Quantum asymmetry and coherence are genuinely quantum resources that are essential to realize quantum advantage in information technologies. However, all quantum processes are fundamentally constrained by quantum speed limits, which raises the question on the corresponding bounds on the rate of consumption of asymmetry and coherence. In the present work, we derive a formulation of the quantum speed limit for observables in terms of the trace-norm asymmetry of the time-dependent quantum state relative to the observable. This version of the quantum speed limit can be shown to be directly relevant in weak measurements and quantum metrology. It can be further related to quantum coherence relative to the observable, and we obtain a complementary relation for the speed of three mutually unbiased observables for a single qubit. As an application, we derive a notion of a *quantum thermodynamic speed limit*.

I. INTRODUCTION

The quantum speed limit is a fundamental restriction on the maximum speed with which a quantum system can evolve [1, 2]. Thus, it sets a crucial constraint in different information-processing protocols leveraging quantum dynamics such as quantum computation and communication [3–5], quantum control [6–8], quantum metrology [9, 10], quantum thermodynamics [11–14], and recently in the speed of quantum resources [15–19].

The first example of a genuine quantum speed limit was derived by Mandelstam and Tamm as an operational and a rigorous reformulation of the Heisenberg uncertainty relation for energy and time [20]. Since then, extensive efforts have been made to study its rich aspects from various angles and along different directions [21–40]. The usual approach to define quantum speed is based on the state distinguishability via some notion of distance in state space. Note, however, that there are pairs of quantum states which are perfectly distinguishable in the state space, but their expectation values relative to a physically relevant observable are equal or very close to each other. This observation has led to recent efforts to study quantum speed and its limit based on using the speed of the quantum expectation value of an observable in a state [41–43]. In the present work, we discuss the relation between the speed of expectation value of an observable and quantumness arising from the noncommutativity between the observable and the state.

Consider a quantum system with a time-dependent state $\varrho(t)$ on a Hilbert space $\mathcal{H}$. Then the quantum "speed" associate with an observable K is given by the rate of change of its expectation value [41–43],

$$v_K \equiv \frac{1}{2} \left|\partial_t \langle K\rangle\right|, \tag{1}$$

where for any operator O the angular brackets denote an average over the time-dependent state, $\langle O\rangle = \mathrm{tr}\{O\,\varrho(t)\}$. In the definition (1), we have included a factor $1/2$ for later convenience. Interestingly, this notion of speed is also the starting point for deriving the Mandelstam-Tamm bound [20], where for pure $\varrho(t)$ the observable K is chosen as the projector onto $\varrho(t)$.

For general K and unitary dynamics $U(t)$, with generator $H(t) = -i\,\dot{U}(t)^\dagger U(t)$, it can be shown that [41, 43],

$$v_K \leq \Delta_K\, \Delta_H\,, \tag{2}$$

where $\Delta_O^2 = \langle O^2\rangle - \langle O\rangle^2$ is the variance of observable O. As usual, we are working in units where $\hbar = 1$. This result has recently been rigorously generalized to arbitrary quantum dynamics [41–43].

For our present purposes, we note that the instantaneous quantum speed (2) under unitary dynamics is governed by the pairwise quantum noncommutativity relations between the generator of the dynamics, the probing observable, and the state. Noncommutativity crucially underlies virtually all of quantum physics, such as quantum coherence and asymmetry [44–53], and general quantum correlation including quantum entanglement between two or more parties [48, 54–56]. It is therefore natural to ask if asymmetry, coherence and general quantum correlation can be used to fully characterize v_K.

Yet, the usual formulation of upper bounds on v_K, such as Eq. (2), obscures this issue. While the instantaneous speed must be vanishing when the probing observable K commutes with the state $\varrho(t)$, the upper bound, which depends on the variance of K, can still be finite [57] Therefore, the present work is dedicated to deriving an upper bound, i.e., a version of the quantum speed limit

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that is explicitly determined by the noncommutativity between the probing observable K and the state, $\varrho(t)$. Through such a formulation, we can then directly relate the quantum speed limit for an observable with quantum asymmetry and coherence carried by a quantum state relative to the observable, K.

II. SPEED AND QUANTUM ASYMMETRY

Consider a quantum system undergoing unitary dynamics within a finite-dimensional Hilbert space $\mathcal{H}$. For such scenarios, if infinite resources are available, also the quantum speed can be infinite. In particular, if the spectrum of the Hamiltonian is unbounded, i.e., $\|H(t)\|_\infty = \infty$, the right hand side of Eq. (2) can diverge. We denote the operator norm by $\|H(t)\|_\infty$, which is identical to the Schatten-∞ norm of $H(t)$. Since this case is not particularly interesting, we restrict ourselves to a set of unitaries with a bounded Hamiltonian generator.

For simplicity and without loss of generality we then consider only Hamiltonians, for which $\|H(t)\|_\infty = 1$. To ease the notation, we denote the corresponding quantum speed limit as

$$v_{\mathrm{QSL}} \equiv \sup_{\|H(t)\|_\infty=1} \{v_K\} \,, \tag{3}$$

which is the maximal quantum speed a system with bounded Hamiltonian can achieve.

Inspecting this quantum speed limit (3), we note that the state $\varrho(t)$ can be understood as a quantum object containing a useful quantum resource. This resource can be harnessed using a probing observable K via a protocol or strategy encoded in the unitary evolution $U(t)$ generated by $H(t)$. We then optimize the instantaneous speed or the power of harnessing or injecting the resource content over all the unitary protocols. Interestingly, the quantum ergotropy is defined in a similar spirit as the maximum amount of work that can be extracted using all allowed unitaries [58]. Therefore, v_{QSL} can be interpreted as the optimal power of harnessing or injecting a certain useful resource captured by the expectation value of K in the state $\varrho(t)$ using any allowed unitary protocols, relative to the worst cost of implementing the unitary protocol.

To derive an upper bound on v_{QSL} we first express the instantaneous speed as

$$v_K = \frac{1}{2}\left|\mathrm{tr}\left\{[H(t),\varrho(t)]K\right\}\right| = \frac{1}{2}\left|\mathrm{tr}\left\{H(t),[\varrho(t),K]\right\}\right| \,. \tag{4}$$

The latter can be further bounded by Hölder's inequality [59] which reads $\left|\mathrm{tr}\left\{A\,B^\dagger\right\}\right| \leq \|A\|_p\|B\|_q$ for all nonnegative p and q with $1/p + 1/q = 1$. Here $\|O\|_p \equiv \mathrm{tr}\left\{|O|^p\right\}^{1/p}$, $|O| = \sqrt{OO^\dagger}$, is the Schatten-p norm of O, which can be expressed in terms of the singular values. For instance, the Schatten-1 norm of O, i.e., the trace norm can be written as $\|O\|_1 = \sum_\nu \sigma_\nu$, where σ_n are the singular values of O. Now choosing $A = H(t)$ with $p = \infty$ and $B = [\varrho(t), K]$ with $q = 1$, we immediately have

$$v_K \leq \frac{1}{2}\|H(t)\|_\infty \, \|[\varrho(t),K]\|_1 \,. \tag{5}$$

Thus, for bounded generators with $\|H(t)\|_\infty = 1$, we obtain the quantum speed limit as the trace-norm noncommutativity between the state and the probing observable,

$$v_{\mathrm{QSL}} \leq \frac{1}{2}\,\|[\varrho(t),K]\|_1, \tag{6}$$

Interestingly, as shown in App. A for a single qubit, the inequality becomes an equality where the supremum is attained when the generator $H(t)$ of the unitary dynamics at time t has eigenvalues $\{-1, 1\}$.

Equation (6) is our first main result. The upper bound, that is trace-norm noncommutativity $\|[\varrho(t),K]\|_1/2$, is a bonafide measure of the asymmetry, called trace-norm asymmetry, of the state $\varrho(t)$ relative to the group of unitary translations $\{U_\theta = e^{-iK\theta}\}_\theta$, $\theta \in \mathbb{R}$, generated by the Hermitian operator K [44]. Hence, we observe that the instantaneous speed of the expectation value of the observable K in the state $\varrho(t)$ under unitary dynamics with Hermitian generators having an operator norm unity, is upper bounded by the trace-norm asymmetry in the state $\varrho(t)$ relative to the group of unitary translations generated by K.

Alternatively, Eq. (6) can also be understood as the maximum speed of harnessing a feature captured by the expectation value of the probing observable K in the state $\varrho(t)$ using any unitary protocol, relative to the worst cost of implementing the unitary. This quantum speed is upper bounded by the quantum asymmetry in the state relative to the group of translation unitaries generated by the probing observable K.

It is also interesting to point out that related results have appeared in the literature. In particular, Ref. [44] derived a quantum speed limit for unitary dynamics in terms of the trace-norm asymmetry $\|[\varrho(t),H]\|_1/2$ relative to the generator of the unitary, $H(t)$. A similar result was obtained in Ref. [60], where the asymmetry is quantified by the Wigner-Yanese skew information [61]. In our result (6) the quantum asymmetry is determined relative to a specific observable K, which is *not* the generator of the dynamics. In the following, we will see that our formulation of the quantum speed limit is, thus, particularly useful in quantum thermodynamic considerations.

A. Quantum speed from weak measurements

We begin by discussing the operational, statistical, and information theoretical meaning of our formulation of the quantum speed limit for observables (6). In previous work [47, 48], it was shown that the trace-norm asymmetry in the state $\varrho(t)$ relative to the group of translation unitaries generated by K can be expressed variationally as

$$\frac{1}{2}\,\|[K,\varrho(t)]\|_1 = \sup_{\mathbb{B}_o(\mathcal{H})}\left\{\sum_x |\Im\{K_{\mathrm{w}}(x|\varrho(t))\}|\;\Pr(x|\varrho(t))\right\}, \tag{7}$$

where we have introduced the weak value of K with the preselected state $\varrho(t)$ and postselected state $|x\rangle$ [62, 63],

$$K_{\mathrm{w}}(x|\varrho(t)) := \frac{\langle x|K\varrho(t)|x\rangle}{\langle x|\varrho(t)|x\rangle}\,. \tag{8}$$

Moreover, $\Pr(x|\varrho(t)) = \operatorname{tr}\{\Pi_x\,\varrho(t)\}$ is the probability to obtain the outcome x in a measurement described by the rank-1 orthogonal PVM (projection-valued measure) $\{\Pi_x := |x\rangle\langle x|\}$. The supremum is taken over all $\{|x\rangle\} \in \mathbb{B}_o(\mathcal{H})$, which is the set of all orthonormal bases of $\mathcal{H}$.

Therefore, we also have

$$v_{\mathrm{QSL}} \le \sup_{\mathbb{B}_o(\mathcal{H})}\left\{\sum_x |\Im\{K_{\mathrm{w}}(x|\varrho(t))\}|\;\Pr(x|\varrho(t))\right\}, \tag{9}$$

and hence the quantum speed is experimentally accessible. In fact, the weak value $K_{\mathrm{w}}(x|\varrho(t))$ can be directly observed in experiments, either using a weak measurement followed by a strong projection measurement [62–65], or using other methods without using weak measurements [66–70].

Hence, both sides of Eq. (9) can be in principle inferred directly in experiment, whilst involving optimization, respectively, over all unitaries on and all orthonormal bases of the Hilbert space. Such optimizations can be implemented using the presently available NISQ hardware using hybrid quantum-classical variational quantum algorithm [71]. Recently, the imaginary part of the weak value has been argued to indicate quantum contextuality via weak measurement with postselection [72]. In this sense, Eq. (9) thus shows that the quantum contextuality is necessary for a nonvanishing quantum speed of expectation value, and conversely, a nonvanishing quantum speed of expectation value is sufficient to indicate quantum contextuality. The imaginary part of the weak value $K_{\mathrm{w}}(x|\varrho(t))$ can also be interpreted as a disturbance due to the weak measurement of K [73, 74]. Hence, according to Eq. (9), the sensitivity of the state due to weak measurement of K gives an upper bound to the speed of the expectation value of K under unitary dynamics.

B. Quantum metrological consequences

Our quantum speed limit for observables (6) can also be directly related to the precision with which the values of K can be estimated in quantum measurements. In fact, in Ref. [47] it was shown that the trace-norm asymmetry is upper bounded by

$$\|[\varrho(t),K]\|_1 \le \sqrt{\mathcal{F}_\theta(\varrho(t),K)}\,, \tag{10}$$

where $\mathcal{F}_\theta(\varrho(t),K)$ is the quantum Fisher information [75] about a parameter θ that is imprinted onto the quantum state $\varrho(t)$ by a unitary translation generated by K, $\varrho_\theta(t) = \exp(-iK\theta)\varrho(t)\exp(iK\theta)$. Note that the inequality in Eq. (10) becomes an equality for pure states and for all states of a single qubit.

Accordingly, our quantum speed limit (6) can also be written as

$$v_{\mathrm{QSL}} \le \frac{1}{2}\sqrt{\mathcal{F}_\theta(\varrho(t),K)}\,. \tag{11}$$

Thus, we conclude that the quantum speed relative to the observable K is upper bounded by the amount of information that can be extracted about a parameter θ conjugate to K.

It has been well established in the literature [32] that $\sqrt{\mathcal{F}_\theta(\varrho(t),K)}/2$ is the natural, geometric quantum speed on the θ-parameterized manifold of states. In fact, this is the maximum quantum speed any dynamics can achieve [37]. We have now shown that the maximum instantaneous speed of the observable K for any unitary dynamics with bounded generators is always lower than the speed of state evolution induced by the unitary generated by K.

However, while our v_{QSL} is the speed associated with the unitary dynamics, the upper bound is the speed in the parameter θ conjugate to K induced by the unitary translation $\exp(-iK\theta)$. In fact, Eq. (11) can be read as a relation between the speed of quantum evolution and the rate with which a parameter θ can be measured.

C. Quantum speed limit and quantum fluctuations

Our quantum speed limit (6) can also be directly related to established formulations for observables (2). In fact, we have

$$\frac{1}{2}\|[K,\varrho(t)]\|_1 \le \Delta_K, \tag{12}$$

where the inequality again becomes an equality for pure states [44]. When K and $\varrho(t)$ commute, the left-hand side vanishes while the right-hand side may remain finite. This fact suggests that the trace-nom asymmetry $\|[K,\varrho(t)]\|/2$ can be understood as the genuine quantum part of the fluctuations of K, while Δ_K quantifies the total (quantum and classical) fluctuations of K relative to $\varrho(t)$ [53, 76–78].

Noting this, Eq. (6) can be interpreted that the maximum speed of the quantum expectation value of an observable K in the state $\varrho(t)$, under any unitary dynamics with a Hermitian generator having operator norm unity, is upper bounded by the genuine quantum fluctuations of K in $\varrho(t)$. By contrast, in classical stochastic mechanics, the rate of change of the average of a random variable is not fundamentally restricted by the fluctuations of the variable.

D. Quantum speed limit time

So far we have focused on the quantum speed limit, i.e., on upper bounds on the rate with which an expectation value can change. Associated with any maximal rate is always a minimal time, which in the present case corresponds to the minimal time that is required to observe a predetermined change in the expectation value of K.

To this end, we now need to specify the parametrization of the time-dependent Hamiltonian generating the unitary dynamics. Thus, we write $H[\gamma(t)]$, where $\gamma(t)$ is a time-dependent protocol. The corresponding time-evolution then becomes

$$U[\gamma(t)] = \mathcal{T}_> \exp\left(-i\int^t \mathrm{d}t'\, H[\gamma(t')]\right), \tag{13}$$

where $\mathcal{T}_>$ is the time ordering operator.

Integrating Eq. (1) and using Eqs. (3) and (6) we obtain that for a change in expectation value $\Delta K_\tau = |\langle K\rangle_\tau - \langle K\rangle_0|$ in time τ the quantum system needs at least

$$\tau \geq \frac{\Delta K_\tau}{\overline{\|[\varrho[\gamma], K]\|_1}} \equiv \tau_{\mathrm{QSL}}\,, \tag{14}$$

where $\overline{f[\gamma]} = 1/\tau \int_0^\tau \mathrm{d}t\, f[\gamma(t)]$ is the time average of the function $f(t)$ along the trajectory $\gamma(t)$.

In other words, given a unitary dynamics $U[\gamma(t)]$ with Hermitian generator $H[\gamma(t)]$ the minimum time needed to transform an initial state $\varrho(0)$ to a final state $\varrho(\tau)$ inducing a difference between the initial and final expectation value of the probing observable ΔK_τ, is inversely proportional to the time-average trace-norm asymmetry $\overline{\|[\varrho[\gamma], K]\|_1/2}$ of the state $\varrho(t)$ relative to the translation unitaries generated by K along the trajectory. In particular, the larger the time-average trace-norm asymmetry generated along the trajectory, the smaller is the minimum time needed to get a targeted change of expectation value of the observable using the unitary transformation.

The obvious question then is whether there is an optimal choice for the parameterization $\gamma(t)$. Indeed, this simply becomes an optimal control problem for $\gamma(t)$, in the sense that the optimal $\gamma(t)$ is the protocol that minimizes the lower bound in Eq. (14).

Moreover, it is worth emphasizing that Eq. (14) holds for any choice of K. In particular, also for K that commute with the quantum state $\varrho(t)$. Hence, one can choose K which solves the variational equation $\|\varrho(\tau) - \varrho(0)\|_1/2 = \sup_{0\leq K\leq \mathbb{I}} \mathrm{tr}\left\{K(\varrho(\tau) - \varrho(0))\right\}$, in which case Eq. (14) becomes the minimal time required for a quantum state $\varrho(0)$ to evolve to a another state $\varrho(\tau)$.

III. COHERENCE AND COMPLEMENTARITY

Our quantum speed limit for observables (6) can also be directly related to measures of coherence. In this context, recall that the asymmetry in the state $\varrho(t)$ relative to the group of unitary translations generated by a Hermitian observable K has been dubbed *unspeakable coherence* relative to the orthonormal eigenbasis $\{|k\rangle\}$ of K [44, 47, 48].

Thus, it is interesting to analyze v_{QSL} (6) for specific choices of $\{|k\rangle\}$. To this end, we now consider bounded K, where again without loss of generality we can set $\|K\|_\infty = 1$. In such cases, it was shown in Ref. [79] that

$$\sup_K \left\{\frac{\|[\varrho(t), K]\|}{2\,\|K\|_\infty}\right\} \leq C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho(t); \{|k\rangle\}) \tag{15}$$

where $C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho(t); \{|k\rangle\})$ is a quantifier of coherence in the state $\varrho(t)$ relative to the orthonormal basis $\{|k\rangle\}$ defined as [52]

$$C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho(t); \{|k\rangle\}) := \sum_k \sup_{\mathbb{B}_{\mathrm{o}}(\mathcal{H})} \sum_c |\mathrm{Im}\{\mathrm{Pr}_{\mathrm{KD}}(k, c|\varrho(t))\}|, \tag{16}$$

and $\mathrm{Pr}_{\mathrm{KD}}(k, c|\varrho(t)) := \langle c|k\rangle \langle k|\varrho(t)|c\rangle$ is the Kirkwood-Dirac (KD) quasiprobability associated with the state $\varrho(t)$ relative to two orthonormal bases $\{|k\rangle\}$ and $\{|c\rangle\}$ of $\mathcal{H}$ [80, 81].

Accordingly, we also have that

$$v_{\mathrm{QSL}} \leq C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho(t); \{|k\rangle\})\,. \tag{17}$$

Thus, the quantum speed limit for observables is upper bounded by the coherence encoded in $\varrho(t)$ relative to bounded K. Interestingly, for single qubits the inequality in Eq. (17) becomes an equality when the two eigenvalues $\{k_+, k_-\}$ of K satisfies $k_+ = -k_-$.

Next, as shown in Ref. [52], the KD-nonreality coherence (16) is upper bounded as

$$C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho(t); \{|k\rangle\}) \leq C_{l_1}(\varrho(t); \{|k\rangle\}), \tag{18}$$

where, $C_{l_1}(\varrho(t); \{|k\rangle\})$ is the l_1-norm coherence in the state $\varrho(t)$ relative to the orthonormal basis $\{|k\rangle\}$ which is defined as $C_{l_1}(\varrho(t); \{|k\rangle\}) := \sum_{k\neq k'} |\langle k|\varrho(t)|k'\rangle|$ [50]. Moreover, for any state of a single qubit, the inequality (18) again becomes an equality.

A. Example: quantum speed of single qubits

Since we have mentioned several times above that the inequalities become tight for single qubits, we pause to illustrate and demonstrate our findings for this scenario. To this end, consider two unit vectors $\vec{n}_K \in \mathbb{R}^3$ and $\vec{n}_U \in \mathbb{R}^3$, and the Pauli vector $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. In this case, the quantum state is written in Bloch representation as $\varrho(t) = \frac{1}{2}(\mathbb{I} + \vec{r}_{\mathrm{S}} \cdot \vec{\sigma})$, and the normalized observable is $K = \vec{n}_K \cdot \vec{\sigma}$. Note that it is easy to see that the eigenvalues of K are $\{1, -1\}$, and hence Eqs. (17) and (18) become equalities.

Thus, we can also find a simple relation for the minimum time (14) for the quantum system to achieve a

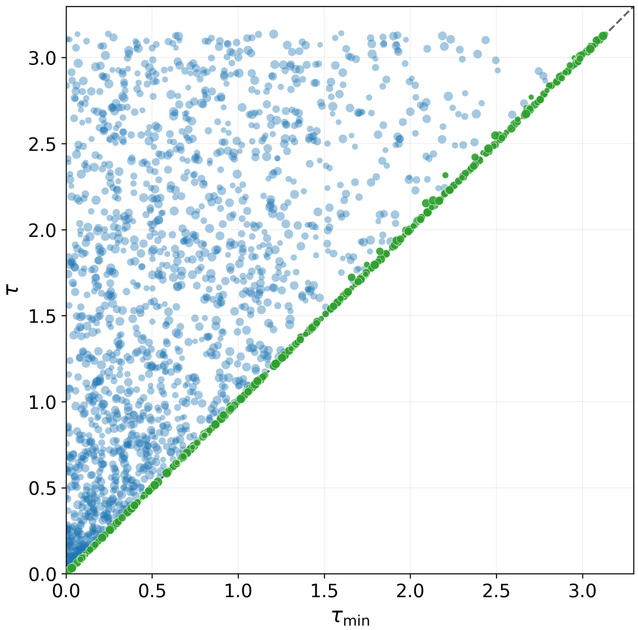

FIG. 1. Realizations of the a quantum process (blue markers), with random initial states and unitary maps. The size of the marker represents the purity of the initial states. Green markers depict realizations that are optimal.

change in expectation value. We have,

$$\tau \geq \frac{|\mathrm{tr}\left\{(\vec{n}_K \cdot \vec{\sigma})\varrho(\tau)\right\} - \mathrm{tr}\left\{(\vec{n}_K \cdot \vec{\sigma})\varrho(0)\right\}|/2}{\overline{C_{l_1}(\varrho[\gamma]; \{|k\rangle\})}} \equiv \tau_{\mathrm{min}}\,, \tag{19}$$

which follows from using the equality of Eqs. (17) and (18) in the expression (14). As mentioned above, this inequality (19) becomes time, for the optimal protocol $\gamma(t)$ determined from solving the corresponding optimal control problem.

This is illustrated for a numerical solution of the problem in Fig. 1. To this end, we randomly chose initial states and unitary maps. Specifically, for each trial we randomly sampled $\vec{n}_K$, $\vec{n}_U(0)$ and $\vec{r}_S(0)$, solve the von-Neumann equation for sufficiently short interval of time, and vary the control unitary by changing randomly $\vec{n}_U(t)$ for each time step. We computed the difference between the expectation value of $\vec{n}_K \cdot \vec{\sigma}$ and the time-average l_1-norm coherence relative to the eigenbasis of $\vec{n}_K \cdot \vec{\sigma}$ to obtain τ_{min} (19). Observe, that all the trials satisfy the inequality. The green markers depict realizations that saturate the lower bound is obtained by randomly choosing $\vec{n}_U(t)$ that are orthogonal to the plane defined by $\vec{n}_{\mathrm{O}}$ and $\vec{r}_{\mathrm{S}}(t)$ for all $t \in [0, \tau]$, in case of which the maximum speed is attained.

B. Complementarity bounds for qubits

Equation (17) further implies the following complementarity relations for the quantum speed of three observables of a single qubit whose eigenbases comprise a set of mutually unbiased bases (MUB) of the two-dimensional Hilbert space $\mathcal{C}^2$ [82–84]. First, consider three mutually complementary Pauli operators, e.g., σ_x, σ_y and σ_z. Note that the eigenbases of σ_x, σ_y and σ_z, respectively, $\mathbb{X} := \{|x_+\rangle, |x_-\rangle\}$, $\mathbb{Y} := \{|y_+\rangle, |y_-\rangle\}$ and $\mathbb{Z} = \{|z_+\rangle, |z_-\rangle\}$, where $|x_\pm\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle \pm |z_-\rangle)$ and $|y_\pm\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle \pm i\,|z_-\rangle)$, are mutually unbiased with each other. Moreover, in this case, the l_1-norm coherence in the state $\varrho(t)$ relative to these MUB bases satisfy the following complementarity relation [85],

$$C_{l_1}(\varrho(t); \mathbb{X})^2 + C_{l_1}(\varrho(t); \mathbb{Y})^2 + C_{l_1}(\varrho(t); \mathbb{Z})^2 = 2|\vec{r}_{\mathrm{S}}|^2. \tag{20}$$

Thus, we can also write,

$$v^{\mathbb{X}}_{\mathrm{QSL}} + v^{\mathbb{Y}}_{\mathrm{QSL}} + v^{\mathbb{Z}}_{\mathrm{QSL}} = 8\left[2P(t) - 1\right]\,, \tag{21}$$

where $P(t) := \mathrm{tr}\left\{\varrho(t)^2\right\} = (1 + |\vec{r}_{\mathrm{S}}|^2)/2$ is the purity.

As a simpler but weaker form of a complementarity relation for our quantum speed limit, we note that as shown in Ref. [86], the l_1-norm coherences in the state relative to the orthonormal bases $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$ satisfy a complementarity relation which is independent of the state purity,

$$C_{l_1}(\varrho(t); \mathbb{X}) + C_{l_1}(\varrho(t); \mathbb{Y}) + C_{l_1}(\varrho(t); \mathbb{Z}) \leq \sqrt{6}, \tag{22}$$

and the inequality becomes an equality for a specific pure state of the form $\varrho(t) = \frac{1}{2}[\mathbb{I} + \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z)]$. Thus, we obtain,

$$v^{\mathbb{X}}_{\mathrm{QSL}} + v^{\mathbb{Y}}_{\mathrm{QSL}} + v^{\mathbb{Z}}_{\mathrm{QSL}} \leq 2\sqrt{6}, \tag{23}$$

The above results in particular shows that for any state $\varrho(t)$ the maximum speed of the expectation value of σ_x, σ_y and σ_z cannot all reach their speed limit.

IV. RELATIVE ENTROPY OF ATHERMALITY

We conclude the discussion with an application of our quantum speed limit for observables (6) in quantum thermodynamics. So far our analysis has been general for any K. In the remainder, we will now identify a specific observable K that is of particular thermodynamic relevance.

Arguably, the central quantity of quantum thermodynamic is the entropy production. Consider a quantum system that is initially prepared in equilibrium and then undergoes a unitary, non-equilibrium process. In this case, the nonequilibrium entropy production is [11, 87, 88]

$$\Sigma = S(\varrho(\tau)||\sigma)\,, \tag{24}$$

where σ is the thermal Gibbs state corresponding to the final reference Hamiltonian H_b, namely $\sigma = \exp\left(-\beta H_b)\right)/Z$ and β is the inverse temperature.

In Eq. (24) the entropy production is expressed as the quantum relative entropy [89], which reads explicitly

$$S(\varrho(\tau)||\sigma) = \mathrm{tr}\left\{\varrho(\tau)\ln(\varrho(\tau))\right\} - \mathrm{tr}\left\{\varrho(\tau)\ln(\sigma)\right\} \,. \quad (25)$$

Note that the relative entropy is a measure of distinguishability of quantum states [89].

Interestingly, Gibbs states are also so-called passive states [90] that carry neither classical nor quantum correlations [91]. Thus, in a certain sense, the entropy production Σ measures the "athermality" of a quantum state $\varrho(t)$ relative to a reference Hamiltonian.

For our present purposes, it is even more interesting that the rate of change of Σ was suggested already by Spohn [92] as the natural notion of entropy production rate. In particular, we have for unitary dynamics

$$\sigma = \dot{\Sigma} = -\mathrm{tr}\left\{\dot{\varrho(t)}\ln(\sigma)\right\} \,. \quad (26)$$

Thus, we identify $K = -\ln(\sigma)$ as the thermodynamically motivated observable.

For this case, our quantum speed limit for observables (6) becomes

$$v_{\mathrm{QSL}} \leq \frac{1}{2}\,\|[\varrho(t), \ln(\sigma)]\|_1 = \frac{\beta}{2}\,\|[\varrho(t), H_b]\|_1 \,, \quad (27)$$

which is strongly reminiscent of quantum thermodynamic formulations of the Bremerman-Bekenstein bound [5, 11] and thermodynamically relevant versions of the quantum speed limit [93]. Therefore, Eq. (27) can also be interpreted as another version of the *quantum thermodynamic speed limit*. Interestingly, we also observe that in the high-temperature, semiclassical limit $\beta \ll 1$ the quantum speed limit vanishes. Note that the quantum speed limit (27) for unitary, isolated processes is measured relative to a thermal equilibrium state. Hence, a vanishing speed in the high temperature limit means nothing but that relative to fully mixed state the quantum speed vanishes.

V. CONCLUDING REMARKS

In the present work, we have derived a quantum speed limit for observables in terms of the quantum asymmetry of the quantum state relative to the observable. This quantum speed limit can be determined from weak quantum measurements, is metrologically relevant and directly related to the coherence encoded in the time-dependent quantum state. Interestingly, this novel formulation of the quantum speed limit can also be expressed as a quantum thermodynamic speed limit for the nonequilibrium entropy production.

Our results strengthen the intuition that quantum asymmetry and coherence are important quantum resources. In future work, it will be interesting to investigate whether other measures of general quantum correlation and entanglement can also be leveraged to derive upper bounds on quantum speed. In particular, it will be interesting to study whether asymmetry, coherence, and entanglement arise in other notions of speed, such as speed of basis rotation [94] which are crucial in different protocols of quantum technology.

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Appendix A: Quantum speed limit for qubits

In this appendix, we show explicitly that the quantum speed limit (6) becomes tight for qubits. To this end, consider the observable K on two-dimensional Hilbert space $\mathbb{C}^2$ with the spectral decomposition,

$$K = k_+ \left|k_+\right\rangle \left\langle k_+\right| + k_- \left|k_-\right\rangle \left\langle k_-\right| , \quad (A1)$$

where $k_+, k_- \in \mathbb{R}$, and the eigenvectors $\{\left|k_+\right\rangle, \left|k_-\right\rangle\}$ comprises an orthonormal basis of $\mathbb{C}^2$.

Generally, the Hermitian generator $H(t)$ of the unitary on $\mathbb{C}^2$ can be written as

$$H(t) = h_+ \left|h_+\right\rangle \left\langle h_+\right| + h_- \left|h_-\right\rangle \left\langle h_-\right| . \quad (A2)$$

Here, we have omitted the trivial dependence on t. The eigenstates, $\left|h_\pm\right\rangle$, depend on the polar $\alpha \in [0, \pi]$ and the azimuthal $\beta \in [0, 2\pi)$ angles of the Bloch sphere, and $\{h\} = \{h_+, h_-\}$, $h_+, h_- \in \mathbb{R}$, are the corresponding eigenvalues.

Thus, we can also write

$$\begin{aligned} \left|h_+\right\rangle &= \cos\left(\alpha/2\right)\,\left|k_+\right\rangle + e^{i\beta}\sin\left(\alpha/2\right)\,\left|k_-\right\rangle \\ \left|h_-\right\rangle &= \sin\left(\alpha/2\right)\,\left|k_+\right\rangle - e^{i\beta}\cos\left(\alpha/2\right)\,\left|k_-\right\rangle . \end{aligned} \quad (A3)$$

We can assume, without loss of generality, that $|h_+| > |h_-|$ so that $\|H(t)\|_\infty = |h_+| = 1$. Then, computing the left-hand side of the inequality in Eq. (6), we obtain

$$
\begin{aligned}
v_{\mathrm{QSL}} =& \max_{\{(h_+,h_-)\},\{(\alpha,\beta)\}} \left\{ \frac{|h_+ - h_-|}{2|h_+|} \left|k_+ - k_-\right| \left|\langle k_-|\varrho|k_+\rangle\right| \left|\sin(\alpha)\sin(\beta+\phi)\right| \right\} \\
=& |k_+ - k_-| \left|\langle k_+|\varrho|k_-\rangle\right| = \|[\varrho, K]\|_1/2.
\end{aligned} \tag{A4}
$$

Here, $\phi = \arg\langle k_+|\varrho|k_-\rangle$, and we have used the fact that $|h_+| = \|H(t)\|_\infty = 1$. The maximum over angular variables $\{(\alpha,\beta) \in [0,\pi] \times [0,2\pi)\}$ is attained for $\alpha = \pi/2$ and $\beta = \pi/2 - \phi$, and the maximum over $\{(h_+,h_-) \in \mathbb{R}^2 \| h_-| \leq |h_+| = 1\}$ is attained when $h_- = -h_+$. This means that the generator $H(t)$ of the unitary must have eigenvalues $\{-1,1\}$, and the associated pair of orthogonal eigenvectors must lie on the xy plane of the Bloch sphere. The last equality can be checked straightforwardly by using the spectral decomposition of K.

[1] M. R. Frey, Quantum speed limits—primer, perspectives, and potential future directions, Quantum Inf. Process. **15**, 3919 (2016).

[2] S. Deffner and S. Campbell, Quantum speed limits: from heisenberg's uncertainty principle to optimal quantum control, J. Phys. A: Math. Theor. **50**, 453001 (2017).

[3] S. Lloyd, Ultimate physical limits to computation, Nature **406**, 1047 (2000).

[4] S. Ashhab, P. C. de Groot, and F. Nori, Speed limits for quantum gates in multiqubit systems, Phys. Rev. A **85**, 052327 (2012).

[5] S. Deffner, Quantum speed limits and the maximal rate of information production, Phys. Rev. Res. **2**, 013161 (2020).

[6] T. Caneva, M. Murphy, T. Calarco, R. Fazio, S. Montangero, V. Giovannetti, and G. E. Santoro, Optimal control at the quantum speed limit, Phys. Rev. Lett. **103**, 240501 (2009).

[7] S. Campbell and S. Deffner, Trade-off between speed and cost in shortcuts to adiabaticity, Phys. Rev. Lett. **118**, 100601 (2017).

[8] M. Aifer and S. Deffner, From quantum speed limits to energy-efficient quantum gates, New J. Phys. **24**, 055002 (2022).

[9] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nature Photon. **5**, 222 (2011).

[10] S. Campbell, M. G. Genoni, and S. Deffner, Precision thermometry and the quantum speed limit, Quantum Sci. Technol. **3**, 025002 (2018).

[11] S. Deffner and E. Lutz, Generalized clausius inequality for nonequilibrium quantum processes, Phys. Rev. Lett. **105**, 170402 (2010).

[12] C. Mukhopadhyay, A. Misra, S. Bhattacharya, and A. K. Pati, Quantum speed limit constraints on a nanoscale autonomous refrigerator, Phys. Rev. E **97**, 062116 (2018).

[13] K. Funo, N. Shiraishi, and K. Saito, Speed limit for open quantum systems, New J. Phys. **21**, 013006 (2019).

[14] S. Deffner, Towards enhanced precision in thermometry with nonlinear qubits, Quantum Sci. Technol. **10**, 025009 (2025).

[15] B. Mohan, S. Das, and A. K. Pati, Quantum speed limits for information and coherence, New J. Phys. **24**, 065003 (2022).

[16] F. Campaioli, C.-s. Yu, F. A. Pollock, and K. Modi, Resource speed limits: maximal rate of resource variation, New J. Phys. **24**, 065001 (2022).

[17] D. Allan, N. Hörnedal, and O. Andersson, Time-optimal quantum transformations with bounded bandwidth, Quantum **5**, 462 (2021).

[18] S. Silva Pratapsi, S. Deffner, and S. Gherardini, Quantum speed limit for kirkwood–dirac quasiprobabilities, Quantum Sci. Technol. **10**, 035019 (2025).

[19] I. Marvian and R. W. Spekkens, How to quantify coherence: Distinguishing speakable and unspeakable notions, Phys. Rev. A **94**, 052324 (2016).

[20] L. Mandelstam and I. Tamm, The uncertainty relation between energy and time in nonrelativistic quantum mechanics, J. Phys. **9**, 249 (1945).

[21] K. Bhattacharyya, Quantum decay and the mandelstam-tamm-energy inequality, J. Phys. A: Math. Gen. **16**, 2993 (1983).

[22] G. N. Fleming, A unitarity bound on the evolution of nonstationary states, Nuov. Cim. A **16**, 232 (1973).

[23] J. Anandan and Y. Aharonov, Geometry of quantum evolution, Phys. Rev. Lett. **65**, 1697 (1990).

[24] A. K. Pati, Relation between "phases" and "distance" in quantum evolution, Phys. Lett. A **159**, 105 (1991).

[25] A. Uhlmann, An energy dispersion estimate, Phys. Lett. A **161**, 329 (1992).

[26] L. Vaidman, Minimum time for the evolution to an orthogonal quantum state, Am. J. Phys. **60**, 182 (1992).

[27] J. Uffink, The rate of evolution of a quantum state, Am. J. Phys. **61**, 935 (1993).

[28] N. Margolus and L. B. Levitin, The maximum speed of dynamical evolution, Physica D **120**, 188 (1998), proceedings of the Fourth Workshop on Physics and Consumption.

[29] J. Kupferman and B. Reznik, Entanglement and the speed of evolution in mixed states, Phys. Rev. A **78**, 042305 (2008).

[30] L. B. Levitin and T. Toffoli, Fundamental limit on the rate of quantum dynamics: The unified bound is tight, Phys. Rev. Lett. **103**, 160502 (2009).

[31] P. J. Jones and P. Kok, Geometric derivation of the quantum speed limit, Phys. Rev. A **82**, 022107 (2010).

[32] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Quantum speed limit for physical processes, Phys. Rev. Lett. **110**, 050402 (2013).

[33] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Quantum speed limits in open system dynamics, Phys. Rev. Lett. **110**, 050403 (2013).

[34] S. Deffner and E. Lutz, Quantum speed limit for non-

markovian dynamics, Phys. Rev. Lett. **111**, 010402 (2013).

[35] D. P. Pires, M. Cianciaruso, L. C. Céleri, G. Adesso, and D. O. Soares-Pinto, Generalized geometric quantum speed limits, Phys. Rev. X **6**, 021031 (2016).

[36] F. Campaioli, F. A. Pollock, F. C. Binder, and K. Modi, Tightening quantum speed limits for almost all states, Phys. Rev. Lett. **120**, 060409 (2018).

[37] E. O'Connor, G. Guarnieri, and S. Campbell, Action quantum speed limits, Phys. Rev. A **103**, 022210 (2021).

[38] P. M. Poggi, S. Campbell, and S. Deffner, Diverging quantum speed limits: A herald of classicality, PRX Quantum **2**, 040349 (2021).

[39] R. Hamazaki, Speed limits for macroscopic transitions, PRX Quantum **3**, 020319 (2022).

[40] S. Deffner, Nonlinear speed-ups in ultracold quantum gases, EPL (Europhys. Lett.) **140**, 48001 (2022).

[41] B. Mohan and A. K. Pati, Quantum speed limits for observables, Phys. Rev. A **106**, 042436 (2022).

[42] L. P. García-Pintos, S. B. Nicholson, J. R. Green, A. del Campo, and A. V. Gorshkov, Unifying quantum and classical speed limits on observables, Phys. Rev. X **12**, 011038 (2022).

[43] D. Shrimali, B. Panda, and A. K. Pati, Stronger speed limit for observables: Tighter bound for the capacity of entanglement, the modular hamiltonian, and the charging of a quantum battery, Phys. Rev. A **110**, 022425 (2024).

[44] I. Marvian, R. W. Spekkens, and P. Zanardi, Quantum speed limits, coherence, and asymmetry, Phys. Rev. A **93**, 052331 (2016).

[45] G. Gour and R. W. Spekkens, The resource theory of quantum reference frames: manipulations and monotones, New J. Phys. **10**, 033023 (2008).

[46] J. A. Vaccaro, F. Anselmi, H. M. Wiseman, and K. Jacobs, Tradeoff between extractable mechanical work, accessible entanglement, and ability to act as a reference system, under arbitrary superselection rules, Phys. Rev. A **77**, 032114 (2008).

[47] A. Budiyono, Operational interpretation and estimation of quantum trace-norm asymmetry based on weak-value measurement and some bounds, Phys. Rev. A **108**, 012431 (2023).

[48] A. Budiyono, M. K. Agusta, B. E. B. Nurhandoko, and H. K. Dipojono, Quantum coherence as asymmetry from complex weak values, J. Phys. A: Math. Theor. **56**, 235304 (2023).

[49] A. Streltsov, G. Adesso, and M. B. Plenio, Colloquium: Quantum coherence as a resource, Rev. Mod. Phys. **89**, 041003 (2017).

[50] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, Phys. Rev. Lett. **113**, 140401 (2014).

[51] D. Girolami, Observable measure of quantum coherence in finite dimensional systems, Phys. Rev. Lett. **113**, 170401 (2014).

[52] A. Budiyono and H. K. Dipojono, Quantifying quantum coherence via kirkwood-dirac quasiprobability, Phys. Rev. A **107**, 022408 (2023).

[53] A. Budiyono, Separation of measurement uncertainty into quantum and classical parts based on kirkwood–dirac quasiprobability and generalized entropy, J. Phys. A: Math. Theor. **57**, 465303 (2024).

[54] G. Adesso, T. R. Bromley, and M. Cianciaruso, Measures and applications of quantum correlations, J. Phys. A: Math. Theor. **49**, 473001 (2016).

[55] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[56] A. Budiyono, Quantum entanglement as an extremal kirkwood-dirac nonreality, Phys. Rev. A **111**, 012216 (2025).

[57] This is the case when the state $\varrho(t)$ is a classical statistical mixture of the eigenbasis $\{|k\rangle\}$ of K, i.e., $\varrho(t) = \sum_k p_k |k\rangle \langle k|$, for some set of normalized probabilities $\{p_k\}$, $p_k \geq 0$, $\sum_k p_k = 1$.

[58] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, EPL (Europhys. Lett.) **67**, 565 (2004).

[59] B. Baumgartner, An inequality for the trace of matrix products, using absolute values, arXiv preprint arXiv:1106.6189 10.48550/arXiv.1106.6189 (2011).

[60] D. Mondal, C. Datta, and S. Sazim, Quantum coherence sets the quantum speed limit for mixed states, Phys. Lett. A **380**, 689 (2016).

[61] E. P. Wigner and M. M. Yanase, Information contents of distributions, PNAS **49**, 910 (1963).

[62] Y. Aharonov, D. Z. Albert, and L. Vaidman, How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100, Phys. Rev. Lett. **60**, 1351 (1988).

[63] H. M. Wiseman, Weak values, quantum trajectories, and the cavity-qed experiment on wave-particle correlation, Phys. Rev. A **65**, 032111 (2002).

[64] J. Lundeen and K. Resch, Practical measurement of joint weak values and their connection to the annihilation operator, Phys. Lett. A **334**, 337 (2005).

[65] R. Jozsa, Complex weak values in quantum measurement, Phys. Rev. A **76**, 044103 (2007).

[66] L. M. Johansen, Reconstructing weak values without weak measurements, Phys. Lett. A **366**, 374 (2007).

[67] G. Vallone and D. Dequal, Strong measurements give a better direct measurement of the quantum wave function, Phys. Rev. Lett. **116**, 040502 (2016).

[68] E. Cohen and E. Pollak, Determination of weak values of quantum operators using only strong measurements, Phys. Rev. A **98**, 042112 (2018).

[69] R. Wagner, Z. Schwartzman-Nowik, I. L. Paiva, A. Te'eni, A. Ruiz-Molero, R. S. Barbosa, E. Cohen, and E. F. Galvão, Quantum circuits for measuring weak values, kirkwood–dirac quasiprobability distributions, and state spectra, Quantum Sci. Technol. **9**, 015030 (2024).

[70] G. Chiribella, K. Simonov, and X. Zhao, Dimension-independent weak value estimation via controlled swap operations, Phys. Rev. Res. **6**, 043043 (2024).

[71] M. Cerezo, A. Arrasmith, R. Babbush, S. C. Benjamin, S. Endo, K. Fujii, J. R. McClean, K. Mitarai, X. Yuan, L. Cincio, and P. J. Coles, Variational quantum algorithms, Nat. Rev. Phys. **3**, 625 (2021).

[72] R. Kunjwal, M. Lostaglio, and M. F. Pusey, Anomalous weak values and contextuality: Robustness, tightness, and imaginary parts, Phys. Rev. A **100**, 042116 (2019).

[73] Y. Aharonov and A. Botero, Quantum averages of weak values, Phys. Rev. A **72**, 052111 (2005).

[74] J. Dressel and A. N. Jordan, Significance of the imaginary part of the weak value, Phys. Rev. A **85**, 012107 (2012).

[75] S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, Phys. Rev. Lett. **72**, 3439 (1994).

[76] S. Luo, Heisenberg uncertainty relation for mixed states, Phys. Rev. A **72**, 042110 (2005).
[77] K. Korzekwa, M. Lostaglio, D. Jennings, and T. Rudolph, Quantum and classical entropic uncertainty relations, Phys. Rev. A **89**, 042122 (2014).
[78] M. J. W. Hall, Asymmetry and tighter uncertainty relations for rényi entropies via quantum-classical decompositions of resource measures, Phys. Rev. A **107**, 062215 (2023).
[79] A. Budiyono, Sufficient conditions, lower bounds, and trade-off relations for quantumness in kirkwood-dirac quasiprobability, Phys. Rev. A **109**, 062405 (2024).
[80] J. G. Kirkwood, Quantum statistics of almost classical assemblies, Phys. Rev. **44**, 31 (1933).
[81] P. A. M. Dirac, On the analogy between classical and quantum mechanics, Rev. Mod. Phys. **17**, 195 (1945).
[82] I. D. Ivonovic, Geometrical description of quantal state determination, J. Phys. A: Math. Gen. **14**, 3241 (1981).
[83] W. K. Wootters, Quantum mechanics without probability amplitudes, Found. Phys. **16**, 391 (1986).
[84] W. K. Wootters and B. D. Fields, Optimal state-determination by mutually unbiased measurements, Annals of Physics **191**, 363 (1989).
[85] S. Cheng and M. J. W. Hall, Complementarity relations for quantum coherence, Phys. Rev. A **92**, 042101 (2015).
[86] D. Mondal, T. Pramanik, and A. K. Pati, Nonlocal advantage of quantum coherence, Phys. Rev. A **95**, 010301 (2017).
[87] S. Deffner and E. Lutz, Nonequilibrium entropy production for open quantum systems, Phys. Rev. Lett. **107**, 140404 (2011).
[88] S. Deffner and E. Lutz, Thermodynamic length for far-from-equilibrium quantum systems, Phys. Rev. E **87**, 022143 (2013).
[89] H. Umegaki, Conditional expectation in an operator algebra, iv (entropy and information), in *Kodai Mathematical Seminar Reports*, Vol. 14 (Department of Mathematics, Tokyo Institute of Technology, 1962) pp. 59–85.
[90] G. Gour, *Quantum Resource Theories* (Cambridge University Press, 2025).
[91] A. Touil, B. Çakmak, and S. Deffner, J. phys. a: Math. theor., Journal of Physics A: Mathematical and Theoretical **55**, 025301 (2021).
[92] H. Spohn, Entropy production for quantum dynamical semigroups, J. Math. Phys. **19**, 1227 (1978).
[93] S.-I. Tang, E. Doucet, A. Touil, S. Deffner, and A. Sone, Information processing in quantum thermodynamic systems: an autonomous hamiltonian approach, arXiv preprint arXiv:2511.08858 (2025).
[94] M. Naseri, C. Macchiavello, D. Bruß, P. Horodecki, and A. Streltsov, Quantum speed limits for change of basis, New J. Phys. **26**, 023052 (2024).