

Zero forcing propagation time intervals and graphs with fixed propagation time

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Abstract

Zero forcing in a graph refers to the evolution of vertex states under repeated application of a color change rule. Typically the states are chosen to be blue and white, and a forcing set is an initial set of blue vertices such that all of the vertices are blue at the end of the process. In this context, the propagation time of a set in a graph is the number of iterations of the color change rule required to have all vertices blue, performing independent color changes simultaneously. Different minimal forcing sets need not have the same propagation time, and we study the realizability of specific integers as propagation times of minimal forcing sets in graphs for two of the most well-studied color change rules (standard and positive semidefinite). Particular attention is paid to the case where all minimal forcing sets have the same propagation time, and we term this phenomenon fixed propagation time. For each of the two variants, we present a general form of graphs all of which have fixed propagation time equal to one. We conjecture that these are the only such graphs and prove the conjectures for joins of graphs. Families of graphs with longer fixed propagation time for standard forcing are exhibited, and it is shown that such graphs do not exist for positive semidefinite forcing.

Keywords propagation time; zero forcing; PSD forcing; fixed propagation time

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1 Introduction

Zero forcing describes a dynamic coloring process on a graph, where initially each vertex is assigned blue or white. White vertices can become blue by applying a color change rule, which is iterated until a final coloring of the graph is obtained. The version of the color change rule determines the variant of zero forcing and here we discuss both the standard and positive semidefinite (PSD) variants of zero forcing. A set B is a (*standard or PSD*) *forcing set* for a graph G if, starting with exactly the vertices in B blue, repeated applications of the (standard or PSD) color change rule can color all vertices of G blue. Precise definitions of the color change rules and other terms used are given later in this introduction.

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Zero forcing arose about twenty years ago in numerous applications, including control of quantum systems, the Inverse Eigenvalue Problem of a Graph, and graph searching, and there has been extensive study of both standard and PSD zero forcing (see [11] and references therein).

Initially, the main focus of the study of zero forcing was to determine the size of a minimum (standard or PSD) forcing set, called the *standard forcing number* $Z(G)$ or the *PSD forcing number* $Z_+(G)$. Subsequently, interest developed in the number of iterations (called *rounds* or *time-steps*) needed to color all vertices blue when performing all independently possible forces simultaneously, called the *propagation time*. When starting with a specific forcing set $B \subseteq V(G)$, the propagation time of this set is denoted by $\text{pt}(G, B)$ or $\text{pt}_+(G, B)$. The *standard propagation time of G* , $\text{pt}(G)$, is the minimum propagation time of $\text{pt}(G, B)$ among all minimum standard forcing sets B , and analogously for $\text{pt}_+(G)$ [10, 13].

In the last five years there has been much more interest in the study of all forcing sets. This includes determination of the maximum cardinality of a minimal forcing set, called the *upper (standard or PSD) forcing number* and the reconfiguration of forcing sets. Reconfiguration examines relationships among solutions to a problem, with solutions considered abstractly as the vertices of a *reconfiguration graph*. A reconfiguration rule is given for the types of small changes that are allowed when trying to transform one solution to another, defining the edges of the reconfiguration graph. Depending on the rule chosen, a given pair of solutions may or may not belong to the same connected component of the reconfiguration graph, and the nature of these connected components, for various rules and for various types of zero forcing, is an active area of study. An introduction to reconfiguration, in both its broader context and its particular application to zero forcing, is provided by the bibliography of [4] for example, which studies certain reconfiguration graphs for PSD forcing and skew forcing.

Propagation time does not lend itself directly to a reconfiguration rule on the solution sets themselves, but an analogous question is raised if one considers a set of abstract vertices defined by the different possible propagation times, with an edge between times that differ exactly by one. Thus when there is a missing value between the maximum and minimum propagation times over minimum forcing sets, i.e., an intermediate value that cannot be realized as the propagation time of a minimum forcing set, this corresponds to a disconnected reconfiguration graph. Similar questions can be asked for minimal forcing sets. Results related to these ideas are presented in Sections 2 and 3.

The *(standard or PSD) propagation time interval* is the interval of integers between the minimum of the (standard or PSD) propagation times of minimum (standard or PSD) forcing sets of G and the maximum of the (standard or PSD) propagation times of minimum (standard or PSD) forcing sets of G ; the propagation time interval is *full* if every integer in the propagation time interval is achievable by some minimum forcing set [10, 13]. If a propagation time interval is full, then one can move between any possible propagation times of minimum forcing sets in increments of one (possibly requiring major changes to the minimum forcing sets).

Definition 1.1. The *(standard or PSD) propagation time set* of a graph G , denoted by $\text{PTS}(G)$ or $\text{PTS}_+(G)$, is the set of (standard or PSD) propagation times realized by minimum (standard or PSD) forcing sets.

For a graph G , $\text{PTS}(G)$ has a *gap* if $k, k+r \in \text{PTS}(G)$ for $r \geq 2$ but $k+1 \notin \text{PTS}(G)$, and similarly for $\text{PTS}_+(G)$.

The propagation time interval of G is full if and only if the set of integers in the propagation time interval equals the propagation time set of G of the same type. It was shown in [10] that the standard propagation time interval need not be full. In 2016 Warnberg conjectured that every PSD propagation time interval is full [13]. In 2020 Ben Small constructed an example of a graph that does not have a full PSD propagation time interval (shown in Figure 2.1), and the existence of such a graph was announced in [11]. In Section 2 this example is presented and it is proved that its PSD propagation time interval is not full.

Now that minimal forcing sets are studied, it is of interest to study their propagation times, and to minimize and maximize those values, as in the next definition.

Definition 1.2. For a graph G , the *upper standard propagation time* $\overline{\text{pt}}(G)$ (respectively, the *upper PSD propagation time* $\overline{\text{pt}}_+(G)$) is the maximum propagation time of any minimal standard (respectively, PSD) zero forcing set. The *lower standard propagation time* $\underline{\text{pt}}(G)$ (respectively, the *lower PSD propagation time* $\underline{\text{pt}}_+(G)$) is the minimum propagation time of any minimal standard (respectively, PSD) zero forcing set.

The *expanded standard propagation time interval* of a graph G is defined to be $[\underline{\text{pt}}(G), \overline{\text{pt}}(G)] = \{\underline{\text{pt}}_+(G), \underline{\text{pt}}(G) + 1, \dots, \overline{\text{pt}}(G) - 1, \overline{\text{pt}}(G)\}$ and the *expanded PSD propagation time interval* is defined analogously. An expanded propagation time interval is *full* if every integer in the propagation time interval is realized by some minimal forcing set.

We denote the maximum propagation time of a *minimal* forcing set by $\overline{\text{pt}}(G)$ and $\overline{\text{pt}}_+(G)$. Note that in the study of propagation time intervals (of minimum forcing sets) in the literature, the notations $\text{PT}(G)$ and $\text{PT}_+(G)$ are used to denote the maximum propagation time of a *minimum* forcing set. However, we follow the reconfiguration literature for zero forcing related parameters in putting a bar over the name of the main propagation parameter (which is the minimum time of a minimum set) to denote the maximum propagation time of a *minimal* forcing set. The minimum propagation time of a minimal forcing set is denoted by $\underline{\text{pt}}(G)$ and $\underline{\text{pt}}_+(G)$, respectively. The possibility that $\underline{\text{pt}}(G) < \text{pt}(G)$ (and similarly for PSD propagation time) occurs because we use minimal rather than minimum forcing sets. A path and standard forcing provides an example illustrating this: It is well-known that $\text{pt}(P_n) = n - 1$, but $\underline{\text{pt}}(P_n) = \lfloor \frac{n-2}{2} \rfloor$ using a set of two adjacent vertices in the center of the path.

Definition 1.3. The *expanded (standard or PSD) propagation time set* of a graph G , denoted by $\text{EPTS}(G)$ or $\text{EPTS}_+(G)$, is the set of (standard or PSD) propagation times realized by minimal (standard or PSD) forcing sets.

For a graph G , $\text{EPTS}(G)$ has a *gap* if $k, k + r \in \text{EPTS}(G)$ for $r \geq 2$ but $k + 1 \notin \text{EPTS}(G)$, and similarly for $\text{EPTS}_+(G)$.

As is the case for the propagation time interval, the expanded propagation time interval of G is full if and only if the set of integers in the propagation time interval equals the expanded propagation time set of G of the same type. The graph in Figure 2.1 does not have a full expanded PSD propagation time interval. A gap in an expanded propagation time set identifies a place where we cannot change from one propagation time to another with an increment of one using minimal forcing sets.

The simplest type of expanded propagation time interval, which is guaranteed to be full, is one where the upper and lower propagation times are equal.

Definition 1.4. A graph G has *fixed propagation time* (respectively, *fixed PSD propagation time*) if $\underline{\text{pt}}(G) = \overline{\text{pt}}(G)$ (respectively, $\underline{\text{pt}}_+(G) = \overline{\text{pt}}_+(G)$).

Equivalently, a graph G has fixed propagation time if and only if the expanded propagation time interval has exactly one element. Note that $|\text{EPTS}(G)| = 1$ implies $|\text{PTS}(G)| = 1$, but the reverse implication does not hold, as evidenced by a path on at least four vertices. As discussed in Section 3, it is easy to find examples of graphs that have fixed standard propagation times greater than one. In that section we also show that a fixed PSD propagation time greater than one is not possible.

Fixed propagation time equal to one, i.e., the study of graphs G such that $\underline{\text{pt}}(G) = \overline{\text{pt}}(G) = 1$ or $\underline{\text{pt}}_+(G) = \overline{\text{pt}}_+(G) = 1$, is of particular interest because it is equivalent to every forcing set $B \neq V(G)$ having propagation time equal to one (not just the minimal forcing sets). It is also simpler, in part because $\overline{\text{pt}}(G) = 1$ implies $\underline{\text{pt}}(G) = 1$, and similarly for PSD. In Section 4 we show that graphs with two specific forms have fixed standard or PSD propagation time equal to one; we conjecture (see

Conjectures 1.7 and 1.8) that every graph having fixed standard or PSD propagation time equal to one has the associated form.

Next we define these specific forms. Let G_1 and G_2 be two vertex disjoint graphs (i.e. $V(G_1) \cap V(G_2) = \emptyset$). The *(disjoint) union* of G_1 and G_2 is the graph $G = G_1 \dot{\cup} G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* of disjoint G_1 and G_2 is the graph $G_1 \vee G_2$ having $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{v_1 v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$.

Definition 1.5. A graph G of order $n \geq 2$ is a *PSD fast join* if $G \cong K_n$ (i.e. $G \cong K_{n-t} \vee K_t$ for $1 \leq t \leq n-1$) or $G \cong (K_{n_1} \dot{\cup} K_{m_1}) \vee \dots \vee (K_{n_t} \dot{\cup} K_{m_t})$ for $t \geq 2$ and positive integers n_i and m_i such that $n = \sum_{i=1}^t n_i + m_i$.

A graph G of order $n \geq 2$ is a *standard fast join* if $G \cong K_n$ or $G = \bigvee_{i=1}^t G_i$ where $t \geq 2$ and each graph G_i satisfies $G_i \cong n_i K_1$ or $G_i \cong K_{n_i} \dot{\cup} K_{m_i}$ with $n_i, m_i \geq 2$.

In Section 4 we establish that fast joins have fixed propagation time equal to one:

Theorem 1.6. *If G is a PSD fast join, then $\overline{\text{pt}}_+(G) = 1$. If G is a standard fast join, then $\overline{\text{pt}}(G) = 1$.*

We conjecture the converses of the statements in this theorem are true and in Section 5 we prove these conjectures for graphs that are joins or have sufficiently large forcing numbers.

Conjecture 1.7. *Let G be a connected graph. If $\overline{\text{pt}}_+(G) = 1$, then G is a PSD fast join.*

Conjecture 1.8. *Let G be a connected graph. If $\overline{\text{pt}}(G) = 1$, then G is a standard fast join.*

We conclude this introduction with precise definitions of the notation and terminology.

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. The elements of $E(G)$ are subsets of vertices of cardinality two; we denote the edge between distinct vertices u and v by uv or vu . If $uv \in E(G)$, then u and v are called *neighbors*. The *open neighborhood* of u in G , denoted $N_G(u)$, is the set of all neighbors of u in G , while the *closed neighborhood* of u in G is $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of u in G is the cardinality of $N_G(u)$ and is denoted by $\deg_G(u)$. A vertex u satisfying $\deg_G(u) = 1$ is called a *leaf*, while if $|V(G)| = n$ and $\deg_G(u) = n-1$, then u is called a *universal vertex*.

We consider two color change rules, CCR-Z for standard zero forcing and CCR-Z₊ for PSD forcing, and use CCR-R to denote either one of these.

- CCR-Z: If a blue vertex u has a unique white neighbor w , then u can force w to become blue.
- CCR-Z₊: If B is the set of currently blue vertices, C is a component of $G - B$, and u is a blue vertex such that $N_G(u) \cap V(C) = \{w\}$, then u can force w to become blue.

For $B \subseteq V(G)$ and a color change rule CCR-R, we define two sequences of sets: $B^{[0]} = B^{(0)} = B$ is the set of initially blue vertices. Assume $B^{(i)}$ and $B^{[i]}$ have been constructed. Then

$$B^{(i+1)} = \{w : w \text{ can be forced by some } v \text{ (given } B^{[i]} \text{ blue)}\} \quad \text{and} \quad B^{[i+1]} = B^{[i]} \cup B^{(i+1)}.$$

The vertices in $B^{(i)}$ are said to be forced in *round i* . Let p denote the greatest integer such that $B^{(p)} \neq \emptyset$ (note $B^{(i)} = \emptyset$ implies $B^{(i+1)} = \emptyset$). Then B is a *forcing set* of G if and only if $B^{[p]} = V(G)$.

A *minimum* forcing set B is a forcing set of minimum cardinality, i.e., if $|B'| < |B|$, then B' is not a forcing set. A *minimal* forcing set B is a forcing set such that no proper subset is a forcing set, i.e., if $B' \subsetneq B$, then B' is not a forcing set. The (standard and PSD) zero forcing numbers of G , denoted by $Z(G)$ and $Z_+(G)$, are the minimum cardinalities among (standard or PSD) forcing sets. Note that every minimum forcing set is a minimal forcing set, but not all minimal forcing sets are minimum forcing sets. Every forcing set contains a subset (not necessarily proper) that is a minimal forcing set, but not necessarily a subset that is a minimum forcing set.

When B is a forcing set and p is the greatest integer such that $B^{(p)} \neq \emptyset$, then p is called the *propagation time* of B in G and denoted $\text{pt}(G, B)$ and $\text{pt}_+(G, B)$ for standard and PSD zero forcing sets, respectively; if B is not a forcing set, then $\text{pt}(G, B) = \infty$ or $\text{pt}_+(G, B) = \infty$.

Let G be a graph. A nonempty set $F \subseteq V(G)$ is a standard fort of G if there does not exist a vertex in $V(G) \setminus F$ that has exactly one neighbor in F . Let $F \subseteq V(G)$ be nonempty, and $\{C_i\}_{i=1}^k$ be the components of $G[F]$. Then F is a *PSD fort* of G if for each $v \in V(G) \setminus F$ and each $i \in \{1, 2, \dots, k\}$, $|N_G(v) \cap V(C_i)| \neq 1$.

Theorem 1.9. *Let G be a graph and $B \subseteq V(G)$.*

- (1) [6, Theorem 8] *The set B is a zero forcing set of G if and only if $B \cap F \neq \emptyset$ for every fort F of G .*
- (2) [12, Theorem 2] *The set B is a PSD forcing set of G if and only if $B \cap F \neq \emptyset$ for each PSD fort F of G .*

2 A graph that does not have a full PSD propagation time interval

In this section we present with proof a family of graphs whose PSD propagation time interval is not full (that includes Small's example), providing a negative resolution to Warnberg's conjecture in [13]. We also show that the expanded PSD propagation time interval of each graph in the family is not full. The smallest graph in the family is shown in Figure 2.1.

Definition 2.1. Let $A = \{a_1, \dots, a_{8+2k}\}$ and $B = \{b_1, \dots, b_{7+2k}\}$, and define a graph of order $15 + 4k$ by $S_k = P_{8+2k} \vee (7 + 2k)K_1$ where $V(P_{8+2k}) = A$ (in path order) and $V((7 + 2k)K_1) = B$.

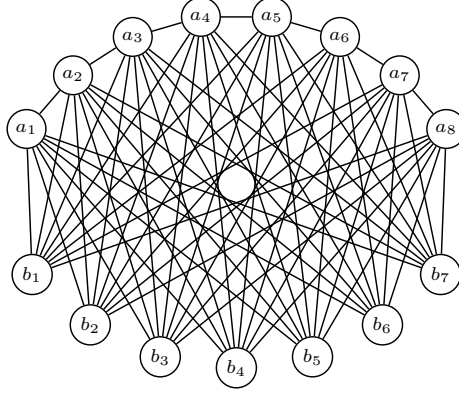


Figure 2.1: The graph S_0 , which does not have a full PSD propagation interval nor a full expanded PSD propagation time interval.

Theorem 2.2. *For $k \geq 0$, the graph S_k of order $15 + 4k$ has $Z_+(S_k) = 8 + 2k$, $\overline{Z}_+(S_k) = 9 + 2k$, and $\text{EPTS}_+(S_k) = \text{PTS}_+(S_k) = \{1, 2, 4 + k, 5 + k, \dots, 7 + 2k\}$.*

Proof. We first note that any induced subgraph of S_k containing at least one vertex each from A and B is connected. Let $X \subsetneq V(S_k)$ be a minimal PSD forcing set and define $W = V(S_k) \setminus X$.

If $|W \cap A| \geq 2$ and $|W \cap B| \geq 2$, then W is a PSD fort (because it induces a connected graph and any vertex in X has two white neighbors in one of the sets A or B) so X is not a PSD forcing set. Therefore $|X \cap A| \geq 7 + 2k$ or $|X \cap B| \geq 6 + 2k$. We analyze cases.

(i) $|X \cap A| = 8 + 2k$: The removal of X results in isolated vertices and $\text{pt}_+(\mathbf{S}_k, X) = 1$. Since X is minimal, this implies $X \cap B = \emptyset$ and $|X| = 8 + 2k$.

(ii) $|X \cap A| = 7 + 2k$:

If $X \cap B = \emptyset$, then $G[W]$ is connected and every vertex in X has at least seven white neighbors in B , so no forcing could take place.

Thus $|X \cap B| \geq 1$. Then $(A \setminus \{a_i\}) \cup \{b_j\} \subseteq X$ for some i, j , so in the first round b_j forces a_i , and $\text{pt}_+(\mathbf{S}_k, X) = 2$. Necessarily $|X \cap B| = 1$ and $|X| = 8 + 2k$ since X is minimal.

(iii) $|X \cap B| = 7 + 2k$:

If $X \cap A = \emptyset$, then $G[W]$ is connected and every vertex in X has at least eight white neighbors in A , so no forcing could take place.

Thus $|X \cap A| = 1$ and $|X| = 8 + 2k$ because adding any one vertex of A to B produces a PSD forcing set. Observe that $\text{pt}_+(\mathbf{S}_k, B \cup \{a_i\}) = 8 + 2k - i$ for $i = 1, \dots, 4 + k$, yielding PSD propagation times $7 + 2k, 6 + 2k, \dots, 4 + k$, respectively. Furthermore, $i \in \{5 + k, \dots, 8 + 2k\}$ yields the same set of PSD propagation times by symmetry.

(iv) $|X \cap B| = 6 + 2k$:

If $|X \cap A| \leq 1$, then $G[W]$ is connected, the blue vertex in $X \cap A$ (if there is one) has at least one white neighbor in A and one white neighbor in B , and every vertex in $X \cap B$ has at least seven white neighbors in A , so no forcing could take place.

Thus $|X \cap A| \geq 2$. Note first that the vertices in B are adjacent to every vertex in A , so as long as there are two white vertices in A , no vertex in B can force. A vertex in A cannot force a vertex in A until the one white vertex in B is forced. In order for vertex a_i to force a vertex in B , we must have a_{i-1}, a_i, a_{i+1} in X (if all three vertices exist). Thus we have two possibilities:

$|X \cap A| = 2$ and $|X| = 8 + 2k$. In this case, $X \cap A = \{a_1, a_2\}$ or $X \cap A = \{a_7 + 2k, a_8 + 2k\}$ and $\text{pt}(\mathbf{S}_k, X) = 7 + 2k$.

$|X \cap A| = 3$ and $|X| = 9 + 2k$. In this case, $X \cap A = \{a_i, a_{i+1}, a_{i+2}\}$ for $i = 2, \dots, 5 + 2k$. Let $X_B = X \cap B$. Then $\text{pt}_+(\mathbf{S}_k, X_B \cup \{a_i, a_{i+1}, a_{i+2}\}) = 7 + 2k - i$ for $i = 2, \dots, 3 + k$, yielding PSD propagation times $5 + 2k, 4 + 2k, \dots, 4 + k$, respectively. Furthermore, $i \in \{5 + k, \dots, 8 + 2k\}$ yields the same set of PSD propagation times by symmetry.

Thus $Z_+(\mathbf{S}_k) = 8 + 2k$ and $\overline{Z}_+(\mathbf{S}_k) = 9 + 2k$, no minimal set has PSD propagation time of $3, 4, \dots, 3 + k$, $\text{EPTS}_+(\mathbf{S}_k) = \{1, 2, 4 + k, 5 + k, \dots, 7 + 2k\}$, and each of these values is realized by a minimum PSD forcing set, giving $\text{PTS}_+(\mathbf{S}_k) = \{1, 2, 4 + k, 5 + k, \dots, 7 + 2k\}$. $\square$

3 Fixed propagation time greater than one

When studying the propagation time set or expanded propagation time set, it is natural to ask what fixed propagation times are possible, in addition to which graphs have fixed propagation time. In Sections 4 and 5 we discuss fixed propagation time equal to one. In this section we present families of graphs that have larger standard fixed propagation time and show that no graph can have fixed PSD propagation time greater than one.

3.1 Fixed standard propagation time greater than one

We begin with examples of graphs that have fixed standard propagation time greater than one.

Example 3.1. For the cycle C_n with $n \geq 5$, a set is a minimal standard zero forcing set if and only if it consists of two adjacent vertices, and thus $\underline{\text{pt}}(C_n) = \overline{\text{pt}}(C_n) = \lceil \frac{n-2}{2} \rceil$.

Example 3.2. For the star $K_{1,n-1}$ with $n \geq 3$, a set is a minimal standard zero forcing set if and only if it consists of $n - 2$ leaves, and thus $\underline{\text{pt}}(K_{1,n-1}) = \overline{\text{pt}}(K_{1,n-1}) = 2$.

The maximum size of a minimal standard forcing set for a graph G is denoted by $\overline{Z}(G)$. If $Z(G) = \overline{Z}(G)$, then every minimal standard forcing set of G is also a minimum standard forcing set of G . Two minimum standard zero forcing sets B_1 and B_2 of a graph G are *isomorphic* if there is a graph automorphism ψ of G such that $\psi(B_1) = B_2$. Note that both of the previous examples are covered by the next observation.

Observation 3.3. *If G is a graph with $Z(G) = \overline{Z}(G)$ and in which all minimum standard zero forcing sets are isomorphic, then G has fixed standard propagation time.*

In the next examples we see examples of graphs where not all minimum standard zero forcing sets are isomorphic.

Example 3.4. The smallest order n for which there is a graph with fixed standard propagation time is greater than one is $n = 3$, namely $K_{1,2} = P_3$. The connected graphs of orders $n = 4$ and $n = 5$ with fixed standard propagation time two are listed in Figure 3.1.

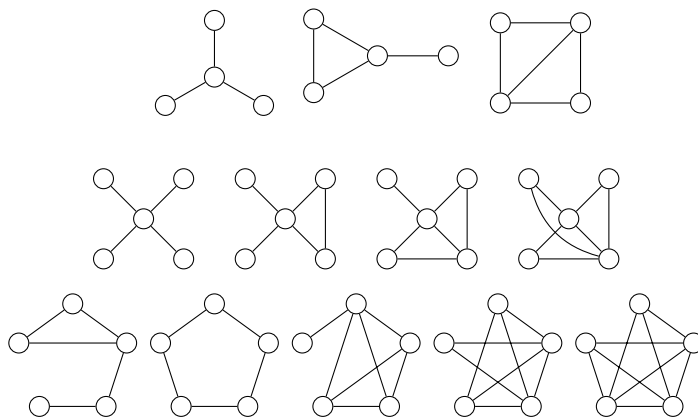


Figure 3.1: The connected graphs of order $n = 4, 5$ with $\underline{\text{pt}}(G) = \overline{\text{pt}}(G) = 2$

The star and all but two of the graphs in Figure 3.1 are threshold graphs. A *threshold graph* is a graph that can be built from isolated vertices by a sequence of disjoint union and join operations, where at each stage one of the two graphs is a single isolated vertex. This can be diagrammed as a rooted binary tree where the leaves are isolated vertices and the vertices of degree two or more are marked as diamonds with $\dot{\cup}$ (for disjoint union) or $\dot{\vee}$ (for join), with the restriction that at least one of the parts added in each operation is a single vertex, as illustrated in Figure 3.2; this is called a *construction tree* of the graph. We show in Theorem 3.8 that every threshold graph has fixed standard propagation time. Furthermore, every connected threshold graph except K_n has fixed standard propagation time greater than one, as shown in Corollary 3.9.

We begin with elementary results on the standard zero forcing number of graphs with a universal vertex, a class that includes connected threshold graphs. These results are known and have been used elsewhere (e.g., [5]), but we do not know of clear sources for some.

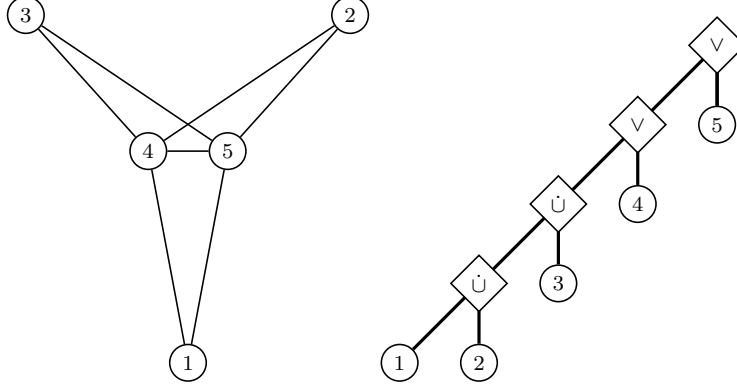


Figure 3.2: A threshold graph and its construction tree

Proposition 3.5. *Let G be a graph and let $H = G \vee K_1$.*

- (1) [11, Proposition 9.16] *If G has no isolated vertices, then $Z(H) = Z(G) + 1$.*
- (2) *If G has at least two isolated vertices, then $Z(H) = Z(G) - 1$.*
- (3) *If G has exactly one isolated vertex, then $Z(H) = Z(G)$.*

Proof. Let $V(K_1) = \{u\}$. For Item (2), let $I = \{v_1, v_2, \dots, v_k\}$ denote the isolated vertices of G , and assume that $k \geq 2$. Choose a minimum standard zero forcing set B for G , which necessarily contains I and let $\mathcal{F}$ be a chronological list of forces for B in G . Let $B' = B \setminus \{v_1\}$. Then $v_2 \rightarrow u$ is a valid first force for B' in G , after which all of the forces of $\mathcal{F}$ are valid in H , and only v_1 remains white. Adding the final force $u \rightarrow v_1$ shows that B' is a standard zero forcing set of H and thus $Z(H) \leq Z(G) - 1$, and it is well known that $Z(H) \geq Z(G) - 1$ since $G = H - u$ [11, Proposition 9.13]

To show Item (3), first let B be a minimum standard zero forcing set of G , $\mathcal{F}$ be a chronological list of forces for B in G , and v be the isolated vertex of G . Then $v \in B$, and since u is the only neighbor of v in H , the force $v \rightarrow u$ is a valid first force in H starting with the vertices in B blue. After this force, every force in $\mathcal{F}$ is valid in H , so B is a standard zero forcing set of H . Thus $Z(H) \leq Z(G)$.

Now suppose B' is a minimum standard zero forcing set of H with corresponding chronological list of forces $\mathcal{F}$. If $v \in B'$, then $u \notin B'$ since B' is minimal and the first force must be $v \rightarrow u$; the remaining forces of $\mathcal{F}$ can take place entirely in G (replacing a last force $u \rightarrow x$ by $y \rightarrow x$ for $y \in N_G(x)$ if necessary). Thus $Z(G) \leq Z(H)$. If $v \notin B'$ and $u \notin B'$, then the first force in $\mathcal{F}$ must be $x \rightarrow u$, where $x \neq v$. Since u is the unique white neighbor of x in H , we have $N_G(x) \subseteq B'$. Thus we may obtain a new minimum standard zero forcing set $(B' \setminus \{y\}) \cup \{u\}$, where $y \in N_G(x)$. Therefore if $v \notin B'$ we may assume $u \in B'$. Under this hypothesis, every force in $\mathcal{F}$ is valid in G except $u \rightarrow v$, and this must be the last force in $\mathcal{F}$. Thus $(B' \setminus \{u\}) \cup \{v\}$ is a standard zero forcing set of G whose chronological list of forces is $\mathcal{F}$ except the last force, so $Z(G) \leq Z(H)$. $\square$

Theorem 3.6. *If G is a threshold graph, then $Z(G) = \bar{Z}(G)$.*

Proof. Let G be a threshold graph with $|G| = n$. We proceed by induction on n . The statement is true if $n \leq 2$, so suppose the statement holds for $k \leq n$. Let H be a threshold graph with $|H| = n+1$. The statement is trivial if H is the form $G \dot{\cup} K_1$, so consider $H = G \vee K_1$, where G is a threshold graph. Let u denote the universal vertex joined to G . Let B be a minimal standard zero forcing set of H of maximum cardinality.

If G has two or more isolated vertices, then $Z(H) = Z(G) - 1$ by Proposition 3.5(2) and B contains all but one of the isolated vertices of G . Furthermore, $u \notin B$ since any of the isolated

vertices of G in B can force u of H in the first round of propagation. Let x be the isolated vertex in G such that $x \notin B$. Then $B' = B \cup \{x\}$ is a standard zero forcing set of G . Suppose B' is not minimal for G and y is a vertex such that $B' \setminus \{y\}$ is a standard zero forcing set of G . Then y cannot be any of the isolated vertices of G , and consider $B \setminus \{y\}$. Let $w \neq x$ be an isolated vertex of G . Then $w \rightarrow u$ is a valid first force for $B \setminus \{y\}$ in H , and each of the forces occurring in a chronological list of forces for $B' \setminus \{y\}$ in G is valid for $B \setminus \{y\}$ in H . Thus $B \setminus \{y\}$ is a standard zero forcing set of H , contradicting the minimality of H . Therefore, $\bar{Z}(H) = |B| = |B'| - 1 = Z(G) - 1 = Z(H)$.

So we assume G has either one or no isolated vertices. Suppose first that G has one isolated vertex x and $x \in B$. Then $u \notin B$ and the only force in the first round is $x \rightarrow u$. Let $B' = (B \setminus \{x\}) \cup \{u\}$ and note that B' is a standard forcing set of the same cardinality because B' uses the forces of B except that the first round force $x \rightarrow u$ is replaced by a last round force $u \rightarrow x$. So henceforth we assume no isolated vertex of G is in B .

If $u \notin B$, then the first force in H must be of the form $z \rightarrow u$, where $z \in V(G)$, $N_G[z] \subseteq B$, and there exists $y \in N_G(z) \cap B$. Then $B' = (B \setminus \{y\}) \cup \{u\}$ is a minimal standard zero forcing set of H . Thus we assume for the remainder of the proof that $u \in B$.

Now suppose G has no isolated vertices (whether in B or not). Then $Z(H) = Z(G) + 1$ by Proposition 3.5(1), and $B' = B \setminus \{u\}$ is a standard zero forcing set of G since $u \rightarrow w$ can only occur as the last force in H , which can be replaced $y \rightarrow w$ for any $y \in N_G(w)$. We claim that B' is a minimal standard zero forcing set of G . Suppose to the contrary that $B' \setminus \{y\}$ is a standard zero forcing set for G , where $y \in B \cap V(G)$. Then $B \setminus \{y\}$ is a standard zero forcing set for H since each of the forces for $B' \setminus \{y\}$ in G is valid for $B \setminus \{y\}$ in H , contradicting the minimality of H . This gives $\bar{Z}(H) = |B| = Z(G) + 1 = Z(H)$.

Finally, suppose G has exactly one isolated vertex x . Then $Z(H) = Z(G)$ by Proposition 3.5(3), and since $u \in B$ it must be the case that $x \notin B$ since B is minimal and $u \rightarrow x$ is the final force performed. Consider $B' = (B \setminus \{u\}) \cup \{x\}$, which is a standard zero forcing set of H , as can be seen by adding $x \rightarrow u$ as the first force, proceeding with the forces for B in H until the final force $u \rightarrow x$, which is removed. As before, if $B' \setminus \{y\}$ is a standard zero forcing set of G , then $B \setminus \{y\}$ is a standard zero forcing set of H , a contradiction (note that $y \neq x$ since x is an isolated vertex in G). This gives $\bar{Z}(H) = |B| = |B'| = Z(G) = Z(H)$. $\square$

The previous theorem combined with the next result establishes threshold graphs have fixed propagation time. The *(standard) throttling number* of B in G is $\text{th}(G, B) = |B| + \text{pt}(G, B)$ and the *(standard) throttling number* of a graph G is $\text{th}(G) = \min_{B \subseteq V(G)} \text{th}(G, B)$. Observe that $\text{th}(G, B) \leq |V(G)|$ for any standard zero forcing B , because at least one vertex must be forced in each round.

Theorem 3.7. [11, Theorem 11.22] *A graph G has $\text{th}(G) = |V(G)|$ if and only if G is a threshold graph.*

Corollary 3.8. *Every threshold graph G has fixed standard propagation time equal to $|V(G)| - Z(G)$.*

Proof. If B is a minimal standard zero forcing set of a threshold graph G , then $|B| = Z(G) = \bar{Z}(G)$ by Theorem 3.6. By Theorem 3.7, $\text{th}(G) = |V(G)|$, so $\text{th}(G, B) = |V(G)|$ for every minimum standard zero forcing set. Thus $\text{pt}(G, B) = |V(G)| - |B| = |V(G)| - Z(G)$. $\square$

The next result is immediate from the previous corollary.

Corollary 3.9. *If G is a connected threshold graph of order $n \geq 2$ with $\bar{\text{pt}}(G) = 1$, then $G \cong K_n$.*

A *cograph* is a graph that can be built from isolated vertices by a sequence of disjoint union and join operations (the restriction that one of the two graphs in each operation is a single vertex is removed). There are cographs that do not have fixed standard propagation time and for which $Z(G) < \bar{Z}(G)$, as seen in the next two examples.

Example 3.10. The wheel graph W_5 is shown in Figure 3.3 together with a construction tree. Note that W_5 does not have fixed standard propagation time because (with the labeling shown), $\text{pt}(W_5, \{1, 2, 5\}) = 1$ and $\text{pt}(W_5, \{1, 2, 3\}) = 2$.

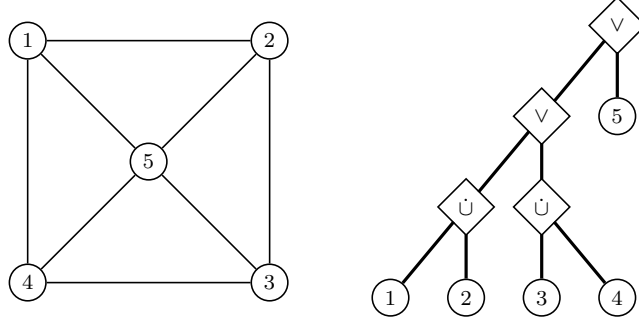


Figure 3.3: The cograph W_5 and a construction tree

Example 3.11. The cograph G shown in Figure 3.4 (together with a construction tree) has $Z(G) = 4 < 5 \leq \bar{Z}(G)$ because $\{1, 3, 5, 7\}$ and $\{1, 2, 3, 4, 5\}$ are minimal standard zero forcing sets. Any standard zero forcing set must contain one of the vertices 1 and 2, one of the vertices 3 and 4, and one of the vertices 5 and 6 because each of these pairs are twins [11, Proposition 9.15]. Furthermore, $\{1, 3, 5\}$ is not a standard zero forcing set. (In fact, $\bar{Z}(G) = 5$ as verified in [8].)

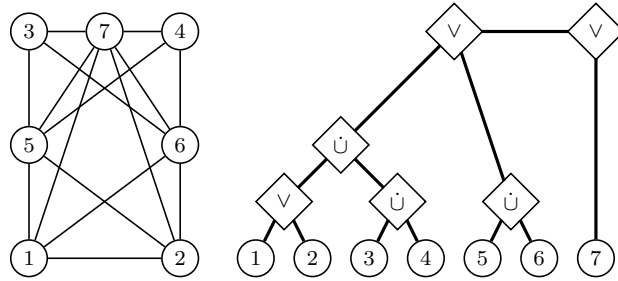


Figure 3.4: A cograph G with $Z(G) \neq \bar{Z}(G)$ and a construction tree

Finally we note that even if a graph G satisfies $\bar{Z}(G) \neq Z(G)$, it may still be the case G has fixed standard propagation time. Two minimal standard zero forcing sets B_1 and B_2 of a graph G are *isomorphic* if there is a graph automorphism ψ of G such that $\psi(B_1) = B_2$ (necessarily $|B_1| = |B_2|$).

Example 3.12. Let G be the graph in Figure 3.5. Then, every minimum standard forcing set is isomorphic to $B_2 = \{1, 2\}$ because such a set must contain a vertex of degree two and a neighbor. Every minimal standard forcing set of size three is isomorphic to $B_3 = \{2, 3, 4\}$ because it can't contain a vertex of degree two and must contain a vertex of degree three and two of its neighbors. Moreover, no set of four or more vertices is a minimal forcing set. Direct computation reveals that $\text{pt}(G, B_2) = 3 = \text{pt}(G, B_3)$.

3.2 No graph has fixed PSD propagation time greater than one

We apply known results to show that the only possible fixed PSD propagation time is one.

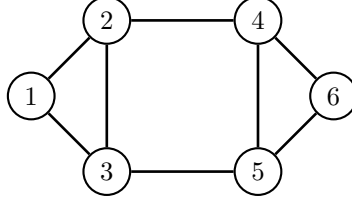


Figure 3.5: A graph G with $Z(G) \neq \bar{Z}(G)$ and fixed standard propagation time

Lemma 3.13. [7, Lemma 4] *If G is a connected graph, then there exists a minimum positive semidefinite forcing set $B \subseteq V(G)$ such that $G - B$ is connected.*

Lemma 3.14. [9, Lemma 3.1] *Let G be a graph with PSD forcing set B such that $G - B$ is connected. Let B' be the endpoints of the PSD forcing trees after the first time step. Then B' is another PSD forcing set of G with $|B| = |B'|$. Furthermore, if $\text{pt}_+(G, B) \geq 2$, then $\text{pt}_+(G, B') = \text{pt}_+(G, B) - 1$.*

Theorem 3.15. *Let G be a connected graph. If G has fixed PSD propagation time, then $\text{pt}_+(G) = 1$.*

Proof. Suppose G has a minimum PSD forcing set B such that B is a minimum forcing set with $\text{pt}_+(G, B) \geq 2$. By Lemma 3.13, there exists a minimum PSD forcing set $\hat{B}$ such that $G - \hat{B}$ is connected. If $\text{pt}_+(G, \hat{B}) \neq \text{pt}_+(G, B)$, then G does not have fixed PSD propagation time. So assume $\text{pt}_+(G, \hat{B}) = \text{pt}_+(G, B) \geq 2$. Then by Lemma 3.14 there is another minimum PSD forcing set B' such that $\text{pt}_+(G, B') < \text{pt}_+(G, \hat{B})$. This implies G does not have fixed PSD propagation time. $\square$

4 Fast joins have fixed propagation time equal to one

In this section we show that fast joins have fixed propagation time equal to one. We also show that fast joins have very large forcing numbers. We begin with some preliminary material.

Observation 4.1. *A graph $G \not\cong K_n$ is a PSD fast join if and only if $\bar{G} \cong \dot{\cup}_{i=1}^t K_{n_i, m_i}$ where $n_i, m_i \geq 1$ for $i = 1, \dots, t$ and $t \geq 2$. A graph $G \not\cong K_n$ is a standard fast join if and only if $\bar{G} \cong \dot{\cup}_{i=1}^t G_i$ where $t \geq 2$ and for each $i = 1, \dots, t$, $G_i \cong K_{n_i}$ or $G_i \cong K_{n_i, m_i}$ with $n_i, m_i \geq 2$.*

Observation 4.2. *Let G be a graph.*

1. *If $v_1, v_2 \in V(G)$ satisfy $N_G[v_1] = N_G[v_2]$ (implying $v_1 v_2 \in E(G)$), then $\{v_1, v_2\}$ is a PSD fort and therefore a standard fort.*
2. *If $v_1, v_2 \in V(G)$ satisfy $N_G(v_1) = N_G(v_2)$ (implying $v_1 v_2 \notin E(G)$), then $\{v_1, v_2\}$ is a standard fort.*

Lemma 4.3. *Let $G = \bigvee_{i=1}^k G_i$, let B be a minimum (standard or PSD) forcing set of G , and let $W = V(G) \setminus B$. Then $W \cap V(G_i) \neq \emptyset$ for at most two values of $i \in \{1, 2, \dots, k\}$.*

Proof. Suppose to the contrary and without loss of generality that $|W \cap V(G_i)| \geq 1$ for $i \in \{1, 2, 3\}$, and choose $v_i \in W \cap V(G_i)$ for each i . Then $G[\{v_1, v_2, v_3\}] \cong K_3$, and any $b \in B$ is adjacent to at least two vertices in $\{v_1, v_2, v_3\}$. Thus no forces (standard or PSD) are possible, so B is not a (standard or PSD) forcing set. $\square$

Theorem 4.4. *If G is a PSD fast join, then $\overline{\text{pt}}_+(G) = 1$ and $Z_+(G) \geq |V(G)| - 2$. If G is a standard fast join, then $\overline{\text{pt}}(G) = 1$ and $Z(G) \geq |V(G)| - 2$.*

Proof. Assume G is a PSD fast join or a standard fast join of order n . If G is a complete graph of order at least two, then $Z_+(G) = Z(G) = n - 1$, and so $\overline{\text{pt}}_+(G) = \overline{\text{pt}}(G) = 1$. So assume $G \not\cong K_n$, which implies $Z_+(G), Z(G) \leq n - 2$. Let B be a minimal (standard or PSD) forcing set of G and let $W = V(G) \setminus B$, so that $|B| \leq n - 2$ and $|W| \geq 2$.

In light of Lemma 4.3, W can contain vertices from at most two components of $\overline{G}$. By Observation 4.2, we may assume G has no white adjacent twins. Thus, W can contain at most one vertex from each partite set of a component of $\overline{G}$ that is (complete) bipartite. Observation 4.2 also allows us to assume no white independent twins if G is a standard fast join; thus W can contain at most one vertex from a component of $\overline{G}$ that is a complete graph ($K_2 = K_{1,1}$ is treated as a complete graph for a standard fast join). We are thus left with the following possibilities for W :

1. W consists of one vertex from each of two components of $\overline{G}$;
2. W consists of one vertex from each partite set of a single component of $\overline{G}$; or
3. W consists of one vertex from each partite set of one component of $\overline{G}$ and at least one vertex from another component of $\overline{G}$.

First we show that the set B in case (3) is neither a PSD nor a standard forcing set. Note that in this case, the white vertices induce a connected graph in G , so only the standard zero forcing rule applies initially.

- Any blue vertex in a component of $\overline{G}$ with two white vertices is adjacent in G to the white vertex in its partite set and a white vertex in the other component of $\overline{G}$ with white vertices,
- Any blue vertex in a component of $\overline{G}$ with one white vertex is adjacent in G to both of the white vertices in the component with two white vertices, and
- Any other blue vertex is adjacent to all of the white vertices.

Because B cannot perform any forces, it is not a (standard or PSD) forcing set. Since case (3) cannot occur for PSD or standard forcing and B is any minimal forcing set, the (standard or PSD) forcing number is at least $n - 2$.

For cases (1) and (2), we consider PSD forcing and standard forcing separately, beginning with PSD forcing. For case (2), both white vertices can be PSD forced in the first round by any vertex in any of the other components, because the two white vertices are in separate components of $G - B$. Now consider case (1). For $i = 1, 2$, let A_i (containing white vertex w_i) and B_i (all vertices blue) denote the partite sets of the two components of $\overline{G}$ containing white vertices. Then in the first round, w_1 can be forced by a vertex in B_2 and w_2 can be forced by a vertex in B_1 . In both cases, $\text{pt}_+(G, B) = 1$. Thus, for all PSD forcing sets $B \subsetneq V(G)$ of G , we have $\text{pt}_+(G, B) = 1$ and therefore $\overline{\text{pt}}_+(G) = 1$.

Finally we consider standard forcing. For case (2), both white vertices can be standard forced in the first round, each by a blue vertex in the same partite set of the component in $\overline{G}$, because each partite set contains at least two vertices. For case (1), denote the vertices of the components of $\overline{G}$ that contain white vertices by U_1 and U_2 and the white vertices by w_i with $w_i \in U_i$ for $i = 1, 2$. Then w_1 can be forced in round one by a blue vertex in U_2 that is not adjacent in G to w_2 , and similarly for w_2 . In both cases, $\text{pt}(G, B) = 1$. Thus, for all standard zero forcing sets $B \subsetneq V(G)$ of G , we have $\text{pt}(G, B) = 1$ and therefore $\overline{\text{pt}}(G) = 1$. $\square$

5 Joins with fixed propagation time equal to one are fast joins

In this section we resolve Conjectures 1.7 and 1.8 under the additional hypotheses that the graph is a join of two graphs, or that the forcing number is sufficiently large. We begin with a few necessary

tools.

Lemma 5.1. *Let G be a graph and let u and v be adjacent vertices of G such that $N_G[u] \subsetneq N_G[v]$. Then $\text{pt}(G, V(G) \setminus \{u, v\}) = \text{pt}_+(G, V(G) \setminus \{u, v\}) = 2$.*

Proof. Start with exactly the vertices in $B = V(G) \setminus \{u, v\}$ blue. Then no vertex can force u in the first round but there is a vertex x adjacent to v and not to u , so $x \rightarrow v$ in the first round and $v \rightarrow u$ in the second round. Thus $\text{pt}(G, B) = \text{pt}_+(G, B) = 2$. $\square$

The next two corollaries are immediate from the previous lemma.

Corollary 5.2. *If G is a connected graph of order $n \geq 3$ with $\overline{\text{pt}}_+(G) = 1$ or $\overline{\text{pt}}(G) = 1$, then G has no leaves.*

Corollary 5.3. *If $G \not\cong K_n$ and $\overline{\text{pt}}_+(G) = 1$ or $\overline{\text{pt}}(G) = 1$, then G cannot have a universal vertex.*

A set $B \subsetneq V(G)$ is a *slow* standard (respectively, PSD) forcing set if $\text{pt}(G, B) \geq 2$ (respectively, $\text{pt}_+(G, B) \geq 2$).

Lemma 5.4. *If G is a join of two graphs and at least one of the two conditions $\overline{\text{pt}}_+(G) = 1$ or $\overline{\text{pt}}(G) = 1$ holds, then none of the graphs P_4 , $P_3 \dot{\cup} K_1$, or $K_2 \dot{\cup} 2K_1$ is an induced subgraph of G .*

Proof. Suppose that $G = G_1 \vee G_2$ and at least one of the conditions $\overline{\text{pt}}_+(G) = 1$ or $\overline{\text{pt}}(G) = 1$ holds. The proof proceeds by assuming that G has an induced subgraph H isomorphic to one of the three listed graphs, and showing that this implies a slow (standard or PSD) forcing set exists. Figure 5.1 illustrates the proof.

In all cases, the four vertices of H must all lie within a single side of the join, without loss of generality in G_1 , because H does not contain a 4-cycle and has no vertex of degree greater than 2. In the case that $H = P_4$, label its vertices (w, x, y, z) in order. In the case that $H = P_3 \dot{\cup} K_1$, let the vertices of the P_3 be labeled (w, x, y) in order and let the vertex of the K_1 be labeled z . In the case that $H = K_2 \dot{\cup} 2K_1$, let the vertices of the induced K_2 be labeled x and y and let the remaining vertices be labeled w and z . Let p be any vertex of G_2 . and let $B = V(G) \setminus \{p, y, z\}$.

No blue vertex can force y or z in the first round, because the set of white vertices is connected and every blue vertex is either in G_1 , and also adjacent to the white vertex p , or is in G_2 , and is adjacent to both white vertices y and z . The blue vertex w is adjacent to p and to no other white vertex, so w forces p in first round, then in the second round the vertex x forces y . The vertex z can be forced by p in the third round if it was not already forced in the second round. Thus B is a slow forcing set of G . $\square$

The next result extends the power of Lemma 5.4.

Theorem 5.5. [2, Theorem 23] *Let H be a connected graph. Then H is $\{P_4, \text{paw}, \text{diamond}\}$ -free. if and only if $H \cong K_n$ for some $n > 0$ or $H \cong K_{m,n}$ for some $m, n > 0$.*

Corollary 5.6. *If G is a join of two graphs and at least one of the two conditions $\overline{\text{pt}}_+(G) = 1$ or $\overline{\text{pt}}(G) = 1$ holds, then all of the components of $\overline{G}$ are complete graphs or complete bipartite graphs.*

Proof. Let G be a join of two graphs and suppose that at least one of $\overline{\text{pt}}_+(G) = 1$ or $\overline{\text{pt}}(G) = 1$ holds. By Lemma 5.4, none of P_4 , $P_3 \dot{\cup} K_1$, or $K_2 \dot{\cup} 2K_1$ is an induced subgraph of G ; therefore $\overline{G}$ is $\{P_4, \text{paw}, \text{diamond}\}$ -free. Therefore, all of its components are as well, so by Theorem 5.5, they are complete graphs or complete bipartite graphs. $\square$

Now that we have preliminary tools that apply to both standard and PSD propagation time, we focus first on the PSD case. A set $B \subseteq V(G)$ is *independent* if $G[B]$ has no edges. The *independence number* of G , denoted by $\alpha(G)$, is the maximum cardinality of an independent set.

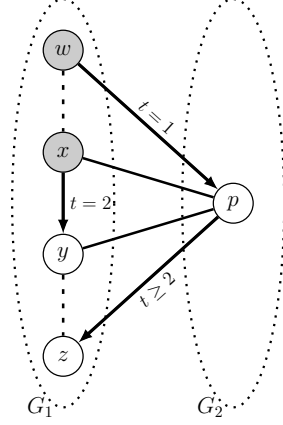


Figure 5.1: Illustration of the proof of Lemma 5.4. The dotted line from w to x indicates an edge except for the case of $K_2 \dot{\cup} 2K_1$. The dashed line from y to z is only an edge in the case of P_4 .

Lemma 5.7. *If $\overline{\text{pt}}_+(G) = 1$ and G is a join of two graphs, then $\alpha(G) \leq 2$ and therefore $\overline{G}$ has no induced subgraphs isomorphic to K_3 .*

Proof. Suppose $G = G_1 \vee G_2$ and there are three independent vertices u, v , and w . These must all lie in one side of the join, without loss of generality in G_1 . Let p be any vertex of G_2 . We show that the complement of the vertex set $\{p, u, v\}$ is a PSD forcing set that forces in time greater than one. No blue vertex can force u in the first round, because the set of white vertices is connected and every blue vertex that is adjacent to u is either in G_1 , and is also adjacent to p , or is in G_2 and is also adjacent to v . However, the blue vertex w is adjacent to p and no other white vertex, and so in the first round it forces p . Now the set of white vertices $\{u, v\}$ is disconnected, and in the second round, p can force both u and v . $\square$

Theorem 5.8. *Let $\overline{\text{pt}}_+(G) = 1$ and let G be a join of two graphs. Then G is a PSD fast join.*

Proof. Since G is the join of two graphs, $\overline{G}$ has at least two connected components. By Corollary 5.6 and Lemma 5.7, every component of $\overline{G}$ is a complete bipartite graph. Consider a component of $\overline{G}$ and let A and B denote the partite sets. If A (or B) is empty, then B (or A) contains an isolated vertex of $\overline{G}$ and therefore a universal vertex of G , implying that $G \cong K_n$ (by Corollary 5.3). $\square$

Next we consider standard forcing.

Lemma 5.9. *Let G be a join of two graphs G_1 and G_2 , and $\overline{G_1} \cong K_{1,r}$ with $r \geq 2$. Then $\overline{\text{pt}}(G) \geq 2$.*

Proof. Let c be the center of the star $\overline{G_1}$, let u and v be leaves of $\overline{G_1}$, let $w \in V(G_2)$, and define $B = V(G) \setminus \{c, v\}$. No vertex in G_2 can perform a force in the first round since such a vertex is adjacent to both v and c in G . Since u is adjacent in G to v but not to c , u can force v in the first round. However, since $N_G(c) = V(G_2)$, c cannot be forced in the first round. After round one, c is the only white vertex, so any of its neighbors (e.g., w) can then force c . $\square$

Theorem 5.10. *If $\overline{\text{pt}}(G) = 1$ and G is a join of two graphs, then G is a standard fast join.*

Proof. Assume $\overline{\text{pt}}(G) = 1$ and G is a join of two graphs, so $\overline{G}$ has at least two connected components. If G is not only a join but a complete graph, then G is indeed a standard fast join. Otherwise it suffices to show, for any component H of $\overline{G}$, that H is either complete of order at least two, or is complete bipartite with each partite set containing at least two vertices.

By Theorem 5.5, H is either complete or complete bipartite. If H is complete, it must be of order at least two so as to avoid the universal vertices that are forbidden by Corollary 5.3. This leaves only the case where H is complete bipartite. The case where one partite set has exactly one vertex and the other has more than one makes H a star graph of the sort forbidden by Lemma 5.9. Thus A and B each contain at least two vertices. $\square$

The next two results allow us to resolve Conjectures 1.7 and 1.8 for graphs G with the associated forcing number at least $n - 2$.

Theorem 5.11. [1, Theorem 5.1], [3, Theorem 14] *A graph G of order n has $Z(G) \geq n - 2$ if and only if the complement of G is of the form*

$$(K_{s_1} \dot{\cup} \cdots \dot{\cup} K_{s_t} \dot{\cup} K_{p_1, q_1} \dot{\cup} \cdots \dot{\cup} K_{p_k, q_k}) \vee K_r$$

for appropriate nonnegative integers $t, k, s_1, \dots, s_t, p_1, q_1, \dots, p_k, q_k, r$ with $p_i + q_i > 0$ for all $i = 1, \dots, k$.

Corollary 5.12. *Let G be a connected graph of order $n \geq 2$. If $Z(G) \geq n - 2$ and $\overline{\text{pt}}(G) = 1$, then G is a standard fast join. If $Z_+(G) \geq n - 2$ and $\overline{\text{pt}}_+(G) = 1$, then G is a PSD fast join.*

Proof. Assume $Z(G) \geq n - 2$ or $Z_+(G) \geq n - 2$; the latter implies $Z(G) \geq n - 2$. We apply Theorem 5.11 to show G is a join of two graphs: Observe first that since G is connected, $r = 0$, i.e., $\overline{G} = K_{s_1} \dot{\cup} \cdots \dot{\cup} K_{s_t} \dot{\cup} K_{p_1, q_1} \dot{\cup} \cdots \dot{\cup} K_{p_k, q_k}$. If $t \geq 1$ and $k \geq 1$, or $t \geq 2$, or $k \geq 2$ then G is a join. If $k = 0$ then $t \geq 2$ since G is connected and $n \geq 2$. So suppose $t = 0$ and $k = 1$. Note that $\overline{G} = K_{p_1, q_1}$ with $p_1, q_1 \geq 1$ would result in a disconnected graph G . So $p_1 \geq 2$ and $q_1 = 0$, and $G = K_{p_1}$ is a complete graph (which is a join since $n \geq 2$). Since G is a join, the statements follow from Theorems 5.8 and 5.10. $\square$

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