NON-SQUEEZING AND OTHER GLOBAL RIGIDITY RESULTS IN LOCALLY CONFORMAL SYMPLECTIC GEOMETRY

MÉLANIE BERTELSON, PRANAV CHAKRAVARTHY, AND SHEILA SANDON

We dedicate this work to Michèle Audin

ABSTRACT. Using generating functions quadratic at infinity for Lagrangian submanifolds of twisted cotangent bundles, we define spectral selectors for compactly supported lcs Hamiltonian diffeomorphisms of the locally conformal symplectizations $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$ of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$, and obtain several applications: the construction of a bi-invariant partial order on the group of compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$, of an integer-valued bi-invariant metric on the group of compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$, and of an integer-valued lcs capacity for domains of $S^1 \times \mathbb{R}^{2n} \times S^1$. The lcs capacity is used to prove a lcs non-squeezing theorem in $S^1 \times \mathbb{R}^{2n} \times S^1$ analogous to the contact non-squeezing theorem in $\mathbb{R}^{2n} \times S^1$ discovered in 2006 by Eliashberg, Kim and Polterovich. Along the way we introduce for Liouville lcs manifolds the notions of essential Lee chords between exact Lagrangian submanifolds and of essential translated points of exact lcs diffeomorphisms. We prove that essential translated points always exist for time-1 maps of sufficiently $\mathcal{C}^0$-small lcs Hamiltonian isotopies of compact Liouville lcs manifolds and for all compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$. We also obtain an existence result for essential Lee chords between the zero section of a twisted cotangent bundle with compact base and its image by any lcs Hamiltonian isotopy, which can be thought of as a lcs analogue of the Lagrangian and Legendrian Arnold conjectures on usual cotangent and 1-jet bundles. Finally, we introduce the notion of orderability for lcs manifolds, and prove that $S^1 \times \mathbb{R}^{2n+1}$, $S^1 \times \mathbb{R}^{2n} \times S^1$ and twisted cotangent bundles are orderable.

1. INTRODUCTION

A locally conformal symplectic (lcs) form on a manifold M of dimension $2n \geq 4$ is a non-degenerate 2-form ω such that, for some cover $\{\mathcal{U}_i\}$ of M, $\omega|_{\mathcal{U}_i} = e^{\mu_i}\,\omega_i$ for a function $\mu_i : \mathcal{U}_i \to \mathbb{R}$ and a symplectic form ω_i on $\mathcal{U}_i$. Equivalently, a non-degenerate 2-form ω is lcs if there exists a closed 1-form η (determined by ω, and called the Lee form) such that $d_\eta \omega = 0$, where d_η is the Lichnerowicz–de Rham differential (defined by $d_\eta \sigma = d\sigma - \eta \wedge \sigma$). A lcs structure is an equivalence class of lcs forms under the equivalence relation $\omega \sim e^f \omega$ for $f : M \to \mathbb{R}$, equivalently an equivalence class $[(\eta, \omega)]$ of pairs formed by a closed 1-form η and a non-degenerate 2-form ω with $d_\eta \omega = 0$, for the equivalence relation $(\eta, \omega) \sim (\eta + df, e^f \omega)$. A lcs manifold is a manifold endowed with a lcs structure. If $\big(M, [(\eta, \omega)]\big)$ is a lcs manifold such that for some (hence any) representative (η, ω) the Lee form η is exact, then for any primitive μ of η the form $e^{-\mu}\omega$ is a symplectic form whose conformal class is determined by the lcs structure $[(\eta, \omega)]$. Conformal symplectic manifolds are thus examples of lcs manifolds. Other important examples of lcs manifolds are the twisted cotangent bundle $T^*_\beta B := \big(T^*B\,,\,[(\pi^*\beta, d_{\pi^*\beta}\lambda)]\big)$ of a manifold B with respect to a closed 1-form β on B, where $\pi : T^*B \to B$ is the projection and λ is the tautological 1-form, and the locally conformal symplectization $\big(S^1 \times Y\,,\,[(-d\theta, d_{-d\theta}\alpha)]\big)$ of a co-oriented contact manifold (Y, ξ) with respect to a contact form α, where θ denotes the coordinate in S^1. In the present article we study in particular the locally conformal symplectization of the standard contact Euclidean space $(\mathbb{R}^{2n+1}, \xi_0)$ with respect to the contact form $\alpha_0 = \sum_{j=1}^n \frac{x_i dy_j - y_j dx_j}{2} - dz$, and of the quotient $\mathbb{R}^{2n} \times S^1 = \mathbb{R}^{2n} \times \mathbb{R}/\mathbb{Z}$, endowed with the induced contact structure and contact form. We denote these two lcs manifolds simply by $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$.

Lcs manifolds have been studied since the 40s by several authors, among others Lee [Lee43], Libermann [Lib55], Vaisman [Vai76, Vai85], Lichnerowicz [Lic83], Banyaga [Ban02], Otiman and Stanciu [OS17], Apostolov and Dloussky [AD15, AD18], Bazzoni and Marrero [BaMa18], Chantraine and Murphy [CM19], Eliashberg and Murphy [EM23], Bertelson and Meigniez [BeMe24], Allais and Arnaud [AArn24], Chantraine and Sackel [CS24] and Currier [Cur24, Cur25]. Most of the basic notions of symplectic topology can be generalized to the lcs case. For instance, Lagrangian submanifolds and Hamiltonian isotopies on a lcs manifold $\big(M, [(\eta,\omega)]\big)$ are defined as follows. A Lagrangian submanifold is a half-dimensional submanifold on which for some (hence any) representative (η,ω) the lcs form ω vanishes. A vector field X_t is said to be Hamiltonian if for some (hence any) representative (η,ω) the form $\iota_{X_t}\omega$ is η-exact. The flow of X_t is then said to be a lcs Hamiltonian isotopy, and its time-1 map a lcs Hamiltonian diffeomorphism. In particular, lcs geometry can be regarded as a generalization of symplectic geometry that allows to study Hamiltonian dynamics is a more general context (and on many more spaces, since the class of manifolds admitting a lcs structure is much bigger than the class of manifolds admitting a global symplectic form [EM23, BeMe24]). The usual local flexibility properties of symplectic and contact topology (in particular the Darboux and Weinstein theorems) have natural lcs analogues. On the other hand, the study of global rigidity phenomena, which is a central theme of research in symplectic and contact topology, is less explored in the lcs case and is the main subject of the present article.

The following global rigidity result in lcs geometry has been proved by Chantraine and Murphy [CM19] (see also [Cur25]). Consider a twisted cotangent bundle $T^*_\beta B$ with compact base B, and a Lagrangian submanifold L in $T^*_\beta B$ that is lcs Hamiltonian isotopic to the zero section. Suppose that the intersections of L with the zero section are transverse. Then their number is at least equal to the rank of the Novikov homology of B with respect to β. Notice that this result does not imply that L must always intersect the zero section, indeed the Novikov homology can vanish (for instance if β has no zeros). In fact, it is easy to see that if β has no zeros then there is a lcs Hamiltonian isotopy (the standard Lee flow) that displaces the zero section. The naive analogue of the Lagrangian Arnold conjecture on usual cotangent bundles, proved by Laudenbach and Sikorav [LS85], thus does not hold in the lcs case. However, the situation is similar to what already happens in the contact case for 1-jet bundles, where the standard Reeb flow also displaces the zero section. In the contact case the following analogue of the Lagrangian Arnold conjecture, where intersections are replaced by Reeb chords, has been proved by Chekanov [Che96] and Chaperon [Cha95]. We endow the 1-jet bundle $J^1B = T^*B \times \mathbb{R}$ with its standard contact structure $\xi = \ker(\lambda - dz)$, where z is the coordinate on $\mathbb{R}$. If B is compact then for any Legendrian submanifold Λ of J^1B that is the image of the zero section by a contact isotopy the number of Reeb chords with respect to $\lambda - dz$ between Λ and the zero section is greater than or equal to the cup-length of B, and greater than or equal to the sum of the Betti numbers if all the Reeb chords are transverse. Our first observation is that a similar result also holds in the lcs case for essential Lee chords on twisted cotangent bundles. Before precisely stating the result, we need to recall or introduce a few more notions, referring to Sections 2 and 3 for more details. The Lee flow on a lcs manifold $\big(M, [(\eta,\omega)]\big)$ with respect to a representative (η,ω) is the flow $\{\varphi^\omega_t\}$ of the vector field R_ω defined by the relation $\omega(R_\omega, \cdot\,) = \eta$ (equivalently, the lcs Hamiltonian isotopy with Hamiltonian function 1 with respect to (η,ω)). For instance, the Lee flow on a locally conformal symplectization $\big(S^1 \times Y\,,\, [(-d\theta, d_{-d\theta}\alpha)]\big)$ with respect to $(-d\theta, d_{-d\theta}\alpha)$ is $\{\mathrm{id} \times \varphi^\alpha_t\}$, where $\{\varphi^\alpha_t\}$ is the Reeb flow on Y with respect to α, and the Lee flow on a twisted cotangent bundle $T^*_\beta B$ with respect to $(\pi^*\beta, d_{\pi^*\beta}\lambda)$ is given by $\varphi^\omega_t(\sigma_q) = \sigma_q + t\beta(q)$. Let L_1 and L_2 be two Lagrangian submanifolds of a lcs manifold $\big(M, [(\eta,\omega)]\big)$. A Lee chord with respect to (η,ω) from L_1 to L_2 is a pair (T,γ) with T a real number (called the time-shift of the Lee chord) and $\gamma : [0,1] \to M$ a curve such that $\gamma(0) \in L_1$, $\gamma(1) \in L_2$ and $\dot\gamma(t) = T\, R_\omega\big(\gamma(t)\big)$ for all t. We say that a Lee chord (T,γ) with respect to (η,ω) from L_1 to L_2 is transverse if $(\varphi^\omega_T)_*\big(T_{\gamma(0)}L_1\big) + T_{\gamma(1)}L_2 = T_{\gamma(1)}M$. A lcs manifold $\big(M, [(\eta,\omega)]\big)$ is said to be exact if for some (hence any) representative (η,ω) the lcs form ω is η-exact. Any 1-form λ such that $\omega = d_\eta\lambda$ is then called a Liouville form for (η,ω). If λ is a Liouville form for (η,ω) then, for every function f, $e^f\lambda$ is a Liouville form for $(\eta + df, e^f\omega)$. A Liouville lcs manifold is a manifold M endowed with an equivalence class $[(\eta,\omega,\lambda)]$ of triples formed by a closed 1-form η, a non-degenerate 2-form ω and a 1-form λ such that $\omega = d_\eta\lambda$, under the equivalence relation $(\eta,\omega,\lambda) \sim (\eta + df, e^f\omega, e^f\lambda)$. For instance, twisted cotangent bundles $\big(T^*B\,,\, [(\pi^*\beta, d_{\pi^*\beta}\lambda, \lambda)]\big)$ and locally conformal symplectizations $\big(S^1 \times Y\,,\, [(-d\theta, d_{-d\theta}\alpha, \alpha)]\big)$ are Liouville lcs manifolds. A Lagrangian submanifold L of a Liouville lcs manifold $\big(M, [(\eta,\omega,\lambda)]\big)$ is said to be exact if the

1-form $i^*\lambda$, where $i : L \to M$ is the inclusion, is $i^*\eta$-exact. In particular, any Lagrangian submanifold of $T^*_\beta B$ that is the image of the zero section by a lcs Hamiltonian diffeomorphism is exact (Corollary 2.32). We say that an exact Lagrangian submanifold L of a Liouville lcs manifold $\big(M, [(\eta,\omega,\lambda)]\big)$ has action $S_L : L \to \mathbb{R}$ with respect to (η,ω,λ) if $i^*\lambda = d_{i^*\eta} S_L$. If L is connected and the restriction of η to L is not exact then such action S_L is uniquely defined (see Remark 2.1). In this case we define the action with respect to (η,ω,λ) of a Lee chord (T,γ) between two exact Lagrangian submanifolds L_1 and L_2 to be the real number

$$\mathcal{A}_{L_1,L_2}(\gamma) = S_{L_1}\big(\gamma(0)\big) - S_{L_2}\big(\gamma(1)\big) + \int_\gamma \lambda\,.$$

This notion generalizes the usual symplectic action of a Lagrangian intersection (see Example 3.1) and the usual contact action of a Reeb chord between two Legendrian submanifolds of a contact manifold (the integral of the contact form along the Reeb chord, see Example 3.2). While in the contact case the action of a Reeb chord is always equal to its time-shift, in the lcs case these two notions differ in general. The Lee chords of time-shift equal to their action play an important role in our work. In analogy with the terminology introduced in [Cur24] for Liouville chords, we call them *essential Lee chords*. Our first result is then the following theorem.

Theorem 1.1 (Essential Lee chords)**.** *Consider a connected compact manifold B, a closed non-exact 1-form β on B and a Lagrangian submanifold L of the twisted cotangent bundle $T^*_\beta B$ that is lcs Hamiltonian isotopic to the zero section. Then the number of essential Lee chords with respect to $(\pi^*\beta, d_{\pi^*\beta}\lambda, \lambda)$ from L to the zero section is greater than or equal to the cup-length of B, and greater than or equal to the sum of the Betti numbers of B if all the essential Lee chords are transverse.*

Similarly to the theorem of Chekanov [Che96] and Chaperon [Cha95] in the contact case, Theorem 1.1 is a direct consequence of existence of generating functions quadratic at infinity. As observed by Chantraine and Murphy [CM19], the theory of generating functions in symplectic and contact topology generalizes quite easily to the lcs case (in contrast to the theory of holomorphic curves, whose generalization seems more delicate). Recall first that in the symplectic and contact cases the easiest examples of generating functions are given by the following observation: for every function $f : B \to \mathbb{R}$, the differential $df : B \to T^*B$ and the 1-jet $j^1 f : B \to J^1 B$ are respectively a Lagrangian and a Legendrian embedding. We say that f is a generating function for the images of these embeddings. Similarly, the twisted differential $d_\beta f : B \to T^*_\beta B$ is a Lagrangian embedding, and we say that f is a generating function for the image of this embedding. As in the symplectic and contact cases, this elementary observation can be generalized as follows. Let $F : E \to \mathbb{R}$ be a function defined on the total space of a trivial vector bundle $E = B \times \mathbb{R}^N \to B$. Suppose that the differential $dF : E \to T^*E$ is transverse to the subbundle N^*_E of T^*E formed by the covectors that vanish on vertical vectors. Then the space $\Sigma_F = dF^{-1}\big(N^*_E \cap \mathrm{im}(dF)\big)$ of fibre critical points of F is a submanifold of E of dimension equal to the dimension of B, and the map $i_{F,\beta} : \Sigma_F \to T^*_\beta B$ that sends $e = (q,\zeta) \in \Sigma_F$ to the covector $i_{F,\beta}(e)$ at q defined by

$$i_{F,\beta}(e)(X) = d_\beta F(\hat{X}),$$

where $\hat{X}$ is any vector in $T_e E$ projecting to X, is a Lagrangian immersion. If $i_{F,\beta}$ is an embedding we say that F is a generating function for the Lagrangian submanifold $L_{F,\beta} = \mathrm{im}(i_{F,\beta})$ of $T^*_\beta B$. Chantraine and Murphy [CM19] proved that if B is compact then every Lagrangian submanifold of $T^*_\beta B$ that is lcs Hamiltonian isotopic to the zero section has a generating function quadratic at infinity, unique up to stabilization and fibre preserving diffeomorphism. This result is in fact a consequence of the corresponding existence and uniqueness theorems for generating functions quadratic at infinity for Legendrian submanifolds of 1-jet bundles, proved by Chekanov [Che96], Chaperon [Cha95], Viterbo [Vit92] and Théret [The99, The95]. Indeed, let L be a Lagrangian submanifold of $T^*_\beta B$ that is lcs Hamiltonian isotopic to the zero section. Then we can consider the image of the lift $\widetilde{L}$ of L to the contactization $J^1_\beta B := \big(J^1 B\,,\, \ker(\lambda - d_\beta z)\big)$ of $T^*_\beta B$ by the strict contactomorphism

$$U_\beta : J^1_\beta B \to J^1 B\,,\ (\sigma_q, z) \mapsto \big(\sigma_q + z\beta(q)\,,\, z\big)\,,$$

and observe that F is a generating function for L if and only if it is a generating function (in the usual sense) for the Legendrian submanifold $U_\beta(\widetilde{L})$ of $J^1 B$ (see [CM19] or Section 4 for more details). To obtain their result on intersections with the zero section of a Lagrangian submanifold L of $T^*_\beta B$ lcs Hamiltonian isotopic to the zero section, Chantraine and Murphy looked at the twisted critical points of a generating function quadratic at infinity F for L (the zeros of the twisted differential $d_\beta F$). These points do not always exist (their number in the non-degenerate case being bounded below by the rank of the Novikov homology [CM19, Cur25]). On the other hand, in order to prove Theorem 1.1 we look at the usual critical points of F, which always exist (since F is quadratic at infinity). We observe that the (non-degenerate) critical points of F are in bijection with the (transverse) essential Lee chords from L to the zero section (Lemma 4.1). Theorem 1.1 is then a consequence of the fact that the number of critical points of a function quadratic at infinity over B is greater than or equal to the cup-length of B, and greater than or equal to the sum of the Betti numbers of B if all the critical points are non-degenerate.

Our second result concerns essential translated points of lcs Hamiltonian diffeomorphisms. Recall that a point p of a co-oriented contact manifold (Y, ξ) is said to be a translated point of a contactomorphism ϕ with respect to a contact form α if p and $\phi(p)$ are in the same orbit of the Reeb flow of α and α is preserved by ϕ at p. This notion has been introduced in [Bhu01, San11a] in the case of the standard contact Euclidean space, and in [San11b, San12] in general. In [San11b, San12], the third author has observed that the time-1 map of a sufficiently $\mathcal{C}^0$-small contact isotopy on a compact contact manifold always has translated points (at least as many as the cup-length of the manifold, and as many as the sum of the Betti numbers in the non-degenerate case) and, in analogy with the Arnold conjecture on fixed points of Hamiltonian diffeomorphisms, proposed to study the question of existence of translated points for more general contactomorphisms contact isotopic of the identity. Existence of translated points (for compactly supported contactomorphisms when the contact manifold is not compact) has been proved beyond the $\mathcal{C}^0$-small case on several contact manifolds [San11b, San12, AM13, AFM15, She17, MN18, MU19, GKPS20, Ter21, All22a, All22b, AArl23]. On the other hand, examples of contactomorphisms contact isotopic to the identity without translated points have also been found on the standard contact sphere [Can24] and on a more general class of contact manifolds [HS24]. The question of existence of translated points is probably related to the notion of orderability, introduced by Eliashberg and Polterovich [EP00]. We say that a contact isotopy on a co-oriented contact manifold (Y, ξ) is non-negative if it is generated by a non-negative Hamiltonian function, and thus moves every point in a direction positively transverse or tangent to the contact distribution. The contact manifold (Y, ξ) is then said to be orderable if the partial relation $\leq$ on the universal cover of the identity component of the group of compactly supported contactomorphisms defined by posing $[\{\phi_t^1\}] \leq [\{\phi_t^2\}]$ if $[\{\phi_t^2\}] \cdot [\{\phi_t^1\}]^{-1}$ can be represented by a non-negative contact isotopy is a bi-invariant partial order. Most contact manifolds for which translated points are proved to exist for all contactomorphisms contact isotopic to the identity are also known to be orderable. On the other hand, the standard contact sphere, on which existence of translated points fail, is not orderable [EKP06], and the results in [HS24, AArl23] also seem to suggest a relation between orderability and existence of translated points. We also note that the standard contact sphere does not admit any non-trivial quasimorphism or bi-invariant metric on the universal cover of the contactomorphism group [FPR18], in contrast to several of the contact manifolds where translated points always exist [San10, CS24, Giv90, GKPS20, FPR18, BZ15]. Existence of translated points of contactomorphisms is a manifestation of global rigidity, as well as orderability and the existence of quasimorphisms and bi-invariant metrics on (the universal cover of) the contactomorphism group.

In analogy with the contact case, we say that a point p on a lcs manifold $\big(M, [(\eta, \omega)]\big)$ is a translated point of a lcs diffeomorphism φ (a diffeomorphism such that $\varphi^*\omega = e^g\omega$ for some function $g : M \to \mathbb{R}$, called the conformal factor) with respect to a representative (η, ω) if p and $\varphi(p)$ are in the same orbit of the Lee flow of (η, ω) and ω is preserved by φ at p (thus $g(p) = 0$). This notion generalizes the notions of fixed points and leafwise intersections of symplectomorphisms (see Example 3.7) and of translated points of contactomorphisms (see Example 3.8). A pair (T, γ) with $T \in \mathbb{R}$ and $\gamma : [0, 1] \to M$ a curve such that $\gamma(0) = \varphi(p)$, $\gamma(1) = p$ and $\dot\gamma(t) = T\, R_\omega\big(\gamma(t)\big)$ for all t is said to be a Lee chord of time-shift T for the translated point p of φ. Such a Lee chord is said to be transverse if there is no non-zero vector $Y \in T_pM$ such that $(\varphi_T^\omega \circ \varphi)_*(Y) = Y$ and $dg(Y) = 0$. A lcs diffeomorphism φ of a Liouville lcs manifold $\big(M, [(\eta, \omega)]\big)$ is said to be exact if for some (hence any) representative (η, ω, λ) the 1-form $\lambda - e^{-g}\varphi^*\lambda$, where g is the conformal factor of φ, is η-exact. In particular, lcs Hamiltonian diffeomorphisms are exact (Proposition 2.24). We say that an exact lcs diffeomorphism φ has action $S_\varphi : M \to \mathbb{R}$ with respect to (η, ω, λ) if $\lambda - e^{-g}\varphi^*\lambda = d_\eta S_\varphi$. If M is connected and η is not exact then such action S_φ is uniquely defined (see

Remark 2.1). In this case, we define the action with respect to (η,ω,λ) of a Lee chord (T,γ) for a translated point p of a lcs Hamiltonian diffeomorphism φ to be the real number

$$\mathcal{A}_\varphi(T,\gamma) = -S_\varphi(p) + \int_\gamma \lambda\,.$$

This notion generalizes the usual symplectic action of fixed points of Hamiltonian diffeomorphisms (see Example 3.9) and the usual contact action of Reeb chords for translated points of contactomorphisms (see Example 3.11). We say that a Lee chord for a translated point is essential if it has time-shift equal to the action, and that a translated point is essential if it admits an essential Lee chord.

Theorem 1.2 (Essential translated points)**.**

(i) *Let φ be the time-1 map of a sufficiently $\mathcal{C}^0$-small lcs Hamiltonian isotopy of a compact connected Liouville lcs manifold $\big(M,[(\eta,\omega,\lambda)]\big)$ with η not exact. Then the number of essential translated points of φ with respect to any representative (η,ω) is greater than or equal to the cup-length of M, and greater than or equal to the sum of the Betti numbers of M if all the essential Lee chords for the essential translated points are non-degenerate.*

(ii) *Every compactly supported lcs Hamiltonian diffeomorphisms of $S^1\times\mathbb{R}^{2n+1}$ or $S^1\times\mathbb{R}^{2n}\times S^1$ with non-empty support has at least one essential translated point with respect to $(-d\theta, d_{-d\theta}\alpha_0,\alpha_0)$ in the interior of its support.*

The first point of Theorem 1.2 is a consequence of Theorem 1.1. The argument is analogue to the one for the corresponding result on translated points of contactomorphisms [San11b, San12]. In the contact case one observes that the translated points of a contactomorphism ϕ of a contact manifold (Y,ξ) with respect to a contact form α are in bijection with the Reeb chords between the contact graph of ϕ in the contact product of (Y,ξ) with respect to α and the contact graph of the identity. By the $\mathcal{C}^0$-smallness assumption, the contact graph of ϕ is contained in a Weinstein neighborhood of the contact graph of the identity, and so it can be identified to a Legendrian submanifold of a 1-jet bundle, Legendrian isotopic to the zero section. Existence of translated points then follows from the existence of Reeb chords between such Legendrian and the zero section. In the lcs case, there is a similar construction. The lcs product of (Liouville) lcs manifolds and the lcs graph of lcs diffeomorphisms have been defined by Chantraine and Sackel [CS24] (although we give a simpler description of the lcs graph of divergence free lcs diffeomorphisms, in particular lcs Hamiltonian diffeomorphisms, see Section 2.11). We observe that the (essential) Lee chords for the translated points of a lcs Hamiltonian diffeomorphism φ are in bijection with the (essential) Lee chords between its lcs graph and the lcs graph of the identity (Proposition 3.13). Using the Weinstein theorem in lcs geometry to identify a neighborhood of the lcs graph of the identity with a neighborhood of the zero section of a twisted cotangent bundle we then deduce Theorem 1.2 (i) from Theorem 1.1 (see Section 4).

Before indicating the main ideas of the proof of Theorem 1.2 (ii), we discuss how the notion of orderability can also be generalized to the lcs case, referring to Section 2.10 for more details. Let $\big(M,[(\eta,\omega)]\big)$ be a connected lcs manifold such that η is not exact. For any lcs Hamiltonian vector field X_t there is then a unique time dependent function H_t such that $\iota_{X_t}\omega = -d_\eta H_t$ (see Remark 2.1); we say that H_t is the Hamiltonian function with respect to (η,ω) of the lcs Hamiltonian isotopy generated by X_t. We then define a lcs Hamiltonian isotopy to be non-negative (respectively positive) if its Hamiltonian function with respect to some (hence any) representative (η,ω) is non-negative (respectively positive), and consider the partial relation $\leq$ on the universal cover of the group of compactly supported lcs Hamiltonian diffeomorphisms defined by posing $[\{\varphi_t^1\}]\leq[\{\varphi_t^2\}]$ if $[\{\varphi_t^2\}]\cdot[\{\varphi_t^1\}]^{-1}$ can be represented by a non-negative lcs Hamiltonian isotopy. We say that the lcs manifold $\big(M,[(\eta,\omega)]\big)$ is orderable if $\leq$ is a bi-invariant partial order, equivalently if there does not exist any non-negative non-constant contractible loop of compactly supported lcs Hamiltonian diffeomorphisms. The locally conformal symplectization of a contact manifold that is not orderable (for instance the standard contact sphere) is also not orderable, because a non-constant non-negative contractible loop of contactomorphisms can be lifted to a non-constant non-negative contractible loop of lcs Hamiltonian diffeomorphisms of the locally conformal symplectization (see Example 2.19). The converse implication is in general not clear, but we show that it holds for $S^1\times\mathbb{R}^{2n+1}$ and $S^1\times\mathbb{R}^{2n}\times S^1$. We also consider the partial relation $\leq$ on the group of compactly supported lcs Hamiltonian diffeomorphisms of a connected lcs manifold $\big(M, [(\eta, \omega)]\big)$ with η not exact defined by posing $\varphi_1 \leq \varphi_2$ if $\varphi_2 \circ \varphi_1^{-1}$ is the time-1 map of a non-negative lcs Hamiltonian isotopy, and say that $\big(M, [(\eta, \omega)]\big)$ is strongly orderable if $\leq$ is a bi-invariant partial order, equivalently if there does not exist any non-negative non-constant loop of compactly supported lcs Hamiltonian diffeomorphisms. In particular, strongly orderable lcs manifolds are orderable.

Theorem 1.3 (Orderability)**.**

(i) *Let B be a compact connected manifold, and β a closed non-exact* 1*-form. Then the twisted cotangent bundle $T^*_\beta B$ is strongly orderable.*
(ii) *The lcs manifolds $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$ are strongly orderable.*

The first point of Theorem 1.3 is an easy consequence of the corresponding result in the contact case for 1-jet bundles, proved by Colin, Ferrand and Pushkar [CFP17] and Chernov and Nemirovsky [CN10]. Indeed, a non-negative non-constant loop of compactly supported lcs Hamiltonian diffeomorphisms of $T^*_\beta B$ would lift to a non-negative non-constant loop of compactly supported contactomorphisms of $J^1_\beta B$, which would induce, after conjugating with the untwisting map $U_\beta : J^1_\beta B \to J^1 B$, a non-negative non-constant loop of compactly supported contactomorphisms of $J^1 B$ (see Section 2.10).

In order to prove Theorem 1.2 (ii) and Theorem 1.3 (ii) we use generating functions to define spectral selectors for compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$, analogous to the spectral selectors for compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ defined by Viterbo [Vit92] and the spectral selectors for compactly supported contactomorphisms contact isotopic to the identity of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$ defined respectively by Bhupal [Bhu01] and the third author [San10, San11a]. The key observation is that for the lcs manifold $S^1 \times \mathbb{R}^{2n+1}$ we do not just have the local identifications of neighborhoods of Lagrangian submanifolds of the lcs product with neighborhoods of the zero section of a twisted cotangent bundle given by the lcs Weinstein theorem, but also a global identification of the whole lcs product with the twisted cotangent bundle $T_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$, where θ denotes the coordinate on S^1. Indeed, the lcs product of $S^1 \times \mathbb{R}^{2n+1}$ can be identified to the locally conformal symplectization $S^1 \times (\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R})$ of the contact product of $\mathbb{R}^{2n+1}$ (Proposition 2.41), and the twisted cotangent bundle $T_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$ can be identified to the locally conformal symplectization $S^1 \times J^1\mathbb{R}^{2n+1}$ of $J^1\mathbb{R}^{2n+1}$ (Lemma 2.13). We can thus lift the strict contactomorphism from $\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$ to $J^1\mathbb{R}^{2n+1}$ used in [Bhu01, San10, San11a] to the locally conformal symplectizations to obtain the desired identification (see equation (9) in Section 5). By considering the image by this identification of the lcs graph of a lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n+1}$ we obtain a Lagrangian submanifold L_φ of $T_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$. If φ is compactly supported then L_φ can be compactified to a Lagrangian submanifold $\overline{L_\varphi}$ of $T_{-d\theta}(S^1 \times S^{2n+1})$, lcs Hamiltonian isotopic to the zero section. Similarly, we can associate to a compactly supported lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n} \times S^1$ a Lagrangian submanifold L_φ of $T_{-d\theta}(S^1 \times \mathbb{R}^{2n} \times S^1)$, which can be compactified to a Lagrangian submanifold $\overline{L_\varphi}$ of $T_{-d\theta}(S^1 \times S^{2n} \times S^1)$, lcs Hamiltonian isotopic to the zero section. Using generating functions quadratic at infinity for $\overline{L_\varphi}$ we define, exactly as in the contact case for contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$ [Bhu01, San11a, San10] spectral selectors $c_+(\varphi) \geq 0$ and $c_-(\varphi) \leq 0$ for any compactly supported lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$. These spectral selectors have the same properties as in the contact case (Proposition 6.3), and thus can be used to obtain similar applications. To start with, spectrality ($c_+(\varphi)$ and $c_-(\varphi)$ are equal to the action of an essential translated point of φ) and non-degeneracy ($c_+(\varphi) = c_-(\varphi) = 0$ if and only if φ is the identity) imply Theorem 1.2 (ii). Moreover, as we will see, the properties listed in Proposition 6.3 imply that the partial relation $\preceq$ on the group of compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$ defined by posing $\varphi_1 \preceq \varphi_2$ if $c_+(\varphi_1 \circ \varphi_2^{-1}) = 0$ is a bi-invariant partial order (Theorem 7.1). By comparing the partial relation $\preceq$ with the one $\leq$ used to define orderability we then obtain a proof of Theorem 1.3 (ii). Finally, we show that the arguments in [San11a, San10] can be generalized to the lcs case to obtain an unbounded integer-valued bi-invariant metric d on the group of compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$, compatible with the partial orders $\preceq$ and $\leq$, by posing

$$d(\varphi_1, \varphi_2) = \lceil c_+(\varphi_1 \circ \varphi_2^{-1}) \rceil - \lfloor c_-(\varphi_1 \circ \varphi_2^{-1}) \rfloor$$

(Theorems 7.3 and 7.4), and an integer-valued lcs capacity for domains, by posing

$$c(\mathcal{U}) = \sup\{\, \lceil c_+(\varphi) \rceil \mid \varphi \in \mathrm{Ham}(\mathcal{U}) \,\}\,,$$

where $\mathrm{Ham}(\mathcal{U})$ denotes the group of lcs diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$ that are the time-1 map of a compactly supported lcs Hamiltonian isotopy supported in $\mathcal{U}$ (see Lemma 8.1 and Proposition 8.2). Using the lcs capacity we then obtain the following lcs non-squeezing result, which is analogous to the contact non-squeezing theorem for integers in $\mathbb{R}^{2n} \times S^1$ discovered by Eliashberg, Kim and Polterovich [EKP06] and reproved in [San11a] using generating functions.

Theorem 1.4 (Lcs non-squeezing for integers). *If $\pi R_2^2 \leq k \leq \pi R_1^2$ for $k \in \mathbb{N}_0$ then there is no compactly supported lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n} \times S^1$ such that*

$$\varphi\big(\overline{S^1 \times B^{2n}(R_1) \times S^1}\big) \subset S^1 \times B^{2n}(R_2) \times S^1\,.$$

In [EKP06] it is also proved that if $\pi R_1^2 < 1$ then $B^{2n}(R_1) \times S^1$ can be squeezed by a contact isotopy into $B^{2n}(R_2) \times S^1$ for R_2 arbitrarily small. The analogous result is true also in the lcs case, indeed (using Example 2.19) the contact isotopy squeezing $B^{2n}(R_1) \times S^1$ into $B^{2n}(R_2) \times S^1$ can be lifted to a lcs Hamiltonian isotopy of $S^1 \times \mathbb{R}^{2n} \times S^1$ squeezing $S^1 \times B^{2n}(R_1) \times S^1$ into $S^1 \times B^{2n}(R_2) \times S^1$. In [Chi17, Fra16, FSZ23] the contact non-squeezing theorem for integers of [EKP06] and [San10] has been generalized to prove that if $1 \leq \pi R_2^2 \leq \pi R_1^2$ then there is no compactly supported contact isotopy of $\mathbb{R}^{2n} \times S^1$ squeezing the closure of $B^{2n}(R_1) \times S^1$ into $B^{2n}(R_2) \times S^1$. We expect that it should be possible to extend to the lcs case the equivariant generating function homology defined in [FSZ23], in order to prove that if $1 \leq \pi R_2^2 \leq \pi R_1^2$ then there is no compactly supported lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n} \times S^1$ such that

$$\varphi\big(\overline{S^1 \times B^{2n}(R_1) \times S^1}\big) \subset S^1 \times B^{2n}(R_2) \times S^1\,.$$

This will be the object of a forthcoming work.

Finally we remark that, as in the work of Serraille and Stojisavljević [SS24], Theorem 1.4 implies that the group of compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$ does not have the Rokhlin property (see Section 9).

The article is organized as follows. In Section 2 we gather the basic notions on lcs manifolds that are needed in the rest of the article. Moreover, we discuss in 2.10 the notion of orderability and prove Theorem 1.3 (i). In Section 3 we define essential Lee chords and translated points. In Section 4 we recall from [CM19] the theory of generating functions for Lagrangian submanifolds of twisted cotangent bundles, and prove Theorem 1.1 and Theorem 1.2 (i). In Section 5 we define generating functions for compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$, and in Section 6 we use them to define the spectral selectors c_+ and c_-. A first application of these spectral selectors, Theorem 1.2 (ii), is also discussed in Section 6. In Section 7 we use the spectral selectors to define a partial order on the group of compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$, and a bi-invariant metric on the group of compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$, and obtain a proof of Theorem 1.3 (ii). In Section 8 we define a lcs capacity for domains of $S^1 \times \mathbb{R}^{2n} \times S^1$, and use it to prove the lcs non-squeezing theorem for integers (Theorem 1.4). Finally, in Section 9 we discuss, following [SS24], how this non-squeezing result implies that the group of compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$ does not have the Rokhlin property.

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2. Locally conformal symplectic manifolds

In this section we present the basic notions and results on lcs geometry that are needed in the rest of the article. Although the material is classic (except for 2.11 on the lcs product and graph, introduced by

Chantraine and Sackel [CS24], and 2.10 on orderability), we include detailed proofs whenever we could not find them in the literature. Among the basic results discussed in this section, Corollary 2.9 is particularly important for us, because it allows to define the (unique) lift to the symplectic cover of a divergence free lcs diffeomorphism. Using this notion we give in Section 2.11 a description of the lcs graph of a divergence free lcs diffeomorphism that seems simpler than the one in [CS24]. In 2.10 we introduce the notion of orderability in lcs geometry, and prove Theorem 1.3 (i).

2.1. **The Lichnerowicz–de Rham differential.** The Lichnerowicz–de Rham differential with respect to a closed 1-form η on a manifold M is the map $\Omega^d(M) \to \Omega^{d+1}(M)$ defined by

$$d_\eta \sigma = d\sigma - \eta \wedge \sigma \,.$$

A form $\sigma \in \Omega^d(M)$ is said to be η-closed if $d_\eta \sigma = 0$, and η-exact if $\sigma = d_\eta \gamma$ for some $\gamma \in \Omega^{d-1}(M)$. The Lichnerowicz–de Rham differential satisfies the relations

$$d_\eta^2 = 0 \,,$$

$$d_{\eta+df}\sigma = e^f \, d_\eta(e^{-f}\sigma)$$

for any function $f : M \to \mathbb{R}$, and

$$(\iota_X \circ d_\eta + d_\eta \circ \iota_X)\sigma = \mathcal{L}_X \sigma - \eta(X)\sigma$$

for any vector field X.

Remark 2.1. *If M is connected and η is not exact then the equation $d_\eta f = 0$ for $f : M \to \mathbb{R}$ has a unique solution $f \equiv 0$. Indeed, suppose by contradiction that the open subset $\mathcal{U}$ on which f does not vanish is not empty. Then $\ln |f|_{\mathcal{U}}|$ is a primitive of η on $\mathcal{U}$, thus $\mathcal{U}$ is a proper subset of M and its closure $\overline{\mathcal{U}}$ has a non-empty boundary $\partial\overline{\mathcal{U}}$. Let p be a point of $\partial\overline{\mathcal{U}}$, and $\mathcal{V}_p$ a contractible neighborhood of p. On $\mathcal{V}_p$ the 1-form η has a primitive, say h_p. On each connected component of $\mathcal{U} \cap \mathcal{V}_p$ the functions $\ln|f|$ and h_p differ by a constant. The point p belongs to the closure of (at least) one of these connected components, which we denote by $\mathcal{U}_p$. Then $\ln |f|_{\mathcal{U}_p}| = h_p + c$, and so*

$$|f|_{\mathcal{U}_p}| = e^{h_p + c} \,.$$

In particular, f does not vanish at p and so p cannot belong to $\partial\overline{\mathcal{U}}$. This contradiction forces $\mathcal{U}$ to be empty, and so f to be constantly zero.

2.2. **Lcs structures.** A locally conformal symplectic (lcs) form on a (necessarily even dimensional) manifold M is a non-degenerate 2-form ω such that, for some open cover $\{\mathcal{U}_i\}$ of M,

$$\omega|_{\mathcal{U}_i} = e^{\mu_i}\, \omega_i$$

for a function $\mu_i : \mathcal{U}_i \to \mathbb{R}$ and a symplectic form ω_i on $\mathcal{U}_i$. If M is 2-dimensional then ω is necessarily closed, hence a symplectic form. From now on we thus assume that the dimension of M is at least 4. The fact that on each intersection $\mathcal{U}_i \cap \mathcal{U}_j$ we have $e^{\mu_i}\, \omega_i = e^{\mu_j}\, \omega_j$, and so

$$0 = d\omega_i = d(e^{\mu_j - \mu_i}\omega_j) = d(e^{\mu_j - \mu_i}) \wedge \omega_j \,,$$

implies then that $d(e^{\mu_j - \mu_i}) = 0$ and so $d\mu_i = d\mu_j$. The 1-forms $d\mu_i$ thus glue to a well-defined closed 1-form η on M. Since

$$d(\omega|_{\mathcal{U}_i}) = e^{\mu_i}\, d\mu_i \wedge \omega_i = d\mu_i \wedge \omega|_{\mathcal{U}_i}$$

for every i, we have $d\omega = \eta \wedge \omega$ and so $d_\eta \omega = 0$. Conversely, if ω is a non-degenerate 2-form on M such that $d_\eta\omega = 0$ for some closed 1-form η then ω is a lcs form. Indeed, there exists an open cover $\{\mathcal{U}_i\}$ such that η is exact on each $\mathcal{U}_i$, and so $\eta|_{\mathcal{U}_i} = d\mu_i$ for functions $\mu_i : \mathcal{U}_i \to \mathbb{R}$. Posing $\omega_i = e^{-\mu_i}\ \omega|_{\mathcal{U}_i}$, since $d\omega = \eta \wedge \omega$ we have

$$d\omega_i = e^{-\mu_i} \left(d(\omega|_{\mathcal{U}_i}) - \eta|_{\mathcal{U}_i} \wedge \omega|_{\mathcal{U}_i} \right) = 0 \,,$$

and so ω_i is a symplectic form. A lcs form on M can thus equivalently be defined as a non-degenerate 2-form ω such that $d_\eta\omega = 0$ for some closed 1-form η. Since ω is non-degenerate, the relation $d_\eta\omega = 0$ uniquely determines the closed 1-form η. We say that η is the Lee form of the lcs form ω.

A locally conformal symplectic (lcs) structure on M is an equivalence class of lcs forms under the equivalence relation
$$\omega \sim e^f \omega \quad \text{for } f : M \to \mathbb{R}\,.$$
If η is the Lee form of ω then the Lee form of $e^f\omega$ is $\eta + df$. We can thus equivalently define a lcs structure as an equivalence class $[(\eta, \omega)]$ of pairs formed by a closed 1-form η and a non-degenerate η-closed 2-form ω, under the equivalence relation
$$(\eta, \omega) \sim (\eta + df\,,\, e^f \omega) \quad \text{for } f : M \to \mathbb{R}\,.$$
A locally conformal symplectic (lcs) manifold is a manifold endowed with a lcs structure.

A lcs manifold $\big(M, [(\eta, \omega)]\big)$ is said to be exact if for some (hence any) representative (η, ω) the 2-form ω is η-exact. In this case, any 1-form λ such that $\omega = d_\eta \lambda$ is said to be a Liouville form for (η, ω). For any $f : M \to \mathbb{R}$, if λ is a Liouville form for (η, ω) then $e^f\lambda$ is a Liouville form for $(\eta + df\,,\, e^f\omega)$. We denote by $[(\eta, \omega, \lambda)]$ the equivalence class of triples (η, ω, λ) under the equivalence relation
$$(\eta, \omega, \lambda) \sim (\eta + df\,,\, e^f\omega\,,\, e^f\lambda) \quad \text{for } f : M \to \mathbb{R}\,,$$
and we say that $\big(M, [(\eta, \omega, \lambda)]\big)$ is a Liouville lcs manifold. The Liouville vector field of a Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$ is the vector field Z on M defined by $\iota_Z\omega = \lambda$ for some (hence any) representative (η, ω, λ). It only depends on the Liouville lcs structure $[(\eta, \omega, \lambda)]$, and not on a specific representative.

Example 2.2 (Conformal symplectic manifolds)**.** *If ω and $e^f\omega$ are symplectic forms on a manifold M for some function $f : M \to \mathbb{R}$, then f is necessarily constant. A conformal symplectic manifold is thus defined to be a manifold M endowed with an equivalence class $[\omega]$ of symplectic forms under the equivalence relation $\omega \sim e^c\omega$ for $c \in \mathbb{R}$. If $\big(M, [(\eta, \omega)]\big)$ is a lcs manifold such that for some (hence any) representative (η, ω) the Lee form η is exact, then for any primitive μ of η we have*
$$(\eta, \omega) = (d\mu, \omega) \sim (0, e^{-\mu}\omega)\,,$$
and so $e^{-\mu}\omega$ is a symplectic form. The conformal class $[e^{-\mu}\omega]$ of this symplectic form only depends on the lcs structure $[(\eta, \omega)]$. We can thus associate to the lcs manifold $\big(M, [(\eta, \omega)]\big)$ the conformal symplectic manifold $(M, [e^{-\mu}\omega])$. Conversely, we can identify a conformal symplectic manifold $(M, [\omega])$ with the lcs manifold $\big(M, [(0, \omega)]\big)$. We say that a conformal symplectic manifold $(M, [\omega])$ is exact if some (hence any) representative ω is exact. In this case, any 1-form λ such that $\omega = d\lambda$ is said to be a Liouville form for ω. If λ is a Liouville form for ω then $e^c\lambda$ is a Liouville form for $e^c\omega$. We denote by $[(\omega, \lambda)]$ the equivalence class of pairs (ω, λ) under the equivalence relation $(\omega, \lambda) \sim (e^c\omega, e^c\lambda)$, and say that $\big(M, [(\omega, \lambda)]\big)$ is a conformal Liouville manifold. If $\big(M, [(\eta, \omega)]\big)$ is an exact lcs manifold with $\eta = d\mu$ then the associated conformal symplectic manifold $(M, [e^{-\mu}\omega])$ is exact, and if λ is a Liouville form for (η, ω) then $e^{-\mu}\lambda$ is a Liouville form for $e^{-\mu}\omega$. We say that $\big(M, [(e^{-\mu}\omega, e^{-\mu}\lambda)]\big)$ is the conformal Liouville manifold associated to the Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$.

Example 2.3 (Twisted cotangent bundles)**.** *Let β be a closed 1-form on a manifold B, and consider the cotangent bundle $\pi : T^*B \to B$, endowed with the tautological 1-form λ. The twisted cotangent bundle of B with respect to β is the Liouville lcs manifold*
$$T^*_\beta B := \big(T^*B\,,\, [(\pi^*\beta\,,\, d_{\pi^*\beta}\lambda\,,\, \lambda)]\,\big).$$
The Liouville vector field of $T^*_\beta B$ coincides with the standard Liouville vector field, which in local coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ is given by $Z = \sum_i p_i \frac{\partial}{\partial p_i}$. If β is exact with primitive μ then
$$(\pi^*\beta\,,\, d_{\pi^*\beta}\lambda\,,\, \lambda) = \big(d(\mu\circ\pi)\,,\, d_{d(\mu\circ\pi)}\lambda\,,\, \lambda\big) \sim (0\,,\, e^{-\mu\circ\pi}d_{d(\mu\circ\pi)}\lambda\,,\, e^{-\mu\circ\pi}\lambda) = \big(0\,,\, d(e^{-\mu\circ\mu}\lambda)\,,\, e^{-\mu\circ\mu}\lambda\big)\,,$$
thus the conformal Liouville manifold associated to $T^*_\beta B$ is $\big(T^*B\,,\, [(d(e^{-\mu\circ\mu}\lambda), e^{-\mu\circ\mu}\lambda)]\big)$. The map $\sigma_q \mapsto e^{\mu(q)}\sigma_q$ is a diffeomorphism of T^*B that pulls back $e^{-\mu\circ\pi}\lambda$ to λ, hence the conformal Liouville structure $[(d(e^{-\mu\circ\mu}\lambda), e^{-\mu\circ\mu}\lambda)]$ to the conformal class $[(d\lambda, \lambda)]$ of the standard Liouville structure.

Example 2.4 (Locally conformal symplectizations)**.** *The locally conformal symplectization of a co-oriented contact manifold (Y, ξ) with respect to a contact form α for ξ is the Liouville lcs manifold*
$$\big(S^1 \times Y\,,\, [(-d\theta\,,\, d_{-d\theta}\alpha\,,\, \alpha)]\,\big),$$

where θ denotes the standard coordinate on S^1 and where we still denote by α its pullback by the projection $S^1 \times Y \to Y$. Its Liouville vector field is $\frac{\partial}{\partial\theta}$.

2.3. **Symplectic covers.** Let $\big(M, [(\eta,\omega)]\big)$ be a lcs manifold. Consider the cover $\pi : \bar{M} \to M$ corresponding to the kernel of the homomorphism

$$\langle [\eta], \cdot \rangle : \pi_1(M) \to \mathbb{R}\,, \ \langle [\eta], [\gamma] \rangle = \int_0^1 \eta\Big(\frac{d\gamma}{dt}\Big)\, dt\,,$$

thus $\bar{M}$ is the space of equivalence classes of paths in M starting at a fixed base point for the equivalence relation

$$\gamma_1 \sim \gamma_2 \quad \text{if } \gamma_1(1) = \gamma_2(1) \text{ and } \int_0^1 \eta\big(\frac{d\gamma_1}{dt}\big)\, dt = \int_0^1 \eta\big(\frac{d\gamma_2}{dt}\big)\, dt\,,$$

and the projection $\pi : \bar{M} \to M$ is given by $\pi([\gamma]) = \gamma(1)$. The 1-form $\pi^*\eta$ is exact, with primitive

$$\bar{M} \to \mathbb{R}\,, \ [\gamma] \mapsto \int_0^1 \eta\Big(\frac{d\gamma}{dt}\Big)\, dt\,.$$

The next lemma follows directly from the definitions.

Lemma 2.5. *Let $\mu : \bar{M} \to \mathbb{R}$ be a primitive of $\pi^*\eta$. If two points x and y of $\bar{M}$ satisfy $\pi(x) = \pi(y)$ and $\mu(x) = \mu(y)$ then $x = y$.*

The form $\pi^*\omega$ is a lcs form on $\bar{M}$, with Lee form $\pi^*\eta$. Since $\pi^*\eta$ is exact, by Example 2.2 we can associate to the lcs manifold $\big(\bar{M}, [(\pi^*\eta, \pi^*\omega)]\big)$ the conformal symplectic manifold $(\bar{M}, [e^{-\mu}\pi^*\omega])$, where μ is any primitive of $\pi^*\eta$. We say that $(\bar{M}, [e^{-\mu}\pi^*\omega])$ is the symplectic cover of the lcs manifold $\big(M, [(\eta,\omega)]\big)$. If the lcs manifold $\big(M, [(\eta,\omega)]\big)$ is exact and λ a Liouville form for (η,ω) then the symplectic cover $(\bar{M}, [e^{-\mu}\pi^*\omega])$ is exact and $e^{-\mu}\pi^*\lambda$ is a Liouville form for $e^{-\mu}\pi^*\omega$. We say that the conformal Liouville manifold $\big(\bar{M}, [(e^{-\mu}\pi^*\omega, e^{-\mu}\pi^*\lambda)]\big)$ is the symplectic cover of the Liouville lcs manifold $\big(M, [(\eta,\omega,\lambda)]\big)$.

Example 2.6. *The symplectic cover of the locally conformal symplectization $\big(S^1 \times Y\,,\, [(-d\theta\,,\, d_{-d\theta}\alpha\,,\, \alpha)]\big)$ of a co-oriented contact manifold $\big(Y, \xi = \ker(\alpha)\big)$ is the conformal class $\big(\mathbb{R}\times Y\,,\, [(d(e^\theta\alpha)\,,\, e^\theta\alpha)]\big)$ of the usual symplectization.*

2.4. **Lcs diffeomorphisms.** A diffeomorphism $\varphi : M_1 \to M_2$ between two lcs manifolds $\big(M_1, [(\eta_1,\omega_1)]\big)$ and $\big(M_2, [(\eta_2,\omega_2)]\big)$ is said to be a locally conformal symplectic (lcs) diffeomorphism if for any representatives (η_1,ω_1) and (η_2,ω_2) there is a function $g : M_1 \to \mathbb{R}$, called the conformal factor of φ with respect to (η_1,ω_1) and (η_2,ω_2), such that $\varphi^*\omega_2 = e^g\omega_1$. The relation $\varphi^*\omega_2 = e^g\omega_1$ implies that $\varphi^*\eta_2 = \eta_1 + dg$. We say that the lcs diffeomorphism φ is strict with respect to (η_1,ω_1) and (η_2,ω_2) (or just that it is strict when the representatives we consider are clear from the context) if $\varphi^*\omega_2 = \omega_1$, and so $\varphi^*\eta_2 = \eta_1$.

Example 2.7. *A conformal symplectomorphism between two conformal symplectic manifolds $(M_1, [\omega_1])$ and $(M_2, [\omega_2])$ is a diffeomorphism $\varphi : M_1 \to M_2$ such that for any representatives ω_1 and ω_2 there is a constant $c \in \mathbb{R}$, called the conformal factor of φ with respect to ω_1 and ω_2, such that $\varphi^*\omega_2 = e^c\,\omega_1$. Seeing $(M_j, [\omega_j])$ for $j = 1,2$ as lcs manifolds $\big(M_j, [(0,\omega_j)]\big)$, any conformal symplectomorphism is a lcs diffeomorphism. Conversely, any lcs diffeomorphism φ between two lcs manifolds $\big(M_1, [(\eta_1,\omega_1)]\big)$ and $\big(M_2, [(\eta_2,\omega_2)]\big)$ with $\eta_1 = d\mu_1$ and $\eta_2 = d\mu_2$ is a conformal symplectomorphism between the associated conformal symplectic manifolds $(M_1, [e^{-\mu_1}\omega_1])$ and $(M_2, [e^{-\mu_2}\omega_2])$. Indeed,*

$$\varphi^*(e^{-\mu_2}\omega_2) = e^{-\mu_2\circ\varphi+g+\mu_1}\, e^{-\mu_1}\omega_1\,,$$

where g is the conformal factor of φ with respect to (η_1,ω_1) and (η_2,ω_2), and

$$d(\mu_2\circ\varphi) = \varphi^*\eta_2 = \eta_1 + dg = d(\mu_1 + g)\,,$$

thus $-\mu_2\circ\varphi + g + \mu_1 = c$ for some $c \in \mathbb{R}$.

Similarly, we have the following result.

Proposition 2.8. *Consider two lcs manifolds $\big(M_1, [(\eta_1, \omega_1)]\big)$ and $\big(M_2, [(\eta_2, \omega_2)]\big)$, and their symplectic covers $(\bar{M}_1, [e^{-\mu_1}\pi_1^*\omega_1])$ and $(\bar{M}_2, [e^{-\mu_2}\pi_2^*\omega_2])$. Suppose that $\varphi : M_1 \to M_2$ and $\bar{\varphi} : \bar{M}_1 \to \bar{M}_2$ are diffeomorphisms such that $\pi_2 \circ \bar{\varphi} = \varphi \circ \pi_1$. Then φ is a lcs diffeomorphism if and only if $\bar{\varphi}$ is a conformal symplectomorphism. In this case, if g is the conformal factor of φ with respect to (η_1, ω_1) and (η_2, ω_2) then*

$$-\mu_2 \circ \bar{\varphi} + g \circ \pi_1 + \mu_1 = c$$

for some $c \in \mathbb{R}$, and c is the conformal factor of $\bar{\varphi}$ with respect to $e^{-\mu_1}\pi_1^*\omega_1$ and $e^{-\mu_2}\pi_2^*\omega_2$.

Proof. We denote $\bar{\omega}_j = e^{-\mu_j}\,\pi_j^*\omega_j$. Suppose that φ is a lcs diffeomorphism, with conformal factor g with respect to (η_1, ω_1) and (η_2, ω_2). Then $\varphi^*\omega_2 = e^g\omega_1$, thus $\pi_1^*(\varphi^*\omega_2) = \pi_1^*(e^g\omega_1)$. But

$$\pi_1^*(\varphi^*\omega_2) = \bar{\varphi}^*(\pi_2^*\omega_2) = \bar{\varphi}^*(e^{\mu_2}\bar{\omega}_2) = e^{\mu_2\circ\bar{\varphi}}\,\bar{\varphi}^*\bar{\omega}_2$$

and

$$\pi_1^*(e^g\omega_1) = e^{g\circ\pi_1}\,\pi_1^*\omega_1 = e^{g\circ\pi_1+\mu_1}\,\bar{\omega}_1\,,$$

thus

$$\bar{\varphi}^*\bar{\omega}_2 = e^{-\mu_2\circ\bar{\varphi}+g\circ\pi_1+\mu_1}\,\bar{\omega}_1\,.$$

Since

$$d\,(-\mu_2 \circ \bar{\varphi} + g \circ \pi_1 + \mu_1) = \pi_1^*(-\varphi^*\eta_2 + dg + \eta_1) = 0\,,$$

we have $-\mu_2 \circ \bar{\varphi} + g \circ \pi_1 + \mu_1 = c$ for some $c \in \mathbb{R}$. We conclude that $\bar{\varphi}^*\bar{\omega}_2 = e^c\,\bar{\omega}_1$, thus $\bar{\varphi}$ is a conformal symplectomorphism with conformal factor c with respect to $\bar{\omega}_1$ and $\bar{\omega}_2$. Conversely, suppose that $\bar{\varphi}$ is a conformal symplectomorphism with conformal factor c with respect to $\bar{\omega}_1$ and $\bar{\omega}_2$. Then

$$\pi_1^*(\varphi^*\omega_2) = \bar{\varphi}^*(\pi_2^*\omega_2) = \bar{\varphi}^*(e^{\mu_2}\bar{\omega}_2) = e^{\mu_2\circ\bar{\varphi}+c-\mu_1}\,\pi_1^*\omega_1\,.$$

This implies that $\mu_2 \circ \bar{\varphi} + c - \mu_1 = g \circ \pi_1$ for some $g : M_1 \to \mathbb{R}$. We thus have

$$\pi_1^*(\varphi^*\omega_2) = e^{g\circ\pi_1}\,\pi_1^*\omega_1 = \pi_1^*(e^g\omega_1)\,,$$

and so $\varphi^*\omega_2 = e^g\omega_1$. This shows that φ is a lcs diffeomorphism with conformal factor g with respect to (η_1, ω_1) and (η_2, ω_2). ☐

We say that a conformal symplectomorphism φ of a conformal symplectic manifold $(M, [\omega])$ is a symplectomorphism if $\varphi^*\omega = \omega$ for some (hence any) representative ω. Proposition 2.8 implies the following result.

Corollary 2.9. *Let φ be a lcs diffeomorphism of a lcs manifold $\big(M, [(\eta, \omega)]\big)$, and suppose that $\bar{\varphi}$ is a conformal symplectomorphism of the symplectic cover $(\bar{M}, [e^{-\mu}\pi^*\omega])$ such that $\pi \circ \bar{\varphi} = \varphi \circ \pi$. Then $\bar{\varphi}$ is a symplectomorphism if and only if for some (hence any) representative (η, ω) and primitive μ of $\pi^*\eta$ we have*

$$\mu \circ \bar{\varphi} - \mu = g \circ \pi\,,$$

where g is the conformal factor of φ with respect to (η, ω).

Proof. Proposition 2.8 implies that $\bar{\varphi}$ is a symplectomorphism if and only if for any representative (η, ω) and primitive μ of $\pi^*\eta$ we have $\mu \circ \bar{\varphi} - \mu = g \circ \pi$. If this relation holds for one primitive μ of $\pi^*\eta$ then it also holds for any other primitive $\mu + c$. Moreover, if the relation holds for one representative (η, ω) then it also holds for any other representative $(\eta + df, e^f\omega)$. Indeed, since $\pi^*(\eta + df) = d(\mu + f \circ \pi)$ and since the conformal factor of φ with respect to $(\eta + df, e^f\omega)$ is $f \circ \varphi - f + g$, the desired relation for $(\eta + df, e^f\omega)$ is

$$(\mu + f \circ \pi) \circ \bar{\varphi} - \mu - f \circ \pi = (f \circ \varphi - f + g) \circ \pi\,,$$

which is equivalent to $\mu \circ \bar{\varphi} - \mu = g \circ \pi$. ☐

We say that a lcs diffeomorphism φ of a lcs manifold $\big(M, [(\eta, \omega)]\big)$ is divergence free if there is a symplectomorphism $\bar{\varphi}$ of the symplectic cover $(\bar{M}, [e^{-\mu}\pi^*\omega])$ such that $\pi \circ \bar{\varphi} = \varphi \circ \pi$. By Lemma 2.5 and Corollary 2.9, such a symplectomorphism is then unique. We call it the lift to the symplectic cover of the divergence free lcs diffeomorphism φ. In particular, the lift of the identity of M is the identity of $\bar{M}$.

Let $\varphi : M_1 \to M_2$ be a lcs diffeomorphism between two Liouville lcs manifolds $\big(M_1, [(\eta_1, \omega_1, \lambda_1)]\big)$ and $\big(M_2, [(\eta_2, \omega_2, \lambda_2)]\big)$. We say that φ is exact if for some (hence any) representatives $(\eta_1, \omega_1, \lambda_1)$ and $(\eta_2, \omega_2, \lambda_2)$

the η_1-closed 1-form $\lambda_1 - e^{-g}\varphi^*\lambda_2$, where g is the conformal factor of φ with respect to (η_1, ω_1) and (η_2, ω_2), is η_1-exact. In this case, we say that φ has action $S : M_1 \to \mathbb{R}$ with respect to $(\eta_1, \omega_1, \lambda_1)$ and $(\eta_2, \omega_2, \lambda_2)$ (or just action S when the representatives we consider are clear from the context) if

$$\lambda_1 - e^{-g}\varphi^*\lambda_2 = d_{\eta_1} S\,.$$

By Remark 2.1, if M is connected and η is not exact then the action S is uniquely defined (otherwise it is just determined up to an additive constant on each connected component).

Example 2.10. *Let $\varphi : M_1 \to M_2$ be a conformal symplectomorphism between two conformal Liouville manifolds $\big(M_1, [(\omega_1, \lambda_1)]\big)$ and $\big(M_2, [(\omega_2, \lambda_2)]\big)$. We say that φ is exact if for some (hence any) representatives (ω_1, λ_1) and (ω_2, λ_2) the closed 1-form $\lambda_1 - e^{-c}\varphi^*\lambda_2$, where c is the conformal factor of φ with respect to ω_1 and ω_2, is exact. We then say that φ has action S with respect to (ω_1, λ_1) and (ω_2, λ_2) if $\lambda_1 - e^{-c}\varphi^*\lambda_2 = dS$. Let $\big(M_1, [(\eta_1, \omega_1, \lambda_1)]\big)$ and $\big(M_2, [(\eta_2, \omega_2, \lambda_2)]\big)$ be Liouville lcs manifolds with exact Lee forms $\eta_j = d\mu_j$, and consider the associated conformal Liouville manifolds $\big(M_j, [(e^{-\mu_j}\omega_j, e^{-\mu_j}\lambda_j)]\big)$. Then $\varphi : M_1 \to M_2$ is an exact lcs diffeomorphism with action S with respect to $(\eta_1, \omega_1, \lambda_1)$ and $(\eta_2, \omega_2, \lambda_2)$ if and only if it is an exact conformal symplectomorphism with action $e^{-\mu_1} S$ with respect to $(e^{-\mu_1}\omega_1, e^{-\mu_1}\lambda_1)$ and $(e^{-\mu_2}\omega_2, e^{-\mu_2}\lambda_2)$.*

Similarly, we have the following result.

Proposition 2.11. *Consider two Liouville lcs manifolds $\big(M_1, [(\eta_1, \omega_1, \lambda_1)]\big)$ and $\big(M_2, [(\eta_2, \omega_2, \lambda_2)]\big)$, and their symplectic covers $\big(\bar{M}_1, [(e^{-\mu_1}\pi_1^*\omega_1, e^{-\mu_1}\pi_1^*\lambda_1)]\big)$ and $\big(\bar{M}_2, [(e^{-\mu_2}\pi_2^*\omega_2, e^{-\mu_2}\pi_2^*\lambda_2)]\big)$. Suppose that $\varphi : M_1 \to M_2$ and $\bar{\varphi} : \bar{M}_1 \to \bar{M}_2$ are a lcs diffeomorphism and a conformal symplectomorphism such that $\pi_2 \circ \bar{\varphi} = \varphi \circ \pi_1$. Then φ is exact with action S with respect to $(\eta_1, \omega_1, \lambda_1)$ and $(\eta_2, \omega_2, \lambda_2)$ if and only if $\bar{\varphi}$ is exact with action $e^{-\mu_1}\,(S \circ \pi_1)$ with respect to $(e^{-\mu_1}\,\pi_1^*\omega_1, e^{-\mu_1}\,\pi_1^*\lambda_1)$ and $(e^{-\mu_2}\,\pi_2^*\omega_2, e^{-\mu_2}\,\pi_2^*\lambda_2)$.*

Proof. We denote $\bar{\omega}_j = e^{-\mu_j}\,\pi_j^*\omega_j$ and $\bar{\lambda}_j = e^{-\mu_j}\,\pi_j^*\lambda_j$. Let c be the conformal factor of $\bar{\varphi}$ with respect to $\bar{\omega}_1$ and $\bar{\omega}_2$, and g the conformal factor of φ with respect to (η_1, ω_1) and (η_2, ω_2). By Proposition 2.8 we have $-\mu_2 \circ \bar{\varphi} + g \circ \pi_1 + \mu_1 = c$, thus

$$\bar{\lambda}_1 - e^{-c}\,\bar{\varphi}^*\bar{\lambda}_2 = e^{-\mu_1}\,\pi_1^*\lambda_1 - e^{-c-\mu_2\circ\bar{\varphi}}\,\pi_1^*(\varphi^*\lambda_2) = e^{-\mu_1}\,\pi_1^*(\lambda_1 - e^{-g}\varphi^*\lambda_2)\,.$$

If φ is exact with action S with respect to $(\eta_1, \omega_1, \lambda_1)$ and $(\eta_2, \omega_2, \lambda_2)$ then

$$\bar{\lambda}_1 - e^{-c}\,\bar{\varphi}^*\bar{\lambda}_2 = e^{-\mu_1}\,\pi_1^*(d_{\eta_1}S) = e^{-\mu_1}\,d_{d\mu_1}(S \circ \pi_1) = d\big(e^{-\mu_1}(S \circ \pi_1)\big)\,,$$

thus $\bar{\varphi}$ is exact with action $e^{-\mu_1}(S \circ \pi_1)$ with respect to $(\bar{\omega}_1, \bar{\lambda}_1)$ and $(\bar{\omega}_2, \bar{\lambda}_2)$. Conversely, if $\bar{\varphi}$ is exact with action $e^{-\mu_1}(S \circ \pi_1)$ with respect to $(\bar{\omega}_1, \bar{\lambda}_1)$ and $(\bar{\omega}_2, \bar{\lambda}_2)$ then

$$e^{-\mu_1}\,\pi_1^*(d_{\eta_1}S) = e^{-\mu_1}\,d_{d\mu_1}(S \circ \pi_1) = d\big(e^{-\mu_1}(S \circ \pi_1)\big) = \bar{\lambda}_1 - e^{-c}\,\bar{\varphi}^*\bar{\lambda}_2 = e^{-\mu_1}\,\pi_1^*(\lambda_1 - e^{-g}\varphi^*\lambda_2)\,.$$

This implies that $\lambda_1 - e^{-g}\varphi^*\lambda_2 = d_{\eta_1}S$, thus φ is exact with action S with respect to $(\eta_1, \omega_1, \lambda_1)$ and $(\eta_2, \omega_2, \lambda_2)$. □

Example 2.12. *Let $\phi : Y_1 \to Y_2$ be a contactomorphism between two contact manifolds $\big(Y_1, \xi_1 = \ker(\alpha_1)\big)$ and $\big(Y_2, \xi_2 = \ker(\alpha_2)\big)$, with $\phi^*\alpha_2 = e^g\alpha_1$. Then*

$$\widetilde{\phi} : S^1 \times Y_1 \to S^1 \times Y_2\,,\ \widetilde{\phi}(\theta, p) = \big(\theta - g(p)\,,\, \phi(p)\big)$$

is an exact lcs diffeomorphism between the locally conformal symplectizations $\big(S^1 \times Y_1\,,\, [(-d\theta_1, d_{-d\theta_1}\alpha_1, \alpha_1)]\big)$ and $\big(S^1 \times Y_2\,,\, [(-d\theta_2, d_{-d\theta_2}\alpha_2, \alpha_2)]\big)$, with conformal factor $g \circ \mathrm{pr}_2$, where $\mathrm{pr}_2 : S^1 \times Y_1 \to Y_1$ denotes the projection to the second factor, and action zero with respect to $(-d\theta_1, d_{-d\theta_1}\alpha_1, \alpha_1)$ and $(-d\theta_2, d_{-d\theta_2}\alpha_2, \alpha_2)$.

The following lemma, whose proof is immediate, will be needed later. Recall that the 1-jet bundle of a manifold B is the manifold $J^1B = T^*B \times \mathbb{R}$. We endow it with the contact structure given by the kernel of the contact form $\lambda - dz$, where λ is the tautological 1-form on T^*B and z the coordinate on $\mathbb{R}$.

Lemma 2.13. *For any manifold B, the map*

$$T^*_{-d\theta}(S^1 \times B) \to S^1 \times J^1B\,,\ (\theta, q, z, p) \mapsto (\theta, q, p, z)$$

is a strict and exact lcs diffeomorphism from the twisted cotangent bundle $T^*_{-d\theta}(S^1 \times B)$ to the locally conformal symplectization

$$\big(S^1 \times J^1 B\,,\,[(-d\theta\,,\,d_{-d\theta}(\lambda - dz)\,,\,\lambda - dz)]\big)$$

of the 1-jet bundle $J^1 B$, of action $S(\theta, q, z, p) = z$.

2.5. **Lcs vector fields and isotopies.** A locally conformal symplectic (lcs) vector field on a lcs manifold $\big(M, [(\eta,\omega)]\big)$ is a vector field X such that for any representative (η,ω) there is a function $h : M \to \mathbb{R}$, called the conformal factor of X with respect to (η,ω), satisfying $\mathcal{L}_X\omega = h\,\omega$. This relation implies that $\mathcal{L}_X\eta = dh$.

Lemma 2.14. *A vector field X on a lcs manifold $\big(M, [(\eta,\omega)]\big)$ is lcs if and only if there is a real number c such that*

$$d_\eta(\iota_X\omega) = c\,\omega$$

for some (hence any) representative (η,ω). In this case, the conformal factor of X with respect to (η,ω) is $h = c + \eta(X)$.

Proof. Suppose that X is lcs, thus $\mathcal{L}_X\omega = h\,\omega$ and $\mathcal{L}_X\eta = dh$ for some $h : M \to \mathbb{R}$. Then

$$d_\eta(\iota_X\omega) = \mathcal{L}_X\omega - \iota_X(d_\eta\omega) - \eta(X)\,\omega = \big(h - \eta(X)\big)\,\omega$$

and

$$dh = \mathcal{L}_X\eta = d(\iota_X\eta) + \iota_X d\eta = d(\iota_X\eta)\,.$$

Thus $h - \eta(X) = c$ for some $c \in \mathbb{R}$, and $d_\eta(\iota_X\omega) = c\,\omega$. Conversely, if $d_\eta(\iota_X\omega) = c\,\omega$ for some $c \in \mathbb{R}$ then

$$\mathcal{L}_X\omega = d_\eta(\iota_X\omega) + \iota_X(d_\eta\omega) + \eta(X)\,\omega = \big(c + \eta(X)\big)\,\omega\,,$$

thus X is lcs with conformal factor $h = c + \eta(X)$. □

A locally conformal symplectic (lcs) isotopy on a lcs manifold $\big(M, [(\eta,\omega)]\big)$ is an isotopy $\{\varphi_t\}$ such that each φ_t is a lcs diffeomorphism. We say that a time dependent vector field X_t is lcs if each X_t is lcs.

Lemma 2.15. *Let X_t be a time dependent vector field on a lcs manifold $\big(M, [(\eta,\omega)]\big)$, with flow $\{\varphi_t\}$. Then X_t is a lcs vector field if and only if $\{\varphi_t\}$ is a lcs isotopy. In this case, if g_t and h_t are the conformal factors of φ_t and X_t with respect to (η,ω) then*

$$h_t = \frac{dg_t}{dt} \circ \varphi_t^{-1}\,.$$

Proof. Suppose that $\{\varphi_t\}$ is a lcs isotopy with conformal factors g_t. Then

$$\varphi_t^*(\mathcal{L}_{X_t}\omega) = \frac{d}{dt}\,\varphi_t^*\omega = \frac{d}{dt}\,(e^{g_t}\omega) = \frac{dg_t}{dt}\,e^{g_t}\omega = \frac{dg_t}{dt}\,\varphi_t^*\omega\,,$$

thus

$$\mathcal{L}_{X_t}\omega = \Big(\frac{dg_t}{dt} \circ \varphi_t^{-1}\Big)\,\omega$$

and so X_t is a lcs vector field with conformal factors $h_t = \frac{dg_t}{dt} \circ \varphi_t^{-1}$. Conversely, suppose that X_t is a lcs vector field with conformal factors h_t. Set $g_t = \int_0^t h_s \circ \varphi_s\,ds$. Then

$$\frac{d}{dt}\,\varphi_t^*\omega = \varphi_t^*(\mathcal{L}_{X_t}\omega) = \varphi_t^*(h_t\,\omega) = (h_t \circ \varphi_t)\,\varphi_t^*\omega = \frac{dg_t}{dt}\,\varphi_t^*\omega\,.$$

This implies that $\varphi_t^*\omega = e^{g_t}\omega$, thus $\{\varphi_t\}$ is a lcs isotopy with conformal factors g_t. □

We say that a lcs vector field X_t on a lcs manifold $\big(M, [(\eta,\omega)]\big)$ is divergence free if $d_\eta(\iota_{X_t}\omega) = 0$ for all t, for some (hence any) representative (η,ω). We say that a lcs isotopy $\{\varphi_t\}$ is divergence free if each φ_t is divergence free.

Lemma 2.16. *Let X_t be a lcs vector field on a lcs manifold $\big(M, [(\eta,\omega)]\big)$, and $\{\varphi_t\}$ the lcs isotopy generated by X_t. Then X_t is divergence free if and only if $\{\varphi_t\}$ is divergence free.*

Proof. Let $\{\bar\varphi_t\}$ be the lift of $\{\varphi_t\}$ to the symplectic cover $(\bar M, [e^{-\mu}\pi^*\eta])$ starting at the identity, and g_t the conformal factors of $\{\varphi_t\}$. Then

$$d(\mu\circ\bar\varphi_t) = \bar\varphi_t^*\,\pi^*\eta = \pi^*\varphi_t^*\eta = \pi^*(\eta + dg_t) = d(\mu + g_t\circ\pi)\,,$$

thus

$$\mu\circ\bar\varphi_t = \mu + g_t\circ\pi + a_t$$

for a 1-parameter family of real numbers a_t. Denote by $\bar X_t$ the vector field generating $\{\bar\varphi_t\}$. Using Lemmas 2.14 and 2.15 we obtain

$$\frac{da_t}{dt} = \frac{d}{dt}\big(\mu\circ\bar\varphi_t - g_t\circ\pi\big) = d\mu\,(\bar X_t\circ\bar\varphi_t) - \frac{dg_t}{dt}\circ\pi$$

$$= \Big(\eta(X_t\circ\varphi_t) - \frac{dg_t}{dt}\Big)\circ\pi = \Big(h_t\circ\varphi_t - c_t - \frac{dg_t}{dt}\Big)\circ\pi = -c_t\,,$$

where c_t is the 1-parameter family of real numbers such that $d_\eta(\iota_{X_t}\omega) = c_t\,\omega$. The vector field X_t is divergence free if and only if $c_t = 0$ for all t, thus if and only if $\frac{da_t}{dt} = 0$ for all t. Since $\bar\varphi_0$ is the identity, we have $a_0 = 0$. We thus conclude that X_t is divergence free if and only if $a_t = 0$ for all t, and so

$$\mu\circ\bar\varphi_t - \mu = g_t\circ\pi$$

for all t. By Corollary 2.9, this happens if and only if $\{\bar\varphi_t\}$ is a symplectic isotopy, and so if and only if $\{\varphi_t\}$ is divergence free. ☐

Example 2.17. *If X_t is a contact vector field on a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$ with $\mathcal{L}_{X_t}\alpha = h_t\,\alpha$ then*

$$\widetilde{X_t}(\theta, p) = X_t(p) - h_t(p)\,\frac{\partial}{\partial\theta}$$

is a divergence free lcs vector field on the locally conformal symplectization $\big(S^1\times Y\,,\,[(-d\theta\,,\,d_{-d\theta}\alpha)]\big)$. The isotopy generated by $\widetilde{X_t}$ is the lift, as defined in Example 2.12, of the contact isotopy generated by X_t.

2.6. **Lcs Hamiltonian isotopies.** A divergence free lcs vector field X_t on a lcs manifold $\big(M, [(\eta,\omega)]\big)$ is said to be Hamiltonian if for some (hence any) representative (η,ω) the η-closed 1-form $\iota_{X_t}\omega$ is η-exact. We then say that X_t has Hamiltonian function $H_t : M\to\mathbb{R}$ with respect to (η,ω) if

$$\iota_{X_t}\omega = -d_\eta H_t\,.$$

By Remark 2.1, if M is connected and η is not exact then such H_t is unique (otherwise it is just determined up to an additive constant on each connected component). By Lemma 2.16, the flow of X_t is a divergence free lcs isotopy. We say that it is a lcs Hamiltonian isotopy, with Hamiltonian function H_t with respect to (η,ω). A lcs diffeomorphism is said to be Hamiltonian if it is the time-1 map of a lcs Hamiltonian isotopy.

Example 2.18. *We say that a symplectic isotopy $\{\varphi_t\}$ of a conformal symplectic manifold $(M, [\omega])$ is Hamiltonian if the closed 1-form $\iota_{X_t}\omega$ is exact for some (hence any) representative ω, where X_t is the vector field generating $\{\varphi_t\}$. We then say that $\{\varphi_t\}$ has Hamiltonian function H_t with respect to ω if $\iota_{X_t}\omega = -dH_t$. Identifying $(M, [\omega])$ with the lcs manifold $\big(M, [(0,\omega)]\big)$, $\{\varphi_t\}$ is a lcs Hamiltonian isotopy with Hamiltonian function H_t with respect to $(0,\omega)$ if and only if it is a Hamiltonian isotopy with Hamiltonian function H_t with respect to ω.*

The description of Example 2.18 always applies locally. Let $\big(M, [(\eta,\omega)]\big)$ be a lcs manifold, and $\{\mathcal{U}_i\}$ an open cover such that $\omega|_{\mathcal{U}_i} = e^{\mu_i}\,\omega_i$ with ω_i symplectic. If X_t is a lcs Hamiltonian vector field with Hamiltonian function H_t with respect (η,ω) then $X|_{\mathcal{U}_i}$ is a Hamiltonian vector field on $(\mathcal{U}_i,\omega_i)$ with Hamiltonian function $e^{-\mu_i}H_t$. Conversely, given a time dependent function $H_t : M\to\mathbb{R}$ we can define the associated lcs vector field as follows. Consider on every $\mathcal{U}_i$ the time dependent function $H_t^{(i)} = e^{-\mu_i}\;H_t|_{\mathcal{U}_i}$, and let $X_t^{(i)}$ be the associated Hamiltonian vector field. Since the equation $\omega_i(X_t^{(i)},\,\cdot\,) = -dH_t^{(i)}$ is equivalent to $\omega(X_t^{(i)},\,\cdot\,) = -d_\eta\;H_t|_{\mathcal{U}_i}$, the vector fields $X_t^{(i)}$ glue to a global vector field X_t, satisfying $\omega(X_t,\,\cdot\,) = -d_\eta H_t$.

Example 2.19. *Recall that for any contact isotopy $\{\phi_t\}$ of a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$ there is a unique time dependent function H_t, called the Hamiltonian function of $\{\phi_t\}$ with respect to α, such that $\alpha(X_t) = H_t$ and $\iota_{X_t} d\alpha = dH_t(R_\alpha)\alpha - dH_t$, where X_t is the the vector field generating the isotopy. Recall also that $\mathcal{L}_{X_t}\alpha = h_t\alpha$ for $h_t = dH_t(R_\alpha)$. The lift $\{\widetilde{\phi_t}\}$ of $\{\phi_t\}$ to the locally conformal symplectization $\big(S^1 \times Y\, ,\, [(-d\theta\, ,\, d_{-d\theta}\alpha)]\big)$ is a lcs Hamiltonian isotopy with Hamiltonian function*

$$\widetilde{H_t} : S^1 \times Y \to \mathbb{R}\, ,\ \widetilde{H_t}(\theta, p) = H_t(p)$$

with respect to $(-d\theta\, ,\, d_{-d\theta}\alpha)$. Indeed, by Example 2.17 the lcs isotopy $\{\widetilde{\phi_t}\}$ is generated by the vector field $\widetilde{X_t} = X_t - h_t \frac{\partial}{\partial\theta}$. Thus

$$\iota_{\widetilde{X_t}}(d_{-d\theta}\alpha) = \iota_{X_t} d\alpha - h_t\alpha - \alpha(X_t)d\theta = dH_t(R_\alpha)\alpha - dH_t - h_t\alpha - H_t d\theta = -d_{-d\theta}\widetilde{H_t}\, .$$

We denote by $\mathrm{Ham}\,\big(M, [(\eta, \omega)]\big)$ the space of lcs Hamiltonian diffeomorphisms of $\big(M, [(\eta, \omega)]\big)$. By the following lemma, whose proof is left to the reader, $\mathrm{Ham}\,\big(M, [(\eta, \omega)]\big)$ is a group.

Lemma 2.20. *Let $\{\varphi_t\}$, $\{\varphi_t^{(1)}\}$ and $\{\varphi_t^{(2)}\}$ be lcs Hamiltonian isotopies on a lcs manifold $\big(M, [(\eta, \omega)]\big)$, with Hamiltonian functions H_t, $H_t^{(1)}$ and $H_t^{(2)}$ and conformal factors $g_t, g_t^{(1)}$ and $g_t^{(2)}$ with respect to (η, ω). Then*

- (i) *$\{\varphi_t^{-1}\}$ is a lcs Hamiltonian isotopy with Hamiltonian function $-e^{-g_t}(H_t \circ \varphi_t)$ with respect to (η, ω);*
- (ii) *$\{\varphi_t^{(1)} \circ \varphi_t^{(2)}\}$ is a lcs Hamiltonian isotopy with Hamiltonian function*

$$H_t^{(1)} + e^{g_t^{(1)} \circ (\varphi_t^{(1)})^{-1}}\, H_t^{(2)} \circ (\varphi_t^{(1)})^{-1}$$

with respect to (η, ω).

We will also need the following lemma, whose proof is again left to the reader.

Lemma 2.21. *Let $\{\varphi_t\}$ be a lcs Hamiltonian isotopy of a lcs manifold $\big(M, [(\eta, \omega)]\big)$ with Hamiltonian function H_t with respect to (η, ω), and $\psi : M' \to M$ a lcs diffeomorphism from another lcs manifold $\big(M', [(\eta', \omega')]\big)$ to $\big(M, [(\eta, \omega)]\big)$. Then $\{\psi^{-1} \circ \varphi_t \circ \psi\}$ is a lcs Hamiltonian isotopy of $\big(M', [(\eta', \omega')]\big)$ with Hamiltonian function $e^{-f}(H_t \circ \psi)$ with respect to (η', ω'), where f is the conformal factor of ψ with respect to (η', ω') and (η, ω).*

Example 2.18 can be reformulated as follows. Let $\big(M, [(\eta, \omega)]\big)$ be a lcs manifold with exact Lee form $\eta = d\mu$, and consider the associated conformal symplectic manifold $\big(M, [e^{-\mu}\omega]\big)$. Then $\{\varphi_t\}$ is a lcs Hamiltonian isotopy with Hamiltonian function H_t with respect to (η, ω) if and only if it is a Hamiltonian isotopy with Hamiltonian function $e^{-\mu}H_t$ with respect to $e^{-\mu}\omega$. Similarly, we have the following result.

Proposition 2.22. *Let $\{\varphi_t\}$ be a divergence free lcs isotopy of a lcs manifold $\big(M, [(\eta, \omega)]\big)$ starting at the identity, and $\{\bar{\varphi}_t\}$ its lift to the symplectic cover $(\bar{M}, [e^{-\mu}\pi^*\omega])$. Then $\{\varphi_t\}$ is a lcs Hamiltonian isotopy if and only if $\{\bar{\varphi}_t\}$ is a Hamiltonian isotopy. In this case, the Hamiltonian function of $\{\bar{\varphi}_t\}$ with respect to $e^{-\mu}\pi^*\omega$ is $e^{-\mu}(H_t \circ \pi)$, where H_t is the Hamiltonian function of $\{\varphi_t\}$ with respect to (η, ω).*

Proof. Let $\bar{X}_t$ be the vector field generating $\{\bar{\varphi}_t\}$. Then $\pi_*(\bar{X}_t) = X_t$, where X_t is the vector field generating $\{\varphi_t\}$. Suppose that $\{\varphi_t\}$ is a lcs Hamiltonian isotopy with Hamiltonian function H_t with respect to (η, ω). For every vector field Y on $\bar{M}$ we then have

$$\begin{aligned} e^{-\mu}\pi^*\omega(\bar{X}_t, Y) &= e^{-\mu}\,\omega\big(X_t\, ,\, \pi_*(Y)\big) = -e^{-\mu}\, d_\eta H_t\big(\pi_*(Y)\big) \\ &= -e^{-\mu}\, d_{d\mu}(H_t \circ \pi)(Y) = -d\big(e^{-\mu}(H_t \circ \pi)\big)(Y)\, , \end{aligned}$$

thus $\{\bar{\varphi}_t\}$ is a Hamiltonian isotopy with Hamiltonian function $e^{-\mu}(H_t \circ \pi)$ with respect to $e^{-\mu}\pi^*\omega$. Conversely, suppose that $\{\bar{\varphi}_t\}$ is a Hamiltonian isotopy with Hamiltonian function $\bar{H}_t$ with respect to $e^{-\mu}\pi^*\omega$. Then

$$e^{-\mu}\,\omega\big(X_t, \pi_*(Y)\big) = e^{-\mu}\,\pi^*\omega(\bar{X}_t, Y) = -d\bar{H}_t(Y)$$

for every vector field Y on $\bar{M}$, thus

$$\pi^*(\iota_{X_t}\omega) = -e^{\mu}\, d\bar{H}_t = -d_{d\mu}(e^{\mu}\bar{H}_t) = -d_{\pi^*\eta}(e^{\mu}\bar{H}_t)\, .$$

This implies that $\bar{H}_t = e^{-\mu}(H_t \circ \pi)$ for some $H_t : M \to \mathbb{R}$ and $\iota_{X_t}\omega = -d_\eta H_t$, thus $\{\varphi_t\}$ is a lcs Hamiltonian isotopy with Hamiltonian function H_t with respect to (η, ω). $\square$

Proposition 2.22 allows to derive the following result from the corresponding one in the symplectic case, which is due to Banyaga [Ban78] (see also [McDS17, Proposition 10.2.12]).

Theorem 2.23. *Let $\{\varphi_t\}$ be a lcs isotopy on a lcs manifold $\big(M, [(\eta, \omega)]\big)$. Suppose that each φ_t is a lcs Hamiltonian diffeomorphism. Then $\{\varphi_t\}$ is a lcs Hamiltonian isotopy.*

Proof. Since each φ_t is Hamiltonian, hence divergence free, $\{\varphi_t\}$ is a divergence free lcs isotopy. By Proposition 2.22, its lift $\{\bar{\varphi}_t\}$ to the symplectic cover is a symplectic isotopy such that each $\bar{\varphi}_t$ is Hamiltonian. Thus $\{\bar{\varphi}_t\}$ is a Hamiltonian isotopy. By Proposition 2.22, $\{\varphi_t\}$ is thus a lcs Hamiltonian isotopy. $\square$

We end this subsection by showing that lcs Hamiltonian diffeomorphisms on Liouville lcs manifolds are exact.

Proposition 2.24. *Let $\{\varphi_t\}$ be a divergence free lcs isotopy of a Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$. Then $\{\varphi_t\}$ is a lcs Hamiltonian isotopy if and only if φ_t is exact for every t. In this case, if H_t and g_t are the Hamiltonian function and the conformal factors of $\{\varphi_t\}$ with respect to (η, ω), then the action of $\{\varphi_t\}$ with respect to (η, ω, λ) is given by*

$$S_t = \int_0^t e^{-g_s} \Big(\big(H_s - \lambda(X_s)\big) \circ \varphi_s \Big)\, ds\,, \tag{1}$$

where X_t is the vector field generating $\{\varphi_t\}$.

Proof. By Lemma 2.16, X_t is divergence free and so $d_\eta(\iota_{X_t}\omega) = 0$ for all t. By Lemmas 2.14 and 2.15 we thus have

$$\eta(X_t) = \frac{dg_t}{dt} \circ \varphi_t^{-1}\,,$$

and so

$$\begin{aligned} \frac{d}{dt}\big(\lambda - e^{-g_t}\,\varphi_t^*\lambda\big) &= \frac{dg_t}{dt}\, e^{-g_t}\,\varphi_t^*\lambda - e^{-g_t}\,\frac{d}{dt}\,\varphi_t^*\lambda = \varphi_t^*\left(\left(\frac{dg_t}{dt} \circ \varphi_t^{-1}\right) e^{-g_t\circ\varphi_t^{-1}}\lambda - e^{-g_t\circ\varphi_t^{-1}}\mathcal{L}_{X_t}\lambda\right) \\ &= \varphi_t^*\Big(\eta(X_t)\, e^{-g_t\circ\varphi_t^{-1}}\lambda - e^{-g_t\circ\varphi_t^{-1}}\big(d_\eta(\iota_{X_t}\lambda) + \iota_{X_t}d_\eta\lambda + \eta(X_t)\lambda\big)\Big) \\ &= \varphi_t^*\Big(- e^{-g_t\circ\varphi_t^{-1}}\big(d_\eta(\iota_{X_t}\lambda) + \iota_{X_t}\omega\big)\Big) \\ &= -e^{-g_t}\,\varphi_t^*\big(d_\eta(\iota_{X_t}\lambda) + \iota_{X_t}\omega\big)\,. \end{aligned}$$

Suppose that $\{\varphi_t\}$ is a lcs Hamiltonian isotopy with Hamiltonian function H_t with respect to (η, ω), and define S_t by (1). Then

$$\begin{aligned} \frac{d}{dt}\big(\lambda - e^{-g_t}\,\varphi_t^*\lambda\big) &= -e^{-g_t}\,\varphi_t^*\big(d_\eta(\iota_{X_t}\lambda - H_t)\big) = -e^{-g_t}\,d_{\varphi_t^*\eta}\big((\iota_{X_t}\lambda - H_t)\circ\varphi_t\big) \\ &= -e^{-g_t}\,d_{\eta+dg_t}\big((\iota_{X_t}\lambda - H_t)\circ\varphi_t\big) = d_\eta\Big(e^{-g_t}\big((H_t - \iota_{X_t}\lambda)\circ\varphi_t\big)\Big) \\ &= d_\eta\,\frac{dS_t}{dt} = \frac{d}{dt}\,d_\eta S_t\,. \end{aligned}$$

Since $\lambda - e^{-g_0}\,\varphi_0^*\lambda = 0$ and $d_\eta S_0 = 0$, this implies that $\lambda - e^{-g_t}\,\varphi_t^*\lambda = d_\eta S_t$, and so each φ_t is exact with action S_t with respect to (η, ω, λ). Conversely, suppose that each φ_t is exact with action S_t with respect to (η, ω, λ), and define H_t by (1). Then

$$d_\eta\Big(e^{-g_t}\big(\big(H_t - \lambda(X_t)\big)\circ\varphi_t\big)\Big) = d_\eta\,\frac{dS_t}{dt} = \frac{d}{dt}\,d_\eta S_t = \frac{d}{dt}\big(\lambda - e^{-g_t}\,\varphi_t^*\lambda\big) = -e^{-g_t}\,\varphi_t^*\big(d_\eta(\iota_{X_t}\lambda) + \iota_{X_t}\omega\big)\,,$$

thus

$$\begin{aligned} \varphi_t^*\big(d_\eta(\iota_{X_t}\lambda) + \iota_{X_t}\omega\big) &= -e^{g_t}\,d_\eta\Big(e^{-g_t}\big((H_t - \iota_{X_t}\lambda)\circ\varphi_t\big)\Big) = -d_{\eta+dg_t}\big((H_t - \iota_{X_t}\lambda)\circ\varphi_t\big) \\ &= -d_{\varphi_t^*\eta}\big((H_t - \iota_{X_t}\lambda)\circ\varphi_t\big) = \varphi_t^*\,d_\eta(\iota_{X_t}\lambda - H_t)\,. \end{aligned}$$

This implies that $\iota_{X_t}\omega = -d_\eta H_t$, thus $\{\varphi_t\}$ is a lcs Hamiltonian isotopy with Hamiltonian function H_t with respect to (η, ω). Alternatively, we can apply the corresponding result in symplectic topology [McDS17, Proposition 9.3.1] to the lift of $\{\varphi_t\}$ to the symplectic cover, and use Corollary 2.9, Proposition 2.11 and Proposition 2.22. $\square$

2.7. **Lee flow.** The Lee vector field of a lcs manifold $\big(M, [(\eta, \omega)]\big)$ with respect to (η, ω) is the vector field R_ω that satisfies

$$\omega(R_\omega\,, \cdot\,) = \eta\,,$$

in other words the lcs Hamiltonian vector field with Hamiltonian function $H_t \equiv 1$ with respect to (η, ω). In particular, R_ω is divergence free. By Lemma 2.14 it thus has conformal factor $h = \eta(R_\omega) = 0$ with respect to (η, ω), and so it satisfies $\mathcal{L}_{R_\omega}\omega = 0$ and $\mathcal{L}_{R_\omega}\eta = 0$. The Lee flow $\{\varphi_t^\omega\}$ of (η, ω) is the flow of R_ω. It satisfies $(\varphi_t^\omega)^*\omega = \omega$ and $(\varphi_t^\omega)^*\eta = \eta$.

Example 2.25. *The Lee flow of a conformal symplectic manifold $(M, [\omega]) = \big(M, [(0, \omega)]\big)$ with respect to $(0, \omega)$ is the identity. More generally, the Lee flow with respect to $(d\mu, e^\mu\omega)$ is the Hamiltonian flow of $e^{-\mu}$ with respect to the symplectic form ω. In particular, the Lee flow with respect to $(d\mu, e^\mu\omega)$ preserves μ.*

Similarly, we have the following result, whose proof is immediate using Example 2.25.

Lemma 2.26. *Let $\big(M, [(\eta, \omega)]\big)$ be a lcs manifold, and $(\bar{M}, [e^{-\mu}\pi^*\omega])$ its symplectic cover. The Lee flow on $\bar{M}$ with respect to $(\pi^*\eta, \pi^*\omega)$ is the Hamiltonian flow of $e^{-\mu}$ with respect to $e^{-\mu}\pi^*\omega$. It preserves μ and commutes with the projection $\pi : \bar{M} \to M$. The induced isotopy on M is the Lee flow with respect to (η, ω).*

Example 2.27. *The Lee vector field of the locally conformal symplectization $\big(S^1 \times Y\,,\ [(-d\theta\,,\ d_{-d\theta}\alpha)]\big)$ of a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$ with respect to $(-d\theta\,,\ d_{-d\theta}\alpha)$ is $(0, R_\alpha)$, where R_α is the Reeb vector field with respect to α.*

Example 2.28. *The Lee vector field of the twisted cotangent bundle $T^*_\beta B$ with respect to $(\pi^*\beta\,,\ d_{\pi^*\beta}\lambda)$ is the vector field that is dual to $\pi^*\beta$ with respect to $d_{\pi^*\beta}\lambda$. The Lee flow $\{\varphi_t\}$ is given by*

$$\varphi_t(\sigma_q) = \sigma_q + t\beta(q)\,.$$

2.8. **Lagrangian submanifolds.** An immersion $i : L \to M$ into a lcs manifold $\big(M, [(\eta, \omega)]\big)$ is said to be Lagrangian if the dimension of L is half the dimension of M and $i^*\omega = 0$ for some (hence any) representative (η, ω). A submanifold L of M is said to be Lagrangian if the inclusion $L \hookrightarrow M$ is Lagrangian. If $i : L \to M$ is a Lagrangian immersion into a Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$ then for any representative (η, ω, λ) the 1-form $i^*\lambda$ is $i^*\eta$-closed. We say that $i : L \to M$ is an exact Lagrangian immersion if for some (hence any) representative (η, ω, λ) the 1-form $i^*\lambda$ is $i^*\eta$-exact. We then say that the exact Lagrangian immersion $i : L \to M$ has action $S : L \to \mathbb{R}$ with respect to (η, ω, λ) (or just action S when the representative we consider is clear from the context) if $i^*\lambda = d_{i^*\eta}S$. By Remark 2.1, if L is connected and $i^*\eta$ is not exact then the action S is unique (otherwise it is just determined up to an additive constant on each connected component). A Lagrangian submanifold L of M is said to be exact and have action S if the inclusion $L \hookrightarrow M$ is exact and has action S.

Example 2.29. *For any function $f : B \to \mathbb{R}$, the β-twisted differential*

$$d_\beta f : B \to T^*B\,,\ d_\beta f(q) = df(q) - f(q)\beta(q)$$

is an exact Lagrangian embedding into the twisted cotangent bundle $T^*_\beta B$, of action f with respect to $(\pi^*\beta\,,\ d_{\pi^*\beta}\lambda\,,\ \lambda)$. Indeed, by the tautological property of λ we have

$$(d_\beta f)^*\lambda = d_\beta f\,.$$

We denote by $L_{f,\beta}$ the image of $d_{f,\beta}$. It is an exact Lagrangian submanifold of $T^*_\beta B$ of action

$$S_{f,\beta} : L_{f,\beta} \to \mathbb{R}\,,\ S_{f,\beta}\big(d_\beta f(q)\big) = f(q)\,.$$

In particular, the zero section $L_0 = L_{0,\beta}$ of $T^*_\beta B$ is an exact Lagrangian submanifold of action zero. Any exact Lagrangian submanifold of $T^*_\beta B$ that is a section with respect to the projection $T^*_\beta B \to B$ is equal to $L_{f,\beta}$ for some function $f : B \to \mathbb{R}$.

Example 2.30. *If Λ is a Legendrian submanifold of a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$ then $S^1 \times \Lambda$ is an exact Lagrangian submanifold of the locally conformal symplectization $\big(S^1 \times Y\,,\, [(-d\theta\,,\, d_{-d\theta}\alpha\,,\, \alpha)]\big)$, of action zero with respect to $(-d\theta\,,\, d_{-d\theta}\alpha\,,\, \alpha)$. We say that $S^1 \times \Lambda$ is the lift of Λ to the locally conformal symplectization. Similarly, the lift of a Legendrian immersion $i : \Lambda \to Y$ is the exact Lagrangian immersion*

$$\widetilde{i} : S^1 \times \Lambda \to S^1 \times Y\,,\ (\theta, x) \mapsto \big(\theta, i(x)\big)\,.$$

The next proposition follows easily from the definitions.

Proposition 2.31. *Let $\big(M, [(\eta, \omega, \lambda)]\big)$ be a Liouville lcs manifold, $i : L \to M$ an exact Lagrangian immersion and $\varphi : M \to M$ an exact lcs diffeomorphism. Then $\varphi \circ i : L \to M$ is an exact Lagrangian immersion.*

In particular, we obtain the following corollary.

Corollary 2.32. *Any Lagrangian submanifold of a twisted cotangent bundle $T^*_\beta B$ that is the image of the zero section by a lcs Hamiltonian diffeomorphism is exact.*

Proof. By Proposition 2.24, any lcs Hamiltonian diffeomorphism is exact. Since the zero section is an exact Lagrangian submanifold of $T^*_\beta B$, the result thus follows from Proposition 2.31. ☐

We have the following lcs analogue of the Weinstein neighborhood theorem for Lagrangians submanifolds. We refer to [OS17, Theorem 3.2] or [CM19, Theorem 2.11] for a proof.

Theorem 2.33. *Let L be a compact Lagrangian submanifold of a lcs manifold $\big(M, [(\eta, \omega)]\big)$. Denote by β the pullback of η by the inclusion $L \hookrightarrow M$. Then there exist a neighborhood $\mathcal{U}$ of L in M, a neighborhood $\mathcal{V}$ of the zero section of $T^*_\beta L$ and a strict lcs diffeomorphism $\psi : \mathcal{U} \to \mathcal{V}$ with respect to $(\eta|_{\mathcal{U}}\,,\, \omega|_{\mathcal{U}})$ and $\big((\pi^*\beta)|_{\mathcal{V}}\,,\, (d_{\pi^*\beta}\lambda)|_{\mathcal{V}}\big)$ such that $\psi(L)$ is the zero section.*

If L is an exact compact Lagrangian submanifold of a Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$ then the strict lcs diffeomorphism $\psi : \mathcal{U} \to \mathcal{V}$ of Theorem 2.33 is actually exact. To see this, we need the following lemma.

Lemma 2.34. *Consider a vector bundle $\pi : E \to B$, and a closed 1-form β on B that is not exact. Identify B with the zero section, and denote by $i : B \hookrightarrow E$ the inclusion. Let $\mathcal{U} \subset E$ be a neighborhood of B that is fibrewise starshaped, i.e. $\mathcal{U} \cap E_x \subset E_x$ is starshaped for all fibres $E_x = \pi^{-1}(x)$. If σ is a $\pi^*\beta$-closed form on $\mathcal{U}$ such that $i^*\sigma$ is β-exact then σ is $\pi^*\beta$-exact.*

Proof. Let $h_t : \mathcal{U} \to \mathcal{U}$ be a homotopy between $h_0 := \mathrm{id}|_{\mathcal{U}}$ and $h_1 := i \circ \pi$, obtained by contracting radially along the fibres. Consider the total map

$$h : \mathcal{U} \times \mathbb{R} \to \mathcal{U}\,,\ h(x, t) = h_t(x)\,,$$

and the operator

$$I : \Omega^k(\mathcal{U} \times \mathbb{R}) \to \Omega^{k-1}(\mathcal{U})\,,\ I(\zeta) = \int_0^1 j_t^*(\iota_{\partial/\partial t}\zeta)\, dt\,,$$

where, for every t, $j_t : \mathcal{U} \to \mathcal{U} \times \mathbb{R}$ is the map $j_t(x) = (x, t)$. Then $H := I \circ h^* : \Omega^k(\mathcal{U}) \to \Omega^{k-1}(\mathcal{U})$ satisfies

$$h_1^* - h_0^* = d_{\pi^*\beta} \circ H + H \circ d_{\pi^*\beta}\,.$$

Indeed, consider the maps

$$\phi_t : \mathcal{U} \times \mathbb{R} \to \mathcal{U} \times \mathbb{R}\,,\ \phi_t(x, s) = (x, s + t)\,.$$

Then $h_t = h \circ j_t = h \circ \phi_t \circ j_0$, thus

$$\begin{aligned}
h_1^*\zeta - h_0^*\zeta &= \int_0^1 \frac{d}{dt}\, h_t^*\zeta \, dt = \int_0^1 j_0^* \frac{d}{dt}\, \phi_t^*(h^*\zeta) \, dt = \int_0^1 j_0^* \phi_t^*(\mathcal{L}_{\partial/\partial t} h^*\zeta) \, dt \\
&= \int_0^1 j_0^* \phi_t^* \Big(d_{\pi^*\beta}(\iota_{\partial/\partial t} h^*\zeta) + \iota_{\partial/\partial t} d_{\pi^*\beta} h^*\zeta + \pi^*\beta\big(\frac{\partial}{\partial t}\big)\, h^*\zeta \Big) \, dt \\
&= \int_0^1 j_0^* \phi_t^* \big(d_{\pi^*\beta}(\iota_{\partial/\partial t} h^*\zeta) + \iota_{\partial/\partial t} d_{\pi^*\beta} h^*\zeta \big) \, dt \\
&= d_{\pi^*\beta} \int_0^1 j_t^*(\iota_{\partial/\partial t} h^*\zeta) \, dt + \int_0^1 j_t^* \big(\iota_{\partial/\partial t} d_{\pi^*\beta} h^*\zeta \big) \, dt \\
&= d_{\pi^*\beta} \circ I \circ h^*(\zeta) + I \circ h^* \circ d_{\pi^*\beta}(\zeta) \\
&= d_{\pi^*\beta} \circ H\,(\zeta) + H \circ d_{\pi^*\beta}\,(\zeta)\,,
\end{aligned}$$

where in the fifth equality we have used that $d_{\pi^*\beta} h^*\zeta = h^* \circ d_{\pi^*\beta}(\zeta)$, since $\pi \circ h = \pi$.

Since $i^*\sigma$ is β-exact, the pullback $\pi^*(i^*\sigma)$ is $\pi^*\beta$-exact. Since moreover σ is $\pi^*\beta$-closed, we conclude that

$$\sigma = h_0^*(\sigma) = h_1^*(\sigma) - d_{\pi^*\beta}\big(H(\sigma)\big) - H(d_{\pi^*\beta}\sigma) = \pi^*(i^*\sigma) - d_{\pi^*\beta}\big(H(\sigma)\big)$$

is $\pi^*\beta$-exact. □

Remark 2.35. *Without loss of generality we can assume that the neighborhood $\mathcal{V}$ of the zero section of $T^*_\beta L$ in Theorem 2.33 is fibrewise starshaped. If L is an exact compact Lagrangian submanifold of a Liouville lcs manifold $\big(M, [(\eta,\omega,\lambda)]\big)$, the strict lcs diffeomorphism $\psi : \mathcal{U} \to \mathcal{V}$ of Theorem 2.33 is then exact. Indeed, denoting now the tautological 1-form on T^*L by λ_0, $(\psi^{-1})^*\lambda - \lambda_0$ is a $\pi^*\beta$-closed 1-form whose pullback by the inclusion of L as the zero section is $\pi^*\beta$-exact. Using Lemma 2.34 we conclude that $(\psi^{-1})^*\lambda - \lambda_0$ is $\pi^*\beta$-exact, and so ψ is exact.*

2.9. **Contactization.** The contactization of a Liouville lcs manifold $\big(M\,,\,[(\eta,\omega,\lambda)]\big)$ with respect to (η,ω,λ) is the contact manifold $\big(M\times\mathbb{R}\,,\,\xi_\lambda = \ker(\alpha_\lambda)\big)$ with $\alpha_\lambda = \lambda - d_\eta z$, where z denotes the coordinate on $\mathbb{R}$ and where we still denote by λ and η their pullbacks by the projection $M\times\mathbb{R}\to M$.

Let $i : L \to M$ be an exact Lagrangian immersion into a Liouville lcs manifold $\big(M, [(\eta,\omega,\lambda)]\big)$, of action S with respect to (η,ω,λ). Then

$$\widetilde{i} : L \to M\times\mathbb{R}\,,\; \widetilde{i}(q) = \big(i(q), S(q)\big)$$

is a Legendrian immersion into $(M\times\mathbb{R}\,,\,\xi_\lambda)$. We say that $\widetilde{i}$ is the lift of i to the contactization.

Let φ be an exact lcs diffeomorphism of a Liouville lcs manifold $\big(M, [(\eta,\omega,\lambda)]\big)$, with conformal factor g and action S with respect to (η,ω,λ). Then the diffeomorphism $\widetilde{\varphi}$ of the contactization $(M\times\mathbb{R}\,,\,\xi_\lambda)$ defined by

$$\widetilde{\varphi}\,(p,z) = \Big(\varphi(p)\,,\, e^{g(p)}\big(z - S(p)\big)\Big)$$

is a contactomorphism, with conformal factor $(p,z)\mapsto g(p)$ with respect to α_λ. We say that $\widetilde{\varphi}$ is the lift of φ to the contactization.

Proposition 2.36. *Let $\{\varphi_t\}$ be a lcs Hamiltonian isotopy of a Liouville lcs manifold $\big(M, [(\eta,\omega,\lambda)]\big)$ with Hamiltonian function H_t with respect to (η,ω). Then its lift $\{\widetilde{\varphi_t}\}$ to the contactization $(M\times\mathbb{R}\,,\,\xi_\lambda)$ is a contact isotopy with Hamiltonian function $(p,z)\mapsto H_t(p)$ with respect to α_λ.*

Proof. The contact isotopy $\{\widetilde{\varphi_t}\}$ is generated by the vector field $\widetilde{X_t}$ defined by

$$\widetilde{X_t}\big(\widetilde{\varphi_t}(p)\big) = \Big(X_t\big(\varphi_t(p)\big)\,,\, \frac{d}{dt}\Big(e^{g_t(p)}\big(z - S_t(p)\big)\Big)\Big)\,,$$

where S_t is the action of $\{\varphi_t\}$ and X_t the vector field generating $\{\varphi_t\}$. By Proposition 2.24 we have

$$\frac{dS_t}{dt} = e^{-g_t}\Big(\big(H_t - \lambda(X_t)\big)\circ\varphi_t\Big)\,.$$

Moreover, by Lemmas 2.14 and 2.15 we have $\frac{dg_t}{dt} = \eta(X_t) \circ \varphi_t$. We thus obtain

$$\begin{aligned}\frac{d}{dt}\Big(e^{g_t(p)}\big(z - S_t(p)\big)\Big) &= -e^{g_t(p)}\,\frac{dS_t}{dt}(p) + e^{g_t(p)}\,\frac{dg_t}{dt}(p)\,\big(z - S_t(p)\big)\\ &= \lambda\Big(X_t\big(\varphi_t(p)\big)\Big) - H_t\big(\varphi_t(p)\big) + e^{g_t(p)}\,\eta\Big(X_t\big(\varphi_t(p)\big)\Big)\,\big(z - S_t(p)\big)\,,\end{aligned}$$

and so

$$\begin{aligned}&\alpha_\lambda\Big(\widetilde{X_t}\big(\widetilde{\varphi_t}(p)\big)\Big) = (\lambda - dz + z\eta)\Big(\widetilde{X_t}\big(\widetilde{\varphi_t}(p)\big)\Big)\\ &= \lambda\Big(X_t\big(\varphi_t(p)\big)\Big) - \lambda\Big(X_t\big(\varphi_t(p)\big)\Big) + H_t\big(\varphi_t(p)\big) - e^{g_t(p)}\,\eta\Big(X_t\big(\varphi_t(p)\big)\Big)\,\big(z - S_t(p)\big)\\ &\quad + e^{g_t(p)}\,\big(z - S_t(p)\big)\,\eta\Big(X_t\big(\varphi_t(p)\big)\Big) = H_t\big(\varphi_t(p)\big)\,.\end{aligned}$$

This shows that the contact isotopy $\{\widetilde{\varphi_t}\}$ has Hamiltonian function $(p, z) \mapsto H_t(p)$ with respect to α_λ. ☐

Example 2.37. *The contactization of the twisted cotangent bundle $T^*_\beta B$ with respect to $(\pi^*\beta\,,\, d_{\pi^*\beta}\lambda\,,\, \lambda)$ is the twisted 1-jet bundle*

$$J^1_\beta B := \big(T^*B \times \mathbb{R}\,,\, \ker(\lambda - d_{\pi^*\beta} z)\big)\,.$$

For any function $f : B \to \mathbb{R}$, the twisted 1-jet

$$j^1_\beta f : B \to J^1_\beta B\,,\ j^1_\beta f(q) = \big(d_\beta f(q)\,,\, f(q)\big)$$

is a Legendrian embedding, which is the lift of the exact Lagrangian embedding $d_\beta f : B \to T^*_\beta B$. Following [CM19]*, we consider the strict contactomorphism*

(2) $$U_\beta : J^1_\beta B \to J^1 B\,,\ U_\beta(\sigma_q, z) = (\sigma_q + z\beta(q), z)\,.$$

This map sends β-twisted 1-jets to usual 1-jets:

$$U_\beta \circ j^1_\beta f = j^1 f\,.$$

In particular, U_β sends the zero section to the zero section. We call it the untwisting map from $J^1_\beta B$ to $J^1 B$.

The untwisting map U_β of Example 2.37 will play an important role in Sections 4 and 5. In particular, we will need the following lemma, whose proof follows directly from the definitions and Example 2.30.

Lemma 2.38. *Let Λ be a Legendriann submanifold of a 1-jet bundle $J^1 B$, and $S^1 \times \Lambda$ its lift to the locally conformal symplectization $S^1 \times J^1 B$. Identify $S^1 \times J^1 B$ with $T^*_{-d\theta}(S^1 \times B)$ by the strict lcs diffeomorphism of Lemma 2.13, and consider the lift $\widetilde{S^1 \times \Lambda}$ of $S^1 \times \Lambda \subset T^*_{-d\theta}(S^1 \times B)$ to the contactization $J^1_{-d\theta}(S^1 \times B)$. Then*

$$U_{-d\theta}(\widetilde{S^1 \times \Lambda}) = 0_{S^1} \times \Lambda\,,$$

where $U_{-d\theta}$ is the untwisting map (2) *from $J^1_{-d\theta}(S^1 \times B)$ to $J^1(S^1 \times B)$ and where we identify $J^1(S^1 \times B)$ with $T^*S^1 \times J^1 B$.*

We will also use the untwisting map in the next section in order to derive (strong) orderability of twisted cotangent bundles from the corresponding result in the contact case for 1-jet bundles.

2.10. **Orderability.** Let $\big(M, [(\eta, \omega)]\big)$ be a connected lcs manifold such that η is not exact. By Remark 2.1, the Hamiltonian function H_t of a lcs Hamiltonian isotopy $\{\varphi_t\}$ with respect to a representative (η, ω) is then uniquely defined. Moreover, it is non-negative (respectively positive) if and only if the same is true for the Hamiltonian function with respect to any other representative (indeed, the Hamiltonian function of $\{\varphi_t\}$ with respect to another representative $(\eta + df, e^f\omega)$ is $e^f H_t$). We say that a lcs Hamiltonian isotopy $\{\varphi_t\}$ is non-negative (respectively positive) if its Hamiltonian function with respect to some (hence any) representative (η, ω) is non-negative (respectively positive)[1].

[1]If η is exact then, by Example 2.18, the lcs Hamiltonian isotopies of $\big(M, [(\eta, \omega)]\big)$ coincide with the Hamiltonian isotopies of the symplectic manifold $(M, e^{-\mu}\omega)$, where μ is any primitive of η. The Hamiltonian function of a Hamiltonian isotopy is in this case only defined up to addition of a constant. If M is not compact then the Hamiltonian functions of compactly supported Hamiltonian isotopies can be normalized by requiring them to have compact support, and we can define a compactly supported Hamiltonian isotopy to be non-negative (respectively positive) if its normalized Hamiltonian function is non-negative

Example 2.39. *Recall from* [EP00] *that a contact isotopy $\{\phi_t\}$ on a co-oriented contact manifold (Y,ξ) is said to be non-negative (respectively positive) if its contact Hamiltonian function H_t with respect to some (hence any) contact form α is non-negative (respectively positive), equivalently if it moves every point in a direction positively transverse or tangent to ξ (respectively positively transverse to ξ). By Example 2.19, $\{\phi_t\}$ is a non-negative (respectively positive) contact isotopy if and only if its lift $\{\widetilde{\phi_t}\}$ to the locally conformal symplectization is a non-negative (respectively positive) lcs Hamiltonian isotopy.*

Recall that a bi-invariant partial order on a group G is a partial relation $\leq$ that satisfies the following properties:

(i) *Reflexivity:* $g \leq g$.
(ii) *Transitivity:* if $g_1 \leq g_2$ and $g_2 \leq g_3$ then $g_1 \leq g_3$.
(iii) *Anti-symmetry:* if $g_1 \leq g_2$ and $g_2 \leq g_1$ then $g_1 = g_2$.
(iv) *Bi-invariance:* if $g_1 \leq g_2$ then $hg_1 \leq hg_2$ and $g_1 h \leq g_2 h$.

Recall also from [EP00] that a compact co-oriented contact manifold (Y,ξ) is said to be orderable if the partial relation $\leq$ on the universal cover of the identity component of the contactomorphism group defined by posing $[\{\phi_t^{(1)}\}] \leq [\{\phi_t^{(2)}\}]$ if $[\{\phi_t^{(2)}\}] \cdot [\{\phi_t^{(1)}\}]^{-1}$ can be represented by a non-negative contact isotopy is a bi-invariant partial order. If (Y,ξ) is not compact, we consider the partial relation $\leq$ on the universal cover of the identity component of the group of compactly supported contactomorphisms defined in the same way, and say that (Y,ξ) is orderable if $\leq$ is a partial order (see [FPR18]). Moreover, a contact manifold (Y,ξ) is said to be strongly orderable if the partial relation $\leq$ on the group of compactly supported contactomorphisms defined by posing $\phi_1 \leq \phi_2$ if $\phi_2 \circ \phi_1^{-1}$ is the time-1 map of a non-negative compactly supported contact isotopy is a partial order.

Similarly, for a connected lcs manifold $\big(M,[(\eta,\omega)]\big)$ such that η is not exact we consider the partial relation $\leq$ on the universal cover of the group of compactly supported lcs Hamiltonian diffeomorphisms defined by posing $[\{\varphi_t^{(1)}\}] \leq [\{\varphi_t^{(2)}\}]$ if $[\{\varphi_t^{(2)}\}] \cdot [\{\varphi_t^{(1)}\}]^{-1}$ can be represented by a non-negative lcs Hamiltonian isotopy. We say that $\big(M,[(\eta,\omega)]\big)$ is orderable if $\leq$ is a bi-invariant partial order. As in the contact case, reflexivity, transitivity and bi-invariance of $\leq$ are always true, and can be easily proved using Lemmas 2.20 and 2.21. On the other hand, anti-symmetry is equivalent to the non-existence of a non-negative non-constant contractible loop of compactly supported lcs Hamiltonian diffeomorphisms, and can fail in general. For instance, by Example 2.19 any non-negative non-constant contractile loop of contactomorphisms on a contact manifold lifts to a non-negative non-constant contractile loop of lcs Hamiltonian diffeomorphisms of the locally conformal symplectization. Thus, the locally conformal symplectization of a contact manifold that is not orderable (for instance the standard contact sphere [EKP06]) is also not orderable. The converse implication is in general not clear, but we will see in Section 7 that it holds for the locally conformal symplectizations of $(\mathbb{R}^{2n+1},\xi_0)$ and $(\mathbb{R}^{2n}\times S^1,\xi_0)$.

Consider now the partial relation $\leq$ on the group of compactly supported lcs Hamiltonian diffeomorphisms of $\big(M,[(\eta,\omega)]\big)$ defined by posing $\varphi_1 \leq \varphi_2$ if $\varphi_2 \circ \varphi_1^{-1}$ is the time-1 map of a non-negative lcs Hamiltonian isotopy. We say that $\big(M,[(\eta,\omega)]\big)$ is strongly orderable if $\leq$ is a bi-invariant partial order (equivalently, if it does not admit any non-negative non-constant loop of compactly supported lcs Hamiltonian diffeomorphisms). In particular, if $\big(M,[(\eta,\omega)]\big)$ is strongly orderable then it is orderable. Again, the locally conformal symplectization of a contact manifold that is not strongly orderable (for instance any contact manifold with periodic Reeb flow) is also not strongly orderable. We will see in Section 7 that the locally conformal symplectizations of $(\mathbb{R}^{2n+1},\xi_0)$ and $(\mathbb{R}^{2n}\times S^1,\xi_0)$ are in fact strongly orderable. Using the untwisting map (2) of Example 2.37, we now derive strong orderability of twisted cotangent bundles with compact base from the corresponding result for 1-jet bundles [CFP17, CN10].

Proof of Theorem 1.3 (i). Let B be a compact connected manifold, and β a closed non-exact 1-form. We want to show that the twisted cotangent bundle $T^*_\beta B$ is strongly orderable, thus that it does not admit any

(respectively positive). However, we do not consider this case in our discussion of orderability because it does not add anything to what is already known for symplectic manifolds.

non-constant non-negative loop $\{\varphi_t\}$ of compactly supported lcs Hamiltonian diffeomorphisms. Suppose by contradiction that such $\{\varphi_t\}$ exists. By Proposition 2.36, the lift $\{\widetilde{\varphi}_t\}$ to the contactization $J^1_\beta B$ is a non-constant non-negative loop of compactly supported contactomorphisms. By Lemma 2.21, the conjugation $\{U_\beta \circ \widetilde{\varphi}_t \circ U_\beta^{-1}\}$ by the untwisting map (2) is thus a non-constant non-negative loop of compactly supported contactomorphisms of $J^1 B$. By [FPR18, Proposition 2.12], $\{U_\beta \circ \widetilde{\varphi}_t \circ U_\beta^{-1}\}$ can be deformed to a loop of contactomorphisms whose restriction to the zero section is positive. But this contradicts [CFP17, Theorem 1] and [CN10, Corollary 5.5]. $\square$

2.11. **Lcs product and graph.** In this section we describe the lcs product of a lcs manifold and the lcs graph of a divergence free lcs diffeomorphism, referring to [CS24] for a more formal point of view on these notions.

Let $\big(M, [(\eta,\omega)]\big)$ be a lcs manifold. Consider the homomorphism

$$\Delta_\eta : \pi_1(M) \times \pi_1(M) \to \mathbb{R}\,,\ \ ([\gamma_1],[\gamma_2]) \mapsto \langle [\eta], [\gamma_1]\rangle - \langle [\eta],[\gamma_2]\rangle\,,$$

and let

$$M \boxtimes M \to M \times M$$

be the cover corresponding to the subgroup $\ker(\Delta_\eta)$ of $\pi_1(M\times M) \cong \pi_1(M)\times\pi_1(M)$. In other words, if we denote by $\pi : \bar M \to M$ the symplectic cover then the manifold $M \boxtimes M$ is the quotient of $\bar M \times \bar M$ by the equivalence relation

$$(x_1,x_2) \sim (y_1,y_2) \quad \text{ if } \pi(x_1) = \pi(y_1)\,,\ \pi(x_2)=\pi(y_2) \text{ and } \mu(x_1)-\mu(y_1) = \mu(x_2)-\mu(y_2)\,, \tag{3}$$

where μ is any primitive of $\pi^*\eta$. Consider the symplectic form $\bar\omega = e^{-\mu}\,\pi^*\omega$ on $\bar M$. Then the symplectic form $-\operatorname{pr}_1^*\bar\omega + \operatorname{pr}_2^*\bar\omega$ on $\bar M\times\bar M$, where pr_1 and pr_2 denote the projections of $\bar M \times \bar M$ to the first and second factors, can be written as

$$\begin{aligned} -\operatorname{pr}_1^*\bar\omega + \operatorname{pr}_2^*\bar\omega &= -e^{-\mu\circ\operatorname{pr}_1}\,(\pi\circ\operatorname{pr}_1)^*\omega + e^{-\mu\circ\operatorname{pr}_2}\,(\pi\circ\operatorname{pr}_2)^*\omega \\ &= e^{-\mu\circ\operatorname{pr}_1}\big(-(\pi\circ\operatorname{pr}_1)^*\omega + e^{\mu\circ\operatorname{pr}_1-\mu\circ\operatorname{pr}_2}\,(\pi\circ\operatorname{pr}_2)^*\omega\big)\,. \end{aligned}$$

Therefore

$$\begin{aligned} \big(0\,,\,-\operatorname{pr}_1^*\bar\omega + \operatorname{pr}_2^*\bar\omega\big) &\sim \big(d(\mu\circ\operatorname{pr}_1)\,,\,-(\pi\circ\operatorname{pr}_1)^*\omega + e^{\mu\circ\operatorname{pr}_1-\mu\circ\operatorname{pr}_2}\,(\pi\circ\operatorname{pr}_2)^*\omega\big) \\ &= \big((\pi\circ\operatorname{pr}_1)^*\eta\,,\,-(\pi\circ\operatorname{pr}_1)^*\omega + e^{\mu\circ\operatorname{pr}_1-\mu\circ\operatorname{pr}_2}\,(\pi\circ\operatorname{pr}_2)^*\omega\big)\,. \end{aligned}$$

Since the function $\mu\circ\operatorname{pr}_1 - \mu\circ\operatorname{pr}_2$ is invariant by the equivalence relation (3), the forms $(\pi\circ\operatorname{pr}_1)^*\eta$ and

$$-(\pi\circ\operatorname{pr}_1)^*\omega + e^{\mu\circ\operatorname{pr}_1-\mu\circ\operatorname{pr}_2}\,(\pi\circ\operatorname{pr}_2)^*\omega$$

descend to forms $\eta\boxtimes\eta$ and $\omega\boxtimes\omega$ on $M\boxtimes M$, defining a lcs structure. We say that

$$\big(M\boxtimes M\,,\ [(\eta\boxtimes\eta, \omega\boxtimes\omega)]\big)$$

is the lcs product of the lcs manifold $\big(M,[(\eta,\omega)]\big)$ with respect to (η,ω). By construction, its symplectic cover is the conformal class $(\bar M\times\bar M\,,\,[-\operatorname{pr}_1^*\bar\omega + \operatorname{pr}_2^*\bar\omega])$ of the symplectic product of the symplectic cover. We denote by

$$\pi\boxtimes\pi : \bar M \times \bar M \to M \boxtimes M$$

the symplectic cover projection. The following diagram of covers summarizes the situation:

$$\begin{tikzcd} \widetilde{M}\times\widetilde{M} \arrow[ddd, "\pi_1(M)\times\pi_1(M)"'] \arrow[dr, "\ker(\eta)\times\ker(\eta)"] \arrow[ddr, "\ker(\Delta_\eta)"'] & \\ & \bar M \times \bar M \arrow[d] \\ & M\boxtimes M \arrow[dl] \\ M\times M\,. & \end{tikzcd}$$

For a Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$ we have

$$-(\pi \circ \mathrm{pr}_1)^*\omega + e^{\mu\circ\mathrm{pr}_1 - \mu\circ\mathrm{pr}_2}(\pi \circ \mathrm{pr}_2)^*\omega = d_{(\pi\circ\mathrm{pr}_1)^*\eta}\big(-(\pi \circ \mathrm{pr}_1)^*\lambda + e^{\mu\circ\mathrm{pr}_1 - \mu\circ\mathrm{pr}_2}(\pi \circ \mathrm{pr}_2)^*\lambda\big)\,.$$

The 1-form $-(\pi \circ \mathrm{pr}_1)^*\lambda + e^{\mu\circ\mathrm{pr}_1 - \mu\circ\mathrm{pr}_2}(\pi \circ \mathrm{pr}_2)^*\lambda$ on $\bar{M} \times \bar{M}$ descends to a 1-form $\lambda \boxtimes \lambda$ on $M \boxtimes M$ with $\omega \boxtimes \omega = d_{\eta\boxtimes\eta}\lambda \boxtimes \lambda$. The lcs structure $[(\eta \boxtimes \eta, \omega \boxtimes \omega)]$ on $M \boxtimes M$ is thus exact, with Liouville form $\lambda \boxtimes \lambda$. We say that

$$\big(M \boxtimes M\,,\ [(\eta \boxtimes \eta, \omega \boxtimes \omega, \lambda \boxtimes \lambda)]\big)$$

is the lcs product of the Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$ with respect to (η, ω, λ).

Lemma 2.40. *Let $\{\bar{\varphi}^{\omega}_t\}$ be the lift to $\bar{M}$ of the Lee flow on M with respect to (η, ω). The isotopy*

$$(x_1, x_2) \mapsto \big(\bar{\varphi}^{\omega}_{-t}(x_1)\,,\ x_2\big)$$

on $\bar{M} \times \bar{M}$ commutes with $\pi \boxtimes \pi : \bar{M} \times \bar{M} \to M \boxtimes M$, and the induced isotopy on $M \boxtimes M$ is the Lee flow with respect to $(\eta \boxtimes \eta, \omega \boxtimes \omega)$.

Proof. By Lemma 2.26, $\{\bar{\varphi}^{\omega}_t\}$ is the Hamiltonian flow of $e^{-\mu}$ with respect to $\bar{\omega}$. The isotopy $(x_1, x_2) \mapsto \big(\bar{\varphi}^{\omega}_{-t}(x_1), x_2\big)$ is thus the Hamiltonian flow of $e^{-\mu\circ\mathrm{pr}_1}$ with respect to $-\mathrm{pr}_1^*\bar{\omega} + \mathrm{pr}_2^*\bar{\omega}$. Since $\mu \circ \mathrm{pr}_1$ is a primitive of the pullback of $\eta\boxtimes\eta$ by $\pi\boxtimes\pi$, and $-\mathrm{pr}_1^*\bar{\omega} + \mathrm{pr}_2^*\bar{\omega}$ is the pullback of $\omega\boxtimes\omega$ by $\pi\boxtimes\pi$, by Lemma 2.26 the isotopy $(x_1, x_2) \mapsto \big(\bar{\varphi}^{\omega}_{-t}(x_1), x_2\big)$ commutes with $\pi \boxtimes \pi$ and induces the Lee flow on $M \boxtimes M$ with respect to $(\eta \boxtimes \eta, \omega \boxtimes \omega)$. ☐

We define the contact product[2] of a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$ with respect to the contact form α to be the contact manifold

$$\big(Y \times Y \times \mathbb{R}\,,\ \ker(e^{-\rho}\,\mathrm{pr}_2^*\,\alpha - \mathrm{pr}_1^*\,\alpha)\big)\,,$$

where ρ denotes the coordinate on $\mathbb{R}$ and pr_1 and pr_2 the projections to the first and second factors. The lcs product and the contact product are related as follows. Consider the symplectic cover $\pi : \mathbb{R} \times Y \to S^1 \times Y$ of the conformal symplectization, and the product $(\mathbb{R} \times Y) \times (\mathbb{R} \times Y)$. Denote by θ_1 and θ_2 the coordinates on the first and second $\mathbb{R}$-factors and by $\mathrm{pr}_j : (\mathbb{R}\times Y)\times(\mathbb{R}\times Y) \to \mathbb{R}\times Y$ for $j = 1, 2$ the projections on the first and second factors. The lcs product of the locally conformal symplectization $\big(S^1 \times Y\,,\ [(-d\theta\,,\ d_{-d\theta}\alpha\,,\ \alpha)]\big)$ is

$$\Big((\mathbb{R} \times Y \times \mathbb{R} \times Y)/\sim,\ \big[\big(-d\theta_1\,,\ d_{-d\theta_1}(e^{\theta_2-\theta_1}\,\mathrm{pr}_2^*\,\alpha - \mathrm{pr}_1^*\,\alpha)\,,\ e^{\theta_2-\theta_1}\,\mathrm{pr}_2^*\,\alpha - \mathrm{pr}_1^*\,\alpha\big)\big]\Big)\,,$$

where $\sim$ is the equivalence relation generated by

$$(\theta_1, p_1, \theta_2, p_2) \sim (\theta_1 + 1, p_1, \theta_2 + 1, p_2)\,.$$

The next result is then immediate.

Proposition 2.41. *For any contact manifold $\big(Y, \xi = \ker(\alpha)\big)$, the map*

$$I : (S^1 \times Y) \boxtimes (S^1 \times Y) \to S^1 \times (Y \times Y \times \mathbb{R})$$

defined by

$$I\,\big([(\theta_1, p_1, \theta_2, p_2)]\big) = ([\theta_1], p_1, p_2, \theta_1 - \theta_2)\,,$$

where we identify as above $(S^1 \times Y) \boxtimes (S^1 \times Y)$ with $(\mathbb{R} \times Y \times \mathbb{R} \times Y)/\sim$, is a strict lcs diffeomorphism between the lcs product of the locally conformal symplectization $\big(S^1 \times Y\,,\ [(-d\theta\,,\ d_{-d\theta}\alpha\,,\ \alpha)]\big)$ and the locally conformal symplectization

$$\big(S^1 \times (Y \times Y \times \mathbb{R})\,,\ [(-d\theta\,,\ d_{-d\theta}(e^{-\rho}\,\mathrm{pr}_2^*\,\alpha - \mathrm{pr}_1^*\,\alpha)\,,\ e^{-\rho}\,\mathrm{pr}_2^*\,\alpha - \mathrm{pr}_1^*\,\alpha)]\big)$$

of the contact product $\big(Y \times Y \times \mathbb{R}\,,\ \ker(e^{-\rho}\,\mathrm{pr}_2^*\,\alpha - \mathrm{pr}_1^*\,\alpha)\big)$, which is exact with action zero.

[2] In [Bhu01, San10, San11a, FSZ23] the contact product of $\big(Y, \xi = \ker(\alpha)\big)$ with respect to α is defined to be $\big(Y \times Y \times \mathbb{R}\,,\ \ker(\mathrm{pr}_2^*\,\alpha - e^{\rho}\,\mathrm{pr}_1^*\,\alpha)\big)$. The formula we use in the present article works better for our purposes, and could also have been used in [Bhu01, San10, San11a, FSZ23] (with a consequently adapted formula for the indentification of $\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$ with $J^1\mathbb{R}^{2n+1}$, as in Section 5) without any inconvenience.

Let φ be a divergence free lcs diffeomorphism of a lcs manifold $\big(M,[(\eta,\omega)]\big)$, and $\bar\varphi:\bar M\to\bar M$ its lift to the symplectic cover $(\bar M,[e^{-\mu}\pi^*\omega])$. By Corollary 2.9, for every primitive μ of $\pi^*\eta$ and $x\in\bar M$ we have

$$\mu\big(\bar\varphi(x)\big)-\mu(x)=g\big(\pi(x)\big)\,,$$

where g is the conformal factor of φ with respect to (η,ω). This implies that the symplectic graph

$$\mathrm{gr}(\bar\varphi):\bar M\to\bar M\times\bar M\,,\ x\mapsto\big(x,\bar\varphi(x)\big)$$

descends to a map

$$\mathrm{gr}_{\mathrm{lcs}}(\varphi):M\to M\boxtimes M\,.$$

Indeed, for $x,y\in\bar M$ with $\pi(x)=\pi(y)$ we have

$$\mu\big(\bar\varphi(x)\big)-\mu(x)=g\big(\pi(x)\big)=g\big(\pi(y)\big)=\mu\big(\bar\varphi(y)\big)-\mu(y)\,,$$

and so $\big(x,\bar\varphi(x)\big)\sim\big(y,\bar\varphi(y)\big)$. The map $\mathrm{gr}_{\mathrm{lcs}}(\varphi):M\to M\boxtimes M$ is a Lagrangian embedding, which we call the lcs graph of the divergence free lcs diffeomorphism φ.

Lemma 2.42. *If φ is an exact divergence free lcs diffeomorphism of a Liouville lcs manifold $\big(M,\,[(\eta,\omega,\lambda)]\big)$ with action S with respect to (η,ω,λ) then $\mathrm{gr}_{\mathrm{lcs}}(\varphi):M\to M\boxtimes M$ is an exact Lagrangian embedding with action $-S$ with respect to $(\eta\boxtimes\eta,\omega\boxtimes\omega,\lambda\boxtimes\lambda)$.*

Proof. Let $\bar\varphi$ be the lift of φ to the symplectic cover. Then

$$\begin{aligned}\mathrm{gr}(\bar\varphi)^*(-(\pi\circ\mathrm{pr}_1)^*\lambda+e^{\mu\circ\mathrm{pr}_1-\mu\circ\mathrm{pr}_2}(\pi\circ\mathrm{pr}_2)^*\lambda)&=-\pi^*\lambda+e^{\mu-\mu\circ\bar\varphi}\,\bar\varphi^*\pi^*\lambda\\ =-\pi^*\lambda+e^{-g\circ\pi}\,\pi^*\varphi^*\lambda=-\pi^*\lambda+e^{-g\circ\pi}\,\pi^*\big(e^{g}(\lambda-d_\eta S)\big)&=d_{\pi^*\eta}(-S\circ\pi)\,,\end{aligned}$$

thus $\mathrm{gr}_{\mathrm{lcs}}(\varphi)^*(\lambda\boxtimes\lambda)=d_\eta(-S)=d_{\mathrm{gr}_{\mathrm{lcs}}(\varphi)^*(\eta\boxtimes\eta)}(-S)$. ∎

Lemma 2.43. *Let φ_1 and φ_2 be divergence free lcs diffeomorphisms a lcs manifold $\big(M,[(\eta,\omega)]\big)$, and $\bar\varphi_1$ and $\bar\varphi_2$ their lifts to the symplectic cover $(\bar M,[e^{-\mu}\pi^*\omega])$. Then the map*

$$\varphi_1\boxtimes\varphi_2:M\boxtimes M\to M\boxtimes M\,,\ [(x_1,x_2)]\mapsto\big[\big(\bar\varphi_1(x_1)\,,\,\bar\varphi_2(x_2)\big)\big]$$

is well-defined, and is a divergence free lcs diffeomorphism. Moreover, the following holds:

(i) *For every divergence free lcs diffeomorphism φ we have*

$$\mathrm{gr}_{\mathrm{lcs}}(\varphi)=(\mathrm{id}\boxtimes\varphi)\circ\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})\,.$$

(ii) *If $\{\varphi_t^{(1)}\}$ and $\{\varphi_t^{(2)}\}$ are lcs Hamiltonian isotopies with Hamiltonian functions $H_t^{(1)}$ and $H_t^{(2)}$ with respect to (η,ω), then $\big\{\varphi_t^{(1)}\boxtimes\varphi_t^{(2)}\big\}$ is a lcs Hamiltonian isotopy with Hamiltonian function*

$$H_t^{(1)}\boxplus H_t^{(2)}\big([(x,y)]\big)=-H_t^{(1)}\big(\pi(x)\big)+e^{\mu(x)-\mu(y)}\,H_t^{(2)}\big(\pi(y)\big)$$

with respect to $(\eta\boxtimes\eta,\omega\boxtimes\omega)$.

Proof. If $(x_1,x_2)\sim(y_1,y_2)$ then $\big(\bar\varphi_1(x_1),\bar\varphi_2(x_2)\big)\sim\big(\bar\varphi_1(y_1),\bar\varphi_2(y_2)\big)$. Indeed, by Corollary 2.9 we have $\mu\big(\bar\varphi_j(x_j)\big)=\mu(x_j)+g\big(\pi(x_j)\big)$ and $\mu\big(\bar\varphi_j(y_j)\big)=\mu(y_j)+g_j\big(\pi(y_j)\big)$. This shows that $\varphi_1\boxtimes\varphi_2$ is well-defined. Since it is covered by the symplectomorphism $\bar\varphi_1\times\bar\varphi_2$ of the symplectic cover, it is divergence free.

Statement (i) follows from the fact that $\mathrm{gr}(\bar\varphi)=(\mathrm{id}\times\bar\varphi)\circ\mathrm{gr}(\mathrm{id})$.

For (ii), by Proposition 2.22 the lifts $\{\bar\varphi_t^{(j)}\}$ are Hamiltonian isotopies with Hamiltonian functions $\bar H_t^{(j)}=e^{-\mu}(H_t^{(j)}\circ\pi)$. Thus $\{\bar\varphi_t^{(1)}\times\bar\varphi_t^{(2)}\}$ is a Hamiltonian isotopy of $\bar M\times\bar M$ with Hamiltonian function

$$(x,y)\mapsto-e^{-\mu(x)}H_t^{(1)}\big(\pi(x)\big)+e^{-\mu(y)}H_t^{(2)}\big(\pi(y)\big)=e^{-\mu(x)}\Big(-H_t^{(1)}\big(\pi(x)\big)+e^{\mu(x)-\mu(y)}H_t^{(2)}\big(\pi(y)\big)\Big)\,.$$

By Proposition 2.22 we conclude that $\big\{\varphi_t^{(1)}\boxtimes\varphi_t^{(2)}\big\}$ is a lcs Hamiltonian isotopy with Hamiltonian function

$$[(x,y)]\mapsto-H_t^{(1)}\big(\pi(x)\big)+e^{\mu(x)-\mu(y)}\,H_t^{(2)}\big(\pi(y)\big)\,.$$

∎

The contact graph of a contactomorphism ϕ of a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$ with respect to α is the Legendrian embedding

$$\mathrm{gr}(\phi) : Y \to Y \times Y \times \mathbb{R}\,,\ x \mapsto \big(x, \phi(x), g(x)\big)$$

into the contact product $\big(Y \times Y \times \mathbb{R}\,, \ker(e^{-\rho}\,\mathrm{pr}_2^*\,\alpha - \mathrm{pr}_1^*\,\alpha)\big)$, where g is the conformal factor of ϕ with respect to α. Recall from Example 2.12 that the lift of ϕ to the locally conformal symplectization $\big(S^1 \times Y\,, [(-d\theta, d_{-d\theta}\alpha)]\big)$ is the lcs diffeomorphism $\widetilde{\phi} : S^1 \times Y \to S^1 \times Y$ defined by $\widetilde{\phi}(\theta, p) = \big(\theta - g(p), \phi(p)\big)$. This lcs diffeomorphism is divergence free, since it is covered by the symplectomorphism of the symplectic cover $\big(\mathbb{R} \times Y\,, [d(e^\theta \alpha)]\big)$ defined by the same formula. Moreover, we have the following relation between the contact graph of ϕ and the lcs graph of its lift $\widetilde{\phi}$.

Proposition 2.44. *Let ϕ be a contactomorphism of a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$, and $\widetilde{\phi}$ its lift to the locally conformal symplectization $\big(S^1 \times Y\,, [(-d\theta, d_{-d\theta}\alpha)]\big)$. Consider the contact graph $\mathrm{gr}(\phi) : Y \to Y \times Y \times \mathbb{R}$ of ϕ, and its lift*

$$\widetilde{\mathrm{gr}(\phi)} : S^1 \times Y \to S^1 \times (Y \times Y \times \mathbb{R})$$

to the locally conformal symplectization of the contact product. Then

$$\widetilde{\mathrm{gr}(\phi)} = I \circ \mathrm{gr}_{\mathrm{lcs}}(\widetilde{\phi})\,,$$

where $I : (S^1 \times Y) \boxtimes (S^1 \times Y) \to S^1 \times (Y \times Y \times \mathbb{R})$ is the lcs diffeomorphism defined in Proposition 2.41.

Proof. We have

$$\mathrm{gr}_{\mathrm{lcs}}(\widetilde{\phi})\,([\theta], x) = \big[\big(\theta, x, \theta - g(x), \phi(x)\big)\big]\,,$$

and so

$$I \circ \mathrm{gr}_{\mathrm{lcs}}(\widetilde{\phi})\,([\theta], x) = \big([\theta], x, \phi(x), g(x)\big) = \widetilde{\mathrm{gr}(\phi)}\,([\theta], x)\,.$$

$\square$

3. Essential Lee chords and translated points

Let p_1 and p_2 be two points of a lcs manifold $\big(M, [(\eta, \omega)]\big)$. A Lee chord with respect to (η, ω) from p_1 to p_2 is a pair (T, γ) with T a real number and $\gamma : [0,1] \to M$ a curve such that $\gamma(0) = p_1$, $\gamma(1) = p_2$ and $\dot{\gamma}(t) = T\,R_\omega\big(\gamma(t)\big)$ for all t. We say that T is the time-shift of the Lee chord (T, γ). Observe that either $R_\omega\big(\gamma(t)\big) = 0$ for all t or $R_\omega\big(\gamma(t)\big) \neq 0$ for all t. In the second case the time-shift T is determined by γ, and we simply say that γ is a Lee chord from p_1 to p_2 of time-shift T. If $R_\omega\big(\gamma(t)\big) = 0$ for all t then γ is the constant curve $\gamma(t) = p_1$ and, for any other T', the pair (T', γ) is also a Lee chord from p_1 to $p_2 = p_1$. It is pertinent to regard (T, γ) and (T', γ) as different Lee chords, indeed in the first case we regard γ as a (constant) orbit of the flow $\{\varphi^\omega_{Tt}\}_{t \in [0,1]}$ while in the second case we regard it as a (constant) orbit of the flow $\{\varphi^\omega_{T't}\}_{t \in [0,1]}$. Although $\varphi^\omega_{Tt}(p_1) = \varphi^\omega_{T't}(p_1) = p_1$ for all t, the two flows do not necessarily coincide on a neighborhood of p_1. In particular, their differentials at p_1 might be different (this will play a role for instance in the definition of transverse Lee chords between Lagrangian submanifolds below).

Let now L_1 and L_2 be two Lagrangian submanifolds of $\big(M, [(\eta, \omega)]\big)$. A Lee chord with respect to (η, ω) from L_1 to L_2 is a Lee chord (T, γ) from $\gamma(0) \in L_1$ to $\gamma(1) \in L_2$. We say that it is transverse if

$$(\varphi^\omega_T)_*\big(T_{\gamma(0)} L_1\big) + T_{\gamma(1)} L_2 = T_{\gamma(1)} M\,.$$

If L_1 and L_2 are connected exact Lagrangian submanifolds of a Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$ and if the pullback of η by the inclusion of L_1 and L_2 into M is not exact then we define the (total) action with respect to (η, ω, λ) of a Lee chord (T, γ) from L_1 to L_2 by

$$\mathcal{A}_{L_1, L_2}(T, \gamma) = S_{L_1}\big(\gamma(0)\big) - S_{L_2}\big(\gamma(1)\big) + \int_\gamma \lambda\,,$$

where S_{L_1} and S_{L_2} are the actions of L_1 and L_2 with respect to (η, ω, λ). We say that $S_{L_1}\big(\gamma(0)\big) - S_{L_2}\big(\gamma(1)\big)$ and $\int_\gamma \lambda$ are respectively the Hamiltonian and Lee actions of the Lee chord (T, γ). We say that the Lee chord (T, γ) from L_1 to L_2 is essential if the action $\mathcal{A}_{L_1, L_2}(T, \gamma)$ is equal to the time-shift T.

Example 3.1. *Let L_1 and L_2 be Lagrangian submanifolds of a conformal symplectic manifold $\big(M,[\omega]\big) = \big(M,[(0,\omega)]\big)$. Since the Lee flow $\{\varphi_t^\omega\}$ with respect to $(0,\omega)$ is the identity, (T,γ) is a Lee chord from L_1 to L_2 with respect to $(0,\omega)$ if and only if γ is a constant curve at an intersection point of L_1 and L_2. In this case there is no reason to distinguish between Lee chords (T,γ) and (T',γ) for $T \neq T'$, so we omit the time-shift from the notation. If L_1 and L_2 are exact Lagrangian submanifolds of a conformal Liouville manifold $\big(M,[(\omega,\lambda)]\big) = \big(M,[(0,\omega,\lambda)]\big)$, then the Lee action with respect to $(0,\omega,\lambda)$ of any Lee chord vanishes. The Hamiltonian action is not well-defined, because S_{L_1} and S_{L_2} are only defined up to the addition of a constant. However, if we restrict to the set $\mathcal{L}_{p_0}$ of Lagrangian submanifolds that contain a certain point p_0 then we can normalize the action S_L of any $L \in \mathcal{L}_{p_0}$ by asking that it vanishes at p_0. For L_1 and L_2 in $\mathcal{L}_{p_0}$, the total action of a Lee chord γ from L_1 to L_2, i.e. an intersection at $\gamma(t) \equiv p$ of L_1 and L_2, is equal to the Hamiltonian action $S_{L_1}(p) - S_{L_2}(p)$ and coincides with the usual (normalized) action of p as a Lagrangian intersection.*

Example 3.2. *Let Λ_1 and Λ_2 be Legendrian submanifolds of a contact manifold $\big(Y,\xi = \ker(\alpha)\big)$, and consider their lifts $S^1 \times \Lambda_1$ and $S^1 \times \Lambda_2$ to the locally conformal symplectization $\big(S^1 \times Y\,,\,[(-d\theta\,,\,d_{-d\theta}\alpha)]\big)$. Since the Lee vector field with respect to $(-d\theta\,,\,d_{-d\theta}\alpha)$ never vanishes, the time-shift of any Lee chord (T,γ) is determined by γ and so we omit it from the notation. The Lee chords with respect to $(-d\theta\,,\,d_{-d\theta}\alpha)$ from $S^1 \times \Lambda_1$ to $S^1 \times \Lambda_2$ are of the form $\gamma = (\theta,\gamma_Y)$ with $\theta \in S^1$ and γ_Y a Reeb chord with respect to α from Λ_1 to Λ_2, i.e. a curve $\gamma_Y : [0,1] \to M$ such that $\gamma_Y(0) \in \Lambda_1$, $\gamma_Y(1) \in \Lambda_2$ and $\dot\gamma_Y(t) = TR_\alpha\big(\gamma_Y(t)\big)$ for all t. By Example 2.30, the Lagrangian submanifolds $S^1 \times \Lambda_1$ and $S^1 \times \Lambda_2$ have action zero with respect to $(-d\theta\,,\,d_{-d\theta}\alpha\,,\,\alpha)$. Thus, the Hamiltonian action with respect to $(-d\theta\,,\,d_{-d\theta}\alpha\,,\,\alpha)$ of any Lee chord from $S^1 \times \Lambda_1$ to $S^1 \times \Lambda_2$ vanishes, and so the total action is given by the Lee action, which coincides with the contact action $\int_{\gamma_Y} \alpha$ of the Reeb chord γ_Y. Since $\lambda(R_\omega) = \alpha(R_\alpha) = 1$, the time-shift of γ is equal to the Lee action, hence to the total action. In particular, all the Lee chords from $S^1 \times \Lambda_1$ to $S^1 \times \Lambda_2$ are essential.*

Example 3.3. *Let L_1 and L_2 be Lagrangian submanifolds of a twisted cotangent bundle $T^*_\beta B$. A Lee chord with respect to $(\pi^*\beta\,,\,d_{\pi^*\beta}\lambda)$ from L_1 to L_2 is a pair (T,γ) with $\gamma : [0,1] \to T^*_\beta B$ of the form*

$$\gamma(t) = \sigma_q + t\,T\,\beta(q)$$

with $\gamma(0) = \sigma_q \in L_1$ and $\gamma(1) = \sigma_q + T\,\beta(q) \in L_2$. Suppose now that the Lagrangian submanifolds L_1 and L_2 are connected and exact, and that the pullback of $\pi^*\beta$ by the inclusions of L_1 and L_2 into $T^*_\beta B$ are not exact. The (total) action with respect to $(\pi^*\beta\,,\,d_{\pi^*\beta}\lambda,\lambda)$ is then equal to the Hamiltonian action, indeed the Lee action vanishes (since λ vanishes on vectors tangent to the fibres). The time-shift is in general not equal to the total action (nor to the Lee action).

In the next lemma and in the rest of the article, the Lee chords between Lagrangian submanifolds of a twisted cotangent bundle $T^*_\beta B$ are meant to be Lee chords with respect to $(\pi^*\beta\,,\,d_{\pi^*\beta}\lambda)$, and all the actions are with respect to $(\pi^*\beta\,,\,d_{\pi^*\beta}\lambda,\lambda)$. Recall from Example 2.29 that for any function $f : B \to \mathbb{R}$ we denote by $L_{f,\beta}$ the image of the Lagrangian embedding $d_\beta f : B \to T^*_\beta B$. It is an exact Lagrangian submanifold of action $S_{f,\beta}$ given by $S_{f,\beta}\big(d_\beta f(q)\big) = f(q)$.

Lemma 3.4. *For any function $f : B \to \mathbb{R}$, there is a bijection between the critical points of f and the essential Lee chords from the Lagrangian submanifold $L_{f,\beta}$ of $T^*_\beta B$ to the zero section. The action (hence the time-shift) of any such Lee chord is equal to the critical value of the corresponding critical point of f. Moreover, the bijection sends non-degenerate critical points to transverse Lee chords.*

Proof. Suppose that $q \in B$ is a critical point of f. Then

$$d_\beta f(q) = df(q) - f(q)\beta(q) = -f(q)\beta(q)\,,$$

thus $\big(f(q),\gamma_q\big)$ with

$$\gamma_q : [0,1] \to T^*_\beta B\,,\ \gamma_q(t) = -f(q)\,\beta(q) + tf(q)\,\beta(q)$$

is a Lee chord from the point $-f(q)\,\beta(q)$ of $L_{f,\beta}$ to the point 0_q of the zero section. The action

$$S_{f,\beta}\big(\gamma(0)\big) - S_{0,\beta}\big(\gamma(1)\big) + \int_{\gamma_q} \lambda = S_{f,\beta}\big(\gamma(0)\big) = f(q)$$

of the Lee chord $(f(q), \gamma_q)$ is equal to the time-shift, and to the critical value of the critical point q. In particular, the Lee chord $(f(q), \gamma_q)$ is essential. The map

$$q \mapsto \big(f(q), \gamma_q\big) \tag{4}$$

from the set of critical points of f to the set of essential Lee chords from $L_{f,\beta}$ to the zero section is injective by definition. We now show that it is also surjective. Suppose that (T, γ) with

$$\gamma : [0,1] \to T^*_\beta B\,,\ \gamma(t) = d_\beta f(q) + t\,T\,\beta(q)$$

is an essential Lee chord from $L_{f,\beta}$ to the zero section. Since the time-shift T is equal to the action $f(q)$, we have

$$0_q = \gamma(1) = df(q) - f(q)\beta(q) + T\beta(q) = df(q)\,.$$

Thus q is a critical point of f, and $\gamma = \gamma_q$. This shows that the map (4) is surjective, hence bijective.

Finally we prove that a critical point q is non-degenerate if and only if the Lee chord $\big(f(q), \gamma_q\big)$ is transverse. By definition, $\big(f(q), \gamma_q\big)$ is not transverse if and only if

$$(\varphi^\omega_{f(q)})_*\big(T_{d_\beta f(q)} L_{f,\beta}\big) + T_{0_q} 0_B \neq T_{0_q}\, T^*B\,,$$

thus if and only if $(\varphi^\omega_{f(q)})_*\big(T_{d_\beta f(q)} L_{f,\beta}\big) \cap T_{0_q} 0_B \neq \{0\}$, and so if and only if there is $Y \in T_q B$ such that $\big(\varphi^\omega_{f(q)} \circ d_\beta f\big)_* (Y) \in T_{0_q} 0_B$. Consider local coordinates $x = (x_1, \cdots, x_n)$ on a neighborhood of q, and write $Y = \sum_{j=1}^n Y_{x_j} \frac{\partial}{\partial x_j}$ and $Y_x = \begin{pmatrix} Y_{x_1} \\ \cdots \\ Y_{x_n} \end{pmatrix}$. Also write $\beta = \sum_{j=1}^n \beta_j\, dx_j$. In the associated coordinates on T^*B we then have

$$d_\beta f(x) = \Big(x\,,\ \frac{\partial f}{\partial x_1}(x) - f(x)\,\beta_1\,,\ \cdots\,,\ \frac{\partial f}{\partial x_n}(x) - f(x)\,\beta_n\Big)\,.$$

Since q is a critical point of f, we have

$$(d_\beta f)_*(Y) = \big(Y_x\,,\ \mathrm{Hess}_q(f) \cdot Y_x - f(q)\, d\beta \cdot Y_x\big)\,,$$

where we write $d\beta = \big(\frac{\partial \beta_i}{\partial x_j}\big)_{ij}$. Thus

$$(\varphi^\omega_{f(q)} \circ d_\beta f)_*\,(Y) = \big(Y_x\,,\ \mathrm{Hess}_q(f) \cdot Y_x\big)\,,$$

and so $(\varphi^\omega_{f(q)} \circ d_\beta f)_*\,(Y) \in T_{0_q} 0_B$ if and only if $\mathrm{Hess}_q(f) \cdot Y_x = 0$. This shows that $\big(f(q), \gamma_q\big)$ is not transverse if and only if the Hessian of f at q is degenerate, thus if and only if q is a degenerate critical point of f. $\square$

We will see in Section 4 that a similar bijection holds for all Lagrangian submanifolds of $T^*_\beta B$ Hamiltonian isotopic to the zero section and their generating functions. We now show that exact strict lcs diffeomorphisms send essential Lee chords to essential Lee chords.

Proposition 3.5. *Let $\psi : M_1 \to M_2$ be an exact lcs diffeomorphism between two Liouville lcs manifolds $\big(M_1, [(\eta_1, \omega_1, \lambda_1)]\big)$ and $\big(M_2, [(\eta_2, \omega_2, \lambda_2)]\big)$ that is strict with respect to (η_1, ω_1) and (η_2, ω_2). Let $L_1^{(j)}$ and $L_2^{(j)}$ be connected exact Lagrangian submanifolds of $\big(M_j, [(\eta_j, \omega_j, \lambda_j)]\big)$ with $\psi(L_1^{(1)}) = L_1^{(2)}$ and $\psi(L_2^{(1)}) = L_2^{(2)}$, and suppose that the pullback of η_j by the inclusion of $L_i^{(j)}$ into M_j is not exact. Let (T, γ) be a Lee chord with respect to (η_1, ω_1) from $L_1^{(1)}$ to $L_2^{(1)}$. Then $(T, \psi \circ \gamma)$ is a Lee chord with respect to (η_2, ω_2) from $L_1^{(2)}$ to $L_2^{(2)}$, and the action of $(T, \psi \circ \gamma)$ with respect to $(\eta_2, \omega_2, \lambda_2)$ is equal to the action of (T, γ) with respect to $(\eta_1, \omega_1, \lambda_1)$. In particular, ψ sends essential Lee chords from $L_1^{(1)}$ to $L_2^{(1)}$ to essential Lee chords from $L_1^{(2)}$ to $L_2^{(2)}$.*

Proof. The fact that $(T, \psi \circ \gamma)$ is a Lee chord with respect to (η_2, ω_2) from $L_1^{(2)}$ to $L_2^{(2)}$ just follows from the fact that $\psi^*\omega_2 = \omega_1$ and $\psi^*\eta_2 = \eta_1$ (since ψ is strict). Let S_ψ be the action of ψ with respect to $(\eta_1, \omega_1, \lambda_1)$, thus $\psi^*\lambda_2 = \lambda_1 - d_{\eta_1} S_\psi$. Then

$$\int_\gamma \lambda_1 = \int_\gamma \psi^*\lambda_2 + \int_\gamma d_{\eta_1} S_\psi = \int_{\psi\circ\gamma} \lambda_2 + \int_\gamma dS_\psi = \int_{\psi\circ\gamma} \lambda_2 + S_\psi\big(\gamma(1)\big) - S_\psi\big(\gamma(0)\big)\,,$$

where in the second equality we have used that $\int_\gamma S_\psi \eta_1 = 0$, because $\dot\gamma(t) = T\, R_{\omega_1}\big(\gamma(t)\big)$ and $\eta_1(R_{\omega_1}) = 0$. Moreover, denoting by $S_{L_i^{(j)}}$ the action of $L_i^{(j)}$ with respect to $(\eta_j, \omega_j, \lambda_j)$, we have

$$S_{L_j^{(1)}}(p) = S_{L_j^{(2)}}\big(\psi(p)\big) + S_\psi(p)$$

for all $p \in L_j^{(1)}$. We thus conclude that

$$\begin{aligned}\mathcal{A}_{L_1^{(1)}, L_2^{(1)}}(T, \gamma) &= S_{L_1^{(1)}}\big(\gamma(0)\big) - S_{L_2^{(1)}}\big(\gamma(1)\big) + \int_\gamma \lambda_1 \\ &= S_{L_1^{(2)}}\big(\psi \circ \gamma\,(0)\big) - S_{L_2^{(2)}}\big(\psi \circ \gamma\,(1)\big) + \int_{\psi\circ\gamma} \lambda_2 = \mathcal{A}_{L_1^{(2)}, L_2^{(2)}}\big(T, \psi \circ \gamma\big)\,.\end{aligned}$$

$\square$

Remark 3.6. *Abouzaid and Kragh* [AK18] *proved that if L is a compact exact Lagrangian submanifold of the cotangent bundle T^*B of a compact manifold B then the restriction to L of the projection $T^*B \to B$ is a simple homotopy equivalence. This result implies that if Λ is a compact Legendrian submanifold of J^1B that has no Reeb chords (and so the projection of Λ to T^*B is an exact embedded Lagrangian submanifold) then the restriction to Λ of the projection $J^1B \to B$ is a simple homotopy equivalence. A similar result also holds for compact exact Lagrangian submanifolds of twisted cotangent bundles that have no essential Lee chords. Indeed, let L be an exact Lagrangian submanifold of $T^*_\beta B$, with $i^*\lambda = d_{i^*(\pi^*\beta)} S$ for $S : L \to \mathbb{R}$, where $i : L \hookrightarrow T^*_\beta B$ denotes the inclusion. A Lee chord of L is a pair (T, γ) with $T \in \mathbb{R}$ and $\gamma : [0,1] \to T^*_\beta B$ a curve of the form $\gamma(t) = \sigma_q + tT\beta(q)$ with $\gamma(0)$ and $\gamma(1)$ in L. We say that the Lee chord (T, γ) is essential if its action*

$$\mathcal{A}_L(T, \gamma) = S\big(\gamma(0)\big) - S\big(\gamma(1)\big) + \int_\gamma \lambda = S\big(\gamma(0)\big) - S\big(\gamma(1)\big)$$

is equal to the time-shift T. Consider the Legendrian submanifold $\widetilde{L} = \{\, \big(\sigma_q, S(\sigma_q)\big)\,,\ \sigma_q \in L \,\}$ of $J^1_\beta B$, and the Legendrian submanifold $U_\beta(\widetilde{L})$ of J^1B, where U_β is the untwisting map (2). *The map $(T, \gamma) \mapsto (T, \gamma')$ with*

$$\gamma' : [0,1] \to J^1B\,,\ \gamma'(t) = \Big(\gamma(0) + S\big(\gamma(0)\big)\, \beta\big(\pi \circ \gamma(0)\big)\,,\ S\big(\gamma(0)\big) - T + tT\Big)$$

is then a bijection between the essential Lee chords of L and the Reeb chords of $U_\beta(\widetilde{L})$. Thus, if L is compact and has no essential Lee chords then $U_\beta(\widetilde{L})$ is a compact Legendrian submanifold of J^1B that has no Reeb chords. We thus conclude that the restriction to $U_\beta(\widetilde{L})$ of the projection $J^1B \to B$ is a simple homotopy equivalence, and so the restriction to L of the projection $T^*_\beta B \to B$ is also a simple homotopy equivalence. This observation should be compared to a similar result of Currier [Cur24, Theorem 1.3] *for essential Liouville chords.*

Let φ be a lcs diffeomorphism of a lcs manifold $\big(M,\, [(\eta, \omega)]\big)$, with conformal factor g with respect to (η, ω). We say that a point p of M is a translated point of φ with respect to (η, ω) if p and $\varphi(p)$ are in the same orbit of the Lee flow of (η, ω) and $g(p) = 0$. If there is only one Lee chord (T, γ) from $\varphi(p)$ to p then we say that the translated point p of φ has time-shift T.

Example 3.7. *Let φ be a symplectomorphism of a conformal symplectic manifold $(M, [\omega]) = \big(M, [(0, \omega)]\big)$. Since the conformal factor of φ is zero and the Lee flow of $(0, \omega)$ is the identity, the translated points of φ with respect to $(0, \omega)$ are the fixed points of φ. More generally, since the Lee flow of $(d\mu, e^\mu\omega)$ is the Hamiltonian flow of $e^{-\mu}$ with respect to ω, the translated points of φ with respect to $(d\mu\,,\, e^\mu\omega)$ are the leafwise intersections of φ with respect to $e^{-\mu}$, i.e. the points p such that p and $\varphi(p)$ belong to the same orbit of the Hamiltonian flow of $e^{-\mu}$. On the other hand, if φ is a conformal symplectomorphism with $\varphi^*\omega = e^c\omega$ for $c \neq 0$ then φ has no translated points with respect to $(0, \omega)$, because the conformal factor never vanishes. Moreover, φ also has no translated points with respect to any other representative $(d\mu\,,\, e^\mu\omega)$. Indeed, since the conformal factor of φ with respect to $(d\mu\,,\, e^\mu\omega)$ is $\mu \circ \varphi - \mu + c$, if p is a translated point of φ with respect to $(d\mu\,,\, e^\mu\omega)$ then p and $\varphi(p)$ are in the same Lee orbit and $\mu\big(\varphi(p)\big) - \mu(p) = -c$. But this is impossible, because the Lee flow of $(d\mu\,,\, e^\mu\omega)$ preserves μ.*

Example 3.8. *Recall that a point p of a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$ is said to be a translated point of a contactomorphism ϕ with respect to α if p and $\phi(p)$ are in the same orbit of the Reeb flow of α and the conformal factor of ϕ with respect to α vanishes at p. Let $\varphi = \widetilde{\phi}$ be the lift of ϕ to the locally conformal symplectization $\big(S^1 \times Y\,,\, [(-d\theta\,,\, d_{-d\theta}\alpha)]\big)$. Then $(\theta, p) \in S^1 \times Y$ is a translated point of φ with respect to $(-d\theta\,,\, d_{-d\theta}\alpha)$ if and only if p is a translated point of ϕ with respect to α. In this case, the Lee chords from $\varphi(\theta, p)$ to (θ, p) are of the form $\gamma = (\theta, \gamma_Y)$ with γ_Y a Reeb chord from $\phi(p)$ to p.*

Suppose now that φ is a lcs Hamiltonian diffeomorphism of a connected Liouville lcs manifold $\big(M, [(\eta, \omega, \lambda)]\big)$ with η not exact. Let p be a translated point of φ, and (T, γ) a Lee chord from $\varphi(p)$ to p. We define the (total) action with respect to (η, ω, λ) of (T, γ) by

$$\mathcal{A}_\varphi(T, \gamma) = -S_\varphi(p) + \int_\gamma \lambda\,,$$

where S_φ is the action of φ with respect to (η, ω, λ). We say that $-S_\varphi(p)$ and $\int_\gamma \lambda$ are respectively the Hamiltonian and Lee actions of (T, γ). If there is only one Lee chord (T, γ) from $\varphi(p)$ to p then we say that $\mathcal{A}_\varphi(\gamma)$, $-S_\varphi(p)$ and $\int_\gamma \lambda$ are the total, Hamiltonian and Lee actions of the translated point p. The Hamiltonian action always just depends on the translated point p, and not on the Lee chord from $\varphi(p)$ to p. By Proposition 2.24, it can also be written as

$$-S_\varphi(p) = -\int_0^1 e^{-g_t(p)} \Big(H_t\big(\varphi_t(p)\big) - \lambda\big(X_{H_t}\big(\varphi_t(p)\big)\big) \Big)\, dt\,,$$

where $\{\varphi_t\}_{t\in[0,1]}$ is any lcs Hamiltonian isotopy with $\varphi_1 = \varphi$ and g_t and H_t its conformal factors and Hamiltonian function with respect to (η, ω). We say that a Lee chord for a translated point is essential if it has time-shift equal to the action, and that a translated point is essential if it admits an essential Lee chord.

Example 3.9. *Let φ be a Hamiltonian symplectomorphism of a conformal Liouville manifold $\big(M, [(\omega, \lambda)]\big) = \big(M, [(0, \omega, \lambda)]\big)$. The action with respect to $(0, \omega, \lambda)$ of a translated (hence fixed) point p of φ is not well-defined, because S_φ is only defined up to the addition of a constant. However, if φ is compactly supported then we can normalize S_φ by asking that it is also compactly supported. The action of p, which is equal to the Hamiltonian action $-S_\varphi(p)$, coincides then with the usual (normalized) action of p as a fixed point of φ.*

Example 3.10. *Let ϕ be a contactomorphism of a contact manifold $\big(Y, \xi = \ker(\alpha)\big)$, and $\varphi = \widetilde{\phi}$ its lift to the locally conformal symplectization $\big(S^1 \times Y\,,\, [(-d\theta\,,\, d_{-d\theta}\alpha\,,\, \alpha)]\big)$. The Hamiltonian action of any translated point of φ vanishes because, by Example 2.12, φ has action zero. Thus, for any translated point (θ, p) of φ, the action of any Lee chord $\gamma = (\theta, \gamma_Y)$ from $\varphi(\theta, p) = \big(\theta, \phi(p)\big)$ to (θ, p) is equal to the Lee action, which coincides with the action and time-shift of the Reeb chord γ_Y from $\phi(p)$ to p, and to the time-shift of γ. In particular, all the Lee chords for the translated points of φ are essential.*

Example 3.11. *Let p be a translated point of a lcs Hamiltonian diffeomorphism φ of a locally conformal symplectization $\big(S^1 \times Y\,,\, [(-d\theta\,,\, d_{-d\theta}\alpha\,,\, \alpha)]\big)$. The time-shift of any Lee chord from $\varphi(p)$ to p is equal to the Lee action, because $\lambda(R_\omega) = \alpha(R_\alpha) = 1$. Thus, a Lee chord for p is essential if and only if $S_\varphi(p) = 0$.*

We remark for completeness the following analogue of Proposition 3.5 (even though we will not need it).

Proposition 3.12. *Let $\psi : M_1 \to M_2$ be an exact lcs diffeomorphism between two Liouville lcs manifolds $\big(M_1, [(\eta_1, \omega_1, \lambda_1)]\big)$ and $\big(M_2, [(\eta_2, \omega_2, \lambda_2)]\big)$ that is strict with respect to (η_1, ω_1) and (η_2, ω_2), and suppose that the Lee forms η_1 and η_2 are not exact. Let φ be a lcs Hamiltonian diffeomorphism of $\big(M_1, [(\eta_1, \omega_1, \lambda_1)]\big)$, p a translated point of φ with respect to (η_1, ω_1) and (T, γ) a Lee chord for p. Then $\psi(p)$ is a translated point of $\psi \circ \varphi \circ \psi^{-1}$ with respect to (η_2, ω_2), and $(T, \psi \circ \gamma)$ is a Lee chord for $\psi(p)$ with*

$$\mathcal{A}_\varphi(T, \gamma) = \mathcal{A}_{\psi\circ\varphi\circ\psi^{-1}}(T, \psi \circ \gamma)\,.$$

In particular, p is an essential translated point of φ if and only if $\psi(p)$ is an essential translated point of $\psi \circ \varphi \circ \psi^{-1}$.

Proof. The facts that $\psi(p)$ is a translated point of $\psi\circ\varphi\circ\psi^{-1}$ with respect to (η_2,ω_2) and that $(T,\psi\circ\gamma)$ is a Lee chord for $\psi(p)$ just follow from the fact that $\psi^*\omega_2=\omega_1$ and $\psi^*\eta_2=\eta_1$ (since ψ is strict). Let S_ψ be the action of ψ, thus $\psi^*\lambda_2=\lambda_1-d_{\eta_1}S_\psi$. As in the proof of Proposition 3.5,

$$\int_\gamma \lambda_1=\int_{\psi\circ\gamma}\lambda_2+S_\psi\big(\gamma(1)\big)-S_\psi\big(\gamma(0)\big)=\int_{\psi\circ\gamma}\lambda_2+S_\psi(p)-S_\psi\big(\varphi(p)\big)\,.$$

Moreover, a direct calculation shows that $\psi\circ\varphi\circ\psi^{-1}$ has action

$$S_{\psi\circ\varphi\circ\psi^{-1}}=(S_\varphi-S_\psi+e^{-g}S_\psi\circ\varphi)\circ\psi^{-1}\,,$$

where S_φ and g are the action and conformal factor of φ, thus

$$S_{\psi\circ\varphi\circ\psi^{-1}}\big(\psi(p)\big)=S_\varphi(p)-S_\psi(p)+S_\psi\big(\varphi(p)\big)\,.$$

We thus conclude that

$$\mathcal{A}_\varphi(T,\gamma)=-S_\varphi(p)+\int_\gamma\lambda_1=-S_{\psi\circ\varphi\circ\psi^{-1}}\big(\psi(p)\big)+\int_{\psi\circ\gamma}\lambda_2=\mathcal{A}_{\psi\circ\varphi\circ\psi^{-1}}(T,\psi\circ\gamma)\,.$$

□

Let p be a translated point of a lcs diffeomorphism φ of a lcs manifold $\big(M,[(\eta,\omega)]\big)$, and (T,γ) a Lee chord with respect to (η,ω) from $\varphi(p)$ to p. We say that (T,γ) is a transverse Lee chord for the translated point p if there is no non-zero vector $Y\in T_pM$ such that $(\varphi_T^\omega\circ\varphi)_*(Y)=Y$ and $dg(Y)=0$, where g is the conformal factor of φ. If there is only one Lee chord (T,γ) from $\varphi(p)$ to p then we say that the translated point p is non-degenerate if (T,γ) is a transverse Lee chord.

Proposition 3.13. *Let φ be a divergence free lcs diffeomorphism of a lcs manifold $\big(M,[(\eta,\omega)]\big)$. Then p is a translated point of φ with respect to (η,ω) if and only if there is a Lee chord in $M\boxtimes M$ with respect to $(\eta\boxtimes\eta,\omega\boxtimes\omega)$ from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)(p)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})\big(\varphi(p)\big)$. In this case, there is a bijection between the Lee chords in M from $\varphi(p)$ to p and the Lee chords in $M\boxtimes M$ from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)(p)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})\big(\varphi(p)\big)$. This bijection preserves the time-shift, and sends transverse Lee chords for the translated point p to transverse Lee chords from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})$. For Liouville lcs manifolds and lcs Hamiltonian diffeomorphisms, the bijection preserves moreover the Hamiltonian and Lee actions, and thus sends essential Lee chords for p to essential Lee chords from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)(p)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})\big(\varphi(p)\big)$.*

Proof. By Lemma 2.40, the Lee flow on $M\boxtimes M$ with respect to $(\eta\boxtimes\eta,\omega\boxtimes\omega)$ is the projection of the isotopy $(x_1,x_2)\mapsto\big(\bar\varphi_{-t}^\omega(x_1),x_2\big)$ on $\bar M\times\bar M$, where $\{\bar\varphi_t^\omega\}$ is the lift to $\bar M$ of the Lee flow $\{\varphi_t^\omega\}$ on M with respect to (η,ω). Thus, since $\mathrm{gr}_{\mathrm{lcs}}(\varphi):M\to M\boxtimes M$ is the map induced by the symplectic graph $\mathrm{gr}(\bar\varphi):\bar M\to\bar M\times\bar M$, there can be a Lee chord from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)(p)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})(x)$ only if $x=\varphi(p)$. Since moreover $\{\bar\varphi_t^\omega\}$ is the Lee flow on $\bar M$ with respect to $(\pi^*\eta,\pi^*\omega)$, for every $p\in M$ and $\bar p\in\bar M$ with $\pi(\bar p)=p$ there is a bijection between the Lee chords in $M\boxtimes M$ from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)(p)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})\big(\varphi(p)\big)$ and the Lee chords in $\bar M$ with respect to $(\pi^*\eta,\pi^*\omega)$ from $\bar\varphi(\bar p)$ to $\bar p$. By Example 2.25, the Lee flow on $\bar M$ with respect to $(\pi^*\eta,\pi^*\omega)$ preserves μ. Thus, by Corollary 2.9, there is a Lee chord with respect to $(\pi^*\eta,\pi^*\omega)$ from $\bar\varphi(\bar p)$ to $\bar p$ if and only if there is a Lee chord with respect to (η,ω) from $\varphi(p)$ to p and $g(p)=0$, where g is the conformal factor of φ, and so if and only if p is a translated point of φ. In this case, there is a bijection between the Lee chords with respect to (η,ω) from $\varphi(p)$ to p and the Lee chords with respect to $(\pi^*\eta,\pi^*\omega)$ from $\bar\varphi(\bar p)$ to $\bar p$, hence with the Lee chords with respect to $(\eta\boxtimes\eta,\omega\boxtimes\omega)$ from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)(p)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})\big(\varphi(p)\big)$. This bijection preserves the time-shift. Suppose now that φ is a lcs Hamiltonian diffeomorphism of a Liouville lcs manifold $\big(M,[(\eta,\omega,\lambda)]\big)$, and consider the lcs product $\big(M\boxtimes M,[(\eta\boxtimes\eta,\omega\boxtimes\omega,\lambda\boxtimes\lambda)]\big)$. It is immediate to see that the above bijection preserves the Lee actions. The fact that it also preserves the Hamiltonian actions follows from Lemma 2.42.

Finally, a Lee chord (T,γ) in $M\boxtimes M$ from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)(p)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})(\varphi(p))$ is transverse if and only if the corresponding Lee chord in $\bar M\times\bar M$ from $\mathrm{gr}(\bar\varphi)(\bar p)$ to $\mathrm{gr}(\mathrm{id})(\bar\varphi(\bar p))$ is transverse for some (hence any) $\bar p$ in $\pi^{-1}(p)$. This happens if and only if there is no non-zero vector tangent to $\mathrm{gr}(\bar\varphi)$ at $\mathrm{gr}(\bar\varphi)(\bar p)=(\bar p,\bar\varphi(\bar p))$

whose image by $(\bar{\varphi}^{\omega}_{-T}\times \mathrm{id})_*$ is tangent to $\mathrm{gr}(\mathrm{id})$ at $\mathrm{gr}(\mathrm{id})(\bar{\varphi}(\bar{p})) = (\bar{\varphi}(\bar{p}),\bar{\varphi}(\bar{p}))$, thus no non-zero vector $X\in T_{\bar{p}}\bar{M}$ such that

$$(\bar{\varphi}^{\omega}_{-T}\times \mathrm{id})_*(X,\bar{\varphi}_*(X)) \in T_{\bar{\varphi}(\bar{p})}\,\mathrm{gr}(\mathrm{id})\,.$$

This is equivalent to $(\bar{\varphi}^{\omega}_{-T})_*(X) = \bar{\varphi}_*(X)$, and so to $(\bar{\varphi}^{\omega}_{T}\circ\bar{\varphi})_*\,(X) = X$. But $(\bar{\varphi}^{\omega}_{T}\circ\bar{\varphi})_*\,(X) = X$ if and only if $(\varphi^{\omega}_{T}\circ\varphi)_*\big(\pi_*(X)\big) = \pi_*(X)$ and $d(\mu\circ\bar{\varphi})(X) = d\mu(X)$. The second equality is equivalent to $dg\big(\pi_*(X)\big) = 0$, because $\mu\circ\bar{\varphi}-\mu = g\circ\pi$ by Corollary 2.9. We conclude that (T,γ) is a transverse Lee chord from $\mathrm{gr}_{\mathrm{lcs}}(\varphi)(p)$ to $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})(\varphi(p))$ if and only if there is no non-zero vector $Y\in T_pM$ such that $(\varphi^{\omega}_{T}\circ\varphi)_*\,(Y) = Y$ and $dg(Y)=0$, and so if and only if the corresponding Lee chord from $\varphi(p)$ to p is a transverse Lee chord for the translated point p of φ. ☐

Remark 3.14. *In* [CS24, Theorem 1.5 (3)] *it is shown that for any lcs Hamiltonian diffeomorphism φ of a lcs manifold $\big(M,[(\eta,\omega)]\big)$ there is a bijection between $\mathrm{gr}_{\mathrm{lcs}}(\varphi)\cap\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})$ and the set of fixed points p of φ such that $\int_{\varphi_t(p)}\eta = 0$ for some (hence any) lcs Hamiltonian isotopy $\{\varphi_t\}_{t\in[0,1]}$ with $\varphi_1=\varphi$. Let $\{\bar{\varphi}_t\}$ be the lift of $\{\varphi_t\}$ to the symplectic cover $(\bar{M},[e^{-\mu}\pi^*\omega])$. For any $\bar{p}\in\bar{M}$ with $\pi(\bar{p}) = p$ we have*

$$\int_{\varphi_t(p)}\eta = \int_{\bar{\varphi}_t(\bar{p})} d\mu = \mu\circ\bar{\varphi}(\bar{p}) - \mu(\bar{p})\,.$$

By Corollary 2.9, the condition $\int_{\varphi_t(p)}\eta = 0$ is thus equivalent to the vanishing of the conformal factor of φ at p. We conclude that the bijection in [CS24, Theorem 1.5 (3)] *can be reformulated as a bijection between $\mathrm{gr}_{\mathrm{lcs}}(\varphi)\cap\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})$ and the set of translated points of φ that are also fixed points, and is thus consistent with Proposition 3.13.*

The next proposition will be generalized in Section 4 to the $\mathcal{C}^0$-small case, to obtain Theorem 1.2 (i).

Proposition 3.15. *Let φ be the time-1 map of a sufficiently $\mathcal{C}^1$-small lcs Hamiltonian isotopy $\{\varphi_t\}$ of a compact Liouville lcs manifold $\big(M,[(\eta,\omega,\lambda)]\big)$ with η not exact. Then φ has at least as many essential translated points with respect to any representative (η,ω,λ) as the minimal number of critical points of a function on M. Moreover, if all the essential Lee chords for the essential translated points are non-degenerate then the number of essential translated points is greater than or equal to the minimal number of critical points of a Morse function on M.*

Proof. Let $\Delta\subset M\boxtimes M$ be the image of $\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id}): M\to M\boxtimes M$, and denote by β the pullback of $\eta\boxtimes\eta$ by the inclusion $\Delta\hookrightarrow M\boxtimes M$. By Lemma 2.42, Δ is an exact Lagrangian submanifold of $M\boxtimes M$. By Theorem 2.33 and Remark 2.35, there are a neighborhood $\mathcal{U}$ of Δ in $M\boxtimes M$, a neighborhood $\mathcal{V}$ of the zero section in $T^*_\beta\Delta$ and an exact lcs diffeomorphism $\psi:\mathcal{U}\to\mathcal{V}$ sending Δ to the zero section that is strict with respect to $(\eta\boxtimes\eta,\omega\boxtimes\omega)$ and $(\pi^*\beta, d_{\pi^*\beta}\lambda)$. Since $\{\varphi_t\}$ is $\mathcal{C}^1$-small, $L_{\varphi_t} := \mathrm{im}\big(\mathrm{gr}_{\mathrm{lcs}}(\varphi_t)\big)$ is contained in $\mathcal{U}$ for all t (in fact, it would be enough here to assume that $\{\varphi_t\}$ is sufficiently $\mathcal{C}^0$-small). By Lemma 2.43, $L_\varphi := L_{\varphi_1}$ is the image of Δ by the time-1 map of a lcs Hamiltonian isotopy $\{\chi_t\}$. Since χ_t sends Δ to L_{φ_t} and L_{φ_t} is contained in $\mathcal{U}$, we can assume that $\{\chi_t\}$ is supported in $\mathcal{U}$. By Lemma 2.21, the conjugation $\{\psi\circ\chi_t\circ\psi^{-1}\}$ is also a lcs Hamiltonian isotopy. Since the time-1 map of this lcs Hamiltonian isotopy sends the zero section to $\psi(L_\varphi)$, by Corollary 2.32 we deduce that $\psi(L_\varphi)$ is an exact Lagrangian submanifold of $T^*_\beta\Delta$. By the $\mathcal{C}^1$-smallness assumption, $\psi(L_\varphi)$ is also a section of $T^*_\beta\Delta$, and so, by Example 2.29, $\psi(L_\varphi) = L_{f,\beta}$ for some function f on Δ. Since there are no closed Lee orbits in $\mathcal{U}$, by Proposition 3.13, Lemma 3.4 and Proposition 3.5 we obtain an injection from the set of critical point of f to the set of essential translated points of φ with respect to (η,ω,λ). Since Δ is diffeomorphic to M, we conclude that φ has at least as many essential translated points with respect to (η,ω,λ) as the minimal number of critical points of a function on M. If all the essential Lee chords for the essential translated points of φ are transverse then, by Proposition 3.13 and Lemma 3.4, all the critical points of f are non-degenerate. We thus conclude that in this case φ has at least as many essential translated points with respect to (η,ω,λ) as the minimal number of critical points of a Morse function on M. ☐

4. Generating functions for Lagrangian submanifolds of twisted cotangent bundles

Before describing generating functions for Lagrangian submanifolds of twisted cotangent bundles, we recall the classical notions of generating functions for Lagrangian and Legendrian submanifolds of usual cotangent and 1-jet bundles. We follow [Vit92, Bhu01, San11a], and invite the reader to consult these sources and the references therein for more details.

A function $F : E \to \mathbb{R}$ defined on the total space of a trivial vector bundle $E = B \times \mathbb{R}^N \to B$ is a *generating function* if the differential $dF : E \to T^*E$ is transverse to the subbundle N_E^* of T^*E formed by the covectors that vanish on vertical vectors. In this case, the space

$$\Sigma_F = dF^{-1}\big(N_E^* \cap \operatorname{im}(dF)\big)$$

of fibre critical points of F is a submanifold of E of dimension equal to the dimension of B. The map $i_F : \Sigma_F \to T^*B$ that associates to $e = (q, \zeta) \in \Sigma_F$ the covector $i_F(e) \in T_q^*B$ defined by $i_F(e)(X) = dF(\hat{X})$, where $\hat{X}$ is any vector in T_eE projecting to X, is an exact Lagrangian immersion, with $i_F^*\lambda = d(F|_{\Sigma_F})$. Its lift to the 1-jet bundle $J^1B = T^*B \times \mathbb{R}$, endowed with the contact structure $\xi = \ker(\lambda - dz)$, is the Legendrian immersion

$$j_F : \Sigma_F \to J^1B\,,\ e \mapsto \big(i_F(e), F(e)\big)\,.$$

If $i_F : \Sigma_F \to T^*B$ is an embedding we say that F is a generating function for the Lagrangian submanifold $\operatorname{im}(i_F)$ of T^*B. The map i_F then induces a bijection between the critical points of F and the intersections of $\operatorname{im}(i_F)$ with the zero section. Similarly, if $j_F : \Sigma_F \to J^1B$ is an embedding we say that F is a generating function for the Legendrian submanifold $\operatorname{im}(j_F)$ of J^1B. The map j_F then induces a bijection between the critical points of F and the Reeb chords from the zero section to $\operatorname{im}(j_F)$.

Suppose now that B is compact. A generating function $F : E = B \times \mathbb{R}^N \to \mathbb{R}$ is said to be *quadratic at infinity* if there exists a non-degenerate quadratic form F_∞ on E, i.e. a map $F_\infty : E \to \mathbb{R}$ whose restriction to every fibre is a non-degenerate quadratic form, such that $\partial_v(F - F_\infty) : E \to E^*$ is bounded, where ∂_v denotes the vertical derivative. It has been proved by Sikorav [Sik87] that every Lagrangian submanifold of T^*B Hamiltonian isotopic to the zero section has a generating function quadratic at infinity, and by Chaperon [Cha95] and Chekanov [Che96] that every Legendrian submanifold of J^1B contact isotopic to the zero section has a generating function quadratic at infinity. Moreover, we have the following uniqueness result. If $F : B \times \mathbb{R}^N \to \mathbb{R}$ is a generating function quadratic at infinity for a Legendrian submanifold Λ of J^1B then for any fibre preserving diffeomorphim Φ of $B \times \mathbb{R}^N$ and non-degenerate quadratic form Q on $\mathbb{R}^{N'}$ the composition $F \circ \Phi : B \times \mathbb{R}^N \to \mathbb{R}$ and the stabilization $F \oplus Q : B \times \mathbb{R}^N \times \mathbb{R}^{N'} \to \mathbb{R}$ are also generating functions quadratic at infinity for Λ. We consider the equivalence relation on the space of generating functions quadratic at infinity generated by these two operations: two generating functions quadratic at infinity F_1 and F_2 are equivalent if there are non-degenerate quadratic forms Q_1 and Q_2 and a fibre preserving diffeomorphism Φ such that $F_1 \oplus Q_1 = (F_2 \oplus Q_2) \circ \Phi$. It has been proved by Viterbo and Théret [Vit92, The99, The95] that if Λ is a Legendrian submanifold of J^1B contact isotopic to the zero section then all the generating functions quadratic at infinity for Λ are equivalent (similarly, all *normalized* generating functions quadratic at infinity for a Lagrangian submanifold of T^*B Hamiltonian isotopic to the identity are equivalent). Finally, we recall that the existence theorems are special cases of the following results: if L is a Lagrangian submanifold of T^*B that has a generating function quadratic at infinity F, then for any Hamiltonian isotopy $\{\varphi_t\}_{t\in[0,1]}$ of T^*B there is a 1-parameter family F_t of generating functions quadratic at infinity for $\{\varphi_t(L)\}$ such that F_0 is a stabilization of F [Sik87], and if Λ is a Legendrian submanifold of J^1B that has a generating function quadratic at infinity F, then for any contact isotopy $\{\phi_t\}_{t\in[0,1]}$ of J^1B there is a 1-parameter family F_t of generating functions quadratic at infinity for $\{\phi_t(\Lambda)\}$ such that F_0 is a stabilization of F [Cha95, Che96].

Following Chantraine and Murphy [CM19], we now describe the lcs analogues of these notions and results. Let $F : E \to \mathbb{R}$ be a function defined on the total space of a trivial vector bundle $E = B \times \mathbb{R}^N \to B$, and β a closed 1-form on B. Notice first that the twisted differential $d_\beta F : E \to T^*E$, where we still denote by β its pullback by the projection $B \times \mathbb{R}^N \to B$, is transverse to N_E^* if and only if the usual differential

$dF : E \to T^*E$ is transverse to N_E^*, and that in this case the space

$$\Sigma_{F,\beta} = (d_{\pi^*\beta}F)^{-1}\big(N_E^* \cap \mathrm{im}(d_{\pi^*\beta}F)\big)$$

of twisted fibre critical points coincides with the space Σ_F of usual fibre critical points. Assume thus that $dF : E \to T^*E$ is transverse to N_E^*, and so $\Sigma_F = \Sigma_{F,\beta}$ is a submanifold of E of dimension equal to the dimension of B. Consider the map

$$i_{F,\beta} : \Sigma_F \to T^*B$$

that associates to $e = (q, \zeta) \in \Sigma_F$ the covector $i_{F,\beta}(e) \in T_q^*B$ defined by

$$i_{F,\beta}(e)(X) = d_\beta F(\hat{X}) = dF(e)(\hat{X}) - F(e)\,\beta(q)(X)\,,$$

where $\hat{X}$ is any vector in T_eE projecting to X. Then $i_{F,\beta}$ is an exact Lagrangian immersion into the twisted cotangent bundle T_β^*B, of action $F|_{\Sigma_F}$. Indeed, by the tautological property of λ we have

$$i_{F,\beta}^*\,\lambda = d_\beta\big(\,F|_{\Sigma_F}\,\big)\,. \tag{5}$$

If $i_{F,\beta} : \Sigma_F \to T_\beta^*B$ is an embedding we say that F is a generating function for the exact Lagrangian submanifold $L_{F,\beta} := \mathrm{im}(i_{F,\beta})$ of T_β^*B. The map $i_{F,\beta}$ then induces a bijection between the twisted critical points of F (the zeros of $d_\beta F$) and the intersections of $L_{F,\beta}$ with the zero section. Moreover we have the following bijection, which generalizes Lemma 3.4.

Lemma 4.1. *Let $F : E \to \mathbb{R}$ be a generating function for a Lagrangian submanifold L of T_β^*B. Then there is a bijection between the critical points of F and the essential Lee chords from L to the zero section. The action (hence the time-shift) of any such Lee chord is equal to the critical value of the corresponding critical point of F. Moreover, the bijection sends non-degenerate critical points to transverse Lee chords.*

Proof. By definition, L is the image of the exact Lagrangian embedding $i_{F,\beta} : \Sigma_F \to T_\beta^*B$. By (5), it has action

$$S : L \to \mathbb{R}\,,\ S\big(i_{F,\beta}(e)\big) = F(e)\,.$$

Suppose that $e = (q, \zeta) \in \Sigma_F$ is a critical point of F. Then

$$i_{F,\beta}(e) = -F(e)\,\beta(q)\,,$$

thus $\big(F(e), \gamma_e\big)$ with

$$\gamma_e : [0,1] \to T_\beta^*B\,,\ \gamma_e(t) = -F(e)\,\beta(q) + t\,F(e)\,\beta(q)$$

is a Lee chord from the point $-F(e)\,\beta(q)$ of L to the point 0_q of the zero section. The action

$$S\big(\gamma(0)\big) - S\big(\gamma(1)\big) + \int_{\gamma_e} \lambda = S\big(\gamma(0)\big) = F(e)$$

of the Lee chord $\big(F(e), \gamma_e\big)$ is equal to the time-shift, and to the critical value of the critical point e. In particular, the Lee chord $\big(F(e), \gamma_e\big)$ is essential. The map

$$e \mapsto \big(F(e), \gamma_e\big) \tag{6}$$

from the set of critical points of F to the set of essential Lee chords from L to the zero section is injective, because $i_{F,\beta}$ is an embedding. We now show that it is also surjective. Suppose that (T, γ) with

$$\gamma : [0,1] \mapsto T_\beta^*B\,,\ \gamma(t) = \sigma_q + t\,T\,\beta(q)$$

is an essential Lee chord from L to the zero section. Since $\sigma_q \in L$, there is $e = (q, \zeta) \in \Sigma_F$ with $i_{F,\beta}(e) = \sigma_q$. Thus

$$dF(e)(\hat{X}) = \sigma_q(X) + F(e)\beta(q)(X)$$

for all $X \in T_qB$ and $\hat{X} \in T_eE$ projecting to X. Since (T, γ) is essential, the time-shift T is equal to the action $F(e)$. Since moreover $\gamma(1) = \sigma_q + T\beta(q)$ belongs to the zero section, we thus have $dF(e)(\hat{X}) = 0$, and so e is a critical point of F. Moreover, $\sigma_q = -T\beta(q) = -F(e)\beta(q)$ and so $(T, \gamma) = \big(F(e), \gamma_e\big)$. This shows that the map (6) is surjective, hence a bijection.

Finally, we show that a critical point e is non-degenerate if and only if the Lee chord $\big(F(e), \gamma_e\big)$ is transverse. By definition, the Lee chord $\big(F(e), \gamma_e\big)$ is not transverse if and only if

$$(\varphi^{\omega}_{F(e)})_* \big(T_{i_{F,\beta}(e)} L\big) + T_{0_q} 0_B \neq T_{0_q} T^* B \,,$$

thus if and only if $(\varphi^{\omega}_{F(e)})_* \big(T_{i_{F,\beta}(e)} L\big) \cap T_{0_q} 0_B \neq \{0\}$, and so if and only if there is $Y \in T_e \Sigma_F$ such that $(\varphi^{\omega}_{F(e)} \circ i_{F,\beta})_* \,(Y) \in T_{0_q} 0_B$. Consider local coordinates $x = (x_1, \cdots, x_n)$ on a neighborhood $\mathcal{U}$ of q in B, and coordinates $(x, \zeta) = (x_1, \cdots, x_n, \zeta_1, \cdots, \zeta_N)$ on $\mathcal{U} \times \mathbb{R}^N$. Write

$$Y = \sum_{j=1}^{n} Y_{x_j} \frac{\partial}{\partial x_j} + \sum_{j=1}^{N} Y_{\zeta_j} \frac{\partial}{\partial \zeta_j} \,,$$

$Y_x = \begin{pmatrix} Y_{x_1} \\ \cdots \\ Y_{x_n} \end{pmatrix}$ and $Y_\zeta = \begin{pmatrix} Y_{\zeta_1} \\ \cdots \\ Y_{\zeta_N} \end{pmatrix}$. In the associated coordinates on T^*B we then have

$$(i_{F,\beta})_*(Y) = \Big(Y_x \,,\, \frac{\partial^2 F}{\partial x^2}(e) \cdot Y_x + \frac{\partial^2 F}{\partial x \partial \zeta}(e) \cdot Y_\zeta - F(e)\, d\beta \cdot Y_x\Big) \,,$$

where we write $\beta = \sum_{j=1}^{n} \beta_j dx_j$ and $d\beta = \big(\frac{\partial \beta_i}{\partial x_j}\big)_{ij}$. We thus have

$$(\varphi^{\omega}_{F(e)} \circ i_{F,\beta})_*(Y) = \Big(Y_x \,,\, \frac{\partial^2 F}{\partial x^2}(e) \cdot Y_x + \frac{\partial^2 F}{\partial x \partial \zeta}(e) \cdot Y_\zeta\Big)$$

and so $(\varphi^{\omega}_{F(e)} \circ i_{F,\beta})_*(Y) \in T_{0_q} 0_B$ if and only if

$$\frac{\partial^2 F}{\partial x^2}(e) \cdot Y_x + \frac{\partial^2 F}{\partial x \partial \zeta}(e) \cdot Y_\zeta = 0 \,.$$

Since $Y \in T_e \Sigma_F$ we also have

$$\frac{\partial^2 F}{\partial x \partial \zeta}(e) \cdot Y_x + \frac{\partial^2 F}{\partial \zeta^2}(e) \cdot Y_\zeta = 0 \,.$$

We conclude that $\big(F(e), \gamma_e\big)$ is not transverse if and only if there is $Y \in T_e E$ such that $\mathrm{Hess}_e(F) \cdot Y = 0$, and so if and only if e is a degenerate critical point of F. $\square$

By (5), the lift of $i_{F,\beta} : \Sigma_F \to T^*_\beta B$ to the contactization $J^1_\beta B$ is the Legendrian immersion

$$\widetilde{i_{F,\beta}} : \Sigma_F \to J^1_\beta B \,,\ \widetilde{i_{F,\beta}}(e) = \big(i_{F,\beta}(e) \,,\, F(e)\big) \,.$$

We have

$$U_\beta \circ \widetilde{i_{F,\beta}} = j_F \,,$$

where U_β is the untwisting map (2) from $J^1_\beta B$ to $J^1 B$ and

$$j_F : \Sigma_F \to J^1 B \,,\ j_F(e) = \big(i_F(e), F(e)\big)$$

the Legendrian immersion generated (in the usual sense) by F. This implies the following result.

Proposition 4.2. *A function $F : B \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for a Lagrangian submanifold L of $T^*_\beta B$ if and only if it is a generating function for the Legendrian submanifold $U_\beta(\widetilde{L})$ of $J^1 B$, where $\widetilde{L}$ is the lift of L to the contactization $J^1_\beta B$ and U_β the untwisting map* (2) *from $J^1_\beta B$ to $J^1 B$.*

Using Proposition 4.2 (and still following [CM19]), we can deduce the following existence and uniqueness results from the corresponding ones on 1-jet bundles.

Theorem 4.3. *Let B be a compact manifold, and β a closed 1-form on B.*

(i) *If L is a Lagrangian submanifold of $T^*_\beta B$ that is lcs Hamiltonian isotopic to the zero section then L has a generating function quadratic at infinity, unique up to equivalence.*
(ii) *If L is a Lagrangian submanifold of $T^*_\beta B$ that has a generating function quadratic at infinity F then for any lcs Hamiltonian isotopy $\{\varphi_t\}_{t \in [0,1]}$ of $T^*_\beta B$ there is a 1-parameter family $F_t : E \to \mathbb{R}$ of generating functions quadratic at infinity for $\{\varphi_t(L)\}$ such that F_0 is a stabilization of F.*

Proof. Suppose that L is a Lagrangian submanifold of $T^*_\beta B$ that is lcs Hamiltonian isotopic to the zero section. By Corollary 2.32, L is exact. Let $\widetilde{L}$ be its lift to $J^1_\beta B$, and consider the image $U_\beta(\widetilde{L})$ by the untwisting map $U_\beta : J^1_\beta B \to J^1 B$. Then $U_\beta(\widetilde{L})$ is a Legendrian submanifold of J^1B, contact isotopic to the zero section. Indeed, it is the image of the zero section by the time-1 map of the contact isotopy $\{U_\beta \circ \widetilde{\varphi_t} \circ U_\beta^{-1}\}$, where $\{\widetilde{\varphi_t}\}$ is the lift to $J^1_\beta B$ of the lcs Hamiltonian isotopy $\{\varphi_t\}$ whose time-1 map sends the zero section to L. By [Cha95, Che96, Vit92, The99], $U_\beta(\widetilde{L})$ has a generating function quadratic at infinity, unique up to equivalence. By Proposition 4.2, we thus conclude that L has a generating function quadratic at infinity, unique up equivalence. This proves (i).

Suppose now that L is a Lagrangian submanifold of $T^*_\beta B$ that has a generating function quadratic at infinity F, and that $\{\varphi_t\}_{t\in[0,1]}$ is a lcs Hamiltonian isotopy of $T^*_\beta B$. Consider the lifts $\widetilde{L}$ and $\{\widetilde{\varphi_t}\}$ of L and $\{\varphi_t\}$ to $J^1_\beta B$. Then $\widetilde{\varphi_t}(\widetilde{L}) = \widetilde{\varphi_t(L)}$, thus

$$U_\beta \circ \widetilde{\varphi_t} \circ U_\beta^{-1}\big(U_\beta(\widetilde{L})\big) = U_\beta(\widetilde{\varphi_t(L)})\,.$$

By Proposition 4.2, F is a generating functions quadratic at infinity for $U_\beta(\widetilde{L})$. By [Cha95, Che96], there is a 1-parameter family of generating functions quadratic at infinity $F_t : E \to \mathbb{R}$ for $\{U_\beta\big(\widetilde{\varphi_t(L)}\big)\}$ such that F_0 is a stabilization of F. By Proposition 4.2, F_t is then also a 1-parameter family of generating functions quadratic at infinity for $\{\varphi_t(L)\}$. This proves (ii). □

We can now prove Theorem 1.1 and Theorem 1.2 (i).

Proof of Theorem 1.1. Let B be a connected compact manifold, β a closed non-exact 1-form on B, and L a Lagrangian submanifold of the twisted cotangent bundle $T^*_\beta B$ that is lcs Hamiltonian isotopic to the zero section. By Theorem 4.3 (i), L has a generating function quadratic at infinity $F : B \times \mathbb{R}^N \to \mathbb{R}$, and by Lemma 4.1 there is a bijection between the (non-degenerate) critical points of F and the (transverse) essential Lee chords from L to the zero section. Since the number of critical points of a function quadratic at infinity over B is greater than or equal to the cup-length of B, and greater than or equal to the sum of the Betti numbers of B if all the critical points are non-degenerate, we conclude that the number of essential Lee chords from L to the zero section is greater than or equal to the cup-length of B, and greater than or equal to the sum of the Betti numbers of B if all essential Lee chords are transverse. □

Proof of Theorem 1.2 (i). Let φ be the time-1 map of a sufficiently $\mathcal{C}^0$-small lcs Hamiltonian isotopy $\{\varphi_t\}$ of a compact Liouville lcs manifold $\big(M, [(\eta,\omega,\lambda)]\big)$ with η not exact. We need to show that the number of essential translated points of φ with respect to any representative (η,ω) is greater than or equal to the cup-length of M, and greater than or equal to the sum of the Betti numbers of M if all the essential Lee chords for the essential translated points are non-degenerate. As in the proof of Proposition 3.15, $L_\varphi = \mathrm{im}\,\big(\mathrm{gr}_{\mathrm{lcs}}(\varphi)\big)$ is contained in a Weinstein neighborhood of $\Delta = \mathrm{im}\,\big(\mathrm{gr}_{\mathrm{lcs}}(\mathrm{id})\big)$ in $M \boxtimes M$, and its image by the strict exact lcs diffeomorphism given by Theorem 2.33 and Remark 2.35 is a Lagrangian submanifold of $T^*_\beta\Delta$, where β is the pullback of $\eta \boxtimes \eta$ by the inclusion $\Delta \hookrightarrow M \boxtimes M$, lcs Hamiltonian isotopic to the zero section. We thus conclude using Theorem 1.1, Proposition 3.5 and Proposition 3.13. □

Remark 4.4. *The conclusions of Theorem 1.2 (i) also hold (with the same proof, without using Proposition 3.5) for translated points of the time-*1 *map of a sufficiently* $\mathcal{C}^0$*-small lcs Hamiltonian isotopy of any (not necessarily Liouville) compact lcs manifold. However, the estimates are less pertinent in this case because translated points are generically not isolated.*

5. Generating functions for lcs diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$

In [Vit92, Bhu01, San11a], Viterbo, Bhupal and the third author associate generating functions quadratic at infinity to compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ and to compactly supported contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$ contact isotopic to the identity. In this section we describe a lcs analogue of these constructions, to associate generating functions quadratic at infinity to compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$. We start by recalling the constructions in the symplectic and contact cases.

Consider the Euclidean space $\mathbb{R}^{2n}$, with coordinates $(x_1, y_1, \cdots, x_n, y_n)$, endowed with the standard symplectic form $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. The map

$$\tau_{\text{symp}} : \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$$

defined by

$$\tau_{\text{symp}}\,(x, y, X, Y) = \Big(\frac{x+X}{2}\,,\,\frac{y+Y}{2}\,,\,y-Y\,,\,X-x\Big)$$

is a symplectomorphism between the symplectic product of $\mathbb{R}^{2n}$ and the cotangent bundle $T^*\mathbb{R}^{2n}$, which sends the graph of the identity to the zero section. Let φ be a symplectomorphism of $\mathbb{R}^{2n}$, and consider its graph

$$\mathrm{gr}(\varphi) : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}\,,\ \mathrm{gr}(\varphi)(p) = \big(p, \varphi(p)\big)$$

and the Lagrangian submanifold $L_\varphi := \mathrm{im}\,\big(\tau_{\text{symp}} \circ \mathrm{gr}(\varphi)\big)$ of $T^*\mathbb{R}^{2n}$. If $F : \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for L_φ then the critical points of F are in bijection with the fixed points of φ, with critical value given by the action of the corresponding fixed point. Suppose now that φ is a compactly supported Hamiltonian symplectomorphism[3]. In order to apply the existence and uniqueness theorems for generating functions quadratic at infinity, the Lagrangian submanifold L_φ of $T^*\mathbb{R}^{2n}$ is compactified to a Lagrangian submanifold $\overline{L_\varphi}$ of T^*S^{2n} as follows. Let $\{\varphi_t\}_{t\in[0,1]}$ be a compactly supported Hamiltonian isotopy with $\varphi_1 = \varphi$, and consider the compactly supported Hamiltonian isotopy $\{\Psi_{\varphi_t}\}$ of $T^*\mathbb{R}^{2n}$ that is the conjugation by τ_{symp} of the compactly supported Hamiltonian isotopy $\{\mathrm{id} \times \varphi_t\}$ of $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$. Then L_φ is the image of the zero section by Ψ_{φ_1}. Since L_φ coincides with the zero section outside a compact set, it extends to a Lagrangian submanifold $\overline{L_\varphi}$ of T^*S^{2n}. Moreover, $\{\Psi_{\varphi_t}\}$ extends to a Hamiltonian isotopy of T^*S^{2n} whose time-1 map sends the zero section to $\overline{L_\varphi}$. Thus, $\overline{L_\varphi}$ is a Lagrangian submanifold of T^*S^{2n} Hamiltonian isotopic to the zero section. We say that $\overline{F} : S^{2n} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for φ if it is a generating function for $\overline{L_\varphi}$. The induced function $F : \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ (which has the same critical values as $\overline{F}$) is then a generating function for L_φ, thus the set of critical values of $\overline{F}$ coincides with the action spectrum of φ. By [Sik87, Vit92, The99] any compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ has a generating function quadratic at infinity, unique up to fibre preserving diffeomorphism and stabilization.

Consider now the Euclidean space $\mathbb{R}^{2n+1}$, with coordinates $(x_1, y_1, \cdots, x_n, y_n, z)$, endowed with the contact structure ξ_0 given by the kernel of the contact form $\alpha_0 = \sum_{j=1}^{n} \frac{x_i dy_j - y_j dx_j}{2} - dz$, and the quotient $\mathbb{R}^{2n} \times S^1$, where $S^1 = \mathbb{R}/\mathbb{Z}$, with the induced contact form and contact structure, still denoted by α_0 and ξ_0. The map

$$\tau_{\text{cont}} : \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R} \to J^1\mathbb{R}^{2n+1}$$

defined by

$$\tau_{\text{cont}}\,(x, y, z, X, Y, Z, \rho) =$$
$$\Big(\frac{x + e^{-\frac{\rho}{2}}X}{2}\,,\,\frac{y + e^{-\frac{\rho}{2}}Y}{2}\,,\,Z\,,y - e^{-\frac{\rho}{2}}Y\,,\,e^{-\frac{\rho}{2}}X - x\,,\,1 - e^{-\rho}\,,\,Z - z + \frac{e^{-\frac{\rho}{2}}(yX - xY)}{2}\Big)$$

is a strict contactomorphism[4] between the contact product of $\mathbb{R}^{2n+1}$ with respect to α_0 and the 1-jet bundle $J^1\mathbb{R}^{2n+1}$, which sends the contact graph of the identity to the zero section. Let ϕ be a contactomorphism of $\mathbb{R}^{2n+1}$ with conformal factor g with respect to α_0, and consider its contact graph

$$\mathrm{gr}(\phi) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}\,,\ \mathrm{gr}(\phi)(p) = \big(p, \phi(p), g(p)\big)$$

and the Legendrian submanifold $\Lambda_\phi := \mathrm{im}\,\big(\tau_{\text{cont}} \circ \mathrm{gr}(\phi)\big)$ of $J^1\mathbb{R}^{2n+1}$. If $F : \mathbb{R}^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for Λ_ϕ then the critical points of F are in bijection with the translated points of ϕ, with critical value

[3] By a compactly supported Hamiltonian symplectomorphism we always mean a symplectomorphism that is the time-1 map of a compactly supported Hamiltonian isotopy.

[4] Several identifications of the contact product of $\mathbb{R}^{2n+1}$ with $J^1\mathbb{R}^{2n+1}$ are used in the literature [Bhu01, San10, San11a, FSZ23], and different choices are made for the contact forms on $\mathbb{R}^{2n+1}$ and on the contact product (cf. footnote 1). In all these references everything could have been done without any inconvenience using the choices of the present article.

given by the action of the corresponding translated point. Suppose now that ϕ is a compactly supported contactomorphism contact isotopic to the identity[5]. In order to apply the existence and uniqueness theorems for generating functions quadratic at infinity, the Legendrian submanifold Λ_ϕ of $J^1\mathbb{R}^{2n+1}$ is compactified to a Legendrian submanifold $\overline{\Lambda_\phi}$ of J^1S^{2n+1} as follows. Let $\{\phi_t\}_{t\in[0,1]}$ be a compactly supported contact isotopy with $\phi_1=\phi$ and conformal factors g_t, and consider the compactly supported contact isotopy $\{\Psi_{\phi_t}\}$ of $J^1\mathbb{R}^{2n+1}$ that is the conjugation by τ_{cont} of the compactly supported contact isotopy $\{\Phi_t\}$ of $\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\times\mathbb{R}$ defined by

$$\Phi_t(p,P,\rho)=\big(p\,,\,\phi_t(P)\,,\,\rho+g_t(P)\big)\,.$$

Then Λ_ϕ is the image of the zero section by Ψ_{ϕ_1}. Since Λ_ϕ coincides with the zero section outside a compact set, it extends to a Legendrian submanifold $\overline{\Lambda_\phi}$ of J^1S^{2n+1}. Moreover, $\{\Psi_{\phi_t}\}$ extends to a contact isotopy of J^1S^{2n+1} whose time-1 map sends the zero section to $\overline{\Lambda_\phi}$. Thus, $\overline{\Lambda_\phi}$ is a Legendrian submanifold of J^1S^{2n+1} contact isotopic to the zero section. We say that $\overline{F}:S^{2n+1}\times\mathbb{R}^N\to\mathbb{R}$ is a generating function for ϕ if it is a generating function for $\overline{\Lambda_\phi}$. The induced function $F:\mathbb{R}^{2n+1}\times\mathbb{R}^N\to\mathbb{R}$ (which has the same critical values as $\overline{F}$) is then a generating function for Λ_ϕ, thus the set of critical values of $\overline{F}$ coincides with the action spectrum of ϕ. Similarly, if ϕ a compactly supported contactomorphism of $\mathbb{R}^{2n}\times S^1$ contact isotopic to the identity then the Legendrian submanifold $\Lambda_{\phi_\mathbb{R}}$ of $J^1\mathbb{R}^{2n+1}$ associated to the lift $\phi_\mathbb{R}$ of ϕ to $\mathbb{R}^{2n+1}$ (the contactomorphism of $\mathbb{R}^{2n+1}$ that projects to ϕ and is the identity over the complement of the support of ϕ) extends to a Legendrian submanifold of $J^1(S^{2n}\times\mathbb{R})$ contact isotopic to the zero section. The last descends to a Legendrian submanifold $\overline{\Lambda_\phi}:=\overline{\Lambda_{\phi_\mathbb{R}}}$ of $J^1(S^{2n}\times S^1)$ contact isotopic to the zero section, indeed

$$\tau_{\text{cont}}(x,y,z+1,X,Y,Z+1,\rho)=\tau_{\text{cont}}(x,y,z,X,Y,Z,\rho)+(0,0,1,0,0,0,0)\,.$$

We say that $\overline{F}:S^{2n}\times S^1\times\mathbb{R}^N\to\mathbb{R}$ is a generating function for ϕ if it is a generating function for $\overline{\Lambda_\phi}$. The induced function $F:\mathbb{R}^{2n+1}\times\mathbb{R}^N\to\mathbb{R}$ (which has the same critical values as $\overline{F}$) is then a generating function for $\Lambda_{\phi_\mathbb{R}}$, thus the set of critical values of $\overline{F}$ coincides with the action spectrum of ϕ (which is defined to be the action spectrum of $\phi_\mathbb{R}$).

By [Cha95, Che96, Vit92, The99] any compactly supported contactomorphism of $\mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n}\times S^1$ contact isotopic to the identity has a generating function quadratic at infinity, unique up to fibre preserving diffeomorphism and stabilization.

In the lcs case, the analogue of the identifications τ_{symp} and τ_{cont} is obtained as follows. By Lemma 2.13 and Proposition 2.41 we have strict lcs diffeomorphisms

$$T^*_{-d\theta}(S^1\times\mathbb{R}^{2n+1})\to S^1\times J^1\mathbb{R}^{2n+1}\tag{7}$$

and

$$(S^1\times\mathbb{R}^{2n+1})\boxtimes(S^1\times\mathbb{R}^{2n+1})\to S^1\times(\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\times\mathbb{R})\,.\tag{8}$$

By Example 2.12, the lift

$$\widetilde{\tau_{\text{cont}}}:S^1\times(\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\times\mathbb{R})\to S^1\times J^1\mathbb{R}^{2n+1}\,,\ (\theta,p)\mapsto(\theta,\tau_{\text{cont}}(p))$$

of the strict contactomorphism $\tau_{\text{cont}}:\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\times\mathbb{R}\to J^1\mathbb{R}^{2n+1}$ to the locally conformal symplectizations is a strict lcs diffeomorphism. Composing it with the strict lcs diffeomorphisms (7) and (8) we obtain a strict lcs diffeomorphism

$$\tau:(S^1\times\mathbb{R}^{2n+1})\boxtimes(S^1\times\mathbb{R}^{2n+1})\to T^*_{-d\theta}(S^1\times\mathbb{R}^{2n+1})\,,\tag{9}$$

defined by

$$\tau\big([(\theta,x,y,z,\Theta,X,Y,Z)]\big)=$$
$$\Big([\theta],\frac{x+e^{\frac{\Theta-\theta}{2}}X}{2}\,,\,\frac{y+e^{\frac{\Theta-\theta}{2}}Y}{2}\,,\,Z\,,\,Z-z+\frac{e^{\frac{\Theta-\theta}{2}}(yX-xY)}{2}\,,\,y-e^{\frac{\Theta-\theta}{2}}Y\,,\,e^{\frac{\Theta-\theta}{2}}X-x\,,\,1-e^{\Theta-\theta}\Big)\,.$$

[5]By a compactly supported contactomorphism contact isotopic to the identity we always mean a contactomorphism that is isotopic to the identity through a compactly supported contact isotopy.

Denote by $\lambda_{S^1\times\mathbb{R}^{2n+1}}$ the tautological 1-form on the cotangent bundle of $S^1\times\mathbb{R}^{2n+1}$. A direct calculation shows that

$$\tau^*\lambda_{S^1\times\mathbb{R}^{2n+1}} = \alpha_0 \boxtimes \alpha_0 - d_{-d\theta}\Big(z - Z + \frac{e^{\frac{\Theta-\theta}{2}}(xY - yX)}{2}\Big)\,. \tag{10}$$

The lcs diffeomorphism τ is thus exact, with action

$$S_\tau = z - Z + \frac{e^{\frac{\Theta-\theta}{2}}(xY - yX)}{2}\,.$$

Consider now a lcs Hamiltonian diffeomorphism φ of $S^1\times\mathbb{R}^{2n+1}$. Its lcs graph is the Lagrangian embedding

$$\mathrm{gr}_{\mathrm{lcs}}(\varphi): S^1\times\mathbb{R}^{2n+1}\to (S^1\times\mathbb{R}^{2n+1})\boxtimes(S^1\times\mathbb{R}^{2n+1})$$

defined by

$$\mathrm{gr}_{\mathrm{lcs}}(\varphi)([\theta],x,y,z) = [(\theta,x,y,z,\bar\varphi(\theta,x,y,z))]\,,$$

where $\bar\varphi$ is the lift of φ to the symplectic cover $\mathbb{R}\times\mathbb{R}^{2n+1}$. We define

$$\Gamma_\varphi : S^1\times\mathbb{R}^{2n+1}\to T^*_{-d\theta}(S^1\times\mathbb{R}^{2n+1})$$

to be the composition of $\mathrm{gr}_{\mathrm{lcs}}(\varphi)$ with τ, thus

$$\Gamma_\varphi([\theta],x,y,z) =$$

$$\Big([\theta]\,,\,\frac{x+e^{\frac{\bar\varphi_\theta-\theta}{2}}\bar\varphi_x}{2}\,,\,\frac{y+e^{\frac{\bar\varphi_\theta-\theta}{2}}\bar\varphi_y}{2}\,,\,\bar\varphi_z\,,\,\bar\varphi_z - z + \frac{e^{\frac{\bar\varphi_\theta-\theta}{2}}(y\bar\varphi_x - x\bar\varphi_y)}{2}\,,\, y - e^{\frac{\bar\varphi_\theta-\theta}{2}}\bar\varphi_y\,,\, e^{\frac{\bar\varphi_\theta-\theta}{2}}\bar\varphi_x - x\,,\, 1-e^{\bar\varphi_\theta-\theta}\Big)\,,$$

where $\bar\varphi_\theta$, $\bar\varphi_x$, $\bar\varphi_y$ and $\bar\varphi_z$ are the components of $\bar\varphi$, evaluated at the point (θ,x,y,z). By Corollary 2.9,

$$\theta - \bar\varphi_\theta(\theta,x,y,z) = g([\theta],x,y,z)\,, \tag{11}$$

where g is the conformal factor of φ. Moreover, $\bar\varphi_x=\varphi_x$, $\bar\varphi_y=\varphi_y$ and $\bar\varphi_z=\varphi_z$, where φ_θ, φ_x, φ_y, φ_z denote the components of φ. Thus

$$\Gamma_\varphi([\theta],x,y,z) = \tag{12}$$

$$\Big([\theta]\,,\,\frac{x+e^{-\frac12 g}\varphi_x}{2}\,,\,\frac{y+e^{-\frac12 g}\varphi_y}{2}\,,\,\varphi_z\,,\,\varphi_z - z + \frac{e^{-\frac12 g}(y\varphi_x - x\varphi_y)}{2}\,,\, y - e^{-\frac12 g}\varphi_y\,,\, e^{-\frac12 g}\varphi_x - x\,,\, 1-e^{-g}\Big)\,.$$

The relation (10) and Lemma 2.42 give

$$\Gamma_\varphi^*\lambda_{S^1\times\mathbb{R}^{2n+1}} = d_{-d\theta}\Big(\varphi_z - z + \frac{e^{-\frac12 g}(y\varphi_x - x\varphi_y)}{2} - S_\varphi\Big)\,, \tag{13}$$

where S_φ is the action of φ. The Lagrangian embedding Γ_φ is thus exact, with action

$$S_{\Gamma_\varphi} = \varphi_z - z + \frac{e^{-\frac12 g}(y\varphi_x - x\varphi_y)}{2} - S_\varphi\,.$$

Its image

$$L_\varphi = \mathrm{im}(\Gamma_\varphi)$$

is an exact Lagrangian submanifold of $T^*_{-d\theta}(S^1\times\mathbb{R}^{2n+1})$.

Proposition 5.1. *Let φ be lcs Hamiltonian diffeomorphism of $S^1\times\mathbb{R}^{2n+1}$, and consider the associated Lagrangian submanifold L_φ of $T^*_{-d\theta}(S^1\times\mathbb{R}^{2n+1})$. Then there is a bijection between the translated points of φ and the Lee chords from L_φ to the zero section. Moreover, this bijection sends non-degenerate translated points to transverse Lee chords, and essential translated points to essential Lee chords.*

Proof. Since there are no closed Lee orbits in $S^1\times\mathbb{R}^{2n+1}$ and $T^*_{-d\theta}(S^1\times\mathbb{R}^{2n+1})$ and since τ is a strict exact lcs diffeomorphism, by Propositions 3.5 and 3.13 there is a bijection between the translated points of φ and the Lee chords from L_φ to the zero section, sending non-degenerate translated points to transverse Lee chords and essential translated points to essential Lee chords. ☐

We say that $F : S^1 \times \mathbb{R}^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for a lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n+1}$ if it is a generating function for the associated Lagrangian submanifold L_φ of $T^*_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$. Then $i_{F,-d\theta} : \Sigma_F \to T^*_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$ induces a diffeomorphism between Σ_F and L_φ. We consider the composition

$$\Phi_F : S^1 \times \mathbb{R}^{2n+1} \xrightarrow{\Gamma_\varphi} L_\varphi \longrightarrow \Sigma_F$$

of the inverse of this diffeomorphism with Γ_φ.

Lemma 5.2. *For $(p,\zeta) \in \Sigma_F$ and $([\theta],x,y,z) = (\Phi_F)^{-1}(p,\zeta)$ we have*

$$p = \Big([\theta]\,,\, \frac{x + e^{-\frac{1}{2}g}\varphi_x}{2}\,,\, \frac{y + e^{-\frac{1}{2}g}\varphi_y}{2}\,,\, \varphi_z\Big)$$

and

$$\begin{cases} d_{p_\theta}F\,(p,\zeta) = \varphi_z - z + \frac{e^{-\frac{1}{2}g}(y\varphi_x - x\varphi_y)}{2} - F(p,\zeta) = S_\varphi([\theta],x,y,z) \\ d_{p_x}F\,(p,\zeta) = y - e^{-\frac{1}{2}g}\varphi_y \\ d_{p_y}F\,(p,\zeta) = e^{-\frac{1}{2}g}\varphi_x - x \\ d_{p_z}F\,(p,\zeta) = 1 - e^{-g}\,, \end{cases}$$

where g, φ_x, φ_y and φ_z are evaluated at $([\theta],x,y,z)$.

Proof. By assumption we have

$$\Gamma_\varphi([\theta],x,y,z) = i_{F,-d\theta}\,(p,\zeta)\,,$$

thus all the equalities except the one involving S_φ follow from the equality

$$i_{F,-d\theta}\,(p,\zeta) = \Big(p\,,\, \frac{\partial F}{\partial p}(p,\zeta) + F(p,\zeta)\,d\theta\Big)$$

and formula (12). For the equality involving S_φ, using (13) and (5) we have

$$d_{-d\theta}\Big(\varphi_z - z + \frac{e^{-\frac{1}{2}g}(y\varphi_x - x\varphi_y)}{2} - S_\varphi\Big) = \Gamma_\varphi^*\,\lambda_{S^1\times\mathbb{R}^{2n+1}}$$

$$= (i_{F,-d\theta}\circ\Phi_F)^*\,\lambda_{S^1\times\mathbb{R}^{2n+1}} = \Phi_F^*\big(d_{-d\theta}(F|_{\Sigma_F})\big) = d_{-d\theta}(F|_{\Sigma_F}\circ\Phi_F)\,.$$

By Remark 2.1 we thus have

$$\varphi_z - z + \frac{e^{-\frac{1}{2}g}(y\varphi_x - x\varphi_y)}{2} - S_\varphi([\theta],x,y,z) = F(p,\zeta)\,,$$

which is equivalent to the desired equality. □

Using Lemma 5.2 we obtain the following result.

Proposition 5.3. *A point $e \in \Sigma_F$ is a (non-degenerate) critical point of F if and only if $(\Phi_F)^{-1}(e)$ is an essential (non-degenerate) translated point of φ. In this case, the action (hence the time-shift) of the translated point $\Phi_F^{-1}(e)$ of φ is equal to the critical value of the critical point e of F.*

Proof. The relations in Lemma 5.2 and Example 3.11 imply that $e \in \Sigma_F$ is a critical point of F if and only if $(\Phi_F)^{-1}(e)$ is an essential translated point of φ, and that the critical value $F(e)$ is equal to the action (hence the time-shift) of the translated point $(\Phi_F)^{-1}(e)$. The bijection $e \mapsto (\Phi_F)^{-1}(e)$ between the critical points of F and the essential translated points of φ coincides with the one induced by Lemma 4.1 and Proposition 5.1, and thus sends non-degenerate critical points to non-degenerate translated points. □

Lemma 5.2 is also used to obtain the following result.

Proposition 5.4. *Let φ_1 and φ_2 be lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$, with generating functions $F_1 : S^1 \times \mathbb{R}^{2n+1} \times \mathbb{R}^{N_1} \to \mathbb{R}$ and $F_2 : S^1 \times \mathbb{R}^{2n+1} \times \mathbb{R}^{N_2} \to \mathbb{R}$ respectively. Then there is a bijection between the critical points of the difference function*

$$F_1 - F_2 : S^1 \times \mathbb{R}^{2n+1} \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}\,,\ (F_1 - F_2)(p, \zeta_1, \zeta_2) = F_1(p, \zeta_1) - F_2(p, \zeta_2)$$

and the essential translated points of $\varphi_2^{-1} \circ \varphi_1$. Moreover, the critical value of a critical point of $F_1 - F_2$ is equal to the action (hence the time-shift) of the corresponding essential translated point of $\varphi_2^{-1} \circ \varphi_1$.

Proof. Suppose that (p, ζ_1, ζ_2) is a critical point of $F_1 - F_2$. Then $\frac{\partial F_1}{\partial p}(p, \zeta_1) = \frac{\partial F_2}{\partial p}(p, \zeta_2)$ and $\frac{\partial F_1}{\partial \zeta_1}(p, \zeta_1) = \frac{\partial F_2}{\partial \zeta_2}(p, \zeta_2) = 0$. In particular, (p, ζ_1) and (p, ζ_2) are fibre critical points of F_1 and F_2 respectively. Let

$$([\theta_j], x_j, y_j, z_j) = (\Phi_{F_j})^{-1}(p, \zeta_j)\,.$$

The relations in Lemma 5.2 and (11) imply that $g_1([\theta_1], x_1, y_1, z_1) = g_2([\theta_2], x_2, y_2, z_2)$, $\varphi_1([\theta_1], x_1, y_1, z_1) = \varphi_2([\theta_2], x_2, y_2, z_2)$, $[\theta_1] = [\theta_2]$, $x_1 = x_2$, $y_1 = y_2$ and $z_2 - z_1 = F_1(p, \zeta_1) - F_2(p, \zeta_2)$. We thus see that

$$([\theta_2], x_2, y_2, z_2) = \varphi_2^{-1} \circ \varphi_1\,([\theta_1], x_1, y_1, z_1)\,,$$

that there is a Lee chord from $([\theta_1], x_1, y_1, z_1)$ to $([\theta_2], x_2, y_2, z_2)$ of time-shift equal to the critical value $(F_1 - F_2)(p, \zeta_1, \zeta_2)$, and that the conformal factor $-g_2 \circ \varphi_2^{-1} \circ \varphi_1 + g_1$ of $\varphi_2^{-1} \circ \varphi_1$ vanishes at $([\theta_1], x_1, y_1, z_1)$. We thus conclude that $([\theta_1], x_1, y_1, z_1)$ is a translated point of $\varphi_2^{-1} \circ \varphi_1$ of time-shift equal to $(F_1 - F_2)(p, \zeta_1, \zeta_2)$. Moreover, Lemma 5.2 also gives

$$S_{\varphi_1}([\theta_1], x_1, y_1, z_1) = S_{\varphi_2}([\theta_2], x_2, y_2, z_2)\,.$$

Since the action of $\varphi_2^{-1} \circ \varphi_1$ is $S_{\varphi_1} - e^{g_2 \circ \varphi_2^{-1} \circ \varphi_1 - g_1}\, S_{\varphi_2} \circ \varphi_2^{-1} \circ \varphi_1$, we conclude that the translated point $([\theta_1], x_1, y_1, z_1)$ of $\varphi_2^{-1} \circ \varphi_1$ has Hamiltonian action zero and so, by Example 3.11, is an essential translated point of $\varphi_2^{-1} \circ \varphi_1$.

Conversely, suppose that $([\theta], x, y, z)$ is an essential translated point of $\varphi_2^{-1} \circ \varphi_1$. Set $([\theta_1], x_1, y_1, z_1) = ([\theta], x, y, z)$ and $([\theta_2], x_2, y_2, z_2) = \varphi_2^{-1} \circ \varphi_1([\theta], x, y, z)$. Then $g_1([\theta_1], x_1, y_1, z_1) = g_2([\theta_2], x_2, y_2, z_2)$, $[\theta_1] = [\theta_2]$, $x_1 = x_2$, $y_1 = y_2$, $\varphi_1([\theta_1], x_1, y_1, z_1) = \varphi_2([\theta_2], x_2, y_2, z_2)$ and

$$S_{\varphi_1}([\theta_1], x_1, y_1, z_1) = S_{\varphi_2}([\theta_2], x_2, y_2, z_2)\,.$$

Let $(p_j, \zeta_j) = \Phi_{F_j}([\theta_j], x_j, y_j, z_j)$. The relations in Lemma 5.2 imply that $p_1 = p_2 =: p$ and $\frac{\partial F_1}{\partial p}(p, \zeta_1) = \frac{\partial F_2}{\partial p}(p, \zeta_2)$, thus (p, ζ_1, ζ_2) is a critical point of $F_1 - F_2$, and that the critical value $(F_1 - F_2)(p, \zeta_1, \zeta_2) = F_1(p, \zeta_1) - F_2(p, \zeta_2)$ is equal to the time-shift $z_2 - z_1$. $\square$

Similarly to the symplectic and contact cases, in order to apply the existence and uniqueness results for generating functions quadratic at infinity (Theorem 4.3) we need to compactify the Lagrangian submanifold associated to a compactly supported lcs Hamiltonian diffeomorphism[6] φ of $S^1 \times \mathbb{R}^{2n+1}$ to a Lagrangian submanifold of the twisted cotangent bundle of a compact manifold. This can be done as follows. Let $\{\varphi_t\}$ be a compactly supported lcs Hamiltonian isotopy with $\varphi_1 = \varphi$, and consider the compactly supported lcs Hamiltonian isotopy $\{\Psi_{\varphi_t}\}$ that is the conjugation by τ of the compactly supported lcs Hamiltonian isotopy $\{\mathrm{id} \boxtimes \varphi_t\}$ of $(S^1 \times \mathbb{R}^{2n+1}) \boxtimes (S^1 \times \mathbb{R}^{2n+1})$ given by Lemma 2.43. Then L_φ is the image of the zero section by Ψ_{φ_1}. Since L_φ coincides with the zero section outside a compact set, it extends to a Lagrangian submanifold $\overline{L_\varphi}$ of $T^*_{-d\theta}(S^1 \times S^{2n+1})$. Moreover, $\{\Psi_{\varphi_t}\}$ extends to a lcs Hamiltonian isotopy of $T^*_{-d\theta}(S^1 \times S^{2n+1})$ whose time-1 map sends the zero section to $\overline{L_\varphi}$. Thus, $\overline{L_\varphi}$ is a Lagrangian submanifold of $T^*_{-d\theta}(S^1 \times S^{2n+1})$ lcs Hamiltonian isotopic to the zero section. We say that $\overline{F} : S^1 \times S^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for φ if it is a generating function for $\overline{L_\varphi}$.

Consider now a compactly supported lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n} \times S^1$. Denote by $\varphi_\mathbb{R}$ the lift of φ to $S^1 \times \mathbb{R}^{2n+1}$, i.e. the lcs Hamiltonian diffeomorphism of $S^1 \times \mathbb{R}^{2n+1}$ that projects to φ and is the identity over the complement of the support of φ. We say that $F : S^1 \times \mathbb{R}^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ is a generating

[6]By a compactly supported lcs Hamiltonian diffeomorphism we always mean the time-1 map of a compactly supported lcs Hamiltonian isotopy.

function for φ if it is a generating function for the Lagrangian submanifold $L_{\varphi_{\mathbb{R}}}$ of $T^*_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$. By similar arguments as before, the Lagrangian submanifold $L_{\varphi_{\mathbb{R}}}$ of $T^*_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$ extends to a Lagrangian submanifold of $T^*_{-d\theta}(S^1 \times S^{2n} \times \mathbb{R})$ lcs Hamiltonian isotopic to the zero section. Moreover, the last descends to a Lagrangian submanifold $\overline{L_\varphi}$ of $T^*_{-d\theta}(S^1 \times S^{2n} \times S^1)$ lcs Hamiltonian isotopic to the zero section, indeed

$$\tau\big([(\theta, x, y, z+1, \Theta, X, Y, Z+1)]\big) = \tau\big([(\theta, x, y, z, \Theta, X, Y, Z)]\big) + (0,0,0,1,0,0,0,0)\,.$$

We say that $\overline{F} : S^1 \times S^{2n} \times S^1 \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for φ if it is a generating function for $\overline{L_\varphi}$.

Applying Theorem 4.3 we obtain the following result.

Proposition 5.5. *Let φ be a compactly supported lcs Hamiltonian diffeomorphism of $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$. Then φ has a generating function quadratic at infinity $\overline{F} : S^1 \times B \times \mathbb{R}^N \to \mathbb{R}$, where B is either S^{2n+1} or $S^{2n} \times S^1$, which is unique up to equivalence.*

We define the action spectrum of a lcs Hamiltonian diffeomorphism of $S^1 \times \mathbb{R}^{2n+1}$ to be the set of actions (hence time-shifts) of its essential translated points. The translated points of a lcs Hamiltonian diffeomorphisms φ of $S^1 \times \mathbb{R}^{2n} \times S^1$ do not have a well-defined time-shift, because the Lee flow is periodic and so for any translated point p of φ there is a $\mathbb{Z}$-family of Lee chords from $\varphi(p)$ to p. However, if φ is compactly supported then it has a unique lift $\varphi_{\mathbb{R}}$ that is the identity over the complement of the support of φ. Any point $\bar{p}$ of $S^1 \times \mathbb{R}^{2n+1}$ projecting to p is a translated point of $\varphi_{\mathbb{R}}$. Moreover, since $\varphi_{\mathbb{R}}$ is equivariant by translation by 1 in the Lee direction, all the points projecting to p have the same time-shift T (as translated points of $\varphi_{\mathbb{R}}$). We say that T is the time-shift of the translated point p of φ, and we define the action spectrum of φ to be the set of actions (hence time-shifts) of its essential translated points (in other words, the action spectrum of $\varphi_{\mathbb{R}}$).

Proposition 5.6. *Let φ, φ_1 and φ_2 be compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$, and $\overline{F} : S^1 \times B \times \mathbb{R}^N \to \mathbb{R}$, $\overline{F_1} : S^1 \times B \times \mathbb{R}^{N_1} \to \mathbb{R}$ and $\overline{F_2} : S^1 \times B \times \mathbb{R}^{N_2} \to \mathbb{R}$ generating functions for φ, φ_1 and φ_2 respectively, where B is either S^{2n+1} or $S^{2n} \times S^1$. Then*

(i) *The action spectrum of φ is equal to the set of critical values of $\overline{F}$.*
(ii) *The action spectrum of $\varphi_2^{-1} \circ \varphi_1$ is equal to the set of critical values of the difference function*

$$\overline{F_1} - \overline{F_2} : S^1 \times B \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}\,,\ (\overline{F_1} - \overline{F_2})(\theta, p, \zeta_1, \zeta_2) = \overline{F_1}(\theta, p, \zeta_1) - \overline{F_2}(\theta, p, \zeta_2)\,.$$

Proof. If $\overline{F} : S^1 \times S^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for a compactly supported lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n+1}$ then the induced function $F : S^1 \times \mathbb{R}^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for L_φ. Similarly, if $\overline{F} : S^1 \times S^{2n} \times S^1 \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for a compactly supported lcs Hamiltonian diffeomorphism φ of $S^1 \times \mathbb{R}^{2n} \times S^1$ then the induced function $F : S^1 \times \mathbb{R}^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for $L_{\varphi_{\mathbb{R}}}$. In both cases, F and $\overline{F}$ have the same critical values. Thus (i) and (ii) follow from Propositions 5.3 and 5.4. □

For compactly supported lcs Hamiltonian diffeomorphisms φ_1 and φ_2 of $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$, we pose $\varphi_1 \leq \varphi_2$ if $\varphi_2 \circ \varphi_1^{-1}$ is the time-1 map of a non-negative compactly supported lcs Hamiltonian isotopy. We will prove in Section 6 that $\leq$ is a partial order. One of the main ingredients will be the next proposition.

Proposition 5.7. *Let φ_1 and φ_2 be compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$ with $\varphi_1 \leq \varphi_2$. Then there are generating functions quadratic at infinity $\overline{F_1}, \overline{F_2} : S^1 \times B \times \mathbb{R}^N \to \mathbb{R}$ for φ_1 and φ_2 respectively, where B is either S^{2n+1} or $S^{2n} \times S^1$, with $\overline{F_1} \leq \overline{F_2}$.*

Before proving Proposition 5.7 we recall from [Cha95, Theorem 3] and [The95, Section III.2] the following construction (see also [San11a, Proposition 3.4], [Gir10, Proposition 3.4] and [FSZ23, Lemma 2.10]). The *transition function* of a $\mathcal{C}^1$-small contactomorphism ϕ of a 1-jet bundle $J^1\mathbb{R}^m$ is the function $G : J^1\mathbb{R}^m \to \mathbb{R}$ defined by

$$G(q, p, z) = g_{p,z}(q) - f_{p,z}(q)\,,$$

where, for every p and z, $f_{p,z}(q) = z + pq$ and $g_{p,z}$ is the function such that $\mathrm{im}(j^1 g_{p,z}) = \phi\big(\mathrm{im}(j^1 f_{p,z})\big)$. In particular, $G = 0$ if and only if ϕ is the identity. Moreover, we have the following facts.

- If ϕ is a $\mathcal{C}^1$-small contactomorphism of $J^1\mathbb{R}^m$ with transition function $G : J^1\mathbb{R}^m \to \mathbb{R}$ and Λ a Legendrian submanifold of $J^1\mathbb{R}^m$ with generating function $F : \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$ then the function
$$G \sharp F : \mathbb{R}^m \times \big((\mathbb{R}^m)^* \times \mathbb{R}^m \times \mathbb{R}^N\big) \to \mathbb{R}\,,$$
$$(\underline{q}; p, q, \zeta) \mapsto G\big(\underline{q}, p, F(\underline{q}+q,\zeta) - p(\underline{q}+q)\big) + F(\underline{q}+q,\zeta) - pq$$
is a generating function for $\phi(\Lambda)$.
- Denote the coordinates in $J^1\mathbb{R}^m$ by $(q_1, \cdots, q_m, p_1, \cdots, p_m, z)$. If ϕ is a $\mathcal{C}^1$-small contactomorphism of $J^1\mathbb{R}^m$ that is equivariant by translation by 1 in the q_j-direction then the transition function $G : J^1\mathbb{R}^m \to \mathbb{R}$ of ϕ satisfies
$$G(q_1, \cdots, q_j + 1, \cdots, q_m, p_1, \cdots, p_m, z - p_j) = G(q_1, \cdots, q_m, p_1, \cdots, p_m, z)\,.$$
Thus, if moreover $F : \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$ is a function that is invariant by translation by 1 in the q_j-direction then $G \sharp F$ is invariant by translation by 1 in the $\underline{q}_j$-direction. This implies the following. We say that $F : \mathbb{R}^{m_1+m_2} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for a Legendrian submanifold Λ of $J^1(\mathbb{R}^{m_1} \times (S^1)^{m_2})$ if it is a generating function for the preimage of Λ by the covering map $J^1\mathbb{R}^{m_1+m_2} \to J^1(\mathbb{R}^{m_1} \times (S^1)^{m_2})$. Let Λ be a Legendrian submanifold of $J^1(\mathbb{R}^{m_1} \times (S^1)^{m_2})$ with generating function $F : \mathbb{R}^{m_1+m_2} \times \mathbb{R}^N \to \mathbb{R}$, $\{\phi_t\}$ a $\mathcal{C}^1$-small contact isotopy of $J^1(\mathbb{R}^{m_1} \times (S^1)^{m_2})$ and $G_t : J^1\mathbb{R}^{m_1+m_2} \to \mathbb{R}$ the transition functions of the lift of $\{\phi_t\}$ to $J^1\mathbb{R}^{m_1+m_2}$. Then $G_t \sharp F$ is a 1-parameter family of generating functions for $\{\phi_t(\Lambda)\}$.
- Let $\{\phi_t\}$ be a $\mathcal{C}^1$-small contact isotopy of $J^1\mathbb{R}^m$ with Hamiltonian function H_t and transition functions G_t. For every p and z, denote by $\{(\phi_t)_{p,z}\}$ the isotopy of $\mathbb{R}^m$ defined by
$$(\phi_t)_{p,z}(q) = \pi \circ \phi_t \circ j^1 f_{p,z}(q)\,,$$
where $\pi : J^1\mathbb{R}^m \to \mathbb{R}^m$ is the projection. Then
$$\left.\frac{dG_t}{dt}\right|_{t=t_0}\big((\phi_t)_{p,z}(q), p, z\big) = H_{t_0} \circ \phi_{t_0} \circ j^1 f_{p,z}(q)\,.$$
- Let Λ be a Legendrian submanifold of $J^1(\mathbb{R}^{m_1} \times (S^1)^{m_2})$ that coincides with the zero section outside a compact set of $\mathbb{R}^{m_1} \times (S^1)^{m_2}$, and $F : \mathbb{R}^{m_1+m_2} \times \mathbb{R}^N \to \mathbb{R}$ a generating function for Λ. We say that F is a special generating function quadratic at infinity for Λ if there is a non-degenerate quadratic form F_∞ on $\mathbb{R}^N$ such that $F(x_1, \cdots, x_{m_1}, \theta_1, \cdots, \theta_{m_2}, \zeta) = F_\infty(\zeta)$ for $(x_1, \cdots, x_{m_1})$ outside a compact set. If $F : \mathbb{R}^{m_1+m_2} \times \mathbb{R}^N \to \mathbb{R}$ is a special generating function quadratic at infinity for Λ and $\{\phi_t\}_{t\in[0,1]}$ a $\mathcal{C}^1$-small compactly supported contact isotopy of $J^1(\mathbb{R}^{m_1} \times (S^1)^{m_2})$ whose lift to $J^1\mathbb{R}^{m_1+m_2}$ has transition functions $G_t : J^1\mathbb{R}^{m_1+m_2} \to \mathbb{R}$, then there is a fibre preserving diffeomorphism Φ such that $F_t := (G_t \sharp F) \circ \Phi$ is a 1-parameter family of special generating functions quadratic at infinity for $\{\phi_t(\Lambda)\}$ with F_0 a stabilization of F. If $\|\phi_t\|_{\mathcal{C}^1} \leq \epsilon$ for all t then $\|G_t\|_{\mathcal{C}^0} \leq \epsilon$ for all t and so, since $G_0 = 0$, F_1 and F_0 are generating functions quadratic at infinity for $\phi_1(\Lambda)$ and Λ respectively with $\|F_1 - F_0\|_{\mathcal{C}^0} \leq \epsilon$.

In particular the above facts imply the following results.

Lemma 5.8. *Let Λ_1 and Λ_2 be Legendrian submanifolds of $J^1(\mathbb{R}^{m_1} \times (S^1)^{m_2})$ that are the image of the zero section by a compactly supported contact isotopy and such that Λ_2 is the image of Λ_1 by the time-1 map of a compactly supported non-negative contact isotopy. Let $\overline{\Lambda_1}$ and $\overline{\Lambda_2}$ be the induced Legendrian submanifolds of $J^1\big(S^{m_1} \times (S^1)^{m_2}\big)$. Then there are generating functions quadratic at infinity $\overline{F_1}, \overline{F_2} : S^{m_1} \times (S^1)^{m_2} \times \mathbb{R}^N \to \mathbb{R}$ for $\overline{\Lambda_1}$ and $\overline{\Lambda_2}$ with $\overline{F_1} \leq \overline{F_2}$.*

Lemma 5.9. *Let ϕ be a compactly supported contactomorphism of $J^1\big(\mathbb{R}^{m_1} \times (S^1)^{m_2}\big)$ with $\|\phi\|_{\mathcal{C}^1} \leq \epsilon$ sufficiently small, and Λ a Legendrian submanifold of $J^1\big(\mathbb{R}^{m_1} \times (S^1)^{m_2}\big)$ that is the image of the zero section by a compactly supported contact isotopy. Let $\overline{\phi}$ and $\overline{\Lambda}$ be the induced contactomorphism and Legendrian submanifold of $J^1\big(S^{m_1} \times (S^1)^{m_2}\big)$. Then there are generating functions quadratic at infinity $\overline{F_0}, \overline{F_1} : S^{m_1} \times (S^1)^{m_2} \times \mathbb{R}^N \to \mathbb{R}$ for $\overline{\Lambda}$ and $\overline{\phi}(\overline{\Lambda})$ with $\|\overline{F_1} - \overline{F_0}\|_{\mathcal{C}^0} \leq \epsilon$.*

Using Lemma 5.8 we can now prove Proposition 5.7.

Proof of Proposition 5.7. Suppose first that φ_1 and φ_2 are compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$. Let $\{\psi_t\}$ be a compactly supported lcs Hamiltonian isotopy with $\psi_1 = \varphi_2 \circ \varphi_1^{-1}$ and with Hamiltonian function $H_t \geq 0$, and consider the compactly supported lcs Hamiltonian isotopy $\{\Psi_{\psi_t}\}$ of $T^*_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$ that is the conjugation by τ by the compactly supported lcs Hamiltonian isotopy $\{\mathrm{id} \boxtimes \psi_t\}$ of $(S^1 \times \mathbb{R}^{2n+1}) \boxtimes (S^1 \times \mathbb{R}^{2n+1})$. Then $L_{\varphi_2} = \Psi_{\psi_1}(L_{\varphi_1})$. Moreover, $\{\Psi_{\psi_t}\}$ is a non-negative lcs Hamiltonian isotopy. Indeed, by Lemma 2.43 the Hamiltonian function of $\{\mathrm{id} \boxtimes \psi_t\}$ is

$$0 \boxplus H_t : \big[\big((\theta_1, p_1), (\theta_2, p_2)\big)\big] \mapsto e^{\theta_2 - \theta_1} H_t(p_2)$$

and so, by Lemma 2.21, the Hamiltonian function of $\{\Psi_{\psi_t}\}$ is $(0 \boxplus H_t) \circ \tau^{-1}$, which is non-negative because so is H_t. By Proposition 2.36, the lift of $\{\Psi_{\psi_t}\}$ to the contactization $J^1_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$ is a non-negative compactly supported contact isotopy $\{\widetilde{\Psi_{\psi_t}}\}$. The conjugation $\{U_{-d\theta} \circ \widetilde{\Psi_{\psi_t}} \circ (U_{-d\theta})^{-1}\}$ by the untwisting map (2) is thus a non-negative compactly supported contact isotopy of $J^1(S^1 \times \mathbb{R}^{2n+1})$, which sends $U_{-d\theta}(\widetilde{L_{\varphi_1}})$ to $U_{-d\theta}(\widetilde{L_{\varphi_2}})$. By Lemma 5.8, there are generating functions quadratic at infinity $\overline{F_1}, \overline{F_2} : S^1 \times S^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ for the compactifications in $J^1(S^1 \times S^{2n+1})$ of $U_{-d\theta}(\widetilde{L_{\varphi_1}})$ and $U_{-d\theta}(\widetilde{L_{\varphi_2}})$ with $\overline{F_1} \leq \overline{F_2}$. Since these compactifications are $U_{-d\theta}(\widetilde{\overline{L_{\varphi_1}}})$ and $U_{-d\theta}(\widetilde{\overline{L_{\varphi_2}}})$, by Proposition 4.2 we conclude that $\overline{F_1}$ and $\overline{F_2}$ are generating functions quadratic at infinity for $\overline{L_{\varphi_1}}$ and $\overline{L_{\varphi_2}}$, hence for φ_1 and φ_2, with $\overline{F_1} \leq \overline{F_2}$. If φ_1 and φ_2 are compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$ a similar proof applies by considering their lifts to $S^1 \times \mathbb{R}^{2n+1}$. $\square$

Using Lemma 5.9 we also obtain the following result, which is needed later.

Lemma 5.10. *Let φ be a compactly supported lcs Hamiltonian diffeomorphism of $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$. For every $\epsilon > 0$ small enough there is $\delta > 0$ such that if ψ is a compactly supported lcs Hamiltonian diffeomorphism with $\|\varphi \circ \psi^{-1}\|_{\mathcal{C}^1} \leq \delta$ then there are generating functions quadratic at infinity $\overline{F_\varphi}, \overline{F_\psi}$: $S^1 \times B \times \mathbb{R}^N \to \mathbb{R}$ for φ and ψ respectively, where B is either S^{2n+1} or $S^{2n} \times S^1$, with $\|\overline{F_\varphi} - \overline{F_\psi}\|_{\mathcal{C}^0} \leq \epsilon$.*

Proof. Suppose first that φ is a compactly supported lcs Hamiltonian diffeomorphism of $S^1 \times \mathbb{R}^{2n+1}$. For every ψ we have $L_\varphi = \Psi_{\varphi\circ\psi^{-1}}(L_\psi)$, where $\Psi_{\varphi\circ\psi^{-1}}$ is the conjugation by τ of $\mathrm{id} \boxtimes (\varphi \circ \psi^{-1})$, and so

$$U_{-d\theta}(\widetilde{L_\varphi}) = U_{-d\theta} \circ \widetilde{\Psi_{\varphi\circ\psi^{-1}}} \circ (U_{-d\theta})^{-1} \left(U_{-d\theta}(\widetilde{L_\psi})\right),$$

where $\widetilde{L_\varphi}$, $\widetilde{L_\psi}$ and $\widetilde{\Psi_{\varphi\circ\psi^{-1}}}$ are the lifts of L_φ, L_ψ and $\Psi_{\varphi\circ\psi^{-1}}$ to the contactization $J^1_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$ and $U_{-d\theta}$ is the untwisting map (2). Take $\delta > 0$ such that if $\|\phi\|_{\mathcal{C}^1} \leq \delta$ then $\|U_{-d\theta} \circ \widetilde{\Psi_\phi} \circ (U_{-d\theta})^{-1}\|_{\mathcal{C}^1} \leq \epsilon$. Let ψ be a compactly supported lcs Hamiltonian diffeomorphism of $S^1 \times \mathbb{R}^{2n+1}$ with $\|\varphi \circ \psi^{-1}\|_{\mathcal{C}^1} \leq \delta$, and let $\{\phi_t\}_{t\in[0,1]}$ be a compactly supported lcs Hamiltonian isotopy of $S^1 \times \mathbb{R}^{2n+1}$ with $\phi_1 = \varphi \circ \psi^{-1}$ and $\|\phi_t\|_{\mathcal{C}^1} \leq \delta$ for all t. Then $\|U_{-d\theta} \circ \widetilde{\Psi_{\phi_t}} \circ (U_{-d\theta})^{-1}\|_{\mathcal{C}^1} \leq \epsilon$ for all t. By Lemma 5.9, there are generating functions quadratic at infinity $\overline{F_\varphi}$ and $\overline{F_\psi}$ for the compactifications of $U_{-d\theta}(\widetilde{L_\varphi})$ and $U_{-d\theta}(\widetilde{L_\psi})$ with $\|\overline{F_\varphi} - \overline{F_\psi}\|_{\mathcal{C}^0} \leq \epsilon$. Since these compactifications are $U_{-d\theta}(\widetilde{\overline{L_\varphi}})$ and $U_{-d\theta}(\widetilde{\overline{L_\psi}})$, by Proposition 4.2 we deduce that $\overline{F_\varphi}$ and $\overline{F_\psi}$ are generating functions quadratic at infinity for $\overline{L_\varphi}$ and $\overline{L_\psi}$, hence of φ and ψ. If φ is a compactly supported lcs Hamiltonian diffeomorphism of $S^1 \times \mathbb{R}^{2n} \times S^1$ the proof is similar. $\square$

We end this section by discussing the relation with generating functions for contactomorphisms. We start with the following lemma.

Lemma 5.11. *Let ϕ be a contactomorphism of $\mathbb{R}^{2n+1}$, and $\widetilde{\phi}$ its lift to $S^1 \times \mathbb{R}^{2n+1}$. Consider the Legendrian submanifold Λ_ϕ of $J^1\mathbb{R}^{2n+1}$, and the exact Lagrangian submanifold $L_{\widetilde{\phi}}$ of $T^*_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$. Let $\widetilde{L_{\widetilde{\phi}}}$ be the lift of $L_{\widetilde{\phi}}$ to the contactization $J^1_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$. Then*

$$U_{-d\theta}\left(\widetilde{L_{\widetilde{\phi}}}\right) = 0_{S^1} \times \Lambda_\phi,$$

where $U_{-d\theta}$ is the untwisting map (2) from $J^1_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$ to $J^1(S^1 \times \mathbb{R}^{2n+1})$ and where we identify $J^1(S^1 \times \mathbb{R}^{2n+1})$ with $T^*S^1 \times J^1\mathbb{R}^{2n+1}$.

Proof. Identify $T^*_{-d\theta}(S^1 \times \mathbb{R}^{2n+1})$ with $S^1 \times J^1\mathbb{R}^{2n+1}$ by the strict lcs diffeomorphism of Lemma 2.13. By Proposition 2.44 we then have $L_{\widetilde{\phi}} = S^1 \times \Lambda_\phi$. We thus conclude using Lemma 2.38. □

Using Lemma 5.11 we now obtain the following relation between generating functions.

Proposition 5.12. *Let ϕ be a compactly supported contactomorphism of $\mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n} \times S^1$ contact isotopic to the identity, and $\widetilde{\phi}$ its lift to $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$ respectively. Let $\overline{F} : B \times \mathbb{R}^N \to \mathbb{R}$, where B is either S^{2n+1} or $S^{2n} \times S^1$, be a generating function for ϕ. Then the function $\overline{F} \circ \mathrm{pr}_2 : S^1 \times (B \times \mathbb{R}^N) \to \mathbb{R}$, where pr_2 is the projection to the second factor, is a generating function for $\widetilde{\phi}$.*

Proof. Suppose first that ϕ is a contactomorphism of $\mathbb{R}^{2n+1}$. By definition, $\overline{F} : S^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for the Legendrian submanifold $\overline{\Lambda_\phi}$ of $J^1 S^{2n+1}$. This implies that $\overline{F} \circ \mathrm{pr}_2$ is a generating function for the Legendrian submanifold $0_{S^1} \times \overline{\Lambda_\phi}$ of $J^1(S^1 \times S^{2n+1}) \equiv T^*S^1 \times J^1 S^{2n+1}$. Lemma 5.11 implies that $0_{S^1} \times \overline{\Lambda_\phi} = U_{-d\theta}(\widetilde{\overline{L_{\widetilde{\phi}}}})$. By Proposition 4.2 we thus conclude that $\overline{F} \circ \mathrm{pr}_2$ is a generating function for $\overline{L_{\widetilde{\phi}}}$, hence for $\widetilde{\phi}$. If ϕ is a contactomorphism of $\mathbb{R}^{2n} \times S^1$ the proof is similar, by applying Lemma 5.11 to its lift to $\mathbb{R}^{2n+1}$. □

6. Spectral selectors for compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$

In this section we define spectral selectors for compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ and $S^1 \times \mathbb{R}^{2n} \times S^1$. Our construction is analogous to those in [Vit92, Bhu01, San10, San11a] of spectral selectors for compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ and for compactly supported contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$ contact isotopic to the identity.

We first recall from [Vit92] the definition of spectral selectors for functions quadratic at infinity. Let B be a compact manifold and $F : E = B \times \mathbb{R}^N \to \mathbb{R}$ a function quadratic at infinity. Let $E^- \to B$ be the subbundle of $E \to B$ on which the quadratic form F_∞ is negative definite, and consider the Thom isomorphism

$$T : H^*(B) \to H^{*+\mathrm{ind}(F_\infty)}\big(D(E^-), S(E^-)\big)\,,$$

where we denote by $D(E^-)$ and $S(E^-)$ the disc and sphere bundles of the vector bundle $E^- \to B$. We consider the sublevel sets

$$E^a = \{\, e \in E \mid F(e) \leq a \,\}\,,$$

and we denote $E^{-\infty} = E^a$ for a smaller than all the critical values of F. We then consider the inclusion

$$i_a : (E^a, E^{-\infty}) \hookrightarrow (E, E^{-\infty})\,,$$

and the induced homomorphism in cohomology

$$i_a^* : H^*(E, E^{-\infty}) \to H^*(E^a, E^{-\infty})\,.$$

Finally we consider the composition

$$i_a^* \circ \mathrm{ex} \circ T : H^*(B) \to H^{*+\mathrm{ind}(F_\infty)}(E^a, E^{-\infty})\,,$$

where

$$\mathrm{ex} : H^*\big(D(E^-), S(E^-)\big) \to H^*(E, E^{-\infty})$$

is the excision isomorphism. For a smaller than all the critical values of F the homomorphism $i_a^* \circ \mathrm{ex} \circ T$ vanishes, while for a bigger than all the critical values of F it is an isomorphism. For any non-zero class u in $H^*(B)$ we define

$$c(u, F) = \inf \{\, a \in \mathbb{R} \mid i_a^* \circ \mathrm{ex} \circ T\,(u) \neq 0 \,\}\,.$$

The properties in the following proposition are proved in [Vit92]. In (iv) and (v), $[B]$ and 1 denote respectively the fundamental and unit classes in $H^*(B)$. For (vii), recall that two functions F_1 and F_2 quadratic at infinity are said to be equivalent if there are non-degenerate quadratic forms Q_1 and Q_2 and a fibre preserving diffeomorphism Φ such that $F_1 \oplus Q_1 = (F_2 \oplus Q_2) \circ \Phi$.

Proposition 6.1. *The spectral selectors $c(u, F)$ satisfy the following properties:*

(i) Spectrality: $c(u,F)$ *is a critical value of* F.
(ii) Continuity: *if* $|F_1 - F_2|_{C^0} \leq \epsilon$ *then*

$$|\,c(u,F_1) - c(u,F_2)\,| \leq \epsilon\,.$$

(iii) Monotonicity: *if* $F_1, F_2 : B \times \mathbb{R}^N \to \mathbb{R}$ *satisfy* $F_1 \leq F_2$ *then* $c(u,F_1) \leq c(u,F_2)$.
(iv) Poincaré duality: $c([B], -F) = -c(1,F)$.
(v) Non-degeneracy: *if* $F : B \times \mathbb{R}^N \to \mathbb{R}$ *is a generating function quadratic at infinity for a Lagrangian submanifold* L *of* T^*B *and if* $c([B],F) = c(1,F)$ *then* L *is the zero section.*
(vi) Triangle inequality: *let* $F_1 : B \times \mathbb{R}^{N_1} \to \mathbb{R}$ *and* $F_2 : B \times \mathbb{R}^{N_2} \to \mathbb{R}$ *be functions quadratic at infinity, and consider the function quadratic at infinity* $F_1 + F_2 : B \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}$ *defined by*

$$(F_1 + F_2)(q, \zeta_1, \zeta_2) = F_1(q, \zeta_1) + F_2(q, \zeta_2)\,.$$

Then

$$c(u \cup v, F_1 + F_2) \geq c(u, F_1) + c(v, F_2)\,.$$

(vii) Equivalence: *if* F_1 *and* F_2 *are equivalent then* $c(u,F_1) = c(u,F_2)$ *for any* u.

Let now φ be a compactly supported lcs Hamiltonian diffeomorphism of $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$, and $\overline{F} : S^1 \times B \times \mathbb{R}^N \to \mathbb{R}$, where B is either S^{2n+1} or $S^{2n} \times S^1$, a generating function quadratic at infinity for φ. We define

$$c_+(\varphi) = c([S^1 \times B], \overline{F}) \quad \text{ and } \quad c_-(\varphi) = c(1, \overline{F})\,,$$

where $[S^1 \times B]$ and 1 are the fundamental and unit classes in $H^*(S^1 \times B)$. By Proposition 6.1 (vii) and the uniqueness statement in Proposition 5.5, the quantities $c_+(\varphi)$ and $c_-(\varphi)$ are well-defined, in the sense that they do not depend on the choice of the generating function quadratic at infinity $\overline{F}$ for φ. Moreover, by Proposition 6.1 (i) and Proposition 5.6 (i) they belong to the action spectrum of φ. In particular, $c_\pm(\mathrm{id}) = 0$.

Before discussing the other properties of c_+ and c_- we prove the following lemma, which will be the key ingredient to obtain the invariance by conjugation property.

Lemma 6.2.

(i) *Let* φ *and* ψ *be compactly supported lcs Hamiltonian diffeomorphisms of* $S^1 \times \mathbb{R}^{2n+1}$. *Suppose that* p *is an essential translated point of* φ *of time-shift equal to zero. Then* $\psi(p)$ *is an essential translated points of* $\psi \circ \varphi \circ \psi^{-1}$ *of time-shift equal to zero. Moreover,* $\psi(p)$ *is a non-degenerate translated point of* $\psi \circ \varphi \circ \psi^{-1}$ *if and only if* p *is a non-degenerate translated point of* φ.
(ii) *Let* φ *and* ψ *be compactly supported lcs Hamiltonian diffeomorphisms of* $S^1 \times \mathbb{R}^{2n} \times S^1$. *Suppose that* p *is an essential translated point of* φ *of time-shift equal to an integer* k. *Then* $\psi(p)$ *is an essential translated points of* $\psi \circ \varphi \circ \psi^{-1}$ *of time-shift equal to* k. *Moreover,* $\psi(p)$ *is a non-degenerate translated point of* $\psi \circ \varphi \circ \psi^{-1}$ *if and only if* p *is a non-degenerate translated point of* φ.

Proof. We prove (ii), the proof of (i) being similar. Since the time-shift of p is an integer, we have $\varphi(p) = p$. Let g and f be the conformal factors of φ and ψ respectively. Then the conformal factor of $\psi \circ \varphi \circ \psi^{-1}$ is $(f \circ \varphi + g - f) \circ \psi^{-1}$. In particular, since $g(p) = 0$ and $\varphi(p) = p$ we have

$$(f \circ \varphi + g - f) \circ \psi^{-1}\big(\psi(p)\big) = 0\,.$$

Let $\varphi_{\mathbb{R}}$ and $\psi_{\mathbb{R}}$ be the lifts of φ and ψ to $S^1 \times \mathbb{R}^{2n+1}$, and $\bar{p}$ a point of $S^1 \times \mathbb{R}^{2n+1}$ projecting to p. Then $\bar{p}$ is a translated point of $\varphi_{\mathbb{R}}$ of time-shift k, in particular there is a Lee chord of time-shift k from $\varphi_{\mathbb{R}}(\bar{p})$ to $\bar{p}$. Since k is an integer and $\psi_{\mathbb{R}}$ is equivariant by translation by 1 in the Lee direction, this implies that there is a Lee chord of time-shift k between $\psi_{\mathbb{R}}\big(\varphi_{\mathbb{R}}(\bar{p})\big) = (\psi_{\mathbb{R}} \circ \varphi_{\mathbb{R}} \circ \psi_{\mathbb{R}}^{-1})\big(\psi_{\mathbb{R}}(\bar{p})\big)$ and $\psi_{\mathbb{R}}(\bar{p})$. Since $\psi_{\mathbb{R}}(\bar{p})$ projects to $\psi(p)$ and $\psi_{\mathbb{R}} \circ \varphi_{\mathbb{R}} \circ \psi_{\mathbb{R}}^{-1} = (\psi \circ \varphi \circ \psi^{-1})_{\mathbb{R}}$, we conclude that $\psi(p)$ is a translated point of $\psi \circ \varphi \circ \psi^{-1}$ of time-shift k. Let now S_φ be the action of φ and S_ψ the action of ψ. Then the action of $\psi \circ \varphi \circ \psi^{-1}$ is

$$S_{\psi \circ \varphi \circ \psi^{-1}} = e^{f \circ \psi^{-1}}(S_\psi - S_\varphi - e^{-g}\, S_\psi \circ \varphi) \circ \psi^{-1}\,.$$

Since p is an essential translated point of φ, by Example 3.11 we have $S_\varphi(p) = 0$. Since moreover $g(p) = 0$ and $\varphi(p) = p$, we conclude that $S_{\psi \circ \varphi \circ \psi^{-1}}\big(\psi(p)\big) = 0$ and so, by Example 3.11, $\psi(p)$ is an essential translated point of $\psi \circ \varphi \circ \psi^{-1}$. We now show that $\psi(p)$ is non-degenerate if and only if so is p. Suppose that p is a degenerate translated point of φ, thus there is a non-zero tangent vector Y at p such that $\varphi_*(Y) = Y$ and $dg(Y) = 0$. Then

$$(\psi \circ \varphi \circ \psi^{-1})_* \big(\psi_*(Y)\big) = \psi_*(Y)$$

and

$$d\big((f \circ \varphi + g - f) \circ \psi^{-1}\big)\big(\psi_*(Y)\big) = df\big(\varphi_*(Y)\big) + dg(Y) - df(Y) = 0\,,$$

thus $\psi(p)$ is a degenerate translated point of $\psi \circ \varphi \circ \psi^{-1}$. The other implication is similar. □

We can now prove the following properties of the spectral selectors c_+ and c_-.

Proposition 6.3. *The spectral selectors c_+ and c_- on $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n+1})$ and $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$ satisfy the following properties:*

(i) Continuity: *c_+ and c_- are continuous with respect to the $\mathcal{C}^1$-topology.*
(ii) Monotonicity: *if $\varphi_1 \leq \varphi_2$ then $c_+(\varphi_1) \leq c_+(\varphi_2)$ and $c_-(\varphi_1) \leq c_-(\varphi_2)$.*
(iii) Poincaré duality: *$c_-(\varphi) = -c_+(\varphi^{-1})$.*
(iv) Positivity: *$c_+(\varphi) \geq 0$ and $c_-(\varphi) \leq 0$.*
(v) Non-degeneracy: *$c_+(\varphi) = c_-(\varphi) = 0$ if and only if φ is the identity.*
(vi) Triangle inequality: *$c_+(\varphi_1 \circ \varphi_2) \leq c_+(\varphi_1) + c_+(\varphi_2)$ and $c_-(\varphi_1 \circ \varphi_2) \geq c_-(\varphi_1) + c_-(\varphi_2)$.*
(vii) Invariance by conjugation: *if φ and ψ are compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ then $c_+(\varphi) = 0$ if and only if $c_+(\psi \circ \varphi \circ \psi^{-1}) = 0$, and similarly for c_-. If φ and ψ are compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$ then $\lceil c_+(\psi \circ \varphi \circ \psi^{-1}) \rceil = \lceil c_+(\phi) \rceil$ and $\lfloor c_+(\psi \circ \varphi \circ \psi^{-1}) \rfloor = \lfloor c_+(\phi) \rfloor$, and similarly for c_-, where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling and floor functions.*

Proof. Continuity (i) follows from Proposition 6.1 (ii) and Lemma 5.10, and monotonicity (ii) from Proposition 6.1 (iii) and Proposition 5.7.

Poincaré duality (iii) can be seen as follows. Let $\overline{F_\varphi}$ and $\overline{F_{\varphi^{-1}}}$ be generating functions quadratic at infinity for φ and φ^{-1} respectively. We first show that, for any non-zero class u,

$$c(u, \overline{F_{\varphi^{-1}}}) = c(u, -\overline{F_\varphi})\,. \tag{14}$$

To see this, let $\{\varphi_t\}$ be a compactly supported lcs Hamiltonian isotopy with $\varphi_1 = \varphi$, let $\overline{F_t}$ and $\overline{G_t}$ be 1-parameter families of generating functions quadratic at infinity for $\{\varphi_t\}$ and $\{\varphi_t \circ \varphi^{-1}\}$ respectively, and set

$$c_t = c(u\,,\, \overline{G_t} - \overline{F_t})\,.$$

By Proposition 5.6 (ii), for every t the set of critical values of $\overline{G_t} - \overline{F_t}$ is equal to the action spectrum of φ^{-1}. The functions $\overline{G_t} - \overline{F_t}$ have thus all the same set S of critical values. By Proposition 6.1 (i), c_t is in S for all t. By Proposition 6.1 (ii), the map $t \mapsto c_t$ is continuous. Since S is nowhere dense we conclude that $t \mapsto c_t$ is constant, and so $c_0 = c(u, \overline{G_0} - \overline{F_0})$ is equal to $c_1 = c(u, \overline{G_1} - \overline{F_1})$. But $\overline{G_0}$ is a generating function quadratic at infinity for φ^{-1}, hence equivalent to $\overline{F_{\varphi^{-1}}}$, and $\overline{F_0}$ is a generating function quadratic at infinity for the identity, hence equivalent to the zero function. Thus $\overline{G_0} - \overline{F_0}$ is equivalent to $\overline{F_{\varphi^{-1}}}$. Similarly, $\overline{G_1} - \overline{F_1}$ is equivalent to $-\overline{F_\varphi}$. By Proposition 6.1 (vii), we thus obtain (14). Using (14) and Proposition 6.1 (iv) we then conclude that

$$c_-(\varphi) = c(1, \overline{F_\varphi}) = -c([S^1 \times B], -\overline{F_\varphi}) = -c([S^1 \times B], \overline{F_{\varphi^{-1}}}) = -c_+(\varphi^{-1})\,.$$

By the Poincaré duality property (iii), positivity (iv) follows if we prove that $c_-(\varphi) \leq 0$. This can be seen as follows. Let $\overline{F} : E = S^1 \times B \times \mathbb{R}^N \to \mathbb{R}$, where B is either S^{2n+1} or $S^{2n} \times S^1$, be a generating function quadratic at infinity for φ. Since

$$c_-(\varphi) = c(1, \overline{F}) = \inf\{\, a \in \mathbb{R} \mid i_a^* \circ \mathrm{ex} \circ T\,(1) \neq 0\,\}\,,$$

we need to show that $i_0^* \circ \mathrm{ex} \circ T\,(1) \neq 0$. Suppose first that B is S^{2n+1}, and denote by $p_\infty \in S^{2n+1}$ the point at infinity when seeing S^{2n+1} as the 1-point compactification of $\mathbb{R}^{2n+1}$. Denote by $E_{S^1 \times \{p_\infty\}}$, $E^0_{S^1 \times \{p_\infty\}}$ and

$E^{-\infty}_{S^1\times\{p_\infty\}}$ the preimages of $S^1\times\{p_\infty\}$ by the projections of E, E^0 and $E^{-\infty}$ to $S^1\times S^{2n+1}$. Consider the commutative diagram

$$\begin{array}{ccccc} H^*(S^1\times S^{2n+1}) & \xrightarrow{\ \mathrm{ex}\circ T\ } & H^{*+\mathrm{ind}(\overline{F}_\infty)}(E,E^{-\infty}) & \xrightarrow{\ i_0\ } & H^{*+\mathrm{ind}(\overline{F}_\infty)}(E^0,E^{-\infty}) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(S^1\times\{p_\infty\}) & \xrightarrow{\mathrm{ex}\circ T} & H^{*+\mathrm{ind}(\overline{F}_\infty)}\big(E_{S^1\times\{p_\infty\}},E^{-\infty}_{S^1\times\{p_\infty\}}\big) & \xrightarrow{i_0} & H^{*+\mathrm{ind}(\overline{F}_\infty)}\big(E^0_{S^1\times\{p_\infty\}},E^{-\infty}_{S^1\times\{p_\infty\}}\big) \end{array}$$

where the vertical arrows are the homomorphisms induced by the inclusions. Since the image of

$$i_{\overline{F},-d\theta}:\Sigma_{\overline{F}}\to T^*_{-d\theta}(S^1\times S^{2n+1})\,,\ (\theta,p,\zeta)\mapsto\Big(\theta\,,\,p\,,\,\frac{\partial\overline{F}}{\partial\theta}(\theta,p,\zeta)+\overline{F}(\theta,p,\zeta)\,,\,\frac{\partial\overline{F}}{\partial p}(\theta,p,\zeta)\Big)$$

is $\overline{L_\varphi}$, which by construction is equal to the zero section on a neighborhood of $S^1\times\{p_\infty\}$, for every $\theta\in S^1$ and $\zeta\in\mathbb{R}^N$ the point $i_{\overline{F},-d\theta}(\theta,p_\infty,\zeta)$ belongs to the zero section. If (θ,p_∞,ζ) is a critical point of $\overline{F}$ it thus has critical value zero. We conclude that zero is the only critical value of $\overline{F}\big|_{E_{S^1\times\{p_\infty\}}}$, and so that the map

$$i_0:H^{*+\mathrm{ind}(\overline{F}_\infty)}\big(E_{S^1\times\{p_\infty\}},E^{-\infty}_{S^1\times\{p_\infty\}}\big)\to H^{*+\mathrm{ind}(\overline{F}_\infty)}\big(E^0_{S^1\times\{p_\infty\}},E^{-\infty}_{S^1\times\{p_\infty\}}\big)$$

is an isomorphism. In particular, the image of the unit class of $H^*(S^1\times\{p_\infty\})$ by

$$i_0^*\circ\mathrm{ex}\circ T:H^*(S^1\times\{p_\infty\})\to H^{*+\mathrm{ind}(\overline{F}_\infty)}\big(E^0_{S^1\times\{p_\infty\}},E^{-\infty}_{S^1\times\{p_\infty\}}\big)$$

is different than zero. Since the first vertical arrow sends the unit class of $H^*(S^1\times S^{2n+1})$ to the unit class of $H^*(S^1\times\{p_\infty\})$, we conclude that the image of the unit class of $H^*(S^1\times S^{2n+1})$ by

$$i_0^*\circ\mathrm{ex}\circ T:H^*(S^1\times S^{2n+1})\to H^{*+\mathrm{ind}(\overline{F}_\infty)}(E^0,E^{-\infty})$$

is different than zero, as we wanted. If B is $S^{2n}\times S^1$ the proof is similar, considering the restriction of E to $S^1\times\{p_\infty\}\times S^1$, where p_∞ is the point at infinity of S^{2n}.

We now prove non-degeneracy (v). Suppose that $c_+(\varphi)=c_-(\varphi)=0$, thus

$$c([S^1\times B],\overline{F})=c(1,\overline{F})=0$$

for any generating function quadratic at infinity $\overline{F}:S^1\times B\times\mathbb{R}^N\to\mathbb{R}$ for φ. By Proposition 6.1 (v), $\overline{F}$ generates the zero section of $T^*(S^1\times B)$, hence the zero section of $J^1(S^1\times B)$. But, by definition, $\overline{F}$ generates the Lagrangian submanifold $\overline{L_\varphi}$ of $T^*_{-d\theta}(S^1\times B)$ and so, by Proposition 4.2, the Legendrian submanifold $U_{-d\theta}(\widetilde{\overline{L_\varphi}})$ of $J^1(S^1\times B)$. We thus conclude that $U_{-d\theta}(\widetilde{\overline{L_\varphi}})$ is the zero section. But this implies that $\widetilde{\overline{L_\varphi}}$ is the zero section, thus $\overline{L_\varphi}$ is the zero section and so φ is the identity.

By the Poincaré duality property (iii), it is enough to prove the triangle inequality (vi) for c_+. This can be done as follows. Let $\overline{F_{\varphi_1}}$, $\overline{F_{\varphi_2}}$ and $\overline{F_{\varphi_1\circ\varphi_2}}$ be generating functions quadratic at infinity for φ_1, φ_2 and $\varphi_1\circ\varphi_2$ respectively. We first show that, for any non-zero class u,

$$c(u,\overline{F_{\varphi_2}})=c(u,\overline{F_{\varphi_1\circ\varphi_2}}-\overline{F_{\varphi_1}})\,. \tag{15}$$

To see this, let $\{\varphi_t\}$ be a lcs Hamiltonian isotopy with time-1 map φ_1, let $\overline{F_t}$ and $\overline{G_t}$ be 1-parameter families of generating functions quadratic at infinity for $\{\varphi_t\}$ and $\{\varphi_t\circ\varphi_2\}$ respectively, and set

$$c_t=c(u\,,\,\overline{G_t}-\overline{F_t})\,.$$

By Proposition 5.6 (ii), for every t the set of critical values of $\overline{G_t}-\overline{F_t}$ is equal to the action spectrum of φ_2. The functions $\overline{G_t}-\overline{F_t}$ have thus all the same set S of critical values. By Proposition 6.1 (i), c_t is in S for all t. By Proposition 6.1 (ii) the map $t\mapsto c_t$ is continuous. Since S is nowhere dense we conclude that $t\mapsto c_t$ is constant, and so $c_0=c(u,\overline{G_0}-\overline{F_0})$ is equal to $c_1=c(u,\overline{G_1}-\overline{F_1})$. Since $\overline{G_0}-\overline{F_0}$ is equivalent to $\overline{F_{\varphi_2}}$ and $\overline{G_1}-\overline{F_1}$ is equivalent to $\overline{F_{\varphi_1\circ\varphi_2}}-\overline{F_{\varphi_1}}$, by Proposition 6.1 (vii) we obtain (15). Using (15) and Proposition 6.1 (iv) and (vi) we thus conclude that

$$\begin{aligned} c_+(\varphi_2)=c([S^1\times B],\overline{F_{\varphi_2}})&=c([S^1\times B],\overline{F_{\varphi_1\circ\varphi_2}}-\overline{F_{\varphi_1}})\geq c([S^1\times B],\overline{F_{\varphi_1\circ\varphi_2}})+c(1,-\overline{F_{\varphi_1}})\\ &=c([S^1\times B],\overline{F_{\varphi_1\circ\varphi_2}})-c([S^1\times B],\overline{F_{\varphi_1}})=c_+(\varphi_1\circ\varphi_2)-c_+(\varphi_1)\,. \end{aligned}$$

The proof of the invariance by conjugation property (vii) is analogous to the proof of [Bhu01, Theorem 1.1 (i)] and [San11a, Lemma 3.16]. Suppose first that φ and ψ are compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$. Let $\{\psi_t\}$ be a compactly supported lcs Hamiltonian isotopy with $\psi_1 = \psi$, let $\overline{F_t} : E \to \mathbb{R}$ a 1-parameter family of generating functions quadratic at infinity for $\{\psi_t \circ \varphi \circ \psi_t^{-1}\}$, and set $c_t = c(u, \overline{F_t})$, where u is either the fundamental or the unit class of $S^1 \times S^{2n} \times S^1$. The result follows if we prove that if $c_{t_0} = k \in \mathbb{Z}$ for some t_0 then $c_t = k$ for all t. Let x_t be a path in E, for t in some subinterval of $[0,1]$ containing t_0, such that each x_t is a critical point of $\overline{F_t}$ with critical value c_t. Let q_t be the corresponding path of essential translated points of $\psi_t \circ \varphi \circ \psi_t^{-1}$. Assume first that q_{t_0} is a non-degenerate translated point of $\psi_{t_0} \circ \varphi \circ \psi_{t_0}^{-1}$. Since q_0 has time-shift $c_{t_0} = k \in \mathbb{Z}$, by Lemma 6.2 for every t the point $\psi_t \circ \psi_{t_0}^{-1}(q_{t_0})$ is an essential non-degenerate translated point of $\psi_t \circ \varphi \circ \psi_t^{-1}$ of time-shift k. Let y_t be the corresponding path of non-degenerate critical points of $\overline{F_t}$ of critical value k. Since $y_{t_0} = x_{t_0}$ we deduce that $x_t = y_t$ for all t, and so $c_t = k$ for all t. This ends the proof for $S^1 \times \mathbb{R}^{2n} \times S^1$ under the assumption that q_{t_0} is a non-degenerate translated point of $\psi_{t_0} \circ \varphi \circ \psi_{t_0}^{-1}$. If q_{t_0} is degenerate, we obtain the result by an approximation argument as in [Bhu01]. For every $\epsilon > 0$ there is a generating function quadratic at infinity $G : E \to \mathbb{R}$ such that $\|G - \overline{F_{t_0}}\|_{\mathcal{C}^\infty} < \epsilon$, $G \equiv \overline{F_{t_0}}$ on an neighborhood of $S^1 \times \{p_\infty\} \times S^1 \times \mathbb{R}^N$, all the critical points of G outside this neighborhood are non-degenerate and $c(u, G) = c(u, \overline{F_{t_0}})$. Since $\|G - \overline{F_{t_0}}\|_{\mathcal{C}^\infty} < \epsilon$, G is a generating function quadratic at infinity for a compactly supported lcs Hamiltonian diffeomorphism ϕ_{t_0} of $S^1 \times \mathbb{R}^{2n} \times S^1$ with $\|\phi_{t_0} \circ (\psi_{t_0} \circ \varphi \circ \psi_{t_0})^{-1}\|_{\mathcal{C}^\infty} < \epsilon$. Let $\phi_t = (\psi_t \circ \psi_{t_0}) \circ \phi_{t_0} \circ (\psi_t \circ \psi_{t_0})^{-1}$. Then $\|\phi_t \circ (\psi_t \circ \varphi \circ \psi_t)^{-1}\|_{\mathcal{C}^\infty} < \epsilon$. Let G_t be a 1-parameter family of generating functions quadratic at infinity for $\{\phi_t\}$. By the first part of the proof, $c(u, G_t) = k$ for all t. But $c(u, G_t)$ is equal to $c_+(\phi_t)$ or $c_-(\phi_t)$. Since ϵ is arbitrary, by continuity (i) we thus obtain that $c_t = k$ for all t. For compactly supported lcs Hamiltonian diffeomorphisms of $S^1 \times \mathbb{R}^{2n+1}$ a similar argument works for $k = 0$, showing that $c_+(\varphi) = 0$ if and only if $c_+(\psi \circ \varphi \circ \psi^{-1}) = 0$ and similarly for c_-. □

Let ϕ be a compactly supported contactomorphism of $\mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n} \times S^1$ contact isotopic to the identity, and $\overline{F} : B \times \mathbb{R}^N \to \mathbb{R}$, where B is either S^{2n+1} or $S^{2n} \times S^1$, a generating function quadratic at infinity for ϕ. As in [Bhu01, San10, San11a], we define $c_+(\phi) = c([B], \overline{F})$ and $c_-(\phi) = c(1, \overline{F})$. The spectral selectors c_+ and c_- on $\mathrm{Cont}_0^c(\mathbb{R}^{2n+1})$ and $\mathrm{Cont}_0^c(\mathbb{R}^{2n} \times S^1)$ enjoy properties similar to those discussed above for the spectral selectors c_+ and c_- on $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n+1})$ and $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$. Except for the analogues of (iii) and (vi), these properties are proved in [Bhu01, San10, San11a]. As discussed in the next remark, the contact analogues of (iii) and (vi) are in fact stronger than the corresponding properties proved in [Bhu01, San10, San11a].

Remark 6.4. *Arguments as in the proof of Proposition 6.3 (iii) and (vi) also apply to the spectral selectors $c_\pm$ of compactly supported contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$, proving in particular that $c_-(\phi) = -c_+(\phi^{-1})$, $c_+(\phi_1 \circ \phi_2) \leq c_+(\phi_1) + c_+(\phi_2)$ and $c_-(\phi_1 \circ \phi_2) \geq c_-(\phi_1) + c_-(\phi_2)$. In* [San11a] *and* [San10] *only weaker versions of these properties are proved: $\lfloor c_-(\phi) \rfloor = -\lceil c_+(\phi^{-1}) \rceil$, $\lceil c_+(\phi_1 \circ \phi_2) \rceil \leq \lceil c_+(\phi_1) \rceil + \lceil c_+(\phi_2) \rceil$ and $\lfloor c_-(\phi_1 \circ \phi_2) \rfloor \geq \lfloor c_-(\phi_1) \rfloor + \lfloor c_-(\phi_2) \rfloor$. It turns out that this difference is due to the fact that in the present article we use a different formula for the identification τ_{cont} of the contact product of $\mathbb{R}^{2n+1}$ with $J^1\mathbb{R}^{2n+1}$. If we would use the identification of* [San11a, San10]*, in the contact analogue of Proposition 5.6 (ii) we would obtain that for any two compactly supported contactomorphisms ϕ_1 and ϕ_2 of $\mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n} \times S^1$ contact isotopic to the identity and generating functions quadratic at infinity $\overline{F_1}$ and $\overline{F_2}$ for ϕ_1 and ϕ_2, the set of critical values of $\overline{F_1} - \overline{F_2}$ is equal to the action spectrum of $\phi_1 \circ \phi_2^{-1}$, not of $\phi_2^{-1} \circ \phi_1$. Thus in the contact analogue of the proof of Proposition 6.3 (iii) the set of critical values of $\overline{G_t} - \overline{F_t}$ would not be equal to the action spectrum of φ^{-1} but to the action spectrum of $\varphi_t \circ \varphi^{-1} \circ \varphi_t^{-1}$ and so we would need to integrate in the proof also arguments as in the proof of Proposition 6.3 (vii), where the floor and ceiling functions are necessary. Something similar also happens for the triangle inequality.*

We now discuss the relation between the spectral selectors of lcs Hamiltonian diffeomorphisms and those of contactomorphisms.

Proposition 6.5. *Let ϕ be a compactly supported contactomorphism of $\mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n} \times S^1$ contact isotopic to the identity, and $\widetilde{\phi}$ its lift to $S^1 \times \mathbb{R}^{2n+1}$ or $S^1 \times \mathbb{R}^{2n} \times S^1$. Then $c_+(\widetilde{\phi}) = c_+(\phi)$ and $c_-(\widetilde{\phi}) = c_-(\phi)$.*

Proof. By the Poincaré duality property, Proposition 6.3 (iii), and the analogous property for the spectral selectors of contactomorphisms (cf. Remark 6.4), it is enough to prove that $c_-(\widetilde{\phi}) = c_-(\phi)$ for any compactly supported contactomorphism ϕ of $\mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n}\times S^1$ contact isotopic to the identity. Let $\overline{F}: B\times\mathbb{R}^N\to\mathbb{R}$ be a generating function quadratic at infinity for ϕ, where B is either S^{2n+1} or $S^{2n}\times S^1$. By Proposition 5.12, the function $\overline{F}\circ \mathrm{pr}_2 : S^1\times(B\times\mathbb{R}^N)\to\mathbb{R}$, where pr_2 is the projection to the second factor, is a generating function quadratic at infinity for $\widetilde{\phi}$. Denote $E_B = B\times\mathbb{R}^N$ and $E = S^1\times B\times\mathbb{R}^N$, and similarly $E_B^a = \{\overline{F}\le a\}$ and $E^a = \{\overline{F}\circ\mathrm{pr}_2\le a\}$. Then $E = S^1\times E_B$ and $E^a = S^1\times E_B^a$. Consider the commutative diagram

$$\begin{tikzcd}
H^*(B) \arrow[r, "\mathrm{ex}\circ T"] \arrow[d] & H^{*+\mathrm{ind}(\overline{F}_\infty)}(E_B, E_B^{-\infty}) \arrow[r, "i_a"] \arrow[d] & H^{*+\mathrm{ind}(\overline{F}_\infty)}(E_B^a, E_B^{-\infty}) \arrow[d] \\
H^*(S^1\times B) \arrow[r, "\mathrm{ex}\circ T"] & H^{*+\mathrm{ind}(\overline{F}_\infty)}(E, E^{-\infty}) \arrow[r, "i_a"] & H^{*+\mathrm{ind}(\overline{F}_\infty)}(E^a, E^{-\infty})
\end{tikzcd}$$

where the vertical arrows are the homomorphisms induced by the projection $S^1\times B\to B$ and the induced projections on fibre bundles. Since the first vertical arrow sends the unit class of $H^*(B)$ to the unit class of $H^*(S^1\times B)$ and since the last vertical arrow is injective, we deduce that the image of the unit class of $H^*(B)$ by $i_a^*\circ\mathrm{ex}\circ T$ is non-zero if and only if the image of the unit class of $H^*(S^1\times B)$ by $i_a^*\circ\mathrm{ex}\circ T$ is non-zero. This implies that $c(1,\overline{F}\circ\mathrm{pr}_2) = c(1,\overline{F})$, and so $c_-(\widetilde{\phi}) = c_-(\phi)$. ☐

Recall that the spectral selectors for compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ are defined in [Vit92] by

$$c_+(\varphi) = c([S^{2n}],\overline{F}) \quad\text{and}\quad c_-(\varphi) = c(1,\overline{F})$$

for any generating function quadratic at infinity $\overline{F}: S^{2n}\times\mathbb{R}^N\to\mathbb{R}$ for φ. For any φ we have

$$c_\pm(\widetilde{\varphi}) = c_\pm(\varphi)\,,$$

where $\widetilde{\varphi}$ denotes the lift of φ to $\mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n}\times S^1$. Indeed, this follows from the fact that the Legendrian submanifold $\Lambda_{\widetilde{\varphi}}$ of $J^1\mathbb{R}^{2n+1}\equiv T^*\mathbb{R}\times J^1\mathbb{R}^{2n}$ is equal to $0_{\mathbb{R}}\times\widetilde{L_\varphi}$, where $\widetilde{L_\varphi}$ is the lift of $L_\varphi\subset T^*\mathbb{R}^{2n}$ to $J^1\mathbb{R}^{2n}$ (cf. [San11a, Proposition 3.18]).

As a first application of our spectral selectors we obtain a proof of Theorem 1.2 (ii).

Proof of Theorem 1.2 (ii). Let φ be a compactly supported lcs Hamiltonian diffeomorphisms of $S^1\times\mathbb{R}^{2n+1}$ or $S^1\times\mathbb{R}^{2n}\times S^1$ with non-empty support. We have to show that φ has at least one essential translated point in the interior of its support. By Proposition 6.3 (v), either $c_+(\varphi)\neq 0$ or $c_-(\varphi)\neq 0$. But, $c_+(\varphi)$ and $c_-(\varphi)$ belong to the action spectrum of φ. We thus conclude that φ has at least one essential translated point that has action (and time-shift) different from zero, and thus belongs to the interior of the support of φ. ☐

7. Bi-invariant partial orders on the Hamiltonian groups of $S^1\times\mathbb{R}^{2n+1}$ and $S^1\times\mathbb{R}^{2n}\times S^1$, and an integer-valued bi-invariant metric on the Hamiltonian group of $S^1\times\mathbb{R}^{2n}\times S^1$

Recall that for compactly supported lcs Hamiltonian diffeomorphisms φ_1 and φ_2 of $S^1\times\mathbb{R}^{2n+1}$ or $S^1\times\mathbb{R}^{2n}\times S^1$ we have posed $\varphi_1\le\varphi_2$ if $\varphi_2\circ\varphi_1^{-1}$ is the time-1 map of a non-negative compactly supported lcs Hamiltonian isotopy. We now show that $\le$ is a bi-invariant partial order, and so $S^1\times\mathbb{R}^{2n+1}$ and $S^1\times\mathbb{R}^{2n}\times S^1$ are strongly orderable. We obtain this by comparing $\le$ with a bi-invariant partial order $\preceq$ defined using the spectral selectors of the previous section, which is the lcs analogue of the partial orders on $\mathrm{Ham}^c(\mathbb{R}^{2n})$, $\mathrm{Cont}_0^c(\mathbb{R}^{2n+1})$ and $\mathrm{Cont}_0^c(\mathbb{R}^{2n}\times S^1)$ defined in [Vit92], [Bhu01] and [San11a] respectively.

For compactly supported lcs Hamiltonian diffeomorphisms φ_1 and φ_2 of $S^1\times\mathbb{R}^{2n+1}$ or $S^1\times\mathbb{R}^{2n}\times S^1$ we pose

$$\varphi_1\preceq\varphi_2 \quad\text{if}\;\; c_+(\varphi_1\circ\varphi_2^{-1}) = 0\,.$$

Theorem 7.1. *The relation $\preceq$ on $\mathrm{Ham}^c(S^1\times\mathbb{R}^{2n+1})$ or $\mathrm{Ham}^c(S^1\times\mathbb{R}^{2n}\times S^1)$ is a bi-invariant partial order.*

Proof. Reflexivity follows from the fact that $c_+(\mathrm{id})=0$. Suppose that $\varphi_1 \preceq \varphi_2$ and $\varphi_2 \preceq \varphi_3$, thus $c_+(\varphi_1\circ\varphi_2^{-1}) = c_+(\varphi_2\circ\varphi_3^{-1}) = 0$. By Proposition 6.3 (iv) and (vi) we then have

$$0 \leq c_+(\varphi_1\circ\varphi_3^{-1}) \leq c_+(\varphi_1\circ\varphi_2^{-1}) + c_+(\varphi_2\circ\varphi_3^{-1}) = 0\,,$$

thus $c_+(\varphi_1\circ\varphi_3^{-1}) = 0$ and so $\varphi_1 \preceq \varphi_3$, showing transitivity. Suppose that $\varphi_1 \preceq \varphi_2$ and $\varphi_2 \preceq \varphi_1$, thus $c_+(\varphi_1\circ\varphi_2^{-1}) = c_+(\varphi_2\circ\varphi_1^{-1}) = 0$. By Proposition 6.3 (iii) we then have

$$c_-(\varphi_1\circ\varphi_2^{-1}) = -c_+(\varphi_2\circ\varphi_1^{-1}) = 0\,.$$

It thus follows from Proposition 6.3 (v) that $\varphi_1\circ\varphi_2^{-1}$ is the identity, and so $\varphi_1=\varphi_2$. This proves anti-symmetry. As for bi-invariance, if $\varphi_1 \preceq \varphi_2$, thus $c_+(\varphi_1\circ\varphi_2^{-1})=0$, then for every ψ we have

$$c_+\big((\varphi_1\circ\psi)\circ(\varphi_2\circ\psi)^{-1}\big) = c_+(\varphi_1\circ\varphi_2^{-1}) = 0\,,$$

thus $\varphi_1\circ\psi \preceq \varphi_2\circ\psi$, and, by Proposition 6.3 (vii),

$$c_+\big((\psi\circ\varphi_1)\circ(\psi\circ\varphi_2)^{-1}\big) = c_+\big(\psi\circ\varphi_1\circ\varphi_2^{-1}\circ\psi^{-1}\big) = 0\,,$$

thus $\psi\circ\varphi_1 \preceq \psi\circ\varphi_2$. ☐

In order to deduce that the relation $\leq$ is also a partial order, and so that $S^1\times\mathbb{R}^{2n+1}$ and $S^1\times\mathbb{R}^{2n}\times S^1$ are strongly orderable (Theorem 1.3 (ii)), we use the following comparison.

Proposition 7.2. *Let φ_1 and φ_2 be compactly supported lcs Hamiltonian diffeomorphisms of $S^1\times\mathbb{R}^{2n+1}$ or $S^1\times\mathbb{R}^{2n}\times S^1$ with $\varphi_1\leq\varphi_2$. Then $\varphi_1\preceq\varphi_2$.*

Proof. Since $\varphi_1\leq\varphi_2$, $\varphi_2\circ\varphi_1^{-1}$ is the time-1 map of a compactly supported non-negative lcs Hamiltonian isotopy. By definition this implies that $\mathrm{id}\leq\varphi_2\circ\varphi_1^{-1}$, and so by Proposition 6.3 (ii) and (iv) we have

$$0 = c_-(\mathrm{id}) \leq c_-(\varphi_2\circ\varphi_1^{-1}) \leq 0\,,$$

thus $c_-(\varphi_2\circ\varphi_1^{-1})=0$. By Proposition 6.3 (iii) this implies that $c_+(\varphi_1\circ\varphi_2^{-1})=0$, and so $\varphi_1\preceq\varphi_2$. ☐

We can now prove that $S^1\times\mathbb{R}^{2n+1}$ and $S^1\times\mathbb{R}^{2n}\times S^1$ are strongly orderable.

Proof of Theorem 1.3 (ii). We have to show that the relation $\leq$ on $\mathrm{Ham}^c(S^1\times\mathbb{R}^{2n+1})$ or $\mathrm{Ham}^c(S^1\times\mathbb{R}^{2n}\times S^1)$ is a bi-invariant partial order. Reflexivity follows from the fact that $\{\mathrm{id}\}$ has Hamiltonian function $H_t\equiv 0$. Transitivity and bi-invariance follow from Lemmas 2.20 and 2.21. As for anti-symmetry, if $\varphi_1\leq\varphi_2$ and $\varphi_2\leq\varphi_1$ then by Proposition 7.2 we also have $\varphi_1\preceq\varphi_2$ and $\varphi_2\preceq\varphi_1$. But, by Theorem 7.1, $\preceq$ is anti-symmetric. We thus conclude that $\varphi_1=\varphi_2$. ☐

Recall that a bi-invariant metric on a group G is a map $d: G\times G\to\mathbb{R}$ that satisfies the following properties:

(i) *Positivity:* $d(g_1,g_2)\geq 0$, and $d(g,g)=0$.
(ii) *Non-degeneracy:* $d(g_1,g_2)=0$ if and only if $g_1=g_2$.
(iii) *Symmetry:* $d(g_1,g_2)=d(g_2,g_1)$.
(iv) *Triangle inequality:* $d(g_1,g_3)\leq d(g_1,g_2)+d(g_2,g_3)$.
(v) *Bi-invariance:* $d(hg_1,hg_2)=d(g_1h,g_2h)=d(g_1,g_2)$.

In [San10] the third author defined an integer valued bi-invariant metric d_{cont} on $\mathrm{Cont}_0^c(\mathbb{R}^{2n}\times S^1)$ by posing

$$d_{\mathrm{cont}}(\phi_1,\phi_2) = \lceil c_+(\phi_1\circ\phi_2^{-1})\rceil - \lfloor c_-(\phi_1\circ\phi_2^{-1})\rfloor\,,$$

and proved that this metric is unbounded. This metric is the contact analogue of the Viterbo metric d_{symp} on $\mathrm{Ham}^c(\mathbb{R}^{2n})$, which is defined by

$$d_{\mathrm{symp}}(\varphi_1,\varphi_2) = c_+(\varphi_1\circ\varphi_2^{-1}) - c_-(\varphi_1\circ\varphi_2^{-1})$$

and is also unbounded. Similarly, for compactly supported lcs Hamiltonian diffeomorphisms φ_1 and φ_2 of $S^1\times\mathbb{R}^{2n}\times S^1$ we define

$$d(\varphi_1,\varphi_2) = \lceil c_+(\varphi_1\circ\varphi_2^{-1})\rceil - \lfloor c_-(\varphi_1\circ\varphi_2^{-1})\rfloor\,.$$

Theorem 7.3. *The map d is an unbounded bi-invariant metric on* $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$.

Proof. Positivity follows from Proposition 6.3 (iv) and the fact that $c_\pm(\mathrm{id}) = 0$. Suppose that

$$d(\varphi_1, \varphi_2) = \lceil c_+(\varphi_1 \circ \varphi_2^{-1}) \rceil - \lfloor c_-(\varphi_1 \circ \varphi_2^{-1}) \rfloor = 0\,.$$

Since $c_+(\varphi_1 \circ \varphi_2^{-1}) \geq 0$ and $c_-(\varphi_1 \circ \varphi_2^{-1}) \leq 0$, this implies that $c_+(\varphi_1 \circ \varphi_2^{-1}) = c_-(\varphi_1 \circ \varphi_2^{-1}) = 0$ and, by Proposition 6.3 (v), that $\varphi_1 \circ \varphi_2^{-1}$ is the identity, or, equivalently, that $\varphi_1 = \varphi_2$. This proves non-degeneracy. Symmetry, the triangle inequality and bi-invariance follow from Proposition 6.3 (iii), (vi) and (vii) respectively. Observe now that, by Proposition 6.5, if ϕ_1 and ϕ_2 are compactly supported contactomorphisms of $\mathbb{R}^{2n} \times S^1$ contact isotopic to the identity and $\widetilde{\phi_1}$ and $\widetilde{\phi_2}$ their lifts to $S^1 \times \mathbb{R}^{2n} \times S^1$ then $d(\widetilde{\phi_1}, \widetilde{\phi_2}) = d_{\mathrm{cont}}(\phi_1, \phi_2)$. Since d_{cont} is unbounded, we conclude d is also unbounded. □

Recall that a partially ordered metric space is a metric space (Z, d) endowed with a partial order $\leq$ that is compatible with d, in the sense that for all $a, b, c \in Z$ with $a \leq b \leq c$ we have $d(a, b) \leq d(a, c)$.

Theorem 7.4. $(\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1), d, \leq)$ *and* $(\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1), d, \preceq)$ *are partially ordered metric spaces.*

Proof. By Proposition 7.2, it is enough to prove that $\preceq$ is compatible with d. For this, suppose that $\varphi_1 \preceq \varphi_2 \preceq \varphi_3$. Then $c_+(\varphi_1 \circ \varphi_2^{-1}) = c_+(\varphi_2 \circ \varphi_3^{-1}) = c_+(\varphi_1 \circ \varphi_3^{-1}) = 0$. Thus, using Proposition 6.3 (iii) and (vi) we get

$$\begin{aligned} d(\varphi_1, \varphi_2) &= \lceil c_+(\varphi_1 \circ \varphi_2^{-1}) \rceil - \lfloor c_-(\varphi_1 \circ \varphi_2^{-1}) \rfloor = -\lfloor c_-(\varphi_1 \circ \varphi_2^{-1}) \rfloor = \lceil c_+(\varphi_2 \circ \varphi_1^{-1}) \rceil \\ &\leq \lceil c_+(\varphi_2 \circ \varphi_3^{-1}) \rceil + \lceil c_+(\varphi_3 \circ \varphi_1^{-1}) \rceil = \lceil c_+(\varphi_3 \circ \varphi_1^{-1}) \rceil \\ &= \lceil c_+(\varphi_1 \circ \varphi_3^{-1}) \rceil + \lceil c_+(\varphi_3 \circ \varphi_1^{-1}) \rceil = \lceil c_+(\varphi_1 \circ \varphi_3^{-1}) \rceil - \lfloor c_-(\varphi_1 \circ \varphi_3^{-1}) \rfloor \\ &= d(\varphi_1, \varphi_3)\,. \end{aligned}$$

□

Remark 7.5. *By an argument of Polterovich, which is written down in* [San15, Proposition 5.1], *the identity component of the group of compactly supported contactomorphisms of any contact manifold does not admit any bi-invariant metric taking values arbitrarily close to zero. A similar argument also works in the lcs case to show that the group of compactly supported lcs diffeomorphisms of any lcs manifold $\big(M, [(\eta, \omega)]\big)$ does not admit any bi-invariant metric that takes values arbitrarily close to zero. Indeed, let B be a Darboux ball in M and let ψ_1 and ψ_2 be two lcs diffeomorphisms that are supported in B and do not commute, i.e.* $[\psi_1, \psi_2] := \psi_1 \circ \psi_2 \circ \psi_1^{-1} \circ \psi_2^{-1} \neq \mathrm{id}$. *Suppose by contradiction that d is bi-invariant metric on the group of compactly supported lcs diffeomorphisms of $\big(M, [(\eta, \omega)]\big)$ that takes values arbitrarily close to zero, and consider the associated conjugation-invariant norm* $\|\cdot\| = d(\,\cdot\,, \mathrm{id})$. *Let φ be a compactly supported lcs diffeomorphism with $\|\varphi\| = \epsilon$ for ϵ arbitrarily small. Since φ is not the identity, there is a small ball B_φ in M that is displaced by φ. Since it is possible to squeeze any given domain of $\big(\mathbb{R}^{2n}, [(0, \omega_0)]\big)$ into an arbitrarily small one by the Liouville flow (which is a lcs isotopy), we can find a compactly supported lcs diffeomorphism θ of M such that $\theta(B) \subset B_\phi$. By Lemma 2.21, $\theta \circ \psi_1 \circ \theta^{-1}$ and $\theta \circ \psi_2 \circ \theta^{-1}$ are lcs diffeomorphisms supported in B_ϕ. Since we assume that $\|\cdot\|$ is conjugation-invariant and* $[\psi_1, \psi_2] \neq \mathrm{id}$, *we have*

$$\|\,[\,\theta \circ \psi_1 \circ \theta^{-1}\,,\, \theta \circ \psi_2 \circ \theta^{-1}\,]\,\| = \|\,\theta \circ [\psi_1, \psi_2] \circ \theta^{-1}\,\| = \|\,[\psi_1, \psi_2]\,\| \neq 0\,.$$

On the other hand, we also have

$$\|\,[\,\theta \circ \psi_1 \circ \theta^{-1}\,,\, \theta \circ \psi_2 \circ \theta^{-1}\,]\,\| \leq 4\,\|\varphi\| = 4\epsilon\,.$$

Indeed, as in [San15, Proposition 5.1], *this follows from the fact that if a_1 and a_2 are supported in a domain $\mathcal{U}$ and b displaces $\mathcal{U}$ then for every conjugation-invariant norm $\|\cdot\|$ we have $\|[a_1, a_2]\| \leq 4\,\|b\|$. Since ϵ was chosen arbitrarily small, we thus obtain a contradiction. As the Liouville flow is not Hamiltonian, it is unclear to us whether the group of compactly supported lcs Hamiltonian diffeomorphisms admits a bi-invariant metric that takes values arbitrarily close to zero.*

8. A lcs capacity for domains of $S^1 \times \mathbb{R}^{2n} \times S^1$, and proof of lcs non-squeezing

We define the *lcs capacity* of an open and bounded domain $\mathcal{U}$ of $S^1 \times \mathbb{R}^{2n} \times S^1$ by

$$c(\mathcal{U}) = \sup\{\, \lceil c_+(\varphi) \rceil \mid \varphi \in \mathrm{Ham}(\mathcal{U}) \,\},$$

where $\mathrm{Ham}(\mathcal{U})$ denotes the group of lcs diffeomorphisms of $S^1 \times \mathbb{R}^{2n} \times S^1$ that are the time-1 map of a lcs Hamiltonian isotopy supported in $\mathcal{U}$. By the following lemma, $c(\mathcal{U})$ is a well-defined integer number.

Lemma 8.1. *For every compactly supported lcs Hamiltonian diffeomorphism ψ of $S^1 \times \mathbb{R}^{2n} \times S^1$ such that $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$ we have*

$$c(\mathcal{U}) \leq d(\mathrm{id}, \psi)\,.$$

Proof. We first prove that

$$\lfloor c_+(\psi \circ \varphi) \rfloor = \lfloor c_+(\psi) \rfloor \tag{16}$$

for every $\varphi \in \mathrm{Ham}(\mathcal{U})$. Let $\{\varphi_t\}$ be a lcs Hamiltonian isotopy supported in $\mathcal{U}$ with $\varphi_1 = \varphi$, let $\overline{F_t} : E \to \mathbb{R}$ be a 1-parameter family of generating functions quadratic at infinity for $\{\psi \circ \varphi_t\}$, and set

$$c_t = c_+(\psi \circ \varphi_t) = c([S^1 \times S^{2n} \times S^1], \overline{F_t})\,.$$

The results follows if we prove that if $c_{t_0} = k \in \mathbb{Z}$ for some t_0 then $c_t = k$ for all t. Let x_t be a path in E, for t in a subinterval of $[0,1]$ containing t_0, such that each x_t is a critical point of $\overline{F_t}$ with critical value c_t. Let q_t be the corresponding path of essential translated points of $\psi \circ \varphi_t$. As in the proof of Proposition 6.3 (vii), without loss of generality we can assume that q_{t_0} is a non-degenerate essential translated point of $\psi \circ \varphi_{t_0}$. Since q_{t_0} has time-shift $k \in \mathbb{Z}$, we have $\psi \circ \varphi_{t_0}(q_{t_0}) = q_{t_0}$. Since φ_{t_0} is supported in $\mathcal{U}$ and $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$, this implies that $q_{t_0} \notin \mathcal{U}$ and so $\varphi_t(q_{t_0}) = q_{t_0}$ for all t and $\psi(q_{t_0}) = q_{t_0}$, thus $\psi \circ \varphi_t(q_{t_0}) = q_{t_0}$ for all t. Let g_t and f be the conformal factors of φ_t and ψ respectively. Then the conformal factor of $\psi \circ \varphi_t$ is $f \circ \varphi_t + g_t$. Since q_{t_0} is a translated point of $\psi \circ \varphi_{t_0}$ and since $g_t(q_{t_0}) = 0$ for all t (because q_{t_0} does not belong to the support of φ_t), we have

$$0 = (f \circ \varphi_{t_0} + g_{t_0})(q_{t_0}) = f(q_{t_0})\,,$$

and so

$$(f \circ \varphi_t + g_t)(q_{t_0}) = f(q_{t_0}) = 0$$

for all t. We conclude that q_{t_0} is a translated point of $\psi \circ \varphi_t$ of time-shift k, for all t. The action of $\psi \circ \varphi_t$ is $S_{\varphi_t} - e^{-g_t} S_\psi \circ \varphi_t$. Since q_{t_0} is an essential translated point of $\psi \circ \varphi_{t_0}$, by Example 3.11 we have

$$0 = (S_{\varphi_{t_0}} - e^{-g_{t_0}} S_\psi \circ \varphi_{t_0})(q_{t_0}) = S_{\varphi_{t_0}}(q_{t_0}) - S_\psi(q_{t_0})\,.$$

Since moreover $S_{\varphi_t}(q_{t_0}) = 0$ for all t (by Remark 2.1, since q_{t_0} does not belong to the support of φ_t and the Lee form $-d\theta$ is not exact on the complement of the support of φ_t), we thus have

$$(S_{\varphi_t} - e^{-g_t} S_\psi \circ \varphi_t)(q_{t_0}) = -S_\psi(q_{t_0}) = 0$$

for all t. By Example 3.11, we conclude that q_{t_0} is an essential translated point of $\psi \circ \varphi_t$ for all t. Moreover, the fact that q_{t_0} is a non-degenerate translated point of $\psi \circ \varphi_{t_0}$ implies that it is a non-degenerate translated point of $\psi \circ \varphi_t$ for all t (using again that q_{t_0} does not belong to the support of φ_t). Let y_t be the corresponding path of non-degenerate critical points of $\overline{F_t}$ of critical value k. Since $y_{t_0} = x_{t_0}$, we deduce that $y_t = x_t$ for all t, and so $c_t = k$ for all t. This proves (16). Using (16) and Proposition 6.3 (iii) and (vi) we then obtain

$$\lceil c_+(\varphi) \rceil \leq \lceil c_+(\psi \circ \varphi) \rceil + \lceil c_+(\psi^{-1}) \rceil = \lceil c_+(\psi) \rceil - \lfloor c_-(\psi) \rfloor = d(\mathrm{id}, \psi)$$

for every $\varphi \in \mathrm{Ham}(\mathcal{U})$, and so $c(\mathcal{U}) \leq d(\mathrm{id}, \psi)$. $\square$

We define the *displacement energy* of an open and bounded domain $\mathcal{U}$ of $S^1 \times \mathbb{R}^{2n} \times S^1$ by

$$E(\mathcal{U}) = \inf\{\, d(\mathrm{id}, \psi) \mid \psi(\mathcal{U}) \cap \mathcal{U} \neq \emptyset \,\}\,.$$

Lemma 8.1 then also implies the energy–capacity inequality

$$c(\mathcal{U}) \leq E(\mathcal{U})\,.$$

Proposition 8.2. *The lcs capacity c satisfies the following properties:*

(i) Monotonicity: *if* $\mathcal{U}_1 \subset \mathcal{U}_2$ *then* $c(\mathcal{U}_1) \leq c(\mathcal{U}_2)$.
(ii) Lcs invariance: *for any compactly supported lcs Hamiltonian diffeomorphism* ψ *of* $S^1 \times \mathbb{R}^{2n} \times S^1$*, we have* $c(\mathcal{U}) = c\big(\psi(\mathcal{U})\big)$.

Proof. Monotonicity holds by definition. Lcs invariance follows from Proposition 6.3 (vii), because $\varphi \in \mathrm{Ham}(\mathcal{U})$ if and only if $\psi \circ \varphi \circ \psi^{-1} \in \mathrm{Ham}\,\big(\psi(\mathcal{U})\big)$. □

As in [Vit92], for any open and bounded domain $\mathcal{U}$ of $\mathbb{R}^{2n}$ we define

$$c_{\mathrm{symp}}(\mathcal{U}) = \sup\{\, c_+(\varphi) \mid \varphi \in \mathrm{Ham}(\mathcal{U}) \,\}\,,$$

where $\mathrm{Ham}(\mathcal{U})$ denotes the group of diffeomorphisms of $\mathbb{R}^{2n}$ that are the time-1 map of a Hamiltonian isotopy supported in $\mathcal{U}$. Similarly, as in [San11a], for any open and bounded domain $\mathcal{U}$ of $\mathbb{R}^{2n} \times S^1$ we define

$$c_{\mathrm{cont}}(\mathcal{U}) = \sup\{\, \lceil c_+(\phi) \rceil \mid \phi \in \mathrm{Cont}(\mathcal{U}) \,\}\,,$$

where $\mathrm{Cont}(\mathcal{U})$ denotes the group of diffeomorphisms of $\mathbb{R}^{2n} \times S^1$ that are the time-1 map of a contact isotopy supported in $\mathcal{U}$. For every open and bounded domain $\mathcal{U}$ of $\mathbb{R}^{2n}$ we then have

$$c_{\mathrm{cont}}(\mathcal{U} \times S^1) = \lceil c_{\mathrm{symp}}(\mathcal{U}) \rceil\,. \tag{17}$$

We also have the following result.

Proposition 8.3. *For every open and bounded domain* $\mathcal{U}$ *of* $\mathbb{R}^{2n} \times S^1$ *we have*

$$c(S^1 \times \mathcal{U}) = c_{\mathrm{cont}}(\mathcal{U})\,.$$

Proof. The result follows from Proposition 6.3 (ii) and Proposition 6.5 if we prove that for every $\varphi \in \mathrm{Ham}(S^1 \times \mathcal{U})$ there is $\phi \in \mathrm{Cont}(\mathcal{U})$ such that $\varphi \leq \widetilde{\phi}$. This can be seen as follows. Let $\{\varphi_t\}$ be a lcs Hamiltonian isotopy of $S^1 \times \mathbb{R}^{2n} \times S^1$ with $\varphi_1 = \varphi$ and with Hamiltonian function H_t supported in $S^1 \times \mathcal{U}$. Let h_t be the conformal factors of $\{\varphi_t\}$, and G a function on $\mathbb{R}^{2n} \times S^1$ supported in $\mathcal{U}$ such that

$$G(p) \leq -e^{-h_t(\theta,p)}\, H_t \circ \varphi_t(\theta,p) \qquad \text{for all } p, \theta \text{ and } t\,.$$

Let $\{\chi_t\}$ be the contact isotopy with Hamiltonian function G, and $\{\widetilde{\chi_t}\}$ its lift to $S^1 \times \mathbb{R}^{2n} \times S^1$. By Example 2.12 and Example 2.19, $\{\widetilde{\chi_t}\}$ has conformal factor $\widetilde{g_t}(\theta,p) = g_t(p)$ and Hamiltonian function $\widetilde{G}(\theta,p) = G(p)$. Let $\{\phi_t\}$ be the contact isotopy defined by $\phi_t = \chi_t^{-1}$. Then $\{\widetilde{\phi_t}\}$ has conformal factor $-\widetilde{g_t} \circ \widetilde{\chi_t}^{-1}$ and, by Lemma 2.20 (i), Hamiltonian function $-e^{-\widetilde{g_t}}\, \widetilde{G} \circ \widetilde{\chi_t}$. By Lemma 2.20, $\{\widetilde{\phi_t} \circ \varphi_t^{-1}\}$ has thus Hamiltonian function

$$-e^{-\widetilde{g_t}}\, \widetilde{G} \circ \widetilde{\chi_t} + e^{-\widetilde{g_t}}\, (-e^{-h_t} H_t \circ \varphi_t) \circ \widetilde{\chi_t} = -e^{-\widetilde{g_t}}\, (\widetilde{G} + e^{-h_t} H_t \circ \varphi_t) \circ \widetilde{\chi_t} \geq 0\,.$$

This shows that $\varphi_1 \leq \widetilde{\phi_1}$, as we wanted. □

In particular, by Proposition 8.3 and (17) we have

$$c(S^1 \times B^{2n}(R) \times S^1) = c_{\mathrm{cont}}(B^{2n}(R) \times S^1) = \lceil c_{\mathrm{symp}}(B^{2n}(R)) \rceil = \lceil \pi R^2 \rceil\,. \tag{18}$$

Using this we now obtain a proof of the lcs non-squeezing theorem for integers.

Proof of Theorem 1.4. Suppose that $\pi R_2^2 \leq k \leq \pi R_1^2$ for $k \in \mathbb{N}_0$. We have to show that there is no lcs squeezing of $S^1 \times B^{2n}(R_1) \times S^1$ into $S^1 \times B^{2n}(R_2) \times S^1$, i.e. no lcs Hamiltonian isotopy $\{\varphi_t\}_{t\in[0,1]}$ of $S^1 \times \mathbb{R}^{2n} \times S^1$ such that

$$\varphi_1\big(\overline{S^1 \times B^{2n}(R_1) \times S^1}\big) \subset S^1 \times B^{2n}(R_2) \times S^1\,.$$

Suppose by contradiction that such a lcs squeezing exists. We then also have a lcs squeezing of a neighborhood of $S^1 \times B^{2n}(R_1) \times S^1$ into $S^1 \times B^{2n}(R_2) \times S^1$, so without loss of generality we can assume that $\pi R_2^2 \leq k < \pi R_1^2$. By (18) and Proposition 8.2 we have

$$c\Big(\varphi_1\big(S^1 \times B^{2n}(R_1) \times S^1\big)\Big) = c\big(S^1 \times B^{2n}(R_1) \times S^1\big) = \lceil \pi R_1^2 \rceil > k$$

and

$$c\Big(\varphi_1\big(S^1 \times B^{2n}(R_1) \times S^1\big)\Big) \leq c\big(S^1 \times B^{2n}(R_2) \times S^1\big) = \lceil \pi R_2^2 \rceil \leq k\,.$$

This contradiction finishes the proof. $\square$

9. The Hamiltonian group of $S^1 \times \mathbb{R}^{2n+1} \times S^1$ does not have the Rokhlin property

Recall that a topological group G is said to have the Rokhlin property if there exists $g \in G$ whose conjugacy class $\{ hgh^{-1} \mid h \in G \}$ is dense in G. This notion has been introduced by Glasner and Weiss [GW01, GW08]. Serraille and Stojisavljević [SS24] discovered a connection between the Rokhlin property for groups of contactomorphisms or contact homeomorphisms and contact non-squeezing. More precisely, they proved that the closure of the identity component of the group of compactly supported contactomorphisms of $\mathbb{R}^{2n+1}$ inside the group of compactly supported homeomorphisms endowed with the $\mathcal{C}^0$-topology (with respect to the Euclidean distance) has the Rokhlin property, while the closure of the identity component of the group of compactly supported contactomorphisms of $\mathbb{R}^{2n} \times S^1$ inside the group of compactly supported homeomorphisms endowed with the $\mathcal{C}^0$-topology (with respect to the distance on $\mathbb{R}^{2n} \times S^1 = \mathbb{R}^{2n} \times \mathbb{R}/\mathbb{Z}$ induced by the Euclidean distance on $\mathbb{R}^{2n+1}$) does not have the Rokhlin property. The proof of the Rokhlin property for $\mathbb{R}^{2n+1}$ uses the fact that any ball in $\mathbb{R}^{2n+1}$ can be squeezed by a compactly supported contact isotopy into an arbitrarily small ball. On the other hand, the proof of the fact that the Rokhlin property does not hold in the case of $\mathbb{R}^{2n} \times S^1$ is based on the contact non-squeezing theorem of Eliashberg, Kim and Polterovich. In fact, the same proof also shows that the identity component of the group of compactly supported contactomorphisms of $\mathbb{R}^{2n} \times S^1$, endowed with the $\mathcal{C}^0$-topology, does not have the Rokhlin property. Similarly, we obtain the following corollary of Theorem 1.4, where $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$ is endowed with the $\mathcal{C}^0$-topology with respect to the distance induced by the Euclidean distance on $\mathbb{R}^{2n+2}$.

Corollary 9.1. *The topological group* $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$ *does not have the Rokhlin property.*

Proof. We proceed exactly as in [SS24]. Suppose by contradiction that there exists $g \in \mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$ whose conjugacy class is dense in $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$. Since g has compact support, we can assume that its support is contained in $S^1 \times B(R) \times S^1$ for some R. Choose $f \in \mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$ such that $f(x) \neq x$ for all $x \in S^1 \times B(R+1) \times S^1$. Such f can be found by considering the time-1 map of a lcs Hamiltonian isotopy with Hamiltonian function that is equal to $\frac{1}{2}$ on $S^1 \times B(R+1) \times S^1$ and vanishes outside a larger set. Since we assume that the conjugacy class of g is dense in $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$, there exists a sequence (h_n) in $\mathrm{Ham}^c(S^1 \times \mathbb{R}^{2n} \times S^1)$ such that $(h_n \circ g \circ h_n^{-1})$ converges to f. This implies that

$$S^1 \times B(R+1) \times S^1 \not\subset \mathrm{supp}(h_n \circ g \circ h_n^{-1}) = h_n(\mathrm{supp}(g)) \tag{19}$$

for all n. Indeed, if $S^1 \times B(R+1) \times S^1 \subset h_n(\mathrm{supp}(g))$ then $S^1 \times B(R+1) \times S^1 \subset h_n(S^1 \times B(R) \times S^1)$, and so $h_n^{-1}(S^1 \times B(R+1) \times S^1) \subset S^1 \times B(R) \times S^1$, contradicting Theorem 1.4. By taking n large enough in (19) we obtain

$$S^1 \times B(R+1) \times S^1 \not\subset \mathrm{supp}(f)\,,$$

and thus there exists $x \in S^1 \times B(R+1) \times S^1$ such that $f(x) = x$, contradicting the construction of f. $\square$

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Département de Mathématiques, C.P. 218 Université Libre de Bruxelles, 1050 Bruxelles, Belgium

Email address: `melanie.bertelson@ulb.be`

Département de Mathématiques, C.P. 218 Université Libre de Bruxelles, 1050 Bruxelles, Belgium

Email address: `pranav.vijay.chakravarthy@ulb.be`

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France

Email address: `sandon@math.unistra.fr`