EXISTENCE AND UNIQUENESS OF THE CONFORMALLY COVARIANT GEODESIC METRIC ON SIMPLE CONFORMAL LOOP ENSEMBLE CARPETS

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ABSTRACT. We prove that for each $\kappa \in (8/3, 4)$ there exists a geodesic metric on the carpet of a CLE_κ which is canonical in the sense that it is characterized by a certain list of axioms. Our metric can be constructed explicitly as the scaling limit of *Minkowski first passage percolation* (MFPP), i.e., the metric obtained by taking the infimum of the Lebesgue measure of the ε-neighborhood of all paths connecting each pair of points. Earlier work by the first co-author [Mil21] showed that MFPP admits nontrivial subsequential limits. The present paper shows that this subsequential limit is unique and is characterized by our list of axioms. We conjecture that our metric describes the scaling limit of the chemical distance metric for discrete loop models that converge to CLE_κ for $\kappa \in (8/3, 4)$ in the scaling limit, e.g., the critical Ising model for $\kappa = 3$. Our argument is inspired by recent works of Gwynne and Miller [GM21b] and Ding and Gwynne [DG23] on the uniqueness of Liouville quantum gravity metrics.

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1. INTRODUCTION

1.1. **Overview.** The *conformal loop ensembles* (CLE_κ, $\kappa \in (8/3, 8)$) make up the canonical conformally invariant family of probability measures on countable families of loops in a simply connected domain in $\mathbb{C}$ which do not cross each other or themselves [She09, SW12]. They are the loop-analogs of Schramm's SLE_κ ($\kappa > 0$) curves [Sch00], and, just like SLE_κ has either been conjectured or shown to describe the scaling limit of a single interface in a number of discrete models from statistical mechanics defined on a planar lattice [Smi01, LSW04, SS09, Smi10], CLE_κ has been conjectured

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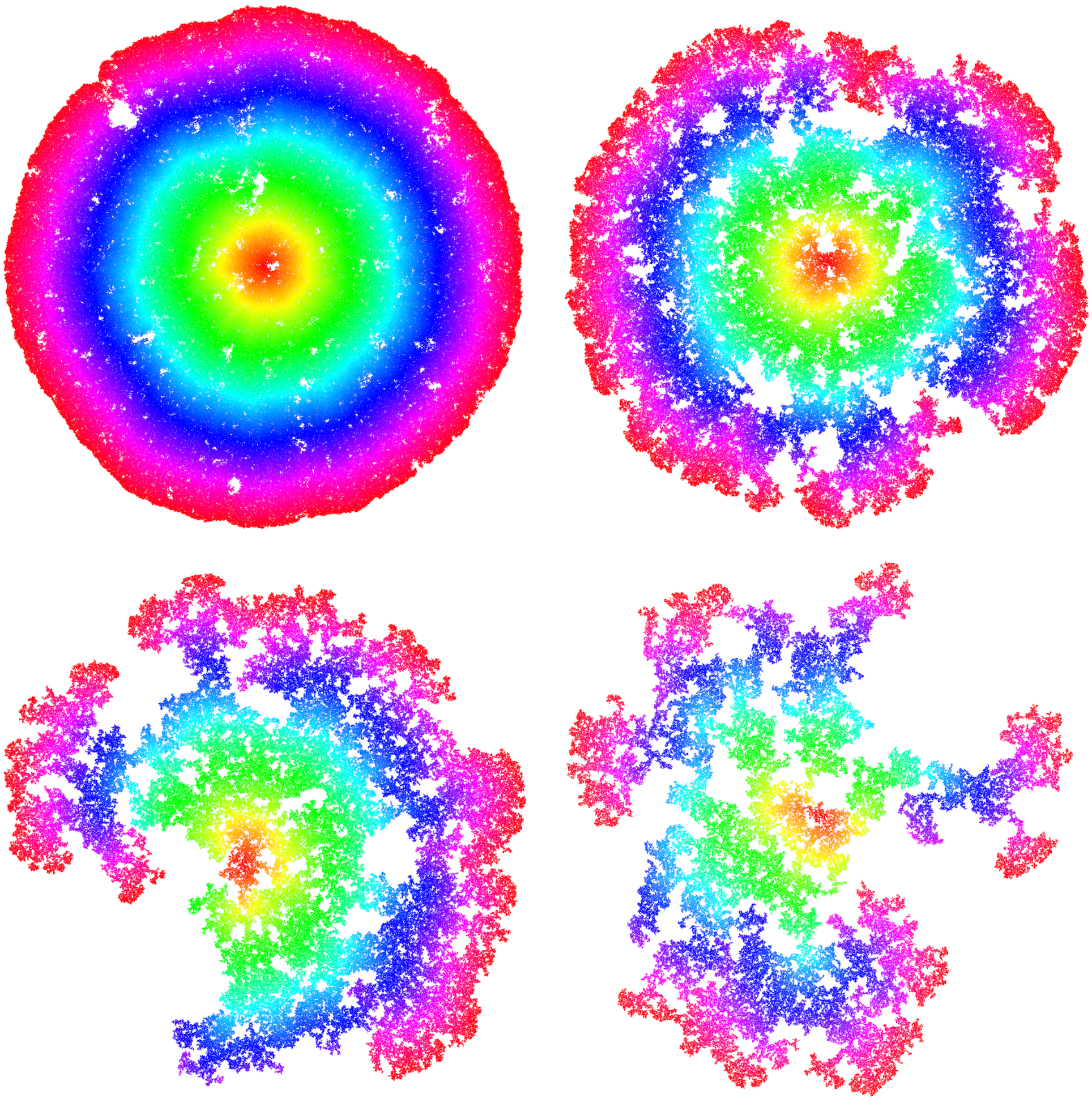

FIGURE 1. Simulation of a metric ball associated with the CLE_κ metric with $\kappa \approx 2.72$ (top left), $\kappa = 3$ (top right), $\kappa = 10/3$ (bottom left), and $\kappa \approx 3.99$ (bottom right). Different colors represent the distance of points in the ball to the center. The κ value associated with the top left simulation is close to the critical value $\kappa = 8/3$ where the CLE carpet is the entire domain and the κ value associated with the bottom right simulation is close to the critical value $\kappa = 4$ above which the loops of a CLE intersect themselves, each other, and the domain boundary and the results of the present paper do not apply.

or shown to describe the joint scaling limit of all of the interfaces [BH19, CN06, KS19, LSW04]. In the setting of random planar lattices (i.e., random planar maps), several similar results have been proved [She16c, KMSW19, GM21c, GM21a, GKMW18, LSW17] using the framework from [She16a, DMS21]. Each loop in a CLE_κ looks locally like an SLE_κ, so their phases coincide. In particular, for $\kappa \in (8/3, 4]$ the loops are simple and do not intersect each other or the domain boundary while for $\kappa \in (4, 8)$ the loops are self-intersecting and intersect each other and the domain boundary [RS05]. For $\kappa \in (8/3, 4]$ (resp. $\kappa \in (4, 8)$), the set of points not surrounded by a loop is referred to as its *carpet* (resp. *gasket*). The reason for the distinction in this terminology is that the CLE_κ carpet is a topological Sierpinski carpet, while the CLE_κ gasket can be thought of as a random analog of the Sierpinski gasket.

The purpose of the present paper is to complete the program of showing that one can associate with a CLE_κ carpet (i.e., focusing on $\kappa \in (8/3, 4)$) a canonical geodesic metric. This is a nontrivial problem since the CLE_κ carpet does not contain any nontrivial rectifiable paths. The first step in this program was carried out in [Mil21] in which the tightness and nontriviality of the subsequential limits associated with a particular approximation scheme (Minkowski first passage percolation, MFPP) to the geodesic metric were established. The contribution of this work is that the subsequential limits exist as true limits, and that the limit is canonical in the sense that it is singled out by a simple list of axioms. Our metric will provide a new tool for studying analysis questions on the CLE_κ carpet, beyond its Hausdorff dimension [SSW11, NW11] and canonical conformally covariant measure [MSW22, MS24]. We also expect that our metric describes the scaling limit of the chemical distance for lattice models which converge to CLE_κ.

Our overall approach mirrors the program used to construct the Liouville quantum gravity (LQG) metric [DDDF20, GM21b, DG23]. Specifically, we will: (1) propose a concise set of axioms that any candidate limit must satisfy; (2) prove that all subsequential limits of MFPP obey these axioms; and (3) show that these axioms determine the metric uniquely. Precise statements appear in Section 1.3.

1.2. **Setup.** Throughout the remainder of the present paper, fix $\kappa \in (8/3, 4)$. In the context of CLE_κ, one can refer to either its *nested* or *non-nested* variant. The latter consists of a countable collection of loops which do not cross themselves or each other in a simply connected domain U where no loop surrounds another. In the case of the former, the outermost loops have the law of a non-nested CLE_κ and the law more generally is characterized by the fact that for each loop $\mathscr{L}$ the conditional law of the outermost loops in the component V surrounded by $\mathscr{L}$ given $\mathscr{L}$ and all of the loops not surrounded by $\mathscr{L}$ is that of non-nested CLE_κ in V. By considering a nested CLE_κ in U and then taking an appropriate limit as U increases to $\mathbb{C}$, it is possible to define the so-called whole-plane nested CLE_κ which turns out to be invariant under translation, scaling, rotation, and inversion [KW16] (see also [GMQ21] for a proof of the inversion symmetry for $\kappa \in (4, 8)$). For the remainder of the paper, we work in this setting. Once the metric is constructed here, the corresponding metrics for CLE_κ on general simply connected domains follow directly by restriction and/or local absolute continuity [MT25b].

Let Γ be a whole-plane nested CLE_κ. For $\mathscr{L} \in \Gamma$, we shall write $X(\mathscr{L}) \subset \mathbb{C}$ for the closed subset of points that are surrounded by $\mathscr{L}$ but by no other loop of Γ inside $\mathscr{L}$. We observe that there is a natural mapping $\wp \colon \Gamma \to \mathbb{Z}/2\mathbb{Z}$ which is well-defined up to composition with an automorphism of $\mathbb{Z}/2\mathbb{Z}$ such that the following is true: For each pair of distinct loops $\mathscr{L}_1, \mathscr{L}_2 \in \Gamma$ such that $\mathscr{L}_2$ is surrounded by $\mathscr{L}_1$ but by no other loop of Γ in between $\mathscr{L}_1$ and $\mathscr{L}_2$, we have $\wp(\mathscr{L}_1) \neq \wp(\mathscr{L}_2)$. Conditionally on Γ, let us sample a uniform choice among the two possible representatives of $\wp$. We shall write

$$\Upsilon \stackrel{\mathrm{def}}{=} \bigcup_{\mathscr{L} \in \Gamma : \wp(\mathscr{L}) = 0} X(\mathscr{L}) \quad \text{and} \quad \Upsilon^\dagger \stackrel{\mathrm{def}}{=} \bigcup_{\mathscr{L} \in \Gamma : \wp(\mathscr{L}) = 1} X(\mathscr{L}).$$

We observe that Υ and $\Upsilon^\dagger$ have the same law, and that Γ is almost surely determined by Υ. For an open subset $U \subset \mathbb{C}$ and points $x, y \in U \cap \Upsilon$, we shall write $x \xleftrightarrow{U \cap \Upsilon} y$ (or, when there is no danger of confusion, $x \overset{U}{\leftrightarrow} y$) if x and y lie in the same connected component of $U \cap \Upsilon$. The use of Υ instead of $\coprod_{\mathscr{L} \in \Gamma} X(\mathscr{L})$ has the advantage that it can be naturally embedded into $\mathbb{C}$ as a subset, and that for any $x, y \in \Upsilon$, it is possible to perform a "local rewiring" of Γ so that $x \leftrightarrow y$. (One may imagine Υ as corresponding to the collection of the plus clusters of a critical Ising model on $\mathbb{Z}^2$. Thus, given two vertices with plus spins, it is possible to flip the spins at some other vertices so that these two vertices are contained in the same cluster.)

Suppose that we are given mappings $D_{X(\mathscr{L})}\colon X(\mathscr{L}) \times X(\mathscr{L}) \to \mathbb{R}$ for all $\mathscr{L} \in \Gamma$ with $\wp(\mathscr{L}) = 0$. Then for each $x, y \in \Upsilon$, we shall write

$$D_\Upsilon(x,y) \overset{\text{def}}{=} \begin{cases} D_{X(\mathscr{L})}(x,y) & \text{if } x, y \in X(\mathscr{L}) \text{ for some } \mathscr{L} \in \Gamma \text{ with } \wp(\mathscr{L}) = 0; \\ \infty & \text{otherwise.} \end{cases} \tag{1.1}$$

Throughout the remainder of the present paper, let Γ*,* $\wp$*, and* Υ *be as above, unless otherwise specified.*

Let us now recall some preliminary definitions concerning metric spaces. Let (X, d) be a metric space. For a continuous path $P\colon [0,1] \to X$, we shall refer to

$$\operatorname{len}(P; d) \overset{\text{def}}{=} \sup_{0 = t_0 < t_1 < \cdots < t_n = 1} \sum_{j=1}^{n} d(P(t_{j-1}), P(t_j))$$

as the *d-length* of P, where the supremum is over all partitions of $[0,1]$. We shall refer to d as a *geodesic metric* if for each $x, y \in X$, there exists a continuous path P in X connecting x and y such that $\operatorname{len}(P; d) = d(x,y)$. Let $Y \subset X$ be a subset. Then we shall refer to

$$d(x, y; Y) \overset{\text{def}}{=} \inf_{\substack{P \subset Y \\ P\colon x \to y}} \operatorname{len}(P; d), \quad \forall x, y \in Y,$$

where P ranges over all continuous paths in Y connecting x and y (with the convention that $\inf \emptyset = \infty$), as the *internal metric* on Y. We shall refer to d as a *length metric* if $d(\bullet, \bullet) = d(\bullet, \bullet; X)$.

Let $n \in \mathbb{N}$. We shall write $\mathscr{H}\!\mathit{aus}_n$ for the metric space consisting of nonempty compact subsets of $\mathbb{R}^n$, and equipped with the Hausdorff distance. We shall write $\mathscr{H}\!\mathit{aus}\mathfrak{U}\mathit{ni}_n$ for the metric space consisting of pairs (X, f), where $X \subset \mathbb{R}^n$ is a nonempty compact subset and $f\colon X \to \mathbb{R}$ is a continuous function, and equipped with the metric given by

$$\begin{aligned} d_{\mathscr{H}\!\mathit{aus}\mathfrak{U}\mathit{ni}_n}((X,f),(Y,g)) \overset{\text{def}}{=} & \inf\{\varepsilon > 0 : \forall x \in X,\ \exists y \in Y \text{ s.t. } \|x - y\| \le \varepsilon \text{ and } |f(x) - g(y)| \le \varepsilon\} \\ & \vee \inf\{\varepsilon > 0 : \forall y \in Y,\ \exists x \in X \text{ s.t. } \|x - y\| \le \varepsilon \text{ and } |f(x) - g(y)| \le \varepsilon\}, \\ & \qquad \forall (X,f),(Y,g) \in \mathscr{H}\!\mathit{aus}\mathfrak{U}\mathit{ni}_n. \end{aligned}$$

One verifies easily that $\mathscr{H}\!\mathit{aus}\mathfrak{U}\mathit{ni}_n$ is a separable (but not complete) metric space. However, we recall from [Bil99, Theorem 5.1] that Prokhorov's theorem holds for all (not necessarily separable or complete) metric spaces. The following lemma follows immediately from a similar argument to the argument applied in the proof of the Arzelà-Ascoli theorem (cf. [Mil21]).

Lemma 1.1. *Let* $F \subset \mathscr{H}\!\mathit{aus}\mathfrak{U}\mathit{ni}_n$ *be a subset. Suppose that the following conditions are satisfied:*

(i) For each $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for each* $(X, f) \in F$ *and* $x, y \in X$ *with* $\|x - y\| < \delta$*, we have* $|f(x) - f(y)| < \varepsilon$*.*
(ii) There exists $C > 0$ *such that* $X \subset B_C(0)$ *and* $\sup_{x \in X} |f(x)| \le C$ *for all* $(X, f) \in F$*.*

Then the closure of F *in* $\mathscr{H}\!\mathit{aus}\mathfrak{U}\mathit{ni}_n$ *is compact.*

Let X be a random variable taking values in $\mathscr{H}aus_2$. Then we shall say that the law of X is *absolutely continuous with respect to the sum of the laws of* $X(\mathscr{L})$ *for* $\mathscr{L} \in \Gamma$ if

$$\text{“}\mathbf{P}[X \in A] > 0\text{''} \;\Rightarrow\; \text{“}\mathbf{P}[\text{there exists } \mathscr{L} \in \Gamma \text{ such that } X(\mathscr{L}) \in A] > 0\text{''}, \quad \forall \text{ Borel subset } A \subset \mathscr{H}aus_2. \tag{1.2}$$

Let us now recall the precise definition of the **Minkowski first passage percolation (MFPP)** approximation procedure. Let $\mathrm{Leb}(\bullet)$ denote the Lebesgue measure. For $\varepsilon > 0$, the ε*-MFPP metric* on X is given by

$$D^\varepsilon_X(x,y) \overset{\mathrm{def}}{=} \inf_{P\colon x \to y} \mathrm{Leb}(B_\varepsilon(P)), \quad \forall x, y \in X,$$

where P ranges over all continuous paths in X connecting x and y. To extract a nontrivial limit from the metrics D^ε_X, we need to renormalize. For $\varepsilon > 0$, we shall write $\mathfrak{a}_\varepsilon$ for the median of the random variable

$$\sup_{x,y \in X(\mathscr{L}_1)} D^\varepsilon_{X(\mathscr{L}_1)}(x,y), \tag{1.3}$$

where $\mathscr{L}_1$ denotes the largest loop of Γ contained in $B_1(0)$ and surrounding the origin.

The existence of subsequential limits of $\{(X \times X, \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_X)\}_{\varepsilon > 0}$ follows essentially from [Mil21]. The main challenge in the present paper is to prove uniqueness of the subsequential limit. We adopt the framework used in the uniqueness proofs for the Liouville quantum gravity metric [GM21b, DG23]: we first formulate a list of axioms for the geodesic CLE_κ carpet metric; then we show that every subsequential limit of $\{(X \times X, \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_X)\}_{\varepsilon > 0}$ satisfies these axioms; and finally we prove that at most one metric can satisfy them. Consequently, a unique geodesic CLE_κ carpet metric exists.

1.3. Main results.

Definition 1.2. We shall refer to as a **(strong) geodesic CLE_κ carpet metric** a measurable mapping

$$D\colon \mathscr{H}aus_2 \to \mathscr{H}aus\mathscr{U}ni_4\colon X \mapsto (X \times X, D_X)$$

such that whenever (1.2) holds, the following conditions are satisfied:

(I) **(Geodesic metric)** D_X is almost surely a geodesic metric on X inducing the Euclidean topology.

(II) **(Locality)** For each deterministic open subset $U \subset \mathbb{C}$, the internal metric $D_X(\bullet, \bullet; U \cap X)$ is almost surely determined by $U \cap X$.

(III) **(Translation invariance and scale covariance)** There is a deterministic constant $\theta \in \mathbb{R}$ (which we shall refer to as the **(chemical) distance exponent**) such that for each deterministic $z \in \mathbb{C}$ and $r > 0$,

$$D_{rX+z}(r \bullet + z, r \bullet + z) = r^\theta D_X(\bullet, \bullet) \quad \text{almost surely.}$$

(IV) **(Tightness)** There is a deterministic constant $\alpha_{\mathrm{KC}} > 0$ such that the following is true: Let $K \subset \mathbb{C}$ be a deterministic compact subset. Then it holds with superpolynomially high probability as $A \to \infty$ that

$$D_\Upsilon(x,y) \le A\,|x-y|^{\alpha_{\mathrm{KC}}}, \quad \forall x, y \in K \cap \Upsilon \text{ with } x \leftrightarrow y.$$

(Here, we note that D_Υ is almost surely well-defined.)

(V) **(Monotonicity)** Let $X_1, X_2 \in \mathscr{H}aus_2$ with $X_1 \subset X_2$. Then $D_{X_1}(x,y) \ge D_{X_2}(x,y)$ for all $x, y \in X_1$.

(The usage of the notation α_{KC} above is that the exponent is derived in [Mil21] using an argument of the same flavor as in the proof of the Kolmogorov-Centsov theorem.)

In the remainder of the present subsection, let X be a random variable taking values in $\mathscr{H}\!aus_2$ that satisfies (1.2).

Our main results establish the existence and uniqueness of the strong geodesic CLE_κ carpet metric.

Theorem 1.3 (Convergence of MFPP). *For each sequence of positive real numbers ε's tending to zero, there exists a strong geodesic* CLE_κ *carpet metric D and a subsequence along which $(X \times X, \mathfrak{a}_\varepsilon^{-1} D_X^\varepsilon)$ converges in probability to $(X \times X, D_X)$ with respect to $\mathscr{H}\!aus\mathscr{U}\!ni_4$.*

Theorem 1.4 (Uniqueness). *Let D and $\widetilde{D}$ be two strong geodesic* CLE_κ *carpet metrics with the same distance exponent θ. Then there is a deterministic constant $M > 0$ such that $\widetilde{D}_X = M D_X$ almost surely.*

Moreover, we prove that the distance exponent θ is uniquely determined by κ.

Theorem 1.5 (Uniqueness of the distance exponent). *There is a unique $\theta = \theta(\kappa)$ for which the strong geodesic* CLE_κ *carpet metric with distance exponent θ exists.*

Finally, we consider the Hausdorff dimension of the carpet and geodesics. We recall that for a metric space (X, d),

$$\inf \left\{ \alpha \geq 0 : \inf \left\{ \sum_{j=1}^{\infty} \operatorname{diam}(U_j; d)^\alpha : X \subset \bigcup_{j=1}^{\infty} U_j \right\} = 0 \right\}$$

is called the *Hausdorff dimension* of (X, d).

Theorem 1.6 (Hausdorff dimension). *Let D be a strong geodesic* CLE_κ *carpet metric with distance exponent θ. Then:*

(i) $\theta \dim(X; D_X) = \dim(X)$ *almost surely, where* $\dim(\bullet; D_X)$ *(resp.* $\dim(\bullet)$*) denotes the Hausdorff dimension with respect to D_X (resp. the Euclidean metric). (Recall from [SSW11, NW11] that* $\dim(X) = 1 + 2/\kappa + 3\kappa/32$ *almost surely.)*

(ii) Almost surely, $\dim(P) = \theta$ *for all (nontrivial) D_X-geodesics P. In particular, it follows that $1 < \theta \leq 1 + 2/\kappa + 3\kappa/32 < 2$. (In fact, by [Hug23], we have a stronger upper bound that $\theta < 1 + \kappa/8$.)*

1.4. **A weaker list of axioms.** It is straightforward to verify that any subsequential limit of $\{(X \times X, \mathfrak{a}_\varepsilon^{-1} D_X^\varepsilon)\}_{\varepsilon > 0}$ satisfies Axioms (I), (II), (IV), and (V) of Definition 1.2, as well as the translation-invariance portion of Axiom (III). However, the scale-covariance portion of Axiom (III) is much harder to verify. To address this, we replace it with a weaker condition that is still sufficient to prove uniqueness.

Definition 1.7. We shall refer to as a *weak geodesic* CLE_κ *carpet metric* a measurable mapping

$$D \colon \mathscr{H}\!aus_2 \to \mathscr{H}\!aus\mathscr{U}\!ni_4 \colon X \mapsto (X \times X, D_X)$$

such that whenever (1.2) holds, the following conditions are satisfied:

(I) **(Geodesic metric)** D_X is almost surely a geodesic metric on X inducing the Euclidean topology.

(II) **(Locality)** For each deterministic open subset $U \subset \mathbb{C}$, the internal metric $D_X(\bullet, \bullet; U \cap X)$ is almost surely determined by $U \cap X$.

(III) **(Translation invariance)** For each deterministic point $z \in \mathbb{C}$, we have $D_{X+z}(\bullet + z, \bullet + z) = D_X(\bullet, \bullet)$ almost surely.

(IV) **(Tightness across scales)** There is a family of deterministic constants $\{\mathfrak{c}_r\}_{r>0}$ satisfying the following conditions:

(i) **(Upper bound)** There is a deterministic constant $\alpha_{\mathrm{KC}} > 0$ such that the following is true: Let $K \subset \mathbb{C}$ be a deterministic compact subset. Then for each $r > 0$, it holds with superpolynomially high probability as $A \to \infty$, at a rate which is uniform in r, that

$$\mathfrak{c}_r^{-1} D_\Upsilon(x,y) \le A \left|\frac{x-y}{r}\right|^{\alpha_{\mathrm{KC}}}, \quad \forall x,y \in rK \cap \Upsilon \text{ with } x \leftrightarrow y. \tag{1.4}$$

(ii) **(Lower bound)** Let $K_1, K_2 \subset \mathbb{C}$ be deterministic and disjoint compact subsets. Then the laws of the random variables

$$\mathfrak{c}_r / D_\Upsilon(rK_1 \cap \Upsilon, rK_2 \cap \Upsilon)$$

for $r > 0$ are tight.

(iii) There is a deterministic constant $\mathfrak{K} > 0$ such that

$$\varepsilon^2 \le \mathfrak{c}_{\varepsilon r}/\mathfrak{c}_r \le \mathfrak{K}\varepsilon, \quad \forall r > 0,\ \forall \varepsilon \in (0,1). \tag{1.5}$$

(V) **(Monotonicity)** Let $X_1, X_2 \in \mathfrak{Haus}_2$ with $X_1 \subset X_2$. Then $D_{X_1}(x,y) \ge D_{X_2}(x,y)$ for all $x, y \in X_1$.

Theorem 1.8. *For each sequence of positive real numbers ε's tending to zero, there exists a weak geodesic* CLE_κ *carpet metric D and a subsequence along which $(X \times X, \mathfrak{a}_\varepsilon^{-1} D_X^\varepsilon)$ converges in probability to $(X \times X, D_X)$ with respect to $\mathfrak{HausUni}_4$.*

Theorem 1.9. *Let D and $\widetilde{D}$ be two weak geodesic* CLE_κ *carpet metrics with the same distance exponent θ. Then there is a deterministic constant $M > 0$ such that $\widetilde{D}_X = M D_X$ almost surely.*

Let us now explain why Theorems 1.5, 1.8 and 1.9 are sufficient to establish our main results, Theorems 1.3 and 1.4.

Lemma 1.10. *Every weak geodesic* CLE_κ *carpet metric is a strong geodesic* CLE_κ *carpet metric.*

Proof of Lemma 1.10 assuming Theorem 1.9. For $\lambda > 0$, write $D_X^{(\lambda)}(\bullet,\bullet) \overset{\mathrm{def}}{=} D_{\lambda X}(\lambda\bullet,\lambda\bullet)$. We *claim* that $D^{(\lambda)}$ is a weak geodesic CLE_κ carpet metric with the same scaling constants as D. Indeed, it is clear that $D^{(\lambda)}$ satisfies Definition 1.7, Axioms (I), (II), (III), and (V). Moreover, since

$$\mathfrak{c}_r^{-1} D_{rX}^{(\lambda)}(r\bullet, r\bullet) = \frac{\mathfrak{c}_{r\lambda}}{\mathfrak{c}_r}\mathfrak{c}_{r\lambda}^{-1} D_{r\lambda X}(r\lambda\bullet, r\lambda\bullet) \quad \text{and} \quad \begin{cases} \lambda^2 \le \mathfrak{c}_{r\lambda}/\mathfrak{c}_r \le \mathfrak{K}\lambda & \text{if } \lambda \in (0,1); \\ \mathfrak{K}^{-1}\lambda \le \mathfrak{c}_{r\lambda}/\mathfrak{c}_r \le \lambda^2 & \text{if } \lambda \ge 1, \end{cases}$$

it follows that $D^{(\lambda)}$ satisfies Definition 1.7, Axiom (IV). Thus, by Theorem 1.9, there exists a deterministic constant $M_\lambda > 0$ such that $D_X^{(\lambda)} = M_\lambda D_X$ almost surely. For $\lambda_1, \lambda_2 > 0$,

$$M_{\lambda_1\lambda_2} D_X(\bullet,\bullet) = D_X^{(\lambda_1\lambda_2)}(\bullet,\bullet) = D_{\lambda_2 X}^{(\lambda_1)}(\lambda_2\bullet,\lambda_2\bullet) = M_{\lambda_1} D_{\lambda_2 X}(\lambda_2\bullet,\lambda_2\bullet) = M_{\lambda_1} M_{\lambda_2} D_X(\bullet,\bullet),$$

which implies that $M_{\lambda_1} M_{\lambda_2} = M_{\lambda_1\lambda_2}$. Since the mapping $X \mapsto (X \times X, D_X)$ is measurable, it follows that the assignment $\lambda \mapsto M_\lambda$ is measurable. Thus, we conclude that there exists $\theta \in \mathbb{R}$ such that $M_\lambda = \lambda^\theta$ for all $\lambda > 0$, i.e., for each $\lambda > 0$, we have $D_{\lambda X}(\lambda\bullet,\lambda\bullet) = \lambda^\theta D_X(\bullet,\bullet)$ almost surely. This completes the proof. □

Proof of Theorems 1.3 and 1.4 assuming Theorems 1.5, 1.8 and 1.9. Theorem 1.3 follows immediately from Theorem 1.8 and Lemma 1.10. By Theorem 1.3, there exists a strong geodesic CLE_κ carpet metric with distance exponent θ. By Theorem 1.5, this θ lies between one and two and is the only possible distance exponent for a strong geodesic CLE_κ carpet metric. Thus, we conclude that every strong geodesic CLE_κ carpet metric satisfies (1.5) with $\{r^\theta\}_{r>0}$ in place of $\{\mathfrak{c}_r\}_{r>0}$, hence is a weak geodesic CLE_κ carpet metric. Thus, Theorem 1.4 follows immediately from Theorem 1.9. $\square$

1.5. **Related works.**

Conformal covariance. In our future paper [MT25b], we will prove that, for each simply connected domain $U \subset \mathbb{C}$, there exists a measurable mapping

$$D^U : \mathscr{Haus}_2 \to \mathscr{HausUni}_4 : X \mapsto (X \times X, D^U_X)$$

satisfying the following conditions:

(I) **(Length metric)** Let Γ be a non-nested CLE_κ in U. Write X for the carpet of Γ. Then D^U_X is almost surely a length metric on $X \setminus \partial U$ inducing the Euclidean topology.

(II) **(Conformal covariance)** Let $\phi\colon U \mapsto \phi(U)$ be a deterministic conformal mapping. Then, almost surely,

$$D^{\phi(U)}_{\phi(X)}(\phi(x), \phi(y)) = \inf_{\substack{P \subset X \setminus \partial U \\ P\colon x \to y}} \int_0^{\mathrm{len}(P; D^U_X)} \left|\phi'(P(t))\right|^\theta \,\mathrm{d}t, \quad \forall x, y \in X \setminus \partial U,$$

where P ranges over all continuous paths in $X \setminus \partial U$ from x to y parameterized by D^U_X-length.

(III) **(Locality)** Let $V \subset U$ be a deterministic simply connected subdomain. Then

$$D^U_X(\bullet, \bullet; V^\star \cap X) = D^{V^\star}_{V^\star \cap X}(\bullet, \bullet) \quad \text{almost surely},$$

where $V^\star \overset{\mathrm{def}}{=} V \setminus \overline{\bigcup_{\mathscr{L} \in \Gamma : \mathscr{L} \not\subset V} \mathrm{int}(\mathscr{L})}$.

Confluence of geodesics. Confluence of geodesics means that if you take two shortest paths that start at different points but head to the same destination, they usually join together before they get there. This does not happen for usual smooth Riemannian manifolds. It was proved for the Brownian map (equivalently, for $\sqrt{8/3}$-LQG) in [LG10]. The subcritical LQG metric also has this property [GM20a], and that fact was crucial for the uniqueness result in [GM21b]. Confluence has likewise been shown for the critical and supercritical LQG metrics [DG24] and for the directed landscape [Dau25]. In our forthcoming paper [MT25a], we will show that geodesics in the geodesic CLE_κ carpet metric also satisfy the confluence property.

The LQG metric on CLE carpets. Set $\gamma \overset{\mathrm{def}}{=} \sqrt{\kappa}$. By a similar argument to the argument applied in [Mil21], the present paper, and [MT25b], one may prove that, for each simply connected domain $U \subset \mathbb{C}$, there exists a measurable mapping

$$D^U : \mathcal{D}'(U) \times \mathscr{Haus}_2 \to \mathscr{HausUni}_4 : (\Phi, X) \mapsto (X \times X, D^U_{\Phi, X})$$

(where $\mathcal{D}'(U)$ denotes the space of Schwartz distributions on U) satisfying the following conditions:

(I) **(Length metric)** Let Φ be a Gaussian free field on U. Let Γ be an independent non-nested CLE_κ in U. Write X for the carpet of Γ. Then $D^U_{\Phi,X}$ is almost surely a length metric on $X \setminus \partial U$ inducing the Euclidean topology.

(II) **(Weyl scaling)** There is a deterministic constant $\zeta = \zeta(\kappa) \in \mathbb{R}$ such that the following is true: Let $f\colon U \to \mathbb{R}$ be a random continuous function that is independent of X. Then, almost surely,

$$D^U_{\Phi+f, X}(x, y) = \inf_{\substack{P \subset X \setminus \partial U \\ P\colon x \to y}} \int_0^{\mathrm{len}(P; D^U_{\Phi,X})} \mathrm{e}^{\zeta f(P(t))} \,\mathrm{d}t, \quad \forall x, y \in X \setminus \partial U, \tag{1.6}$$

where P ranges over all continuous paths in $X \setminus \partial U$ from x to y parameterized by $D^U_{\Phi,X}$-length.

(III) **(Conformal covariance)** Let $\phi\colon U \to \phi(U)$ be a deterministic conformal mapping. Then

$$D^{\phi(U)}_{\phi_*(\Phi),\phi(X)}(\phi(\bullet),\phi(\bullet)) = D^U_{\Phi,X}(\bullet,\bullet) \quad \text{almost surely,}$$

where $\phi_*(\Phi) \stackrel{\text{def}}{=} \Phi \circ \phi^{-1} + Q\log(|(\phi^{-1})'|)$.

(IV) **(Locality)** Let $V \subset U$ be a deterministic simply connected subdomain. Then

$$D^U_{\Phi,X}(\bullet,\bullet; V^\star \cap X) = D^{V^\star}_{\Phi|_{V^\star},V^\star\cap X}(\bullet,\bullet) \quad \text{almost surely,}$$

where $V^\star \stackrel{\text{def}}{=} V \setminus \overline{\bigcup_{\mathcal{L}\in\Gamma:\mathcal{L}\not\subset V}\operatorname{int}(\mathcal{L})}$.

We expect that the above metric describes the scaling limit of the chemical distance for statistical mechanics models on random planar maps.

CLE(4) carpets and non-simple CLE gaskets. We expect that the the analogs of our main results still hold for the critical value $\kappa = 4$. However, in this case, additional challenges need to be addressed since the tightness argument given in [Mil21] does not apply and since the four-arm exponent for CLE_4 is equal to 2. When $\kappa \in (4,8)$, the analogs of our main results are also true [AMY25b, MY25]. However, in this case, the proofs are completely different due to the fact that the loops of a CLE_κ in this case intersect each other. The construction of the canonical metric for CLE_8 (i.e., space-filling SLE_8) and how it arises as the scaling limit of the chemical distance on the uniform spanning tree has been established in [BCK17, HS18]. In this case, the metric space has the topology of the a tree where the branches are given by the SLE_2 branches of a CLE_8 so to define a metric one only needs to assign a length to these branches. This length is supplied by the natural parameterization of SLE_2 [LS11, LR15] and the convergence of loop-erased random walk to SLE_2 with respect to the natural parameterization [LV21] plays a key role in [BCK17].

Other random fractal metrics. There are also several other papers that prove tightness and/or uniqueness for different random fractal metrics. Examples include the critical long-range percolation metric [DFH23] and exponential metrics built from higher-dimensional log-correlated Gaussian fields [DGZ23]. We also note the *directed landscape*, a random directed metric on $\mathbb{R}^2$ connected to the KPZ universality class [DOV22].

1.6. **Open problems.**

The Distance Exponent.

Conjecture 1.11. *The assignment $(8/3,4) \mapsto \mathbb{R}\colon \kappa \mapsto \theta(\kappa)$ is continuous and strictly increasing. Moreover, $\theta(\kappa) \to 1$ as $\kappa \to (8/3)^+$.*

As mentioned earlier, there are similarities between the geodesic CLE_κ carpet metric and the LQG metric. We suspect that computing the exact value of $\theta(\kappa)$ is highly nontrivial, in analogy with the difficulty of computing the exact value of the LQG dimension d_γ. However, we expect that Conjecture 1.11 is tractable (cf. [DG20]). Let us also mention that based on Monte Carlo simulations, [DZG+10] predicts that $\theta(\kappa) = (9\kappa+8)(\kappa+8)/(128\kappa)$.

Geodesics. Let X be the carpet of a non-nested CLE_κ. Given X, let x and y be independent samples from the canonical conformally covariant measure μ_X (renormalized to be a probability measure) (cf. [MS24]).

Conjecture 1.12. *There is almost surely a unique D_X-geodesic connecting x and y.*

Conjecture 1.12 is related to the following.

Conjecture 1.13. *For each deterministic $a \geq 0$, we have $D_X(x, y) \neq a$ almost surely.*

We remark that the analogue of Conjecture 1.13 for the LQG metric is relatively easy to prove, since one can perturb the Gaussian free field by adding a smooth function without significantly altering its probability density. In contrast, in the case of CLE, the randomness is inherently discrete, which makes the conjecture far from obvious (see also [DFH25]).

Problem 1.14. *Does the D_X-geodesic connecting x and y almost surely touch the CLE_κ loops?*

Stable Carpets. Recent works of Le Gall and Miermont [LGM11] and Curien, Miermont, and Riera [CMR25] proved that, as the number of vertices tends to infinity, the rescaled Boltzmann planar maps of index $\alpha \in (1, 2)$ converge to an explicit metric measure space $(\mathcal{S}_\alpha, D_\alpha, \mu_\alpha)$, which is called the *$\alpha$-stable carpet/gasket*, with respect to the Gromov-Hausdorff-Prokhorov topology. Boltzmann planar maps arise naturally. For instance, the carpet (resp. gasket) of a random planar map decorated by a critical Ising model (resp. a critical percolation model) is a Boltzmann planar map of index 11/6 (resp. 7/6). This links the stable carpets/gaskets and the LQG metric on CLE carpets/gaskets. More precisely:

Conjecture 1.15. *Set $\alpha \overset{\text{def}}{=} 1/2 + 4/\kappa$. Let $(\mathfrak{D}, \Phi, 0, \infty)$ be a γ-LQG disk. Let Γ be an independent non-nested CLE_κ in $\mathfrak{D}$. Write X for the carpet of Γ. Then, given $\mu_{\Phi,X}(X) = 1$ (cf. [MSW22]), the metric measure space $(X, D_{\Phi,X}, \mu_{\Phi,X})$ is conditionally a disk variant of the α-stable carpet.*

Conjecture 1.16. *Recall from (1.6) the Weyl scaling exponent ζ. We have $\zeta = \gamma/4$.*

The intuitive reason for Conjecture 1.16 is as follows: We may scale the measure $\mu_{\Phi,X}$ by a factor of $C^\alpha > 0$ by adding $(2/\gamma)\log(C)$ to Φ. This results in scaling the metric $D_{\Phi,X}$ by $C^{2\zeta/\gamma}$. The Hausdorff dimension of the α-stable carpet is almost surely equal to 2α. Since

> "the dimension times the scaling exponent of the metric is equal to the scaling exponent of the measure",

we expect that $(2\alpha) \cdot (2\zeta/\gamma) = \alpha$.

1.7. **Organization of the paper.** The remainder of the paper is structured as follows. In Section 2, we establish notational conventions and review the necessary background material. This section also contains several quantitative estimates that will be used repeatedly throughout the paper. In Section 3, we define the concept of local metrics on CLE carpets, prove that any two such metrics are bi-Lipschitz equivalent, and show that the local metric is almost surely determined by the carpet itself. Section 4 is devoted to proving the existence of a weak geodesic CLE_κ carpet metric (Theorem 1.8). In Section 5, we establish that the distance exponent θ is uniquely determined by κ (Theorem 1.5) and compute the Hausdorff dimensions of both the carpet and its geodesics (Theorem 1.6). Finally, Section 6, Section 7, and Section 8 contain the proof of the uniqueness of weak geodesic CLE_κ carpet metrics (Theorem 1.9).

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2. Preliminaries

2.1. Notation and conventions.

Numbers. The notation $\mathbb{C}$ (resp. $\mathbb{R}$; $\mathbb{Q}$) will be used to denote the field of complex (resp. real; rational) numbers. The notation $\mathbb{Z}$ (resp. $\mathbb{N}$) will be used to denote the set of integers (resp. positive integers). For $a < b$, we shall write $[a,b]_{\mathbb{Z}} \stackrel{\text{def}}{=} [a,b] \cap \mathbb{Z}$.

Geometry. The notation $\mathbb{H}$ (resp. $\mathbb{D}$) will be used to denote the upper half-plane (resp. the open unit disk). For $z \in \mathbb{C}$ and $0 < s < t$, we shall write $B_t(z)$ for the open Euclidean ball of radius t centered at z; we shall write $A_{s,t}(z)$ for the open Euclidean annulus of inner radius s and outer radius t centered at z. We shall refer to as a *rectangle* (resp. *square*) an open subset of the form $(a, a+s) \times (b, b+t) \subset \mathbb{C}$ (resp. $(a, a+t) \times (b, b+t) \subset \mathbb{C}$), where $a, b \in \mathbb{R}$ and $s, t > 0$.

Let (X, d) be a metric space. Then we shall write $\operatorname{diam}(X; d) \stackrel{\text{def}}{=} \sup_{x,y \in X} d(x,y)$. For $x \in X$ and $r > 0$, we shall write $B_r(x; d) \stackrel{\text{def}}{=} \{y \in X : d(x,y) < r\}$ and $\overline{B}_r(x; d) \stackrel{\text{def}}{=} \{y \in X : d(x,y) \leq r\}$.

We shall refer to as a *conformal mapping* a holomorphic bijection between two Riemann surfaces. We shall refer to two Riemann surfaces as *conformally equivalent* if they are biholomorphic. We shall refer to as a *domain* a connected open subset of $\mathbb{C}$. We shall refer to as a *simply connected domain* an open subset of $\mathbb{C}$ that is conformally equivalent to $\mathbb{D}$. We shall refer to as a *doubly connected domain* an open subset of $\mathbb{C}$ that is conformally equivalent to $A_{s,t}(0)$ for some $0 < s < t$.

We shall refer to as a *dyadic square* a square with side length 2^n and vertices in $(2^n\mathbb{Z})^2$ for some $n \in \mathbb{Z}$. We shall write $\mathfrak{DySq}$ for the collection of dyadic squares. We shall refer to a domain $U \subset \mathbb{C}$ as *dyadic* if there exists a finite collection $\mathcal{S}$ of dyadic squares such that U is the interior of $\bigcup_{S \in \mathcal{S}} \overline{S}$. We shall write $\mathfrak{DyDo}$ for the collection of dyadic domains.

We shall refer to as a *path* a continuous mapping from $[0,1]$ to $\mathbb{C}$. We shall refer to as a *parameterized loop* a continuous mapping from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{C}$. The group of orientation-preserving homeomorphisms of $\mathbb{R}/\mathbb{Z}$ acts on the space of parameterized loops via pre-composition. We shall refer to as a *loop* an orbit of this action. We shall write $\mathfrak{Loop}$ for the metric space consisting of loops in $\mathbb{C}$, and equipped with the metric given by

$$d_{\mathfrak{Loop}}(\mathfrak{L}_1, \mathfrak{L}_2) \stackrel{\text{def}}{=} \inf_{P_1, P_2} \sup_{t \in \mathbb{R}/\mathbb{Z}} |P_1(t) - P_2(t)|, \quad \forall \mathfrak{L}_1, \mathfrak{L}_2 \in \mathfrak{Loop},$$

where P_1 (resp. P_2) ranges over all parameterizations of $\mathfrak{L}_1$ (resp. $\mathfrak{L}_2$). One verifies immediately that $\mathfrak{Loop}$ is a separable and complete metric space.

Let $U \subset \mathbb{C}$ be an open subset. Then we shall refer to as a *U-excursion* a path $P\colon [0,1] \to \overline{U}$ such that $P(0), P(1) \in \partial U$ and $P((0,1)) \subset U$. Let $K \subset U$ be a compact subset. Then we shall refer to as a *(U,K)-excursion* a U-excursion that intersects K. Let $P\colon [0,1] \to \mathbb{C}$ be a path. Then we shall refer to as a *U-excursion of P* times $0 \leq s < t \leq 1$ such that $P|_{[s,t]}$ forms a U-excursion. We shall refer to as a *(U,K)-excursion of P* times $0 \leq s < \sigma < \tau < t \leq 1$ such that $P|_{[s,t]}$ forms a U-excursion, σ is the first time after s at which P hits K, and τ is the last time before t at which P hits K. Let $\mathfrak{L} \subset \mathbb{C}$ be a loop. Then we shall refer to as a *complementary (U,K)-excursion of $\mathfrak{L}$* a connected arc of $\mathfrak{L}$ that does not overlap with any (U,K)-excursion of $\mathfrak{L}$ and is not contained in any larger connected arc of $\mathfrak{L}$ with such property.

Probability. Let $\{E_t\}_{t>0}$ be a family of events. Then:

- We shall say that E_t occurs with *polynomially high probability* as $t \to 0$ (resp. $t \to \infty$) if there exists $\alpha > 0$ and $C > 0$ such that

$$\mathbf{P}[E_t] \geq 1 - Ct^{\alpha} \quad (\text{resp. } \mathbf{P}[E_t] \geq 1 - Ct^{-\alpha}), \quad \forall t > 0. \tag{2.1}$$

- We shall say that E_t occurs with *superpolynomially high probability* as $t \to 0$ (resp. $t \to \infty$) if for each $\alpha > 0$, there exists $C = C(\alpha) > 0$ such that (2.1) holds.

2.2. **Simple Conformal Loop Ensembles.** In the present subsection, we review the constructions and properties of simple conformal loop ensembles.

Definition 2.1. For each simply connected domain $U \subset \mathbb{C}$, a *non-nested simple conformal loop ensemble (CLE)* in U is a random collection Γ_U of non-nested disjoint simple loops in U satisfying the following conditions:

- **(Conformal invariance)** For each conformal mapping $\phi\colon U \to \phi(U)$, the collection $\phi(\Gamma_U)$ is a non-nested CLE in $\phi(U)$.
- **(Restriction)** Let $V \subset U$ be a simply connected subdomain. Write

$$V^\star \stackrel{\text{def}}{=} V \setminus \overline{\bigcup_{\mathscr{L} \in \Gamma_U : \mathscr{L} \not\subset V} \operatorname{int}(\mathscr{L})}.$$

Then the restrictions of Γ_U to the connected components of $V^\star$ are conditionally independent non-nested CLEs given $V^\star$.

We review two different constructions of simple CLE, one using Brownian loop-soup [SW12], the other using boundary conformal loop ensembles (BCLE) [MSW17], both of which will be relevant for this work.

Brownian loop-soup. The *Brownian loop measure* μ^{loop} is a σ-finite measure on the metric space $\mathscr{Loop}$ which is uniquely (up to a multiplicative constant) characterized by the following property [LW04, Wer08]: For each conformal mapping $\phi\colon U \to \phi(U)$, if we write μ_U^{loop} for the restriction of μ^{loop} on $\{\mathscr{L} \in \mathscr{Loop} : \mathscr{L} \subset U\}$, then $\phi_*(\mu_U^{\text{loop}}) = \mu_{\phi(U)}^{\text{loop}}$.

Let $c \in (0,1]$. A *Brownian loop-soup* of intensity c in U is a Poisson point process with intensity measure given by $c\mu_U^{\text{loop}}$. Let Ξ_U be a Brownian loop-soup of intensity c in U. Write

$$\kappa(c) \stackrel{\text{def}}{=} \frac{13 - c - \sqrt{(1-c)(25-c)}}{3} \in (8/3, 4].$$

Write Γ_U for the collection of the outer boundaries of the outermost clusters of the Brownian loops of Ξ_U. Then Γ_U is called a *non-nested* CLE_κ in U.

The following is proved in [SW12].

Theorem 2.2. *Each non-nested* CLE_κ *for* $\kappa \in (8/3, 4]$ *is a non-nested simple CLE. Conversely, each non-nested simple CLE is a non-nested* CLE_κ *for some* $\kappa \in (8/3, 4]$.

CLE percolation. Let $\kappa \in (2, 4]$; $\rho \in (-2, \kappa - 4)$; Γ a branching $\mathrm{SLE}_\kappa(\rho; \kappa - 6 - \rho)$ process in $\mathbb{H}$ starting from 0 and targeting all other boundary points. If we equip Γ with the orientation from 0 towards other boundary points, then we shall refer to Γ as a *BCLE*$_\kappa^{\circlearrowright}(\rho)$. Moreover, we shall refer to the clockwise (resp. counterclockwise) loops formed from Γ as the *true loops* (resp. *false loops*) of the $\mathrm{BCLE}_\kappa^{\circlearrowright}(\rho)$. (Note that any point $z \in \mathbb{H} \setminus \Gamma$ is surrounded by either a true loop or a false loop.) We shall refer to a $\mathrm{BCLE}_\kappa^{\circlearrowright}(\kappa - 6 - \rho)$ as a *BCLE*$_\kappa^{\circlearrowleft}(\rho)$; to the true loops (resp. false loops) of $\mathrm{BCLE}_\kappa^{\circlearrowright}(\kappa - 6 - \rho)$ as the *false loops* (resp. *true loops*) of $\mathrm{BCLE}_\kappa^{\circlearrowleft}(\rho)$.

Let $\kappa' \in (4,8)$; $\rho' \in [\kappa'/2-4, \kappa'/2-2]$; Γ' a branching $\mathrm{SLE}_{\kappa'}(\rho'; \kappa'-6-\rho')$ process in $\mathbb{H}$ starting from 0 and targeting all other boundary points. If we equip Γ' with the orientation from 0 towards other boundary points, then we shall refer to Γ' as a $BCLE^{\circlearrowright}_{\kappa'}(\rho')$. Moreover, we shall refer to the clockwise (resp. counterclockwise) loops formed from Γ' as the *true loops* (resp. *false loops*) of the $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')$. (Note that any point $z \in \mathbb{H} \setminus \Gamma'$ is surrounded by either a true loop or a false loop.) We shall refer to a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\kappa'-6-\rho')$ as a $BCLE^{\circlearrowleft}_{\kappa'}(\rho')$; to the true loops (resp. false loops) of $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\kappa'-6-\rho')$ as the *false loops* (resp. *true loops*) of $\mathrm{BCLE}^{\circlearrowleft}_{\kappa'}(\rho')$.

Note that by "forgetting the orientations", a $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)$ and a $\mathrm{BCLE}^{\circlearrowleft}_{\kappa}(\rho)$ have the same law, which we shall simply denote by $BCLE_{\kappa}(\rho)$. In a similar vein, we shall use the notation $BCLE_{\kappa'}(\rho')$.

Let $\kappa \in (2,4]$; $\rho \in (-2, \kappa-4)$. Then:

(i) The law of a $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)$ is invariant under any conformal automorphism of $\mathbb{H}$.
(ii) The image of a $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)$ under any anticonformal automorphism of $\mathbb{H}$ is a $\mathrm{BCLE}^{\circlearrowleft}_{\kappa}(\rho)$.
(iii) The orientation-reversal of a $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)$ is a $\mathrm{BCLE}^{\circlearrowleft}_{\kappa}(\rho)$.
(iv) The collection of loops of a $\mathrm{BCLE}_{\kappa}(\rho)$ is locally finite.

Let $\kappa' \in (4,8)$; $\rho' \in [\kappa'/2-4, \kappa'/2-2]$. Then:

(i) The law of a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')$ is invariant under any conformal automorphism of $\mathbb{H}$.
(ii) The image of a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')$ under any anticonformal automorphism of $\mathbb{H}$ is a $\mathrm{BCLE}^{\circlearrowleft}_{\kappa'}(\rho')$.
(iii) The orientation-reversal of a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')$ is a $\mathrm{BCLE}^{\circlearrowleft}_{\kappa'}(\rho')$.
(iv) The collection of loops of a $\mathrm{BCLE}_{\kappa'}(\rho')$ is locally finite.

Let $\kappa \in (8/3, 4)$; $\beta \in [-1,1]$. Then we shall refer to as the *trunk* of an $\mathrm{SLE}^{\beta}_{\kappa}(\kappa-6)$ the locus of its force point.

The following follows from [MSW17, MSW22].

Theorem 2.3. *Let*

$$\kappa \in (8/3,4); \quad \kappa' \stackrel{\text{def}}{=} 16/\kappa; \quad \rho' \in [\kappa'-6, 0]; \quad \rho_{\mathrm{R}} \stackrel{\text{def}}{=} -(\kappa/4)(\rho'+2); \quad \rho_{\mathrm{L}} \stackrel{\text{def}}{=} \kappa/2-4-\rho_{\mathrm{R}}.$$

Let Γ' be a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')$ in $\mathbb{H}$. Inside of each connected component of each true (resp. false) loop of Γ', we sample an independent $\mathrm{BCLE}^{\circlearrowleft}_{\kappa}(\rho_{\mathrm{R}})$ (resp. $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho_{\mathrm{L}})$), and we write Γ for the collection of all the true loops of all these simple BCLEs. Write η' for the loop from 0 to 0 that traces the domain boundary counterclockwise except that each time we encounter a true loop of Γ' for the first time, we trace the entire loop before continuing. Write η for the loop from 0 to 0 that traces η' except that each time we encounter a true loop of Γ for the first time, we trace the entire loop before continuing. For each $x \in \partial\mathbb{H}$, write η'_x (resp. η_x) for η' (resp. η) parameterized by the half-plane capacity seen from x. Let $\beta = \beta(\kappa', \rho') \in [-1,1]$ be so that

$$\frac{1-\beta}{2} = \frac{\sin(\pi\rho'/2)}{\sin(\pi\rho'/2) - \sin(\pi(\kappa'-\rho')/2)}.$$

Then for each $x \in \partial\mathbb{H}$, η_x has the law of an $\mathrm{SLE}^{\beta}_{\kappa}(\kappa-6)$ targeting x and η'_x is the trunk of η_x; moreover, η_x is equal to the path that traces η'_x except that each time we encounter a true loop of Γ for the first time, we trace the entire loop before continuing. In particular, we conclude that an $\mathrm{SLE}^{\beta}_{\kappa}(\kappa-6)$ is almost surely generated by a continuous curve, and its trunk has the law of an $\mathrm{SLE}_{\kappa'}(\rho'; \kappa'-6-\rho')$.

Corollary 2.4. *Consider the following algorithm:*

- *One samples a $\mathrm{BCLE}_{\kappa'}(0)$. Then, inside each connected component of each true loop of this $\mathrm{BCLE}_{\kappa'}(0)$, one samples an independent $\mathrm{BCLE}_{\kappa}(-\kappa/2)$.*

⋮

- *Inside each connected component of each false loop of the* $\mathrm{BCLE}_{\kappa'}(0)$*'s and* $\mathrm{BCLE}_\kappa(-\kappa/2)$*'s of the previous step, one samples an independent* $\mathrm{BCLE}_{\kappa'}(0)$*. Then, inside each connected component of each true loop of these* $\mathrm{BCLE}_{\kappa'}(0)$*'s, one samples an independent* $\mathrm{BCLE}_\kappa(-\kappa/2)$*.*

⋮

Then the collection of all the true loops of all the $\mathrm{BCLE}_\kappa(-\kappa/2)$*'s has the law of a non-nested* CLE_κ*. (Here, we note that the collection of all the outermost true loops of all the* $\mathrm{BCLE}_{\kappa'}(0)$*'s has the law of a non-nested* $\mathrm{CLE}_{\kappa'}$*.)*

Arm events. Finally, we collect several results concerning arm events for CLE.

We shall write

$$\alpha_{4\mathrm{A}} \stackrel{\mathrm{def}}{=} \frac{(12-\kappa)(4+\kappa)}{8\kappa} > 2. \tag{2.2}$$

for the *four-arm exponent* for CLE_κ.

Lemma 2.5. *Let* U *be either* $\mathbb{C}$ *or a simply connected domain. Let* Γ *be a nested* CLE_κ *in* U*. Then:*

(i) For each $\alpha \in (0, \alpha_{4\mathrm{A}})$*, there exists* $C = C(\alpha) > 0$ *such that the following is true: Let* $z \in U$ *and* $0 < s < t < \mathrm{dist}(z, \partial U)$*. Then it holds with probability at least* $1 - C(s/t)^\alpha$ *that there are at most two connected arcs of* Γ *in* $A_{s,t}(z)$ *connecting its inner and outer boundaries.*

(ii) For each $\alpha \in (0, \alpha_{4\mathrm{A}})$*, there exists* $\lambda_* = \lambda_*(\alpha) \in (0,1)$ *such that for each* $0 < \lambda_1 < \lambda_2 < 1$ *with* $\lambda_1/\lambda_2 \le \lambda_*$*, there exists* $b = b(\alpha, \lambda_1, \lambda_2) \in (0,1)$ *and* $C = C(\alpha, \lambda_1, \lambda_2) > 0$ *such that the following is true: Let* $z \in U$*. Let* $\{r_j\}_{j \in \mathbb{N}}$ *be a decreasing sequence of positive real numbers such that* $r_1 \le \mathrm{dist}(z, \partial U)$ *and* $r_{j+1}/r_j \le \lambda_1$ *for all* $j \in \mathbb{N}$*. Then for each* $n \in \mathbb{N}$*, it holds with probability at least* $1 - C(\lambda_1/\lambda_2)^{\alpha n}$ *that there are at least* bn *values of* $j \in [1,n]_{\mathbb{Z}}$ *for which there are at most two connected arcs of* Γ *in* $A_{\lambda_1 r_j, \lambda_2 r_j}(z)$ *connecting its inner and outer boundaries.*

Proof. See [AMY25b, Appendix C]. □

Lemma 2.6. *Let* Γ *be a whole-plane nested* CLE_κ*. Let* $z \in \mathbb{C}$ *and* $0 < s < t$*. Then it holds with superpolynomially high probability as* $N \to \infty$ *that there are at most* $2N$ *connected arcs of* Γ *in* $A_{s,t}(z)$ *connecting its inner and outer boundaries.*

Proof. Write $\kappa' \stackrel{\mathrm{def}}{=} 16/\kappa$. Let η' be a whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve from ∞ to ∞. Recall from [MSW17] that it is possible to couple η' with Γ so that it traces the loops of Γ. We observe that, if there are at least $2N$ connected arcs of Γ in $A_{s,t}(z)$ connecting its inner and outer boundaries, then η' makes at least $2N$ crossings between the inner and outer boundaries of $A_{s,t}(z)$. By [GHM20, Proposition 3.4 and Remark 3.9], it holds with superpolynomially high probability as $\varepsilon \to 0$ that for each $a < b$ with $\eta'([a,b]) \subset B_t(z)$ and $\mathrm{diam}(\eta'([a,b])) \ge \varepsilon^{1/2}$, the set $\eta'([a,b])$ contains a ball of radius at least ε. Suppose that this event occurs and $\varepsilon^{1/2} \le t - s$. Then η' makes at most $\mathrm{Leb}(A_{s,t}(z))/(\pi\varepsilon^2)$ crossings between the inner and outer boundaries of $A_{s,t}(z)$. This completes the proof. □

2.3. **Liouville quantum gravity.** Let $\gamma \in (0,2)$ and $Q \stackrel{\mathrm{def}}{=} 2/\gamma + \gamma/2$. Then a *$\gamma$-Liouville quantum gravity surface* (γ-LQG surface) is an equivalence class of pairs (U, Φ), where $U \subset \mathbb{C}$ is a simply connected domain, Φ is an instance of the Gaussian free field (GFF) on U, and (U_1, Φ_1) and (U_2, Φ_2) are equivalent if there exists a conformal mapping $\phi\colon U_1 \to U_2$ such that

$$\Phi_1 = \Phi_2 \circ \phi + Q \log(|\phi'|). \tag{2.3}$$

More generally, we can consider a γ-LQG surface with marked points $x_1, \ldots, x_n \in U$. Two γ-LQG surfaces with marked points are considered to be equivalent if the fields are related as in (2.3) where the conformal map ϕ takes the marked points for one surface to the marked points for the other.

The volume form (which is formally given by "$e^{\gamma\Phi} dx dy$") was constructed in [DS11] (see also the references therein). If Φ is (some form of) the GFF with Neumann boundary conditions, then one can also make sense of a boundary length measure.

The recent works [DDDF20, GM21b] constructed the *γ-LQG metric D_Φ* on the γ-LQG surface (U, Φ) which is formally given by "$e^{\gamma\Phi}(dx^2 + dy^2)$". Moreover, two equivalent γ-LQG surfaces are isometric. The γ-LQG metric is conjectured to be the scaling limit of the graph distance on random planar maps with respect to the Gromov-Hausdorff topology.

The γ-LQG half-plane and γ-LQG disk are γ-LQG surfaces with two boundary marked points. They are in some sense the most natural infinite-volume and finite-volume γ-LQG surfaces, respectively, due to the following properties:

- Let $(\mathbb{H}, \Phi, 0, \infty)$ be a γ-LQG half-plane. Then for each deterministic $r > 0$ and $x \in \mathbb{R}$ so that the boundary length measure of $[0, x]$ is equal to r we have that $(\mathbb{H}, \Phi, x, \infty)$ is again a γ-LQG half-plane.
- Let $(\mathbb{H}, \Phi, 0, \infty)$ be a γ-LQG disk. Given Φ, let x and y be independent samples from the γ-LQG length measure on $\partial\mathbb{H}$ associated with Φ (renormalized to be a probability measure). Then $(\mathbb{H}, \Phi, x, y)$ is again a γ-LQG disk.

The following are proved in [MSW22].

When $\kappa = \gamma^2 \in (0, 4)$, we say that an SLE_κ (or a CLE_κ) and a γ-LQG surface have the same "central charge", in the sense that an independent SLE_κ (or CLE_κ) on a γ-LQG surface has a well-defined γ-LQG length and cuts the γ-LQG surface into independent γ-LQG surfaces [She16b, DMS21, MSW22].

Theorem 2.7. *Let $\gamma \in (\sqrt{8/3}, 2)$; $\kappa \stackrel{\text{def}}{=} \gamma^2$; $\beta \in [-1, 1]$; $\mathcal{H} = (\mathbb{H}, \Phi, 0, \infty)$ a γ-LQG half-plane; η an independent $\mathrm{SLE}_\kappa^\beta(\kappa - 6)$ on $\mathcal{H}$ from 0 to ∞ that is parameterized by the generalized γ-LQG length of its trunk η'. Write $\Gamma_{\mathrm{L},+}$ (resp. $\Gamma_{\mathrm{R},+}$) for the collection of pairs $(\mathcal{S}, t)$, where $\mathcal{S}$ is the γ-LQG surface parameterized by a connected component of $\mathbb{H} \setminus \eta$ that is encircled by a CLE_κ loop that lies to the left (resp. right) of η' and is first visited by η' at time t, equipped with a marked point $\eta'(t)$. Write $\Gamma_{\mathrm{L},-}$ (resp. $\Gamma_{\mathrm{R},-}$) for the collection of pairs $(\mathcal{S}, t)$, where $\mathcal{S}$ is the γ-LQG surface parameterized by a connected component of $\mathbb{H} \setminus \eta$ that is disconnected from ∞ by η' at time t and lies to the left (resp. right) of η', equipped with a marked point $\eta'(t)$. Then there are deterministic constants $a_{\mathrm{L},+}, a_{\mathrm{R},+}, a_{\mathrm{L},-}, a_{\mathrm{R},-} > 0$ with*

$$a_{\mathrm{L},+} : a_{\mathrm{R},+} : a_{\mathrm{L},-} : a_{\mathrm{R},-} = -(1-\beta)\cos(4\pi/\kappa) : -(1+\beta)\cos(4\pi/\kappa) : 1 : 1.$$

such that $\Gamma_{\mathrm{L},+}$ (resp. $\Gamma_{\mathrm{R},+}$; $\Gamma_{\mathrm{L},-}$; $\Gamma_{\mathrm{R},-}$) is a Poisson point process with intensity measure given by the product of the Lebesgue measure on $\mathbb{R}_{\geq 0}$ and $a_{\mathrm{L},+}$ (resp. $a_{\mathrm{R},+}$; $a_{\mathrm{L},-}$; $a_{\mathrm{R},-}$) times the one-pointed γ-LQG disk measure. Moreover, $\Gamma_{\mathrm{L},+}$, $\Gamma_{\mathrm{R},+}$, $\Gamma_{\mathrm{L},-}$, $\Gamma_{\mathrm{R},-}$ are independent.

Corollary 2.8. *Let $\gamma \in (\sqrt{8/3}, 2)$; $\kappa \stackrel{\text{def}}{=} \gamma^2$; $\beta \in [-1, 1]$; $\mathcal{H} = (\mathbb{H}, \Phi, 0, \infty)$ a γ-LQG half-plane; η an independent $\mathrm{SLE}_\kappa^\beta(\kappa - 6)$ on $\mathcal{H}$ from 0 to ∞ that is parameterized by the generalized γ-LQG length of its trunk. For each $t \geq 0$, write $\mathcal{H}_t$ for the γ-LQG surface parameterized by the unbounded connected component of $\mathbb{H} \setminus \eta([0, t])$, equipped with marked points $\eta(t)$ and ∞; $L_t^{\mathcal{H}}$ (resp. $R_t^{\mathcal{H}}$) for the*

γ-LQG length of the left (resp. right) boundary of $\mathcal{H}_t$ minus the γ-LQG length of the left (resp. right) boundary of $\mathcal{H}$. Then $L^{\mathcal{H}}$ and $R^{\mathcal{H}}$ are independent $4/\kappa$-stable Lévy processes.

Theorem 2.9. *Let $\gamma \in (\sqrt{8/3}, 2)$; $\kappa \stackrel{\text{def}}{=} \gamma^2$; $\mathfrak{D} = (\mathbb{H}, \Phi)$ a γ-LQG disk of fixed boundary length; $\Gamma_{\mathfrak{D}}$ an independent CLE_κ on $\mathfrak{D}$. Then the γ-LQG surfaces encircled by the loops of $\Gamma_{\mathfrak{D}}$ are conditionally independent γ-LQG disks given their boundary lengths.*

2.4. **Multichordal SLE.** In the present subsection, we review the theory of multichordal SLE [MS16b, MW18, MSW20, AMY25a].

Let $U \subset \mathbb{C}$ be a deterministic simply connected domain. Let $x_1, \ldots, x_{2N}$ be deterministic and distinct points of ∂U. A *link pattern* on $(U; x_1, \ldots, x_{2N})$ is a partition $\{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$ of $\{x_1, \ldots, x_{2N}\}$ such that there exist N disjoint simple paths in $\overline{U}$ connecting a_j and b_j, respectively. Let $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$ be a deterministic link pattern on $(U; x_1, \ldots, x_{2N})$. A *multichordal* SLE_κ in $(U; x_1, \ldots, x_{2N})$ with *interior* link pattern α is a random collection $\{\eta_1, \ldots, \eta_N\}$ of disjoint simple paths in $\overline{U}$ such that for each $j \in [1, N]_{\mathbb{Z}}$, given $\{\eta_k : k \neq j\}$, the path η_j is conditionally a chordal SLE_κ in U_j connecting a_j and b_j, where U_j denotes the connected component of $U \setminus \bigcup_{k: k \neq j} \eta_k$ having a_j and b_j on its boundary.

Theorem 2.10. *The law of a multichordal SLE_κ in $(U; x_1, \ldots, x_{2N})$ with interior link pattern α exists and is unique. Moreover, if $\{\eta_1, \ldots, \eta_N\}$ is a multichordal SLE_κ in $(U; x_1, \ldots, x_{2N})$ with interior link pattern α, then the following conditions are satisfied:*

(i) *Let $\phi \colon U \to \phi(U)$ be a deterministic conformal mapping. Then $\{\phi(\eta_1), \ldots, \phi(\eta_N)\}$ is a multichordal SLE_κ in $(\phi(U); \phi(x_1), \ldots, \phi(x_{2N}))$ with interior link pattern $\phi(\alpha)$.*
(ii) *Let $j \in [1, N]_{\mathbb{Z}}$. Let τ be a stopping time for η_j. Then, given $\eta_j|_{[0,\tau]}$, the collection $\{\eta_j|_{[\tau,1]}\} \cup \{\eta_k : k \neq j\}$ is conditionally a multichordal SLE_κ in $(U \setminus \eta([0, \tau]); \{x_1, \ldots, x_{2N}, \eta(\tau)\} \setminus \{\eta(0)\})$ with interior link pattern obtained from α by replacing $\eta(0)$ with $\eta(\tau)$.*

Lemma 2.11. *Let $\{\eta_1, \ldots, \eta_N\}$ be a multichordal SLE_κ in $(U; x_1, \ldots, x_{2N})$ with interior link pattern α. Let $\gamma_1, \ldots, \gamma_N$ be disjoint simple paths in $\overline{U}$ inducing the link pattern α. Then for each $\varepsilon > 0$, it holds with positive probability that $\eta_j \subset B_\varepsilon(\gamma_j)$ for all $j \in [1, N]_{\mathbb{Z}}$.*

Proof. We apply induction on N. The case where $N = 1$ follows immediately from [MS16a, Proposition 3.4]. Suppose that $N \geq 2$. Write $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$ such that for each $j \in [1, N]_{\mathbb{Z}}$, γ_j is a simple path connecting a_j and b_j. By possibly decreasing ε, we may assume without loss of generality that $B_\varepsilon(\gamma_j) \cap B_\varepsilon(\gamma_k) = \emptyset$ for all distinct $j, k \in [1, N]_{\mathbb{Z}}$. One verifies immediately that there exists $j \in [1, N]_{\mathbb{Z}}$ such that either $(a_j, b_j)^{\circlearrowleft}_{\partial U} \cap \{x_1, \ldots, x_{2N}\} = \emptyset$ or $(a_j, b_j)^{\circlearrowright}_{\partial U} \cap \{x_1, \ldots, x_{2N}\} = \emptyset$. We may assume without loss of generality that the former is true. Again, by possibly decreasing ε, we may assume without loss of generality that $B_\varepsilon(\gamma_k) \cap B_\varepsilon((a_j, b_j)^{\circlearrowleft}_{\partial U}) = \emptyset$ for all $k \in [1, N]_{\mathbb{Z}}$ with $k \neq j$. First, it follows from the case where $N = 1$ that it holds with positive probability that $\eta_j \subset B_\varepsilon((a_j, b_j)^{\circlearrowleft}_{\partial U})$. Then, by the induction hypothesis, it holds with positive probability that $\eta_j \subset B_\varepsilon((a_j, b_j)^{\circlearrowleft}_{\partial U})$ and $\eta_k \subset B_\varepsilon(\gamma_k)$ for all $k \in [1, N]_{\mathbb{Z}}$ with $k \neq j$. Finally, it follows again from the case where $N = 1$ that it holds with positive probability that $\eta_k \subset B_\varepsilon(\gamma_k)$ for all $k \in [1, N]_{\mathbb{Z}}$. This completes the proof. $\square$

Write

$$\begin{cases} Z(x) \stackrel{\text{def}}{=} x^{2/\kappa}(1-x)^{1-6/\kappa} {}_2F_1(4/\kappa, 1-4/\kappa, 8/\kappa; x); \\ H(x) \stackrel{\text{def}}{=} \dfrac{Z(x)}{Z(x) - 2\cos(4\pi/\kappa) Z(1-x)}, \end{cases} \qquad \forall x \in (0,1),$$

where ${}_2F_1$ denotes the hypergeometric function. Choose $x \in (0,1)$ so that $(U; x_1, x_2, x_3, x_4)$ is conformally equivalent to $(\mathbb{H}; \infty, 0, 1-x, 1)$. A *bichordal* SLE_κ in $(U; x_1, x_2, x_3, x_4)$ with *exterior* link pattern $\{\{x_1, x_2\}, \{x_3, x_4\}\}$ is obtained by first sampling α with

$$\mathbf{P}[\alpha = \{\{x_1, x_4\}, \{x_2, x_3\}\}] = H(x) \quad \text{and} \quad \mathbf{P}[\alpha = \{\{x_1, x_2\}, \{x_3, x_4\}\}] = 1 - H(x),$$

and then sampling a bichordal SLE_κ in $(U; x_1, x_2, x_3, x_4)$ with interior link pattern α. Let β be a link pattern on $(U; x_1, \ldots, x_{2N})$. A *multichordal* SLE_κ in $(U; x_1, \ldots, x_{2N})$ with *exterior* link pattern β is a random collection $\{\eta_1, \ldots, \eta_N\}$ of disjoint simple paths in $\overline{U}$ such that for each pair of distinct $j, k \in [1, N]_{\mathbb{Z}}$, on the event that η_j and η_k are contained in the same connected component $U_{j,k} \subset U \setminus \bigcup_{l: l \neq j,k}$, given $\{\eta_l : l \neq j, k\}$, the pair $\{\eta_j, \eta_k\}$ is conditionally a multichordal SLE_κ in $(U_{j,k}; \eta_j(0), \eta_j(1), \eta_k(0), \eta_k(1))$ with the exterior link pattern induced by $\{\eta_l : l \neq j, k\}$ and β.

Theorem 2.12. *The law of a multichordal* SLE_κ *in* $(U; x_1, \ldots, x_{2N})$ *with exterior link pattern* β *exists and is unique. Moreover, if* $\{\eta_1, \ldots, \eta_N\}$ *is a multichordal* SLE_κ *in* $(U; x_1, \ldots, x_{2N})$ *with exterior link pattern* β*, then the following conditions are satisfied:*

(i) Let $\phi \colon U \to \phi(U)$ *be a deterministic conformal mapping. Then* $\{\phi(\eta_1), \ldots, \phi(\eta_N)\}$ *is a multichordal* SLE_κ *in* $(\phi(U); \phi(x_1), \ldots, \phi(x_{2N}))$ *with exterior link pattern* $\phi(\beta)$*.*
(ii) Let $j \in [1, N]_{\mathbb{Z}}$*. Let* τ *be a stopping time for* η_j*. Then, given* $\eta_j|_{[0,\tau]}$*, the collection* $\{\eta_j|_{[\tau,1]}\} \cup \{\eta_k : k \neq j\}$ *is conditionally a multichordal* SLE_κ *in* $(U \setminus \eta([0, \tau]); \{x_1, \ldots, x_{2N}, \eta(\tau)\} \setminus \{\eta(0)\})$ *with exterior link pattern obtained from* β *by replacing* $\eta(0)$ *with* $\eta(\tau)$*.*

Lemma 2.13. *Let* $\{\eta_1, \ldots, \eta_N\}$ *be a multichordal* SLE_κ *in* $(U; x_1, \ldots, x_{2N})$ *with exterior link pattern* β*. Then* $\mathbf{P}[$*the link pattern induced by* $\{\eta_1, \ldots, \eta_N\}$ *is equal to* $\alpha] > 0$*. Moreover, the mapping*

$$(U; x_1, \ldots, x_{2N}) \mapsto \mathbf{P}[\textit{the link pattern induced by } \{\eta_1, \ldots, \eta_N\} \textit{ is equal to } \alpha]$$

is continuous with respect to the Carathéodory topology for simply connected domains, together with the uniform topology for boundary points.

Let $\{\eta_1, \ldots, \eta_N\}$ be a multichordal SLE_κ in $(U; x_1, \ldots, x_{2N})$ with exterior link pattern β. Given $\{\eta_1, \ldots, \eta_N\}$, let Γ be conditionally independent nested CLE_κ's in each connected component of $U \setminus \bigcup_j \eta_j$. Then $\{\eta_1, \ldots, \eta_N\} \cup \Gamma$ has the law of a *multichordal* CLE_κ in $(U; x_1, \ldots, x_{2N})$ with exterior link pattern β.

Theorem 2.14. *Let* U *be either* $\mathbb{C}$ *or a deterministic simply connected domain. Let* $V \subset U$ *be a deterministic simply connected subdomain. Let* $K \subset V$ *be a deterministic connected and compact subset. Let* Γ *be a nested* CLE_κ *in* U*. Write* $V^\star$ *for the connected component containing* K *of the open subset obtained by removing from* V *the closure of the union of the* $(U \setminus K, U \setminus V)$*-excursions of* Γ *and all the other loops of* Γ *that intersect with* $U \setminus V$*. Write* $X \stackrel{\text{def}}{=} \partial V^\star \cap K$ *for the collection of the endpoints of the* $(U \setminus K, U \setminus V)$*-excursions of* Γ*. Write* β *for the link pattern on* $(V^\star; X)$ *induced by the* $(U \setminus K, U \setminus V)$*-excursions of* Γ*. Then, given* $(V^\star; X)$ *and* β*, the collection of the complementary* $(U \setminus K, U \setminus V)$*-excursions of* Γ*, together with the collection of the loops of* Γ *that are contained in* $V^\star$*, is conditionally a multichordal* CLE_κ *in* $(V^\star; X)$ *with exterior link pattern* β*.*

Lemma 2.15. *Let* U *be a deterministic simply connected domain. Let* $V \Subset U$ *be a deterministic simply connected subdomain. Let* $K \subset V$ *be a deterministic connected and compact subset. Let* Γ *be a multichordal* CLE_κ *in* $(U; x_1, \ldots, x_{2N})$ *with exterior link pattern* β*. Write* $\Gamma(V, K)$ *for the union of the collection of the* (V, K)*-excursions of* Γ *and the collection of the loops of* Γ *that are contained in* V *and intersect with* K*. Then the mapping*

$$(U; x_1, \ldots, x_{2N}) \mapsto (\textit{the law of } \Gamma(V, K))$$

is continuous with respect to the Carathéodory topology for simply connected domains, together with the uniform topology for boundary points, and the total variation distance for probability measures.

2.5. **Independence for CLE.** A key tool in the present paper is the fact that the restrictions of a nested CLE_κ to disjoint concentric annuli or disjoint balls are nearly independent [AMY25a].

Lemma 2.16. *For each $\lambda \in (0,1)$, $\alpha > 0$, and $b \in (0,1)$, there exists $p = p(\lambda, \alpha, b) \in (0,1)$ and $C = C(\lambda, \alpha, b) > 0$ such that the following is true:*

> *Let U be either $\mathbb{C}$ or a simply connected domain. Let Γ be a nested CLE_κ in U. Let $z \in U$. Let $\{r_j\}_{j\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $r_1 \le \mathrm{dist}(z, \partial U)$ and $r_{j+1}/r_j \le \lambda$ for all $j \in \mathbb{N}$. Let $\{E_j\}_{j\in\mathbb{N}}$ be a sequence of events such that*
> $$E_j \in \sigma\left(A_{r_{j+1},r_j}(z) \cap \Gamma\right) \quad \text{and} \quad \mathbf{P}[E_j] \ge p, \quad \forall j \in \mathbb{N}, \tag{2.4}$$
> *where $A_{r_{j+1},r_j}(z) \cap \Gamma \stackrel{\text{def}}{=} \{A_{r_{j+1},r_j}(z) \cap \mathscr{L} : \mathscr{L} \in \Gamma\}$. Then*
> $$\mathbf{P}[\#\{j \in [1,n]_{\mathbb{Z}} : E_j \text{ occurs}\} \ge bn] \ge 1 - C\mathrm{e}^{-\alpha n}, \quad \forall n \in \mathbb{N}.$$

Lemma 2.17. *For each $\lambda \in (0,1)$ and $p \in (0,1)$, there exists $\alpha = \alpha(\lambda, p) > 0$, $b = b(\lambda, p) \in (0,1)$, and $C = C(\lambda, p) > 0$ such that (2.4) is true.*

Lemma 2.18. *For each $t > 0$ and $p \in (0,1)$, there exists $\alpha = \alpha(t,p) > 0$ and $C = C(t,p) > 0$ such that the following is true: Let U be either $\mathbb{C}$ or a simply connected domain. Let Γ be a nested CLE_κ in U. Let $r > 0$. Let $z_1, \ldots, z_n \in U$ be a sequence of points such that*
$$\mathrm{dist}(z_j, \partial U) \ge (1+t)r \quad \text{and} \quad |z_j - z_k| \ge (2+t)r, \quad \forall \text{ distinct } j,k \in [1,n]_{\mathbb{Z}}.$$
Let $E_1, \ldots, E_n$ be a sequence of events such that
$$E_j \in \sigma(B_r(z_j) \cap \Gamma) \quad \text{and} \quad \mathbf{P}[E_j] \ge p, \quad \forall j \in [1,n]_{\mathbb{Z}}.$$
Then
$$\mathbf{P}\left[\bigcup_{j=1}^n E_j\right] \ge 1 - C\mathrm{e}^{-\alpha n}.$$

Remark 2.19. Let Γ be a whole-plane nested CLE_κ. Let Υ and $\Upsilon^\dagger$ be as in Section 1.2. Let $U \subset \mathbb{C}$ be an open subset. Since Υ and $\Upsilon^\dagger$ have the same law, it follows that for each event $E \in \sigma(U \cap \Upsilon)$, we have
$$\{\Upsilon \in E\} \cap \{\Upsilon^\dagger \in E\} \in \sigma(U \cap \Gamma); \quad \mathbf{P}[\{\Upsilon \in E\} \cap \{\Upsilon^\dagger \in E\}] \ge 2\mathbf{P}[\Upsilon \in E] - 1;$$
$$\{\Upsilon \in E\} \cup \{\Upsilon^\dagger \in E\} \in \sigma(U \cap \Gamma); \quad \mathbf{P}[\{\Upsilon \in E\} \cup \{\Upsilon^\dagger \in E\}] \le 2\mathbf{P}[\Upsilon \in E].$$
Thus, we conclude that Lemmas 2.16, 2.17 and 2.18 also hold with
$$\text{“}E_j \in \sigma(A_{r_{j+1},\lambda r_j}(z) \cap \Upsilon)\text{”} \quad \text{and} \quad \text{“}E_j \in \sigma(B_r(z_j) \cap \Upsilon)\text{”}$$
in place of
$$\text{“}E_j \in \sigma(A_{r_{j+1},\lambda r_j}(z) \cap \Gamma)\text{”} \quad \text{and} \quad \text{“}E_j \in \sigma(B_r(z_j) \cap \Gamma)\text{”},$$
respectively.

2.6. **Lower bounds.** Let D be a weak geodesic CLE_κ carpet metric. In the present subsection, we prove several lower bounds for D_Υ. First, we prove that the D_Υ-distance between two disjoint compact subsets has finite moments of all negative orders (Lemma 2.20). Next, we prove that the identity mapping from (Υ, D_Υ) to Υ, equipped with the Euclidean metric, is locally Hölder continuous (Lemma 2.21).

Lemma 2.20. *Let $K_1, K_2 \subset \mathbb{C}$ be deterministic and disjoint compact subsets. Then for each $\mathbb{r} > 0$, it holds with superpolynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that*
$$D_\Upsilon(\mathbb{r}K_1 \cap \Upsilon, \mathbb{r}K_2 \cap \Upsilon) \ge \varepsilon \mathfrak{c}_{\mathbb{r}}.$$

Proof. Fix a deterministic bounded open subset $U \subset \mathbb{C}$ and $\delta > 0$ such that

$$K_1, K_2 \subset U, \quad \mathrm{dist}(K_1, K_2) \geq \delta, \quad \mathrm{dist}(K_1, \partial U) \geq \delta.$$

Fix $\mathbb{r} > 0$ and $\alpha > 0$. By Definition 1.7, Axioms (III) and (IV) (translation invariance and tightness across scales), for each $p \in (0,1)$, there exists $a = a(p) > 0$ such that

$$\mathbf{P}[D_\Upsilon(\partial B_{r/2}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq a\mathfrak{c}_r] \geq p, \quad \forall z \in \mathbb{C}, \ \forall r > 0.$$

Thus, by Lemma 2.16 and a union bound, there exists a sufficiently small $a > 0$ such that it holds with probability at least $1 - O(\varepsilon^\alpha)$ as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that for each $z \in \left(\frac{1}{100}\varepsilon^2\mathbb{r}\mathbb{Z}\right)^2 \cap \mathbb{r}U$, there exists $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ such that

$$D_\Upsilon(\partial B_{r/2}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq a\mathfrak{c}_r. \tag{2.5}$$

Henceforth assume that $\varepsilon \leq \delta/100$ and the above event occurs. Then each path in Υ connecting $\mathbb{r}K_1 \cap \Upsilon$ and $\mathbb{r}K_2 \cap \Upsilon$ must cross between the inner and outer boundaries of $A_{r/2,r}(z)$ for some $z \in \left(\frac{1}{100}\varepsilon^2\mathbb{r}\mathbb{Z}\right)^2 \cap \mathbb{r}U$ and $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ for which (2.5) is true. Combining this with (1.5), we obtain that

$$D_\Upsilon(\mathbb{r}K_1 \cap \Upsilon, \mathbb{r}K_2 \cap \Upsilon) \geq \inf\left\{a\mathfrak{c}_r : r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]\right\} \geq a\varepsilon^4\mathfrak{c}_\mathbb{r}.$$

Since α is arbitrary, this completes the proof. □

Lemma 2.21. *Let $\chi > 2$. Let $K \subset \mathbb{C}$ be a deterministic compact subset. Then for each $\mathbb{r} > 0$, it holds with probability tending to one as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that*

$$\mathfrak{c}_\mathbb{r}^{-1} D_\Upsilon(x,y) \geq \left|\frac{x-y}{\mathbb{r}}\right|^\chi, \quad \forall x, y \in \mathbb{r}K \cap \Upsilon \text{ with } |x-y| \leq \varepsilon\mathbb{r}.$$

Lemma 2.22. *Let $\alpha > 0$. Let $K \subset \mathbb{C}$ be a deterministic compact subset. Then for each $\mathbb{r} > 0$, it holds with probability tending to one as $\varepsilon_* \to 0$, at a rate which is uniform in $\mathbb{r}$, that*

$$D_\Upsilon(\partial B_{\varepsilon\mathbb{r}/2}(z) \cap \Upsilon, \partial B_{\varepsilon\mathbb{r}}(z) \cap \Upsilon) \geq \varepsilon^\alpha \mathfrak{c}_{\varepsilon\mathbb{r}}, \quad \forall z \in \mathbb{r}K, \ \forall \varepsilon \in (0, \varepsilon_*]. \tag{2.6}$$

Proof. By Lemma 2.20, for each $r > 0$, it holds with superpolynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in r, that

$$D_\Upsilon(\partial B_{5r/8}(0) \cap \Upsilon, \partial B_{7r/8}(0) \cap \Upsilon) \geq \varepsilon^\alpha \mathfrak{c}_r.$$

Thus, by Definition 1.7, Axiom (III) (translation invariance) and a union bound, for each $\mathbb{r} > 0$, it holds with probability tending to one as $\varepsilon_* \to 0$, at a rate which is uniform in $\mathbb{r}$, that

$$\begin{aligned} &D_\Upsilon(\partial B_{5\varepsilon\mathbb{r}/8}(z) \cap \Upsilon, \partial B_{7\varepsilon\mathbb{r}/8}(z) \cap \Upsilon) \geq \varepsilon^\alpha \mathfrak{c}_{\varepsilon\mathbb{r}}, \\ &\qquad\qquad \forall \varepsilon \in (0, \varepsilon_*] \cap \left\{\left(\frac{99}{100}\right)^k\right\}_{k \in \mathbb{Z}}, \ \forall z \in \left(\frac{1}{100}\varepsilon\mathbb{r}\mathbb{Z}\right)^2 \cap B_\mathbb{r}(\mathbb{r}K). \end{aligned} \tag{2.7}$$

Now, (2.6) follows immediately from (2.7), together with the fact that for each $z \in \mathbb{r}K$ and $\varepsilon \in (0, \varepsilon_*]$, there exists $\varepsilon' \in (0, \varepsilon_*] \cap \left\{(99/100)^k\right\}_{k \in \mathbb{Z}}$ and $z' \in \left(\frac{1}{100}\varepsilon'\mathbb{r}\mathbb{Z}\right)^2 \cap B_\mathbb{r}(\mathbb{r}K)$ such that $B_{\varepsilon\mathbb{r}/2}(z) \subset B_{5\varepsilon'\mathbb{r}/8}(z') \subset B_{7\varepsilon'\mathbb{r}/8}(z') \subset B_{\varepsilon\mathbb{r}}(z)$. □

Proof of Lemma 2.21. Fix $\alpha \in (0, \chi - 2)$. Then, by Lemma 2.22, for each $\mathbb{r} > 0$, it holds with probability tending to one as $\varepsilon_* \to 0$ that

$$D_\Upsilon(\partial B_{\varepsilon\mathbb{r}/2}(z) \cap \Upsilon, \partial B_{\varepsilon\mathbb{r}}(z) \cap \Upsilon) \geq \varepsilon^\alpha \mathfrak{c}_{\varepsilon\mathbb{r}}, \quad \forall z \in \mathbb{r}K, \ \forall \varepsilon \in (0, \varepsilon_*]. \tag{2.8}$$

Henceforth assume that (2.8) holds. Let $x, y \in \mathbb{r}K \cap \Upsilon$ with $|x-y| \leq \varepsilon_*\mathbb{r}$. Combining (1.5) and (2.8), we obtain that

$$D_\Upsilon(x,y) \geq D_\Upsilon(\partial B_{|x-y|/2}(x) \cap \Upsilon, \partial B_{|x-y|}(x) \cap \Upsilon) \geq \left|\frac{x-y}{\mathbb{r}}\right|^\alpha \mathfrak{c}_{|x-y|} \geq \left|\frac{x-y}{\mathbb{r}}\right|^{2+\alpha} \mathfrak{c}_\mathbb{r}.$$

Since $\alpha + 2 < \chi$, this completes the proof. □

Remark 2.23. By a similar argument to the argument applied in the proof of Lemma 2.21, the following is true: Let D be a strong geodesic CLE_κ carpet metric with distance exponent θ. Let $\chi > \theta$. Let $K \subset \mathbb{C}$ be a deterministic compact subset. Then it holds with probability tending to one as $\varepsilon \to 0$ that

$$D_\Upsilon(x, y) \geq |x - y|^\chi, \quad \forall x, y \in K \cap \Upsilon \text{ with } |x - y| \leq \varepsilon.$$

2.7. **Upper bounds.** Let D be a weak geodesic CLE_κ carpet metric. In the present subsection, we prove several upper bounds for D_Υ. First, we prove that the diameter with respect to the internal metric of D_Υ has finite moments of positive orders (Lemma 2.24). Next, we prove a result which roughly speaking states the existence of a ball whose D_Υ-diameter is not too large, among a sequence of concentric ones (Lemma 2.25). Finally, we prove that the identity mapping from Υ, equipped with the Euclidean metric, to (Υ, D_Υ) is locally Hölder continuous (Lemma 2.26). (This is analogous to (1.4) but concerning the internal metric of D_Υ.)

Lemma 2.24. *Let $V \Subset U \subset \mathbb{C}$ be deterministic bounded open subsets. Then for each $\mathbb{r} > 0$, it holds with superpolynomially high probability as $A \to \infty$, at a rate which is uniform in $\mathbb{r}$, that*

$$\sup\left\{D_\Upsilon(x, y; \mathbb{r}U \cap \Upsilon) : x, y \in \mathbb{r}V \cap \Upsilon \text{ with } x \overset{\mathbb{r}V}{\longleftrightarrow} y\right\} \leq A\mathfrak{c}_\mathbb{r}.$$

Proof. Let α_{NET} be as in Lemma 4.4. By Remark 4.10 (there is no circular reasoning), it holds with superpolynomially high probability as $\varepsilon_* \to 0$ that for each $\varepsilon \in (0, \varepsilon_*]$ and $x, y \in \mathbb{r}V \cap \Upsilon$ with $x \overset{\mathbb{r}V}{\longleftrightarrow} y$, there exist points $z_1, \ldots, z_{n-1} \in \Upsilon$ with $n \leq \lfloor \varepsilon^{-\alpha_{\mathrm{NET}}} \rfloor$ such that

$$z_{j-1} \overset{\mathbb{r}V}{\longleftrightarrow} z_j \quad \text{and} \quad |z_{j-1} - z_j| \leq \varepsilon\mathbb{r}, \quad \forall j \in [1, n]_\mathbb{Z}, \tag{2.9}$$

where $z_0 \overset{\mathrm{def}}{=} x$ and $z_n \overset{\mathrm{def}}{=} y$. Write $E_1^\mathbb{r}(\varepsilon)$ for this event.

By Definition 1.7, Axiom (IV) (tightness across scales), it holds with superpolynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that

$$\sup\left\{\mathfrak{c}_\mathbb{r}^{-1} D_\Upsilon(x, y) \left|\frac{x - y}{\mathbb{r}}\right|^{-\alpha_{\mathrm{KC}}} : x, y \in \mathbb{r}V \cap \Upsilon \text{ with } x \leftrightarrow y\right\} \leq \varepsilon^{-\alpha_{\mathrm{KC}}/2}. \tag{2.10}$$

Write $E_2^\mathbb{r}(\varepsilon)$ for this event.

By Lemma 2.20, it holds with superpolynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that

$$D_\Upsilon(\mathbb{r}V \cap \Upsilon, (\mathbb{C} \setminus \mathbb{r}U) \cap \Upsilon) \geq \varepsilon^{\alpha_{\mathrm{KC}}/2}\mathfrak{c}_\mathbb{r}. \tag{2.11}$$

Write $E_3^\mathbb{r}(\varepsilon)$ for this event.

Henceforth assume that $E_1^\mathbb{r}(\varepsilon) \cap E_2^\mathbb{r}(\varepsilon) \cap E_3^\mathbb{r}(\varepsilon)$ occurs. Let $x, y \in \mathbb{r}V \cap \Upsilon$ such that $x \overset{\mathbb{r}V}{\longleftrightarrow} y$. Let $x = z_0, z_1, \ldots, z_n = y$ be as in the definition of $E_1^\mathbb{r}(\varepsilon)$. Combining (2.9) and (2.10), we obtain that

$$D_\Upsilon(z_{j-1}, z_j) \leq \varepsilon^{\alpha_{\mathrm{KC}}/2}\mathfrak{c}_\mathbb{r}, \quad \forall j \in [1, n]_\mathbb{Z}.$$

This, together with (2.11), implies that

$$D_\Upsilon(z_{j-1}, z_j; \mathbb{r}U \cap \Upsilon) = D_\Upsilon(z_{j-1}, z_j) \leq \varepsilon^{\alpha_{\mathrm{KC}}/2}\mathfrak{c}_\mathbb{r}, \quad \forall j \in [1, n]_\mathbb{Z}.$$

Thus, we conclude that

$$D_\Upsilon(x, y; \mathbb{r}U \cap \Upsilon) \leq \sum_{j=1}^n D_\Upsilon(z_{j-1}, z_j; \mathbb{r}U \cap \Upsilon) \leq n\varepsilon^{\alpha_{\mathrm{KC}}/2}\mathfrak{c}_\mathbb{r} \leq \varepsilon^{\alpha_{\mathrm{KC}}/2 - \alpha_{\mathrm{NET}}}\mathfrak{c}_\mathbb{r}.$$

This completes the proof. □

Lemma 2.25. *For each $\lambda \in (0,1)$, $\alpha > 0$, and $b \in (0,1)$, there exists $A = A(\lambda,\alpha,b) > 0$ and $C = C(\lambda,\alpha,b) > 0$ such that the following is true: Let $z \in \mathbb{C}$. Let $\{r_j\}_{j\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $r_{j+1}/r_j \le \lambda$ for all $j \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, it holds with probability at least $1 - C\mathrm{e}^{-\alpha n}$ that there are at least bn values of $j \in [1,n]_{\mathbb{Z}}$ for which we have*

$$D_\Upsilon(x,y;B_{2r_j}(z)\cap\Upsilon) \le A\mathfrak{c}_{r_j}, \quad \forall x,y \in B_{r_j}(z)\cap\Upsilon \text{ with } x \xleftrightarrow{B_{r_j}(z)} y.$$

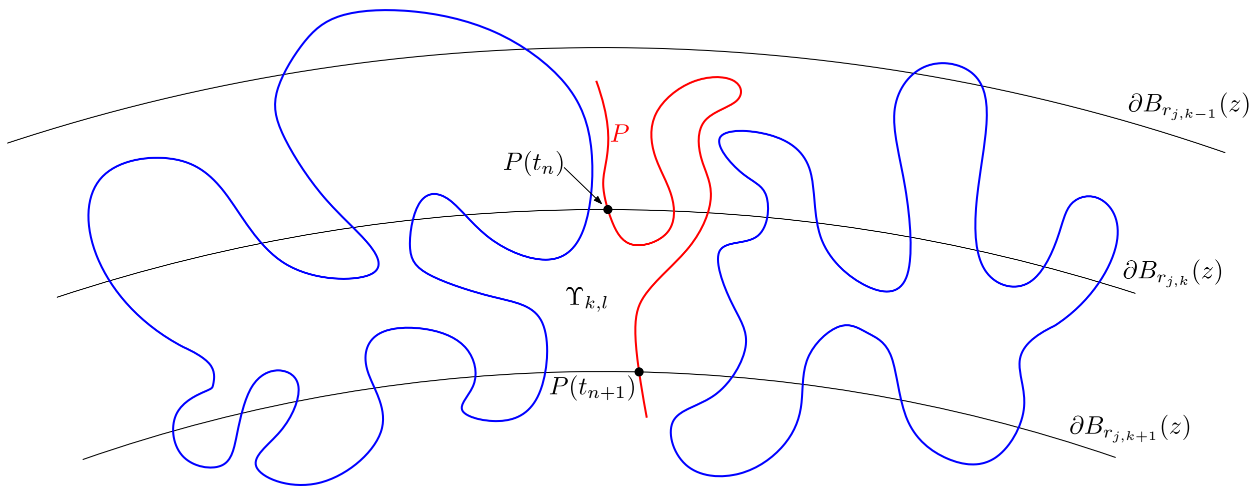

FIGURE 2. Illustration of the proof of Lemma 2.25. The red path P is contained in $B_{r_j}(z)\cap\Upsilon$. The times $\{t_n : n \in [1,N]_{\mathbb{Z}}\}$ are defined inductively so that for each $n \in [1,N-1]_{\mathbb{Z}}$, there exists $k \in \mathbb{N}_{>j}$ such that $P(t_n) \in \partial B_{r_{j,k}}(z)$ and t_{n+1} is the first time after t_n at which P hits $\partial A_{r_{j,k-1},r_{j,k+1}}(z)$. The two blue loops are two loops of Γ that cross between $\partial B_{r_{j,k}}(z)$ and $\partial B_{r_{j,k+1}}(z)$. The set $\Upsilon_{k,l}$ is the connected component of $\overline{A_{r_{j,k},r_{j,k+1}}(z)}\cap\Upsilon$ that lies between the two blue loops. We may assume without loss of generality that for each $\Upsilon_{k,l}$, there exists at most one $n \in [1,N]_{\mathbb{Z}}$ such that $P(t_n) \in \partial B_{r_{j,k}}(z)\cap\Upsilon_{k,l}$, and at most one $n \in [1,N]_{\mathbb{Z}}$ such that $P(t_n) \in \partial B_{r_{j,k+1}}(z)\cap\Upsilon_{k,l}$. Then, since for each $k \in \mathbb{N}_{\ge j}$, we are able to bound the number of connected components of $\overline{A_{r_{j,k},r_{j,k+1}}(z)}\cap\Upsilon$, and the D_Υ-distance between any points that lie on the same such connected component, we are able to bound the D_Υ-distance between $P(0)$ and $P(1)$.

Proof. See Figure 2 for an illustration of the proof. By possibly increasing λ, we may assume without loss of generality that $\lambda \in [3/4,1)$. Fix $A > 0$ to be chosen later. For $j,k \in \mathbb{N}$ with $j \le k$, write $r_{j,k} \stackrel{\text{def}}{=} \lambda^{k-j}r_j$ and $F_{j,k}$ for the event that the following are true:

(i) We have

$$D_\Upsilon(x,y;A_{r_{j,k+3},r_{j,k-2}}(z)\cap\Upsilon) \le A\lambda^{(k-j)/2}\mathfrak{c}_{r_j},$$
$$\forall x,y \in \overline{A_{r_{j,k+2},r_{j,k-1}}(z)}\cap\Upsilon \text{ with } x \xleftrightarrow{A_{r_{j,k+2},r_{j,k-1}}(z)} y$$

with the convention that $r_{j,j-1} \stackrel{\text{def}}{=} \lambda^{-1}r_j$ and $r_{j,j-2} \stackrel{\text{def}}{=} \lambda^{-2}r_j$.

(ii) There are at most $2A\lambda^{(j-k)/4}$ connected arcs of Γ in $A_{r_{j,k+1},r_{j,k}}(z)$ connecting its inner and outer boundaries.

By (1.5) and Lemma 2.24, for each $q > 0$, there exists $C_q > 0$ such that for each $j, k \in \mathbb{N}$ with $j \le k$, condition (i) holds with probability at least

$$1 - \frac{C_q}{A^q \lambda^{(k-j)q/2} (\mathfrak{c}_{r_j}/\mathfrak{c}_{r_{j,k}})^q} \ge 1 - C_p A^{-q} \mathfrak{K}^q \lambda^{(k-j)q/2}.$$

By Lemma 2.6 and the scale invariance of the law of Γ, for each $q > 0$, there exists $C'_q > 0$ such that for each $j, k \in \mathbb{N}$ with $j \le k$, condition (ii) holds with probability at least $1 - C'_q A^{-q} \lambda^{(k-j)q/4}$. Thus, we conclude that for each $q > 0$, there exists $C''_q > 0$ such that

$$\mathbf{P}[F_{j,k}] \ge 1 - C''_q A^{-q} \lambda^{(k-j)q/4}, \quad \forall j, k \in \mathbb{N} \text{ with } j \le k.$$

Thus, it follows from a similar argument to the argument applied in the proof of [MY25, Proposition 3.1] that there exists $A = A(\lambda, \alpha, b) > 0$ and $C = C(\lambda, \alpha, b) > 0$ such that

$$\mathbf{P}\left[\#\left\{j \in [1,n]_{\mathbb{Z}} : \bigcap_{k \in \mathbb{N} : k \ge j} F_{j,k} \text{ occurs}\right\} \ge bn\right] \ge 1 - C\mathrm{e}^{-\alpha n}, \quad \forall n \in \mathbb{N}.$$

Henceforth assume that the event $\bigcap_{k \in \mathbb{N}: k \ge j} F_{j,k}$ occurs. We *claim* that

$$D_\Upsilon(x, y; B_{r_{j,j-2}}(z) \cap \Upsilon) \le \left(\frac{2A^2}{1 - \lambda^{1/4}} + A\right) \mathfrak{c}_{r_j}, \quad \forall x, y \in B_{r_j}(z) \cap \Upsilon \text{ with } x \xleftrightarrow{B_{r_j}(z)} y.$$

Fix $x, y \in B_{r_j}(z) \cap \Upsilon$ and a path $P\colon [0,1] \to B_{r_j}(z) \cap \Upsilon$ from x to y. For each $k \in \mathbb{N}_{\ge j}$, by condition (ii), there are at most $A\lambda^{(j-k)/4}$ connected components of $\overline{A_{r_{j,k+1}, r_{j,k}}(z)} \cap \Upsilon$ connecting its inner and outer boundaries; write $\Upsilon_{k,1}, \Upsilon_{k,2}, \dots$ for these connected components. Set $t_0 \overset{\text{def}}{=} 0$. If $P \cap \partial B_{r_{j,k}}(z) \ne \emptyset$ for some $k \in \mathbb{N}_{>j}$, then set t_1 to be the first time at which P hits $\partial B_{r_{j,k}}(z)$ for some $k \in \mathbb{N}_{>j}$; otherwise set $t_1 \overset{\text{def}}{=} 1$. Then, inductively, for each $n \in \mathbb{N}$, if $t_n \ne 1$, $P(t_n) \in \partial B_{r_{j,k}}(z)$, and $P|_{[t_n,1]} \cap \partial A_{r_{j,k+1}, r_{j,k-1}}(z) \ne \emptyset$, then set t_{n+1} to be the first time after t_n at which P hits $\partial A_{r_{j,k+1}, r_{j,k-1}}(z)$; otherwise set $t_{n+1} \overset{\text{def}}{=} 1$. Write $N \overset{\text{def}}{=} \inf\{n \in \mathbb{N} : t_n = 1\} - 1$. By definition, for each $n \in [1, N]_{\mathbb{Z}}$, we have $P(t_n) \in \partial A_{r_{j,k+1}, r_{j,k}}(z) \cap \Upsilon_{k,l}$ for some $k \in \mathbb{N}_{\ge j}$ and $l \in [1, A\lambda^{(j-k)/4}]_{\mathbb{Z}}$. Suppose that there exist $a, b \in [1, N]_{\mathbb{Z}}$ with $a < b$, $k \in \mathbb{N}_{\ge j}$, and $l \in [1, A\lambda^{(j-k)/4}]_{\mathbb{Z}}$ such that either both $P(t_a)$ and $P(t_b)$ lie in $\partial B_{r_{j,k}}(z) \cap \Upsilon_{k,l}$ or both $P(t_a)$ and $P(t_b)$ lie in $\partial B_{r_{j,k+1}}(z) \cap \Upsilon_{k,l}$. In both cases, $P(t_a)$ and $P(t_b)$ lie in the same connected component of $A_{r_{j,k+1}, r_{j,k}}(z) \cap \Upsilon$, write P' for the concatenation of $P|_{[0,t_a]}$, a path in $A_{r_{j,k+1}, r_{j,k}}(z) \cap \Upsilon$ from $P(t_a)$ to $P(t_b)$, and $P|_{[t_b,1]}$. If we define $\{t'_n\}_{n \in \mathbb{N}}$ and N' in the same manner as $\{t_n\}_{n \in \mathbb{N}}$ and N, resp. but with P' in place of P, then one verifies immediately that

$$P'(t'_n) = \begin{cases} P(t_n) & \text{if } n \le a; \\ P(t_{n+b-a}) & \text{otherwise,} \end{cases}$$

and $N' = N + a - b < N$. Thus, by performing this operation finitely many times, we may assume without loss of generality that such a and b do not exist. On the other hand, for each $n \in [1, N+1]_{\mathbb{Z}}$, by definition, $P(t_{n-1})$ and $P(t_n)$ lie in the same connected component of $A_{r_{j,k+2}, r_{j,k-1}}(z) \cap \Upsilon$ for some $k \in \mathbb{N}_{\ge j}$, in which case we have

$$D_\Upsilon(P(t_{n-1}), P(t_n); B_{r_{j,j-2}}(z) \cap \Upsilon) \le D_\Upsilon(P(t_{n-1}), P(t_n); A_{r_{j,k+3}, r_{j,k-2}}(z) \cap \Upsilon) \le A\lambda^{(k-j)/2} \mathfrak{c}_{r_j}$$

by condition (i). Thus,

$$
\begin{aligned}
D_\Upsilon(x,y;B_{r_{j,j-1}}(z)\cap\Upsilon) &\le \sum_{n=1}^{N} D_\Upsilon(P(t_{n-1}),P(t_n);B_{r_{j,j-1}}(z)\cap\Upsilon)\\
&\quad + D_\Upsilon(P(t_N),P(1);B_{r_{j,j-1}}(z)\cap\Upsilon)\\
&\le \sum_{k=j}^{\infty} 2A\lambda^{(j-k)/4}A\lambda^{(k-j)/2}\mathfrak{c}_{r_j} + A\mathfrak{c}_{r_j}\\
&= \left(\frac{2A^2}{1-\lambda^{1/4}}+A\right)\mathfrak{c}_{r_j}.
\end{aligned}
$$

This completes the proof. □

Lemma 2.26. *Let $\alpha_{4\mathrm{A}}$ be as in (2.2). Let $\chi \in (0, 1-2/\alpha_{4\mathrm{A}})$. Let $U \subset \mathbb{C}$ be a deterministic open subset. Let $K \subset U$ be a deterministic compact subset. Then for each $\mathbb{r} > 0$, it holds with probability tending to one as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that the following is true: Let $x, y \in \mathbb{r}K \cap \Upsilon$ with $|x-y| \le \varepsilon\mathbb{r}$. Then either $x \xleftrightarrow{\mathbb{r}U} y$ or there exists a loop of Γ contained in $\mathbb{r}U$ and disconnecting x and y. Moreover, in the former case, we have*

$$
\mathfrak{c}_{\mathbb{r}}^{-1} D_\Upsilon(x,y;\mathbb{r}U\cap\Upsilon) \le \left|\frac{x-y}{\mathbb{r}}\right|^{\chi}.
$$

Proof. Fix $\chi < \mu < \nu < 1$ such that $\alpha_{4\mathrm{A}}(1-\nu) > 2$.

By possibly shrinking U, we may assume without loss of generality that it is bounded. Then, by Lemma 2.5, (i) and a union bound, for each $\mathbb{r} > 0$, it holds with polynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that for each $z \in \left(\frac{1}{100}\varepsilon\mathbb{r}\mathbb{Z}\right)^2 \cap \mathbb{r}U$, there are at most two connected arcs of Γ in $A_{\varepsilon\mathbb{r},\varepsilon^\nu\mathbb{r}}(z)$ connecting its inner and outer boundaries. Thus, by the Borel-Cantelli lemma, it holds with probability tending to one as $\varepsilon_* \to 0$, at a rate which is uniform in $\mathbb{r}$, that the above event occurs for all $\varepsilon \in (0,\varepsilon_*] \cap \{2^k\}_{k\in\mathbb{Z}}$. Henceforth assume that this holds for a sufficiently small $\varepsilon_* \in (0,1)$ (how small does not depend on $\mathbb{r}$).

Let $x, y \in \mathbb{r}K \cap \Upsilon$ with $|x-y| \le \varepsilon_*\mathbb{r}/100$. Choose $\varepsilon \in (0,\varepsilon_*] \cap \{2^k\}_{k\in\mathbb{Z}}$ and $z \in \left(\frac{1}{100}\varepsilon\mathbb{r}\mathbb{Z}\right)^2 \cap \mathbb{r}U$ such that $x, y \in B_{\varepsilon\mathbb{r}}(z)$ and $\varepsilon\mathbb{r} \le 100|x-y|$. We *claim* that either $x \xleftrightarrow{B_{\varepsilon^\nu\mathbb{r}}(z)} y$ or there exists a loop of Γ contained in $B_{\varepsilon^\nu\mathbb{r}}(z)$ and disconnecting x and y. Suppose by way of contradiction that the *claim* is not true. Since both x and y lie in Υ, there must exist at least two $B_{\varepsilon^\nu\mathbb{r}}(z)$-excursions of Γ disconnecting x and y in $B_{\varepsilon^\nu\mathbb{r}}(z)$. However, this implies that there are at least four connected arcs of Γ in $A_{\varepsilon\mathbb{r},\varepsilon^\nu\mathbb{r}}(z)$ connecting its inner and outer boundaries, a contradiction. This completes the proof of the *claim*.

By Lemma 2.25 and a union bound, there exists a sufficiently large $A > 0$ such that for each $\mathbb{r} > 0$, it holds with polynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that

$$
\begin{aligned}
&D_\Upsilon(x,y;B_{\varepsilon^\mu\mathbb{r}}(z)\cap\Upsilon) \le A\varepsilon^\mu\mathfrak{c}_{\mathbb{r}},\\
&\qquad \forall z \in \left(\frac{1}{100}\varepsilon^\nu\mathbb{r}\mathbb{Z}\right)^2 \cap \mathbb{r}U,\ \forall x,y \in B_{\varepsilon^\nu\mathbb{r}}(z)\cap\Upsilon \text{ with } x \xleftrightarrow{B_{\varepsilon^\nu\mathbb{r}}(z)} y.
\end{aligned} \tag{2.12}
$$

Thus, by the Borel-Cantelli lemma, it holds with probability tending to one as $\varepsilon_* \to 0$, at a rate which is uniform in $\mathbb{r}$, that (2.12) is true for all $\varepsilon \in (0,\varepsilon_*] \cap \{2^k\}_{k\in\mathbb{Z}}$. Henceforth assume that this holds for a sufficiently small $\varepsilon_* \in (0,1)$ (how small does not depend on $\mathbb{r}$). Combining the *claim* of the preceding paragraph and (2.12), we obtain that either there exists a loop of Γ contained in

$B_{\varepsilon^\nu \mathfrak{r}}(z)$ and disconnecting x and y or

$$D_\Upsilon(x,y;\mathfrak{r}U\cap\Upsilon) \le D_\Upsilon(x,y;B_{\varepsilon^\mu \mathfrak{r}}(z)\cap\Upsilon) \le A\varepsilon^\mu \mathfrak{c}_\mathfrak{r} \le A\cdot 100^\mu \cdot \left|\frac{x-y}{\mathfrak{r}}\right|^\mu \mathfrak{c}_\mathfrak{r}.$$

This completes the proof. □

Remark 2.27. By a similar argument to the argument applied in the proof of Lemma 2.26, the following is true: Let D be a strong geodesic CLE_κ carpet metric with distance exponent θ. Let $\chi \in (0,(1-2/\alpha_{4\mathrm{A}})\theta)$. Let $U\subset\mathbb{C}$ be a deterministic open subset. Let $K\subset U$ be a deterministic compact subset. Then it holds with probability tending to one as $\varepsilon\to 0$ that

$$D_\Upsilon(x,y;U\cap\Upsilon) \le |x-y|^\chi, \quad \forall x,y\in K\cap\Upsilon \text{ with } x \overset{U}{\leftrightarrow} y \text{ and } |x-y|\le\varepsilon.$$

2.8. **Estimates for the scaling constants.** In the present subsection, we prove a basic estimate for the scaling constants $\{\mathfrak{c}_r\}_{r>0}$ (Lemma 2.28) which will be useful in Section 7.

Lemma 2.28. *For each $\mathfrak{r}>0$, we have $\mathfrak{c}_{\varepsilon\mathfrak{r}}/(\varepsilon\mathfrak{c}_\mathfrak{r})\to 0$ as $\varepsilon\to 0$, at a rate which is uniform in $\mathfrak{r}$.*

Lemma 2.29. *There exists a deterministic constant $\xi>1$ such that the following is true: Let $\mathfrak{r}>0$. Write $S_\mathfrak{r}$ for the open square of side length $\mathfrak{r}$ centered at 0. For $n\in\mathbb{N}$, write $\mathcal{S}_\mathfrak{r}^n$ for the collection of open squares $S\subset S_\mathfrak{r}$ that have side length $2^{-n}\mathfrak{r}$ and vertices in $(2^{-n}\mathfrak{r}\mathbb{Z})^2$. Write $\mathscr{L}_\mathfrak{r}$ for the largest loop of Γ contained in $B_{4\mathfrak{r}}(0)$ and surrounding 0, and $\Upsilon_\mathfrak{r}\subset\mathbb{C}$ for the closed subset of points that are surrounded by $\mathscr{L}_\mathfrak{r}$ but by no other loop of Γ inside $\mathscr{L}_\mathfrak{r}$. Write $F_\mathfrak{r}$ for the event that $\mathscr{L}_\mathfrak{r}$ surrounds $B_{3\mathfrak{r}}(0)$. Then, on the event $F_\mathfrak{r}$, there almost surely exists $\delta>0$ such that*

$$\inf_P \#\{S\in\mathcal{S}_\mathfrak{r}^n : S\cap P\neq\emptyset\} \ge \delta 2^{\xi n}, \quad \forall n\in\mathbb{N},$$

where P ranges over all continuous paths in $S_\mathfrak{r}\cap\Upsilon_\mathfrak{r}$ that crosses between the left and right sides of $S_\mathfrak{r}$ (with the convention that $\inf\emptyset=\infty$).

Proof. By the scale invariance of the law of Γ, we may assume without loss of generality that $\mathfrak{r}=1$. Write Γ_1 for the collection of loops of Γ that are surrounded by $\mathscr{L}_1$ but by no other loop of Γ inside $\mathscr{L}_1$. Then, given $\mathscr{L}_1$, Γ_1 is conditionally a non-nested CLE_κ inside $\mathscr{L}_1$. Given $\mathscr{L}_1$, let Ξ_1 be a Brownian loop-soup of intensity $c(\kappa)$ inside $\mathscr{L}_1$. Suppose that Γ_1 and Ξ_1 are coupled so that Γ_1 is given by the outer boundaries of the outermost clusters of the Brownian loops of Ξ_1 (cf. [SW12]). For $n\in\mathbb{N}$ and $S\in\mathcal{S}_1^n$, we set $X(S)\overset{\mathrm{def}}{=}0$ if S is surrounded by a Brownian loop of Ξ_1 that is contained in the union of its eight neighbors, and $X(S)\overset{\mathrm{def}}{=}1$ otherwise. Then, given $\mathscr{L}_1$ and the event F_1, the random variables $X(S)$ for $n\in\mathbb{N}$ and $S\in\mathcal{S}_1^n$ are conditionally independent Bernoulli random variables. Thus, given $\mathscr{L}_1$ and the event F_1,

$$\mathscr{C}\overset{\mathrm{def}}{=}\overline{S_1}\setminus\bigcup_{S:X(S)=0} S$$

is conditionally a fractal percolation (with retention probability given by a deterministic constant depending only on κ) such that $S_1\cap\Upsilon_1\subset\mathscr{C}$ almost surely. Thus, Lemma 2.29 follows immediately from [Orz98, (2.32)]. □

Lemma 2.30. *In the notation of Lemma 2.29, there exists $\mathfrak{p}\in(0,1)$ and $A>0$ such that the following is true: Let $\mathfrak{r}>0$. Write $E_\mathfrak{r}$ for the event that $F_\mathfrak{r}$ occurs and there exists a continuous path $P\subset S_\mathfrak{r}\cap\Upsilon_\mathfrak{r}$ that crosses between the left and right sides of $S_\mathfrak{r}$ such that $\mathrm{len}(P;D_\Upsilon)\le A\mathfrak{c}_\mathfrak{r}$. Then $\mathbf{P}[E_\mathfrak{r}]\ge\mathfrak{p}$.*

Proof. This follows immediately from Definition 1.7, Axiom (IV) (tightness across scales). □

Proof of Lemma 2.28. In the notation of Lemmas 2.29 and 2.30, fix $\nu \in (0, \xi - 1)$. By Definition 1.7, Axioms (III) and (IV) (translation invariance and tightness across scales), for each $p \in (0,1)$, there exists $a = a(p) > 0$ such that

$$\mathbf{P}[D_\Upsilon(\partial B_{r/2}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq a\mathfrak{c}_r] \geq p, \quad \forall z \in \mathbb{C},\ \forall r > 0.$$

Thus, by Lemma 2.16 and a union bound, there exists a sufficiently small $a > 0$ such that it holds with polynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that for each $z \in \left(\frac{1}{100}\varepsilon^{1+\nu}\mathbb{r}\mathbb{Z}\right)^2 \cap S_{\mathbb{r}}$, there exists $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]$ such that

$$D_\Upsilon(\partial B_{r/2}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq a\mathfrak{c}_r. \tag{2.13}$$

Thus, by the Borel-Cantelli lemma, almost surely, there exists $\varepsilon_* \in (0,1)$ such that the above event occurs for all $\varepsilon \in (0, \varepsilon_*] \cap \{2^k\}_{k \in \mathbb{Z}}$.

Henceforth assume that $E_{\mathbb{r}}$ occurs. Then there exists a path $P \subset S_{\mathbb{r}} \cap \Upsilon_{\mathbb{r}}$ that crosses between the left and right sides of $S_{\mathbb{r}}$ such that $\operatorname{len}(P; D_\Upsilon) \leq A\mathfrak{c}_{\mathbb{r}}$. On the other hand, by Lemma 2.29, there exists $\delta > 0$ such that

$$\#\{S \in \mathcal{S}_{\mathbb{r}}^n : S \cap P \neq \emptyset\} \geq \delta 2^{\xi n}, \quad \forall n \in \mathbb{N}.$$

By the scale invariance of the law of Γ, we may choose δ so that its law does not depend on $\mathbb{r}$. Write

$$\begin{aligned}
\mathcal{S}_{\mathbb{r},0}^n &\stackrel{\text{def}}{=} \left\{S \in \mathcal{S}_{\mathbb{r}}^n : (\text{the bottom-left vertex of } S) \in (0,0) + (2^{-n+1}\mathbb{r}\mathbb{Z})^2\right\};\\
\mathcal{S}_{\mathbb{r},1}^n &\stackrel{\text{def}}{=} \left\{S \in \mathcal{S}_{\mathbb{r}}^n : (\text{the bottom-left vertex of } S) \in (2^{-n}\mathbb{r},0) + (2^{-n+1}\mathbb{r}\mathbb{Z})^2\right\};\\
\mathcal{S}_{\mathbb{r},2}^n &\stackrel{\text{def}}{=} \left\{S \in \mathcal{S}_{\mathbb{r}}^n : (\text{the bottom-left vertex of } S) \in (0,2^{-n}\mathbb{r}) + (2^{-n+1}\mathbb{r}\mathbb{Z})^2\right\};\\
\mathcal{S}_{\mathbb{r},3}^n &\stackrel{\text{def}}{=} \left\{S \in \mathcal{S}_{\mathbb{r}}^n : (\text{the bottom-left vertex of } S) \in (2^{-n}\mathbb{r},2^{-n}\mathbb{r}) + (2^{-n+1}\mathbb{r}\mathbb{Z})^2\right\}.
\end{aligned}$$

Then $\mathcal{S}_{\mathbb{r}}^n = \mathcal{S}_{\mathbb{r},0}^n \coprod \mathcal{S}_{\mathbb{r},1}^n \coprod \mathcal{S}_{\mathbb{r},2}^n \coprod \mathcal{S}_{\mathbb{r},3}^n$. By the pigeonhole principle, for each $n \in \mathbb{N}$, there exists $\bullet \in \{0,1,2,3\}$ such that

$$\#\{S \in \mathcal{S}_{\mathbb{r},\bullet}^n : S \cap P \neq \emptyset\} \geq (\delta/4)2^{\xi n}. \tag{2.14}$$

Note that $\operatorname{dist}(S_1, S_2) \geq 2^{-n}\mathbb{r}$ for all distinct $S_1, S_2 \in \mathcal{S}_{\mathbb{r},\bullet}^n$. Setting $\varepsilon \stackrel{\text{def}}{=} 2^{-n-2}$ and combining (2.13) and (2.14), we obtain that

$$A\mathfrak{c}_{\mathbb{r}} \geq \operatorname{len}(P; D_\Upsilon) \geq (\delta/4)2^{\xi n} a\mathfrak{c}_{2^{-(1+\nu)(n+2)}\mathbb{r}}$$

for all sufficiently large $n \in \mathbb{N}$. This completes the proof. □

2.9. **Estimates within thin annuli.** In the present subsection, we prove some estimates concerning the intersection of an annulus with Υ as the width of the annulus goes to zero. The results of the present subsection will be used in Sections 7 and 8 to control the amount of time that a D_Υ-geodesic can spend in a thin annulus.

Lemma 2.31. *Let $\nu \in (0,1)$. Let $U \subset \mathbb{C}$ be a deterministic bounded simply connected domain with smooth boundary. Then it holds with superpolynomially high probability as $\varepsilon \to 0$ that every connected component of $(B_\varepsilon(U) \setminus U) \cap \Upsilon$ has Euclidean diameter at most ε^ν.*

Proof. For $z \in \mathbb{C}$ and $r > 0$, write $E_r(z)$ for the event that there exists a loop of Γ in $A_{r,2r}(z)$ disconnecting the inner and outer boundaries of $A_{r,2r}(z)$. By the scaling and translation invariance of the law of Γ, there exists $p \in (0,1)$ such that

$$\mathbf{P}[E_r(z)] = p, \quad \forall z \in \mathbb{C},\ \forall r > 0.$$

Since ∂U is smooth, by the tubular neighborhood theorem (cf. [Lee13, Theorem 6.24]), there exists $\varepsilon_* > 0$ such that the mapping $\partial U \times [0, \varepsilon_*) \to \mathbb{C}\colon (z, \varepsilon) \mapsto z + \varepsilon n_z$ is a diffeomorphism — where n_z denotes the unit normal vector to ∂U at z. In particular, for each $z \in \partial U$ and $\varepsilon \in (0, \varepsilon_*)$, we have $\partial B_\varepsilon(z) \cap \partial B_\varepsilon(U) \neq \emptyset$. Since ∂U is smooth, there exists a constant $A > 0$ such that for each $\varepsilon > 0$, if we choose $z_1, \ldots, z_n \in \partial U$ in counterclockwise order such that $\mathrm{len}([z_j, z_{j+1}]_{\partial U}^{\circlearrowleft}) = A\varepsilon$ for all $j \in [1, n-1]_{\mathbb{Z}}$ and $A\varepsilon \leq \mathrm{len}([z_n, z_1]_{\partial U}^{\circlearrowleft}) < 2A\varepsilon$, then $|z_j - z_k| \geq 5\varepsilon$ for all distinct $j, k \in [1, n]_{\mathbb{Z}}$. Thus, by Lemma 2.18, there exists $\alpha > 0$ and $C > 0$ such that for each connected arc $I \subset \partial U$ of Euclidean length $\varepsilon^\nu/4$, we have

$$\mathbf{P}[\text{there exists } j \in [1, n]_{\mathbb{Z}} \text{ such that } z_j \in I \text{ and } E_\varepsilon(z_j) \text{ occurs}] \geq 1 - C\exp(-\alpha\varepsilon^{\nu-1}). \tag{2.15}$$

Since there exist $\lceil 4\varepsilon^{-\nu}\,\mathrm{len}(\partial U)\rceil$ connected arcs of ∂U of Euclidean length $\varepsilon^\nu/4$ such that every connected arc of ∂U of Euclidean length $\varepsilon^\nu/2$ contains at least one of them, by (2.15) and a union bound, it holds with superpolynomially high probability as $\varepsilon \to 0$ that for each connected arc $I \subset \partial U$ of Euclidean length $\varepsilon^\nu/2$, there exists $j \in [1, n]_{\mathbb{Z}}$ such that $z_j \in I$ and $E_\varepsilon(z_j)$ occurs. Since $\partial B_\varepsilon(z) \cap \partial B_\varepsilon(U) \neq \emptyset$ by the above discussion, it follows that the loop described in the event $E_\varepsilon(z_j)$ must cross between the inner and outer boundaries of $B_\varepsilon(U) \setminus U$. This completes the proof. ☐

Lemma 2.32. *There exists a deterministic constant $\zeta \in (0,1)$ such that the following is true: Let $U \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary. Let $\nu \in (0,1)$ and $\delta > 0$. Then it holds with probability tending to one as $\varepsilon \to 0$ that there exists a subset $Z \subset \left(\frac{1}{100}\varepsilon^\nu\mathbb{Z}\right)^2$ with $\#Z \leq \varepsilon^{-\zeta\nu}$ such that for each connected component $X \subset \Upsilon$ that has Euclidean diameter at least δ and each connected component $\mathscr{C}$ of $(B_\varepsilon(U) \setminus U) \cap X$, there exists $z \in Z$ such that $\mathscr{C} \subset B_{\varepsilon^\nu}(z)$.*

Proof. Since the boundary ∂U is bounded and smooth, for each $\varepsilon > 0$, there exists a deterministic subset $S \subset \left(\frac{1}{100}\varepsilon^\nu\mathbb{Z}\right)^2$ with $\#S = O(\varepsilon^{-\nu})$ as $\varepsilon \to 0$ such that

$$B_\varepsilon(U) \setminus U \subset \bigcup_{z \in S} B_{\varepsilon^\nu/2}(z). \tag{2.16}$$

For $z \in \mathbb{C}$ and $r > 0$, write $E_r(z)$ for the event that there exists a loop of Γ in $A_{r,2r}(z)$ disconnecting the inner and outer boundaries of $A_{r,2r}(z)$. By the scaling and translation invariance of the law of Γ, there exists $p \in (0,1)$ such that

$$\mathbf{P}[E_r(z)] = p, \quad \forall z \in \mathbb{C},\ \forall r > 0.$$

Thus, by Lemma 2.17, there exists $\alpha > 0$ and $C > 0$ such that

$$\mathbf{P}\Big[\text{there exists } r \in (\varepsilon^\nu, \delta/2) \cap \{2^k\}_{k\in\mathbb{Z}} \text{ such that } E_r(z) \text{ occurs}\Big] \geq 1 - C\varepsilon^{\alpha\nu}, \quad \forall \varepsilon > 0,\ \forall z \in \mathbb{C}. \tag{2.17}$$

Write

$$Z \overset{\mathrm{def}}{=} \Big\{z \in S : \text{there does not exist } r \in (\varepsilon^\nu, \delta/2) \cap \{2^k\}_{k\in\mathbb{Z}} \text{ such that } E_r(z) \text{ occurs}\Big\}.$$

Then, by (2.17), we have $\mathbf{E}[\#Z] \leq C\varepsilon^{\alpha\nu}\#S = O(\varepsilon^{(\alpha-1)\nu})$ as $\varepsilon \to 0$. (In particular, we must have $\alpha \in (0,1)$.) Choose $\zeta \in (1-\alpha, 1)$. Then it follows from Markov's inequality that $\mathbf{P}[\#Z \leq \varepsilon^{-\zeta\nu}] \to 1$ as $\varepsilon \to 0$. On the other hand, by Lemma 2.31, it holds with probability tending to one as $\varepsilon \to 0$ that every connected component of $(B_\varepsilon(U) \setminus U) \cap \Upsilon$ has Euclidean diameter at most $\varepsilon^\nu/2$. Henceforth assume that $\#Z \leq \varepsilon^{-\zeta\nu}$ and every connected component of $(B_\varepsilon(U) \setminus U) \cap \Upsilon$ has Euclidean diameter at most $\varepsilon^\nu/2$. It suffices to verify that for each connected component $X \subset \Upsilon$ that has Euclidean diameter at least δ and each connected component of $\mathscr{C} \subset (B_\varepsilon(U) \setminus U) \cap X$, there exists $z \in Z$ such that $\mathscr{C} \subset B_{\varepsilon^\nu}(z)$. By (2.16), we may choose $z \in S$ such that $\mathrm{dist}(\mathscr{C}, z) \leq \varepsilon^\nu/2$. Since the Euclidean diameter of $\mathscr{C}$ is at most $\varepsilon^\nu/2$, it follows that $\mathscr{C} \subset B_{\varepsilon^\nu}(z)$. Since $\mathscr{C}$ is contained in a connected component of Υ that has Euclidean diameter at least δ, it follows that $E_r(z)$ cannot occur for all $r \in (\varepsilon^\nu, \delta/2)$, hence that $z \in Z$. This completes the proof. ☐

3. Local metrics

In the present section, we consider local metrics, which are, a priori, even weaker metrics than the weak geodesic CLE_κ carpet metric. We first prove that any two conditionally independent local metrics are bi-Lipschitz equivalent (Proposition 3.2). We then prove that any local metric is almost surely determined by the CLE_κ carpet (Proposition 3.5). Proposition 3.5 will be applied in Section 4 to prove that any subsequential limit is a weak geodesic CLE_κ carpet metric. The argument of the present section is similar to the argument of [GM20b].

Definition 3.1. Let (Υ, D_Υ) be a coupling between Υ and a metric D_Υ on Υ. Then we shall refer to D_Υ as a *local* metric for Υ if Axioms (I), (III), (IV), (V) of Definition 1.7, together with the following condition, are satisfied:

(II') Let $U \subset \mathbb{C}$ be a deterministic open subset. Then $D_\Upsilon(\bullet, \bullet; U \cap \Upsilon)$ is conditionally independent of the pair $\big((\mathbb{C} \setminus \overline{U}) \cap \Upsilon, D_\Upsilon(\bullet, \bullet; (\mathbb{C} \setminus \overline{U}) \cap \Upsilon)\big)$ given $U \cap \Upsilon$.

3.1. Bi-Lipschitz equivalence.

Proposition 3.2. *Let D_Υ and $\widetilde{D}_\Upsilon$ be local metrics for Υ with the same scaling constants $\{\mathfrak{c}_r\}_{r>0}$. Suppose that D_Υ and $\widetilde{D}_\Upsilon$ are conditionally independent given Υ. Then there is a deterministic constant $M \geq 1$ such that $M^{-1} D_\Upsilon \leq \widetilde{D}_\Upsilon \leq M D_\Upsilon$ almost surely.*

Lemma 3.3. *Let D_Υ and $\widetilde{D}_\Upsilon$ be as in Proposition 3.2. Then for each $\lambda \in (0,1)$, $\alpha > 0$, and $b \in (0,1)$, there exists $p = p(\lambda, \alpha, b) \in (0,1)$ and $C = C(\lambda, \alpha, b) > 0$ such that the following is true: Let $z \in \mathbb{C}$. Let $\{r_j\}_{j \in \mathbb{N}}$ be a decreasing sequence of positive real numbers such that $r_{j+1}/r_j \leq \lambda$ for all $j \in \mathbb{N}$. Let $\{E_j\}_{j\in\mathbb{N}}$ be a sequence of events such that*

$$E_j \in \sigma\left(A_{r_{j+1},r_j}(z) \cap \Upsilon,\ D_\Upsilon(\bullet,\bullet; A_{r_{j+1},r_j}(z)\cap\Upsilon),\ \widetilde{D}_\Upsilon(\bullet,\bullet; A_{r_{j+1},r_j}(z)\cap\Upsilon)\right) \quad \text{and} \quad \mathbf{P}[E_j] \geq p.$$

for all $j \in \mathbb{N}$. Then

$$\mathbf{P}[\#\{j \in [1,n]_{\mathbb{Z}} : E_j \text{ occurs}\} \geq bn] \geq 1 - C\mathrm{e}^{-\alpha n}, \quad \forall n \in \mathbb{N}.$$

Proof. This follows from Definition 3.1, Axiom (II'), together with the same argument as the argument applied in the proof of Lemma 2.16. ☐

Lemma 3.4. *Let D_Υ and $\widetilde{D}_\Upsilon$ be as in Proposition 3.2. Then for each $\lambda \in (0,1)$, $\alpha > 0$, and $b \in (0,1)$ there exists $A = A(\lambda,\alpha,b) > 0$ and $C = C(\lambda,\alpha,b) > 0$ such that the following is true: Let $z \in \mathbb{C}$. Let $\{r_j\}_{j\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $r_{j+1}/r_j \leq \lambda$ for all $j \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, it holds with probability at least $1 - C\mathrm{e}^{-\alpha n}$ that there are at least bn values of $j \in [1,n]_{\mathbb{Z}}$ for which we have*

$$D_\Upsilon\left(x,y; B_{2r_j}(z)\cap\Upsilon\right) \vee \widetilde{D}_\Upsilon\left(x,y; B_{2r_j}(z)\cap\Upsilon\right) \leq A\mathfrak{c}_{r_j}, \quad \forall x,y \in B_{r_j}(z)\cap\Upsilon \text{ with } x \xleftrightarrow{B_{r_j}(z)} y.$$

Proof. This follows from the same argument as the argument applied in the proof of Lemma 2.25. ☐

Proof of Proposition 3.2. By symmetry, it suffices to consider the inequality "$\widetilde{D}_\Upsilon \leq M D_\Upsilon$".

By Definition 1.7, Axiom (IV) (tightness across scales), for each $p \in (0,1)$, there exists $a = a(p) > 0$ such that for each $r > 0$, it holds with probability at least p that

$$D_\Upsilon(\partial B_{r/2}(0)\cap\Upsilon, \partial B_r(0)\cap\Upsilon) \geq a\mathfrak{c}_r.$$

Thus, by Definition 1.7, Axiom (III) (translation invariance), Lemmas 3.3 and 3.4, and a union bound, there exists a sufficiently small $a > 0$ and a sufficiently large $A > 0$ such that it holds with polynomially high probability as $\varepsilon \to 0$ that for each $z \in \left(\frac{1}{100}\varepsilon^2\mathbb{Z}\right)^2 \cap B_{1/\varepsilon}(0)$, there exists $r \in [\varepsilon^2, \varepsilon]$ such that the following are true:

(i) $D_\Upsilon(\partial B_{r/2}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq a\mathfrak{c}_r$.

(ii) We have

$$\widetilde{D}_\Upsilon(x, y) \leq A\mathfrak{c}_r, \quad \forall x, y \in B_r(z) \cap \Upsilon \text{ with } x \xleftrightarrow{B_r(z)} y.$$

Thus, by the Borel-Cantelli lemma, almost surely, there exists $\varepsilon_* \in (0, 1)$ such that the above event occurs for all $\varepsilon \in (0, \varepsilon_*] \cap \{2^k\}_{k\in\mathbb{Z}}$.

Let $x, y \in \Upsilon$ with $x \leftrightarrow y$. Let $P\colon [0, 1] \to \Upsilon$ be a D_Υ-geodesic from x to y. Choose a sufficiently small $\varepsilon \in (0, \varepsilon_*] \cap \{2^k\}_{k\in\mathbb{Z}}$ such that $P \subset B_{1/\varepsilon}(0)$. Set $t_0 \stackrel{\text{def}}{=} 0$ and choose $z_1 \in \left(\frac{1}{100}\varepsilon^2\mathbb{Z}\right)^2 \cap B_{1/\varepsilon}(0)$ and $r_1 \in [\varepsilon^2, \varepsilon]$ such that $x \in B_{\varepsilon^2/2}(z_1)$ and conditions (i) and (ii) (with (z_1, r_1) in place of (z, r)) hold. Then, inductively, for each $j \in \mathbb{N}$, if $y \notin B_{r_j}(z_j)$, then set t_j to be the first time after t_{j-1} at which P hits $\partial B_{r_j}(z_j)$, and choose $z_{j+1} \in \left(\frac{1}{100}\varepsilon^2\mathbb{Z}\right)^2 \cap B_{1/\varepsilon}(0)$ and $r_{j+1} \in [\varepsilon^2, \varepsilon]$ such that $P(t_j) \in B_{\varepsilon^2/2}(z_{j+1})$ and conditions (i) and (ii) (with (z_{j+1}, r_{j+1}) in place of (z, r)) hold; otherwise set $t_j \stackrel{\text{def}}{=} 1$. Write $n \stackrel{\text{def}}{=} \inf\{j \in \mathbb{N} : t_j = 1\}$. Since $P(t_{j-1}) \in B_{r_j/2}(z_j)$ and $P(t_j) \in \partial B_{r_j}(z_j)$ for all $j \in [1, n-1]_\mathbb{Z}$, it follows from condition (i) that

$$D_\Upsilon(P(t_{j-1}), P(t_j)) \geq a\mathfrak{c}_{r_j}, \quad \forall j \in [1, n-1]_\mathbb{Z}. \tag{3.1}$$

Since $P([t_{j-1}, t_j]) \subset B_{r_j}(z_j)$ for all $j \in [1, n]_\mathbb{Z}$, it follows from condition (ii) that

$$\widetilde{D}_\Upsilon(P(t_{j-1}), P(t_j)) \leq A\mathfrak{c}_{r_j}, \quad \forall j \in [1, n]_\mathbb{Z}. \tag{3.2}$$

Combining (1.5), (3.1), and (3.2), we obtain that

$$\begin{aligned}\widetilde{D}_\Upsilon(x, y) \leq \sum_{j=1}^n \widetilde{D}_\Upsilon(P(t_{j-1}), P(t_j)) \leq \sum_{j=1}^n A\mathfrak{c}_{r_j} &\leq \frac{A}{a}\sum_{j=1}^{n-1} D_\Upsilon(P(t_{j-1}), P(t_j)) + A\mathfrak{c}_{r_n} \\ &\leq \frac{A}{a} D_\Upsilon(x, y) + A\mathfrak{K}\varepsilon\mathfrak{c}_1.\end{aligned}$$

Since ε may be chosen to be arbitrarily small, we conclude that $\widetilde{D}_\Upsilon(x, y) \leq (A/a)D_\Upsilon(x, y)$. This completes the proof. ☐

3.2. **Measurability.**

Proposition 3.5. *Let D_Υ be a local metric for Υ. Then D_Υ is almost surely determined by Υ. In particular, D determines a weak geodesic CLE_κ carpet metric.*

The remainder of the present subsection is devoted to the proof of Proposition 3.5.

Lemma 3.6. *There is a deterministic constant $M \geq 1$ such that the following is true: Suppose that distinct points $x, y \in \Upsilon$ are chosen in a manner that is almost surely determined by Υ such that $x \leftrightarrow y$ almost surely. Then*

$$M^{-1}\mathbf{E}[D_\Upsilon(x, y) \mid \Upsilon] \leq D_\Upsilon(x, y) \leq M\mathbf{E}[D_\Upsilon(x, y) \mid \Upsilon] \quad \textit{almost surely.}$$

Proof. Given Υ, let $\widetilde{D}_\Upsilon$ be a conditionally independent copy of D_Υ. Then, by Proposition 3.2, there exists a deterministic constant $M \geq 1$ such that, almost surely,

$$M^{-1}D_\Upsilon(u, v) \leq \widetilde{D}_\Upsilon(u, v) \leq MD_\Upsilon(u, v), \quad \forall u, v \in \Upsilon \text{ with } u \leftrightarrow v. \tag{3.3}$$

Write

$$a(\Upsilon) \stackrel{\text{def}}{=} \inf\{t > 0 : \mathbf{P}[D_\Upsilon(x, y) \leq t \mid \Upsilon] > 0\}; \quad A(\Upsilon) \stackrel{\text{def}}{=} \sup\{t > 0 : \mathbf{P}[D_\Upsilon(x, y) \geq t \mid \Upsilon] > 0\}.$$

Then $a(\Upsilon)$ and $A(\Upsilon)$ are almost surely determined by Υ, and

$$D_\Upsilon(x,y), \mathbf{E}[D_\Upsilon(x,y) \mid \Upsilon] \in [a(\Upsilon), A(\Upsilon)] \quad \text{almost surely.}$$

Thus, it suffices to show that $A(\Upsilon)/a(\Upsilon) \le M$ almost surely. (In particular, $a(\Upsilon) > 0$ and $A(\Upsilon) < \infty$ almost surely.) Choose $\lambda \in (0, A(\Upsilon)/a(\Upsilon))$, in a manner that is almost surely determined by Υ. Then there exists $a' > a(\Upsilon)$ and $A' < A(\Upsilon)$ such that $A'/a' \ge \lambda$. Then

$$\begin{aligned} \mathbf{P}\left[\widetilde{D}_\Upsilon(x,y)/D_\Upsilon(x,y) \ge \lambda \,\middle|\, \Upsilon\right] &\ge \mathbf{P}\left[\widetilde{D}_\Upsilon(x,y) \ge A',\ D_\Upsilon(x,y) \le a' \,\middle|\, \Upsilon\right] \\ &= \mathbf{P}\left[\widetilde{D}_\Upsilon(x,y) \ge A' \,\middle|\, \Upsilon\right] \mathbf{P}[D_\Upsilon(x,y) \le a' \mid \Upsilon] > 0. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we obtain that $\lambda \le M$ almost surely. By letting $\lambda \to A(\Upsilon)/a(\Upsilon)$, we conclude that $A(\Upsilon)/a(\Upsilon) \le M$ almost surely. This completes the proof. □

Lemma 3.7. *There is a deterministic constant $M \ge 1$ such that the following is true: Suppose that D_Υ and $\widetilde{D}_\Upsilon$ have the same conditional law given Υ. (D_Υ and $\widetilde{D}_\Upsilon$ are not necessarily conditionally independent given Υ.) Then $M^{-1} D_\Upsilon \le \widetilde{D}_\Upsilon \le M D_\Upsilon$ almost surely.*

Proof. Let M be as in Lemma 3.6. Label the connected components of Υ as $\Upsilon_1, \Upsilon_2, \ldots$, in a manner that is almost surely determined by Υ. For each $j \in \mathbb{N}$, choose a pairwise distinct sequence $x_{j,1}, x_{j,2}, \ldots \in \Upsilon_j$, in a manner that is almost surely determined by Υ, such that $\{x_{j,k}\}_{k \in \mathbb{N}}$ is almost surely dense in Υ_j. Then it follows from Lemma 3.6 that, almost surely,

$$\begin{aligned} &M^{-1}\mathbf{E}[D_\Upsilon(x_{j,k}, x_{j,l}) \mid \Upsilon] \le D_\Upsilon(x_{j,k}, x_{j,l}) \le M\mathbf{E}[D_\Upsilon(x_{j,k}, x_{j,l}) \mid \Upsilon] \\ &\qquad \text{and} \quad M^{-1}\mathbf{E}[D_\Upsilon(x_{j,k}, x_{j,l}) \mid \Upsilon] \le \widetilde{D}_\Upsilon(x_{j,k}, x_{j,l}) \le M\mathbf{E}[D_\Upsilon(x_{j,k}, x_{j,l}) \mid \Upsilon], \quad \forall j,k,l \in \mathbb{N}. \end{aligned}$$

This implies that, almost surely,

$$M^{-2} D_\Upsilon(x_{j,k}, x_{j,l}) \le \widetilde{D}_\Upsilon(x_{j,k}, x_{j,l}) \le M^2 D_\Upsilon(x_{j,k}, x_{j,l}), \quad \forall j,k,l \in \mathbb{N}.$$

Thus, we conclude from the continuity of D_Υ and $\widetilde{D}_\Upsilon$ that, almost surely,

$$M^{-2} D_\Upsilon(x,y) \le \widetilde{D}_\Upsilon(x,y) \le M^2 D_\Upsilon(x,y), \quad \forall x, y \in \Upsilon \text{ with } x \leftrightarrow y.$$

This completes the proof. □

For $O \in \mathbb{C}/\mathbb{Z}^2$ and $\varepsilon \in (0,1)$, we shall write

$$\mathbb{G}_\varepsilon(O) \overset{\text{def}}{=} \{z \in \mathbb{C} : \Re(z - \varepsilon O) \in \varepsilon\mathbb{Z} \text{ or } \Im(z - \varepsilon O) \in \varepsilon\mathbb{Z}\},$$

and $\mathbb{S}_\varepsilon(O)$ for the collection of connected components of $\mathbb{C} \setminus \mathbb{G}_\varepsilon(O)$.

Let O be sampled uniformly from $\mathbb{C}/\mathbb{Z}^2$, independently of everything else.

Lemma 3.8. *Let $P \colon [0,1] \to \Upsilon$ be a path of finite D_Υ-length, chosen in a manner that is almost surely determined by Υ and D_Υ. Then for each $\varepsilon \in (0,1)$,*

$$\operatorname{len}(P; D_\Upsilon) = \sum_{S \in \mathbb{S}_\varepsilon(O)} \operatorname{len}(P \cap S; D_\Upsilon) \quad \textit{almost surely.} \tag{3.5}$$

Proof. Since O is independent of P, it follows that for each deterministic $t \in [0,1]$, we have $\mathbf{P}[P(t) \in \mathbb{G}_\varepsilon(O) \mid \Upsilon,\ D_\Upsilon] = 0$. This implies that the Lebesgue measure of the set $\{t \in [0,1] : P(t) \in \mathbb{G}_\varepsilon(O)\}$ is almost surely zero, which implies (3.5). □

Lemma 3.9. *For each $\varepsilon \in (0,1)$, the metric D_Υ is almost surely determined by Υ, O, and the internal metrics $D_\Upsilon(\bullet, \bullet; S \cap \Upsilon)$ for $S \in \mathbb{S}_\varepsilon(O)$.*

Proof. Given Υ, O, and the internal metrics $D_\Upsilon(\bullet,\bullet;S\cap\Upsilon)$ for $S\in\mathcal{S}_\varepsilon(O)$, let $\widetilde{D}_\Upsilon$ be a conditionally independent copy of D_Υ. Thus, it suffices to show that $D_\Upsilon=\widetilde{D}_\Upsilon$ almost surely. By definition, almost surely,

$$D_\Upsilon(\bullet,\bullet;S\cap\Upsilon)=\widetilde{D}_\Upsilon(\bullet,\bullet;S\cap\Upsilon),\quad\forall S\in\mathcal{S}_\varepsilon(O). \tag{3.6}$$

By Lemma 3.7, there exists a deterministic constant $M\geq 1$ such that, almost surely,

$$M^{-1}D_\Upsilon(u,v)\leq\widetilde{D}_\Upsilon(u,v)\leq MD_\Upsilon(u,v),\quad\forall u,v\in\Upsilon\text{ with }u\leftrightarrow v. \tag{3.7}$$

Choose $x,y\in\Upsilon$ with $x\leftrightarrow y$ and a D_Υ-geodesic P from x to y, in a manner that is almost surely determined by Υ and D_Υ. By (3.7), the $\widetilde{D}_\Upsilon$-length of P is almost surely finite. Thus, we conclude from Lemma 3.8 and (3.6) that, almost surely,

$$D_\Upsilon(x,y)=\sum_{S\in\mathcal{S}_\varepsilon(O)}\mathrm{len}(P\cap S;D_\Upsilon)=\sum_{S\in\mathcal{S}_\varepsilon(O)}\mathrm{len}(P\cap S;\widetilde{D}_\Upsilon)=\mathrm{len}(P;\widetilde{D}_\Upsilon)\geq\widetilde{D}_\Upsilon(x,y).$$

By symmetry, it follows that $D_\Upsilon(x,y)\leq\widetilde{D}_\Upsilon(x,y)$ almost surely, hence that $D_\Upsilon(x,y)=\widetilde{D}_\Upsilon(x,y)$ almost surely. Let $\{x_{j,k}\}_{j,k\in\mathbb{N}}$ be as in the proof of Lemma 3.7. Then it follows that, almost surely, $D_\Upsilon(x_{j,k},x_{j,l})=\widetilde{D}_\Upsilon(x_{j,k},x_{j,l})$ for all $j,k,l\in\mathbb{N}$. Thus, we conclude from the continuity of D_Υ and $\widetilde{D}_\Upsilon$ that $D_\Upsilon=\widetilde{D}_\Upsilon$ almost surely. This completes the proof. □

Lemma 3.10. *For each $\varepsilon\in(0,1)$, the internal metrics $D_\Upsilon(\bullet,\bullet;S\cap\Upsilon)$ for $S\in\mathcal{S}_\varepsilon(O)$ are conditionally independent given Υ and O.*

Proof. This follows immediately from Definition 3.1, Axiom (II'). □

Before proceeding, we recall the following well-known concentration result.

Lemma 3.11 (Efron-Stein)**.** *Let $X_1,\dots,X_n$ be independent random variables and $(X_1',\dots,X_n')$ be an independent copy of $(X_1,\dots,X_n)$. Then*

$$\mathrm{Var}[F(X_1,\dots,X_n)]\leq\frac{1}{2}\sum_{j=1}^n\mathbf{E}\Big[\big(F(X_1,\dots,X_n)-F(X_1,\dots,X_{j-1},X_j',X_{j+1},\dots,X_n)\big)^2\Big]$$

for all measurable functionals F.

Proof of Proposition 3.5. Fix $\varepsilon>0$. By Lemma 3.9, there exists a measurable mapping F such that

$$D_\Upsilon=F(\Upsilon,O,\{D_\Upsilon(\bullet,\bullet;S\cap\Upsilon):S\in\mathcal{S}_\varepsilon(O)\})\quad\text{almost surely.}$$

Given Υ and O, let $\{D_\Upsilon'(\bullet,\bullet;S\cap\Upsilon):S\in\mathcal{S}_\varepsilon(O)\}$ be a conditionally independent copy of $\{D_\Upsilon(\bullet,\bullet;S\cap\Upsilon):S\in\mathcal{S}_\varepsilon(O)\}$. Given Υ and O, for each $S\in\mathcal{S}_\varepsilon(O)$, write

$$D_\Upsilon^S\stackrel{\mathrm{def}}{=}F\left(\Upsilon,O,\{D_\Upsilon'(\bullet,\bullet;S\cap\Upsilon)\}\cup\{D_\Upsilon(\bullet,\bullet;\widetilde{S}):\widetilde{S}\in\mathcal{S}_\varepsilon(O),\ \widetilde{S}\neq S\}\right).$$

Choose $x,y\in\Upsilon$ with $x\leftrightarrow y$, in a manner that is almost surely determined by Υ. Then it follows from Lemmas 3.10 and 3.11 that, almost surely,

$$\mathrm{Var}[D_\Upsilon(x,y)\mid\Upsilon]=\mathrm{Var}[D_\Upsilon(x,y)\mid\Upsilon,\ O]\leq\frac{1}{2}\sum_{S\in\mathcal{S}_\varepsilon(O)}\mathbf{E}\Big[\big(D_\Upsilon^S(x,y)-D_\Upsilon(x,y)\big)^2\,\Big|\,\Upsilon,\ O\Big]. \tag{3.8}$$

We *claim* that, almost surely,

$$\sum_{S\in\mathcal{S}_\varepsilon(O)}\mathbf{E}\Big[\big(D_\Upsilon^S(x,y)-D_\Upsilon(x,y)\big)^2\,\Big|\,\Upsilon,\ O\Big]\to 0\quad\text{as }\varepsilon\to 0. \tag{3.9}$$

By Lemma 3.7, there exists a deterministic constant $M \geq 1$ such that, almost surely,

$$M^{-1} D_\Upsilon(u,v) \leq D_\Upsilon^S(u,v) \leq M D_\Upsilon(u,v), \quad \forall S \in \mathcal{S}_\varepsilon(O),\ \forall u,v \in \Upsilon \text{ with } u \leftrightarrow v. \tag{3.10}$$

Choose a D_Υ-geodesic P from x to y, in a manner that is almost surely determined by Υ and D_Υ. By Lemma 3.8 and (3.10), we have

$$\begin{aligned} D_\Upsilon^S(x,y) \leq \operatorname{len}(P; D_\Upsilon^S) &= \sum_{\widetilde{S} \in \mathcal{S}_\varepsilon(O)} \operatorname{len}(P \cap \widetilde{S}; D_\Upsilon^S) \\ &= \sum_{\widetilde{S} \in \mathcal{S}_\varepsilon(O) \setminus \{S\}} \operatorname{len}(P \cap \widetilde{S}; D_\Upsilon) + \operatorname{len}(P \cap S; D_\Upsilon^S) \leq D_\Upsilon(x,y) + M \operatorname{len}(P \cap S; D_\Upsilon). \end{aligned} \tag{3.11}$$

By symmetry, we have

$$\mathbf{E}\Big[\big(D_\Upsilon^S(x,y) - D_\Upsilon(x,y)\big)^2 \,\Big|\, \Upsilon,\ O\Big] = 2\mathbf{E}\Big[\big(D_\Upsilon^S(x,y) - D_\Upsilon(x,y)\big)_+^2 \,\Big|\, \Upsilon,\ O\Big], \tag{3.12}$$

where $(D_\Upsilon^S(x,y) - D_\Upsilon(x,y))_+ \overset{\text{def}}{=} (D_\Upsilon^S(x,y) - D_\Upsilon(x,y)) \vee 0$. Combining (3.11) and (3.12), we obtain that

$$\begin{aligned} &\sum_{S \in \mathcal{S}_\varepsilon(O)} \mathbf{E}\Big[\big(D_\Upsilon^S(x,y) - D_\Upsilon(x,y)\big)^2 \,\Big|\, \Upsilon,\ O\Big] \\ &= 2 \sum_{S \in \mathcal{S}_\varepsilon(O)} \mathbf{E}\Big[\big(D_\Upsilon^S(x,y) - D_\Upsilon(x,y)\big)_+^2 \,\Big|\, \Upsilon,\ O\Big] \\ &\leq 2M^2 \sum_{S \in \mathcal{S}_\varepsilon(O)} \mathbf{E}\big[\operatorname{len}(P \cap S; D_\Upsilon)^2 \,\big|\, \Upsilon,\ O\big] \\ &\leq 2M^2 \mathbf{E}\left[\left(\max_{S \in \mathcal{S}_\varepsilon(O)} \operatorname{len}(P \cap S; D_\Upsilon)\right) \sum_{S \in \mathcal{S}_\varepsilon(O)} \operatorname{len}(P \cap S; D_\Upsilon) \,\middle|\, \Upsilon,\ O\right] \\ &\leq 2M^2 \mathbf{E}\left[\left(\max_{S \in \mathcal{S}_\varepsilon(O)} \operatorname{len}(P \cap S; D_\Upsilon)\right)^2 \,\middle|\, \Upsilon,\ O\right]^{1/2} \mathbf{E}\big[D_\Upsilon(x,y)^2 \,\big|\, \Upsilon\big]^{1/2}. \end{aligned} \tag{3.13}$$

By Lemma 3.6 and possibly increasing M, we may assume without loss of generality that

$$M^{-1}\mathbf{E}[D_\Upsilon(x,y) \mid \Upsilon] \leq D_\Upsilon(x,y) \leq M\mathbf{E}[D_\Upsilon(x,y) \mid \Upsilon] \quad \text{almost surely.} \tag{3.14}$$

This implies that

$$\mathbf{E}\big[D_\Upsilon(x,y)^2 \,\big|\, \Upsilon\big] \leq M^2 \mathbf{E}[D_\Upsilon(x,y) \mid \Upsilon]^2 \leq M^4 D_\Upsilon(x,y) < \infty \quad \text{almost surely,} \tag{3.15}$$

and

$$\max_{S \in \mathcal{S}_\varepsilon(O)} \operatorname{len}(P \cap S; D_\Upsilon) \leq D_\Upsilon(x,y) \leq M\mathbf{E}[D_\Upsilon(x,y) \mid \Upsilon] \quad \text{almost surely.} \tag{3.16}$$

On the other hand, it follows from the continuity of D_Υ that

$$\lim_{\varepsilon \to 0} \max_{S \in \mathcal{S}_\varepsilon(O)} \operatorname{len}(P \cap S; D_\Upsilon) = 0 \quad \text{almost surely.} \tag{3.17}$$

Combining (3.16) and (3.17), it follows from the bounded convergence theorem that, almost surely,

$$\mathbf{E}\left[\left(\max_{S \in \mathcal{S}_\varepsilon(O)} \operatorname{len}(P \cap S; D_\Upsilon)\right)^2 \,\middle|\, \Upsilon,\ O\right] \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.18}$$

Thus, we conclude from (3.13), (3.15), and (3.18) that (3.9) is true. This completes the proof of the *claim*.

Finally, we conclude from (3.8) and (3.9) that

$$\mathrm{Var}[D_\Upsilon(x,y) \mid \Upsilon] = 0 \quad \text{almost surely.}$$

This implies that $D_\Upsilon(x,y)$ is almost surely determined by Υ. Let $\{x_{j,k}\}_{j,k\in\mathbb{N}}$ be as in the proof of Lemma 3.7. Then it follows that, almost surely, $D_\Upsilon(x_{j,k},x_{j,l})$ is determined by Υ for all $j,k,l\in\mathbb{N}$. Thus, we conclude from the continuity of D_Υ that D_Υ is almost surely determined by Υ. This completes the proof. □

4. Convergence of MFPP

In the present section, we aim to deduce Theorem 1.8 from the results of [Mil21]. In Section 4.1, we prove that the laws of the rescaled ε-MFPP metrics $(X\times X, \mathfrak{a}_\varepsilon^{-1}D_X^\varepsilon)$ are tight (where X is as in (1.2)), and that any subsequential limit satisfies Axioms (I) and (IV) of Definition 1.2. In Section 4.2, we prove that any subsequential limit satisfies Axiom (IV) of Definition 1.7. In Section 4.3, we prove that any subsequential limit satisfies Axiom (II') of Definition 3.1. We then conclude the proof of Theorem 1.8 using Proposition 3.5.

4.1. Subsequential limits.

In the present subsection, we explain why the results of [Mil21] imply that the family $\{(X\times X, \mathfrak{a}_\varepsilon^{-1}D_X^\varepsilon)\}_{\varepsilon>0}$ is tight, and that any subsequential limit satisfies Axioms (I) and (IV) of Definition 1.2. This is not entirely immediate, since [Mil21] deals only with non-nested CLE_κ's, whereas Axiom (IV) concerns the whole-plane nested CLE_κ. We also include the corresponding results for the internal metrics on dyadic domains, which will be needed in Section 4.3.

Recall from Section 2.1 the definition of dyadic domains. For $n\in\mathbb{N}$, we shall write $\mathfrak{DyDo}_n$ for the collection of sequences $(W_1,\ldots,W_n)\in\mathfrak{DyDo}^n$ such that $W_j \Subset W_{j+1}$ for all $j\in[1,n-1]_\mathbb{Z}$. For an open subset $U\subset\mathbb{C}$, we shall write $\mathfrak{DyDo}_n(U) \overset{\text{def}}{=} \{(W_1,\ldots,W_n)\in\mathfrak{DyDo}_n : W_n\Subset U\}$.

We shall write $\mathfrak{HausUniEns}_4$ for the metric space consisting of discrete subsets of $\mathfrak{HausUni}_4$, and equipped with the metric given by the Hausdorff distance induced by $d_{\mathfrak{HausUni}_4}$.

Proposition 4.1. *Let $\Gamma_\mathbb{D}$ be a non-nested CLE_κ in $\mathbb{D}$. Write $\mathfrak{L}$ for the loop of $\Gamma_\mathbb{D}$ surrounding 0. Given $\mathfrak{L}$, let $\Gamma_\mathfrak{L}$ be a non-nested CLE_κ inside $\mathfrak{L}$. Write $X_\mathfrak{L}$ for the carpet of $\Gamma_\mathfrak{L}$. For $\varepsilon>0$, write $\mathfrak{b}_\varepsilon$ for the median of the random variable $\sup_{x,y\in\mathfrak{L}} D_{X_\mathfrak{L}}^\varepsilon(x,y)$. Then for every sequence of positive real numbers ε's tending to zero, there exists $D_{X_\mathfrak{L}}$, $\{D_{X_\mathfrak{L},W_1,W_2} : (W_1,W_2)\in\mathfrak{DyDo}_2(\mathbb{D})\}$, and a subsequence $\mathcal{E}$ along which we have the convergence of the joint laws*

$$\begin{aligned}&\Bigg(\Big(X_\mathfrak{L}\times X_\mathfrak{L}, \mathfrak{b}_\varepsilon^{-1}D_{X_\mathfrak{L}}^\varepsilon\Big), \Big\{\Big((\overline{W_1}\cap X_\mathfrak{L})\times(\overline{W_1}\cap X_\mathfrak{L}),\\ &\qquad\qquad \mathfrak{b}_\varepsilon^{-1}D_{X_\mathfrak{L}}^\varepsilon(\bullet,\bullet;W_2\cap X_\mathfrak{L})\big|_{(\overline{W_1}\cap X_\mathfrak{L})\times(\overline{W_1}\cap X_\mathfrak{L})}\Big) : (W_1,W_2)\in\mathfrak{DyDo}_2(\mathbb{D})\Big\}\Bigg)\\ &\to \Bigg(\Big(X_\mathfrak{L}\times X_\mathfrak{L}, D_{X_\mathfrak{L}}\Big), \Big\{\Big((\overline{W_1}\cap X_\mathfrak{L})\times(\overline{W_1}\cap X_\mathfrak{L}), D_{X_\mathfrak{L},W_1,W_2}(\bullet,\bullet)\Big) : (W_1,W_2)\in\mathfrak{DyDo}_2(\mathbb{D})\Big\}\Bigg),\end{aligned}$$

with respect to the metric space $\mathfrak{HausUniEns}_4$. Moreover:

(i) $D_{X_\mathfrak{L}}$ is almost surely a geodesic metric on $X_\mathfrak{L}$ inducing the Euclidean topology.
(ii) Almost surely,

$$D_{X_\mathfrak{L},W_1,W_2}(\bullet,\bullet) = D_{X_\mathfrak{L}}(\bullet,\bullet;W_2\cap X_\mathfrak{L})\big|_{(\overline{W_1}\cap X_\mathfrak{L})\times(\overline{W_1}\cap X_\mathfrak{L})}, \quad \forall (W_1,W_2)\in\mathfrak{DyDo}_2(\mathbb{D}).$$

Proof. See [Mil21, Theorem 1.1 and Section 7]. □

In the remainder of the present subsection, let $\{\mathfrak{b}_\varepsilon\}_{\varepsilon>0}$ be as in Proposition 4.1.

Set $\gamma \stackrel{\mathrm{def}}{=} \sqrt{\kappa}$ and $Q \stackrel{\mathrm{def}}{=} 2/\gamma + \gamma/2$. Let $\ell > 0$. Let $(\mathfrak{D}, \Phi, x, y)$ be a γ-LQG disk of boundary length ℓ (cf. Section 2.3). We shall write μ_Φ (resp. ν_Φ) for the γ-LQG area (resp. length) measure associated with Φ. Suppose that the law of $\partial\mathfrak{D}$ is absolutely continuous with respect to the SLE_κ loop measure, i.e.,

$$“\mathbf{P}[\partial\mathfrak{D} \in A] > 0'' \;\Rightarrow\; “\mathbf{P}[\text{there exists } \mathfrak{L} \in \Gamma \text{ such that } \mathfrak{L} \in A] > 0'', \quad \forall \text{ Borel subset } A \subset \mathfrak{Loop}$$

(where Γ is a whole-plane nested CLE_κ). Given $\mathfrak{D}$, let $\Gamma_{\mathfrak{D}}$ be a nested CLE_κ in $\mathfrak{D}$, conditionally independent of Φ. We shall write $\Gamma_{\mathfrak{D}}^\circ$ for the collection of the outermost loops of $\Gamma_{\mathfrak{D}}$. Thus, $\Gamma_{\mathfrak{D}}^\circ$ is conditionally a non-nested CLE_κ in $\mathfrak{D}$ given $\mathfrak{D}$. We shall write $\widehat{\Gamma}_{\mathfrak{D}} \stackrel{\mathrm{def}}{=} \Gamma_{\mathfrak{D}} \cup \{\partial\mathfrak{D}\}$. For $\mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}$, we shall write $X(\mathfrak{L}) \subset \overline{\mathfrak{D}}$ for the closed subset consisting of points that are surrounded by $\mathfrak{L}$ but by no other loop of $\Gamma_{\mathfrak{D}}$ inside $\mathfrak{L}$. To lighten notation, we shall write $X_{\mathfrak{D}} \stackrel{\mathrm{def}}{=} X(\partial\mathfrak{D})$.

Lemma 4.2. *There are deterministic constants $0 < \alpha_{\mathrm{UBD}} < 2 < \alpha_{\mathrm{LBD}}$ such that the following is true: Let $(\mathbb{D}, \Phi_{\mathbb{D}}, -\mathrm{i}, \mathrm{i})$ be a γ-LQG disk of unit boundary length. Then it holds with superpolynomially high probability as $\varepsilon_* \to 0$ that*

$$\varepsilon^{\alpha_{\mathrm{LBD}}} \le \mu_{\Phi_{\mathbb{D}}}(B_\varepsilon(z)) \le \varepsilon^{\alpha_{\mathrm{UBD}}}, \quad \forall z \in \mathbb{D},\ \forall \varepsilon \in (0, \mathrm{dist}(z, \partial\mathbb{D}) \wedge \varepsilon_*]. \tag{4.1}$$

Proof. See [Mil21, Lemma A.6 and Lemma B.3]. □

Lemma 4.3. *Let α_{UBD} and α_{LBD} be as in Lemma 4.2. Then there almost surely exists $\varepsilon_* \in (0,1)$ such that*

$$\varepsilon^{\alpha_{\mathrm{LBD}}} \le \mu_{\Phi}(B_\varepsilon(z)) \le \varepsilon^{\alpha_{\mathrm{UBD}}}, \quad \forall z \in \mathfrak{D},\ \forall \varepsilon \in (0, \mathrm{dist}(z, \partial\mathfrak{D}) \wedge \varepsilon_*]. \tag{4.2}$$

Proof. Let $\mathfrak{L}$ be as in Proposition 4.1. It suffices to consider the case where $\mathfrak{D}$ is the domain surrounded by $\mathfrak{L}$. Let $(\mathbb{D}, \Phi_{\mathbb{D}}, -\mathrm{i}, \mathrm{i})$ be a γ-LQG disk of unit boundary length. Then Lemma 4.3 follows immediately from Lemma 4.2, together with the fact that $(\mathfrak{D}, \Phi_{\mathbb{D}}|_{\mathfrak{D}}, x, y)$ is conditionally a γ-LQG disk given its boundary length (cf. [MSW22]). □

For each $\varepsilon_* \in (0,1)$, we shall write $E(\varepsilon_*)$ for the event that (4.2) holds.

Lemma 4.4. *There is a deterministic constant $\alpha_{\mathrm{NET}} > 0$ such that the following is true: Let $(\mathbb{D}, \Phi_{\mathbb{D}}, -\mathrm{i}, \mathrm{i})$ be a γ-LQG disk of unit boundary length. Let $\Gamma_{\mathbb{D}}$ be an independent non-nested* CLE_κ *in $\mathbb{D}$. Write $X_{\mathbb{D}}$ for the carpet of $\Gamma_{\mathbb{D}}$. Given $\Phi_{\mathbb{D}}$ and $\Gamma_{\mathbb{D}}$, let $\{x_n\}_{n\in\mathbb{N}}$ be conditionally independent samples from $\mu_{\Phi_{\mathbb{D}}, X_{\mathbb{D}}}$ (renormalized to be a probability measure) (cf. [MSW22]). Then it holds with superpolynomially high probability as $\delta_* \to 0$ that*

$$X_{\mathbb{D}} \subset \bigcup_{j \in [1, \delta^{-\alpha_{\mathrm{NET}}}]_{\mathbb{Z}}} B_\delta(x_j), \quad \forall \delta \in (0, \delta_*].$$

Proof. See [Mil21, Appendix B]. □

Lemma 4.5. *Let α_{NET} be as in Lemma 4.4. Then there is a deterministic constant $\alpha_{\mathrm{PERIM}} > 0$ such that the following is true: Given Φ and $\Gamma_{\mathfrak{D}}$, let $\{x_n\}_{n\in\mathbb{N}}$ be conditionally independent samples from $\mu_{\Phi, X_{\mathfrak{D}}}$ (renormalized to be a probability measure). Let $\varepsilon_* \in (0,1)$. Then, on the event $E(\varepsilon_*)$, it holds with superpolynomially high probability as $\delta_* \to 0$, at a rate depending only on ε_*, that*

$$X_{\mathfrak{D}} \subset \bigcup_{j \in [1, \delta^{-\alpha_{\mathrm{NET}}}]_{\mathbb{Z}}} B_{\ell^{\alpha_{\mathrm{PERIM}}}\delta}(x_j), \quad \forall \delta \in (0, \delta_*]. \tag{4.3}$$

Proof. Let ϕ be a conformal mapping from $\mathfrak{D}$ onto $\mathbb{D}$. Write

$$\Phi_{\mathbb{D}} \overset{\text{def}}{=} \Phi \circ \phi^{-1} + Q \log(|(\phi^{-1})'|) - (2/\gamma)\log(\ell); \quad \Gamma_{\mathbb{D}} \overset{\text{def}}{=} \phi(\Gamma_{\mathfrak{D}}); \quad X_{\mathbb{D}} \overset{\text{def}}{=} \phi(X_{\mathfrak{D}}).$$

Then $(\mathbb{D}, \Phi_{\mathbb{D}}, -\mathrm{i}, \mathrm{i})$ is a γ-LQG disk of unit boundary length, and $\Gamma_{\mathbb{D}}$ is an independent non-nested CLE_κ in $\mathbb{D}$. Write $E_{\mathbb{D}}(\delta_*)$ for the event that (4.1) holds. Then it follows from Lemma 4.2 that $E_{\mathbb{D}}(\delta_*)$ holds with superpolynomially high probability as $\delta_* \to 0$. Write $F_{\mathbb{D}}(\delta_*)$ for the event that

$$X_{\mathbb{D}} \subset \bigcup_{j \in [1, \delta^{-\alpha_{\mathrm{NET}}}]_{\mathbb{Z}}} B_\delta(\phi(x_j)), \quad \forall \delta \in (0, \delta_*]. \tag{4.4}$$

Since $\{\phi(x_j)\}_{j\in\mathbb{N}}$ has the law of conditionally independent samples from $\mu_{\Phi_{\mathbb{D}}, X_{\mathbb{D}}}$ (renormalized to be a probability measure), it follows from Lemma 4.5 that $F_{\mathbb{D}}(\delta_*)$ holds with superpolynomially high probability as $\delta_* \to 0$.

Now, suppose that we are working on the event $E(\varepsilon_*) \cap E_{\mathbb{D}}(\delta_*) \cap F_{\mathbb{D}}(\delta_*)$ for some $\delta_* \in (0, \varepsilon_*]$. For each $\delta \in (0, \delta_*]$, by Koebe's quarter theorem and the definitions of $E(\varepsilon_*)$ and $E_{\mathbb{D}}(\delta_*)$, we have

$$\begin{aligned}\delta^{\alpha_{\mathrm{UBD}}} &\geq \mu_{\Phi_{\mathbb{D}}}(B_\delta(z)) = \ell^{-2}\mu_\Phi\left(\phi^{-1}(B_\delta(z))\right) \\ &\geq \ell^{-2}\mu_\Phi\left(B_{|(\phi^{-1})'(z)|\delta/4}(z)\right) \geq \ell^{-2}\left(|(\phi^{-1})'(z)|\delta/4\right)^{\alpha_{\mathrm{LBD}}}, \quad \forall z \in B_{1-\delta}(0).\end{aligned}$$

This implies that

$$\sup_{z \in B_{1-\delta}(0)} \left|(\phi^{-1})'(z)\right| \leq 4\ell^{2/\alpha_{\mathrm{LBD}}}\delta^{\alpha_{\mathrm{UBD}}/\alpha_{\mathrm{LBD}}-1}, \quad \forall \delta \in (0, \delta_*].$$

Since $\alpha_{\mathrm{UBD}}/\alpha_{\mathrm{LBD}} - 1 > -1$, this implies that ϕ^{-1} extends to an $\alpha_{\mathrm{UBD}}/\alpha_{\mathrm{LBD}}$-Hölder continuous mapping from $\overline{\mathbb{D}}$ onto $\overline{\mathfrak{D}}$ with Hölder coefficient equal to $\ell^{2/\alpha_{\mathrm{LBD}}}$ times some constant depending only on α_{UBD} and α_{LBD}. Thus, by decreasing α_{NET}, (4.3) follows immediately from (4.4). This completes the proof. □

Proposition 4.6. *There are deterministic constants $\alpha_{\mathrm{KC}}, \alpha_{\mathrm{PERIM}} > 0$ such that the following is true: Let $\varepsilon_* \in (0,1)$. Then, on the event $E(\varepsilon_*)$, it holds with superpolynomially high probability as $A \to 0$, at a rate depending only on ε_*, that*

$$\mathfrak{b}_\varepsilon^{-1} D^\varepsilon_{X_{\mathfrak{D}}}(u,v) \leq A\ell^{\alpha_{\mathrm{PERIM}}}|u-v|^{\alpha_{\mathrm{KC}}}, \quad \forall u,v \in X_{\mathfrak{D}}, \ \forall \varepsilon \in (0, \varepsilon_*].$$

Proof. See [Mil21, Section 6]. □

Lemma 4.7. *There is a constant $\mathfrak{k} > 0$ such that for each connected and compact subset $K \subset \mathbb{C}$,*

$$\mathrm{Leb}(B_{\varepsilon r}(K)) \geq \mathfrak{k}\varepsilon\,\mathrm{Leb}(B_r(K)), \quad \forall r > 0, \ \forall \varepsilon \in (0,1).$$

In particular,

$$\mathfrak{a}_{\varepsilon r}/\mathfrak{a}_r \geq \mathfrak{k}\varepsilon \quad \textit{and} \quad \mathfrak{b}_{\varepsilon r}/\mathfrak{b}_r \geq \mathfrak{k}\varepsilon, \quad \forall r > 0, \ \forall \varepsilon \in (0,1).$$

Proof. See [Mil21, Lemma 5.4]. □

Lemma 4.8. *(i)* $\mathbf{E}[\mu_\Phi(\mathfrak{D})^p] < \infty$ *for all* $p \in (-\infty, 4/\gamma^2)$.

(ii) $\mathbf{E}\left[\sum_{\mathfrak{L}\in\Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L})^p\right] < \infty$ *for all* $p \in (2, 8/\gamma^2)$.

(iii) Let $\varepsilon_* \in (0,1)$. Then

$$\mathbf{P}\left[\sup_{\mathfrak{L}\in\Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L}) \geq A \,\middle|\, \mu_\Phi(\mathfrak{D}) \leq 1/\varepsilon_*\right] = O(A^{-\infty}) \quad \textit{as } A \to \infty.$$

(iv) Let $\varepsilon_* \in (0,1)$. Then

$$\mathbf{E}\left[\sum_{\mathfrak{L}\in\Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L})^p \,\middle|\, \mu_\Phi(\mathfrak{D}) \leq 1/\varepsilon_*\right] < \infty, \quad \forall p > 2.$$

Proof. Assertion (i) follows, e.g., from [AG21].

Next, we consider assertion (ii). Since $X_{\mathfrak{D}}$ is conditionally independent of Φ given $\mathfrak{D}$ and has zero Lebesgue measure, it follows that $\mu_\Phi(X_{\mathfrak{D}}) = 0$ almost surely. Thus, almost surely,

$$\mu_\Phi(\mathfrak{D}) = \sum_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}^\circ} \mu_\Phi(\mathfrak{D}(\mathfrak{L})),$$

where $\mathfrak{D}(\mathfrak{L})$ denotes the domain surrounded by $\mathfrak{L}$. This implies that, almost surely,

$$\mu_\Phi(\mathfrak{D})^p > \sum_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}^\circ} \mu_\Phi(\mathfrak{D}(\mathfrak{L}))^p, \quad \forall p > 1. \tag{4.5}$$

By [MSW22], for each $\mathfrak{L} \in \Gamma_{\mathfrak{D}}^\circ$, the γ-LQG surface parameterized by $\mathfrak{D}(\mathfrak{L})$ is conditionally a γ-LQG disk given its boundary length. Thus, it follows from assertion (i) that

$$\mathbf{E}[\mu_\Phi(\mathfrak{D}(\mathfrak{L}))^p \mid \nu_\Phi(\mathfrak{L})] = \nu_\Phi(\mathfrak{L})^{2p}\mathbf{E}[\mu_\Phi(\mathfrak{D})^p] < \infty, \quad \forall \mathfrak{L} \in \Gamma_{\mathfrak{D}}^\circ,\ \forall p \in (-\infty, 4/\gamma^2). \tag{4.6}$$

Combining (4.5) and (4.6), we obtain that

$$\mathbf{E}\left[\sum_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}^\circ} \nu_\Phi(\mathfrak{L})^{2p}\right] < 1, \quad \forall p \in (1, 4/\gamma^2). \tag{4.7}$$

Now, assertion (ii) follows immediately from (4.7), together with the fact that the γ-LQG lengths of the loops of $\Gamma_{\mathfrak{D}}$ form a multiplicative cascade with offspring distribution given by the law of the collection $\{\nu_\Phi(\mathfrak{L}) : \mathfrak{L} \in \Gamma_{\mathfrak{D}}^\circ\}$.

Next, we consider assertion (iii). By Bayes' theorem,

$$\begin{aligned}
&\mathbf{P}\left[\sup_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L}) \geq A \,\middle|\, \mu_\Phi(\mathfrak{D}) \leq 1/\varepsilon_*\right] \\
&\quad = \mathbf{P}\left[\mu_\Phi(\mathfrak{D}) \leq 1/\varepsilon_* \,\middle|\, \sup_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L}) \geq A\right] \mathbf{P}\left[\sup_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L}) \geq A\right] \bigg/ \mathbf{P}[\mu_\Phi(\mathfrak{D}) \leq 1/\varepsilon_*] \\
&\qquad \leq \mathbf{P}\left[\mu_\Phi(\mathfrak{D}) \leq 1/\varepsilon_* \,\middle|\, \sup_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L}) \geq A\right] \bigg/ \mathbf{P}[\mu_\Phi(\mathfrak{D}) \leq 1/\varepsilon_*] \,.
\end{aligned} \tag{4.8}$$

Since for each $\mathfrak{L} \in \Gamma_{\mathfrak{D}}$, the γ-LQG surface parameterized by the domain surrounded by $\mathfrak{L}$ is conditionally a γ-LQG disk given its boundary length, it follows from Markov's inequality that

$$\mathbf{P}\left[\mu_\Phi(\mathfrak{D}) \leq 1/\varepsilon_* \,\middle|\, \sup_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L}) \geq A\right] \leq \mathbf{P}[\mu_\Phi(\mathfrak{D}) \leq 1/(A^2\varepsilon_*)] \leq \mathbf{E}[\mu_\Phi(\mathfrak{D})^{-p}](A^2\varepsilon_*)^{-p}. \tag{4.9}$$

for all $p > 0$. Combining (4.8), (4.9), and assertion (i), we complete the proof of assertion (iii).

Finally, assertion (iv) follows immediately from assertions (ii) and (iii). □

Lemma 4.9. *Let α_{KC} be as in Proposition 4.6. Then for every sequence of positive real numbers ε's tending to zero, there exists $\{D_{X(\mathfrak{L})} : \mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}\}$ and a subsequence $\mathcal{E}$ along which we have the convergence of the joint laws*

$$\left\{\left(X(\mathfrak{L}) \times X(\mathfrak{L}), \mathfrak{b}_\varepsilon^{-1} D^\varepsilon_{X(\mathfrak{L})}\right) : \mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}\right\} \to \left\{\left(X(\mathfrak{L}) \times X(\mathfrak{L}), D_{X(\mathfrak{L})}\right) : \mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}\right\}, \tag{4.10}$$

with respect to the metric space $\mathfrak{HausUniEns}_4$. Moreover:

(i) *Almost surely, for each $\mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}$, $D_{X(\mathfrak{L})}$ is a geodesic metric on $X(\mathfrak{L})$ inducing the Euclidean topology.*

(ii) *Given $\mathfrak{D}$, it holds with superpolynomially high conditional probability as $A \to \infty$, at a rate which is uniform in the choice of the subsequential limit, that*

$$D_{X(\mathfrak{L})}(u,v) \le A|u-v|^{\alpha_{\mathrm{KC}}}, \quad \forall \mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}.$$

Proof. It follows immediately from Proposition 4.1 that for each $\mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}$, the family $\{(X(\mathfrak{L}) \times X(\mathfrak{L}), \mathfrak{b}_\varepsilon^{-1} D^\varepsilon_{X(\mathfrak{L})})\}_{\varepsilon>0}$ is tight. Thus, by Tikhonov's theorem and Prokhorov's theorem, the family of the joint laws

$$\left\{ \left(X(\mathfrak{L}) \times X(\mathfrak{L}), \mathfrak{b}_\varepsilon^{-1} D^\varepsilon_{X(\mathfrak{L})} \right) : \mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}} \right\}$$

for $\varepsilon > 0$ is tight. Thus, we conclude that there exists $\{D_{X(\mathfrak{L})} : \mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}\}$ and a subsequence $\mathcal{E}$ along which (4.10) holds. Moreover, assertion (i) follows immediately from Proposition 4.1. Thus, it suffices to verify assertion (ii).

To lighten notation, we may assume that $\mathfrak{D}$ is deterministic. Let $\varepsilon_* > 0$. Write $F(\varepsilon_*)$ for the event that $\mu_\Phi(\mathfrak{D}) \le 1/\varepsilon_*$. By Proposition 4.6, for each $p > 0$, there exists $C_p = C_p(\varepsilon_*) > 0$ such that

$$\mathbf{P}\left[E(\varepsilon_*) \cap \left\{ \sup_{u,v \in X_{\mathfrak{D}}} \frac{\mathfrak{b}_\varepsilon^{-1} D^\varepsilon_{X_{\mathfrak{D}}}(u,v)}{|u-v|^{\alpha_{\mathrm{KC}}}} \ge A \right\} \right] \le C_p A^{-p} \ell^{\alpha_{\mathrm{PERIM}} p}, \quad \forall \varepsilon \in (0, \varepsilon_*], \ \forall A > 0,$$

hence that

$$\mathbf{P}\left[E(\varepsilon_*) \cap F(\varepsilon_*) \cap \left\{ \sup_{u,v \in X_{\mathfrak{D}}} \frac{\mathfrak{b}_\varepsilon^{-1} D^\varepsilon_{X_{\mathfrak{D}}}(u,v)}{|u-v|^{\alpha_{\mathrm{KC}}}} \ge A \right\} \right] \le C_p A^{-p} \ell^{\alpha_{\mathrm{PERIM}} p}, \quad \forall \varepsilon \in (0, \varepsilon_*], \ \forall A > 0. \tag{4.11}$$

Let $\mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}$. Write $\mathfrak{D}(\mathfrak{L})$ for the simply connected domain surrounded by $\mathfrak{L}$. By [MSW22], the γ-LQG surface parameterized by $\mathfrak{D}(\mathfrak{L})$ is conditionally a γ-LQG disk given its boundary length. Note that if $E(\varepsilon_*) \cap F(\varepsilon_*)$ occurs, then $E(\varepsilon_*) \cap F(\varepsilon_*)$ also occurs with $\mathfrak{D}(\mathfrak{L})$ and $\Phi|_{\mathfrak{D}(\mathfrak{L})}$ in place of $\mathfrak{D}$ and Φ, respectively. Thus, we conclude from (4.11) that

$$\mathbf{P}\left[E(\varepsilon_*) \cap F(\varepsilon_*) \cap \left\{ \sup_{u,v \in X(\mathfrak{L})} \frac{\mathfrak{b}_\varepsilon^{-1} D^\varepsilon_{X(\mathfrak{L})}(u,v)}{|u-v|^{\alpha_{\mathrm{KC}}}} \ge A \right\} \,\middle|\, \nu_\Phi(\mathfrak{L}) \right] \le C_p A^{-p} \nu_\Phi(\mathfrak{L})^{\alpha_{\mathrm{PERIM}} p}, \quad \forall \varepsilon \in (0, \varepsilon_*], \ \forall p > 0, \ \forall A > 0. \tag{4.12}$$

Thus, we conclude from (4.12) that

$$\begin{aligned}
&\mathbf{P}\left[E(\varepsilon_*) \cap F(\varepsilon_*) \cap \bigcup_{\mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}} \left\{ \sup_{u,v \in X(\mathfrak{L})} \frac{D_{X(\mathfrak{L})}(u,v)}{|u-v|^{\alpha_{\mathrm{KC}}}} \ge A \right\} \right] \\
&\le \sum_{\mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}} \mathbf{P}\left[E(\varepsilon_*) \cap F(\varepsilon_*) \cap \left\{ \sup_{u,v \in X(\mathfrak{L})} \frac{D_{X(\mathfrak{L})}(u,v)}{|u-v|^{\alpha_{\mathrm{KC}}}} \ge A \right\} \right] \\
&\le \sum_{\mathfrak{L} \in \widehat{\Gamma}_{\mathfrak{D}}} \liminf_{\mathcal{E} \ni \varepsilon \to 0} \mathbf{P}\left[E(\varepsilon_*) \cap F(\varepsilon_*) \cap \left\{ \sup_{u,v \in X(\mathfrak{L})} \frac{\mathfrak{b}_\varepsilon^{-1} D^\varepsilon_{X(\mathfrak{L})}(u,v)}{|u-v|^{\alpha_{\mathrm{KC}}}} \ge A \right\} \right] \\
&\le C_p A^{-p} \left(\ell + \mathbf{E}\left[\sum_{\mathfrak{L} \in \Gamma_{\mathfrak{D}}} \nu_\Phi(\mathfrak{L})^{\alpha_{\mathrm{PERIM}} p} \,\middle|\, E(\varepsilon_*) \cap F(\varepsilon_*) \right] \right), \quad \forall p > 0, \ \forall A > 0.
\end{aligned}$$

Combining this with Lemma 4.8, (iv), we obtain that

$$\mathbf{P}\left[E(\varepsilon_*)\cap F(\varepsilon_*)\cap \bigcup_{\mathscr{L}\in\widehat{\Gamma}_{\mathfrak{D}}}\left\{\sup_{u,v\in X(\mathscr{L})}\frac{D_{X(\mathscr{L})}(u,v)}{|u-v|^{\alpha_{\mathrm{KC}}}}\geq A\right\}\right]=O(A^{-\infty})\quad \text{as } A\to\infty, \tag{4.13}$$

at a rate which is uniform in the choice of the subsequential limit. Since there almost surely exists $\varepsilon_*\in(0,1)$ such that $E(\varepsilon_*)\cap F(\varepsilon_*)$ occurs (cf. Lemma 4.3), and Φ and $\Gamma_{\mathfrak{D}}$ are conditionally independent given $\mathfrak{D}$, we conclude from (4.13) that

$$\mathbf{P}\left[\bigcup_{\mathscr{L}\in\widehat{\Gamma}_{\mathfrak{D}}}\left\{\sup_{u,v\in X(\mathscr{L})}\frac{D_{X(\mathscr{L})}(u,v)}{|u-v|^{\alpha_{\mathrm{KC}}}}\geq A\right\}\right]=O(A^{-\infty})\quad \text{as } A\to\infty.$$

This completes the proof. □

Remark 4.10. It follows from Lemma 4.5, together with a similar argument to the argument applied in the proof of Lemma 4.9, that the following is true: Let α_{NET} be as in Lemma 4.4. Then, given $\mathfrak{D}$, it holds with superpolynomially high conditional probability as $\varepsilon_*\to\infty$ that for each $\varepsilon\in(0,\varepsilon_*]$ and $\mathscr{L}\in\widehat{\Gamma}_{\mathfrak{D}}$, there exists a finite subset $Z\subset X(\mathscr{L})$ with $\#Z\leq\lfloor\varepsilon^{-\alpha_{\mathrm{NET}}}\rfloor$ such that $X(\mathscr{L})\subset\bigcup_{z\in Z}B_\varepsilon(z)$.

Lemma 4.11. *Let α_{KC} be as in Proposition 4.6. Then for every sequence of positive real numbers ε's tending to zero, there exists $\{D_{X(\mathscr{L})}:\mathscr{L}\in\Gamma\}$ and a subsequence $\mathcal{E}$ along which we have the convergence of the joint laws*

$$\left\{\left(X(\mathscr{L})\times X(\mathscr{L}),\mathfrak{b}_\varepsilon^{-1}D^\varepsilon_{X(\mathscr{L})}\right):\mathscr{L}\in\Gamma\right\}\to\left\{\left(X(\mathscr{L})\times X(\mathscr{L}),D_{X(\mathscr{L})}\right):\mathscr{L}\in\Gamma\right\}, \tag{4.14}$$

with respect to the metric space $\mathfrak{HausUniEns}_4$. (Here, we recall from Section 1.2 the definition of $X(\mathscr{L})$ for $\mathscr{L}\in\Gamma$.) Moreover:

- *(i) Almost surely, for each $\mathscr{L}\in\Gamma$, $D_{X(\mathscr{L})}$ is a geodesic metric on $X(\mathscr{L})$ inducing the Euclidean topology.*
- *(ii) Let $K\subset\mathbb{C}$ be a deterministic compact subset. Then it holds with superpolynomially high probability as $A\to\infty$, at a rate which is uniform in the choice of the subsequential limit, that*

$$D_{X(\mathscr{L})}(u,v)\leq A|u-v|^{\alpha_{\mathrm{KC}}},\quad \forall\mathscr{L}\in\Gamma,\ \forall u,v\in K\cap X(\mathscr{L}).$$

Proof. This follows immediately from Lemma 4.9 by letting $\mathfrak{D}\to\mathbb{C}$. □

Lemma 4.12. *$\mathfrak{a}_\varepsilon\asymp\mathfrak{b}_\varepsilon$ as $\varepsilon\to 0$. (Here, we recall from (1.3) the definition of $\mathfrak{a}_\varepsilon$.)*

Proof. For $\varepsilon>0$, write

$$X_\varepsilon\overset{\text{def}}{=}\sup_{u,v\in X(\mathscr{L}_1)}D^\varepsilon_{X(\mathscr{L}_1)}(u,v),$$

where $\mathscr{L}_1$ denotes the larges loop of Γ contained in $B_1(0)$ and surrounding the origin. By definition,

$$\mathbf{P}\left[\mathfrak{a}_\varepsilon^{-1}X_\varepsilon\geq 1\right]\geq 1/2\quad\text{and}\quad\mathbf{P}\left[\mathfrak{a}_\varepsilon^{-1}X_\varepsilon\leq 1\right]\geq 1/2. \tag{4.15}$$

Suppose that there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\lim_{n\to\infty}\varepsilon_n=0$ such that $\lim_{n\to\infty}\mathfrak{b}_{\varepsilon_n}/\mathfrak{a}_{\varepsilon_n}=0$. By possibly passing to a subsequence, we may assume without loss of generality that (4.14) holds. This implies that $\mathfrak{b}_\varepsilon^{-1}X_\varepsilon$ converges in law to a positive and finite random variable. Thus,

$$\mathbf{P}\left[\mathfrak{a}_{\varepsilon_n}^{-1}X_{\varepsilon_n}\geq 1\right]=\mathbf{P}\left[(\mathfrak{b}_{\varepsilon_n}/\mathfrak{a}_{\varepsilon_n})\mathfrak{b}_{\varepsilon_n}^{-1}X_{\varepsilon_n}\geq 1\right]\to 0\quad\text{as } n\to\infty,$$

in contradiction to (4.15). Similarly, if $\lim_{n\to\infty}\mathfrak{b}_{\varepsilon_n}/\mathfrak{a}_{\varepsilon_n}=\infty$, then

$$\mathbf{P}\left[\mathfrak{a}_{\varepsilon_n}^{-1}X_{\varepsilon_n}\leq 1\right]=\mathbf{P}\left[(\mathfrak{b}_{\varepsilon_n}/\mathfrak{a}_{\varepsilon_n})\mathfrak{b}_{\varepsilon_n}^{-1}X_{\varepsilon_n}\leq 1\right]\to 0\quad\text{as } n\to\infty.$$

This completes the proof. $\square$

Proposition 4.13. *Let α_{KC} be as in Proposition 4.6. Then for every sequence of positive real numbers ε's tending to zero, there exists $\{D_{X(\mathscr{L})} : \mathscr{L} \in \Gamma\}$ and a subsequence $\mathcal{E}$ along which we have the convergence of the joint laws*

$$\left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{X(\mathscr{L})}\right) : \mathscr{L} \in \Gamma\right\} \to \left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), D_{X(\mathscr{L})}\right) : \mathscr{L} \in \Gamma\right\}, \tag{4.16}$$

with respect to the metric space $\mathscr{HausUniEns}_4$. (Here, we recall from Section 1.2 the definition of $X(\mathscr{L})$ for $\mathscr{L} \in \Gamma$.) Moreover:

(i) Almost surely, for each $\mathscr{L} \in \Gamma$, $D_{X(\mathscr{L})}$ is a geodesic metric on $X(\mathscr{L})$ inducing the Euclidean topology.
(ii) Let $K \subset \mathbb{C}$ be a deterministic compact subset. Then it holds with superpolynomially high probability as $A \to \infty$, at a rate which is uniform in the choice of the subsequential limit, that

$$D_{X(\mathscr{L})}(u,v) \le A|u-v|^{\alpha_{\mathrm{KC}}}, \quad \forall \mathscr{L} \in \Gamma,\ \forall u,v \in K \cap X(\mathscr{L}).$$

Proof. This follows immediately from Lemmas 4.9 and 4.12. $\square$

Lemma 4.14. *For every sequence of positive real numbers ε's tending to zero, there exists*

$$\left\{\left(D_{X(\mathscr{L})}, D_{X(\mathscr{L}),W_1,W_2}\right) : \mathscr{L} \in \Gamma,\ (W_1,W_2) \in \mathscr{DyDo}_2\right\}$$

and a subsequence $\mathcal{E}$ along which we have the convergence of the joint laws

$$\begin{aligned}
&\Bigg(\left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{X(\mathscr{L})}\right) : \mathscr{L} \in \Gamma\right\}, \Bigg\{\Big((\overline{W_1} \cap X(\mathscr{L})) \times (\overline{W_1} \cap X(\mathscr{L})),\\
&\qquad \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{X(\mathscr{L})}(\bullet,\bullet; W_2 \cap X(\mathscr{L}))|_{(\overline{W_1}\cap X(\mathscr{L}))\times(\overline{W_1}\cap X(\mathscr{L}))}\Big) : \mathscr{L} \in \Gamma,\ (W_1,W_2) \in \mathscr{DyDo}_2\Bigg\}\Bigg)\\
&\quad \to \Bigg(\left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), D_{X(\mathscr{L})}\right) : \mathscr{L} \in \Gamma\right\}, \Bigg\{\Big((\overline{W_1} \cap X(\mathscr{L})) \times (\overline{W_1} \cap X(\mathscr{L})),\\
&\qquad\qquad D_{X(\mathscr{L}),W_1,W_2}(\bullet,\bullet)\Big) : \mathscr{L} \in \Gamma,\ (W_1,W_2) \in \mathscr{DyDo}_2\Bigg\}\Bigg),
\end{aligned}$$

with respect to the metric space $\mathscr{HausUniEns}_4$. Moreover, almost surely,

$$D_{X(\mathscr{L}),W_1,W_2}(\bullet,\bullet) = D_{X(\mathscr{L})}(\bullet,\bullet; W_2 \cap X(\mathscr{L}))|_{(\overline{W_1}\cap X(\mathscr{L}))\times(\overline{W_1}\cap X(\mathscr{L}))}, \quad \forall \mathscr{L} \in \Gamma,\ \forall (W_1,W_2) \in \mathscr{DyDo}_2.$$

Proof. This follows immediately from Proposition 4.1 and Lemma 4.12. $\square$

4.2. Tightness across scales.

In the present subsection, we prove that any subsequential limit satisfies Axiom (IV) of Definition 1.7.

Lemma 4.15. *Let $\mathfrak{k}$ be as in Lemma 4.7. Let $\mathcal{E}$ be a sequence of positive real numbers ε's tending to zero along which (4.16) holds. Then the limits*

$$\mathfrak{c}_r = \mathfrak{c}_r(\mathcal{E}) \stackrel{\text{def}}{=} \lim_{\mathcal{E} \ni \varepsilon \to 0} \frac{r^2 \mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon}, \quad \forall r > 0$$

exist and satisfy (1.5) with $\mathfrak{K} \stackrel{\text{def}}{=} \mathfrak{k}^{-1}$.

Proof. Fix $r > 0$. We observe that

$$\begin{aligned}
&\left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), \mathfrak{a}_{\varepsilon/r}^{-1} D_{X(\mathscr{L})}^{\varepsilon/r}(\bullet,\bullet)\right) : \mathscr{L} \in \Gamma\right\} \\
&= \left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), \frac{\mathfrak{a}_\varepsilon}{r^2 \mathfrak{a}_{\varepsilon/r}} \mathfrak{a}_\varepsilon^{-1} D_{rX(\mathscr{L})}^{\varepsilon}(r\bullet, r\bullet)\right) : \mathscr{L} \in \Gamma\right\} \\
&\overset{d}{=} \left\{\left((X(\mathscr{L})/r) \times (X(\mathscr{L})/r), \frac{\mathfrak{a}_\varepsilon}{r^2 \mathfrak{a}_{\varepsilon/r}} \mathfrak{a}_\varepsilon^{-1} D_{X(\mathscr{L})}^{\varepsilon}(r\bullet, r\bullet)\right) : \mathscr{L} \in \Gamma\right\}, \quad \forall \varepsilon > 0.
\end{aligned}$$

In particular, by the definition of $\mathfrak{a}_{\varepsilon/r}$ (cf. (1.3)),

$$1 = \mathfrak{a}_{\varepsilon/r}^{-1} \operatorname{Med}\left(\sup_{x,y \in X(\mathscr{L}_1)} D_{X(\mathscr{L}_1)}^{\varepsilon/r}(x,y)\right) = \frac{\mathfrak{a}_\varepsilon}{r^2 \mathfrak{a}_{\varepsilon/r}} \mathfrak{a}_\varepsilon^{-1} \operatorname{Med}\left(\sup_{x,y \in X(\mathscr{L}_r)} D_{X(\mathscr{L}_r)}^{\varepsilon/r}(x,y)\right), \tag{4.17}$$

where $\mathscr{L}_r$ denotes the largest loop of Γ contained in $B_r(0)$ and surrounding the origin. Write $\mathfrak{c}_r$ for the median of the random variable

$$\sup_{x,y \in X(\mathscr{L}_r)} D_{X(\mathscr{L}_r)}^{\varepsilon/r}(x,y).$$

Then it follows from (4.17) that $r^2 \mathfrak{a}_{\varepsilon/r}/\mathfrak{a}_\varepsilon \to \mathfrak{c}_r$ as $\mathcal{E} \ni \varepsilon \to 0$. This completes the proof of the first assertion.

Let $\delta \in (0,1)$. Then

$$\mathfrak{c}_{\delta r} = \lim_{\mathcal{E} \ni \varepsilon \to 0} \frac{(\delta r)^2 \mathfrak{a}_{\varepsilon/(\delta r)}}{\mathfrak{a}_\varepsilon} \geq \lim_{\mathcal{E} \ni \varepsilon \to 0} \frac{(\delta r)^2 \mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} = \delta^2 \mathfrak{c}_r$$

and

$$\mathfrak{c}_{\delta r} = \lim_{\mathcal{E} \ni \varepsilon \to 0} \frac{(\delta r)^2 \mathfrak{a}_{\varepsilon/(\delta r)}}{\mathfrak{a}_\varepsilon} \leq \lim_{\mathcal{E} \ni \varepsilon \to 0} \frac{\mathfrak{k}^{-1} \delta r^2 \mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} = \mathfrak{k}^{-1} \delta \mathfrak{c}_r.$$

This completes the proof of the second assertion. □

Proposition 4.16. *Let $\mathcal{E}$ be a sequence of positive real numbers ε's tending to zero along which (4.16) holds. Let $\{\mathfrak{c}_r\}_{r>0}$ be as in Lemma 4.15. Then for each $r > 0$, the law of the collection*

$$\left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), \mathfrak{c}_r^{-1} D_{rX(\mathscr{L})}(r\bullet, r\bullet)\right) : \mathscr{L} \in \Gamma\right\} \tag{4.18}$$

is contained in the closure of the family of the collections

$$\left\{\left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), \mathfrak{a}_\varepsilon^{-1} D_{X(\mathscr{L})}^{\varepsilon}\right) : \mathscr{L} \in \Gamma\right\} : \varepsilon > 0\right\} \tag{4.19}$$

with respect to the Prokhorov topology for probability measures on $\mathfrak{HausUniEns}_4$.

Proof. To lighten notation, write $\mathcal{F}$ for the family (4.19) and $\overline{\mathcal{F}}$ for the closure of $\mathcal{F}$ with respect to the Prokhorov topology for probability measures on $\mathfrak{HausUniEns}_4$. Note that

$$\begin{aligned}
&\left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), \mathfrak{a}_{\varepsilon/r}^{-1} D_{X(\mathscr{L})}^{\varepsilon/r}(\bullet,\bullet)\right) : \mathscr{L} \in \Gamma\right\} \\
&= \left\{\left(X(\mathscr{L}) \times X(\mathscr{L}), \frac{\mathfrak{a}_\varepsilon}{r^2 \mathfrak{a}_{\varepsilon/r}} \mathfrak{a}_\varepsilon^{-1} D_{rX(\mathscr{L})}^{\varepsilon}(r\bullet, r\bullet)\right) : \mathscr{L} \in \Gamma\right\} \\
&\overset{d}{=} \left\{\left((X(\mathscr{L})/r) \times (X(\mathscr{L})/r), \frac{\mathfrak{a}_\varepsilon}{r^2 \mathfrak{a}_{\varepsilon/r}} \mathfrak{a}_\varepsilon^{-1} D_{X(\mathscr{L})}^{\varepsilon}(r\bullet, r\bullet)\right) : \mathscr{L} \in \Gamma\right\}, \quad \forall r > 0,\ \forall \varepsilon > 0.
\end{aligned}$$

This implies that for each $r > 0$ and $\varepsilon > 0$, the law of the collection

$$\left\{\left((X(\mathscr{L})/r) \times (X(\mathscr{L})/r), \frac{\mathfrak{a}_\varepsilon}{r^2 \mathfrak{a}_{\varepsilon/r}} \mathfrak{a}_\varepsilon^{-1} D_{X(\mathscr{L})}^{\varepsilon}(r\bullet, r\bullet)\right) : \mathscr{L} \in \Gamma\right\} \tag{4.20}$$

is contained in $\mathcal{F}$. Fix $r > 0$. It follows from (4.16) and the definition of $\mathfrak{c}_r$ that (4.20) converges in law to

$$\big\{\big((X(\mathscr{L})/r) \times (X(\mathscr{L})/r), \mathfrak{c}_r^{-1} D_{X(\mathscr{L})}(r\bullet, r\bullet)\big) : \mathscr{L} \in \Gamma\big\} \tag{4.21}$$

as $\varepsilon \to 0$. Thus, we conclude that the law of (4.21) is contained in $\overline{\mathcal{F}}$. Since (4.18) and (4.21) have the same law, we complete the proof. □

4.3. **Locality.** In the present subsection, we prove that any subsequential limit satisfies Axiom (II') of Definition 3.1, and then conclude the proof of Theorem 1.8 using Proposition 3.5.

Proposition 4.17. *Let $\mathcal{E}$ be a sequence of positive real numbers ε's tending to zero along which (4.16) holds. Then D_Υ satisfies Axiom (II') of Definition 3.1. (Here, D_Υ is defined as in (1.1).)*

The main difficulty in the proof of Proposition 4.17 is that, in general, conditional independence is not preserved under taking weak limits. This difficulty will be resolved using Lemma 2.15, together with the following lemma.

Lemma 4.18. *Let $\{X_n\}_{n\in\mathbb{N}}$, X, $\{Y_n\}_{n\in\mathbb{N}}$, Y, Z be random variables taking values in complete and separable metric spaces $(\mathcal{X}, d_{\mathcal{X}})$, $(\mathcal{Y}, d_{\mathcal{Y}})$, $(\mathcal{Z}, d_{\mathcal{Z}})$, respectively. Suppose that the following conditions are satisfied:*

- *(i) For each $n \in \mathbb{N}$, X_n and Y_n are conditionally independent given Z.*
- *(ii) (X_n, Y_n, Z) converges in law to (X, Y, Z) as $n \to \infty$.*
- *(iii) For each $\varepsilon > 0$, there exists $\delta > 0$ such that the following is true: For each $n \in \mathbb{N}$ and $z_1, z_2 \in \mathcal{Z}$ with $d_{\mathcal{Z}}(z_1, z_2) \le \delta$, the total variation distance between the conditional law of X_n given $Z = z_1$ and the conditional law of X_n given $Z = z_2$ is at most ε, and the same is true with Y_n in place of X_n.*

Then X and Y are conditionally independent given Z.

Proof. See [AMY25b, Corollary B.5 and Lemma B.7]. □

Proof of Proposition 4.17. By Lemma 4.14 and possibly passing to a subsequence, we may assume without loss of generality that along $\mathcal{E}$ we have the convergence of the joint laws

$$\begin{aligned}
&\bigg(\bigg\{\Big(X(\mathscr{L}) \times X(\mathscr{L}), \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{X(\mathscr{L})}\Big) : \mathscr{L} \in \Gamma\bigg\}, \bigg\{\Big((\overline{W_1} \cap X(\mathscr{L})) \times (\overline{W_1} \cap X(\mathscr{L})), \\
&\qquad \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{X(\mathscr{L})}(\bullet, \bullet; W_2 \cap X(\mathscr{L}))|_{(\overline{W_1}\cap X(\mathscr{L}))\times(\overline{W_1}\cap X(\mathscr{L}))}\Big) : \mathscr{L} \in \Gamma,\ (W_1, W_2) \in \mathfrak{DyDo}_2\bigg\}\bigg) \\
&\quad \to \bigg(\Big\{\big(X(\mathscr{L}) \times X(\mathscr{L}), D_{X(\mathscr{L})}\big) : \mathscr{L} \in \Gamma\Big\}, \Big\{\big((\overline{W_1} \cap X(\mathscr{L})) \times (\overline{W_1} \cap X(\mathscr{L})), \\
&\qquad\quad D_{X(\mathscr{L})}(\bullet, \bullet; W_2 \cap X(\mathscr{L}))|_{(\overline{W_1}\cap X(\mathscr{L}))\times(\overline{W_1}\cap X(\mathscr{L}))}\big) : \mathscr{L} \in \Gamma,\ (W_1, W_2) \in \mathfrak{DyDo}_2\Big\}\bigg).
\end{aligned} \tag{4.22}$$

Let $U \subset \mathbb{C}$ be a deterministic open subset. Fix

$$(W_1, W_2, W_3, W_4) \in \mathfrak{DyDo}_4(U) \quad \text{and} \quad \big(W_1', W_2', W_3', W_4'\big) \in \mathfrak{DyDo}_4(\mathbb{C} \setminus \overline{U}).$$

Write $U^\star$ for the connected component containing W_4 of the open subset obtained by removing from U the closure of the union of the $(\mathbb{C} \setminus \overline{W_4}, \mathbb{C} \setminus U)$-excursions of Γ and all the other loops of Γ that intersect with $\mathbb{C} \setminus U$. Write $X \stackrel{\text{def}}{=} \partial U^\star \cap \partial W_4$ for the collection of the endpoints of the $(\mathbb{C} \setminus \overline{W_4}, \mathbb{C} \setminus U)$-excursions of Γ. Write α for the link pattern on $(U^\star; X)$ induced by the $(\mathbb{C} \setminus \overline{W_4}, \mathbb{C} \setminus U)$-excursions of Γ. Let $(V^\star; Y, \beta)$ be defined in the same manner as $(U^\star; X, \alpha)$ but with $(W_4', \mathbb{C} \setminus \overline{U})$ in place of (W_4, U). Then it follows from Theorem 2.14 that, given $(U^\star; X, \alpha)$ and $(V^\star; Y, \beta)$, the following two collections are conditionally independent:

(i) The collection of the complementary $(\mathbb{C} \setminus \overline{W_4}, \mathbb{C} \setminus U)$-excursions of Γ, together with the collection of the loops of Γ that are contained in $U^\star$.
(ii) The collection of the complementary $(\mathbb{C}\setminus\overline{W_4'}, \overline{U})$-excursions of Γ, together with the collection of the loops of Γ that are contained in $V^\star$.

Write $\Gamma(W_3, W_2)$ for the union of the collection of the (W_3, W_2)-excursions of Γ and the collection of the loops of Γ that are contained in W_3 and intersect with W_2. Let $\Gamma(W_3', W_2')$ be defined in the same manner. Since $\Gamma(W_3, W_2)$ (resp. $\Gamma(W_3', W_2')$) is contained in the collection of (i) (resp. (ii)), it follows that

$$\Gamma(W_3, W_2) \text{ and } \Gamma(W_3', W_2') \text{ are conditionally independent given } (U^\star; X, \alpha) \text{ and } (V^\star; Y, \beta). \tag{4.23}$$

On the other hand, it follows from Lemma 2.15 that the mappings

$$\begin{aligned}(U^\star; X, \alpha) &\mapsto (\text{the conditional law of } \Gamma(W_3, W_2) \text{ given } (U^\star; X, \alpha)) \\ &\text{and} \quad (V^\star; Y, \beta) \mapsto \big(\text{the conditional law of } \Gamma(W_3', W_2') \text{ given } (V^\star; Y, \beta)\big)\end{aligned} \tag{4.24}$$

are continuous with respect to the Carathéodory topology for simply connected domains, together with the uniform topology for boundary points, and the total variation distance for conditional laws. Now, we observe that for each $\varepsilon > 0$, the metrics

$$\Big\{D^\varepsilon_{X(\mathscr{L})}(\bullet, \bullet; W_2 \cap X(\mathscr{L})) : \mathscr{L} \in \Gamma\Big\} \quad \text{and} \quad \Big\{D^\varepsilon_{X(\mathscr{L})}(\bullet, \bullet; W_2' \cap X(\mathscr{L})) : \mathscr{L} \in \Gamma\Big\}$$

are almost surely determined by $\Gamma(W_3, W_2)$ and $\Gamma(W_3', W_2')$, respectively. Since the mappings of (4.24) do not depend on ε, it follows that the mappings

$$\begin{aligned}(U^\star; X, \alpha) \mapsto \Big(&\text{the conditional law of} \\ &\Big\{\mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{X(\mathscr{L})}(\bullet, \bullet; W_2 \cap X(\mathscr{L}))|_{(\overline{W_1} \cap X(\mathscr{L})) \times (\overline{W_1} \cap X(\mathscr{L}))} : \mathscr{L} \in \Gamma\Big\} \text{ given } (U^\star; X, \alpha)\Big)\end{aligned}$$

and

$$\begin{aligned}(V^\star; Y, \beta) \mapsto \Big(&\text{the conditional law of} \\ &\Big\{\mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{X(\mathscr{L})}(\bullet, \bullet; W_2' \cap X(\mathscr{L}))|_{(\overline{W_1'} \cap X(\mathscr{L})) \times (\overline{W_1'} \cap X(\mathscr{L}))} : \mathscr{L} \in \Gamma\Big\} \text{ given } (V^\star; Y, \beta)\Big)\end{aligned}$$

for $\varepsilon > 0$ are uniformly equicontinuous (i.e., satisfy condition (iii) of Lemma 4.18). Combining this with (4.22), (4.23), and Lemma 4.18, we conclude that

$$\begin{aligned}&\Big(\Gamma(W_3, W_2), \Big\{D_{X(\mathscr{L})}(\bullet, \bullet; W_2 \cap X(\mathscr{L}))|_{(\overline{W_1} \cap X(\mathscr{L})) \times (\overline{W_1} \cap X(\mathscr{L}))} : \mathscr{L} \in \Gamma\Big\}\Big) \\ &\qquad \text{and} \quad \Big(\Gamma(W_3', W_2'), \Big\{D_{X(\mathscr{L})}(\bullet, \bullet; W_2' \cap X(\mathscr{L}))|_{(\overline{W_1'} \cap X(\mathscr{L})) \times (\overline{W_1'} \cap X(\mathscr{L}))} : \mathscr{L} \in \Gamma\Big\}\Big)\end{aligned}$$

are conditionally independent given $(U^\star; X, \alpha)$ and $(V^\star; Y, \beta)$. This implies that, given the $(\mathbb{C} \setminus \overline{W_4'}, \overline{U})$-excursions of Γ and all the other loops of Γ that intersect with $\overline{U}$,

$$\begin{aligned}&\Big\{D_{X(\mathscr{L})}(\bullet, \bullet; W_2 \cap X(\mathscr{L}))|_{(\overline{W_1} \cap X(\mathscr{L})) \times (\overline{W_1} \cap X(\mathscr{L}))} : \mathscr{L} \in \Gamma\Big\} \\ &\qquad \text{and} \quad \Big(\Gamma(W_3', W_2'), \Big\{D_{X(\mathscr{L})}(\bullet, \bullet; W_2' \cap X(\mathscr{L}))|_{(\overline{W_1'} \cap X(\mathscr{L})) \times (\overline{W_1'} \cap X(\mathscr{L}))} : \mathscr{L} \in \Gamma\Big\}\Big)\end{aligned}$$

are conditionally independent. By increasing W_1, W_2, W_3, W_4 to U and W_1', W_2', W_3', W_4' to $\mathbb{C} \setminus \overline{U}$, we complete the proof. □

Proof of Theorem 1.8. By Propositions 4.13 and 4.16, for every sequence of positive real numbers ε's tending to zero, there exists $\{D_{X(\mathscr{L})} : \mathscr{L} \in \Gamma\}$ and a subsequence $\mathcal{E}$ along which we have the convergence of the joint laws

$$\left\{ \left(X(\mathscr{L}) \times X(\mathscr{L}), \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{X(\mathscr{L})} \right) : \mathscr{L} \in \Gamma \right\} \to \left\{ \left(X(\mathscr{L}) \times X(\mathscr{L}), D_{X(\mathscr{L})} \right) : \mathscr{L} \in \Gamma \right\},$$

and Axioms (I) and (IV) of Definition 1.7 are satisfied (with D_Υ defined as in (1.1)). By Proposition 4.17, the coupling (Υ, D_Υ) satisfies Axiom (II') of Definition 3.1. The translation invariance and the monotonicity (cf. Axioms (III) and (V) of Definition 1.7) are immediate. Thus, we conclude from Proposition 3.5 that D determines a weak geodesic CLE_κ carpet metric. This completes the proof. □

5. Distance exponent and Hausdorff dimension

In the present section, we prove Theorems 1.5 and 1.6.

5.1. **Proof of Theorem 1.5 (uniqueness of the distance exponent).** Suppose by way of contradiction that there exist $\theta < \widetilde{\theta}$ and strong geodesic CLE_κ carpet metrics D and $\widetilde{D}$ with distance exponent θ and $\widetilde{\theta}$, respectively. By Theorem 1.8 and Lemma 1.10, there exists a weak geodesic CLE_κ carpet metric which is also strong. Moreover, by (1.5), the distance exponent of this metric > 0. Thus, we may assume without loss of generality that $\widetilde{\theta} > 0$. By Definition 1.2, Axiom (III) (translation invariance and scale covariance), for each $p \in (0,1)$, there exists $a = a(p) > 0$ such that

$$\mathbf{P}\left[D_\Upsilon(\partial B_{r/2}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq a r^\theta \right] \geq p, \quad \forall z \in \mathbb{C}, \ \forall r > 0.$$

Thus, by applying Lemmas 2.16 and 2.25 with $\{r^{\widetilde{\theta}}\}_{r>0}$ in place of $\{\mathfrak{c}_r\}_{r>0}$ and a union bound, there exists a sufficiently small $a > 0$ and a sufficiently large $A > 0$ such that it holds with polynomially high probability as $\varepsilon \to 0$ that for each $z \in \left(\frac{1}{100}\varepsilon^2 \mathbb{Z}\right)^2 \cap B_{1/\varepsilon}(0)$, there exists $r \in [\varepsilon^2, \varepsilon]$ such that the following are true:

(i) $D_\Upsilon(\partial B_{r/2}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq a r^\theta$.
(ii) We have
$$\widetilde{D}_\Upsilon(x, y) \leq A r^{\widetilde{\theta}}, \quad \forall x, y \in B_r(z) \cap \Upsilon \text{ with } x \xleftrightarrow{B_r(z)} y.$$

Thus, by the Borel-Cantelli lemma, almost surely, there exists $\varepsilon_* \in (0,1)$ such that the above event occurs for all $\varepsilon \in (0, \varepsilon_*] \cap \{2^k\}_{k \in \mathbb{Z}}$.

Let $x, y \in \Upsilon$ with $x \leftrightarrow y$. Let $P \colon [0,1] \to \Upsilon$ be a D_Υ-geodesic from x to y. Choose a sufficiently small $\varepsilon \in (0, \varepsilon_*] \cap \{2^k\}_{k \in \mathbb{Z}}$ so that $P \subset B_{1/\varepsilon}(0)$. Set $t_0 \stackrel{\text{def}}{=} 0$ and choose $z_1 \in \left(\frac{1}{100}\varepsilon^2 \mathbb{Z}\right)^2 \cap B_{1/\varepsilon}(0)$ and $r_1 \in [\varepsilon^2, \varepsilon]$ such that $x \in B_{\varepsilon^2/2}(z_1)$ and conditions (i) and (ii) (with (z_1, r_1) in place of (z, r)) hold. Then, inductively, for each $j \in \mathbb{N}$, if $y \notin B_{r_j}(z_j)$, then set t_j to be the first time $t \geq t_{j-1}$ at which P hits $\partial B_{r_j}(z_j)$, and choose $z_{j+1} \in \left(\frac{1}{100}\varepsilon^2 \mathbb{Z}\right)^2 \cap B_{1/\varepsilon}(0)$ and $r_{j+1} \in [\varepsilon^2, \varepsilon]$ such that $P(t_j) \in B_{\varepsilon^2/2}(z_{j+1})$ and conditions (i) and (ii) (with (z_{j+1}, r_{j+1}) in place of (z, r)) hold; otherwise set $t_j \stackrel{\text{def}}{=} 1$. Write $n \stackrel{\text{def}}{=} \inf\{j \in \mathbb{N} : t_j = 1\}$. Since $P(t_{j-1}) \in B_{r_j/2}(z_j)$ and $P(t_j) \in \partial B_{r_j}(z_j)$ for all $j \in [1, n-1]_\mathbb{Z}$, it follows from condition (i) that

$$D_\Upsilon(P(t_{j-1}), P(t_j)) \geq a r_j^\theta, \quad \forall j \in [1, n-1]_\mathbb{Z}. \tag{5.1}$$

Since $P([t_{j-1}, t_j]) \subset B_{r_j}(z_j)$ for all $j \in [1, n]_{\mathbb{Z}}$, it follows from condition (ii) that

$$\widetilde{D}_\Upsilon(P(t_{j-1}), P(t_j)) \le A r_j^{\widetilde{\theta}}, \quad \forall j \in [1, n]_{\mathbb{Z}}. \tag{5.2}$$

Combining (5.1) and (5.2), we obtain that

$$\begin{aligned}\widetilde{D}_\Upsilon(x, y) \le \sum_{j=1}^{n} \widetilde{D}_\Upsilon(P(t_{j-1}), P(t_j)) \le \sum_{j=1}^{n} A r_j^{\widetilde{\theta}} \le \frac{A}{a} \sum_{j=1}^{n-1} r_j^{\widetilde{\theta}-\theta} D_\Upsilon(P(t_{j-1}), P(t_j)) + A r_n^{\widetilde{\theta}} \\ \le \frac{A}{a} \varepsilon^{\widetilde{\theta}-\theta} D_\Upsilon(x, y) + A \varepsilon^{\widetilde{\theta}}.\end{aligned}$$

Since $\widetilde{\theta} > \theta \vee 0$ and ε may be chosen to be arbitrarily small, we must have $\widetilde{D}_\Upsilon(x, y) = 0$, a contradiction. This completes the proof. □

5.2. **Hausdorff dimension of CLE carpet.** In the present subsection, we prove Theorem 1.6, (i). We will respectively prove the upper and lower bounds in Lemmas 5.1 and 5.3. Let D be a strong geodesic metric with distance exponent θ. Let X be as in (1.2).

Lemma 5.1. *We have $\theta \dim(X; D_X) \le \dim(X)$ almost surely.*

Lemma 5.2. *Let Y be a non-negative random variable. Then*

$$\mathbf{E}[Y \mathbf{1}(Y \ge a)] \le a^{1-p} \mathbf{E}[Y^p], \quad \forall a > 0, \ \forall p \ge 1.$$

Proof. This follows immediately from the inequality

$$Y \mathbf{1}(Y \ge a) \le Y \left(\frac{Y}{a}\right)^{p-1} \mathbf{1}(Y \ge a) = a^{1-p} Y^p \mathbf{1}(Y \ge a) \le a^{1-p} Y^p.$$

□

Proof of Lemma 5.1. Write $\mathcal{L}_1$ for the largest loop of Γ contained in $B_1(0)$ and surrounding the origin. We may assume without loss of generality that $X = X(\mathcal{L}_1)$. (Recall from Section 1.2 the definition of $X(\mathcal{L}_1)$.) Fix $\ell > \theta^{-1} \dim(X)$. It suffices to show that

$$\dim(X; D_X) \le \ell. \tag{5.3}$$

Choose $d > 2$ to be sufficiently close to 2, $c \in (0, 2 - \dim(X))$ to be sufficiently close to $2 - \dim(X)$, and $\vartheta \in (0, \theta)$ to be sufficiently close to θ so that

$$\vartheta \ell + c > d. \tag{5.4}$$

For each $p > 0$, write

$$C_p \overset{\text{def}}{=} \sup_{z \in \mathbb{C}} \sup_{r > 0} \mathbf{E}\Big[\Big(r^{-\theta} \sup\left\{D_X(x, y) : x, y \in B_r(z) \cap X\right\}\Big)^p\Big]. \tag{5.5}$$

By Definition 1.2, Axioms (III) and (IV), we have $C_p < \infty$ for all $p > 0$. Choose a sufficiently large $p > 0$ so that

$$(\theta - \vartheta) \ell p + \vartheta \ell > d. \tag{5.6}$$

Fix a sufficiently small $\varepsilon \in (0, 1)$. Since $d > 2$, there is a (deterministic) covering $\mathbb{D} \subset \bigcup_{j=1}^{\infty} B_{r_j}(z_j)$ such that

$$\sum_{j=1}^{\infty} r_j^d \le \varepsilon. \tag{5.7}$$

Note that $r_j \le \varepsilon^{1/d}$ for all $j \in \mathbb{N}$. Since $c < 2 - \dim(X)$, if ε is sufficiently small, then

$$\mathbf{P}[B_{r_j}(z_j) \cap X \ne \emptyset] \le r_j^c, \quad \forall j \in \mathbb{N} \tag{5.8}$$

(cf. [SSW11]). Write $J \overset{\text{def}}{=} \{j \in \mathbb{N} : B_{r_j}(z_j) \cap X \neq \emptyset\}$. Then $X \subset \bigcup_{j\in J} B_{r_j}(z_j)$. For each $j \in \mathbb{N}$, write $D_j \overset{\text{def}}{=} \sup\{D_X(x,y) : x, y \in B_{r_j}(z_j) \cap X\}$. We have

$$\begin{aligned}
\mathbf{E}\left[\sum_{j\in J} D_j^\ell\right] &= \sum_{j=1}^\infty \mathbf{E}\Big[D_j^\ell \mathbf{1}(B_{r_j}(z_j) \cap X \neq \emptyset)\Big] \\
&\leq \sum_{j=1}^\infty \Big(r_j^{\vartheta\ell+c} + \mathbf{E}\Big[D_j^\ell \mathbf{1}(D_j \geq r_j^\vartheta)\Big]\Big) \quad \text{(by (5.8))} \\
&\leq \varepsilon + \sum_{j=1}^\infty \mathbf{E}\Big[D_j^\ell \mathbf{1}(D_j \geq r_j^\vartheta)\Big] \quad \text{(by (5.4), (5.7)).}
\end{aligned} \tag{5.9}$$

On the other hand, we have

$$\begin{aligned}
\sum_{j=1}^\infty \mathbf{E}\Big[D_j^\ell \mathbf{1}(D_j \geq r_j^\vartheta)\Big] &\leq \sum_{j=1}^\infty r_j^{\vartheta\ell(1-p)} \mathbf{E}\Big[D_j^{\ell p}\Big] \quad \text{(by Lemma 5.2)} \\
&\leq C_{\ell p} \sum_{j=1}^\infty r_j^{(\theta-\vartheta)\ell p + \vartheta\ell} \quad \text{(by (5.5))} \\
&\leq C_{\ell p}\varepsilon \quad \text{(by (5.6), (5.7)).}
\end{aligned} \tag{5.10}$$

Combining (5.9) and (5.10), we obtain that

$$\mathbf{E}\left[\sum_{j\in J} D_j^\ell\right] \leq (1 + C_{\ell p})\varepsilon.$$

This implies that

$$\mathbf{P}\left[\sum_{j\in J} D_j^\ell \geq \varepsilon^{1/2}\right] \leq (1 + C_{\ell p})\varepsilon^{1/2}.$$

By setting $\varepsilon = 2^{-k}$ for all sufficiently large $k \in \mathbb{N}$ and applying the Borel-Cantelli lemma, we have completed the proof of (5.3). ☐

Lemma 5.3. *We have $\theta \dim(X; D_X) \geq \dim(X)$ almost surely.*

Lemma 5.4. *Let $\vartheta > \theta$. Let $K \subset \mathbb{C}$ be a deterministic compact subset. Then it holds with probability tending to one as $r_* \to 0$ that*

$$D_\Upsilon(\partial B_{r/2}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq r^\vartheta, \quad \forall z \in K,\ \forall r \in (0, r_*].$$

Proof. This follows immediately from Lemma 2.22, together with the fact that we may take $\mathfrak{c}_r = r^\theta$ for all $r > 0$. ☐

Proof of Lemma 5.3. We may assume without loss of generality that $X = X(\mathscr{L}_1)$ is as in the proof of Lemma 5.1. Fix $\ell > \dim(X; D_X)$ and $\vartheta > \theta$. Then it suffices to show that

$$\dim(X) \leq \vartheta\ell. \tag{5.11}$$

By Lemma 5.4, almost surely, there exists $r_* > 0$ such that

$$D_X(\partial B_{r/2}(z) \cap X, \partial B_r(z) \cap X) \geq r^\vartheta, \quad \forall z \in \mathbb{D},\ \forall r \in (0, r_*]. \tag{5.12}$$

Henceforth assume that (5.12) holds. Since $\ell > \dim(X; D_X)$, for each $\varepsilon > 0$, there exists a covering of X by D_X-balls $\{B_{t_j}(z_j; D_X)\}_{j\in\mathbb{N}}$ such that $\sum_{j=1}^\infty t_j^\ell \le \varepsilon$. If ε is sufficiently small, then it follows from (5.12) that

$$B_{t_j}(z_j; D_X) \subset B_{t_j^{1/\vartheta}}(z_j), \quad \forall j \in \mathbb{N}.$$

Thus, there exists a covering $X \subset \bigcup_{j=1}^\infty B_{t_j^{1/\vartheta}}(z_j)$ such that $\sum_{j=1}^\infty t_j^\ell \le \varepsilon$. By sending $\varepsilon \to 0$, we obtain (5.11). This completes the proof. $\square$

Proof of Theorem 1.6, (i). This follows immediately from Lemmas 5.1 and 5.3. $\square$

5.3. **Hausdorff dimension of geodesics.** In the present subsection, we prove Theorem 1.6, (ii). We will break the proof into Lemmas 5.5 and 5.6, which respectively give the lower and upper bounds. Let D be a strong geodesic CLE$_\kappa$ carpet metric with distance exponent θ. Let X be as in (1.2).

Lemma 5.5. *Almost surely,* $\dim(P) \ge \theta$ *for all* D_X*-geodesics* P*.*

Proof. Let $X = X(\mathscr{L}_1)$ be as in the proof of Lemma 5.1. Fix $\vartheta \in (0, \theta)$. Then it suffices to show that, almost surely, $\dim(P) \ge \vartheta$ for all D_X-geodesics $P \subset X$. By the mass distribution principle (cf. [MP10, Theorem 4.19]), it suffices to show that, almost surely, for each D_X-geodesic $P\colon [0,1] \to X$ parameterized with constant speed, there exists $C > 0$ and $\delta > 0$ such that

$$\mathrm{Leb}(V) \le C \,\mathrm{diam}(P(V))^\vartheta, \quad \forall \text{ closed subsets } V \subset [0,1] \text{ with } \mathrm{diam}(P(V)) \le \delta.$$

Choose a sufficiently small $\nu > 0$ so that $\vartheta(1+\nu) < \theta$. By the translation and scale invariance of the law of Γ, for each $p \in (0,1)$, there exists $A = A(p) > 0$ such that for each $z \in \mathbb{C}$ and $r > 0$, it holds with probability at least p that there are at most $2A$ connected arcs of Γ in $A_{r/2,r}(z)$ connecting its inner and outer boundaries (in which case there are at most A connected components of $A_{r/2,r}(z) \cap \Upsilon$ connecting $\partial B_{r/2}(z)$ and $\partial B_r(z)$). Thus, by Lemmas 2.16 and 2.25 and a union bound, there exists a sufficiently large $A = A(\nu) > 0$ such that it holds with polynomially high probability as $\varepsilon \to 0$ that for each $z \in \left(\frac{1}{100}\varepsilon^{1+\nu}\mathbb{Z}\right)^2 \cap \mathbb{D}$, there exists $r \in [\varepsilon^{1+\nu}, \varepsilon]$ such that the following are true:

(i) There are at most A connected components of $A_{r/2,r}(z) \cap X$ connecting $\partial B_{r/2}(z)$ and $\partial B_r(z)$ (in which case there are at most A connected components of $B_r(z) \cap X$ intersecting with $B_{r/2}(z)$).
(ii) We have

$$D_X(x,y) \le A r^\theta, \quad \forall x, y \in B_r(z) \cap X \text{ with } x \xleftrightarrow{B_r(z)\cap X} y.$$

Thus, by the Borel-Cantelli lemma, almost surely, there exists $\varepsilon_* \in (0,1)$ such that the above event occurs for all $\varepsilon \in (0, \varepsilon_*] \cap \{2^k\}_{k\in\mathbb{Z}}$. By possibly decreasing ε_*, we may assume without loss of generality that $\mathrm{diam}(X) \ge 2\varepsilon_*$.

Fix a D_X-geodesic $P\colon [0,1] \to X$ parameterized with constant speed. Let $V \subset [0,1]$ be a closed subset such that $\mathrm{diam}(P(V)) \le \varepsilon_*^{1+\nu}/100$. Then there exists $\varepsilon \in (0, \varepsilon_*] \cap \{2^k\}_{k\in\mathbb{Z}}$ such that

$$(\varepsilon/2)^{1+\nu}/4 < \mathrm{diam}(P(V)) \le \varepsilon^{1+\nu}/4. \tag{5.13}$$

This implies that there exists $z \in \left(\frac{1}{100}\varepsilon^{1+\nu}\mathbb{Z}\right)^2 \cap \mathbb{D}$ and $r \in [\varepsilon^{1+\nu}, \varepsilon]$ such that $P(V) \subset B_{r/2}(z)$ and conditions (i) and (ii) hold. Write $\mathscr{C}_1, \dots, \mathscr{C}_n$ for the connected components of $B_r(z) \cap X$ that intersect with $B_{r/2}(z)$. Then $n \le A$ by condition (i) and $P(V) \subset \bigcup_{j=1}^n \mathscr{C}_j$. By condition (ii), we have

$$D_X(x,y) \le A r^\theta, \quad \forall j \in [1,n]_{\mathbb{Z}},\ \forall x, y \in \mathscr{C}_j. \tag{5.14}$$

For each $j \in [1,n]_{\mathbb{Z}}$, write s_j for the first time at which P enters $\mathscr{C}_j$ and t_j for the last time at which P exits $\mathscr{C}_j$. Then it follows from (5.14) that

$$t_j - s_j \le \frac{Ar^\theta}{\operatorname{len}(P; D_X)}, \quad \forall j \in [1,n]_{\mathbb{Z}}. \tag{5.15}$$

Combining (5.13) and (5.15), we obtain that

$$\begin{aligned}\operatorname{Leb}(V) \le \sum_{j=1}^{n}(t_j - s_j) \le \frac{A^2 r^\theta}{\operatorname{len}(P; D_X)} \le \frac{A^2 \varepsilon^\theta}{\operatorname{len}(P; D_X)} \le \frac{2^{100} A^2 \operatorname{diam}(P(V))^{\theta/(1+\nu)}}{\operatorname{len}(P; D_X)} \\ \le \frac{2^{100} A^2 \operatorname{diam}(P(V))^{\vartheta}}{\operatorname{len}(P; D_X)}.\end{aligned}$$

This completes the proof. ☐

Lemma 5.6. *Almost surely,* $\dim(P) \le \theta$ *for all* D_X*-geodesics* P*.*

Proof. This follows immediately from the fact that for each $\chi > \theta$, almost surely, all D_X-geodesics are Hölder continuous of order $1/\chi$ (cf. Remark 2.23). ☐

Proof of Theorem 1.6, (ii). This follows immediately from Lemmas 5.5 and 5.6. (Since every nontrivial path in X has Hausdorff dimension > 1 (cf., e.g., [Orz98] and the proof of Lemma 2.29), it follows that $\theta > 1$.) ☐

6. Optimal bi-Lipschitz constants

6.1. **Setup and statements.** Fix two weak geodesic CLE_κ carpet metrics D and $\widetilde{D}$ with the same scaling constants $\{\mathfrak{c}_r\}_{r>0}$.

We shall write

$$M_* \overset{\text{def}}{=} \sup\left\{M > 0 : \mathbf{P}\left[\exists x, y \in \Upsilon \text{ such that } x \leftrightarrow y \text{ and } \frac{\widetilde{D}_\Upsilon(x,y)}{D_\Upsilon(x,y)} \le M\right] = 0\right\}; \tag{6.1}$$

$$M^* \overset{\text{def}}{=} \inf\left\{M > 0 : \mathbf{P}\left[\exists x, y \in \Upsilon \text{ such that } x \leftrightarrow y \text{ and } \frac{\widetilde{D}_\Upsilon(x,y)}{D_\Upsilon(x,y)} \ge M\right] = 0\right\}. \tag{6.2}$$

It follows from Proposition 3.2 that M_* and M^* are positive and finite. By definition, for each $M \in (0, M^*)$ (resp. $M > M_*$),

$$\mathbf{P}\left[\text{there are distinct } x, y \in \Upsilon \text{ such that } x \leftrightarrow y \text{ and } \frac{\widetilde{D}_\Upsilon(x,y)}{D_\Upsilon(x,y)} \ge M\right] > 0 \tag{6.3}$$

$$\left(\text{resp. } \mathbf{P}\left[\text{there are distinct } x, y \in \Upsilon \text{ such that } x \leftrightarrow y \text{ and } \frac{\widetilde{D}_\Upsilon(x,y)}{D_\Upsilon(x,y)} \le M\right] > 0\right).$$

The primary objective of this section is to establish several quantitative refinements of (6.3) — specifically, to demonstrate that there exist many values of $\mathfrak{r} > 0$ for which

(6.4) it holds with uniformly positive probability that there are points $x, y \in \Upsilon$ with $x \leftrightarrow y$ such that $|x|$, $|y|$, and $|x-y|$ are all of order $\mathfrak{r}$ and $\widetilde{D}_\Upsilon(x,y)/D_\Upsilon(x,y)$ is close to M^* (resp. M_*).

Such pairs of points will be called "shortcuts". Our results will be formulated in terms of two types of events, defined in Definitions 6.1 and 6.2 below. In Section 7, we will use the results of the present section to show that, with high probability, an annulus of suitable radius contains many "shortcuts".

In Section 8, under the assumption that $M_* < M^*$, we will find many values of $\mathbb{r} > 0$ satisfying a condition that contradicts (6.4), leading to the conclusion that $M_* = M^*$. The argument in the present section follows the same general structure as in [GM21b, Section 3] and [DG23, Section 3], though the definitions of the relevant events are necessarily different.

Definition 6.1. Let $M, \mathbb{r} > 0$ and $b \in (0,1)$. Then:

(i) We shall write $\overline{G}_{\mathbb{r}}(M,b)$ for the event that there exist $x, y \in \overline{B_{\mathbb{r}}(0)} \cap \Upsilon$ such that

$$|x-y| \geq b\mathbb{r}, \quad x \leftrightarrow y, \quad \text{and} \quad \widetilde{D}_\Upsilon(x,y) \geq M D_\Upsilon(x,y).$$

(ii) We shall write $\underline{G}_{\mathbb{r}}(M,b)$ for the event that there exist $x, y \in \overline{B_{\mathbb{r}}(0)} \cap \Upsilon$ such that

$$|x-y| \geq b\mathbb{r}, \quad x \leftrightarrow y, \quad \text{and} \quad \widetilde{D}_\Upsilon(x,y) \leq M D_\Upsilon(x,y).$$

The other version of events has a more complicated definition, and includes several regularity conditions on the "shortcut". See Figure 3 for an illustration.

Definition 6.2. Let $M, r > 0$ and $0 < \alpha < a < 1$. Then:

(i) We shall write $\overline{H}_r(M,a,\alpha)$ for the event that there exists a D_Υ-geodesic

$$P \colon [0,1] \to \overline{A_{(1-a)r,(1+a)r}(0)} \cap \Upsilon$$

from $\partial B_{(1-a)r}(0) \cap \Upsilon$ to $\partial B_{(1+a)r}(0) \cap \Upsilon$ and times $0 < s < t < 1$ satisfying the following conditions:
 (a) $\operatorname{len}(P; D_\Upsilon) \leq a^{-1}\mathfrak{c}_r$.
 (b) The Euclidean diameter of P is at most $r/100$.
 (c) $P(s) \in \partial B_{(1-\alpha)r}(0) \cap \Upsilon$, $P(t) \in \partial B_r(0) \cap \Upsilon$, and $P|_{[s,t]} \subset \overline{A_{(1-\alpha)r,r}(0)} \cap \Upsilon$.
 (d) $D_\Upsilon(P(s),P(t)) \geq \alpha^3 \mathfrak{c}_r$ and $\widetilde{D}_\Upsilon(P(s),P(t)) \geq M D_\Upsilon(P(s),P(t))$.

(ii) We shall write $\underline{H}_r(M,a,\alpha)$ for the event that there exists a $\widetilde{D}_\Upsilon$-geodesic

$$\widetilde{P} \colon [0,1] \to \overline{A_{(1-a)r,(1+a)r}(0)} \cap \Upsilon$$

from $\partial B_{(1-a)r}(0) \cap \Upsilon$ to $\partial B_{(1+a)r}(0) \cap \Upsilon$ and times $0 < s < t < 1$ satisfying the following conditions:
 (a) $\operatorname{len}(\widetilde{P}; \widetilde{D}_\Upsilon) \leq a^{-1}\mathfrak{c}_r$.
 (b) The Euclidean diameter of $\widetilde{P}$ is at most $r/100$.
 (c) $\widetilde{P}(s) \in \partial B_{(1-\alpha)r}(0) \cap \Upsilon$, $\widetilde{P}(t) \in \partial B_r(0) \cap \Upsilon$, and $\widetilde{P}|_{[s,t]} \subset \overline{A_{(1-\alpha)r,r}(0)} \cap \Upsilon$.
 (d) $\widetilde{D}_\Upsilon(\widetilde{P}(s),\widetilde{P}(t)) \geq \alpha^3 \mathfrak{c}_r$ and $\widetilde{D}_\Upsilon(\widetilde{P}(s),\widetilde{P}(t)) \leq M D_\Upsilon(\widetilde{P}(s),\widetilde{P}(t))$.

Lemma 6.3. *(i) For each $M \in (0, M^*)$ and $\mathbb{r} > 0$, there exists $b = b(M,\mathbb{r}) \in (0,1)$ such that $\mathbf{P}[\overline{G}_{\mathbb{r}}(M,b)] \geq b$.*
(ii) For each $M > M_*$ and $\mathbb{r} > 0$, there exists $b = b(M,\mathbb{r}) \in (0,1)$ such that $\mathbf{P}[\underline{G}_{\mathbb{r}}(M,b)] \geq b$.

Proof. First, we consider assertion (i). Suppose by way of contradiction that $\mathbf{P}[\overline{G}_{\mathbb{r}}(M,b)] < b$ for all $b \in (0,1)$. Then, almost surely,

$$\widetilde{D}_\Upsilon(x,y) \leq M D_\Upsilon(x,y), \quad \forall x, y \in \overline{B_{\mathbb{r}}(0)} \cap \Upsilon \text{ with } x \leftrightarrow y. \tag{6.5}$$

By Definition 1.7, Axiom (III) (translation invariance), (6.5) implies that, almost surely,

$$\widetilde{D}_\Upsilon(x,y) \leq M D_\Upsilon(x,y), \quad \forall z \in \mathbb{Q}^2,\ \forall x, y \in \overline{B_{\mathbb{r}}(z)} \cap \Upsilon \text{ with } x \leftrightarrow y.$$

hence that, almost surely,

$$\widetilde{D}_\Upsilon(x,y) \leq M D_\Upsilon(x,y), \quad \forall x, y \in \Upsilon \text{ with } |x-y| \leq \mathbb{r}/2 \text{ and } x \leftrightarrow y. \tag{6.6}$$

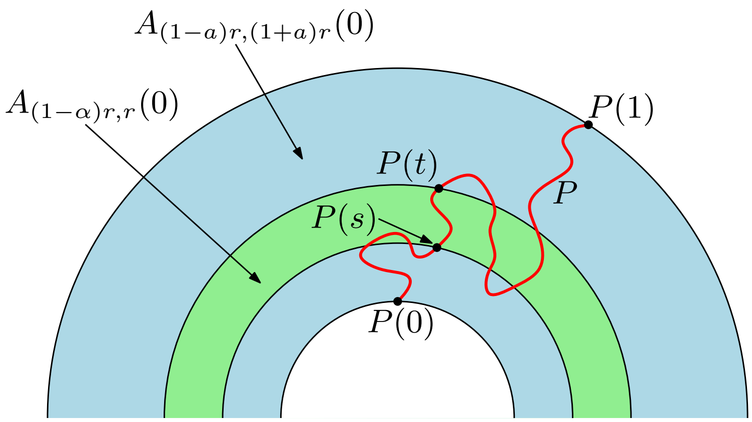

FIGURE 3. Illustration of the event $\overline{H}_r(M,a,\alpha)$ of Definition 6.2. P is a D_Υ-geodesic in $\overline{A_{(1-a)r,(1+a)r}(0)}$ from $\partial B_{(1-a)r}(0)\cap\Upsilon$ to $\partial B_{(1+a)r}(0)\cap\Upsilon$. There are times $0<s<t<1$ such that $P(s)\in\partial B_{(1-\alpha)r}(0)\cap\Upsilon$, $P(t)\in\partial B_r(0)\cap\Upsilon$, and $P|_{[s,t]}\subset\overline{A_{(1-\alpha)r,r}(0)}\cap\Upsilon$. Also, $P|_{[s,t]}$ is a "shortcut", in the sense that $\widetilde{D}_\Upsilon(P(s),P(t))\geq MD_\Upsilon(P(s),P(t))$.

Since for each $x,y\in\Upsilon$ with $x\leftrightarrow y$ and each D_Υ-geodesic $P\colon[0,1]\to\Upsilon$ from x to y, there exist times $0=t_0<t_1<\cdots<t_n=1$ such that $|P(t_{j-1})-P(t_j)|\leq\mathfrak{r}/2$ for all $j\in[1,n]_{\mathbb{Z}}$, it follows from (6.6) that, almost surely,

$$\widetilde{D}_\Upsilon(x,y)\leq\sum_{j=1}^n\widetilde{D}_\Upsilon(P(t_{j-1}),P(t_j))\leq M\sum_{j=1}^n D_\Upsilon(P(t_{j-1}),P(t_j))=MD_\Upsilon(x,y),\qquad\forall x,y\in\Upsilon\text{ with }x\leftrightarrow y.$$

This contradicts the assumption that $M<M^*$ and the definition of M^* (cf. (6.2)). This completes the proof of assertion (i). Assertion (ii) from an argument which is analogous to that used to prove assertion (i). □

Proposition 6.4. *For each $0<\mu<\nu$ and $\Lambda>1$, there exists $p=p(\mu,\nu,\Lambda)\in(0,1)$ and $a=a(\mu,\nu,\Lambda)\in(0,1)$ such that for each $\alpha\in(0,a)$ and $M\in(0,M^*)$, there exists $M'=M'(\mu,\nu,\Lambda,\alpha,M)\in(M,M^*)$ such that for each $b\in(0,1)$, there exists $\varepsilon_*=\varepsilon_*(\mu,\nu,\Lambda,\alpha,M,b)\in(0,1)$ such that for each $\mathfrak{r}>0$ for which $\mathbf{P}[\overline{G}_{\mathfrak{r}}(M',b)]\geq b$ and each $\varepsilon\in(0,\varepsilon_*]$, the following is true:*

(6.7) *There are at least $\mu\log_\Lambda(1/\varepsilon)$ values of $r\in[\varepsilon^{1+\nu}\mathfrak{r},\varepsilon\mathfrak{r}]\cap\{\Lambda^k\}_{k\in\mathbb{Z}}$ for which $\mathbf{P}[\overline{H}_r(M,a,\alpha)]\geq p$.*

We will give the proof of Proposition 6.4 in Section 6.2 just below.

Proposition 6.5. *For each $0<\mu<\nu$ and $\Lambda>1$, there exists $p=p(\mu,\nu,\Lambda)\in(0,1)$ and $a=a(\mu,\nu,\Lambda)\in(0,1)$ such that for each $\alpha\in(0,a)$, $M\in(0,M^*)$, and $\mathfrak{r}>0$, there exists $\varepsilon_*=\varepsilon_*(\mu,\nu,\Lambda,\alpha,M,\mathfrak{r})\in(0,1)$ such that (6.7) holds for all $\varepsilon\in(0,\varepsilon_*]$.*

Proof. This follows immediately from Lemma 6.3 and Proposition 6.4. □

Proposition 6.6. *For each $0<\mu<\nu$ and $\Lambda>1$, there exists $b=b(\mu,\nu,\Lambda)\in(0,1)$ such that for each $M\in(0,M^*)$ and $\mathfrak{r}>0$, there exists $\varepsilon_*=\varepsilon_*(\mu,\nu,\Lambda,M,\mathfrak{r})\in(0,1)$ such that for each $\varepsilon\in(0,\varepsilon_*]$, the following is true:*

There are at least $\mu\log_\Lambda(1/\varepsilon)$ values of $r\in[\varepsilon^{1+\nu}\mathfrak{r},\varepsilon\mathfrak{r}]\cap\{\Lambda^k\}_{k\in\mathbb{Z}}$ for which $\mathbf{P}[\overline{G}_r(M,b)]\geq b$.

Proof. Let p and a be as in Proposition 6.5. Set $b \stackrel{\text{def}}{=} (a/2) \wedge p$. Then Proposition 6.6 follows immediately from Proposition 6.5, together with the fact that $\overline{H}_r(M,a,b) \subset \overline{G}_r(M,b)$. □

By interchanging the roles of D and $\widetilde{D}$, we obtain from Propositions 6.4 and 6.6 the following statements.

Proposition 6.7. *For each $0 < \mu < \nu$ and $\Lambda > 1$, there exists $p = p(\mu,\nu,\Lambda) \in (0,1)$ and $a = a(\mu,\nu,\Lambda) \in (0,1)$ such that for each $\alpha \in (0,a)$ and $M > M_*$, there exists $M' = M'(\mu,\nu,\Lambda,\alpha,M) \in (M_*,M)$ such that for each $b \in (0,1)$, there exists $\varepsilon_* = \varepsilon_*(\mu,\nu,\Lambda,\alpha,M,b) \in (0,1)$ such that for each $\mathbb{r} > 0$ for which $\mathbf{P}[\underline{G}_{\mathbb{r}}(M',b)] \geq b$ and each $\varepsilon \in (0,\varepsilon_*]$, the following is true:*

> *There are at least $\mu \log_\Lambda(1/\varepsilon)$ values of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ for which $\mathbf{P}[\underline{H}_r(M,a,\alpha)] \geq p$.*

Proposition 6.8. *For each $0 < \mu < \nu$ and $\Lambda > 1$, there exists $b = b(\mu,\nu,\Lambda) \in (0,1)$ such that for each $M > M_*$ and $\mathbb{r} > 0$, there exists $\varepsilon_* = \varepsilon_*(\mu,\nu,\Lambda,M,\mathbb{r}) \in (0,1)$ such that for each $\varepsilon \in (0,\varepsilon_*]$, the following is true:*

> *There are at least $\mu \log_\Lambda(1/\varepsilon)$ values of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ for which $\mathbf{P}[\underline{G}_r(M,b)] \geq b$.*

6.2. **Proof of Proposition 6.4.** In order to prove Proposition 6.4, it suffices to prove the following contrapositive.

Lemma 6.9. *For each $0 < \mu < \nu$ and $\Lambda > 1$, there exists $p = p(\mu,\nu,\Lambda) \in (0,1)$ and $a = a(\mu,\nu,\Lambda) \in (0,1)$ such that for each $\alpha \in (0,a)$ and $M \in (0,M^*)$, there exists $M' = M'(\mu,\nu,\Lambda,\alpha,M) \in (M,M^*)$ such that for each $b \in (0,1)$, there exists $\varepsilon_* = \varepsilon_*(\mu,\nu,\Lambda,\alpha,M,b) \in (0,1)$ such that for each $\mathbb{r} > 0$, if there exists $\varepsilon \in (0,\varepsilon_*]$ satisfying (6.8) just below, then $\mathbf{P}[\overline{G}_{\mathbb{r}}(M',b)] < b$.*

(6.8) *There are at least $(\nu-\mu)\log_\Lambda(1/\varepsilon)$ values of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ for which we have $\mathbf{P}[\overline{H}_r(M,a,\alpha)] < p$.*

The main idea behind the proof of Lemma 6.9 is as follows. Assume that (6.8) holds for a sufficiently small universal constant $p \in (0,1)$. Then, by applying Lemmas 2.16 and 2.25 together with a union bound, we can cover the plane with balls of the form $B_{r/2}(z)$ such that the complement of the event $\overline{H}_r(M,a,\alpha)$ occurs and the D_Υ-diameter of $B_{2r}(z) \cap \Upsilon$ remains controlled. In particular, for every pair of points $u \in \partial B_{(1-\alpha)r}(z) \cap \Upsilon$ and $v \in \partial B_r(z) \cap \Upsilon$ that are connected within $A_{(1-\alpha)r,r}(z)$ and satisfy certain regularity properties, we have $\widetilde{D}_\Upsilon(P(s),P(t)) \leq M D_\Upsilon(P(s),P(t))$. Consequently, for any D_Υ-geodesic P whose D_Υ-length is not too small, by examining the moments when it crosses such annuli $A_{(1-\alpha)r,r}(z)$, we can deduce that $\widetilde{D}_\Upsilon(P(0),P(1)) \leq M' D_\Upsilon(P(0),P(1))$ for some suitable constant $M' \in (M,M^*)$.

In the remainder of the present subsection, fix $M \in (0,M^*)$.

Let us now define the event to which we will apply Lemma 2.16. For $z \in \mathbb{C}$, $r > 0$, and $0 < \alpha < a < 1$, write $E_r(z) = E_r(z;M,a,\alpha)$ for the event that the following are true:

(i) We have

$$D_\Upsilon(x,y) \leq \frac{M_*}{M^*}\left(D_\Upsilon(\partial B_{r/2}(z)\cap\Upsilon, \partial B_{3r/4}(z)\cap\Upsilon) \wedge D_\Upsilon(\partial B_{3r/2}(z)\cap\Upsilon, \partial B_{2r}(z)\cap\Upsilon)\right),$$
$$\forall x,y \in A_{(1-a)r,(1+a)r}(z)\cap\Upsilon \text{ with } x \xleftrightarrow{A_{(1-a)r,(1+a)r}(z)} y.$$

(ii) Let $P\colon [0,1] \to \overline{A_{(1-a)r,(1+a)r}(z)} \cap \Upsilon$ be a D_Υ-geodesic connecting $\partial B_{(1-a)r}(z) \cap \Upsilon$ and $\partial B_{(1+a)r}(z) \cap \Upsilon$. Suppose that there are times $0 < s < t < 1$ such that the following conditions are satisfied:
   (a) $\operatorname{len}(P; D_\Upsilon) \leq a^{-1}\mathfrak{c}_r$.
   (b) The Euclidean diameter of P is at most $r/100$.
   (c) $P(s) \in \partial B_{(1-\alpha)r}(z) \cap \Upsilon$, $P(t) \in \partial B_r(z) \cap \Upsilon$, and $P|_{[s,t]} \subset \overline{A_{(1-\alpha)r,r}(z)} \cap \Upsilon$.
   (d) $D_\Upsilon(P(s), P(t)) \geq \alpha^3 \mathfrak{c}_r$.

   Then $\widetilde{D}_\Upsilon(P(s), P(t)) \leq M D_\Upsilon(P(s), P(t))$.

(iii) $D_\Upsilon(\partial B_{(1-\alpha)r}(z) \cap \Upsilon, \partial B_r(z) \cap \Upsilon) \geq \alpha^3 \mathfrak{c}_r$.

(iv) Every connected component of $A_{(1-a)r,(1+a)r}(z) \cap \Upsilon$ has Euclidean diameter at most $r/100$.

The key requirement in the definition of $E_r(z)$ is condition (ii). Note that this condition is precisely the complement of the event $\overline{H}_r(M, a, \alpha)$. Condition (i) is included to guarantee that $E_r(z)$ is almost surely determined by the configuration of $A_{r/2,2r}(z) \cap \Upsilon$. Finally, conditions (iii) and (iv) are introduced so that conditions (ii.d) and (ii.b), respectively, can be verified later.

Lemma 6.10. *Let $z \in \mathbb{C}$ and $r > 0$. Then $E_r(z)$ is almost surely determined by $A_{r/2,2r}(z) \cap \Upsilon$.*

Proof. It is clear that conditions (i), (iii), (iv) are almost surely determined by $A_{r/2,2r}(z) \cap \Upsilon$. If condition (i) holds, then for each $x, y \in A_{(1-a)r,(1+a)r}(z) \cap \Upsilon$ with $x \xleftrightarrow{A_{(1-a)r,(1+a)r}(z)} y$, any path in Υ between x and y that exits $A_{r/2,2r}(z)$ is neither a D_Υ-geodesic nor a $\widetilde{D}_\Upsilon$-geodesic. This implies that, on the event that condition (i) holds, condition (ii) is almost surely determined by $A_{r/2,2r}(z) \cap \Upsilon$. This completes the proof. □

Lemma 6.11. *For each $p \in (0,1)$, there exists $a = a(p) \in (0,1)$ such that for each $\alpha \in (0,a)$ and each $r > 0$ for which $\mathbf{P}[\overline{H}_r(M, a, \alpha)] < p$, we have*

$$\mathbf{P}[E_r(z)] \geq 1 - 2p, \quad \forall z \in \mathbb{C}.$$

Proof. By Definition 1.7, Axiom (III) (translation invariance), it suffices to consider the case $z = 0$. It follows immediately from the definition of $\overline{H}_r(M, a, \alpha)$ that, if $\mathbf{P}[\overline{H}_r(M, a, \alpha)] < p$, then condition (ii) holds with probability at least $1 - p$. Thus, it suffices to show that there exists $a = a(p) \in (0,1)$ such that for each $\alpha \in (0,a)$ and $r > 0$, conditions (i), (iii), (iv) hold with probability at least $1 - p$. First, we consider condition (i). By Definition 1.7, Axiom (IV) (tightness across scales), we may choose a sufficiently small $\delta > 0$ such that for each $r > 0$, it holds with probability at least $1 - p/4$ that

$$D_\Upsilon(x,y) \leq \frac{M_*}{M^*} \left(D_\Upsilon(\partial B_{r/2}(0) \cap \Upsilon, \partial B_{3r/4}(0) \cap \Upsilon) \wedge D_\Upsilon(\partial B_{3r/2}(0) \cap \Upsilon, \partial B_{2r}(0) \cap \Upsilon) \right), \quad \forall x, y \in A_{3r/4,3r/2}(0) \cap \Upsilon \text{ with } x \leftrightarrow y \text{ and } |x - y| \leq \delta r. \tag{6.9}$$

We may choose $a \in (0,1)$ to be sufficiently small so that for each $r > 0$, it holds with probability at least $1 - p/4$ that

(6.10) the Euclidean diameter of each connected component of $A_{(1-a)r,(1+a)r}(z) \cap \Upsilon$ is at most δr.

Combining (6.9) and (6.10), we obtain that for each $r > 0$, condition (i) holds with probability at least $1 - p/2$. Next, it follows immediately from Lemma 2.21 that there exists $a = a(p) \in (0,1)$ such that for each $\alpha \in (0,a)$ and $r > 0$, condition (iii) holds with probability at least $1 - p/4$. Finally, by (6.10), for each $r > 0$, condition (iv) holds with probability at least $1 - p/4$. This completes the proof. □

Lemma 6.12. *For each $0 < \mu < \nu$ and $\Lambda > 1$, there exists $p = p(\mu,\nu,\Lambda) \in (0,1)$ and $a = a(\mu,\nu,\Lambda) \in (0,1)$ such that for each $\alpha \in (0,a)$ and each bounded open subset $U \subset \mathbb{C}$, the following is true: For each $\varepsilon \in (0,1)$ and $\mathbb{r} > 0$ for which (6.8) holds, it holds with probability tending to one as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that for each $z \in \left(\frac{1}{100}\varepsilon^{1+\nu}\mathbb{r}\mathbb{Z}\right)^2 \cap (\mathbb{r}U)$, there exists $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ such that the following are true:*

(i) $E_r(z)$ occurs.
(ii) We have

$$D_\Upsilon(x,y) \le a^{-1}\mathfrak{c}_r, \quad \forall x,y \in B_{2r}(z) \cap \Upsilon \text{ with } x \xleftrightarrow{B_{2r}(z)} y.$$

Proof. This follows immediately from Lemmas 2.16, 2.25, 6.10 and 6.11, together with a union bound. □

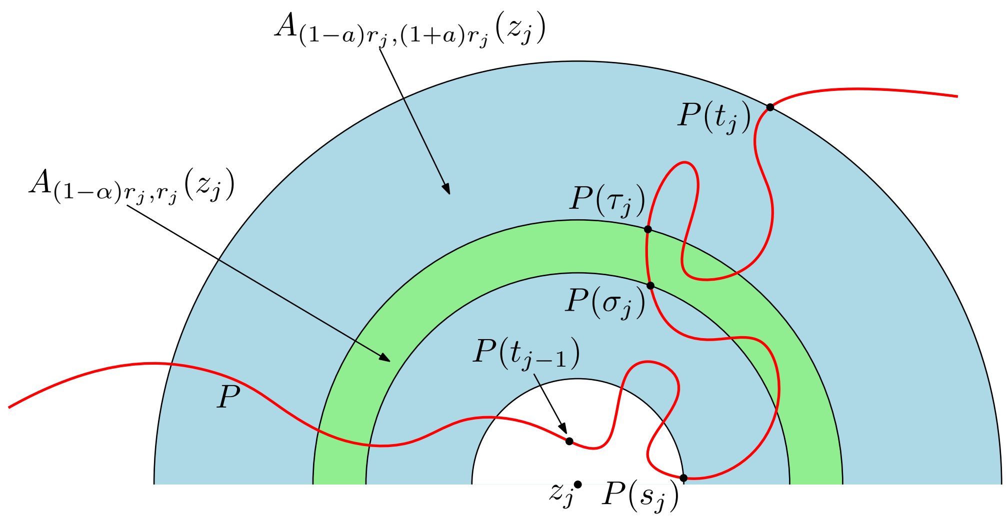

FIGURE 4. Illustration of the definition of the times s_j, σ_j, τ_j, t_j in the proof of Lemma 6.9. t_j is the first time after t_{j-1} at which P hits $\partial B_{(1+a)r_j}(z_j)$; s_j is the last time before t_j at which P hits $\partial B_{(1-a)r_j}(z_j)$; σ_j is the last time before t_j at which P hits $\partial B_{(1-\alpha)r_j}(z_j)$; τ_j is the first time after σ_j at which P hits $\partial B_{r_j}(z_j)$.

Proof of Lemma 6.9. Let p and a be as in Lemma 6.12. Fix $\alpha \in (0,a)$. Choose

$$M' \in \left(M^* - \alpha^4(M^* - M), M^*\right).$$

It suffices to show that for each $b \in (0,1)$, there exists $\varepsilon_* = \varepsilon_*(\mu,\nu,\Lambda,\alpha,M,b) \in (0,1)$ such that for each $\varepsilon \in (0,\varepsilon_*]$ and $\mathbb{r} > 0$ for which (6.8) holds, it holds with probability greater than $1-b$ that

$$\widetilde{D}_\Upsilon(x,y)_\Upsilon < M' D_\Upsilon(x,y), \quad \forall x,y \in \overline{B_{\mathbb{r}}(0)} \cap \Upsilon \text{ with } |x-y| \ge b\mathbb{r} \text{ and } x \leftrightarrow y. \tag{6.11}$$

For $\varepsilon \in (0,1)$, $\mathbb{r} > 0$, and $R > 1$, write $F_\varepsilon^{\mathbb{r}} = F_\varepsilon^{\mathbb{r}}(R)$ for the event that the following are true:

(i) There exists a loop of Γ in $A_{\mathbb{r},R\mathbb{r}}(0)$ disconnecting the inner and outer boundaries of $A_{\mathbb{r},R\mathbb{r}}(0)$. In particular, every D_Υ-geodesic between two points of $\overline{B_{\mathbb{r}}(0)} \cap \Upsilon$ is contained in $B_{R\mathbb{r}}(0)$.
(ii) The event of Lemma 6.12, with $B_R(0)$ in place of U, occurs.
(iii) We have

$$D_\Upsilon(x,y) \ge \varepsilon^{1/2}\mathfrak{c}_{\mathbb{r}}, \quad \forall x,y \in \overline{B_{\mathbb{r}}(0)} \cap \Upsilon \text{ with } |x-y| \ge b\mathbb{r} \text{ and } x \leftrightarrow y.$$

We *claim* that there exists $R = R(b) > 1$ and $\varepsilon_* = \varepsilon_*(\mu, \nu, \Lambda, \alpha, M, b, R) \in (0,1)$ such that for each $\varepsilon \in (0, \varepsilon_*]$ and $\mathbb{r} > 0$, we have $\mathbf{P}[F_\varepsilon^{\mathbb{r}}] > 1 - b$. Since the law of Υ is scaling invariant, it follows that there exists $R = R(b) > 1$ such that for each $\mathbb{r} > 0$, condition (i) holds with probability greater than $1 - b/3$. By Lemma 6.12, there exists $\varepsilon_0 \in (0,1)$ such that for each $\varepsilon \in (0, \varepsilon_0]$ and $\mathbb{r} > 0$, condition (ii) holds with probability greater than $1 - b/3$. By Lemma 2.21, there exists $\varepsilon_1 \in (0,1)$ such that for each $\mathbb{r} > 0$, condition (iii) holds with probability greater than $1 - b/3$. By setting $\varepsilon_* \overset{\text{def}}{=} \varepsilon_0 \wedge \varepsilon_1$, we complete the proof of the *claim*.

Henceforth assume that $F_\varepsilon^{\mathbb{r}}$ occurs. Let $x, y \in \overline{B_{\mathbb{r}}(0)} \cap \Upsilon$ with $|x - y| \geq b\mathbb{r}$ and $x \leftrightarrow y$. Let $P\colon [0,1] \to \Upsilon$ be a D_Υ-geodesic from x to y. It follows from condition (i) that P is contained in $B_{R\mathbb{r}}(0)$. Set $t_0 \overset{\text{def}}{=} 0$ and choose $z_1 \in \left(\frac{1}{100}\varepsilon^{1+\nu}\mathbb{r}\mathbb{Z}\right)^2 \cap B_{R\mathbb{r}}(0)$ and $r_1 \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ such that $x \in B_{r_1/2}(z_1)$ and conditions (i) and (ii) of Lemma 6.12 (with (z_1, r_1) in place of (z, r)) hold. Then, inductively, for each $j \in \mathbb{N}$, if $y \notin B_{(1+a)r_j}(z_j)$, then set t_j to be the first time after t_{j-1} at which P hits $\partial B_{(1+a)r_j}(z_j)$, and choose $z_{j+1} \in \left(\frac{1}{100}\varepsilon^{1+\nu}\mathbb{r}\mathbb{Z}\right)^2 \cap B_{R\mathbb{r}}(0)$ and $r_{j+1} \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ such that $P(t_j) \in B_{r_{j+1}/2}(z_{j+1})$ and conditions (i) and (ii) in Lemma 6.12 (with (z_{j+1}, r_{j+1}) in place of (z, r)) hold; otherwise set $t_j \overset{\text{def}}{=} 1$. Write $n \overset{\text{def}}{=} \inf\{j \in \mathbb{N} : t_j = 1\}$. For each $j = [1, n-1]_{\mathbb{Z}}$, write

- s_j for the last time before t_j at which P hits $\partial B_{(1-a)r_j}(z_j)$;
- σ_j for the last time before t_j at which P hits $\partial B_{(1-\alpha)r_j}(z_j)$;
- τ_j for the first time after σ_j at which P hits $\partial B_{r_j}(z_j)$.

See Figure 4 for an illustration. Thus $t_{j-1} < s_j < \sigma_j < \tau_j < t_j$ for all $j \in [1, n-1]_{\mathbb{Z}}$. Fix $j \in [1, n-1]_{\mathbb{Z}}$. It follows immediately from the above definitions that condition (ii.c) is satisfied with z_j, r_j, $P|_{[s_j, t_j]}$, σ_j, τ_j in place of z, r, P, s, t, respectively. It follows immediately from condition (iii) (resp. (iv)) that condition (ii.d) (resp. (ii.b)) is satisfied with z_j, r_j, $P|_{[s_j, t_j]}$, σ_j, τ_j in place of z, r, P, s, t, respectively. It follows immediately from condition (ii) in Lemma 6.12 that condition (ii.a) is satisfied with z_j, r_j, $P|_{[s_j, t_j]}$, σ_j, τ_j in place of z, r, P, s, t, respectively. Thus, we conclude from condition (ii) that

$$\widetilde{D}_\Upsilon(P(\sigma_j), P(\tau_j)) \leq M D_\Upsilon(P(\sigma_j), P(\tau_j)), \quad \forall j \in [1, n-1]_{\mathbb{Z}}. \tag{6.12}$$

By condition (iii), we have

$$D_\Upsilon(P(\sigma_j), P(\tau_j)) \geq \alpha^3 \mathfrak{c}_{r_j}, \quad \forall j \in [1, n-1]_{\mathbb{Z}}. \tag{6.13}$$

By condition (ii) of Lemma 6.12, we have

$$D_\Upsilon(P(t_{j-1}), P(t_j)) \leq \alpha^{-1} \mathfrak{c}_{r_j}, \quad \forall j \in [1, n]_{\mathbb{Z}}. \tag{6.14}$$

Combining (6.13) and (6.14), we obtain that

$$D_\Upsilon(P(\sigma_j), P(\tau_j)) \geq \alpha^4 D_\Upsilon(P(t_{j-1}), P(t_j)), \quad \forall j \in [1, n-1]_{\mathbb{Z}}. \tag{6.15}$$

By (1.5), condition (iii), and (6.14), we have

$$D_\Upsilon(P(t_{n-1}), P(t_n)) \leq \alpha^{-1} \mathfrak{c}_{r_n} \leq \alpha^{-1} \mathfrak{K} \varepsilon \mathfrak{c}_{\mathbb{r}} \leq \alpha^{-1} \mathfrak{K} \varepsilon^{1/2} D_\Upsilon(x, y). \tag{6.16}$$

Combining (6.12), (6.15), and (6.16), we obtain that

$$
\begin{aligned}
\widetilde{D}_\Upsilon(x,y) \le & \sum_{j=1}^{n-1} \Big(\widetilde{D}_\Upsilon(P(t_{j-1}),P(\sigma_j)) + \widetilde{D}_\Upsilon(P(\sigma_j),P(\tau_j)) + \widetilde{D}_\Upsilon(P(\tau_j),P(t_j)) \Big) \\
& + \widetilde{D}_\Upsilon(P(t_{n-1}),P(t_n)) \\
\le & \sum_{j=1}^{n-1} (M^* D_\Upsilon(P(t_{j-1}),P(\sigma_j)) + M D_\Upsilon(P(\sigma_j),P(\tau_j)) + M^* D_\Upsilon(P(\tau_j),P(t_j))) \\
& + M^* D_\Upsilon(P(t_{n-1}),P(t_n)) \\
= & \sum_{j=1}^{n-1} (M^* D_\Upsilon(P(t_{j-1}),P(t_j)) - (M^*-M) D_\Upsilon(P(\sigma_j),P(\tau_j))) + M^* D_\Upsilon(P(t_{n-1}),P(t_n)) \\
\le & \left(M^* - \alpha^4(M^*-M)\right) \sum_{j=1}^{n-1} D_\Upsilon(P(t_{j-1}),P(t_j)) + M^* D_\Upsilon(P(t_{n-1}),P(t_n)) \\
\le & \left(M^* - \alpha^4(M^*-M)\right) D_\Upsilon(x,y) + M^* D_\Upsilon(P(t_{n-1}),P(t_n)) \\
\le & \left(M^* - \alpha^4(M^*-M) + \alpha^{-1}\mathfrak{K}\varepsilon^{1/2}\right) D_\Upsilon(x,y).
\end{aligned}
$$

Recall that $M' > M^* - \alpha^4(M^*-M)$. Thus, we conclude that, if $F_\varepsilon^{\mathfrak{r}}$ occurs and ε is sufficiently small so that $M^* - \alpha^4(M^*-M) + \alpha^{-1}\mathfrak{K}\varepsilon^{1/2} \le M'$, then (6.11) holds. Combining this with the *claim* of the preceding paragraph, we complete the proof. ☐

7. Preliminary constructions

7.1. **Setup and outline.** The goal of this section is to construct events that satisfy certain conditions. Then, in Section 8, we will use these objects to prove Theorem 1.9. Fix two weak geodesic CLE_κ carpet metrics D and $\widetilde{D}$ with the same scaling constants $\{\mathfrak{c}_r\}_{r>0}$.

Let M_* and M^* be as in (6.1) and (6.2), respectively. Our ultimate goal is to prove that $M_* = M^*$. Suppose by way of eventual contradiction that $M_* < M^*$. To state the conditions which our events need to satisfy, we shall write

$$M_1 \overset{\text{def}}{=} (2M_* + M^*)/3 \quad \text{and} \quad M_2 \overset{\text{def}}{=} (M_* + 2M^*)/3.$$

Fix a sufficiently small deterministic constant $\lambda > 0$, in a manner depending only on M_* and M^*, whose particular value is unimportant. Fix parameters

$$\mathbb{p} \in (0,1), \quad \mathsf{A} > 0, \quad 0 < \mu < \nu, \quad \Lambda \in \mathbb{N} \tag{7.1}$$

to be chosen later (cf. Lemma 8.17), in a manner depending only on D and $\widetilde{D}$. (In particular, $\mathbb{p}$ will be chosen to be close to one, A, ν, and Λ will be chosen to be large, and μ will be chosen to be close to ν.) Let $\underline{p} = p(\mu,\nu,\Lambda)$ and $a = a(\mu,\nu,\Lambda)$ be as in Proposition 6.7.

In the present section, we construct a set of admissible radii $\mathcal{R} \subset \{\Lambda^k\}_{k\in\mathbb{Z}}$ and, for each $r \in \rho^{-1}\mathcal{R}$ (where $\rho \in (0,1)$ is a deterministic constant),

- events E_r and F_r,
- random open subsets $V_r \subset U_r \subset A_{2r,4r}(0)$, and
- a resampling Υ_r of Υ

that roughly speaking satisfy the following conditions:

- The event E_r is almost surely determined by $A_{r,5r}(0) \cap \Upsilon$, and $\mathbf{P}[\mathsf{E}_r] \geq \mathbb{p}$.
- The number of possibilities for (U_r, V_r) is finite.
- Let (U'_r, V'_r) be sampled uniformly from all the possibilities for (U_r, V_r), independently of everything else. Write $\Gamma(\mathbb{C} \setminus \overline{V'_r}, \mathbb{C} \setminus U'_r)$ for the union of the collection of the $(\mathbb{C} \setminus \overline{V'_r}, \mathbb{C} \setminus U'_r)$-excursions of Γ and the collection of the loops of Γ that are contained in $\mathbb{C} \setminus \overline{V'_r}$ and intersect with $\mathbb{C} \setminus U'_r$. Then Υ_r is a conditionally independent copy of Υ given $\Gamma(\mathbb{C} \setminus \overline{V'_r}, \mathbb{C} \setminus U'_r)$ and $\wp$. The purpose of the use of (U'_r, V'_r) is to ensure that Υ_r also has the law of a whole-plane nested CLE_κ. We will ultimately only be interested in the case where $(U_r, V_r) = (U'_r, V'_r)$.
- There is a deterministic constant $\mathsf{a}_9 \in (0,1)$ such that $\mathbf{P}[\mathsf{F}_r \mid \mathsf{E}_r, \ \Upsilon] \geq \mathsf{a}_9$ almost surely.
- Write Γ_r for the whole-plane nested CLE_κ corresponding to Υ_r. Then, on the event $\mathsf{E}_r \cap \mathsf{F}_r$, there is a long narrow "tube" between loops of Γ_r such that if P_r is a D_{Υ_r}-geodesic that passes though the "tube", then there are times $\sigma < \tau$ such that

$$P_r([\sigma, \tau]) \subset B_{4r}(0) \cap \Upsilon_r \quad \text{and} \quad \widetilde{D}_{\Upsilon_r}(P_r(\sigma), P_r(\tau)) \leq M_2 D_{\Upsilon_r}(P_r(\sigma), P_r(\tau)). \tag{7.2}$$

The existence of these objects will be established through a detailed and explicit construction. To provide some intuition, we now give a rough overview of how this construction works; see Figure 5 for an illustration. We begin by considering a large collection of "test balls" within $A_{2r,4r}(0)$. Inside each of these test balls, there is a uniformly positive probability that one can find a "good" pair of points $u, v \in \Upsilon$ with $u \leftrightarrow v$ that satisfy the following conditions:

- $\widetilde{D}_\Upsilon(u, v) \leq M_1 D_\Upsilon(u, v)$.
- $|u - v|$ is bounded below by a constant times r.
- There is a $\widetilde{D}_\Upsilon$-geodesic from u to v that is contained in this "test ball", and there are two connected arcs of Γ — an "inner" one and an "outer" one — such that this $\widetilde{D}_\Upsilon$-geodesic lies between them.

The event E_r will correspond, roughly speaking, to the event that many of the "test balls" succeed. The random open subset U_r will then be the union of long narrow "tubes" that link the "test balls" that succeed, and V_r will be a slightly smaller open subset of U_r. (Thus, the number of possibilities for (U_r, V_r) is finite.) Moreover, the "good" pairs of points described above, together with the $\widetilde{D}_\Upsilon$-geodesics connecting them, will not intersect U_r. The event F_r will correspond, roughly speaking, to the event that the following are true:

- We have $(U_r, V_r) = (U'_r, V'_r)$. In this case, Υ_r is a resampling of Υ within U_r, i.e., $(\mathbb{C} \setminus \overline{U_r}) \cap \Upsilon_r = (\mathbb{C} \setminus \overline{U_r}) \cap \Upsilon$.
- The "inner" connected arcs of Γ mentioned above are linked together to form a single loop $\mathscr{L}_r$ of Γ_r. The "outer" connected arcs of Γ mentioned above are linked together to form two connected arcs γ_r and γ'_r of Γ_r. Thus, the "good" pairs of points described above lie in the long narrow "tube" between $\mathscr{L}_r$, γ_r, and γ'_r.
- The D_{Υ_r}-distances inside the intersection of the "tube" between $\mathscr{L}_r$, γ_r, and γ'_r with each connected component of U_r will be very small.

The existence of the times σ and τ of (7.2) will be an immediate consequence of the definitions of E_r and F_r.

The main ideas in this section are broadly similar to those in [GM21b, Section 5] and [DG23, Section 5], which carry out an analogous construction for the Liouville quantum gravity (LQG) metric. However, there are two key differences. First, since a D_{Υ_r}-geodesic cannot cross the loops of Γ_r, once it enters the "tube" bounded by $\mathscr{L}_r$, γ_r, and γ'_r, it must remain within this tube. In the LQG setting, by contrast, additional work is needed to show that a geodesic entering the tube will spend a long time there. Second, the LQG metric enjoys the Weyl scaling property, which precisely

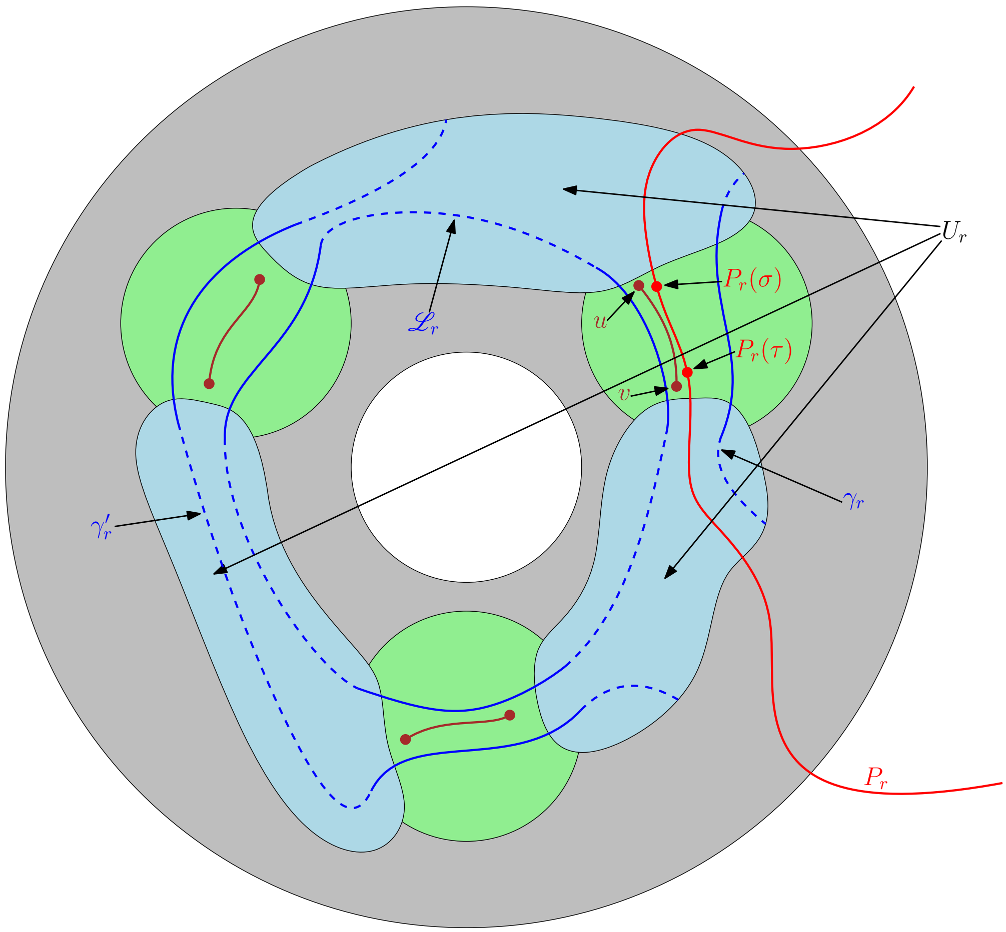

FIGURE 5. Illustration of the objects constructed in this section. The light green balls are the "test balls" that succeed, i.e., each of the light green balls contains a "good" pair of points u and v, together with the $\widetilde{D}_\Upsilon$-geodesic connecting them (brown), and two connected arcs of Γ (blue and solid). These connected arcs of Γ are "rewired" in Γ_r (blue and dashed) to form $\mathscr{L}_r$, γ_r, and γ_r'. The red path P_r is a D_{Υ_r}-geodesic that passes though the "tube" between $\mathscr{L}_r$, γ_r, and γ_r'. The times σ and τ of (7.2) are so that $P_r(\sigma)$ is close to u and $P_r(\tau)$ is close to v.

describes how the metric changes when a smooth function is added to the underlying field. This allows one to subtract a large bump function supported on the tube to "attract" the geodesic into it. In our case, the absence of the Weyl scaling property introduces significant new challenges. Roughly speaking, our method of attracting the geodesic relies instead on removing Brownian loops from the underlying Brownian loop-soup (cf. Definition 1.7, Axiom (V) (monotonicity)).

We now provide a more detailed overview of our construction.

In Section 7.2, we will introduce an event involving a single "good" pair of points u and v. A precise definition is given in Lemma 7.1, and an illustration is provided in Figure 6. Specifically, we require that $u \in \partial B_{(1-\alpha)r}(0) \cap \Upsilon$, $v \in \partial B_r(0) \cap \Upsilon$, $u \leftrightarrow v$, $\widetilde{D}_\Upsilon(u,v) \le M_1 D_\Upsilon(u,v)$, and that the $\widetilde{D}_\Upsilon$-geodesic from u to v is contained in $\mathsf{H}_r \cap \overline{A_{(1-\alpha)r,r}(0)} \cap \Upsilon$, where $\mathsf{H}_r \subset A_{(1-a)r,(1+a)r}(0)$ is a *deterministic* half-annulus. We also require that there exist two connected arcs of Γ in H_r joining $\partial B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1-a)r}(0)$ to $\partial B_{\alpha r}(\mathsf{y}_r) \cap \partial B_{(1+a)r}(0)$ such that the $\widetilde{D}_\Upsilon$-geodesic from u to v lies between them, where $\mathsf{x}_r \in \partial B_{(1-a)r}(0)$ and $\mathsf{y}_r \in \partial B_{(1+a)r}(0)$ are *deterministic*. This deterministic condition is imposed so that, later, when we define the long narrow "tube" $U_{\rho^{-1}r}$ with $B_{\alpha r}(\mathsf{x}_r)$ and $B_{\alpha r}(\mathsf{y}_r)$ serving as the "entry" and "exit" of $U_{\rho^{-1}r}$, respectively, the number of possible choices for $U_{\rho^{-1}r}$ remains finite. These two connected arcs of Γ will be obtained using the coupling between CLE_κ and the Brownian loop-soup by adding Brownian loops. Furthermore, we require that the "tube" between these arcs has "bottlenecks" at u and v, which will allow us to upper-bound the $\widetilde{D}_\Upsilon$-distance from a D_Υ-geodesic to u (resp. v), once we have forced the geodesic to pass through this "tube".

This event is closely related to the event $\underline{H}_r(M_1, a, \alpha)$ defined in Definition 6.2. Using the results from Section 6, we will show that for each

$$r \in \mathcal{R} \overset{\text{def}}{=} \left\{ r \in \{\Lambda^k\}_{k\in\mathbb{Z}} : \mathbf{P}[\underline{H}_r(M_1, a, \alpha)] \geq p \right\},$$

the probability of this event is bounded below by a deterministic constant $\mathsf{p} \in (0,1)$. We then define $\mathsf{G}_{z,r}$, for $z \in \mathbb{C}$ and $r \in \mathcal{R}$, to be the translated version of the event in Lemma 7.1, so that we work with annuli centered at z rather than at the origin (cf. Definition 7.2).

In Section 7.3, we will give a careful definition of the tubes that we will force the geodesic to pass through.

In Section 7.4, we will introduce several positive parameters

$$1 \gg \mathsf{a}_1 \gg \mathsf{a}_2 \gg \mathsf{a}_3 \gg \mathsf{a}_4 \gg \mathsf{a}_5 \gg \mathsf{a}_6 \gg \mathsf{a}_7 \gg \mathsf{a}_8 \gg \mathsf{a}_9,$$

to be chosen later, including the parameter $\rho \overset{\text{def}}{=} \mathsf{a}_6 \in (0,1)$ mentioned above. We then define the random open subsets U_r and V_r, the resampling Υ_r of Υ, and the events E_r and F_r for $r \in \rho^{-1}\mathcal{R}$ in terms of these parameters. More precisely:

- The set $\mathsf{Z}_r \subset \partial B_{3r}(0)$ (the collection of "test points") will be a deterministic finite subset, and
$$Z_r \overset{\text{def}}{=} \{z \in \mathsf{Z}_r : \mathsf{G}_{z,\rho r} \text{ occurs}\}.$$
- The open subset V_r will be defined as the union of finitely many long, narrow "tubes" linking the half-annuli $\mathsf{H}_{z,\rho r}$ for $z \in Z_r$ into an annular region, in a way that is almost surely determined by Z_r. The set U_r will be a small Euclidean neighborhood of V_r. The sets U_r and V_r are chosen so that, for each $z \in Z_r$, the deterministic balls $B_{\alpha\rho r}(\mathsf{x}_{z,\rho r})$ and $B_{\alpha\rho r}(\mathsf{y}_{z,\rho r})$ introduced earlier are contained in V_r, while the points $u, v \in \mathsf{H}_{z,\rho r}$ and the $\widetilde{D}_\Upsilon$-geodesic connecting them remain disjoint from U_r.
- Let (U_r', V_r') be sampled uniformly from all possible pairs (U_r, V_r), independently of everything else. Write $\Gamma(\mathbb{C}\setminus\overline{V_r'}, \mathbb{C}\setminus U_r')$ for the union of the $(\mathbb{C}\setminus\overline{V_r'}, \mathbb{C}\setminus U_r')$-excursions of Γ and the loops of Γ contained in $\mathbb{C}\setminus\overline{V_r'}$ that intersect $\mathbb{C}\setminus U_r'$. Then Υ_r is defined as a conditionally independent copy of Υ given $\Gamma(\mathbb{C}\setminus\overline{V_r'}, \mathbb{C}\setminus U_r')$ and $\wp$. The use of (U_r', V_r') ensures that Υ_r also has the law of a whole-plane nested CLE_κ. Ultimately, we will focus on the case $(U_r, V_r) = (U_r', V_r')$.
- The event E_r will include the condition that $\mathsf{G}_{z,\rho r}$ occurs for many points $z \in \mathsf{Z}_r$, along with several additional regularity requirements. The event F_r will include the condition $(U_r, V_r) = (U_r', V_r')$, a condition on the link pattern induced by the complementary $(\mathbb{C}\setminus\overline{V_r}, \mathbb{C}\setminus U_r)$-excursions of Γ_r (see the blue and dashed arcs in Figure 5), and a condition roughly stating that the D_{Υ_r}-distances inside the "tubes" between these blue and dashed arcs are very small. The event E_r will also require that $\mathbf{P}[\mathsf{F}_r \mid \mathsf{E}_r, \Upsilon] \geq \mathsf{a}_9$. Note that E_r depends only on Υ, while F_r depends on Υ, Υ_r, and (U_r', V_r').

In Section 7.5, we will show that we can choose the parameters of Section 7.4 in such a way that $\mathbf{P}[\mathsf{E}_r] \geq \mathbb{p}$. In Section 7.6, we will check that our objects satisfy (7.2).

7.2. **Existence of a shortcut with positive probability.** In this subsection, we will prove that for each $r \in \mathcal{R}$ (cf. (7.3)), there is a uniformly positive probability that a "good" pair of points $u, v \in \overline{B_r(0)} \cap \Upsilon$ exists with $u \leftrightarrow v$, such that $\widetilde{D}_\Upsilon(u,v) \le M_1 D_\Upsilon(u,v)$ and certain additional regularity conditions are satisfied. In later subsections, we will use the long-range independence property of CLE (cf. Lemma 2.18) to show that, with high probability, many such pairs of points occur within an annulus at a larger scale.

We shall refer to as a *half-annulus* an open subset of the form

$$\{x \in A_{s,t}(z) : \Re(x) > \Re(z)\}, \quad \{x \in A_{s,t}(z) : \Im(x) > \Im(z)\}, \quad \{x \in A_{s,t}(z) : \Re(x) < \Re(z)\}, \quad \text{or} \quad \{x \in A_{s,t}(z) : \Im(x) < \Im(z)\},$$

where $z \in \mathbb{C}$ and $0 < s < t$.

Recall that we let $\underline{p} = p(\mu, \nu, \Lambda)$ and $a = a(\mu, \nu, \Lambda)$ be as in Proposition 6.7.

Lemma 7.1. *There exists $\alpha \in (0, \lambda a]$ and $\mathsf{p} \in (0,1)$ such that for each*

$$r \in \mathcal{R} \overset{\text{def}}{=} \left\{ r \in \{\Lambda^k\}_{k \in \mathbb{Z}} : \mathbf{P}[\underline{H}_r(M_1, a, \alpha)] \ge p \right\}, \tag{7.3}$$

there are

- *a deterministic half-annulus H_r of inner (resp. outer) radius $(1-a)r$ (resp. $(1+a)r$) centered at 0, and*
- *deterministic points*

$$\mathsf{x}_r \in \left\{ (1-a) r e^{\mathrm{i}k} : k \in \lambda\alpha\mathbb{Z} \right\} \cap \partial \mathsf{H}_r \quad \text{and} \quad \mathsf{y}_r \in \left\{ (1+a) r e^{\mathrm{i}k} : k \in \lambda\alpha\mathbb{Z} \right\} \cap \partial \mathsf{H}_r \tag{7.4}$$

such that it holds with probability at least p that there exists $u \in \mathsf{H}_r \cap \partial B_{(1-\alpha)r}(0) \cap \Upsilon$ and $v \in \mathsf{H}_r \cap \partial B_r(0) \cap \Upsilon$ satisfying the following conditions:

(i) There is a $\widetilde{D}_\Upsilon$-geodesic $\widetilde{P}$ from u to v that is contained in $\mathsf{H}_r \cap \overline{A_{(1-\alpha)r,r}(0)} \cap \Upsilon$.

(ii) $\widetilde{D}_\Upsilon(u,v) \ge \alpha^3 \mathfrak{c}_r$ and $\widetilde{D}_\Upsilon(u,v) \le M_1 D_\Upsilon(u,v)$.

(iii) There are two connected arcs η_{L} and η_{R} of Γ in H_r from $B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1-a)r}(0)$ to $B_{\alpha r}(\mathsf{y}_r) \cap \partial B_{(1+a)r}(0)$ such that
- *η_{L} lies to the left of η_{R},*
- *there is no other connected arc of Γ in H_r from $B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1-a)r}(0)$ to $B_{\alpha r}(\mathsf{y}_r) \cap \partial B_{(1+a)r}(0)$ that lies between η_{L} and η_{R},*
- *the right side of η_{L} and the left side of η_{R} are contained in Υ, and*
- *$\widetilde{P}$ lies between η_{L} and η_{R}.*

(iv) Write O_r for the connected component of $\mathsf{H}_r \setminus (\eta_{\mathrm{L}} \cup \eta_{\mathrm{R}})$ containing $\widetilde{P}$. Then

$$\widetilde{D}_\Upsilon(\partial B_{(1-a)r}(0) \cap \partial O_r \cap \Upsilon, \partial B_{(1+a)r}(0) \cap \partial O_r \cap \Upsilon; O_r \cap \Upsilon) \le 2a^{-1} \mathfrak{c}_r.$$

(v) There are paths P_s and P_t in $\overline{\mathsf{H}_r} \cap \Upsilon$ from η_{L} to η_{R} that have $\widetilde{D}_\Upsilon$-lengths at most $\lambda\alpha^3\mathfrak{c}_r$, and there are times $0 < s < t < 1$ such that $\widetilde{P}(s) \in P_s$, $\widetilde{P}(t) \in P_t$, and $\widetilde{D}_\Upsilon(u, \widetilde{P}(s)) \vee \widetilde{D}_\Upsilon(\widetilde{P}(t), v) \le \lambda\alpha^3\mathfrak{c}_r$.

(vi) We have $D_\Upsilon(u,v) \le \lambda D_\Upsilon(u, \partial A_{(1-a/2)r,(1+a/2)r}(0) \cap \Upsilon)$.

An illustration of the objects appearing in Lemma 7.1 can be found in Figure 6. The rest of this subsection is primarily concerned with establishing Lemma 7.1. Before presenting the proof itself, we begin by explaining the motivation behind the formulation of the statement.

In Section 7.4, we will "link up" the half-annuli $\mathsf{H}_{\rho r} + z$ for varying choices of $z \in \partial B_{3r}(0)$ with long narrow "tubes". If we label these z's in counterclockwise order as $z_1, z_2, \ldots$, then the long narrow

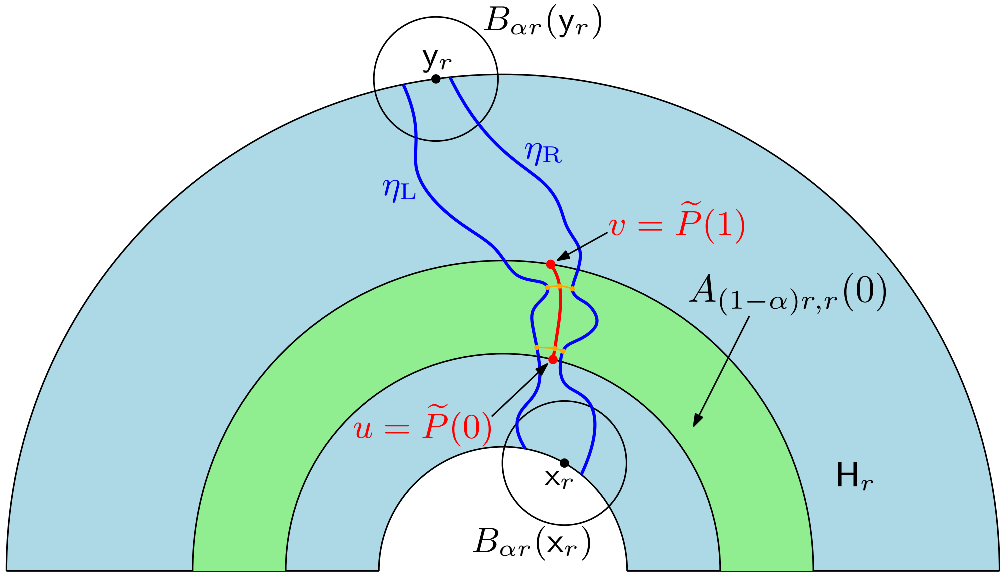

FIGURE 6. Illustration of the objects involved in Lemma 7.1. H_r is a deterministic half-annulus. Both $\mathsf{x}_r \in \partial B_{(1-a)r}(0) \cap \partial \mathsf{H}_r$ and $\mathsf{y}_r \in \partial B_{(1+a)r}(0) \cap \partial \mathsf{H}_r$ are deterministic points. The blue paths $\eta_{\rm L}$ and $\eta_{\rm R}$ are connected arcs of Γ in H_r from $B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1-a)r}(0)$ to $B_{\alpha r}(\mathsf{y}_r) \cap \partial B_{(1+a)r}(0)$. The red path $\widetilde{P}$ is a $\widetilde{D}_\Upsilon$-geodesic in $\mathsf{H}_r \cap \overline{A_{(1-\alpha)r,r}(0)} \cap \Upsilon$ from $u \in \partial B_{(1-\alpha)r}(0) \cap \Upsilon$ to $v \in \partial B_r(0) \cap \Upsilon$ that lies between $\eta_{\rm L}$ and $\eta_{\rm R}$. The pair (u, v) is a "shortcut" in the sense that $\widetilde{D}_\Upsilon(u, v) \leq M_1 D_\Upsilon(u, v)$. There are times $0 < s < t < 1$ and an orange path P_s (resp. P_t) in Υ connecting $\eta_{\rm L}$, $\widetilde{P}(s)$, and $\eta_{\rm R}$ (resp. $\eta_{\rm L}$, $\widetilde{P}(t)$, and $\eta_{\rm R}$). The $\widetilde{D}_\Upsilon$-lengths of P_s, P_t, $\widetilde{P}|_{[0,s]}$, and $\widetilde{P}|_{[t,1]}$ are much smaller than the $\widetilde{D}_\Upsilon$-length of $\widetilde{P}$. This implies that every path in Υ that passes between $\eta_{\rm L}$ and $\eta_{\rm R}$ will have to get close to the "shortcut".

"tube" linking $\mathsf{H}_{\rho r} + z_j$ and $\mathsf{H}_{\rho r} + z_{j+1}$ is given by the Euclidean neighborhood of a deterministic smooth simple path connecting $\mathsf{y}_{\rho r} + z_j$ and $\mathsf{x}_{\rho r} + z_{j+1}$. We want there to be only finitely many possibilities for the set $r^{-1} U_r$, which allows us to get certain estimates trivially by taking a maximum over the possibilities. This is why we require that H_r is either vertical or horizontal and the points x_r and y_r belong to the finite sets in (7.4). We will then resample the configurations inside U_r. The requirement that $\widetilde{P} \subset \mathsf{H}_r \cap \overline{A_{(1-\alpha)r,r}(0)} \cap \Upsilon$ in condition (i), together with condition (vi), ensures that the condition that $\widetilde{P}$ is a $\widetilde{D}_\Upsilon$-geodesic and $\widetilde{D}_\Upsilon(u, v) \leq M_1 D_\Upsilon(u, v)$ is preserved after resampling. Condition (vi) also ensures that the event in the lemma statement depends locally on Υ (cf. Lemma 7.3). Condition (iv) is needed to upper-bound the amount of time a D_Υ-geodesic needs to spend to pass through the the "tube" between $\eta_{\rm L}$ and $\eta_{\rm R}$. The purpose of condition (v) is to ensure the existence of "bottlenecks" at u and v, so that any path in Υ that passes through the "tube" between $\eta_{\rm L}$ and $\eta_{\rm R}$ will have to get close to the "shortcut".

We now proceed to the proof of Lemma 7.1.

Proof of Lemma 7.1. See Figure 7 for an illustration. By Lemma 2.5, (i) and a union bound, we may choose $\delta \in (0, \lambda a]$ to be sufficiently small so that for each $r > 0$, it holds with probability at least $1 - \underline{p}/2$ that the following is true.

(7.5) For each $z \in \{r e^{\mathrm{i}k} : k \in \lambda\delta\mathbb{Z} \cap [0, 2\pi)\}$, there are at most two connected arcs of Γ in $A_{\delta r, ar/2}(z)$ connecting its inner and outer boundaries.

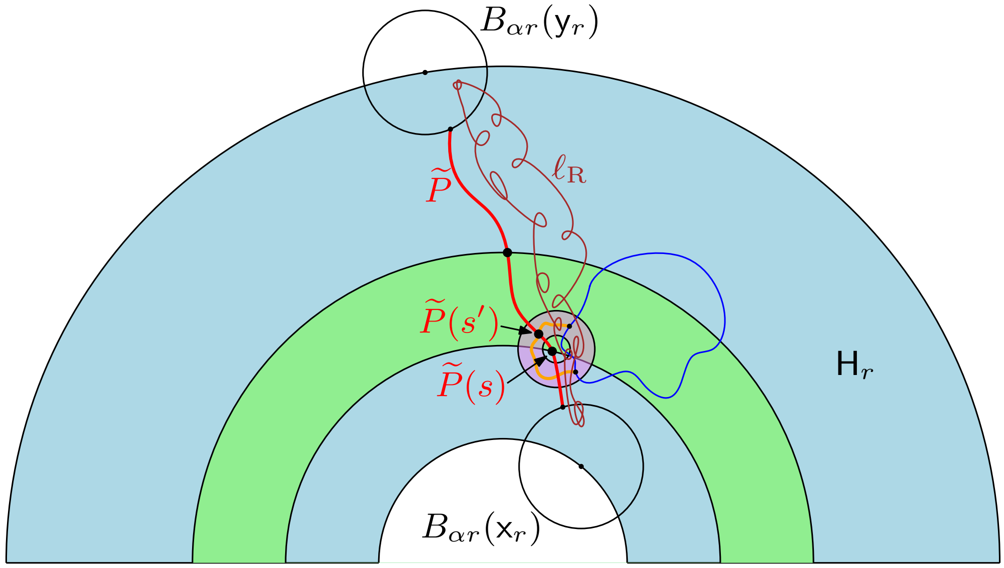

FIGURE 7. Illustration of the proof of Lemma 7.1. The red path $\widetilde{P}$ is obtained by deforming the $\widetilde{D}_\Upsilon$-geodesic described in the definition of $\underline{H}_r(M_1, a, \alpha)$ (cf. Definition 6.2) relative to $\widetilde{P}|_{[s,t]}$. There exists a violet annulus $A_{\delta r, \delta^\omega r}(u)$ and an orange path P_u such that $\widetilde{P}(s) \in B_{\delta r}(u)$ and every path in Υ that crosses between the inner and outer boundaries of $A_{\delta r, \delta^\omega r}(u)$ intersects P_u. The time s' denotes the first time after s at which $\widetilde{P}$ hits P_u. Similarly, we have $A_{\delta r, \delta^\omega r}(v)$ and P_v (not shown). The brown loop ℓ_{R} is a Brownian loop in H_r that intersects $\partial B_{\alpha r}(\mathsf{x}_r)$, $\partial B_{\alpha r}(\mathsf{y}_r)$, $\partial B_{\delta r}(u)$, and $\partial B_{\delta r}(v)$, and lies to the right of $\widetilde{P}$. Similarly, we have ℓ_{L} (not shown). After adding these two Brownian loops, the existence of η_{L} and η_{R} follows from the coupling between CLE_κ and Brownian loop-soup (cf. [SW12]). We remark that $\widetilde{P}|_{[s,t]}$, s', and t' in the proof will play the role of $\widetilde{P}$, s, and t in the lemma statement, respectively.

Moreover, by Definition 1.7, Axiom (IV) (tightness across scales) and possibly decreasing δ, we may arrange that for each $r > 0$, it holds with probability at least $1 - \underline{p}/4$ that the following is true.

$$D_\Upsilon(x, y) \le \lambda D_\Upsilon(y, \partial B_{ar/2}(y) \cap \Upsilon), \quad \forall x, y \in \overline{B_r(0)} \cap \Upsilon \text{ with } x \leftrightarrow y \text{ and } |x - y| \le \frac{1}{2}\delta r. \tag{7.6}$$

By the scale invariance of the law of Υ, we may choose $\alpha \in (0, \lambda\delta]$ to be sufficiently small so that for each $r > 0$, it holds with probability at least $1 - \underline{p}/8$ that the following is true.

(7.7) The Euclidean diameter of each connected component of $A_{(1-\alpha)r,r}(0) \cap \Upsilon$ is at most $\delta r/2$.

Let $\alpha_{4\mathrm{A}}$ be as in (2.2). Fix $\omega \in (0, 1)$ such that $\alpha_{4\mathrm{A}}(1 - \omega) > 1$. Then, by Lemma 2.5, (ii), Lemma 2.16, (1.5), and a union bound, we may choose a sufficiently small $\delta \in (0, \lambda\alpha]$ and a sufficiently large $A > 0$ such that for each $r > 0$, it holds with probability at least $1 - \underline{p}/16$ that the following is true.

(7.8) For each $z \in \{(1-\alpha)re^{\mathrm{i}k} : k \in \lambda\delta\mathbb{Z} \cap [0, 2\pi)\} \cup \{re^{\mathrm{i}k} : k \in \lambda\delta\mathbb{Z} \cap [0, 2\pi)\}$, there are at most two connected arcs of Γ in $A_{\delta r, \delta^\omega r}(z)$ connecting its inner and outer boundaries, and there is a path in $A_{\delta r, \delta^\omega r}(z) \cap \Upsilon$ that has $\widetilde{D}_\Upsilon$-length at most $A\delta^\omega \mathfrak{c}_r$ such that every path in Υ that crosses between the inner and outer boundaries of $A_{\delta r, \delta^\omega r}(z)$ intersects P.

Moreover, by possibly decreasing δ, we may arrange so that $A\delta^\omega \le \lambda\alpha^3$.

By the definition of the event $\underline{H}_r(M_1, a, \alpha)$ (cf. Definition 6.2) and the definition of $\mathcal{R}$ (cf. (7.3)), for each $r \in \mathcal{R}$, it holds with probability at least $\underline{p}$ that the following is true.

$$\text{There exists a } \widetilde{D}_\Upsilon\text{-geodesic } \widetilde{P}\colon [0,1] \to \overline{A_{(1-a)r,(1+a)r}(0)} \cap \Upsilon \text{ from } \partial B_{(1-a)r}(0) \cap \Upsilon \text{ to } \partial B_{(1+a)r}(0) \cap \Upsilon \text{ and times } 0 < s < t < 1 \text{ satisfying conditions (ii.a), (ii.b), (ii.c), and (ii.d) of Definition 6.2.} \tag{7.9}$$

Thus, we conclude that for each $r \in \mathcal{R}$, it holds with probability at least $\underline{p}/16$ that (7.5), (7.6), (7.7), (7.8), and (7.9) are true. Since the Euclidean diameter of $\widetilde{P}$ is at most $r/100$ (cf. condition (ii.b)), there is a half-annulus H_r of inner (resp. outer) radius $(1-a)r$ (resp. $(1+a)r$) centered at 0, and points

$$x_r \in \left\{(1-a)r\mathrm{e}^{\mathrm{i}k} : k \in \lambda\alpha\mathbb{Z}\right\} \cap \partial H_r \quad \text{and} \quad y_r \in \left\{(1+a)r\mathrm{e}^{\mathrm{i}k} : k \in \lambda\alpha\mathbb{Z}\right\} \cap \partial H_r$$

such that $\widetilde{P} \subset \overline{H_r} \cap \Upsilon$, $\widetilde{P}(0) \in B_{\alpha r}(x_r) \cap \partial B_{(1-a)r}(0) \cap \Upsilon$, and $\widetilde{P}(1) \in B_{\alpha r}(y_r) \cap \partial B_{(1+a)r}(0) \cap \Upsilon$. Since the number of possible choices for such H_r, x_r, y_r is at most a deterministic constant not depending on r, it follows that there exists $p' \in (0,1)$ such that for each $r \in \mathcal{R}$, there exists a deterministic choice H_r, x_r, y_r of H_r, x_r, y_r, resp. such that it holds with probability at least p' that (7.5), (7.6), (7.7), (7.8), and (7.9) are true, $\widetilde{P} \subset \overline{\mathsf{H}_r} \cap \Upsilon$, $\widetilde{P}(0) \in B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1-a)r}(0) \cap \Upsilon$, and $\widetilde{P}(1) \in B_{\alpha r}(\mathsf{y}_r) \cap \partial B_{(1+a)r}(0) \cap \Upsilon$.

By (7.8), there exists $u \in \{(1-\alpha)r\mathrm{e}^{\mathrm{i}k} : k \in \lambda\delta\mathbb{Z} \cap [0,2\pi)\}$ and $v \in \{r\mathrm{e}^{\mathrm{i}k} : k \in \lambda\delta\mathbb{Z} \cap [0,2\pi)\}$ such that $\widetilde{P}(s) \in B_{\delta r}(u)$ and $\widetilde{P}(t) \in B_{\delta r}(v)$, and there is a path P_u in $A_{\delta r, \delta^\omega r}(u) \cap \Upsilon$ (resp. P_v in $A_{\delta r, \delta^\omega r}(v) \cap \Upsilon$) that has $\widetilde{D}_\Upsilon$-length at most $A\delta^\omega \mathfrak{c}_r \le \lambda\alpha^3\mathfrak{c}_r$ such that every path in Υ that crosses between the inner and outer boundaries of $A_{\delta r, \delta^\omega r}(u)$ (resp. $A_{\delta r, \delta^\omega r}(v)$) intersects P_u (resp. P_v). Write s' for the first time after s at which $\widetilde{P}$ hits P_u, and t' for the last time before t at which $\widetilde{P}$ hits P_v. Since $\widetilde{P}$ is a $\widetilde{D}_\Upsilon$-geodesic and also hits P_u before s (resp. P_v after t), it follows that

$$\widetilde{D}_\Upsilon(\widetilde{P}(s), \widetilde{P}(s')) \le \operatorname{len}(P_u, \widetilde{D}_\Upsilon) \le A\delta^\omega \mathfrak{c}_r \le \lambda\alpha^3\mathfrak{c}_r \quad \text{and}$$
$$\widetilde{D}_\Upsilon(\widetilde{P}(t'), \widetilde{P}(t)) \le \operatorname{len}(P_v, \widetilde{D}_\Upsilon) \le A\delta^\omega \mathfrak{c}_r \le \lambda\alpha^3\mathfrak{c}_r.$$

Moreover, there are at most two connected arcs of Γ in $A_{\delta r, \delta^\omega r}(u)$ (resp. $A_{\delta r, \delta^\omega r}(v)$) connecting its inner and outer boundaries. If there are exactly two such arcs, write $\mathcal{L}_s$ (resp. $\mathcal{L}_t$) for the loop of Γ containing them; otherwise write $\mathcal{L}_s \stackrel{\text{def}}{=} \emptyset$ (resp. $\mathcal{L}_t \stackrel{\text{def}}{=} \emptyset$). (At this point, we remark that, eventually, $\widetilde{P}|_{[s,t]}$, s', and t' will play the role of $\widetilde{P}$, s, and t in the lemma statement, respectively.)

We observe that for each $\varepsilon > 0$, it is always possible to deform $\widetilde{P}$ relative to $\widetilde{P}|_{[s,t]}$ so that we obtain a path (not necessarily a $\widetilde{D}_\Upsilon$-geodesic) $\widetilde{P}'\colon [0,1] \to \overline{\mathsf{H}_r} \cap \Upsilon$ from $B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1-a)r}(0) \cap \Upsilon$ to $B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1+a)r}(0) \cap \Upsilon$ satisfying the following conditions:

- $\widetilde{P}'|_{[s,t]} \equiv \widetilde{P}|_{[s,t]}$.
- $|\widetilde{P}'(\tau) - \widetilde{P}(\tau)| \le \varepsilon r$ for all $\tau \in [0,1]$.
- $\operatorname{len}(\widetilde{P}'; \widetilde{D}_\Upsilon) \le (1+\varepsilon)\operatorname{len}(\widetilde{P}; \widetilde{D}_\Upsilon)$.
- Every loop of Γ that touches $\widetilde{P}'|_{[0,s]}$ or $\widetilde{P}'|_{[t,1]}$ has Euclidean diameter at most εr.

We note that conditions (ii.c) and (ii.d) of Definition 6.2 are preserved when replacing $\widetilde{P}$ with $\widetilde{P}'$, and it follows from condition (ii.a) that $\operatorname{len}(\widetilde{P}', \widetilde{D}_\Upsilon) \le 2a^{-1}\mathfrak{c}_r$. Write $\mathcal{L}$ for the smallest loop surrounding $\widetilde{P}$ (hence also $\widetilde{P}'$). Write μ^{loop} for the Brownian loop measure (cf. [LW04]). Write A_{L} (resp. A_{R}) for the collection of Brownian loops ℓ satisfying the following conditions:

(a) ℓ is surrounded by $\mathscr{L}$ and contained in H_r.
(b) ℓ intersects $\partial B_{\alpha r}(\mathsf{x}_r)$, $\partial B_{\alpha r}(\mathsf{y}_r)$, $\partial B_{\delta r}(u)$, and $\partial B_{\delta r}(v)$.
(c) ℓ lies to the left (resp. right) of $\widetilde{P}'$.
(d) $\ell \cap B_{\varepsilon r}(\widetilde{P}') = \emptyset$.
(e) $\ell \cap \mathscr{L}_s \subset B_{\delta^\omega r}(u)$, and $\ell \cap \mathscr{L}_s \neq \emptyset$ if and only if $\mathscr{L}_s \neq \emptyset$ and crosses between the inner and outer boundaries of $A_{\delta r, \delta^\omega r}(u)$ on the left (resp. right) side of $\widetilde{P}$.
(f) $\ell \cap \mathscr{L}_t \subset B_{\delta^\omega r}(v)$, and $\ell \cap \mathscr{L}_t \neq \emptyset$ if and only if $\mathscr{L}_t \neq \emptyset$ and crosses between the inner and outer boundaries of $A_{\delta r, \delta^\omega r}(v)$ on the left (resp. right) side of $\widetilde{P}$.
(g) Every loop of Γ that intersects ℓ except possibly $\mathscr{L}_s$ and $\mathscr{L}_t$ has Euclidean diameter at most $\varepsilon r/100$.

We observe that, almost surely, $\mu^{\mathrm{loop}}(A_{\mathrm{L}}) \wedge \mu^{\mathrm{loop}}(A_{\mathrm{R}}) > 0$ whenever ε is sufficiently small. Thus, by the scale invariance of the law of Υ and replacing $\widetilde{P}$ with $\widetilde{P}'$, we conclude that there exists a sufficiently small $\varepsilon > 0$ such that for each $r \in \mathcal{R}$, it holds with probability at least $p'/2$ that (7.5), (7.6), (7.7), (7.8), together with the following (7.10), are true.

(7.10) There is a path (not necessarily a $\widetilde{D}_\Upsilon$-geodesic) $\widetilde{P}\colon [0,1] \to \overline{\mathsf{H}_r} \cap \Upsilon$ from $B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1-a)r}(0) \cap \Upsilon$ to $B_{\alpha r}(\mathsf{x}_r) \cap \partial B_{(1+a)r}(0) \cap \Upsilon$ with $\mathrm{len}(\widetilde{P}, \widetilde{D}_\Upsilon) \leq 2a^{-1}\mathfrak{c}_r$ and times $0 < s < t < 1$ satisfying conditions (ii.c) and (ii.d) of Definition 6.2. Moreover, every loop of Γ that touches $\widetilde{P}|_{[0,s]}$ or $\widetilde{P}|_{[t,1]}$ has Euclidean diameter at most εr, and if we define A_{L} and A_{R} in the same manner as above but with $\widetilde{P}$ in place of $\widetilde{P}'$, then $\mu^{\mathrm{loop}}(A_{\mathrm{L}}) \wedge \mu^{\mathrm{loop}}(A_{\mathrm{R}}) > \varepsilon$.

We *claim* that for each $\ell_{\mathrm{L}} \in A_{\mathrm{L}}$ and $\ell_{\mathrm{R}} \in A_{\mathrm{R}}$, there is no loop of Γ that intersects both ℓ_{L} and ℓ_{R}. First, we observe that $\mathscr{L}_s$ (resp. $\mathscr{L}_t$) cannot intersect both ℓ_{L} and ℓ_{R}. Indeed, this follows immediately from conditions (e), (f), together with the fact that $\mathscr{L}_s$ (resp. $\mathscr{L}_t$) can only cross between the inner and outer boundaries of $A_{\delta r, \delta^\omega r}(u)$ on either the left or the right side of $\widetilde{P}$, but not both (otherwise there would be at least four connected arcs of Γ in $A_{\delta r, \delta^\omega r}(u)$ (resp. $A_{\delta r, \delta^\omega r}(v)$) connecting its inner and outer boundaries). Now, since $\ell_{\mathrm{L}} \cap B_{\varepsilon r}(\widetilde{P}) = \emptyset$ and $\ell_{\mathrm{L}} \cap B_{\varepsilon r}(\widetilde{P}) = \emptyset$ (cf. condition (d)) and ℓ_{L} and ℓ_{R} lie to the different sides of $\widetilde{P}$, it is easy to see that $\mathrm{dist}(\ell_{\mathrm{L}}, \ell_{\mathrm{R}}) > \varepsilon r/100$, which implies that every loop of Γ with Euclidean diameter at most $\varepsilon r/100$ cannot intersect both ℓ_{L} and ℓ_{R}. Thus, by condition (g), we complete the proof of the *claim*.

Let Ξ be a Brownian loop-soup of intensity $c(\kappa)$ inside $\mathscr{L}$. Suppose that Γ and Ξ are coupled so that the outermost loops of Γ inside $\mathscr{L}$ are given by the outer boundaries of the outermost clusters of the Brownian loops of Ξ [SW12]. Given A_{L} (resp. A_{R}), let ℓ_{L} (resp. ℓ_{R}) be sampled uniformly from $\mu^{\mathrm{loop}}|_{A_{\mathrm{L}}}$ (resp. $\mu^{\mathrm{loop}}|_{A_{\mathrm{R}}}$), independently of everything else. Write $\Xi' \stackrel{\mathrm{def}}{=} \Xi \cup \{\ell_{\mathrm{L}}, \ell_{\mathrm{R}}\}$. Then, (by, e.g., Mecke's formula) on the event that $\mu^{\mathrm{loop}}(A_{\mathrm{L}}) \wedge \mu^{\mathrm{loop}}(A_{\mathrm{R}}) > \varepsilon$, the law of Ξ' is absolutely continuous with respect to the law of Ξ and the Radon-Nikodym derivative is bounded above by a deterministic constant.

Since for each $r \in \mathcal{R}$, it holds with probability at least $p'/2$ that (7.5), (7.6), (7.7), (7.8), and (7.10) are true, it suffices to show that, on this event, conditions (i), (ii), (iii), (iv), (v), and (vi) are satisfied for Ξ'. By the above *claim* and the fact that $\ell_{\mathrm{L}} \cap \widetilde{P} = \emptyset$ and $\ell_{\mathrm{R}} \cap \widetilde{P} = \emptyset$ (cf. condition (d)), after adding ℓ_{L} and ℓ_{R}, the path $\widetilde{P}$ is still contained in the carpet of the non-nested CLE_κ inside $\mathscr{L}$. One verifies immediately that conditions (i) and (ii) are satisfied with $\widetilde{P}|_{[s,t]}$, $\widetilde{P}(s)$, and $\widetilde{P}(t)$ in place of $\widetilde{P}$, u, and v, respectively. Condition (iv) follows immediately from the fact that $\mathrm{len}(\widetilde{P}, \widetilde{D}_\Upsilon) \leq 2a^{-1}\mathfrak{c}_r$. Condition (vi) follows immediately from (7.6). By conditions (a), (b), and (c), it is easy to see that condition (iii) is satisfied after adding ℓ_{L} and ℓ_{R}. Thus, it suffices to verify that condition (v) is satisfied for Ξ'. Indeed, write P_s (resp. P_t) for the connected component of $P_u \setminus (\ell_{\mathrm{L}} \cup \ell_{\mathrm{R}})$

(resp. $P_v \setminus (\ell_{\mathrm{L}} \cup \ell_{\mathrm{R}})$) that contains $\widetilde{P}(s)$ (resp. $\widetilde{P}(t)$). Then

$$\mathrm{len}(P_s; \widetilde{D}_\Upsilon) \le \mathrm{len}(P_u; \widetilde{D}_\Upsilon) \le \lambda \alpha^3 \mathfrak{c}_r \quad \text{and} \quad \mathrm{len}(P_t; \widetilde{D}_\Upsilon) \le \mathrm{len}(P_v; \widetilde{D}_\Upsilon) \le \lambda \alpha^3 \mathfrak{c}_r.$$

Moreover, by conditions (b), (e), and (f), one verifies immediately that P_s and P_t must connect the arcs η_{L} and η_{R} of condition (iii). This completes the proof. ☐

The event of Lemma 7.1 will be the "building block" for the event E_r. In the remainder of the present section, let α, p, $\mathcal{R}$, and $\{(\mathsf{H}_r, \mathsf{x}_r, \mathsf{y}_r) : r \in \mathcal{R}\}$ be as in Lemma 7.1. For $z \in \mathbb{C}$ and $r \in \mathcal{R}$, we shall write

$$\mathsf{H}_{z,r} \stackrel{\mathrm{def}}{=} \mathsf{H}_r + z; \quad \mathsf{x}_{z,r} \stackrel{\mathrm{def}}{=} \mathsf{x}_r + z; \quad \mathsf{y}_{z,r} \stackrel{\mathrm{def}}{=} \mathsf{y}_r + z.$$

Definition 7.2. Let $z \in \mathbb{C}$ and $r \in \mathcal{R}$. Then we shall write $\mathsf{G}_{z,r}$ for the event of Lemma 7.1 with $\Upsilon - z$ in place of Υ, i.e., the event that there exists $u \in \mathsf{H}_{z,r} \cap \partial B_{(1-\alpha)r}(z) \cap \Upsilon$ and $v \in \mathsf{H}_{z,r} \cap \partial B_r(z) \cap \Upsilon$ satisfying the following conditions:

(i) There is a $\widetilde{D}_\Upsilon$-geodesic $\widetilde{P}$ from u to v that is contained in $\mathsf{H}_{z,r} \cap \overline{A_{(1-\alpha)r,r}(z)} \cap \Upsilon$.
(ii) $\widetilde{D}_\Upsilon(u,v) \ge \alpha^3 \mathfrak{c}_r$ and $\widetilde{D}_\Upsilon(u,v) \le M_1 D_\Upsilon(u,v)$.
(iii) There are two connected arcs η_{L} and η_{R} of Γ in $\mathsf{H}_{z,r}$ from $B_{\alpha r}(\mathsf{x}_{z,r}) \cap \partial B_{(1-a)r}(z)$ to $B_{\alpha r}(\mathsf{y}_{z,r}) \cap \partial B_{(1+a)r}(z)$ such that
  - η_{L} lies to the left of η_{R},
  - there is no other connected arc of Γ in $\mathsf{H}_{z,r}$ from $B_{\alpha r}(\mathsf{x}_{z,r}) \cap \partial B_{(1-a)r}(z)$ to $B_{\alpha r}(\mathsf{y}_{z,r}) \cap \partial B_{(1+a)r}(z)$ that lies between η_{L} and η_{R},
  - the right side of η_{L} and the left side of η_{R} are contained in Υ, and
  - $\widetilde{P}$ lies between η_{L} and η_{R}.
(iv) Write $O_{z,r}$ for the connected component of $\mathsf{H}_{z,r} \setminus (\eta_{\mathrm{L}} \cup \eta_{\mathrm{R}})$ containing $\widetilde{P}$. Then

$$\widetilde{D}_\Upsilon(\partial B_{(1-a)r}(z) \cap \partial O_{z,r} \cap \Upsilon, \partial B_{(1+a)r}(z) \cap \partial O_{z,r} \cap \Upsilon; O_{z,r} \cap \Upsilon) \le 2a^{-1} \mathfrak{c}_r.$$

(v) There are paths P_s and P_t in $\overline{\mathsf{H}_{z,r}} \cap \Upsilon$ from η_{L} to η_{R} that have $\widetilde{D}_\Upsilon$-lengths at most $\lambda \alpha^3 \mathfrak{c}_r$, and there are times $0 < s < t < 1$ such that $\widetilde{P}(s) \in P_s$, $\widetilde{P}(t) \in P_t$, and $\widetilde{D}_\Upsilon(u, \widetilde{P}(s)) \vee \widetilde{D}_\Upsilon(\widetilde{P}(t), v) \le \lambda \alpha^3 \mathfrak{c}_r$.
(vi) We have $D_\Upsilon(u,v) \le \lambda D_\Upsilon(u, \partial A_{(1-a/2)r,(1+a/2)r}(z) \cap \Upsilon)$.

By Definition 1.7, Axiom (III) (translation invariance) and Lemma 7.1,

$$\mathbf{P}[\mathsf{G}_{z,r}] \ge \mathsf{p}, \quad \forall z \in \mathbb{C}, \ \forall r \in \mathcal{R}. \tag{7.11}$$

Lemma 7.3. *Let $z \in \mathbb{C}$ and $r \in \mathcal{R}$. Then the event is almost surely determined by $B_{(1+a)r}(z) \cap \Upsilon$.*

Proof. It is clear that conditions (iii), (iv), (v), and (vi) are almost surely determined by $B_{(1+a)r}(z) \cap \Upsilon$. Note that if condition (vi) holds, then any path from u to v that exits $A_{(1-a/2)r,(1+a/2)r}(z)$ is neither a D_Υ-geodesic nor a $\widetilde{D}_\Upsilon$-geodesic. This implies that, on the event that condition (vi) holds, conditions (i) and (ii) are almost surely determined by $B_{(1+a)r}(z) \cap \Upsilon$. This completes the proof. ☐

7.3. **Definition of the tubes.** Recall the parameters λ, $\mathbb{p}$, A, and a from Section 7.1, and α and p from Section 7.2. The definitions of U_r, V_r, Υ_r, E_r, and F_r will depend on sufficiently small positive parameters

$$1 \gg \mathsf{a}_1 \gg \mathsf{a}_2 \gg \mathsf{a}_3 \gg \mathsf{a}_4 \gg \mathsf{a}_5 \gg \mathsf{a}_6 \gg \mathsf{a}_7 \gg \mathsf{a}_8 \gg \mathsf{a}_9 \tag{7.12}$$

which will be chosen later in Section 7.5. These parameters will be chosen in a manner so that each a_j depends only on λ, $\mathbb{p}$, A, a, α, p, and $\mathsf{a}_1, \ldots, \mathsf{a}_{j-1}$. To lighten notation, we shall write $\rho \stackrel{\mathrm{def}}{=} \mathsf{a}_6$.

In the present subsection, we define the "tubes" U_r and V_r. We will then define the events E_r and F_r in the next subsection.

Fix $r \in \rho^{-1}\mathcal{R}$. We shall write

$$\mathsf{Z}_r \stackrel{\text{def}}{=} \left\{ 3r\mathrm{e}^{\mathrm{i}k} : k \in (\mathsf{a}_5\mathbb{Z}) \cap [0, 2\pi) \right\}. \tag{7.13}$$

for the set of "test points". The event E_r will include the condition that the event $\mathsf{G}_{z,\rho r}$ occurs for "many" of the points $z \in \mathsf{Z}_r$. To quantify this, we shall write

$$Z_r \stackrel{\text{def}}{=} \{ z \in \mathsf{Z}_r : \mathsf{G}_{z,\rho r} \text{ occurs} \}. \tag{7.14}$$

Recall from Section 7.2 the half-annuli $\mathsf{H}_{z,\rho r}$, and the points $\mathsf{x}_{z,\rho r}$ and $\mathsf{y}_{z,\rho r}$. We will now construct the "tubes" which link up the half-annuli $\mathsf{H}_{z,\rho r}$ for $z \in \mathsf{Z}_r$. Let

$$\{ \mathsf{P}_{w_1,w_2} : w_1, w_2 \in \mathsf{Z}_r,\ w_1 \neq w_2 \}$$

be a family of deterministic smooth simple paths satisfying the following conditions:

(A) Let $w_1, w_2 \in \mathsf{Z}_r$ be distinct. Then P_{w_1,w_2} is a smooth simple path from $\mathsf{y}_{w_1,\rho r}$ to $\mathsf{x}_{w_2,\rho r}$ that does not intersect $\mathsf{H}_{z,\rho r}$ (except at the endpoints). Moreover, if we write I_{w_1,w_2} for the counterclockwise connected arc of $\partial B_{3r}(0)$ from w_1 to w_2, then $\mathsf{P}_{w_1,w_2} \subset B_{100\rho r}(I_{w_1,w_2})$.

(B) Let $w_1, w_2, w_3 \in \mathsf{Z}_r$ be distinct points in counterclockwise order. Then the Euclidean distance between P_{w_1,w_2} and P_{w_2,w_3} is at least $\alpha \rho r$. The Euclidean distance between P_{w_1,w_2} and $\mathsf{H}_{w_3,\rho r}$ is at least $\mathsf{a}_5 r$.

(C) Let $w_1, w_2, w_3, w_4 \in \mathsf{Z}_r$ be distinct points in counterclockwise order. Then the Euclidean distance between P_{w_1,w_2} and P_{w_3,w_4} is at least $\mathsf{a}_5 r$.

(D) The set
$$\left\{ r^{-1}\mathsf{P}_{w_1,w_2} : w_1, w_2 \in \mathsf{Z}_r,\ w_1 \neq w_2 \right\}$$
depends only on a_5, ρ, $(\rho r)^{-1}\mathsf{H}_{\rho r}$, $(\rho r)^{-1}\mathsf{x}_{\rho r}$, $(\rho r)^{-1}\mathsf{y}_{\rho r}$ (but not on r). Thus, the number of possibilities for this set is at most a constant depending only on λ, a, α, a_5, ρ.

We shall write

$$\mathsf{U}_{w_1,w_2} \stackrel{\text{def}}{=} B_{(1+\mathsf{a}_7)\alpha\rho r}(\mathsf{P}_{w_1,w_2}) \quad \text{and} \quad \mathsf{V}_{w_1,w_2} \stackrel{\text{def}}{=} B_{\alpha\rho r}(\mathsf{P}_{w_1,w_2}), \quad \forall \text{ distinct } w_1, w_2 \in \mathsf{Z}_r. \tag{7.15}$$

Let $z_1, \ldots, z_{\#Z_r} \in \partial B_{3r}(0)$ be the points of Z_r in counterclockwise order. Let $j \in [1, \#Z_r]_{\mathbb{Z}}$. Then we shall write

$$P_j \stackrel{\text{def}}{=} \mathsf{P}_{z_j,z_{j+1}}; \quad U_j \stackrel{\text{def}}{=} \mathsf{U}_{z_j,z_{j+1}}; \quad V_j \stackrel{\text{def}}{=} \mathsf{V}_{z_j,z_{j+1}}, \tag{7.16}$$

with the convention that $z_{\#Z_r+1} \stackrel{\text{def}}{=} z_1$. We shall write (by abuse of notation)

$$U_r \stackrel{\text{def}}{=} \bigcup_{j=1}^{\#Z_r} U_j \quad \text{and} \quad V_r \stackrel{\text{def}}{=} \bigcup_{j=1}^{\#Z_r} V_j.$$

See Figure 8 for an illustration of P_j, U_j, and V_j.

7.4. **A prerequisite event and a postrequisite event.** In the present subsection, we define the events E_r and F_r. Recall the parameters of (7.12).

Before proceeding, we introduce some notation. See Figure 9 for an illustration. Let $r \in \rho^{-1}\mathcal{R}$ and $j \in [1, \#Z_r]_{\mathbb{Z}}$. Recall the point z_j and the open subsets U_j and V_j from Section 7.3. We shall write $U_j^\star$ for the connected component containing V_j of the open subset obtained by removing from U_j the closure of the union of all the $(\mathbb{C} \setminus \overline{V_j}, \mathbb{C} \setminus U_j)$-excursions of Γ and the loops of Γ that are contained

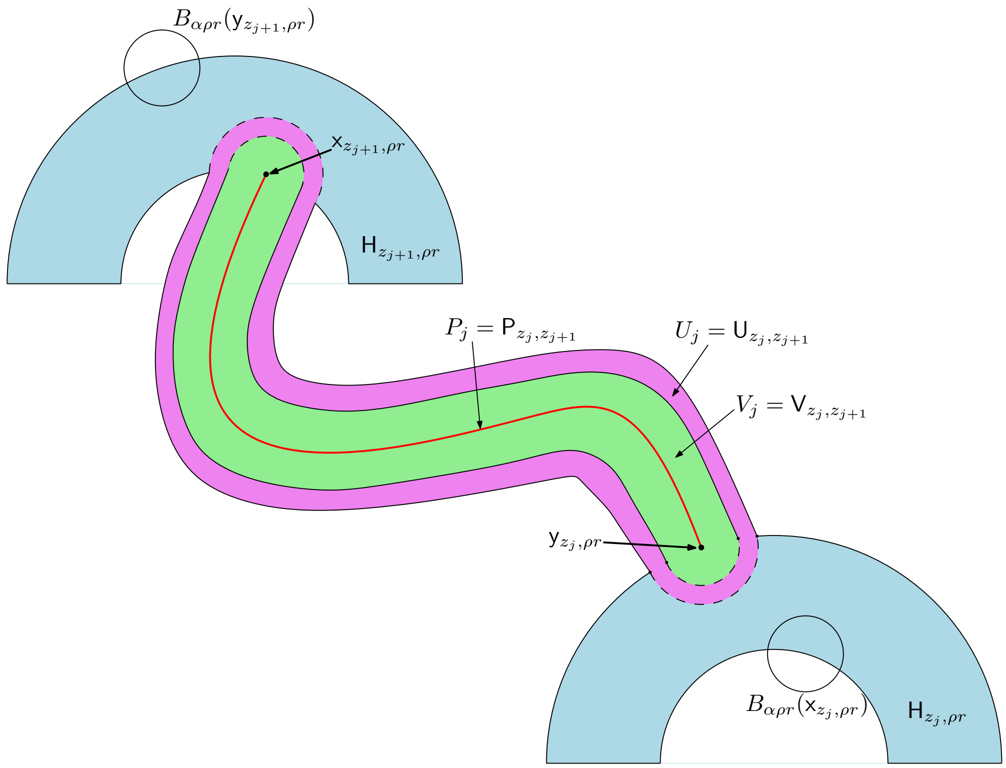

FIGURE 8. Illustration of $P_j = \mathsf{P}_{z_j,z_{j+1}}$, $U_j = \mathsf{U}_{z_j,z_{j+1}}$, and $V_j = \mathsf{V}_{z_j,z_{j+1}}$, where z_j and z_{j+1} are neighboring points in Z_r. The path P_j is a smooth simple path from $\mathsf{y}_{z_j,\rho r}$ to $\mathsf{x}_{z_{j+1},\rho r}$ that are almost surely determined by z_j and z_{j+1}. The open subset U_j (resp. V_j) is the Euclidean $(1+\mathsf{a}_7)\alpha\rho r$- (resp. $\alpha\rho r$-) neighborhood of P_j.

in $\mathbb{C} \setminus \overline{V_j}$ and intersect $\mathbb{C} \setminus U_j$. We shall write $X_j \overset{\text{def}}{=} \partial U_j^\star \cap \partial V_j$ for the collection of the endpoints of the $(\mathbb{C} \setminus \overline{V_j}, \mathbb{C} \setminus U_j)$-excursions of Γ. We shall write

$$U_r^\star \overset{\text{def}}{=} \bigcup_{j=1}^{\#Z_r} U_j^\star. \tag{7.17}$$

Let $\eta_{j,\mathrm{L}}$ and $\eta_{j,\mathrm{R}}$ be as in the definition of $\mathsf{G}_{z_j,\rho r}$ (cf. Definition 7.2). We shall write $x_{j,\mathrm{L}} \in X_{j-1}$ and $y_{j,\mathrm{L}} \in X_j$ (resp. $x_{j,\mathrm{R}} \in X_{j-1}$ and $y_{j,\mathrm{R}} \in X_j$) for the endpoints of $\eta_{j,\mathrm{L}}$ (resp. $\eta_{j,\mathrm{R}}$). Since the right side of $\eta_{j,\mathrm{L}}$ and the left side of $\eta_{j,\mathrm{R}}$ are contained in Υ, it follows that

$$\#\left(X_j \cap [y_{j,\mathrm{L}}, y_{j,\mathrm{R}}]^{\circlearrowleft}_{\partial U_j^\star}\right), \quad \#\left(X_j \cap [y_{j,\mathrm{R}}, x_{j+1,\mathrm{R}}]^{\circlearrowleft}_{\partial U_j^\star}\right),$$
$$\#\left(X_j \cap [x_{j+1,\mathrm{R}}, x_{j+1,\mathrm{L}}]^{\circlearrowleft}_{\partial U_j^\star}\right), \quad \#\left(X_j \cap [x_{j+1,\mathrm{L}}, y_{j,\mathrm{L}}]^{\circlearrowleft}_{\partial U_j^\star}\right)$$

(i.e., the number of boundary marked points of $(U_j^\star; X_j)$ that lie on the counterclockwise arc of $\partial U_j^\star$ from $y_{j,\mathrm{L}}$ to $y_{j,\mathrm{R}}$ (resp. from $y_{j,\mathrm{R}}$ to $x_{j+1,\mathrm{R}}$; from $x_{j+1,\mathrm{R}}$ to $x_{j+1,\mathrm{L}}$; from $x_{j+1,\mathrm{L}}$ to $y_{j,\mathrm{L}}$)) are all even. Thus, we may label the points of X_j in counterclockwise order as

$$\begin{aligned} &y_{j,\mathrm{L}},\ y_{j,1},\ldots,\ y_{j,2n_j},\ y_{j,\mathrm{R}},\ w_{j,1},\ \ldots,\ w_{j,2r_j},\ x_{j+1,\mathrm{R}},\ x_{j+1,1},\ \ldots,\ x_{j+1,2m_j},\ x_{j+1,\mathrm{L}}, \\ &\qquad z_{j,1},\ \ldots,\ z_{j,2l_j}. \end{aligned} \tag{7.18}$$

In this situation, we shall introduce the link patterns α_j and β_j (see Figure 9 for an illustration). The link pattern α_j serves to create an opening in the chain of tubes which the geodesic is able to enter and the link pattern β_j serves to create a tube which the geodesic cannot leave. When we perform the resampling procedure later on, we aim to create a chain of tubes where two of them end up with link pattern α_j (so the geodesic has a place to enter and also a place to leave) and the rest end up with link pattern β_j (so the geodesic has no option but to traverse the tube once it has entered). Concretely, we let

$$\begin{aligned}(7.19)\quad \alpha_j \overset{\text{def}}{=} &\{(x_{j+1,\mathrm{L}}, y_{j,\mathrm{L}})\} \cup \{(z_{j,2k-1}, z_{j,2k}) : k \in [1, l_j]_{\mathbb{Z}}\} \\ &\cup \big\{(y_{j,\mathrm{R}}, w_{j,1}), (w_{j,2r_j}, x_{j+1,\mathrm{R}})\big\} \cup \{(w_{j,2k}, w_{j,2k+1}) : k \in [1, r_j - 1]_{\mathbb{Z}}\} \\ &\cup \{(y_{j,2k-1}, y_{j,2k}) : k \in [1, n_j]_{\mathbb{Z}}\} \cup \{(x_{j+1,2k-1}, x_{j+1,2k}) : k \in [1, m_j]_{\mathbb{Z}}\}\end{aligned}$$

(i.e., α_j is the link pattern on $(U_j^\star; X_j)$ that "links up" the endpoints of $\eta_{j,\mathrm{L}}$ and $\eta_{j+1,\mathrm{L}}$, and all other boundary marked points of $(U_j^\star; X_j)$ to their nearest neighbors);

$$\begin{aligned}\beta_j \overset{\text{def}}{=} &\{(x_{j+1,\mathrm{L}}, y_{j,\mathrm{L}}), (y_{j,\mathrm{R}}, x_{j+1,\mathrm{R}})\} \cup \{(z_{j,2k-1}, z_{j,2k}) : k \in [1, l_j]_{\mathbb{Z}}\} \cup \{(w_{j,2k-1}, w_{j,2k}) : k \in [1, r_j]_{\mathbb{Z}}\} \\ &\cup \{(y_{j,2k-1}, y_{j,2k}) : k \in [1, n_j]_{\mathbb{Z}}\} \cup \{(x_{j+1,2k-1}, x_{j+1,2k}) : k \in [1, m_j]_{\mathbb{Z}}\}\end{aligned}$$

(i.e., β_j is the link pattern on $(U_j^\star; X_j)$ that "links up" the endpoints of $\eta_{j,\mathrm{L}}$ and $\eta_{j+1,\mathrm{L}}$ and the endpoints of $\eta_{j,\mathrm{R}}$ and $\eta_{j+1,\mathrm{R}}$, resp. and all other boundary marked points of $(U_j^\star; X_j)$ to their nearest neighbors). We shall also introduce collections of counterclockwise arcs of $\partial U_j^\star$ with endpoints contained in X_j

$$\begin{aligned}\mathcal{I}_j \overset{\text{def}}{=} &\Big\{[y_{j,\mathrm{L}}, y_{j,1}]^{\circlearrowleft}_{\partial U_j^\star}, [y_{j,2n_j}, y_{j,\mathrm{R}}]^{\circlearrowleft}_{\partial U_j^\star}\Big\} \cup \Big\{[y_{j,2k}, y_{j,2k+1}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, n_j - 1]_{\mathbb{Z}}\Big\} \\ &\cup \Big\{[x_{j+1,\mathrm{R}}, x_{j+1,1}]^{\circlearrowleft}_{\partial U_j^\star}, [x_{j+1,2m_j}, x_{j+1,\mathrm{L}}]^{\circlearrowleft}_{\partial U_j^\star}\Big\} \cup \Big\{[x_{j+1,2k}, x_{j+1,2k+1}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, m_j - 1]_{\mathbb{Z}}\Big\} \\ &\cup \Big\{[z_{j,2k-1}, z_{j,2k}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, l_j]_{\mathbb{Z}}\Big\} \cup \Big\{[w_{j,2k-1}, w_{j,2k}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, r_j]_{\mathbb{Z}}\Big\};\end{aligned}$$

(i.e., $\mathcal{I}_j$ is the collection of all counterclockwise arcs of $\partial U_j^\star$ such that their endpoints are nearest neighbors of X^j and they are contained in Υ when viewed from the inside of $U_j^\star$, this implies that if P is a path in Υ that enters $U_j^\star$ from outside of $U_j^\star$, then the first point at which P enters $U_j^\star$ must belong to $I \cap \partial U_j$ for some $I \in \mathcal{I}_j$);

$$\begin{aligned}\mathcal{I}_j^\alpha \overset{\text{def}}{=} &\Big\{[y_{j,\mathrm{L}}, y_{j,1}]^{\circlearrowleft}_{\partial U_j^\star}, [y_{j,2n_j}, y_{j,\mathrm{R}}]^{\circlearrowleft}_{\partial U_j^\star}\Big\} \cup \Big\{[y_{j,2k}, y_{j,2k+1}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, n_j - 1]_{\mathbb{Z}}\Big\} \\ &\cup \Big\{[x_{j+1,\mathrm{R}}, x_{j+1,1}]^{\circlearrowleft}_{\partial U_j^\star}, [x_{j+1,2m_j}, x_{j+1,\mathrm{L}}]^{\circlearrowleft}_{\partial U_j^\star}\Big\} \cup \Big\{[x_{j+1,2k}, x_{j+1,2k+1}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, m_j - 1]_{\mathbb{Z}}\Big\} \\ &\cup \Big\{[w_{j,2k-1}, w_{j,2k}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, r_j]_{\mathbb{Z}}\Big\} \subset \mathcal{I}_j;\end{aligned}$$

(i.e., $\mathcal{I}_j^\alpha$ is the subcollection of $\mathcal{I}_j$ consisting of the arcs that are not disconnected from the "center" of $U_j^\star$ by the link pattern α_j);

$$\begin{aligned}(7.20)\quad \mathcal{I}_j^\beta \overset{\text{def}}{=} &\Big\{[y_{j,\mathrm{L}}, y_{j,1}]^{\circlearrowleft}_{\partial U_j^\star}, [y_{j,2n_j}, y_{j,\mathrm{R}}]^{\circlearrowleft}_{\partial U_j^\star}\Big\} \cup \Big\{[y_{j,2k}, y_{j,2k+1}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, n_j - 1]_{\mathbb{Z}}\Big\} \\ &\cup \Big\{[x_{j+1,\mathrm{R}}, x_{j+1,1}]^{\circlearrowleft}_{\partial U_j^\star}, [x_{j+1,2m_j}, x_{j+1,\mathrm{L}}]^{\circlearrowleft}_{\partial U_j^\star}\Big\} \cup \Big\{[x_{j+1,2k}, x_{j+1,2k+1}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, m_j - 1]_{\mathbb{Z}}\Big\} \subset \mathcal{I}_j^\alpha;\end{aligned}$$

(i.e., $\mathcal{I}_j^\beta$ is the subcollection of $\mathcal{I}_j$ consisting of the arcs that are not disconnected from the "center" of $U_j^\star$ by the link pattern β_j);

$$(7.21)\qquad \mathcal{I}_j^{\alpha\setminus\beta} \overset{\text{def}}{=} \Big\{[w_{j,2k-1}, w_{j,2k}]^{\circlearrowleft}_{\partial U_j^\star} : k \in [1, r_j]_{\mathbb{Z}}\Big\} = \mathcal{I}_j^\alpha \setminus \mathcal{I}_j^\beta.$$

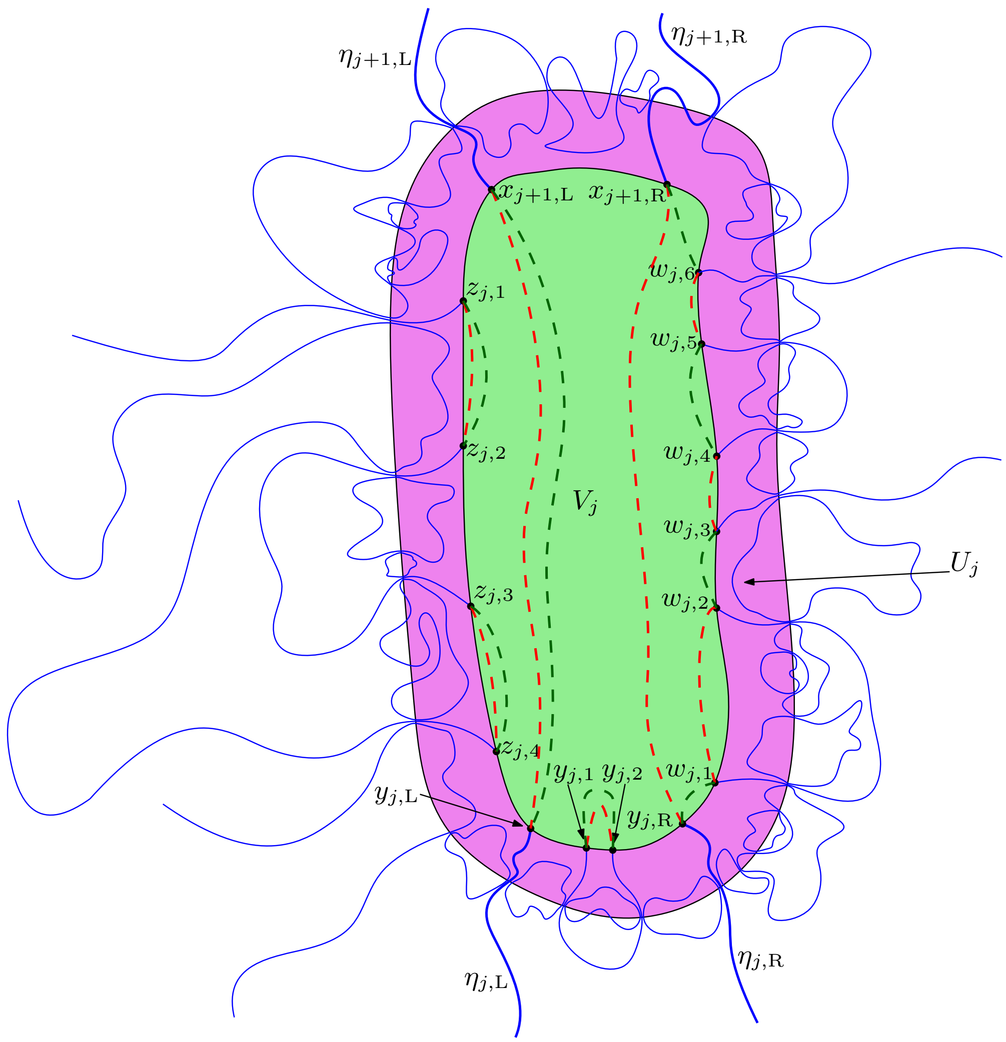

FIGURE 9. Illustration of U_j, V_j, X_j, and the labeling of X_j as in (7.18) for $j \in [1, \#Z_r]_{\mathbb{Z}}$. The blue paths are (connected arcs of) loops of Γ. The dark green dashed paths induce the link pattern α_j, and the red dashed paths induce the link pattern β_j. The purpose of the link pattern α_j is to create an opening that the geodesic can either enter or leave (and we aim to have two of these in the chain of tubes) and the purpose of the link pattern β_j is to make it impossible for the geodesic to leave.

We will now perform a resampling operation. Let (U_r', V_r') be sampled uniformly from all the possibilities for (U_r, V_r), independently of everything else. We shall write $\Gamma(\mathbb{C} \setminus \overline{V_r'}, \mathbb{C} \setminus U_r')$ for the union of the collection of the $(\mathbb{C} \setminus \overline{V_r'}, \mathbb{C} \setminus U_r')$-excursions of Γ and the collection of the loops of Γ that are contained in $\mathbb{C} \setminus \overline{V_r'}$ and intersect $\mathbb{C} \setminus U_r'$. Let Υ_r be a conditionally independent copy of Υ given $\Gamma(\mathbb{C} \setminus \overline{V_r'}, \mathbb{C} \setminus U_r')$ and $\wp$. (In particular, we have $(\mathbb{C} \setminus \overline{U_r'}) \cap \Upsilon_r = (\mathbb{C} \setminus \overline{U_r'}) \cap \Upsilon$ almost surely.) The purpose of the use of (U_r', V_r') is to ensure that Υ_r also has the law of a whole-plane nested CLE_κ. We will ultimately only be interested in the case where $(U_r, V_r) = (U_r', V_r')$. It follows from (D) that the number of possibilities for $r^{-1}(U_r, V_r)$ is at most a deterministic constant not depending on r. Thus, the probability that $(U_r, V_r) = (U_r', V_r')$ is uniformly positive.

For a simply connected domain $U \subset \mathbb{C}$, we shall write

$$d_U(x, y) \stackrel{\text{def}}{=} \inf \left\{ \operatorname{diam}(X) : X \text{ is a connected subset of } \overline{U} \text{ with } x, y \in X \right\}, \quad \forall x, y \in \overline{U}.$$

We will now define the events E_r and F_r.

Definition 7.4. Let $r \in \rho^{-1}\mathcal{R}$. Then we shall write E_r for the event that the following conditions are satisfied:

(i) We have

$$\min\big(D_\Upsilon(\partial B_r(0) \cap \Upsilon, \partial B_{2r}(0) \cap \Upsilon), D_\Upsilon(\partial B_{4r}(0) \cap \Upsilon, \partial B_{5r}(0) \cap \Upsilon)\big) \geq \mathsf{a}_1 \mathfrak{c}_r.$$

(ii) Let $x, y \in A_{2r,4r}(0) \cap \Upsilon$ with $|x-y| \leq \mathsf{a}_3 r$. Then either $x \xleftrightarrow{A_{r,5r}(0)} y$ or there exists a loop of Γ contained in $A_{r,5r}(0)$ and disconnecting x and y. Moreover, in the former case, we have

$$D_\Upsilon(x, y; A_{r,5r}(0) \cap \Upsilon) \leq \mathsf{a}_2 \mathfrak{c}_r.$$

(iii) Let $x, y \in A_{2r,4r}(0) \cap \Upsilon$ with $|x-y| \leq 100\mathsf{a}_4 r$. Then either $x \xleftrightarrow{A_{r,5r}(0)} y$ or there exists a loop of Γ with Euclidean diameter at most $\mathsf{a}_3 r/2$, contained in $A_{r,5r}(0)$, and disconnecting x and y.

(iv) For each connected arc $I \subset \partial B_{3r}(0)$ of Euclidean length at least $\mathsf{a}_4 r$, we have $I \cap Z_r \neq \emptyset$.

(v) There exists a finite subset $\mathcal{S} \subset A_{2r,4r}(0) \times (0, \lambda r]$ such that the following are true:
  (a) For each connected component $\Upsilon_\bullet \subset \Upsilon$ with $(\mathbb{C} \setminus B_{4r}(0)) \cap \Upsilon_\bullet \neq \emptyset$ and each connected component $\mathscr{C} \subset (U_r \setminus V_r) \cap \Upsilon_\bullet$, there exists $(z,t) \in \mathcal{S}$ such that $\mathscr{C} \subset B_t(z)$.
  (b) There exists $N \in \mathbb{N}$ such that for each $(z,t) \in \mathcal{S}$, there are at most $2N$ connected arcs of Γ in $A_{t,2t}(z)$ connecting its inner and outer boundaries.
  (c) We have

$$N \sum_{(z,t) \in \mathcal{S}} \sup\left\{D_\Upsilon(x, y; A_{r,5r}(0) \cap \Upsilon) : x, y \in B_{2t}(z),\ x \xleftrightarrow{B_{2t}(z)} y\right\} \leq \mathsf{a}_2 \mathfrak{c}_r.$$

(vi) For each $j \in [1, \#Z_r]_\mathbb{Z}$, the following are true:
  (a) There are at most $1/\mathsf{a}_8$ connected arcs of Γ in $U_j \setminus V_j$ connecting ∂U_j and ∂V_j.
  (b) Each pair of distinct points of X_j has Euclidean distance at least $\mathsf{a}_8 r$.
  (c) For each $I \in \mathcal{I}_j$, $I \cap \partial U_j$ and P_j are contained in the same connected component of $U_j^\star \setminus \mathscr{B}_{100\mathsf{a}_8 r}(\partial U_j^\star \setminus I; d_{U_j^\star})$, where $\mathscr{B}_{100\mathsf{a}_8 r}(\partial U_j^\star \setminus I; d_{U_j^\star}) \subset U_j^\star$ denotes the $100\mathsf{a}_8 r$-neighborhood of $\partial U_j^\star \setminus I$ with respect to the metric $d_{U_j^\star}$. (Here, we recall the definition of P_j from (7.16).)

(vii) For each subset $J_r \subset [1, \#Z_r]_\mathbb{Z}$ with $\#J_r = 2$, the following is true: Given Υ and the event that condition (vi) holds, the following conditions (ix), (x), and (xi) of Definition 7.5 hold with conditional probability at least a_9. (Here, we observe that condition (vi) depends only on $\Gamma(\mathbb{C} \setminus \overline{V_r}, \mathbb{C} \setminus U_r)$ and $\wp$. Thus, if condition (vi) holds for Υ, then it also holds for Υ_r.)

Definition 7.5. Let $J_r \subset [1, \#Z_r]_\mathbb{Z}$ be a subset with $\#J_r = 2$ (chosen in a manner that is almost surely determined by Υ). Then we shall write $\mathsf{F}_r(J_r)$ for the event that the following conditions are satisfied:

(viii) We have $(U_r, V_r) = (U_r', V_r')$.

(ix) For each $j \in J_r$ (resp. $j \in [1, \#Z_r]_\mathbb{Z} \setminus J_r$), the link pattern induced by the complementary $(\mathbb{C} \setminus \overline{V_j}, \mathbb{C} \setminus U_j)$-excursions of Γ_r is given by α_j (resp. β_j).

(x) For each $j \in J_r$ and $(x,y) \in \alpha_j$ (resp. $j \in [1, \#Z_r]_\mathbb{Z} \setminus J_r$ and $(x,y) \in \beta_j$), the complementary $(\mathbb{C} \setminus \overline{V_j}, \mathbb{C} \setminus U_j)$-excursion of Γ_r connecting x and y is contained in $\mathscr{B}_{\mathsf{a}_8 r}([x,y]^{\circlearrowleft}_{\partial U_j^\star}; d_{U_j^\star})$.

(xi) We have

$$D_{\Upsilon_r}(x_1, x_2; A_{r,5r}(0) \cap \Upsilon_r) \leq \rho \mathfrak{c}_r, \quad \forall j \in J_r,\ \forall I_1, I_2 \in \mathcal{I}_j^\alpha,\ \forall x_1 \in I_1 \cap \partial U_j,\ \forall x_2 \in I_2 \cap \partial U_j.$$

and

$$D_{\Upsilon_r}(x_1, x_2; A_{r,5r}(0) \cap \Upsilon_r) \le \rho \mathfrak{c}_r,$$
$$\forall j \in [1, \#Z_r]_{\mathbb{Z}} \setminus J_r,\ \forall I_1, I_2 \in \mathcal{I}_j^{\beta},\ \forall x_1 \in I_1 \cap \partial U_j,\ \forall x_2 \in I_2 \cap \partial U_j.$$

We note that, by condition (vii), for each $r \in \rho^{-1}\mathcal{R}$ and each $J_r \subset [1, \#Z_r]_{\mathbb{Z}}$ with $\#J_r = 2$ chosen in a manner that is almost surely determined by Υ, we have

$$\mathbf{P}[\mathsf{F}_r(J_r) \mid \mathsf{E}_r,\ \Upsilon] \ge \mathsf{a}_9 \quad \text{almost surely.} \tag{7.22}$$

We next explain the roles of the different conditions appearing in Definitions 7.4 and 7.5. Condition (i) occurs with high probability due to Definition 1.7, Axiom (IV) (tightness across scales). This condition can be used to lower-bound the amount of time a D_Υ-geodesic between points outside of $B_{5r}(0)$ has to spend in $A_{r,5r}(0)$ if it enters $B_r(0)$. Condition (ii) occurs with high probability due to Lemma 2.26. Condition (ii) together with condition (i) implies that if a D_Υ-geodesic between points outside of $B_{5r}(0)$ enters $B_r(0)$, then the points at which it enters and exits $B_{3r}(0)$ cannot be too close to each other. Condition (iii) occurs with high probability due to a similar reason to the proof of Lemma 2.26. Condition (v) occurs with high probability due to Lemma 2.32. Conditions (iii) and (v) will be used to show that a D_Υ-geodesic cannot spend a long time in $U_r \setminus V_r$ (cf. Lemma 8.12). Condition (iv) is in some sense the most important condition in the definition of E_r. Due to the definition of $\mathsf{G}_{z,\rho r}$ from Section 7.2, this condition provides a large collection of "good" pairs of points $u, v \in A_{2r,4r}(0)$ such that $\widetilde{D}_\Upsilon(u, v) \le M_1 D_\Upsilon(u, v)$. The fact that we consider the event $\mathsf{G}_{z,\rho r}$ in this condition is the reason why we need to require that $r \in \rho^{-1}\mathcal{R}$. We will apply Lemma 2.18 to say that condition (iv) occurs with high probability. Condition (vi) occurs with high probability due the scale invariance of the law of Υ. The purpose of condition (vi) is to ensure that conditions (ix), (x), (xi) occur with uniformly positive conditional probability (i.e., condition (vii) occurs). The purpose of conditions (ix), (x), (xi) is to ensure that the "tubes" U_j for $j \in [1, \#Z_r]_{\mathbb{Z}}$ are "penetrated", and it takes a D_{Υ_r}-geodesic a small amount of time to run though them. The "tubes" U_j for $j \in J_r$ (i.e., the "tubes" U_j for which the corresponding link pattern is given by α_j (cf. condition (ix))) will be the place where the D_{Υ_r}-geodesic enters and exits. Note that in the definition of α_j (cf. (7.19)), the endpoints of $\eta_{j,\mathrm{R}}$ and $\eta_{j+1,\mathrm{R}}$ are not "linked up". This will allow the D_{Υ_r}-geodesic to enter and exit the "tube". While in the "tubes" U_j for $j \in [1, \#Z_r]_{\mathbb{Z}} \setminus J_r$ (i.e., the "tubes" U_j for which the corresponding link pattern is given by β_j), the the complementary $(\mathbb{C} \setminus \overline{V_j}, \mathbb{C} \setminus U_j)$-excursions of Γ_r that "link up" the endpoints of $\eta_{j,\mathrm{L}}$ and $\eta_{j+1,\mathrm{L}}$ and the endpoints of $\eta_{j,\mathrm{R}}$ and $\eta_{j+1,\mathrm{R}}$, resp. will be the "wall" that prevents the D_{Υ_r}-geodesic from exiting the "tube" (see Section 7.6 for more details).

7.5. **Properties of the events.** We begin by verifying that E_r satisfies the required measurability property.

Lemma 7.6. *For each $r \in \rho^{-1}\mathcal{R}$, the event E_r is almost surely determined by $A_{r,5r}(0) \cap \Upsilon$.*

Proof. It follows immediately from the definition that conditions (i), (ii), (iii), (v), (vi), (vii) are almost surely determined by $A_{r,5r}(0) \cap \Upsilon$. It follows from Lemma 7.3 that condition (iv) is almost surely determined by $A_{r,5r}(0) \cap \Upsilon$. □

The remainder of the present subsection is devoted to proving the following.

Lemma 7.7. *One may choose the parameters of (7.12) in such a way that*

$$\mathbf{P}[\mathsf{E}_r] \ge \mathbb{p}, \quad \forall r \in \rho^{-1}\mathcal{R}.$$

To establish Lemma 7.7, we will address the conditions of Definition 7.4 sequentially.

Lemma 7.8. *One may choose the parameters* a_1, a_2, a_3, *and* a_4 *of* (7.12) *in such a way that for each* $r > 0$, *the probability of each of conditions* (i), (ii), *and* (iii) *of Definition 7.4 is at least* $1 - (1 - \mathbb{p})/100$.

Proof. By Definition 1.7, Axiom (IV) (tightness across scales), if $\mathsf{a}_1 \in (0,1)$ is chosen to be sufficiently small, then for each $r > 0$, condition (i) holds with probability at least $1 - (1 - \mathbb{p})/100$. Choose $\mathsf{a}_2 \in (0, \mathsf{a}_1)$ to be sufficiently small ($\mathsf{a}_2 \stackrel{\text{def}}{=} \lambda \mathsf{A}^{-1} \mathsf{a}_1^2$ would be sufficient for later use). By Lemma 2.26, if $\mathsf{a}_3 \in (0, \mathsf{a}_2)$ is chosen to be sufficiently small, then for each $r > 0$, condition (ii) holds with probability at least $1 - (1 - \mathbb{p})/100$. Next, we consider condition (iii). By Lemma 2.5, (i) and a union bound, if $\mathsf{a}_4 \in (0, \mathsf{a}_3)$ is chosen to be sufficiently small, then for each $r > 0$, it holds with probability at least $1 - (1 - \mathbb{p})/100$ for each $z \in (\mathsf{a}_4 r\mathbb{Z})^2 \cap B_{4r}(0)$, there are at most two connected arcs of Γ in $A_{200\mathsf{a}_4 r, \mathsf{a}_3 r/4}(z)$ connecting its inner and outer boundaries. Suppose that this event occurs. Let $x, y \in A_{2r,4r}(0) \cap \Upsilon$ with $|x - y| \le 100\mathsf{a}_4 r$. Choose $z \in (\mathsf{a}_4 r\mathbb{Z})^2 \cap B_{4r}(0)$ such that $x, y \in B_{200\mathsf{a}_4 r}(z)$. Suppose by way of contradiction that x and y do not lie in the same connected component of $B_{\mathsf{a}_3 r/4}(z) \cap \Upsilon$. Then either there is a loop of Γ contained in $B_{\mathsf{a}_3 r/4}(z)$ and disconnecting x and y, or there are at least two $B_{\mathsf{a}_3 r/4}(z)$-excursions of Γ disconnecting x and y in $B_{\mathsf{a}_3 r/4}(z)$. However, if the latter is true, since $x, y \in B_{200\mathsf{a}_4 r}(z)$, there would be at least four connected arcs of Γ in $A_{200\mathsf{a}_4 r, \mathsf{a}_3 r/4}(z)$ connecting its inner and outer boundaries, a contradiction. Thus, we conclude that the former is true. This completes the proof. □

Henceforth fix a_1, a_2, a_3, and a_4 as in Lemma 7.8. We next consider condition (iv).

Lemma 7.9. *One may choose the parameters* a_5 *and* ρ *of* (7.12) *in such a way that for each* $r \in \rho^{-1}\mathcal{R}$, *the probability of condition* (iv) *of Definition 7.4 is at least* $1 - (1 - \mathbb{p})/100$.

Proof. For each $r \in \rho^{-1}\mathcal{R}$, we may choose a deterministic collection $\mathcal{J}_r$ of connected arcs of $\partial B_{3r}(0)$ satisfying the following conditions:

(i) $\#\mathcal{J}_r$ is a constant depending only on a_4.
(ii) Each arc $J \in \mathcal{J}_r$ has Euclidean length $\mathsf{a}_4 r/2$.
(iii) Each connected arc $I \subset \partial B_{3r}(0)$ of Euclidean length at least $\mathsf{a}_4 r$ contains come $J \in \mathcal{J}_r$.

Then, by Lemma 2.18, (7.11), and a union bound, if $\mathsf{a}_5 \in (0, \mathsf{a}_4)$ and $\rho \in (0, \mathsf{a}_5)$ are chosen to be sufficiently small, then for each $r \in \rho^{-1}\mathcal{R}$, it holds with probability at least $1 - (1 - \mathbb{p})/100$ that $J \cap Z_r \neq \emptyset$ for all $J \in \mathcal{J}_r$, in which case condition (iv) holds. □

Henceforth fix a_5 and ρ as in Lemma 7.9. We next deal with conditions (v) and (vi).

Lemma 7.10. *One may choose the parameters* a_7 *and* a_8 *of* (7.12) *in such a way that for each* $r \in \rho^{-1}\mathcal{R}$, *the probability of each of conditions* (v) *and* (vi) *of Definition 7.4 is at least* $1-(1-\mathbb{p})/100$.

Proof. Let ζ be as in Lemma 2.32. Then, by Lemma 2.32, for each $r > 0$, it holds with probability tending to one as $\mathsf{a}_7 \to 0$, at a rate which is uniform in r, that there exists a subset $Z \subset \left(\frac{1}{100}\mathsf{a}_7^{1/2}\alpha\rho r\mathbb{Z}\right)^2 \cap A_{2r,4r}(0)$ with $\#Z \le \mathsf{a}_7^{-\zeta/2}$ such that for each connected component $\Upsilon_\bullet \subset \Upsilon$ with $(\mathbb{C} \setminus A_{2r,4r}(0)) \cap \Upsilon_\bullet \neq \emptyset$ and each connected component $\mathscr{C} \subset (U_r \setminus V_r) \cap \Upsilon_\bullet$, there exists $z \in Z$ such that $\mathscr{C} \subset B_{\mathsf{a}_7^{1/2}\alpha\rho r}(z)$. Fix $\xi \in (\zeta, 1)$. By Lemmas 2.16 and 2.25 and a union bound, we may choose a sufficiently large $A > 0$ such that for each $r > 0$, it holds with probability tending to one as $\mathsf{a}_7 \to 0$, at a rate which is uniform in r, that for each $z \in \left(\frac{1}{100}\mathsf{a}_7^{1/2}\alpha\rho r\mathbb{Z}\right)^2 \cap B_{4r}(0)$, there exists

$t_z \in [\mathsf{a}_7^{1/2}\alpha\rho r, \mathsf{a}_7^{\xi/2}\alpha\rho r] \cap \{2^k\}_{k\in\mathbb{Z}}$ such that there are at most $2A$ connected arcs of Γ in $A_{t_z,2t_z}(z)$ connecting its inner and outer boundaries, and

$$D_\Upsilon(x,y;A_{r,5r}(0)\cap\Upsilon) \le A\mathfrak{c}_{t_z}, \quad \forall x,y\in B_{2t_z}(z) \text{ with } x \xleftrightarrow{B_{2t_z}(z)} y.$$

Assume that the above two events occur. Write $\mathcal{S} \overset{\text{def}}{=} \{(z,t_z) : z\in Z\}$. Then, by (1.5), we have

$$A\sum_{(z,t)\in\mathcal{S}} \sup\left\{D_\Upsilon(x,y;A_{r,5r}(0)\cap\Upsilon) : x,y\in B_{2t}(z),\ x \xleftrightarrow{B_{2t}(z)} y\right\} \le A\mathsf{a}_7^{-\zeta/2}A\mathfrak{K}\mathsf{a}_7^{\xi/2}\alpha\rho\mathfrak{c}_r$$

Since $\zeta<\xi$, we conclude that, if $\mathsf{a}_7\in(0,\rho)$ is chosen to be sufficiently small, then for each $r>0$, condition (v) holds with probability at least $1-(1-\mathbb{p})/100$. By the scale invariance of the law of Γ, together with the fact that the collection of all possibilities for $r^{-1}(U_r,V_r)$ is a finite set not depending on r, if $\mathsf{a}_8\in(0,\mathsf{a}_7)$ is chosen to be sufficiently small, then condition (vi) holds with probability at least $1-(1-\mathbb{p})/100$. This completes the proof. □

Henceforth fix a_7 and a_8 as in Lemma 7.9. It remains to deal with condition (vii). Let us first recall the well-known FKG inequality for Poisson point processes.

Lemma 7.11. *Let X be a locally compact Hausdorff space. Write $\mathcal{M}(X)$ for the Prokhorov space of Radon measures on X. Let $F_1,F_2\colon\mathcal{M}(X)\to\mathbb{R}$ be increasing measurable mappings, i.e., for any $\mu,\nu\in\mathcal{M}(X)$ such that $\mu(A)\le\nu(A)$ for all Borel subsets $A\subset X$, we have $F_1(\mu)\le F_1(\nu)$ and $F_2(\mu)\le F_2(\nu)$. Let Ξ be a Poisson point process on X. Then*

$$\mathbf{E}[F_1(\Xi)F_2(\Xi)] \ge \mathbf{E}[F_1(\Xi)]\mathbf{E}[F_2(\Xi)].$$

Proof. See [KS96]. □

Lemma 7.12. *One may choose the parameter a_9 of (7.12) in such a way that for each $r\in\rho^{-1}\mathcal{R}$, the probability of condition (vii) of Definition 7.4 is at least $1-(1-\mathbb{p})/10$.*

Proof. Since the number of possibilities for (U_r,V_r) is at most a deterministic constant not depending on r, it follows from Lemmas 2.11 and 2.13 that there exists $\mathsf{a}\in(0,1)$ such that conditions (viii), (ix), and (x) hold with conditional probability at least a given Υ and condition (vi). Given conditions (vi), (viii), (ix), and (x), write $U_j^\diamond$ for the connected component containing P_j of the open subset obtained by removing from $U_j^\star$ the complementary $(\mathbb{C}\setminus\overline{V_j},\mathbb{C}\setminus U_j)$-excursions of Γ_r. By the Markov property of CLE, the loops of Γ_r contained in $U_j^\diamond$ is conditionally a nested CLE_κ in $U_j^\diamond$. Write $\Gamma_j\subset\Gamma_r$ for the non-nested CLE_κ in $U_j^\diamond$ induced by Γ_r and Υ_j for the carpet of Γ_j. We observe that Υ_j is always contained in Υ_r (Υ_j is a proper subset of a connected component of Υ_r). Given $U_j^\diamond$, let Ξ_j be a Brownian loop-soup of intensity $c(\kappa)$ in $U_j^\diamond$. Suppose that Γ_j and Ξ_j are coupled so that Γ_j is given by the outer boundaries of the outermost clusters of the Brownian loops of Ξ_j (cf. [SW12]).

Fix $\delta>0$ to be chosen in the next paragraph. We observe that there is almost surely a smooth path L_j of finite Euclidean length and $\varepsilon_*>0$ such that $B_{\varepsilon_* r}(L_j)\subset U_j^\diamond$ and $B_{\delta r}(x)\cap L_j\ne\emptyset$ for all $x\in\partial U_j\cap\Upsilon_j$. Thus, we may choose $A=A(\delta)>0$ to be sufficiently large and $\varepsilon_*=\varepsilon_*(\delta)>0$ to be sufficiently small so that it holds with probability at least $1-(1-\mathbb{p})/99$ that condition (vi) occurs, and given Υ and condition (vi), it holds with conditional probability at least $\mathsf{a}/2$ that conditions (viii), (ix), and (x) occur and

(7.23) there exists a smooth path L_j of Euclidean length at most Ar such that $B_{\varepsilon_* r}(L_j)\subset U_j^\diamond$ and $B_{\delta r}(x)\cap L_j\ne\emptyset$ for all $x\in\partial U_j\cap\Upsilon_j$.

By Lemma 2.26, for each $p\in(0,1)$, there exists a sufficiently small $\delta=\delta(p)>0$ such that it holds with probability at least $1-(1-\mathbb{p})/100$ that it holds with conditional probability at least p given

Υ that

$$D_{\Upsilon_r}(x,y;A_{r,5r}(0)\cap\Upsilon_r)\le\rho\mathfrak{c}_r/100,\quad \forall x,y\in A_{2r,4r}(0)\cap\Upsilon_r \text{ with } x\xleftrightarrow{A_{2r,4r}(0)\cap\Upsilon_r} y \text{ and } |x-y|\le\delta r. \tag{7.24}$$

We observe that, if (7.24) holds, then

$$D_{\Upsilon_r}(x,y;A_{r,5r}(0)\cap\Upsilon_r)\le\rho\mathfrak{c}_r/100,\quad \forall x\in\partial U_j\cap\Upsilon_j,\ \forall y\in\Upsilon_j \text{ with } |x-y|\le\delta r. \tag{7.25}$$

Thus, by choosing p to be sufficiently close to one, we conclude that there exists $\mathsf{a}'\in(0,1)$ and $\delta>0$ such that given Υ, the complementary $(\mathbb{C}\setminus\overline{V_j},\mathbb{C}\setminus U_j)$-excursions of Γ_r, and conditions (vi), (viii), (ix), (x), and (7.23), it holds with conditional probability at least a' that condition (7.25) occurs. (Here, we note that, on the event that condition (viii) holds, conditions (vi), (ix), (x), and (7.23) are almost surely determined by Υ and the complementary $(\mathbb{C}\setminus\overline{V_j},\mathbb{C}\setminus U_j)$-excursions of Γ_r.) Moreover, we observe that, by Definition 1.7, Axiom (V) (monotonicity), condition (7.25) is decreasing with respect to Ξ_j.

Next, we observe that for each $\varepsilon>0$, there exists $\mathsf{a}''(\varepsilon)\in(0,1)$ such that, given $U_j^\diamond$, for each z with $B_{\varepsilon r}(z)\subset U_j^\diamond$, it holds with conditional probability at least $\mathsf{a}''(\varepsilon)$ that $\Upsilon_j\cap B_{\varepsilon r/2}(z)\ne\emptyset$ and there is no loop of Γ_j that crosses between the inner and outer boundaries of $A_{\varepsilon r/2,\varepsilon r}(z)$. Then, by choosing $A'=A'(\varepsilon)>0$ to be sufficiently large, we may arrange so that, given $U_j^\diamond$, for each z with $B_{\varepsilon r}(z)\subset U_j^\diamond$, it holds with conditional probability at least $\mathsf{a}''(\varepsilon)$ that

(7.26) $\Upsilon_j\cap B_{\varepsilon r/2}(z)\ne\emptyset$, there is no loop of Γ_j that crosses between the inner and outer boundaries of $A_{\varepsilon r/2,\varepsilon r}(z)$, and there is a loop in $A_{\varepsilon/2,\varepsilon}(z)\cap\Upsilon_j$ of D_{Υ_r}-length at most $A'\mathfrak{c}_{\varepsilon r}$ disconnecting the inner and outer boundaries of $A_{\varepsilon r/2,\varepsilon r}(z)$.

Moreover, we observe that, by Definition 1.7, Axiom (V) (monotonicity), condition (7.26) is decreasing with respect to Ξ_j.

One verifies immediately that for each smooth path $L\subset\mathbb{C}$ of Euclidean length ℓ, and each $\varepsilon>0$, we have

$$\#\left\{z\in\left(\frac{1}{100}\varepsilon\mathbb{Z}\right)^2:\operatorname{dist}(z,L)\le\varepsilon/2\right\}\le 10^{100}\ell/\varepsilon.$$

Thus, by applying Lemma 7.11 to Ξ_j and the preceding discussion, it follows that for each $\varepsilon\in(0,\varepsilon_*]$, given Υ, the complementary $(\mathbb{C}\setminus\overline{V_j},\mathbb{C}\setminus U_j)$-excursions of Γ_r, and conditions (vi), (viii), (ix), (x), and (7.23), it holds with conditional probability at least

$$\mathsf{a}'\mathsf{a}''(\varepsilon)^{10^{100}A/\varepsilon}$$

that condition (7.25) occurs, and condition (7.26) occurs for all $z\in\left(\frac{1}{100}\varepsilon r\mathbb{Z}\right)^2$ with $\operatorname{dist}(z,L_j)\le\varepsilon r/2$. Henceforth assume that this event occurs. We note that, for each $x_1,x_2\in\partial U_j\cap\Upsilon_j$, it is possible to connect $B_{\delta r}(x_1)$ and $B_{\delta r}(x_2)$ by the union of the loops described in condition (7.26) over all $z\in\left(\frac{1}{100}\varepsilon r\mathbb{Z}\right)^2$ with $\operatorname{dist}(z,L_j)\le\varepsilon r/2$. Thus, we conclude that

$$D_{\Upsilon_r}(x_1,x_2;A_{r,5r}(0)\cap\Upsilon_r)\le\rho\mathfrak{c}_r/100+10^{100}AA'\mathfrak{c}_{\varepsilon r}/\varepsilon+\rho\mathfrak{c}_r/100,\quad\forall x_1,x_2\in\partial U_j\cap\Upsilon_j.$$

Finally, by Lemma 2.28, we may choose ε to be sufficiently small so that $10^{100}AA'\mathfrak{c}_{\varepsilon r}/\varepsilon\le\rho\mathfrak{c}_r/2$. This completes the proof. ☐

Proof of Lemma 7.7. Combine Lemmas 7.8, 7.9, 7.10 and 7.12. ☐

7.6. **Forcing a geodesic to take a shortcut.** In the present subsection, fix $r \in \rho^{-1}\mathcal{R}$ and $J_r \subset [1, \#Z_r]_{\mathbb{Z}}$ with $\#J_r = 2$. To lighten notation, we may assume without loss of generality that $J_r = \{1, j\}$. Suppose that the event $\mathsf{E}_r \cap \mathsf{F}_r(J_r)$ occurs. For $k \in [1, \#Z_r]_{\mathbb{Z}}$, let the points of X_k be labeled as in (7.18). We observe that the concatenation of

- $\eta_{2,\mathrm{R}}$,
- the complementary $(\mathbb{C} \setminus V_2, \mathbb{C} \setminus U_2)$-excursion of Γ_r from $y_{2,\mathrm{R}}$ to $x_{3,\mathrm{R}}$,
- $\eta_{3,\mathrm{R}}$,

$\vdots$

- the complementary $(\mathbb{C} \setminus V_{j-1}, \mathbb{C} \setminus U_{j-1})$-excursion of Γ_r from $y_{j-1,\mathrm{R}}$ to $x_{j,\mathrm{R}}$, and
- $\eta_{j,\mathrm{R}}$

is a connected arc of a loop of Γ_r. We shall write γ_r for this connected arc. Similarly, the concatenation of

- $\eta_{j+1,\mathrm{R}}$,
- the complementary $(\mathbb{C} \setminus V_{j+1}, \mathbb{C} \setminus U_{j+1})$-excursion of Γ_r from $y_{j+1,\mathrm{R}}$ to $x_{j+2,\mathrm{R}}$,
- $\eta_{j+2,\mathrm{R}}$,

$\vdots$

- the complementary $(\mathbb{C} \setminus V_{\#Z_r}, \mathbb{C} \setminus U_{\#Z_r})$-excursion of Γ_r from $y_{\#Z_r,\mathrm{R}}$ to $x_{1,\mathrm{R}}$, and
- $\eta_{1,\mathrm{R}}$

is a connected arc of a loop of Γ_r. We shall write γ'_r for this connected arc. Moreover, we observe that the concatenation of

- $\eta_{1,\mathrm{L}}$,
- the complementary $(\mathbb{C} \setminus V_1, \mathbb{C} \setminus U_1)$-excursion of Γ_r from $y_{1,\mathrm{L}}$ to $x_{2,\mathrm{L}}$,

$\vdots$

- $\eta_{\#Z_r,\mathrm{L}}$, and
- the complementary $(\mathbb{C} \setminus V_{\#Z_r}, \mathbb{C} \setminus U_{\#Z_r})$-excursion of Γ_r from $y_{\#Z_r,\mathrm{L}}$ to $x_{1,\mathrm{L}}$

is a loop of Γ_r. We shall write $\mathscr{L}_r$ for this loop. See Figure 10 for an illustration.

Lemma 7.13. *There exists a deterministic constant* $\mathsf{b} > 0$ *such that the following is true: Suppose that the event* $\mathsf{E}_r \cup \mathsf{F}_r(J_r)$ *occurs. Let* $P_r \colon [0,1] \to \Upsilon_r$ *be a* D_{Υ_r}*-geodesic between two points outside of* $B_{4r}(0)$*. Suppose that* P_r *passes through either the tube between* $\mathscr{L}_r$ *and* γ_r *or the tube between* $\mathscr{L}_r$ *and* γ'_r*. (By passing through either the tube between* $\mathscr{L}_r$ *and* γ_r *or the tube between* $\mathscr{L}_r$ *and* γ'_r*, we mean that* P_r *(regarded as a path in the prime end closure of the unbounded connected component of* $\mathbb{C} \setminus (\mathscr{L}_r \cup \gamma_r \cup \gamma'_r)$*) is not homotopic (relative to its endpoints) to a path outside of* $B_{4r}(0)$*.) Then there are times* $0 < \sigma < \tau < 1$ *such that*

$$P_r([\sigma,\tau]) \subset B_{4r}(0) \cap \Upsilon_r, \quad D_{\Upsilon_r}(P_r(\sigma), P_r(\tau)) \geq \mathsf{b}\mathfrak{c}_r, \quad \textit{and}$$
$$\widetilde{D}_{\Upsilon_r}(P_r(\sigma), P_r(\tau)) \leq M_2 D_{\Upsilon_r}(P_r(\sigma), P_r(\tau)).$$

Proof. We may assume without loss of generality that P_r passes through the tube between $\mathscr{L}_r$ and γ_r. In particular, P_r passes through the tube between $\eta_{2,\mathrm{L}}$ and $\eta_{2,\mathrm{R}}$. Let u, v, $\widetilde{P}$, s, t, P_s, P_t be as in the definition of $\mathsf{G}_{z_2,\rho r}$ (cf. Definition 7.2). Then we must have $P_r \cap P_s \neq \emptyset$ and $P_r \cap P_t \neq \emptyset$. Choose times $0 < \sigma < \tau < 1$ such that either $P_r(\sigma) \in P_s$ and $P_r(\tau) \in P_t$ or $P_r(\sigma) \in P_t$ and $P_r(\tau) \in P_s$. We may assume without loss of generality that the former is true. By Definition 7.2, (ii), (v) and the triangle inequality, we have

$$\begin{aligned}\widetilde{D}_{\Upsilon_r}(P_r(\sigma),P_r(\tau)) &\le \widetilde{D}_{\Upsilon_r}(u,v)+\widetilde{D}_{\Upsilon_r}(u,\widetilde{P}(s))+\widetilde{D}_{\Upsilon_r}(\widetilde{P}(t),v)+\mathrm{len}(P_s;\widetilde{D}_{\Upsilon_r})+\mathrm{len}(P_t;\widetilde{D}_{\Upsilon_r})\\ &= \widetilde{D}_{\Upsilon}(u,v)+\widetilde{D}_{\Upsilon}(u,\widetilde{P}(s))+\widetilde{D}_{\Upsilon}(\widetilde{P}(t),v)+\mathrm{len}(P_s;\widetilde{D}_{\Upsilon})+\mathrm{len}(P_t;\widetilde{D}_{\Upsilon})\\ &\le (1+4\lambda)\widetilde{D}_{\Upsilon}(u,v). \end{aligned}\tag{7.27}$$

Again, by Definition 7.2, (ii), (v) and the triangle inequality, we have

$$\begin{aligned}D_{\Upsilon_r}(P_r(\sigma),P_r(\tau)) &\ge D_{\Upsilon_r}(u,v)-D_{\Upsilon_r}(u,\widetilde{P}(s))-D_{\Upsilon_r}(\widetilde{P}(t),v)-\mathrm{len}(P_s;D_{\Upsilon_r})-\mathrm{len}(P_t;D_{\Upsilon_r})\\ &= D_{\Upsilon}(u,v)-D_{\Upsilon}(u,\widetilde{P}(s))-D_{\Upsilon}(\widetilde{P}(t),v)-\mathrm{len}(P_s;D_{\Upsilon})-\mathrm{len}(P_t;D_{\Upsilon})\\ &\ge \frac{1}{M_1}\widetilde{D}_{\Upsilon}(u,v)-\frac{1}{M_*}\Big(\widetilde{D}_{\Upsilon}(u,\widetilde{P}(s))+\widetilde{D}_{\Upsilon}(\widetilde{P}(t),v)+\mathrm{len}(P_s;\widetilde{D}_{\Upsilon})+\mathrm{len}(P_t;\widetilde{D}_{\Upsilon})\Big)\\ &\ge \left(\frac{1}{M_1}-\frac{4\lambda}{M_*}\right)\widetilde{D}_{\Upsilon}(u,v). \end{aligned}\tag{7.28}$$

Since λ may be chosen to be sufficiently small so that $M_2(1/M_1-4\lambda/M_*)\ge(1+4\lambda)$, we conclude from (7.27) and (7.28) that

$$\widetilde{D}_{\Upsilon_r}(P_r(\sigma),P_r(\tau))\le M_2 D_{\Upsilon_r}(P_r(\sigma),P_r(\tau)).$$

Finally, by (1.5), Definition 7.2, (ii), and (7.28), we have

$$D_{\Upsilon_r}(P_r(\sigma),P_r(\tau))\ge\left(\frac{1}{M_1}-\frac{4\lambda}{M_*}\right)\widetilde{D}_{\Upsilon}(u,v)\ge\left(\frac{1}{M_1}-\frac{4\lambda}{M_*}\right)\alpha^3\mathfrak{c}_{\rho r}\ge\left(\frac{1}{M_1}-\frac{4\lambda}{M_*}\right)\alpha^3\rho^2\mathfrak{c}_r.$$

This completes the proof. □

8. Uniqueness

In the present section, we use the objects of Section 7 to prove Theorem 1.9. Fix two weak geodesic CLE_κ carpet metrics D and $\widetilde{D}$ with the same scaling constants $\{\mathfrak{c}_r\}_{r>0}$. Let M_* and M^* be as in (6.1) and (6.2), respectively. Suppose by way of contradiction that $M_*<M^*$. Recall that $M_1=(2M_*+M^*)/3$, $M_2=(M_*+2M^*)/3$, and $\mathbb{p}\in(0,1)$, $\mathsf{A}>0$, $0<\mu<\nu$, $\Lambda\in\mathbb{N}$ are fixed parameters to be chosen later (cf. Lemma 8.17), in a manner depending only on D and $\widetilde{D}$.

Let $z\in\mathbb{C}$ and $r\in\rho^{-1}\mathcal{R}$. Recall from Section 7.2 that $\mathsf{G}_{z,\rho r}$ is the event that roughly speaking says that $B_{\rho r}(z)$ contains a "shortcut", and that $\mathsf{G}_{z,\rho r}$ holds with uniformly positive probability. Recall from (7.13) and (7.14) that $\mathsf{Z}_r\subset\partial B_{3r}(0)$ is the deterministic finite subset of "test points" and that $Z_r\subset\mathsf{Z}_r$ is the random subset consisting of points z for which the event $\mathsf{G}_{z,\rho r}$ occurs. Recall from (7.15) and (7.16) that U_r and V_r are long narrow "tubes" that "link up" the half-annuli $\mathsf{H}_{z,\rho r}\subset A_{(1-a)r,(1+a)r}(z)$ for $z\in Z_r$. The random open subsets U_r and V_r are almost surely determined by Z_r. Recall from (7.17) that $U_r^\star$ is the open subset obtained from the *partial exploration* of Γ in $\mathbb{C}\setminus U_r$ until hitting V_r. Recall from Section 7.4 that E_r is the event which roughly speaking says that Z_r is large and that Υ_r is a resampling of Υ. Recall from Sections 7.4 and 7.6 that F_r is the event which roughly speaking says that after resampling, the "shortcuts" of $\mathsf{G}_{z,\rho r}$ for $z\in Z_r$ are preserved, and there are (connected arcs of) loops $\mathscr{L}_r$, γ_r, γ_r' of Γ_r that form a "tube" that forces the D_{Υ_r}-geodesics to take the "shortcuts". Recall from Section 7.5 that E_r holds with high probability, and that F_r almost surely holds with uniformly positive conditional probability given E_r and Υ.

We shall write

$$Z_{z,r}\overset{\mathrm{def}}{=}Z_r(\Upsilon-z)+z;\quad U_{z,r}\overset{\mathrm{def}}{=}U_r(\Upsilon-z)+z;\quad V_{z,r}\overset{\mathrm{def}}{=}V_r(\Upsilon-z)+z;\quad U_{z,r}^\star\overset{\mathrm{def}}{=}U_r^\star(\Upsilon-z)+z,$$

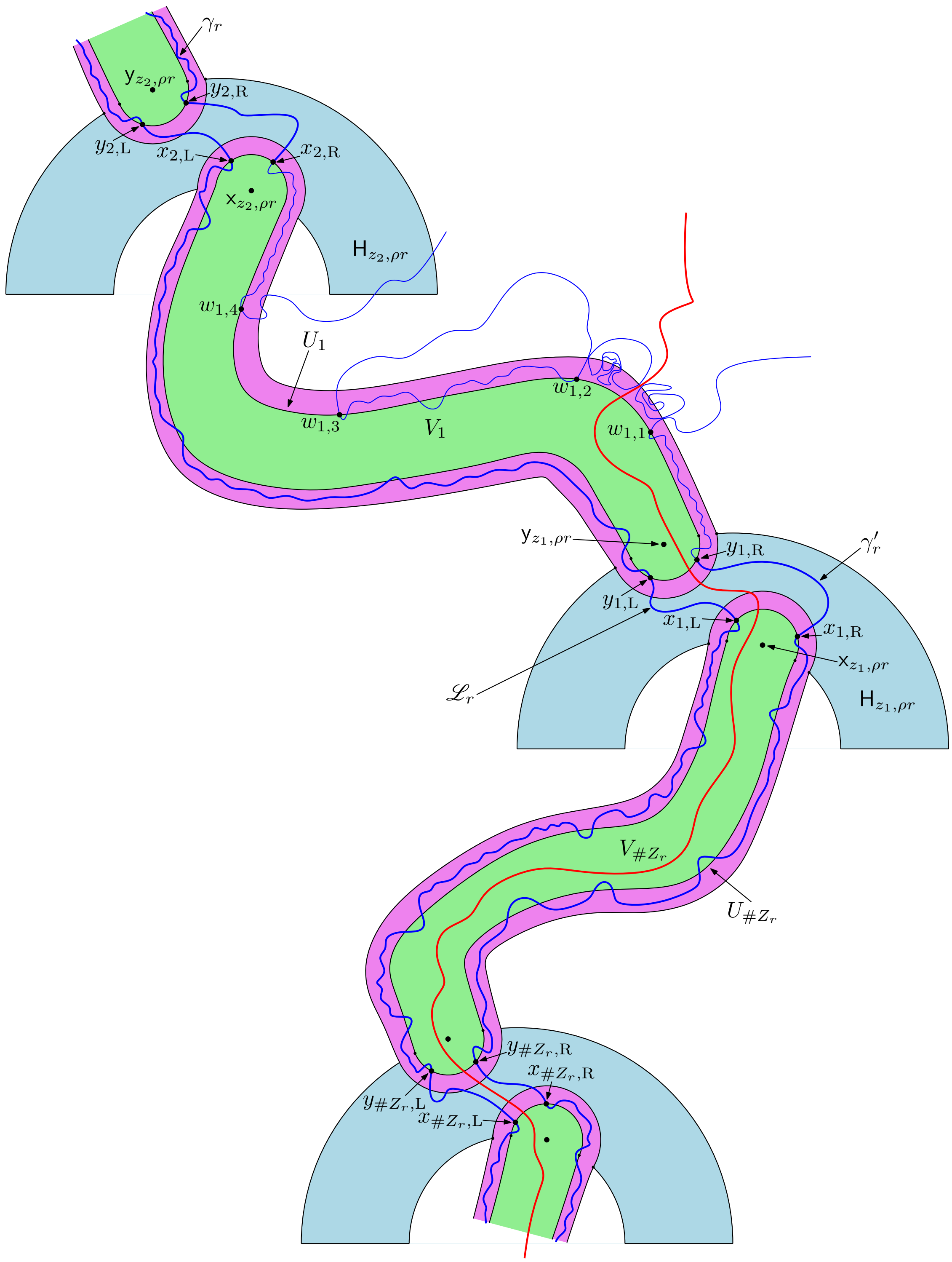

FIGURE 10. Illustration of $\mathscr{L}_r$, γ_r, and γ_r'. The blue paths are (connected arcs of) loops of Γ_r. The red path is a D_{Υ_r}-geodesic that passes through the tube between $\mathscr{L}_r$ and γ_r'.

where $Z_r(\Upsilon - z)$ (resp. $U_r(\Upsilon - z)$; $V_r(\Upsilon - z)$; $U_r^\star(\Upsilon - z)$) is defined in the same manner as Z_r (resp. U_r; V_r; $U_r^\star$) but with $\Upsilon - z$ in place of Υ. Let $J_{z,r} \subset [1, \#Z_{z,r}]_{\mathbb{Z}}$ be a subset with $\#J_{z,r} = 2$

(chosen in a manner that is almost surely determined by Υ; these will be the indices of the two places where the geodesic will enter and leave the tube). Similarly, let $U'_{z,r}$, $V'_{z,r}$, and $\Upsilon_{z,r}$ be defined in the same manner as U'_r, V'_r, and Υ_r, i.e., $(U'_{z,r}, V'_{z,r})$ is sampled uniformly from all the possibilities for $(U_{z,r}, V_{z,r})$, independently of everything else, and $\Upsilon_{z,r}$ is a conditionally independent copy of Υ given $\Gamma(\mathbb{C} \setminus \overline{V'_r}, \mathbb{C} \setminus U'_r)$ and $\wp$. Again, we will only be interested in the case where $(U_{z,r}, V_{z,r}) = (U'_{z,r}, V'_{z,r})$. Moreover, we shall assume that $\Upsilon_{z,r}$ for different z's are conditionally independent given Υ (we will only be considering countably many z's simultaneously). We shall write $\mathsf{E}_{z,r}$ (resp. $\mathsf{F}_{z,r}(J_{z,r})$) for the event defined in the same manner as E_r (resp. $\mathsf{F}_r(J_r)$) but with $\Upsilon - z$ (resp. $\Upsilon_{z,r} - z$) in place of Υ (resp. Υ_r). On the event $\mathsf{E}_{z,r} \cap \mathsf{F}_{z,r}(J_{z,r})$, we shall write

$$\mathscr{L}_{z,r} \stackrel{\text{def}}{=} \mathscr{L}_r(\Upsilon - z) + z; \quad \gamma_{z,r} \stackrel{\text{def}}{=} \gamma_r(\Upsilon - z) + z; \quad \gamma'_{z,r} \stackrel{\text{def}}{=} \gamma'_r(\Upsilon - z) + z,$$

where $\mathscr{L}_r(\Upsilon - z)$ (resp. $\gamma_r(\Upsilon - z)$; $\gamma'_r(\Upsilon - z)$) is defined in the same manner as $\mathscr{L}_r$ (resp. γ_r; γ'_r) of Section 7.6 but with $\Upsilon - z$ and $\Upsilon_{z,r} - z$ in place of Υ and Υ_r, respectively.

We observe that whenever Z is a collection of points such that the Euclidean distance between distinct points are large enough, then we may construct a resampling Υ_Z of Υ out of $\{\Upsilon_{z,r}\}_{z \in Z}$. More precisely: Let $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$. Then we shall write

$$\mathbb{Z}_{\mathbf{e},r} \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : z - \frac{1}{100}\mathbf{e}r \in (3\Lambda r \mathbb{Z})^2 \right\}.$$

Thus, $\mathbb{Z}_{\mathbf{e},r}$ for $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$ form a partition of $(\mathbb{Z}/(r/100)\mathbb{Z})^2$ such that each pair of distinct points of $\mathbb{Z}_{\mathbf{e},r}$ has Euclidean distance at least $3\Lambda r$. For each deterministic subset $Z \subset \mathbb{Z}_{\mathbf{e},r}$, let Υ_Z be such that

$$\left(\mathbb{C} \setminus \bigcup_{z \in Z} B_{5r}(z) \right) \cap \Upsilon_Z = \left(\mathbb{C} \setminus \bigcup_{z \in Z} B_{5r}(z) \right) \cap \Upsilon;$$
$$B_{5r}(z) \cap \Upsilon_Z = B_{5r}(z) \cap \Upsilon_{z,r} \quad \forall z \in Z.$$

(Here, we recall that $(\mathbb{C} \setminus B_{5r}(z)) \cap \Upsilon_{z,r} = (\mathbb{C} \setminus B_{5r}(z)) \cap \Upsilon$.) Note that (Υ, Υ_Z) and (Υ_Z, Υ) have the same law. (This follows from the fact that $(U_{z,r}, V_{z,r})$'s are independent and independent of everything else.)

The idea of the proof of Theorem 1.9 is to show that

(8.1) the probability of the event $\overline{G}_{\mathbb{r}}(M^* - \delta, b)$ of Section 6 goes to 0 as δ goes to 0, at a rate which is uniform in $\mathbb{r}$

(cf. Lemma 8.19). This, together with Proposition 6.8, will lead to a contradiction of Proposition 6.6.

We will call a ball $B_{5r}(z)$ *good* if the event $\mathsf{E}_{z,r} \cap \mathsf{F}_{z,r}(J_{z,r})$ occurs, and *very good* if the event $(\mathsf{E}_{z,r} \cap \mathsf{F}_{z,r}(J_{z,r}))^\vee$ occurs, which is defined in the same manner as $\mathsf{E}_{z,r} \cap \mathsf{F}_{z,r}(J_{z,r})$ but with the roles of Υ and $\Upsilon_{z,r}$ interchanged, occurs.

To prove (8.1), it suffices to show that with high probability, the D_Υ-geodesic P connecting two given compact subsets K_1 and K_2 has to hit a lot of "very good" balls. Lemma 7.13 will then give us a lot of pairs of times $\sigma < \tau$ such that $\widetilde{D}_\Upsilon(P(\sigma), P(\tau)) \le M_2 D_\Upsilon(P(\sigma), P(\tau))$, which in turn will show that $\widetilde{D}_\Upsilon(P(0), P(1))$ is bounded away of $M^* D_\Upsilon(P(0), P(1))$. We will show this using an argument based on counting the number of occurrences of certain events (cf. Definitions 8.3 and 8.4). More precisely, for $r \in \rho^{-1}\mathcal{R}$ and a finite collection of points Z, we will consider the event $\mathfrak{F}_Z$ which roughly speaking says that the following are true (cf. Definition 8.3):

- Each ball $B_{5r}(z)$ for $z \in Z$ is "good".
- The D_Υ-geodesic P from K_1 to K_2 enters $B_r(z)$ for each $z \in Z$.

- The D_{Υ_Z}-geodesic P_Z from K_1 to K_2 passes through the tube between $\mathscr{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$ (as described in Section 7.6) for each $z \in Z$.

We will also consider the event $\mathfrak{F}_Z^\vee$ which is defined in the same manner as $\mathfrak{F}_Z$ but with the roles of Υ and Υ_Z interchanged (cf. Definition 8.4):

- Each ball $B_{5r}(z)$ for $z \in Z$ is "very good".
- The D_{Υ_Z}-geodesic P_Z from K_1 to K_2 enters $B_r(z)$ for each $z \in Z$.
- The D_Υ-geodesic P from K_1 to K_2 passes through the tube between $\mathscr{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$ (as described in Section 7.6) for each $z \in Z$.

Since (Υ, Υ_Z) and (Υ_Z, Υ) have the same law, it follows that $\mathbf{P}[\mathfrak{F}_Z] = \mathbf{P}[\mathfrak{F}_Z^\vee]$. This suggests that the number of Z for which $\mathfrak{F}_Z$ holds should be roughly comparable to the number of Z for which $\mathfrak{F}_Z^\vee$ holds.

Furthermore, we will show that for each $N \in \mathbb{N}$ and $r \in [\varepsilon^{1+\nu}, \varepsilon]$, the number of Z with $\#Z \le N$ for which $\mathfrak{F}_Z$ holds increases proportionally to a positive power of ε^{-N} (cf. Proposition 8.5). Indeed, since the event $\mathsf{E}_{z,r}$ occurs with high probability, it follows from independence across scales (cf. Lemma 2.16) that with high probability, the D_Υ-geodesic P can be covered by balls $B_r(z)$ for which the event $\mathsf{E}_{z,r}$ occurs. Combining with the fact that $\mathsf{F}_{z,r}(J_{z,r})$ occurs with uniformly positive conditional probability given $\mathsf{E}_{z,r}$ and Υ (cf. (7.22)), we have found many Z_0's for which the first two conditions of the definition of $\mathfrak{F}_{Z_0}$ described above are satisfied. In order to produce sets Z for which the third condition described above is also satisfied (i.e., the D_{Υ_Z}-geodesic P_Z from K_1 to K_2 passes through the tube between $\mathscr{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$ for each $z \in Z$), we start with a set Z_0 as above and iteratively construct Z_j by removing the "bad" points $z \in Z_{j-1}$ such that the $D_{\Upsilon_{Z_{j-1}}}$-geodesic $P_{Z_{j-1}}$ from K_1 to K_2 does not pass through the tube between $\mathscr{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$. We will show that this iterative procedure has to terminate before Z_j becomes too small. More precisely, we will show that $Z_j = Z_{j+1} = \cdots$ for some j (in which case the event $\mathfrak{F}_{Z_j}$ occurs) such that $\#Z_j$ is at least a constant times $\#Z_0$. See Section 8.2 for details.

By combining the previous two paragraphs with a deterministic observation (cf. Proposition 8.6) and a straightforward calculation (cf. the end of Section 8.1), we conclude that, with high probability, many sets Z satisfy the event $\mathfrak{F}_Z^\vee$. In particular, there are numerous "very good" balls $B_{5r}(z)$ through which P passes via the tube bounded by $\mathscr{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$. As discussed earlier, this establishes (8.1).

8.1. The core argument.

In the present subsection, we state the main estimates that we will prove. Fix $\mathfrak{r} > 0$ and disjoint compact subsets $K_1, K_2 \subset B_{2\mathfrak{r}}(0)$.

Definition 8.1. Let $\varepsilon \in (0,1)$. Then we shall write $\mathfrak{G}_\varepsilon^{\mathfrak{r}}(K_1, K_2)$ for the event that the following are true:

(i) $K_1 \overset{\Upsilon}{\leftrightarrow} K_2$ and $\widetilde{D}_\Upsilon(K_1, K_2) \ge M^* D_\Upsilon(K_1, K_2) - \varepsilon^{2(1+\nu)} \mathfrak{c}_{\mathfrak{r}}$.

(ii) For each $x \in B_{2\mathfrak{r}}(0)$, there exists $r \in \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathfrak{r}, \varepsilon\mathfrak{r}]$ and $z \in \left(\frac{1}{100} r\mathbb{Z}\right)^2 \cap B_{2\mathfrak{r}}(0)$ such that $|x - z| \le r$ and the following are true:
   (a) $\mathsf{E}_{z,r}(\Upsilon)$ occurs.
   (b) There are at most two connected arcs of Γ in $A_{5r,\Lambda r}(z)$ connecting its inner and outer boundaries.
   (c) There exists a path P_z in $A_{5r,\Lambda r}(z) \cap \Upsilon$ of D_Υ-length at most $\mathsf{A}\mathfrak{c}_r$ such that every path in Υ that crosses between the inner and outer boundaries of $A_{5r,\Lambda r}(z)$ intersects P_z.

The key requirement in Definition 8.1 is condition (i). Our goal is to show that if $M_* < M^*$, then condition (i) is highly unlikely to occur. The motivation for this lies in its later use in Section 8.4, where it will help produce a contradiction to Proposition 6.6. Indeed, Proposition 6.6 provides a lower bound on the probability that there exist points $x, y \in \overline{B_{\mathbb{r}}(0)} \cap \Upsilon$ satisfying certain properties such that $\widetilde{D}_\Upsilon(x,y)$ is approximately $M^* D_\Upsilon(x,y)$. We will prove that this lower bound conflicts with our upper bound on the probability of condition (i). Condition (ii) serves as a global regularity assumption; we will establish in Lemma 8.17 that it holds with high probability. Consequently, an upper bound on $\mathbf{P}[\mathfrak{G}_\varepsilon^{\mathbb{r}}(K_1, K_2)]$ will directly yield an upper bound for the probability of condition (i).

Proposition 8.2. *Let $\eta \in (0,1)$. Suppose that $\mathrm{dist}(K_1, K_2) \geq \eta \mathbb{r}$ and $\mathrm{dist}(K_1, \partial B_{2\mathbb{r}}(0)) \geq \eta \mathbb{r}$. Then it holds with superpolynomially high probability as $\varepsilon \to 0$, at a rate depending only on η (but not on $\mathbb{r}$, K_1, K_2), that $\mathfrak{G}_\varepsilon^{\mathbb{r}}(K_1, K_2)$ does not occur.*

It is essential for our purposes that the bound in Proposition 8.2 be uniform in $\mathbb{r}$, K_1, and K_2. Indeed, we will eventually choose K_1 and K_2 to be Euclidean balls whose radii are $\mathbb{r}$ times a power of ε (cf. Lemma 8.18). In proving Proposition 8.2, we do not need to control the size of $\mathcal{R}$; it suffices that, for every $r \in \rho^{-1}\mathcal{R}$, the objects $U_{z,r}$, $V_{z,r}$, $\Upsilon_{z,r}$, $\mathsf{E}_{z,r}$, and $\mathsf{F}_{z,r}$ are defined and satisfy the required properties. In fact, decreasing $\mathcal{R}$ only makes condition (ii) in the definition of $\mathfrak{G}_\varepsilon^{\mathbb{r}}(K_1, K_2)$ less likely to occur, while increasing $\mathcal{R}$ makes condition (i) less likely, as will become clear in the proof of Proposition 8.2. Either way, we will show that the event $\mathfrak{G}_\varepsilon^{\mathbb{r}}(K_1, K_2)$ is extremely unlikely. The size of $\mathcal{R}$ (cf. Lemma 8.16) will be used only later, to verify that condition (ii) holds with high probability; this, in turn, will force condition (i) to be extremely unlikely. The proof of Theorem 1.9 will be given in Section 8.4 using Proposition 8.2 and may be read without first reading the details of the proof of Proposition 8.2.

The proof of Proposition 8.2 will rely on counting how often certain events occur. We now proceed to define these events.

For $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$ and $r \in \rho^{-1}\mathcal{R}$, we shall write

$$\mathbb{Z}_{\mathbf{e},r}^{\mathbb{r}}(K_1, K_2) \stackrel{\text{def}}{=} \left\{ z \in \mathbb{Z}_{\mathbf{e},r} \cap B_{2\mathbb{r}}(0) : B_{\Lambda r}(z) \cap (K_1 \cup K_2) = \emptyset \right\}.$$

Definition 8.3. Let $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R}$, and $Z \subset \mathbb{Z}_{\mathbf{e},r}^{\mathbb{r}}(K_1, K_2)$. Then we shall write $\mathfrak{F}_Z^{\mathbb{r}}(K_1, K_2)$ for the event that the following are true:

(i) $K_1 \stackrel{\Upsilon}{\leftrightarrow} K_2$ and $\widetilde{D}_\Upsilon(K_1, K_2) \geq M^* D_\Upsilon(K_1, K_2) - \mathfrak{c}_r$.
(ii) For each $z \in Z$, the event $\mathsf{E}_{z,r}(\Upsilon)$ occurs.
(iii) For each $z \in Z$, there are at most two connected arcs of Γ in $A_{5r,\Lambda r}(z)$ connecting its inner and outer boundaries.
(iv) For each $z \in Z$, there exists a path P_z in $A_{5r,\Lambda r}(z) \cap \Upsilon$ of D_Υ-length at most $\mathsf{A}\mathfrak{c}_r$ such that every path in Υ that crosses between the inner and outer boundaries of $A_{5r,\Lambda r}(z)$ intersects P_z.
(v) Write $P \colon [0,1] \to \Upsilon$ for the D_Υ-geodesic from K_1 to K_2. For each $z \in Z$, the D_Υ-geodesic P enters $B_r(z)$.

Before proceeding, for each $z \in Z$, we shall write s_z (resp. t_z) for the first (resp. last) time at which P hits $U_{z,r}^\star$. Write $i_z, j_z \in [1, \# Z_{z,r}]_\mathbb{Z}$ such that $P(s_z) \in \partial U_{i_z}^\star$ and $P(t_z) \in \partial U_{j_z}^\star$. (Note that necessarily $P(s_z) \in \partial U_{i_z}$ and $P(t_z) \in \partial U_{j_z}$. We also recall that $U_1, \dots, U_{\# Z_{z,r}}$ are the connected components of $U_{z,r}$, and $U_1^\star, \dots, U_{\# Z_{z,r}}^\star$ are the connected components of $U_{z,r}^\star$.)

(vi) For each $z \in Z$, the event $\mathsf{F}_{z,r}(\Upsilon_Z; \{i_z, j_z\})$ occurs.
(vii) Write $P_Z \colon [0,1] \to \Upsilon_Z$ for the D_{Υ_Z}-geodesic from K_1 to K_2. For each $z \in Z$, the D_{Υ_Z}-geodesic P_Z passes through the tube between either $\mathfrak{L}_{z,r}$ and $\gamma_{z,r}$ or $\mathfrak{L}_{z,r}$ and $\gamma_{z,r}'$.

An illustration of Definition 8.3 is provided in Figure 11. Condition (i) is closely connected to condition (i) from Definition 8.1, while conditions (ii), (iii), and (iv) correspond closely to condition (ii) in the same reference. Also, we note that the event $\mathfrak{G}_\varepsilon^{\mathfrak{r}}(K_1, K_2)$ depends only on Υ, while the event $\mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2)$ depends on both Υ and Υ_Z.

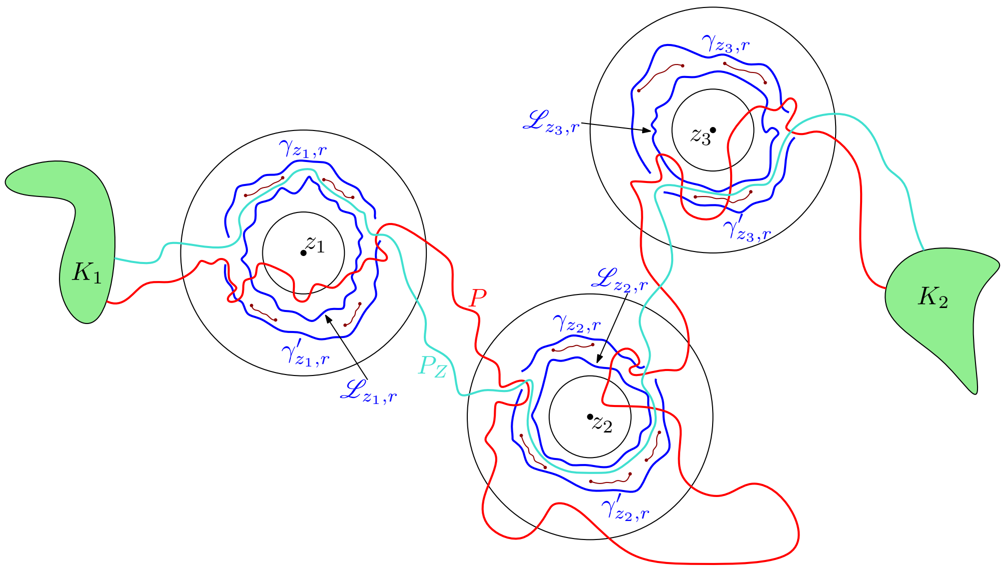

FIGURE 11. Illustration of the event $\mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2)$ of Definition 8.3 for $Z = \{z_1, z_2, z_3\}$. The red path P is a D_Υ-geodesic from K_1 to K_2. The turquoise path P_Z is a D_{Υ_Z}-geodesic from K_1 to K_2. For each $z \in \{z_1, z_2, z_3\}$, the blue paths $\mathscr{L}_{z,r}$, $\gamma_{z,r}$, $\gamma'_{z,r}$ are (connected arcs of) loops of Γ_Z; P_Z passes through the tube between either $\mathscr{L}_{z,r}$ and $\gamma_{z,r}$ or $\mathscr{L}_{z,r}$ and $\gamma'_{z,r}$. The short dark red paths are "shortcuts" (corresponding to $\widetilde{P}$ described in Definition 7.2), in the sense that each of them is a $\widetilde{D}_\Upsilon$-geodesic whose $\widetilde{D}_\Upsilon$-length is at most M_1 times the D_Υ-distance between its endpoints. For each $z \in \{z_1, z_2, z_3\}$, since P_Z passes through the tube between either $\mathscr{L}_{z,r}$ and $\gamma_{z,r}$ or $\mathscr{L}_{z,r}$ and $\gamma'_{z,r}$, it must get close to at least one "shortcut" in that tube.

Definition 8.4. Let $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R}$, and $Z \subset \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2)$. Then we shall write $\mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2)^\vee$ for the event defined in the same manner as $\mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2)$ but with the roles of Υ and Υ_Z interchanged, i.e., the event that the following are true:

(i) $K_1 \overset{\Upsilon_Z}{\longleftrightarrow} K_2$ and $\widetilde{D}_{\Upsilon_Z}(K_1, K_2) \geq M^* D_{\Upsilon_Z}(K_1, K_2) - \mathfrak{c}_r$.
(ii) For each $z \in Z$, the event $\mathsf{E}_{z,r}(\Upsilon_Z)$ occurs.
(iii) For each $z \in Z$, there are at most two connected arcs of Γ_Z in $A_{5r,\Lambda r}(z)$ connecting $\partial B_{5r}(z)$ and $\partial B_{\Lambda r}(z)$.
(iv) For each $z \in Z$, there exists a path P_z in $A_{5r,\Lambda r}(z) \cap \Upsilon_Z$ of D_{Υ_Z}-length at most $\mathsf{A}\mathfrak{c}_r$ such that every path in Υ_Z that crosses between the inner and outer boundaries of $A_{5r,\Lambda r}(z)$ intersects P_z.
(v) Write $P_Z\colon [0,1] \to \Upsilon_Z$ for the D_{Υ_Z}-geodesic from K_1 to K_2. For each $z \in Z$, the D_{Υ_Z}-geodesic P_Z hits $B_r(z)$.

Before proceeding, for each $z \in Z$, we shall write s_z (resp. t_z) for the first (resp. last) time at which P_Z hits $U_{z,r}^\star$. Write $i_z, j_z \in [1, \#Z_{z,r}]_{\mathbb{Z}}$ such that $P_Z(s_z) \in \partial U_{i_z}^\star$ and $P_Z(t_z) \in \partial U_{j_z}^\star$.

(vi) For each $z \in Z$, the event $\mathsf{F}_{z,r}(\Upsilon; \{i_z, j_z\})$ occurs.

(vii) Write $P\colon [0,1] \to \Upsilon$ for the D_Υ-geodesic from K_1 to K_2. For each $z \in Z$, the D_Υ-geodesic P passes through the tube between either $\mathscr{L}_{z,r}$ and $\gamma_{z,r}$ or $\mathscr{L}_{z,r}$ and $\gamma'_{z,r}$. (In this situation, $\mathscr{L}_{z,r}$, $\gamma_{z,r}$, $\gamma'_{z,r}$ are (connected arcs of) loops of Γ instead of $\Gamma_{z,r}$.)

Since (Υ, Υ_Z) and (Υ_Z, Υ) have the same law, we have

$$\mathbf{P}[\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)] = \mathbf{P}[\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)^\vee], \quad \forall \mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2,\ \forall r \in \rho^{-1}\mathcal{R},\ \forall Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2). \tag{8.2}$$

The following estimate tells us that, given Υ and the event $\mathfrak{G}^{\mathfrak{r}}_\varepsilon(K_1,K_2)$, with high probability, there are many $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$ for which $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)$ occurs.

Proposition 8.5. *For each $\eta \in (0,1)$, there exists $\mathfrak{c} = \mathfrak{c}(\eta) > 0$ such that for each $N \in \mathbb{N}$, there exists $\varepsilon_*(\eta, N) \in (0,1)$ such that for each $\varepsilon \in (0,\varepsilon_*]$, the following is true: Suppose that* $\operatorname{dist}(K_1,K_2) \geq \eta\mathfrak{r}$ *and* $\operatorname{dist}(K_1, \partial B_{2\mathfrak{r}}(0)) \geq \eta\mathfrak{r}$*. Then, on the event $\mathfrak{G}^{\mathfrak{r}}_\varepsilon(K_1,K_2)$, there exists $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$ and $r \in \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathfrak{r}, \varepsilon\mathfrak{r}]$ — which are almost surely determined by Υ — such that*

$$\mathbf{E}\big[\#\big\{Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2) : \#Z \leq N \text{ and } \mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2) \text{ occurs}\big\} \,\big|\, \Upsilon\big] \geq \varepsilon^{-\mathfrak{c}N}.$$

Both the proofs of Propositions 8.5 and 8.6 (stated just below) will be proved via elementary deterministic considerations based on the definitions of the events $\mathsf{E}_{z,r}$, $\mathsf{F}_{z,r}$, $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)$, and $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)^\vee$. The proof of Proposition 8.5 is postponed until Section 8.2. The guiding idea is as follows. We have already observed that condition (i) of the event $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)$ is closely related to condition (i) of $\mathfrak{G}^{\mathfrak{r}}_\varepsilon(K_1,K_2)$, while conditions (ii)–(iv) of $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)$ correspond to condition (ii) of $\mathfrak{G}^{\mathfrak{r}}_\varepsilon(K_1,K_2)$. Hence, by the pigeonhole principle, on the event $\mathfrak{G}^{\mathfrak{r}}_\varepsilon(K_1,K_2)$ there exist $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathfrak{r}, \varepsilon\mathfrak{r}]$, and a set $\mathcal{Z}_1 \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$ with $\#\mathcal{Z}_1 = \varepsilon^{-1+o(1)}$ such that conditions (i)–(v) of $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)$ hold for every subset $Z \subset \mathcal{Z}_1$. Invoking (7.22), we further obtain a subset $\mathcal{Z}_2 \subset \mathcal{Z}_1$ with $\mathbf{E}[\#\mathcal{Z}_2 \mid \Upsilon] = \varepsilon^{-1+o(1)}$ almost surely, and for which conditions (i)–(vi) of $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)$ hold for all $Z \subset \mathcal{Z}_2$. Thus it remains to check (vii). We will show that for each $Z \subset \mathcal{Z}_2$ there is a subcollection $Z' \subset Z$ with $\#Z' \geq c\,\#Z$ for some deterministic $c > 0$ such that condition (vii) holds for $\mathfrak{F}^{\mathfrak{r}}_{Z'}(K_1,K_2)$ (hence $\mathfrak{F}^{\mathfrak{r}}_{Z'}(K_1,K_2)$ occurs) (cf. Lemma 8.13 and see also the beginning of Section 8.2 for the idea of the construction of Z'). Consequently,

$$\begin{aligned}
&\mathbf{E}\big[\#\big\{Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2) : \#Z \leq N \text{ and } \mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2) \text{ occurs}\big\} \,\big|\, \Upsilon\big] \\
&\qquad \geq \mathbf{E}\left[\binom{\#\mathcal{Z}_2}{N}\binom{\#\mathcal{Z}_2}{(1-c)N}^{-1} \,\middle|\, \Upsilon\right] \succeq \mathbf{E}\big[(\#\mathcal{Z}_2)^{cN} \,\big|\, \Upsilon\big] = \varepsilon^{-(c+o(1))N} \quad \text{almost surely.}
\end{aligned}$$

Our next result provides an unconditional upper bound on the number of sets Z for which the event $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)^\vee$ occurs.

Proposition 8.6. *There exists a deterministic constant $\mathfrak{C} > 0$ such that for each $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R}$, and $N \in \mathbb{N}$, we have*

$$\#\big\{Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2) : \#Z \leq N \text{ and } \mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)^\vee \text{ occurs}\big\} \leq \mathfrak{C}^N \quad \text{almost surely.} \tag{8.3}$$

The proof of Proposition 8.6 is postponed until Section 8.3. The intuition behind Proposition 8.6 is as follows. Fix $Z_0 \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$ with $\#Z_0 \leq N$ such that $\mathfrak{F}^{\mathfrak{r}}_{Z_0}(K_1,K_2)^\vee$ occurs (if no such Z_0 exists, then (8.3) is immediate). Let Z'_0 be the set of points $z \in \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2) \setminus Z_0$ for which there exists $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$ with $\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)^\vee$ occurring. We *claim* that $\#Z'_0 \leq C\,\#Z_0$ for some deterministic constant C (cf. Lemma 8.15). Indeed, each $z \in Z'_0$ must satisfy the z-localized parts of conditions (ii), (iii), (iv), (vi), (vii) of Definition 8.4, namely:

- $\mathsf{E}_{z,r}(\Upsilon_{z,r})$ occurs.
- At most two connected arcs of $\Gamma_{z,r}$ in $A_{5r,\Lambda r}(z)$ connect the inner and outer boundaries.

- There exists a path $P_z \subset A_{5r,\Lambda r}(z) \cap \Upsilon_{z,r}$ of $D_{\Upsilon_{z,r}}$-length at most $\mathsf{A}\mathfrak{c}_r$ such that every path in $\Upsilon_{z,r}$ that crosses between the inner and outer boundaries of $A_{5r,\Lambda r}(z)$ intersects P_z.
- $\mathsf{F}_{z,r}(\Upsilon; \{i_z, j_z\})$ occurs for some distinct $i_z, j_z \in [1, \#Z_{z,r}]_{\mathbb{Z}}$.
- The D_Υ-geodesic P passes through the tube between either $\mathscr{L}_{z,r}$ and $\gamma_{z,r}$ or $\mathscr{L}_{z,r}$ and $\gamma'_{z,r}$.

The last item, together with Lemma 7.13 (applied with the roles of Υ and $\Upsilon_{z,r}$ interchanged), implies that

(8.4) $\widetilde{D}_\Upsilon(K_1, K_2)$ is at most $M^* D_\Upsilon(K_1, K_2)$ minus a deterministic positive constant multiple of $\#Z_0'$.

On the other hand, since $\mathfrak{F}^{\mathbb{r}}_{Z_0}(K_1, K_2)^\vee$ occurs, condition (i) of $\mathfrak{F}^{\mathbb{r}}_{Z_0}(K_1, K_2)^\vee$ yields

$$\widetilde{D}_{\Upsilon_{Z_0}}(K_1, K_2) \geq M^* D_{\Upsilon_{Z_0}}(K_1, K_2) - \mathfrak{c}_r. \tag{8.5}$$

Since Υ and Υ_{Z_0} differ only inside the Euclidean balls centered at $z \in Z_0$, the differences between $\widetilde{D}_\Upsilon(K_1, K_2)$ and $\widetilde{D}_{\Upsilon_{Z_0}}(K_1, K_2)$ (resp. $D_\Upsilon(K_1, K_2)$ and $D_{\Upsilon_{Z_0}}(K_1, K_2)$) are controlled by a deterministic positive constant multiple of $\#Z_0$. Hence, if $\#Z_0'$ were much larger than $\#Z_0$, (8.4) and (8.5) would be incompatible, proving the *claim* (see also the paragraph after Lemma 8.15). Finally, we conclude Proposition 8.6 by noting that

$$\#\left\{Z \subset \mathbb{Z}^{\mathbb{r}}_{\mathbf{e},r}(K_1, K_2) : \#Z \leq N \text{ and } \mathfrak{F}^{\mathbb{r}}_Z(K_1, K_2)^\vee \text{ occurs}\right\} \leq 2^{\#Z_0 + \#Z_0'} \leq 2^{(1+C)N}.$$

We now show how Proposition 8.2 can be proved under the assumptions of Propositions 8.5 and 8.6. The idea is that, on the event $\mathfrak{G}^{\mathbb{r}}_\varepsilon(K_1, K_2)$, Proposition 8.6 yields a deterministic conclusion that contradicts the requirement of Proposition 8.5 as $N \to \infty$. Consequently, $\mathfrak{G}^{\mathbb{r}}_\varepsilon(K_1, K_2)$ cannot occur with high probability.

Proof of Proposition 8.2 assuming Propositions 8.5 and 8.6. Since $\mathcal{R} \subset \{\Lambda^k\}_{k\in\mathbb{Z}}$, we have

$$\begin{aligned}\#\left(\rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]\right) &= \#\left(\mathcal{R} \cap [\varepsilon^{1+\nu}\rho\mathbb{r}, \varepsilon\rho\mathbb{r}]\right) \leq \#\left(\mathcal{R} \cap [(\varepsilon\rho)^{1+\nu}\mathbb{r}, \varepsilon\rho\mathbb{r}]\right) \\ &\leq \nu \log_\Lambda(1/(\varepsilon\rho)) \leq 2\nu \log_\Lambda(1/\varepsilon), \quad \forall \varepsilon \in (0, \rho].\end{aligned} \tag{8.6}$$

To lighten notation, write $\mathcal{R}_\varepsilon \stackrel{\text{def}}{=} \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]$. Thus, by (8.6) and Proposition 8.6, for each $N \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_*(\eta, N) \wedge \rho]$, we have

$$\begin{aligned}&\mathfrak{C}^N \cdot 9 \cdot 10^4 \cdot \Lambda^2 \cdot 2\nu \log_\Lambda(1/\varepsilon) \\ &\geq \mathfrak{C}^N \sum_{\mathbf{e}\in(\mathbb{Z}/(300\Lambda\mathbb{Z}))^2} \sum_{r\in\mathcal{R}_\varepsilon} 1 \\ &\geq \sum_{\mathbf{e}\in(\mathbb{Z}/(300\Lambda\mathbb{Z}))^2} \sum_{r\in\mathcal{R}_\varepsilon} \mathbf{E}\left[\#\left\{Z \subset \mathbb{Z}^{\mathbb{r}}_{\mathbf{e},r}(K_1, K_2) : \#Z \leq N,\ \mathfrak{F}^{\mathbb{r}}_Z(K_1, K_2)^\vee \text{ occurs}\right\}\right] \\ &= \sum_{\mathbf{e}\in(\mathbb{Z}/(300\Lambda\mathbb{Z}))^2} \sum_{r\in\mathcal{R}_\varepsilon} \sum_{\substack{Z\subset\mathbb{Z}^{\mathbb{r}}_{\mathbf{e},r}(K_1,K_2) \\ \#Z\leq N}} \mathbf{P}\left[\mathfrak{F}^{\mathbb{r}}_Z(K_1, K_2)^\vee\right].\end{aligned} \tag{8.7}$$

By (8.2) and Proposition 8.5, we have

$$
\begin{aligned}
(8.8)\qquad & \sum_{\mathbf{e}\in(\mathbb{Z}/(300\Lambda\mathbb{Z}))^2}\sum_{r\in\mathfrak{R}_\varepsilon}\sum_{\substack{Z\subset\mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)\\ \#Z\le N}} \mathbf{P}\left[\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)^{\vee}\right]\\
&= \sum_{\mathbf{e}\in(\mathbb{Z}/(300\Lambda\mathbb{Z}))^2}\sum_{r\in\mathfrak{R}_\varepsilon}\sum_{\substack{Z\subset\mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)\\ \#Z\le N}} \mathbf{P}[\mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)]\\
&= \sum_{\mathbf{e}\in(\mathbb{Z}/(300\Lambda\mathbb{Z}))^2}\sum_{r\in\mathfrak{R}_\varepsilon}\mathbf{E}\left[\#\left\{Z\subset\mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2) : \#Z\le N,\ \mathfrak{F}^{\mathfrak{r}}_Z(K_1,K_2)\text{ occurs}\right\}\right]\\
&\ge \varepsilon^{-\mathfrak{c}N}\mathbf{P}[\mathfrak{G}^{\mathfrak{r}}_\varepsilon(K_1,K_2)].
\end{aligned}
$$

Combining (8.7) and (8.8), for each $N\in\mathbb{N}$, there exists $\varepsilon_*=\varepsilon_*(\eta,N)\in(0,1)$ such that

$$\mathbf{P}[\mathfrak{G}^{\mathfrak{r}}_\varepsilon(K_1,K_2)]\le 9\cdot10^4\cdot\Lambda^2\cdot\mathfrak{C}^N\cdot 2\nu\log_\Lambda(1/\varepsilon)\cdot\varepsilon^{\mathfrak{c}N},\quad \forall\varepsilon\in(0,\varepsilon_*\wedge\rho].$$

This completes the proof. □

8.2. **Proof of Proposition 8.5.** Fix $\eta\in(0,1)$ such that $\mathrm{dist}(K_1,K_2)\wedge\mathrm{dist}(K_1,\partial B_{2\mathfrak{r}}(0))\ge\eta\mathfrak{r}$.

For $\mathbf{e}\in(\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r\in\rho^{-1}\mathfrak{R}$, and $Z\subset\mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$, we shall write E_Z (resp. F_Z) for the event that conditions (i), (ii), (iii), (iv), (v) (resp. (i), (ii), (iii), (iv), (v), (vi)) of Definition 8.3 hold.

It follows directly from the definition of $\mathfrak{G}^{\mathfrak{r}}_\varepsilon(K_1,K_2)$ that, whenever this event occurs, there exists some $\mathbf{e}\in(\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$ and $r\in\rho^{-1}\mathfrak{R}$ such that (with high conditional probability) many sets $Z\subset\mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$ satisfy the event F_Z. The main challenge in verifying condition (vii) lies in the fact that the D_{Υ_Z}-geodesic between K_1 and K_2 might entirely avoid some of the balls $B_{5r}(z)$ for $z\in Z$. To handle this issue, we will prove that if $Z\subset\mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$ and F_Z holds, then there exists a subset $Z'\subset Z$ such that $\#Z'$ is at least a fixed positive fraction of $\#Z$, and $\mathfrak{F}^{\mathfrak{r}}_{Z'}(K_1,K_2)$ occurs (cf. Lemma 8.13).

The construction of Z' proceeds as follows. In Lemma 8.8, we establish that $D_{\Upsilon_Z}(K_1,K_2)$ is significantly smaller than $D_\Upsilon(K_1,K_2)$. This means that the D_{Υ_Z}-geodesic P_Z from K_1 to K_2 must spend a considerable amount of time in regions where Υ and Υ_Z differ. For each $z\in Z$, there are several types of such regions inside $B_{5r}(z)$ where P_Z could potentially enter; one important example is the tube bounded by $\mathfrak{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$ (see Figure 12 for an illustration). In Lemma 8.12, we show, roughly speaking, that if P_Z enters $B_{4r}(z)$ but does not travel through the tube between $\mathfrak{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$, then the time it spends inside $B_{\Lambda r}(z)$ cannot be much shorter than the time the D_Υ-geodesic P spends there. We then iteratively remove from Z the "bad" points z for which P_Z avoids the tube between $\mathfrak{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$. The key observation is that eliminating these "bad" points does not significantly increase $D_{\Upsilon_Z}(K_1,K_2)$. Hence, throughout the iteration, $D_{\Upsilon_Z}(K_1,K_2)$ remains much smaller than $D_\Upsilon(K_1,K_2)$. This fact ensures that the iterative process must stop before too many points are removed from Z.

We start by proving an upper bound showing that, on the event F_Z (cf. Lemmas 8.7 and 8.8), the quantity $D_{\Upsilon_Z}(K_1,K_2)$ is at most $D_\Upsilon(K_1,K_2)$ minus a deterministic positive constant multiple of $\mathfrak{c}_r\#Z$. This bound arises because the D_Υ-geodesic P must cross each annulus $A_{r,2r}(z)$ for $z\in Z$, which incurs a total cost proportional to $\mathfrak{c}_r\#Z$ (cf. Definition 7.4, (i)). In contrast, the D_{Υ_Z}-geodesic P_Z can take advantage of the tube bounded by $\mathfrak{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$ to bypass this cost. Indeed, by Definition 7.2, (iv) and Definition 7.4, (xi), traveling through this tube requires less time than crossing the corresponding annulus $A_{r,2r}(z)$.

Lemma 8.7. *Let* $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$*,* $r \in \rho^{-1}\mathcal{R}$*, and* $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1, K_2)$*. Suppose that the event* F_Z *occurs. Let* $\{(s_z, t_z)\}_{z\in Z}$ *be as in Definition 8.3. Then*

$$D_{\Upsilon_Z}(P(s_z), P(t_z)) \le \mathsf{a}_5 \mathfrak{c}_r, \quad \forall z \in Z.$$

Proof. Fix $z \in Z$. Let i_z and j_z be as in Definition 8.3. We observe that there exists $I \in \mathfrak{I}^{\alpha\setminus\beta}_{i_z}$ and $J \in \mathfrak{I}^{\alpha\setminus\beta}_{j_z}$ (cf. (7.21)) such that $P(s_z) \in I \cap \partial U_{z,r}$ and $P(t_z) \in J \cap \partial U_{z,r}$. Note that $P(s_z)$ and $P(t_z)$ are connected inside the tube between $\mathfrak{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$ (cf., e.g., Figure 10). By (1.5), Definition 7.2, (iv), and Definition 7.4, (xi)

$$D_{\Upsilon_Z}(P(s_z), P(t_z)) \le (\#Z_{z,r}) \left(2M_*^{-1}a^{-1}\mathfrak{c}_{\rho r} + \rho\mathfrak{c}_r\right) \le 2\pi\mathsf{a}_5^{-1}\left(2M_*^{-1}a^{-1}\mathfrak{K} + 1\right)\rho\mathfrak{c}_r.$$

Since we may choose $\rho = \mathsf{a}_6$ to be sufficiently small so that $2\pi\mathsf{a}_5^{-1}(2M_*^{-1}a^{-1}\mathfrak{K} + 1)\rho \le \mathsf{a}_5$, we complete the proof. □

Lemma 8.8. *Let* $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$*,* $r \in \rho^{-1}\mathcal{R}$*, and* $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1, K_2)$*. Suppose that the event* F_Z *occurs. Then*

$$D_{\Upsilon_Z}(K_1, K_2) \le D_\Upsilon(K_1, K_2) - \mathsf{a}_1\mathfrak{c}_r\#Z.$$

Proof. Let $\{(s_z, t_z)\}_{z\in Z}$ be as in Definition 8.3. By Lemma 8.7, we have

$$D_{\Upsilon_Z}(P(s_z), P(t_z)) \le \mathsf{a}_5\mathfrak{c}_r, \quad \forall z \in Z. \tag{8.9}$$

Since Υ_Z coincides with Υ outside of $\bigcup_{z\in Z} U^\star_{z,r}$, it follows that

$$\operatorname{len}(P|_{[0,1]\setminus\bigcup_{z\in Z}[s_z,t_z]}; D_{\Upsilon_Z}) = \operatorname{len}(P|_{[0,1]\setminus\bigcup_{z\in Z}[s_z,t_z]}; D_\Upsilon). \tag{8.10}$$

Combining (8.9) and (8.10), we obtain that

$$\begin{aligned} D_{\Upsilon_Z}(K_1, K_2) &\le \operatorname{len}(P|_{[0,1]\setminus\bigcup_{z\in Z}[s_z,t_z]}; D_\Upsilon) + \mathsf{a}_5\mathfrak{c}_r\#Z \\ &= D_\Upsilon(K_1, K_2) - \operatorname{len}(P|_{\bigcup_{z\in Z}[s_z,t_z]}; D_\Upsilon) + \mathsf{a}_5\mathfrak{c}_r\#Z. \end{aligned}$$

Thus, it suffices to show that

$$\operatorname{len}(P|_{\bigcup_{z\in Z}[s_z,t_z]}; D_\Upsilon) \ge 2\mathsf{a}_1\mathfrak{c}_r\#Z.$$

Indeed, since P enters $B_r(z)$ for all $z \in Z$. We conclude that there are times $s'_z < t'_z$ such that $P|_{[s'_z,t'_z]}$ forms a $(B_{2r}(z), B_r(z))$-excursion. By Definition 7.4, (i), we have

$$\operatorname{len}(P|_{[s'_z,t'_z]}; D_\Upsilon) \ge 2\mathsf{a}_1\mathfrak{c}_r, \quad \forall z \in Z.$$

It is clear that the intervals $[s'_z, t'_z]$ for $z \in Z$ are pairwise disjoint and $\bigcup_{z\in Z}[s'_z, t'_z] \subset \bigcup_{z\in Z}[s_z, t_z]$. Thus,

$$\operatorname{len}(P|_{\bigcup_{z\in Z}[s_z,t_z]}; D_\Upsilon) \ge \sum_{z\in Z} \operatorname{len}(P|_{[s'_z,t'_z]}; D_\Upsilon) \ge 2\mathsf{a}_1\mathfrak{c}_r\#Z.$$

This completes the proof. □

We now prove an inequality in the reverse direction of that in Lemma 8.8, namely, an upper bound on $D_\Upsilon(K_1, K_2)$ expressed in terms of $D_{\Upsilon_Z}(K_1, K_2)$ (cf. Lemma 8.10). To carry out this proof, we will first need an upper bound on the number of $(B_{\Lambda r}(z), B_{4r}(z))$-excursions that the path P_Z can make.

Lemma 8.9. *Let* $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$*,* $r \in \rho^{-1}\mathcal{R}$*, and* $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1, K_2)$*. Suppose that the event* F_Z *occurs. Write* $P_Z\colon [0,1] \to \Upsilon_Z$ *for the* D_{Υ_Z}*-geodesic from* K_1 *to* K_2*. For each* $z \in Z$*, write* $\mathcal{E}_{z,r}(P_Z)$ *for the collection of* $(B_{\Lambda r}(z), B_{4r}(z))$*-excursions of* P_Z*. Then*

$$\#\mathcal{E}_{z,r}(P_Z) \le \mathsf{A}/\mathsf{a}_1.$$

Proof. Fix $z \in Z$. Write $n \stackrel{\text{def}}{=} \#\mathcal{E}_{z,r}(P_Z)$. Write $\mathcal{E}_{z,r}(P_Z) = \{(\sigma_k', \sigma_k, \tau_k, \tau_k') : k \in [1,n]_{\mathbb{Z}}\}$ in chronological order, i.e.,

$$\sigma_1' < \tau_1' < \sigma_2' < \tau_2' < \cdots < \sigma_n' < \tau_n'.$$

Since $P_Z|_{[\sigma_1',\sigma_1]}$ and $P_Z|_{[\tau_n,\tau_n']}$ intersect the path P_z described in Definition 8.3, (iv), it follows that

$$\operatorname{len}(P_Z|_{[\sigma_1,\tau_n]}; D_{\Upsilon_Z}) \le \operatorname{len}(P_z; D_{\Upsilon_Z}) = \operatorname{len}(P_z; D_\Upsilon) \le \mathsf{A}\mathfrak{c}_r. \tag{8.11}$$

On the other hand, by Definition 7.4, (i),

$$\begin{aligned}\operatorname{len}(P_Z|_{[\sigma_1,\tau_n]}; D_{\Upsilon_Z}) &\ge \sum_{k=1}^{n-1} \operatorname{len}(P_Z|_{[\tau_k,\tau_k']}; D_{\Upsilon_Z}) + \operatorname{len}(P_Z|_{[\sigma_n',\sigma_n]}; D_{\Upsilon_Z}) \\ &= \sum_{k=1}^{n-1} \operatorname{len}(P_Z|_{[\tau_k,\tau_k']}; D_\Upsilon) + \operatorname{len}(P_Z|_{[\sigma_n',\sigma_n]}; D_\Upsilon) \ge n\mathsf{a}_1\mathfrak{c}_r.\end{aligned} \tag{8.12}$$

Combining (8.11) and (8.12), we complete the proof. ☐

Lemma 8.10. *Let* $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R}$, *and* $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$. *Suppose that the event* F_Z *occurs. Then*

$$D_\Upsilon(K_1,K_2) \le D_{\Upsilon_Z}(K_1,K_2) + (\mathsf{A}^2/\mathsf{a}_1)\mathfrak{c}_r\#Z.$$

Proof. For each $z \in Z$, let $\mathcal{E}_{z,r}(P_Z)$ be as in Lemma 8.9. Since Υ_Z coincides with Υ outside of $\bigcup_{z\in Z} U_{z,r}^\star$, it follows that the D_{Υ_Z}-length and the D_Υ-length of the restriction of P_Z to

$$[0,1] \setminus \bigcup_{z\in Z} \bigcup_{(\sigma',\sigma,\tau,\tau')\in\mathcal{E}_{z,r}(P_Z)} [\sigma,\tau]$$

are equal. On the other hand, for each $z \in Z$ and $(\sigma',\sigma,\tau,\tau') \in \mathcal{E}_{z,r}(P_Z)$, the paths $P_Z|_{[\sigma',\sigma]}$ and $P_Z|_{[\tau,\tau']}$ intersect the path P_z described in Definition 8.3, (iv). Thus, we conclude from Lemma 8.9 that

$$D_\Upsilon(K_1,K_2) \le D_{\Upsilon_Z}(K_1,K_2) + \sum_{z\in Z} \sum_{(\sigma',\sigma,\tau,\tau')\in\mathcal{E}_{z,r}(P_Z)} \operatorname{len}(P_z; D_\Upsilon) \le D_{\Upsilon_Z}(K_1,K_2) + (\mathsf{A}^2/\mathsf{a}_1)\mathfrak{c}_r\#Z.$$

This completes the proof. ☐

The following estimate tells us the points at which P enters and exits $U_{z,r}^\star$ cannot be too close to each other.

Lemma 8.11. *Let* $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R}$, *and* $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1,K_2)$. *Suppose that the event* E_Z *occurs. For each* $z \in Z$, *let* i_z *and* j_z *be as in Definition 8.3. Then for each* $z \in Z$, *the Euclidean distance between* U_{i_z} *and* U_{j_z} *is greater than* $\mathsf{a}_3 r/2$.

Proof. Fix $z \in Z$. Let s_z and t_z be as in Definition 8.3. Since $P|_{[s_z,t_z]}$ enters $B_r(z)$ by Definition 8.3, (v), it follows from Definition 7.4, (i) that

$$D_\Upsilon(P(s_z),P(t_z)) \ge D_\Upsilon(\partial B_r(z)\cap\Upsilon, \partial B_{2r}(z)\cap\Upsilon) \ge \mathsf{a}_1\mathfrak{c}_r. \tag{8.13}$$

By Definition 7.4, (iv) and the definition of $\{U_j\}_{j\in[1,\#Z_{z,r}]_{\mathbb{Z}}}$, each of them has Euclidean diameter at most $100\mathsf{a}_4 r$. Thus, if $\operatorname{dist}(U_{i_z},U_{j_z}) \le \mathsf{a}_3 r/2$, then $|P(s_z) - P(t_z)| \le \mathsf{a}_3 r$. In this case, it follows from Definition 7.4, (ii) that $D_\Upsilon(P(s_z),P(t_z)) \le \mathsf{a}_2\mathfrak{c}_r$ (since $P(s_z) \stackrel{\Upsilon}{\leftrightarrow} P(t_z)$, no loop of Γ can disconnect $P(s_z)$ and $P(t_z)$), in contradiction to (8.13). This completes the proof. ☐

The following estimate tells us if a $(B_{\Lambda r}(z), B_{4r}(z))$-excursion of P_Z does not pass through the tube between either $\mathcal{L}_{z,r}$ and $\gamma_{z,r}$ or $\mathcal{L}_{z,r}$ and $\gamma_{z,r}'$, then the D_{Υ_Z}-distance between its endpoints cannot be much shorter than the D_Υ-distance.

Lemma 8.12. *Let* $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$*,* $r \in \rho^{-1}\mathcal{R}$*, and* $Z \subset \mathbb{Z}^{\mathrm{r}}_{\mathbf{e},r}(K_1, K_2)$*. Suppose that the event* F_Z *occurs. Write* $P_Z \colon [0,1] \to \Upsilon_Z$ *for the* D_{Υ_Z}*-geodesic from* K_1 *to* K_2*. Let* $z \in Z$*. Let* $(\sigma', \sigma, \tau, \tau')$ *be a* $(B_{\Lambda r}(z), B_{4r}(z))$*-excursion of* P_Z*. Suppose that* $P_Z|_{[\sigma,\tau]}$ *does not pass through the tube between either* $\mathscr{L}_{z,r}$ *and* $\gamma_{z,r}$ *or* $\mathscr{L}_{z,r}$ *and* $\gamma'_{z,r}$*. Then*

$$D_\Upsilon(P_Z(\sigma), P_Z(\tau)) - D_{\Upsilon_Z}(P_Z(\sigma), P_Z(\tau)) \le 100\mathsf{a}_2\mathfrak{c}_r.$$

See Figure 12 for an illustration of the proof of Lemma 8.12. The idea is as follows. Write S_{i_z} (resp. S_{j_z}) for the first time at which $P_Z|_{[\sigma,\tau]}$ hits I for some $I \in \mathfrak{I}^{\alpha\setminus\beta}_{i_z}$ (resp. $I \in \mathfrak{I}^{\alpha\setminus\beta}_{j_z}$). Write T_{i_z} (resp. T_{j_z}) for the last time at which $P_Z|_{[\sigma,\tau]}$ hits J for some $J \in \mathfrak{I}^{\alpha\setminus\beta}_{i_z}$ (resp. $I \in \mathfrak{I}^{\alpha\setminus\beta}_{j_z}$). We first show that $P_Z(S_{i_z})$, $P_Z(T_{i_z})$, $P_Z(S_{j_z})$, and $P_Z(T_{j_z})$ are contained in the same connected component of Υ (cf. (8.14)). Since it is clear that the endpoints of each small dashed subpath as illustrated in Figure 12 are contained in the same connected component of Υ, it follows that all the solid subpaths of P_Z are contained in the same connected component of Υ. Thus, roughly speaking, $D_\Upsilon(P_Z(\sigma), P_Z(\tau))$ is bounded above by the D_Υ-length (or, equivalently, the D_{Υ_Z}-length) of the solid subpaths of P_Z plus the D_Υ-distance between the endpoints of the dashed subpaths. The D_Υ-distance between $P_Z(S_{i_z})$ and $P_Z(T_{i_z})$ (resp. $P_Z(S_{j_z})$ and $P_Z(T_{j_z})$) is upper-bounded using Definition 7.4, (ii). The D_Υ-distances between the endpoints of the small dashed subpaths are upper-bounded using Definition 7.4, (iii), (v).

Proof of Lemma 8.12. Let i_z and j_z be as in Definition 8.3. Since P_Z does not pass through the tube between either $\mathscr{L}_{z,r}$ and $\gamma_{z,r}$ or $\mathscr{L}_{z,r}$ and $\gamma'_{z,r}$, we observe that for each $U^\star_{z,r}$-excursion (S,T) of $P_Z|_{[\sigma,\tau]}$, there are three possibilities:

(i) There exist $I, J \in \mathfrak{I}^{\alpha\setminus\beta}_{i_z}$ such that $P_Z(S) \in I \cap \partial U_{z,r}$ and $P_Z(T) \in J \cap \partial U_{z,r}$.
(ii) There exist $I, J \in \mathfrak{I}^{\alpha\setminus\beta}_{j_z}$ such that $P_Z(S) \in I \cap \partial U_{z,r}$ and $P_Z(T) \in J \cap \partial U_{z,r}$.
(iii) There exists $j \in [1, \#Z_r]_{\mathbb{Z}} \setminus \{i_z, j_z\}$ and $I \in \mathfrak{I}^{\alpha\setminus\beta}_j$ such that $P_Z(S), P_Z(T) \in I \cap \partial U_{z,r}$.

As was introduced just after the statement of the lemma, write S_{i_z} (resp. S_{j_z}) for the first time at which $P_Z|_{[\sigma,\tau]}$ hits I for some $I \in \mathfrak{I}^{\alpha\setminus\beta}_{i_z}$ (resp. $I \in \mathfrak{I}^{\alpha\setminus\beta}_{j_z}$). Write T_{i_z} (resp. T_{j_z}) for the last time at which $P_Z|_{[\sigma,\tau]}$ hits J for some $J \in \mathfrak{I}^{\alpha\setminus\beta}_{i_z}$ (resp. $I \in \mathfrak{I}^{\alpha\setminus\beta}_{j_z}$). We *claim* that

$$P_Z(S_{i_z}) \overset{\Upsilon}{\leftrightarrow} P_Z(T_{i_z}) \overset{\Upsilon}{\leftrightarrow} P_Z(S_{j_z}) \overset{\Upsilon}{\leftrightarrow} P_Z(T_{j_z}). \tag{8.14}$$

Note that either $S_{i_z} < T_{i_z} < S_{j_z} < T_{j_z}$ or $S_{i_z} < S_{j_z} < T_{i_z} < T_{j_z}$. We only consider the first case, the second case follows from a similar argument. First, we note that the restriction of $P_Z|_{[\sigma',\tau']}$ to the complement of the union of the intervals $[S,T]$ for all $U^\star_{z,r}$-excursions (S,T) of $P_Z|_{[\sigma,\tau]}$ is contained in Υ. Moreover, for any times S and T such that condition (iii) holds, it is clear that $P_Z(S) \overset{\Upsilon}{\leftrightarrow} P_Z(T)$. Thus, we conclude that

$$P_Z|_{[\sigma',\sigma]} \overset{\Upsilon}{\leftrightarrow} P_Z(S_{i_z}), \quad P_Z(T_{i_z}) \overset{\Upsilon}{\leftrightarrow} P_Z(S_{j_z}), \quad \text{and} \quad P_Z(T_{j_z}) \overset{\Upsilon}{\leftrightarrow} P_Z|_{[\tau,\tau']}.$$

Since both $P_Z|_{[\sigma',\sigma]}$ and $P_Z|_{[\tau,\tau']}$ intersect the path P_z described in Definition 8.3, (iv), it follows that $P_Z(S_{i_z}) \overset{\Upsilon}{\leftrightarrow} P_Z(T_{j_z}) \overset{\Upsilon}{\leftrightarrow} P_z$. Thus, it suffices to show that $P_Z(S_{i_z}) \overset{\Upsilon}{\leftrightarrow} P_Z(T_{i_z})$. Suppose by way of contradiction that this is not true. Since the Euclidean diameter of U_{i_z} is at most $100\mathsf{a}_4 r$ (cf. Definition 7.4, (iv)), it follows that $|P_Z(S_{i_z}) - P_Z(T_{i_z})| \le 100\mathsf{a}_4 r$. Then, by Definition 7.4, (iii), there exists a loop $\mathscr{L} \in \Gamma$ with Euclidean diameter at most $\mathsf{a}_3 r/2$, contained in $A_{r,5r}(z)$, and disconnecting $P_Z(S_{i_z})$ and $P_Z(T_{i_z})$. Since $P_Z(S_{i_z})$ is connected to $P_Z|_{[\sigma',\sigma]}$ in Υ, the loop $\mathscr{L}$ cannot surround $P_Z(S_{i_z})$. Thus, $\mathscr{L}$ must surround $P_Z(T_{i_z})$. However, since $P_Z(T_{i_z}) \overset{\Upsilon}{\leftrightarrow} P_Z(S_{j_z})$, and, by Lemma 8.11, the Euclidean distance between $P_Z(T_{i_z})$ and $P_Z(S_{j_z})$ is greater than $\mathsf{a}_3 r/2$,

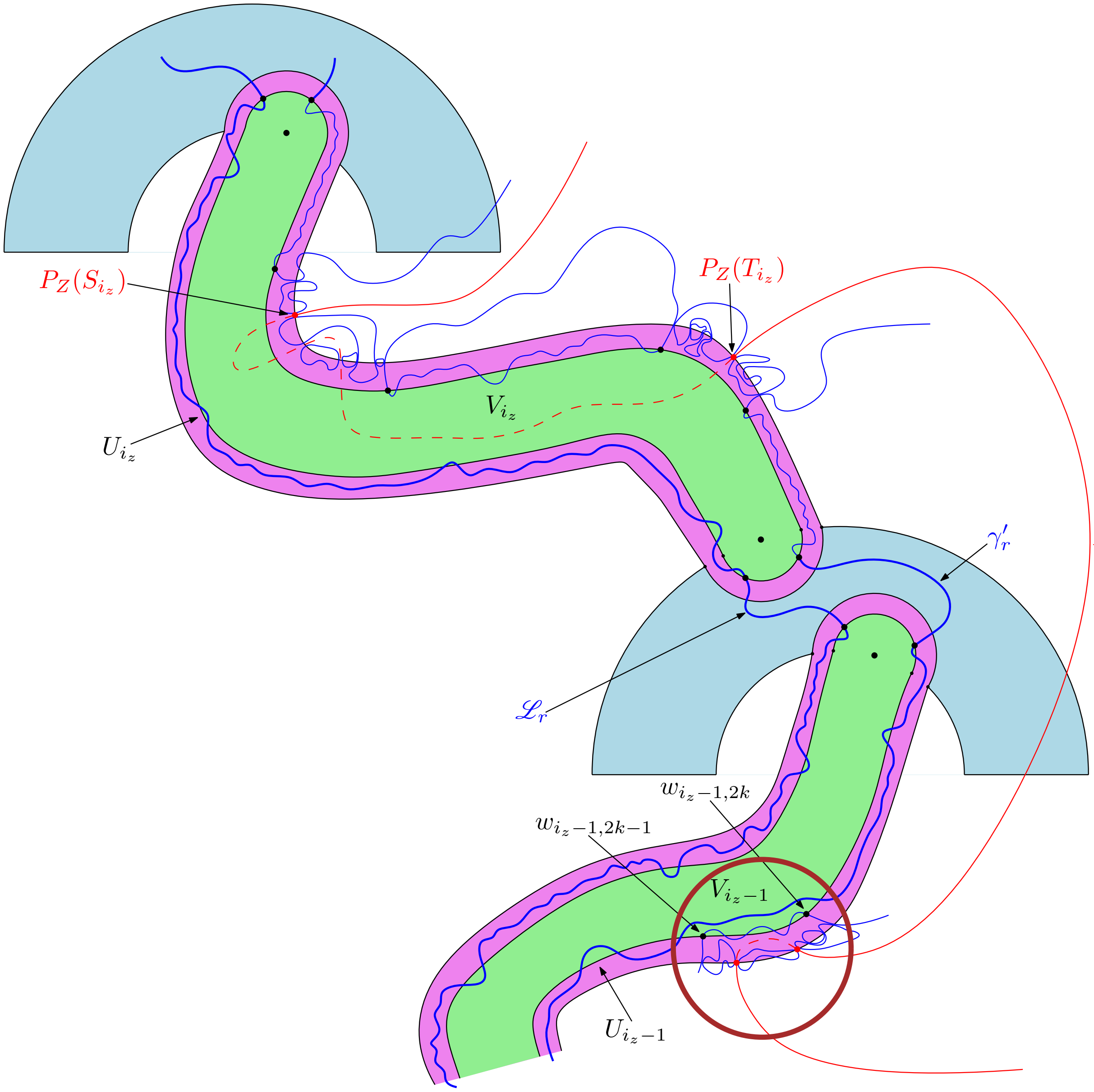

FIGURE 12. Illustration of the proof of Lemma 8.12. The red path P_Z is a D_{Υ_Z}-geodesic that enters the "places" in $B_{5r}(z)$ where Υ and Υ_Z differ (the dashed subpaths of P_Z) but does not passes through the tube between $\mathscr{L}_{z,r}$, $\gamma_{z,r}$, and $\gamma'_{z,r}$. In this situation, there are three different types of such "bad" subpaths: $P_Z|_{[S_{i_z},T_{i_z}]}$, $P_Z|_{[S_{j_z},T_{j_z}]}$ (not shown), and the small dashed subpaths that enters the small "bad places" corresponding to $w_{n,2k-1}$ and $w_{n,2k}$ for some $n \in [1, \#Z_{z,r}]_{\mathbb{Z}} \setminus \{i_z, j_z\}$ and $k \in \mathbb{N}$. (Recall from Figure 9 the definition of the points $w_{n,2k-1}$ and $w_{n,2k}$.)

in contradiction to the fact that $P_Z(T_{i_z})$ is surrounded by a loop of Γ with Euclidean diameter at most $\mathsf{a}_3 r/2$. This completes the proof of (8.14).

Let $\mathcal{S} \subset A_{2r,4r}(0) \times (0, \lambda r]$ be as in Definition 7.4, (v). For each $(w,t) \in \mathcal{S}$, since there are at most $2N$ connected arcs of Γ in $A_{t,2t}(w)$ connecting its inner and outer boundaries, there are at most N

$(B_{2t}(w), B_t(w))$-excursions of Γ. These $(B_{2t}(w), B_t(w))$-excursions divide $B_{2t}(w)$ into at most $N+1$ connected components. Among these $N+1$ connected components, at most N of them, which we shall denote by $A_{w,t,1}, \dots, A_{w,t,N}$, correspond to Υ, i.e., $A_{w,t,j} \cap B_t(w) \cap \Upsilon \overset{\Upsilon}{\leftrightarrow} (\mathbb{C} \setminus B_{2t}(w)) \cap \Upsilon$ for all j. Suppose that S and T are times such that condition (iii) holds. It is clear that $P_Z(S)$ and $P_Z(T)$ are contained in the same connected component of $(U_{z,r} \setminus V_{z,r}) \cap \Upsilon$. Denote this connected component by $\mathscr{C}$. Since P_Z started from outside of $B_{4r}(z)$, it follows that $\mathscr{C} \overset{\Upsilon}{\leftrightarrow} (\mathbb{C} \setminus B_{4r}(z)) \cap \Upsilon$. Thus, it follows from Definition 7.4, (v) that there exists (w,t) such that $\mathscr{C} \subset B_t(w)$, and it follows from the above discussion that there exists $j \in [1,N]_{\mathbb{Z}}$ such that $\mathscr{C} \subset A_{w,t,j}$. We *observe* that for each pair of points $x, y \in A_{w,t,j} \cap \Upsilon$ with $x \overset{\Upsilon}{\leftrightarrow} (\mathbb{C} \setminus B_{2t}(w)) \cap \Upsilon$ and $y \overset{\Upsilon}{\leftrightarrow} (\mathbb{C} \setminus B_{2t}(w)) \cap \Upsilon$, we have $x \xleftrightarrow{B_{2t}(w) \cap \Upsilon} y$. For each $(w,t) \in \mathcal{S}$ and $j \in [1,N]_{\mathbb{Z}}$, write $S_{w,t,j}$ (resp. $T_{w,t,j}$) for the first (resp. last) time at which P_Z hits a connected component $\mathscr{C}$ of $(U_{z,r} \setminus V_{z,r}) \cap \Upsilon$ with $\mathscr{C} \subset B_t(w)$. It follows from the above *observation* that

$$P_Z(S_{w,t,j}) \xleftrightarrow{B_{2t}(w) \cap \Upsilon} P_Z(T_{w,t,j}), \quad \forall (w,t) \in \mathcal{S},\ \forall j \in [1,N]_{\mathbb{Z}}. \tag{8.15}$$

Now, we conclude from the discussion of the preceding paragraphs that for each $U^{\star}_{z,r}$-excursion (S,T) of $P_Z|_{[\sigma,\tau]}$, we have

$$[S,T] \subset [S_{i_z}, T_{i_z}] \cup [S_{j_z}, T_{j_z}] \cup \bigcup_{(w,t)\in\mathcal{S}} \bigcup_{j=1}^{N} [S_{w,t,j}, T_{w,t,j}].$$

This implies that

$$\begin{aligned} D_\Upsilon(P_Z(\sigma), P_Z(\tau)) &\le D_{\Upsilon_Z}(P_Z(\sigma), P_Z(\tau)) \\ &\quad + D_\Upsilon(P_Z(S_{i_z}), P_Z(T_{i_z})) + D_\Upsilon(P_Z(S_{j_z}), P_Z(T_{j_z})) + \sum_{(w,t)\in\mathcal{S}} \sum_{j=1}^{N} D_\Upsilon(P_Z(S_{w,t,j}), P_Z(T_{w,t,j})). \end{aligned} \tag{8.16}$$

Since $P_Z(S_{i_z}), P_Z(T_{i_z}) \in \partial U_{i_z}$, we have $|P_Z(S_{i_z}) - P_Z(T_{i_z})| \le 100\mathsf{a}_4 r$ (cf. Definition 7.4, (iv)). Since $P_Z(S_{i_z}) \overset{\Upsilon}{\leftrightarrow} P_Z(T_{i_z})$ by (8.14), it follows from Definition 7.4, (ii) that

$$D_\Upsilon(P_Z(S_{i_z}), P_Z(T_{i_z})) \le \mathsf{a}_2 \mathfrak{c}_r. \tag{8.17}$$

Similarly, we have

$$D_\Upsilon(P_Z(S_{j_z}), P_Z(T_{j_z})) \le \mathsf{a}_2 \mathfrak{c}_r. \tag{8.18}$$

By Definition 7.4, (v) and (8.15), we have

$$\begin{aligned} &\sum_{(w,t)\in\mathcal{S}} \sum_{j=1}^{N} D_\Upsilon(P_Z(S_{w,t,j}), P_Z(T_{w,t,j})) \\ &\qquad \le N \sum_{(w,t)\in\mathcal{S}} \sup\left\{ D_\Upsilon(x,y) : x, y \in B_t(w),\ x \xleftrightarrow{B_{2t}(w)\cap\Upsilon} y \right\} \le \mathsf{a}_2 \mathfrak{c}_r. \end{aligned} \tag{8.19}$$

Combining (8.16), (8.17), (8.18), and (8.19), we complete the proof. □

The following lemma is the main input in the proof of Proposition 8.5.

Lemma 8.13. *There is a deterministic constant $c \in (0,1)$ such that for each $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R}$, and $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1, K_2)$, the following is true: Suppose that the event F_Z occurs. Then there is a subset $Z' \subset Z$ with $\#Z' \ge c\#Z$ such that the event $\mathfrak{F}^{\mathfrak{r}}_{Z'}(K_1, K_2)$ occurs.*

Proof. Set $Z_0 \overset{\text{def}}{=} Z$. Inductively, for each $n \in \mathbb{N}$, set

$$Z_n \overset{\text{def}}{=} \left\{ z \in Z_{n-1} : P_{Z_{n-1}} \text{ passes through the tube between either } \mathfrak{L}_{z,r} \text{ and } \gamma_{z,r} \text{ or } \mathfrak{L}_{z,r} \text{ and } \gamma'_{z,r} \right\},$$

where $P_{Z_{n-1}}$ denotes the $D_{\Upsilon_{Z_{n-1}}}$-geodesic from K_1 to K_2. It suffices to show that there exists a deterministic constant $c \in (0,1)$ such that $\#Z_n \geq c\#Z$ for all $n \in \mathbb{N}$. Since F_Z occurs, it follows immediately from the definitions that $F_{Z'}$ occurs for all subsets $Z' \subset Z$. Since Υ_{Z_n} and $\Upsilon_{Z_{n-1}}$ coincide outside of $\bigcup_{z \in Z_{n-1} \setminus Z_n} U^\star_{z,r}$, it follows that

$$\begin{aligned} D_{\Upsilon_{Z_n}}(K_1, K_2) &\leq D_{\Upsilon_{Z_{n-1}}}(K_1, K_2) \\ &+ \sum_{z \in Z_{n-1} \setminus Z_n} \sum_{(\sigma', \sigma, \tau, \tau') \in \mathcal{E}_{z,r}(P_{Z_{n-1}})} \Big(D_{\Upsilon_{Z_n}}(P_{Z_{n-1}}(\sigma), P_{Z_{n-1}}(\tau)) - D_{\Upsilon_{Z_{n-1}}}(P_{Z_{n-1}}(\sigma), P_{Z_{n-1}}(\tau)) \Big) \end{aligned} \tag{8.20}$$

for all $n \in \mathbb{N}$, where $\mathcal{E}_{z,r}(P_{Z_{n-1}})$ denotes the collection of $(B_{\Lambda r}(z), B_{4r}(z))$-excursions of $P_{Z_{n-1}}$. By Lemma 8.12, we have

$$\begin{aligned} &D_{\Upsilon_{Z_n}}(P_{Z_{n-1}}(\sigma), P_{Z_{n-1}}(\tau)) - D_{\Upsilon_{Z_{n-1}}}(P_{Z_{n-1}}(\sigma), P_{Z_{n-1}}(\tau)) \leq 100\mathsf{a}_2\mathfrak{c}_r, \\ &\qquad \forall n \in \mathbb{N},\ \forall z \in Z_{n-1} \setminus Z_n,\ \forall (\sigma', \sigma, \tau, \tau') \in \mathcal{E}_{z,r}(P_{Z_{n-1}}). \end{aligned} \tag{8.21}$$

By Lemma 8.9, we have

$$\#\mathcal{E}_{z,r}(P_{Z_{n-1}}) \leq \mathsf{A}/\mathsf{a}_1, \quad \forall n \in \mathbb{N},\ \forall z \in Z_{n-1}. \tag{8.22}$$

Combining (8.20), (8.21), and (8.22), we obtain

$$D_{\Upsilon_{Z_n}}(K_1, K_2) \leq D_{\Upsilon_{Z_{n-1}}}(K_1, K_2) + \frac{100\mathsf{A}\mathsf{a}_2}{\mathsf{a}_1}\mathfrak{c}_r(\#Z_{n-1} - \#Z_n), \quad \forall n \in \mathbb{N}.$$

Thus,

$$\begin{aligned} D_{\Upsilon_{Z_n}}(K_1, K_2) - D_{\Upsilon_Z}(K_1, K_2) &= \sum_{k=1}^n \Big(D_{\Upsilon_{Z_k}}(K_1, K_2) - D_{\Upsilon_{Z_{k-1}}}(K_1, K_2) \Big) \\ &\leq \sum_{k=1}^n \frac{100\mathsf{A}\mathsf{a}_2}{\mathsf{a}_1}\mathfrak{c}_r(\#Z_{k-1} - \#Z_k) \\ &\leq \frac{100\mathsf{A}\mathsf{a}_2}{\mathsf{a}_1}\mathfrak{c}_r\#Z, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{8.23}$$

Combining Lemma 8.8 and (8.23), we obtain

$$D_{\Upsilon_{Z_n}}(K_1, K_2) \leq D_\Upsilon(K_1, K_2) - \left(\mathsf{a}_1 - \frac{100\mathsf{A}\mathsf{a}_2}{\mathsf{a}_1} \right) \mathfrak{c}_r\#Z, \quad \forall n \in \mathbb{N}. \tag{8.24}$$

On the other hand, by Lemma 8.10, we have

$$D_\Upsilon(K_1, K_2) \leq D_{\Upsilon_{Z_n}}(K_1, K_2) + (\mathsf{A}^2/\mathsf{a}_1)\mathfrak{c}_r\#Z_n, \quad \forall n \in \mathbb{N}. \tag{8.25}$$

Combining (8.24) and (8.25), we obtain

$$\#Z_n \geq \frac{\mathsf{a}_1}{\mathsf{A}^2} \left(\mathsf{a}_1 - \frac{100\mathsf{A}\mathsf{a}_2}{\mathsf{a}_1} \right) \#Z, \quad \forall n \in \mathbb{N}.$$

(Here, we recall from the proof of Lemma 7.8 that we may have $\mathsf{a}_1 > 100\mathsf{A}\mathsf{a}_2/\mathsf{a}_1$.) This completes the proof. $\square$

Proof of Proposition 8.5. Suppose that the event $\mathfrak{G}_\varepsilon^{\mathfrak{r}}(K_1, K_2)$ occurs. It follows immediately from (1.5) and Definition 8.1, (i) that condition (i) of Definition 8.3 holds for all $r \in \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathfrak{r}, \varepsilon\mathfrak{r}]$.

Since $\operatorname{dist}(K_1, K_2) \geq \eta\mathfrak{r}$ and $\operatorname{dist}(K_1, \partial B_{2\mathfrak{r}}(0)) \geq \eta\mathfrak{r}$, there exists a subpath of P with Euclidean diameter at least $\eta\mathfrak{r}$ that is contained in $B_{2\mathfrak{r}}(0)$. Thus, by Definition 8.1, (ii), there exists a deterministic constant $\delta > 0$, depending only on η, such that there exists a set $\mathcal{S}$ of pairs (z, r) satisfying the following conditions:

(i) $\#\mathcal{S} \geq \delta/\varepsilon$.
(ii) For each $(z, r) \in \mathcal{S}$, we have $r \in \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathfrak{r}, \varepsilon\mathfrak{r}]$ and $z \in \left(\frac{1}{100}r\mathbb{Z}\right)^2 \cap B_{2\mathfrak{r}}(0)$.
(iii) For each $(z, r) \in \mathcal{S}$, conditions (ii), (iii), (iv), (v) of Definition 8.3 hold.

By the pigeonhole principle and (8.6), if $\varepsilon \leq \rho$, then there exists $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$ and $r \in \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathfrak{r}, \varepsilon\mathfrak{r}]$ such that

$$\# \left\{ z \in \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2) : (z, r) \in \mathcal{S} \right\} \geq \frac{\delta}{\varepsilon} \cdot \frac{1}{(300\Lambda)^2} \cdot \frac{1}{2\nu \log_\Lambda(1/\varepsilon)} \tag{8.26}$$

Henceforth fix $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$ and $r \in \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathfrak{r}, \varepsilon\mathfrak{r}]$ so that (8.26) holds. Write

$$\mathcal{Z}_1 \stackrel{\text{def}}{=} \{z \in \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2) : (z, r) \in \mathcal{S}\}.$$

By the above discussion, the event E_Z occurs for all subsets $Z \subset \mathcal{Z}_1$ (i.e., conditions (i), (ii), (iii), (iv), (v) of Definition 8.3 hold for all subsets $Z \subset \mathcal{Z}_1$). Write

$$\mathcal{Z}_2 \stackrel{\text{def}}{=} \{z \in \mathcal{Z}_1 : \mathsf{F}_{z,r}(\Upsilon_Z; \{i_z, j_z\}) \text{ occurs}\}.$$

Since the set $\mathcal{Z}_1$ is almost surely determined by Υ, it follows from (7.22) and (8.26) that

$$\mathbf{E}[\#\mathcal{Z}_2 \mid \Upsilon] \geq \mathsf{a}_9 \#\mathcal{Z}_1 \geq \mathsf{a}_9 \cdot \frac{\delta}{\varepsilon} \cdot \frac{1}{(300\Lambda)^2} \cdot \frac{1}{2\nu \log_\Lambda(1/\varepsilon)} \quad \text{almost surely.} \tag{8.27}$$

By definition, the event F_Z occurs for all subsets $Z \subset \mathcal{Z}_2$. Thus, by Lemma 8.13, there is a mapping $Z \mapsto Z'$ from the collection of subsets of $\mathcal{Z}_2$ to itself such that $\#Z' \geq c\#Z$ and $\mathfrak{F}_{Z'}^{\mathfrak{r}}(K_1, K_2)$ occurs for all subsets $Z \subset \mathcal{Z}_2$. Thus,

$$\begin{aligned} &\# \{Z \subset \mathcal{Z}_2 : \#Z \leq N,\ \mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2) \text{ occurs}\} \\ &\geq \# \{Z \subset \mathcal{Z}_2 : cN \leq \#Z \leq N,\ \mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2) \text{ occurs}\} \\ &\geq \#\{Z \subset \mathcal{Z}_2 : \#Z = N\} \left(\max_{\substack{Z_0 \subset \mathcal{Z}_2 \\ cN \leq \#Z_0 \leq N}} \#\{Z \subset \mathcal{Z}_2 : \#Z = N,\ Z' = Z_0\} \right)^{-1} \\ &\geq \binom{\#\mathcal{Z}_2}{N} \binom{\#\mathcal{Z}_2}{(1-c)N}^{-1} \\ &\succeq (\#\mathcal{Z}_2)^{cN}. \end{aligned} \tag{8.28}$$

Combining (8.27) and (8.28), we complete the proof. ☐

8.3. Proof of Proposition 8.6.

The proof of Proposition 8.6 relies on estimating how many points $z \in \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2)$ could belong to some set $Z \subset \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2)$ for which the event $\mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2)^\vee$ occurs. For this purpose, we introduce the following definition.

Let $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$ and $r \in \rho^{-1}\mathcal{R}$. Then we shall refer to a point $z \in \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2)$ as $(\mathbf{e}, r)$-*good* if the following are true:

(A) The event $\mathsf{E}_{z,r}(\Upsilon_{z,r})$ occurs.

(B) There are at most two connected arcs of $\Gamma_{z,r}$ in $A_{5r,\Lambda r}(z)$ connecting its inner and outer boundaries.
(C) There exists a path P_z in $A_{5r,\Lambda r}(z) \cap \Upsilon_{z,r}$ of $D_{\Upsilon_{z,r}}$-length at most $\mathsf{A}\mathfrak{c}_r$ such that every path in $\Upsilon_{z,r}$ that crosses between the inner and outer boundaries of $A_{5r,\Lambda r}(z)$ intersects P_z.
(D) The event $\mathsf{F}_{z,r}(\Upsilon; \{i_z, j_z\})$ occurs for some distinct $i_z, j_z \in [1, \#Z_{z,r}]_{\mathbb{Z}}$.
(E) The D_Υ-geodesic P passes through the tube between either $\mathscr{L}_{z,r}$ and $\gamma_{z,r}$ or $\mathscr{L}_{z,r}$ and $\gamma'_{z,r}$.

Lemma 8.14. *Let* $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R}$, *and* $Z \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1, K_2)$. *Suppose that* $\mathfrak{F}^{\mathfrak{r}}_Z(K_1, K_2)^\vee$ *occurs. Then each* $z \in Z$ *is* $(\mathbf{e}, r)$*-good.*

Proof. We observe that conditions (A), (B), (C), (D), (E) correspond to conditions (ii), (iii), (iv), (vi), (vii) of Definition 8.4, respectively. Moreover, conditions (A), (B), (C), (D), (E) are almost surely determined by Υ and $B_{\Lambda r}(z) \cap \Upsilon_{z,r}$; conditions (ii), (iii), (iv), (vi), (vii) in Definition 8.4 are almost surely determined by Υ and $B_{\Lambda r}(z) \cap \Upsilon_Z$. Thus, Lemma 8.14 follows immediately from the fact that $B_{\Lambda r}(z) \cap \Upsilon_{z,r} = B_{\Lambda r}(z) \cap \Upsilon_Z$ almost surely. □

The next lemma provides the key ingredient used in the proof of Proposition 8.6.

Lemma 8.15. *There exists a deterministic constant* $C > 0$ *such that for each* $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$ *and* $r \in \rho^{-1}\mathcal{R}$, *the following is true: Suppose that there are disjoint subsets* $Z, Z' \subset \mathbb{Z}^{\mathfrak{r}}_{\mathbf{e},r}(K_1, K_2)$ *such that* $\mathfrak{F}^{\mathfrak{r}}_Z(K_1, K_2)^\vee$ *occurs and each* $z' \in Z'$ *is* $(\mathbf{e}, r)$*-good. Then* $\#Z' \le C\#Z$.

The idea of the proof of Lemma 8.15 is as follows. Let P be a D_Υ-geodesic from K_1 to K_2. By Lemma 7.13 (with the roles of Υ and $\Upsilon_{z,r}$ interchanged) together with condition (E), the portion of P spent inside the Euclidean balls centered at points $z' \in Z'$ is forced to run near certain "shortcuts" so that the $\widetilde{D}_\Upsilon$-length of this portion falls short of M^* times its D_Υ-length by at least a deterministic positive constant times $\#Z'$. Since Z and Z' are disjoint, Υ and Υ_Z agree on the balls centered at $z' \in Z'$. Hence the same conclusion applies to the segments of P that lie in Υ_Z (i.e., outside the balls centered at $z \in Z$). We now replace the segments of P lying inside the balls centered at $z \in Z$ by the loops provided by condition (iv) of the event $\mathfrak{F}^{\mathfrak{r}}_Z(K_1, K_2)^\vee$, producing a path entirely contained in Υ_Z. A parallel estimate shows that the $\widetilde{D}_{\Upsilon_Z}$-length of this modified path exceeds the $\widetilde{D}_\Upsilon$-length of P by at most a deterministic positive constant multiple of $\#Z$. Moreover, $D_\Upsilon(K_1, K_2) \le D_{\Upsilon_Z}(K_1, K_2)$ (indeed, by the same reasoning as in Lemma 8.8, $D_\Upsilon(K_1, K_2)$ is at most $D_{\Upsilon_Z}(K_1, K_2)$ minus a deterministic positive constant times $\#Z$). Combining these observations yields

$$\widetilde{D}_{\Upsilon_Z}(K_1, K_2) \le M^* D_{\Upsilon_Z}(K_1, K_2) + \mathrm{const} \cdot \#Z - \mathrm{const} \cdot \#Z'.$$

Together with condition (i) of $\mathfrak{F}^{\mathfrak{r}}_{Z_0}(K_1, K_2)^\vee$, this implies that $\#Z'$ is at most a constant multiple of $\#Z$.

Proof of Lemma 8.15. By Lemma 7.13 (applied with the roles of Υ and $\Upsilon_{z,r}$ interchanged) and condition (E), for each $z' \in Z'$, there are times $\sigma_{z'} < \tau_{z'}$ such that

$$P([\sigma_{z'}, \tau_{z'}]) \subset B_{4r}(z) \cap \Upsilon, \quad D_\Upsilon(P(\sigma_{z'}), P(\tau_{z'})) \ge \mathsf{b}\mathfrak{c}_r, \quad \text{and} \quad \widetilde{D}_\Upsilon(P(\sigma_{z'}), P(\tau_{z'})) \le M_2 D_\Upsilon(P(\sigma_{z'}), P(\tau_{z'})). \tag{8.29}$$

On the other hand, by a similar argument to the argument applied in the proof of Lemma 8.10, for each $z \in Z$ and $(\sigma', \sigma, \tau, \tau') \in \mathcal{E}_{z,r}(P)$, there are times $\sigma'' \in (\sigma', \sigma)$ and $\tau'' \in (\tau, \tau')$ such that

$$\sum_{z \in Z} \sum_{(\sigma', \sigma, \tau, \tau') \in \mathcal{E}_{z,r}(P)} D_{\Upsilon_Z}(P(\sigma''), P(\tau'')) \le (\mathsf{A}^2/\mathsf{a}_1)\mathfrak{c}_r \#Z, \tag{8.30}$$

where $\mathcal{E}_{z,r}(P)$ denotes the collection of $(B_{\Lambda r}(z), B_{4r}(z))$-excursions of P. Since Z and Z' are disjoint, it follows that

$$[\sigma'', \tau''] \cap [\sigma_{z'}, \tau_{z'}] = \emptyset, \quad \forall z \in Z,\ \forall (\sigma', \sigma, \tau, \tau') \in \mathcal{E}_Z,\ \forall z' \in Z'.$$

Write

$$I \stackrel{\text{def}}{=} [0,1] \setminus \left(\bigcup_{z \in Z} \bigcup_{(\sigma', \sigma, \tau, \tau') \in \mathcal{E}_Z} [\sigma'', \tau''] \cup \bigcup_{z' \in Z'} [\sigma_{z'}, \tau_{z'}] \right).$$

Combining (8.29) and (8.30), we obtain that

$$\begin{aligned}
(8.31) \quad & \widetilde{D}_{\Upsilon_Z}(K_1, K_2) \\
& \leq \sum_{z \in Z} \sum_{(\sigma', \sigma, \tau, \tau') \in \mathcal{E}_{z,r}(P)} \widetilde{D}_{\Upsilon_Z}(P(\sigma''), P(\tau'')) + \sum_{z' \in Z'} \widetilde{D}_{\Upsilon_Z}(P(\sigma_{z'}), P(\tau_{z'})) + \operatorname{len}(P|_I, \widetilde{D}_{\Upsilon_Z}) \\
& = \sum_{z \in Z} \sum_{(\sigma', \sigma, \tau, \tau') \in \mathcal{E}_{z,r}(P)} \widetilde{D}_{\Upsilon_Z}(P(\sigma''), P(\tau'')) + \sum_{z' \in Z'} \widetilde{D}_{\Upsilon}(P(\sigma_{z'}), P(\tau_{z'})) + \operatorname{len}(P|_I, \widetilde{D}_{\Upsilon}) \\
& \leq M^* \sum_{z \in Z} \sum_{(\sigma', \sigma, \tau, \tau') \in \mathcal{E}_{z,r}(P)} D_{\Upsilon_Z}(P(\sigma''), P(\tau'')) + M_2 \sum_{z' \in Z'} D_{\Upsilon}(P(\sigma_{z'}), P(\tau_{z'})) \\
& + M^* \operatorname{len}(P|_I, D_{\Upsilon}) \\
& \leq M^* (\mathsf{A}^2/\mathsf{a}_1) \mathfrak{c}_r \# Z + M_2 \sum_{z' \in Z'} D_{\Upsilon}(P(\sigma_{z'}), P(\tau_{z'})) + M^* \operatorname{len}(P|_I, D_{\Upsilon}) \\
& \leq M^* (\mathsf{A}^2/\mathsf{a}_1) \mathfrak{c}_r \# Z + M^* D_{\Upsilon}(K_1, K_2) - (M^* - M_2) \sum_{z' \in Z'} D_{\Upsilon}(P(\sigma_{z'}), P(\tau_{z'})) \\
& \leq M^* (\mathsf{A}^2/\mathsf{a}_1) \mathfrak{c}_r \# Z + M^* D_{\Upsilon}(K_1, K_2) - (M^* - M_2) \mathsf{b} \mathfrak{c}_r \# Z'.
\end{aligned}$$

Since $D_{\Upsilon}(K_1, K_2) \leq D_{\Upsilon_Z}(K_1, K_2)$ on the event $\mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2)^{\vee}$ (cf., e.g., Lemma 8.8 (applied with the roles of Υ and Υ_Z interchanged)), (8.31) implies that

$$(8.32) \qquad \widetilde{D}_{\Upsilon_Z}(K_1, K_2) \leq M^* (\mathsf{A}^2/\mathsf{a}_1) \mathfrak{c}_r \# Z + M^* D_{\Upsilon_Z}(K_1, K_2) - (M^* - M_2) \mathsf{b} \mathfrak{c}_r \# Z'.$$

Combining Definition 8.4, (i) and (8.32), we obtain

$$(M^* - M_2) \mathsf{b} \mathfrak{c}_r \# Z' \leq M^* (\mathsf{A}^2/\mathsf{a}_1) \mathfrak{c}_r \# Z + \mathfrak{c}_r \leq \left(M^* \mathsf{A}^2/\mathsf{a}_1 + 1 \right) \mathfrak{c}_r \# Z.$$

This completes the proof. □

Proof of Proposition 8.6. Fix $\mathbf{e} \in (\mathbb{Z}/(300\Lambda\mathbb{Z}))^2$, $r \in \rho^{-1}\mathcal{R}$, and $N \in \mathbb{N}$. Choose $Z_0 \subset \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2)$ with $\# Z_0 \leq N$ such that $\mathfrak{F}_{Z_0}^{\mathfrak{r}}(K_1, K_2)^{\vee}$ occurs. (If such Z_0 does not exist, then (8.3) holds trivially.) Write

$$Z_0' \stackrel{\text{def}}{=} \left\{ z \in \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2) \setminus Z_0 : z \text{ is } (\mathbf{e}, r)\text{-good} \right\}.$$

Then, by Lemma 8.15, we have $\# Z_0' \leq C \# Z_0$. Let $Z \subset \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2)$ such that $\mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2)^{\vee}$ occurs. Then, by Lemma 8.14, each $z \in Z$ is $(\mathbf{e}, r)$-good. Thus, by definition, we must have $Z \subset Z_0 \cup Z_0'$. Thus,

$$\# \left\{ Z \subset \mathbb{Z}_{\mathbf{e},r}^{\mathfrak{r}}(K_1, K_2) : \# Z \leq N \text{ and } \mathfrak{F}_Z^{\mathfrak{r}}(K_1, K_2)^{\vee} \text{ occurs} \right\} \leq 2^{\# Z_0 + \# Z_0'} \leq 2^{(1+C)N}.$$

This completes the proof. □

8.4. **Proof of Theorem 1.9.** In the present subsection, we establish Theorem 1.9. It is enough to prove that $M_* = M^*$. To this end, we argue by contradiction and assume that $M_* < M^*$. Using Proposition 8.2, we will then derive a contradiction with Propositions 6.6 and 6.8. As a first step, we obtain a lower bound for $\#(\rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}])$ under the assumption that $\mathbf{P}[\underline{G}_{\mathbb{r}}(M_1', b)] \geq b$.

Lemma 8.16. *There exists $M_1' = M_1'(\mu, \nu, \Lambda, \alpha) \in (M_*, M_1)$ such that for each $b \in (0,1)$, there exists $\varepsilon_* = \varepsilon_*(\mu, \nu, \Lambda, \alpha, b, \rho) \in (0,1)$ such that for each $\mathbb{r} > 0$ for which $\mathbf{P}[\underline{G}_{\mathbb{r}}(M_1', b)] \geq b$, we have*

$$\#\left(\rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]\right) \geq \mu \log_\Lambda(1/\varepsilon), \quad \forall \varepsilon \in (0, \varepsilon_*].$$

Proof. Write $\mu' \stackrel{\text{def}}{=} (\mu+\nu)/2$. By Proposition 6.7, there exists $M_1' = M_1'(\mu, \nu, \Lambda, \alpha) \in (M_*, M_1)$ such that for each $b \in (0,1)$, there exists $\varepsilon_0 = \varepsilon_0(\mu, \nu, \Lambda, \alpha, b) \in (0,1)$ such that for each $\mathbb{r} > 0$ for which $\mathbf{P}[\underline{G}_{\mathbb{r}}(M_1', b)] \geq b$, we have

$$\#\left(\mathcal{R} \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]\right) \geq \mu' \log_\Lambda(1/\varepsilon), \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{8.33}$$

Since $\mathcal{R} \subset \{\Lambda^k\}_{k\in\mathbb{Z}}$, we conclude from (8.33) that

$$\begin{aligned}\#\left(\rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]\right) &= \#\left(\mathcal{R} \cap [\varepsilon^{1+\nu}\rho\mathbb{r}, \varepsilon\rho\mathbb{r}]\right) \\ &= \#\left(\mathcal{R} \cap [(\varepsilon\rho)^{1+\nu}\mathbb{r}, \varepsilon\rho\mathbb{r}]\right) - \#\left(\mathcal{R} \cap [(\varepsilon\rho)^{1+\nu}\mathbb{r}, \varepsilon^{1+\nu}\rho\mathbb{r}]\right) \\ &\geq \mu' \log_\Lambda(1/(\varepsilon\rho)) - \nu \log_\Lambda(1/\rho) \\ &\geq \mu \log_\Lambda(1/\varepsilon), \quad \forall \varepsilon \in (0, \varepsilon_0 \wedge \rho].\end{aligned}$$

This completes the proof. □

In the remainder of the present subsection, let M_1' be as in Lemma 8.16. We now confirm that the auxiliary condition (ii) in the definition of the event $\mathfrak{G}_\varepsilon^{\mathbb{r}}(K_1, K_2)$ holds with high probability for sufficiently small ε.

Lemma 8.17. *Let $\alpha_{4\mathrm{A}}$ be as in (2.2). Then we may choose the parameters of (7.1) in such a way that*

$$\alpha_{4\mathrm{A}}\mu > 2(1+\nu), \tag{8.34}$$

and the following is true: Let $b \in (0,1)$ and $\mathbb{r} > 0$. Suppose that $\mathbf{P}[\underline{G}_{\mathbb{r}}(M_1', b)] \geq b$. Then it holds with probability tending to one as $\varepsilon \to 0$ (at a rate which is uniform in $\mathbb{r}$) that condition (ii) in Definition 8.1 is true, i.e., for each $x \in B_{2\mathbb{r}}(0)$, there exists $r \in \rho^{-1}\mathcal{R} \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]$ and $z \in \left(\frac{1}{100}r\mathbb{Z}\right)^2 \cap B_{2\mathbb{r}}(0)$ such that $|x - z| \leq r$ and the following are true:

(i) $\mathsf{E}_{z,r}(\Upsilon)$ occurs.
(ii) There are at most two connected arcs of Γ in $A_{5r,\Lambda r}(z)$ connecting its inner and outer boundaries.
(iii) There exists a path P_z in $A_{5r,\Lambda r}(z) \cap \Upsilon$ of D_Υ-length at most $\mathsf{A}\mathfrak{c}_r$ such that every path in Υ that crosses between the inner and outer boundaries of $A_{5r,\Lambda r}(z)$ intersects P_z.

Proof. This follows immediately from Lemmas 2.5, 2.16 and 8.16, Lemma 8.16, and a union bound. □

In the remainder of the present subsection, let the parameters of (7.1) be as in Lemma 8.17. By combining Proposition 8.2 and Lemma 8.17, we obtain the following.

Lemma 8.18. *Let $b,\eta \in (0,1)$ and $\omega, \mathbbm{r} > 0$. Suppose that $\mathbf{P}[\underline{G}_{\mathbbm{r}}(M_1', b)] \ge b$. Then it holds with probability tending to one as $\varepsilon \to 0$ (at a rate which is uniform in $\mathbbm{r}$) that*

$$\widetilde{D}_\Upsilon(B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}) \cap \Upsilon, B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{y}) \cap \Upsilon) \le M^* D_\Upsilon(B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}) \cap \Upsilon, B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{y}) \cap \Upsilon) - \varepsilon^{2(1+\nu)} \mathfrak{c}_{\mathbbm{r}},$$
$$\forall \mathbbm{x}, \mathbbm{y} \in \left(\frac{1}{100}\varepsilon^\omega \mathbbm{r}\mathbb{Z}\right)^2 \cap B_{2\mathbbm{r}}(0) \text{ such that } |\mathbbm{x}-\mathbbm{y}| \ge \eta \mathbbm{r} \text{ and } B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}) \overset{\Upsilon}{\longleftrightarrow} B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}).$$

Proof. By applying a union bound, it follows from Proposition 8.2 that it holds with probability tending to one as $\varepsilon \to 0$ that the event $\mathfrak{G}_\varepsilon^{\mathbbm{r}}(B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}), B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{y}))$ does not occur for all $\mathbbm{x}, \mathbbm{y} \in \left(\frac{1}{100}\varepsilon^\omega \mathbbm{r}\mathbb{Z}\right)^2 \cap B_{2\mathbbm{r}}(0)$ with $|\mathbbm{x}-\mathbbm{y}| \ge \eta\mathbbm{r}$. Combining this with Lemma 8.17, we obtain that it holds with probability tending to one as $\varepsilon \to 0$ that condition (i) of the definition of $\mathfrak{G}_\varepsilon^{\mathbbm{r}}(B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}), B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{y}))$ does not occur for all $\mathbbm{x}, \mathbbm{y} \in \left(\frac{1}{100}\varepsilon^\omega \mathbbm{r}\mathbb{Z}\right)^2 \cap B_{2\mathbbm{r}}(0)$ with $|\mathbbm{x}-\mathbbm{y}| \ge \eta\mathbbm{r}$. This completes the proof. ☐

Recall the events $\overline{G}_{\mathbbm{r}}(M,b)$ and $\underline{G}_{\mathbbm{r}}(M,b)$ from Definition 6.1. Combining Lemma 8.18 with a few geometric considerations, we derive the following result.

Lemma 8.19. *Let $b, b' \in (0,1)$ and $\mathbbm{r} > 0$. Suppose that $\mathbf{P}[\underline{G}_{\mathbbm{r}}(M_1', b)] \ge b$. Then*

$$\lim_{\delta \to 0} \mathbf{P}\left[\overline{G}_{\mathbbm{r}}(M^* - \delta, b')\right] = 0$$

at a rate which is uniform in $\mathbbm{r}$.

Proof. Fix $\chi \in (0, 1 - 2/\alpha_{4\mathrm{A}})$ and $\omega > 2(1+\nu)/\chi$. By Lemma 8.18, it holds with probability tending to one as $\varepsilon \to 0$ (at a rate which is uniform in $\mathbbm{r}$) that

$$\widetilde{D}_\Upsilon(B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}) \cap \Upsilon, B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{y}) \cap \Upsilon) \le M^* D_\Upsilon(B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}) \cap \Upsilon, B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{y}) \cap \Upsilon) - \varepsilon^{2(1+\nu)} \mathfrak{c}_{\mathbbm{r}},$$
$$\forall \mathbbm{x}, \mathbbm{y} \in \left(\frac{1}{100}\varepsilon^\omega \mathbbm{r}\mathbb{Z}\right)^2 \cap B_{2\mathbbm{r}}(0) \text{ such that } |\mathbbm{x}-\mathbbm{y}| \ge \frac{1}{2}b'\mathbbm{r} \text{ and } B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}) \overset{\Upsilon}{\longleftrightarrow} B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x}). \tag{8.35}$$

By Lemma 2.26, it holds with probability tending to one as $\varepsilon \to 0$ (at a rate which is uniform in $\mathbbm{r}$) that

$$\mathfrak{c}_{\mathbbm{r}}^{-1} D_\Upsilon(x,y) \le \left|\frac{x-y}{\mathbbm{r}}\right|^\chi, \quad \forall x, y \in B_{2\mathbbm{r}}(0) \cap \Upsilon \text{ with } x \overset{\Upsilon}{\longleftrightarrow} y \text{ and } |x-y| \le \varepsilon\mathbbm{r}. \tag{8.36}$$

Since $\omega > 1+\nu$ (hence $\alpha_{4\mathrm{A}}(\omega-1) > 2\omega$ (cf. (8.34))), by Lemma 2.5, (i) and a union bound, it holds with probability tending to one as $\varepsilon \to 0$ (at a rate which is uniform in $\mathbbm{r}$) that

(8.37) for each $\mathbbm{x} \in \left(\frac{1}{100}\varepsilon^\omega \mathbbm{r}\mathbb{Z}\right)^2 \cap B_{2\mathbbm{r}}(0)$, there are at most two connected arcs of Γ in $A_{\varepsilon^\omega \mathbbm{r}, \varepsilon\mathbbm{r}}(\mathbbm{x})$ connecting $\partial B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x})$ and $\partial B_{\varepsilon\mathbbm{r}}(\mathbbm{x})$.

By Definition 1.7, Axiom (IV) (tightness across scales), it holds with probability tending to one as $\varepsilon \to 0$ (at a rate which is uniform in $\mathbbm{r}$) that

$$D_\Upsilon(x,y) \le \varepsilon^{-1}\mathfrak{c}_{\mathbbm{r}}, \quad \forall x, y \in B_{2\mathbbm{r}}(0) \cap \Upsilon \text{ with } x \overset{\Upsilon}{\longleftrightarrow} y. \tag{8.38}$$

Henceforth assume that ε is sufficiently small and (8.35), (8.36), (8.37), (8.38) hold. Let $x, y \in \overline{B_{\mathbbm{r}}(0)} \cap \Upsilon$ with $x \overset{\Upsilon}{\longleftrightarrow} y$ and $|x-y| \ge b'\mathbbm{r}$. Choose $\mathbbm{x}, \mathbbm{y} \in \left(\frac{1}{100}\varepsilon^\omega \mathbbm{r}\mathbb{Z}\right)^2 \cap B_{2\mathbbm{r}}(0)$ such that $x \in B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x})$ and $y \in B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{y})$.

We *claim* that for any $x' \in B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x})$ and $y' \in B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{y})$ such that $x' \overset{\Upsilon}{\longleftrightarrow} y'$, we have $x \overset{\Upsilon}{\longleftrightarrow} x'$ and $y \overset{\Upsilon}{\longleftrightarrow} y'$. Indeed, if the connected component of Υ containing x and y is not the same as the connected component of Υ containing x' and y', then both of these two connected components intersect $\partial B_{\varepsilon^\omega \mathbbm{r}}(\mathbbm{x})$ and $\partial B_{\varepsilon\mathbbm{r}}(\mathbbm{x})$. This implies that there are at least four connected arcs of Γ in

$A_{\varepsilon^\omega \mathfrak{r},\varepsilon\mathfrak{r}}(\mathbb{x})$ connecting $\partial B_{\varepsilon^\omega \mathfrak{r}}(\mathbb{x})$ and $\partial B_{\varepsilon\mathfrak{r}}(\mathbb{x})$, in contradiction to (8.37). This completes the proof of the *claim*.

By (8.36) and the above *claim*, we conclude that

$$|D_\Upsilon(B_{\varepsilon^\omega \mathfrak{r}}(\mathbb{x}) \cap \Upsilon, B_{\varepsilon^\omega \mathfrak{r}}(\mathbb{y}) \cap \Upsilon) - D_\Upsilon(x,y)| \le 2\varepsilon^{\chi\omega}\mathfrak{c}_\mathfrak{r}. \tag{8.39}$$

Combining (8.35) and (8.39), we obtain that

$$\widetilde{D}_\Upsilon(x,y) \le M^* D_\Upsilon(x,y) - \varepsilon^{2(1+\nu)}\mathfrak{c}_\mathfrak{r} - 2\varepsilon^{\chi\omega}\mathfrak{c}_\mathfrak{r}.$$

Since $\chi\omega > 2(1+\nu)$, it follows that

$$\widetilde{D}_\Upsilon(x,y) \le M^* D_\Upsilon(x,y) - 2\varepsilon^{2(1+\nu)}\mathfrak{c}_\mathfrak{r} \tag{8.40}$$

whenever ε is sufficiently small. Combining (8.38) and (8.40), we obtain that

$$\widetilde{D}_\Upsilon(x,y) \le \left(M^* - 2\varepsilon^{2(1+\nu)+1}\right) D_\Upsilon(x,y).$$

Thus, we conclude that, if ε is sufficiently small and (8.35), (8.36), (8.37), (8.38) are true, then the event $\overline{G}_\mathfrak{r}(M^* - 2\varepsilon^{2(1+\nu)+1}, b')$ does not occur. This completes the proof. □

Proof of Theorem 1.9. It suffices to show that $M_* = M^*$. Suppose by way of contradiction that $M_* < M^*$. By applying Proposition 6.8 with M_1' in place of M and 1 in place of $\mathfrak{r}$, there exists $\underline{b} \in (0,1)$ and $\varepsilon_0 \in (0,1)$ such that for each $\varepsilon \in (0, \varepsilon_0]$,

(8.41) there are at least $\mu \log_\Lambda(1/\varepsilon)$ values of $\mathfrak{r} \in [\varepsilon^{1+\nu}, \varepsilon] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ for which $\mathbf{P}[\underline{G}_\mathfrak{r}(M, \underline{b})] \ge \underline{b}$.

Let $\overline{b} = b(\mu, \nu, \Lambda)$ be as in Proposition 6.6. By applying Lemma 8.19 with $\underline{b}$ in place of b and $\overline{b}$ in place of b', we conclude from (8.41) that there exists $\delta \in (0, M^*)$ such that for each $\varepsilon \in (0, \varepsilon_0]$,

(8.42) there are at least $\mu \log_\Lambda(1/\varepsilon)$ values of $\mathfrak{r} \in [\varepsilon^{1+\nu}, \varepsilon] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ for which $\mathbf{P}[\overline{G}_\mathfrak{r}(M^* - \delta, \overline{b})] < \overline{b}$.

On the other hand, by applying Proposition 6.6 with $M^* - \delta$ in place of M and 1 in place of $\mathfrak{r}$, there exists $\varepsilon_1 \in (0,1)$ such that for each $\varepsilon \in (0, \varepsilon_1]$,

(8.43) there are at least $\mu \log_\Lambda(1/\varepsilon)$ values of $\mathfrak{r} \in [\varepsilon^{1+\nu}, \varepsilon] \cap \{\Lambda^k\}_{k\in\mathbb{Z}}$ for which $\mathbf{P}[\overline{G}_\mathfrak{r}(M^* - \delta, \overline{b})] \ge \overline{b}$.

Since $\mu > \nu/2$ (cf. (8.34)), (8.42) and (8.43) contradict each other. This completes the proof. □

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