A NOVEL WAY OF COMPUTING THE SHAPE DERIVATIVE FOR A CLASS OF NON-SMOOTH PDES AND ITS IMPACT ON DERIVING NECESSARY CONDITIONS FOR LOCALLY OPTIMAL SHAPES

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Abstract. We derive necessary conditions for locally optimal shapes of a design problem governed by a non-smooth PDE. The main particularity of the state system is the lack of differentiability of the nonlinearity. We work in the framework of the functional variational approach (FVA) [39], which has the capacity to transfer geometric optimization problems into optimal control problems, the set of admissible shapes being parametrized by a large class of continuous mappings. In the FVA setting, we introduce a sensitivity analysis technique that is novel even for smooth PDEs. We emphasize that we do not resort to extensions on the hold-all domain or any kind of approximation of the original PDE. The computation of the directional derivative of the state w.r.t. functional variations results in a new way of computing the shape derivative. The presented approach allows us to handle in the objective pointwise observation and derivatives of the state on an observation set as well as distributed observation terms. In addition, we introduce the concept of locally optimal shapes and we put into evidence its connection to locally minimizers of the corresponding control problem. With directional differentiability results for the control-to-state map at our disposal, we can then state necessary conditions for locally optimal shapes in general non-smooth settings.

Key words. Optimal control of non-smooth PDEs, shape and topology optimization, functional variational approach, Hamiltonian systems, pointwise observation, distributed observation, shape derivative, differential geometry.

AMS subject classifications. 49Q10, 35Q93, 49Q12, 49K20.

1. Introduction.

Shape and topology optimization problems have been extensively addressed in the past decades by different types of methods, see e.g. the classical contributions [2, 7, 12, 30, 32, 36]. Given a hold-all domain D, an optimal design problem, defined over a given family $\mathcal{O}$ of subdomains of D may have the following structure

$$\begin{aligned} \min_{\Omega \in \mathcal{O}} \quad & \mathcal{J}(\Omega, y_\Omega) \\ \text{s.t.} \quad & y_\Omega \text{ solves } (Q_\Omega), \end{aligned} \tag{1.1}$$

where $\mathcal{J}$ is a given shape functional, and (Q_Ω) is the state system (typically a PDE or a VI) posed in Ω with solution y_Ω (the state). As one is interested in finding information about the optimal shape, there is a zoo of works addressing the sensitivity analysis of $\Omega \mapsto \mathcal{J}(\Omega, y_\Omega)$ see e.g. [11, 15–18, 28, 34, 35, 37] and the references therein. The most common differentiability notions appearing in all these papers are based on variations of the geometry.

A meanwhile well-established strategy for handling shape and topology optimization problems is the so-called functional variational approach (FVA), which was first introduced in [20]. This technique will be employed in the present paper too. One of the essential aspects about the FVA is that shape perturbations do not have a prescribed geometric form, as it is usual in the literature. Hence, there is no reason to work with differentiability concepts such as shape/topology derivative [30, 36]. This methodology has proven its viability in numerous contributions that deal with various optimal

*Faculty of Mathematics, University of Würzburg, Germany design problems [10,21–27,39,40]. We refer here also to the recents works [4,5] where convex approximation schemes and optimality systems for optimal design problems governed by non-smooth PDEs were provided. As a possible drawback of the FVA we mention its limitation to dimension two (which stems from the fact that it uses a Hamiltonian technique based on the Poincaré-Bendixson theory [14, Ch. 10]). We point to [38] where some hints on the possible relaxation of the dimensional restriction are given.

In the context of the FVA, the boundaries of the domains appear as zero level sets of certain shape functions defined over a larger fixed domain. This idea is mostly associated with the prominent level set method (LSM) [31, 42]. However, the last-mentioned approach is concerned with the time evolution of curves and/or surfaces and we note that LSMs are mainly employed for numerical purposes; cf. also [3, 19] for rigorous mathematical investigations. While our admissible shapes shall arise as sublevel sets as well, the FVA is essentially different from the LSM, as it is strongly based on an optimal control framework, where the level set function is the control.

The FVA allows one to switch from the shape optimization problem to a control problem in a function space. The advantage of the latter is that it is more amenable to the application of tools from optimal control theory. The new admissible set consists of shape functions (also termed level functions or parametrizations). These belong to a certain family $\mathcal{F}$ that has the capacity to generate all admissible domains $\Omega \in \mathcal{O}$ [4, Prop. 2.8], [7, Thm. 4.1 and 4.2].

If $\mathcal{O}$ is the set of all non-empty $\mathcal{C}^2$ subdomains of $D \subset \mathbb{R}^2$ whose boundaries do not intersect ∂D, this transfer takes place via the relation

$$\Omega = \Omega_g := \{x \in D : g(x) < 0\}, \quad g \in \mathcal{F},$$

where $\mathcal{F}$ is the set of shape functions defined as

$$\mathcal{F} := \{g \in C^2(\bar{D}) : |\nabla g(x)| + |g(x)| > 0 \,\forall\, x \in D,\ g(x) > 0 \,\forall\, x \in \partial D,\ g^{-1}((-\infty, 0]) \neq \emptyset\}. \tag{1.2}$$

Note that, given $g \in \mathcal{F}$, the $\mathcal{C}^2$ set Ω_g may have many components (their number is finite by [39, Prop. 2]), but it is always possible to redefine g so that Ω_g is a domain [4, Lem. 2.9] (in section 4 we select a component by asking that $E \subset \Omega_g$, where E is an observation subdomain). However, the boundary of this particular domain Ω_g may be a union of several closed curves, that is, Ω_g may have holes. Hence, the FVA is related to topological optimization too and we want to underline this aspect.

The *main aim of this paper* is to introduce a direct approach for differentiating solutions of *non-smooth* PDEs with respect to perturbations of $\mathcal{C}^2$ planar domains in the context of the FVA. We focus on the following Dirichlet problem:

$$\begin{aligned} -\triangle y + \beta(y) &= f \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where β is a not necessarily differentiable function, cf. Assumption 3.1 for details. As already mentioned, $D \subset \mathbb{R}^2$ is a fixed bounded hold-all domain. For the right-hand side of the PDE we assume $f \in L^p(D),\ p > 2$, which ensures the maximal regularity of the state $y \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$, since $\Omega \in \mathcal{C}^2$.

Given $g \in \mathcal{F}$, we consider so-called *functional variations* $g + \lambda h$, $\lambda > 0$ "small", where $h \in \mathcal{F}$ is fixed [20]. That is, the perturbation of the domain $\Omega = \Omega_g$ is $\Omega_{g+\lambda h}$. We are thus concerned with the investigation of

$$\lim_{\lambda \searrow 0} \frac{\mathcal{S}(g + \lambda h) - \mathcal{S}(g)}{\lambda}, \tag{1.4}$$

where $\mathcal{S} : g \mapsto y_g$ is the solution operator of (1.3) with $\Omega = \Omega_g$. Because of the non-smooth structure of (1.3), $\mathcal{S}$ is not expected to be Gâteaux-differentiable and we remark here that the sensitivity analysis of non-smooth problems on perturbed domains has been addressed in the literature only for VIs so far [1,6,11,13,28,34,36].

We point out that, to the best of our knowledge, the procedure in the present manuscript is entirely novel even for smooth problems. A less complex situation for $\mathcal{C}^\infty$ domains in arbitrary dimension was discussed in [19, Sec. 3]. There the connection between the shape derivative of $\mathcal{J}$ and the derivative of $g \mapsto \mathcal{J}(\Omega_g)$ was established, but only for shape functionals that *do not* depend on the state. We emphasize that once we compute and analyse (1.4), we are in the position to calculate the directional derivative of $g \mapsto \mathcal{J}(\Omega_g, y_g)$ and since we deal with a non-smooth PDE, the word "directional" is essential here.

When examining (1.4), the first question that will arise is how does the directional derivative of $\mathcal{S}$ look like. It turns out that the limit in (1.4) yields in fact the shape derivative of y_{Ω_g}, see Remark 3.10. Another fundamental question is in which space does the convergence (1.4) hold, the challenge here being that $\mathcal{S}(g)$ maps to $W^{2,p}(\Omega_g) \cap H_0^1(\Omega_g)$, which changes when perturbing the domain. Simply extending $\mathcal{S}(g)$ by zero is not a solution, as the $W^{2,p}$ regularity of the state gets lost outside Ω_g. We underline that this regularity is important as it ensures that ∇y_g is continuous and the boundary condition in the 'linearized' PDE (3.25)-(3.26) is well defined. To tackle these issues we shall inspect the convergence of the difference quotient in (1.4) mainly in $\Omega_g \cap \Omega_{g+\lambda h}$ and in particular on the boundary of $\Omega_g \cap \Omega_{g+\lambda h}$, since it turns out that this dominates the convergence in the interior of the common domain (Proposition 3.14). Our key tools to begin with are the representation of $\mathcal{C}^2$ planar closed curves via Hamiltonian systems [39], their approximating behaviour [5] and differentiability properties [40]. When we go deep in the analysis we see that an essential point is to observe that a certain part of the boundary of $\Omega_g \cap \Omega_{g+\lambda h}$ remains fixed with varying λ, see (3.40) and figure 3.2. We emphasize that the $W^{2,p}$ regularity of the state as well as the stability of the boundary perturbations $\partial\Omega_{g+\lambda h}$ play at different stages of the manuscript fundamental roles.

The first two main results of the manuscript are Theorems 3.18 and 3.20, where we establish two different types of differentiability results for $\mathcal{S}$. These will allow us to consider in the objective pointwise observation and derivatives of the state on an observation set (due to Theorem 3.18, cf. subsection 5.1) as well as distributed observation terms (thanks to Theorem 3.20, see subsection 5.2).

A second key contribution of this paper is the introduction of the notion of *locally optimal domain* and the derivation of necessary optimality conditions therefor. This concept shall be presented in the framework of the optimal design problem

$$\left.\begin{aligned} \min_{\Omega \in \mathcal{O}, E \subset\subset \Omega} \quad & \mathcal{J}(\Omega, y_\Omega) \\ \text{s.t.} \quad & (1.3) \end{aligned}\right\} \tag{P_Ω}$$

where the observation set $E \subset\subset D$ is a non-empty subdomain of D. The problem (P_Ω) can be rewritten as an optimal control problem in function space, which in its reduced from reads

$$\min_{g \in \mathcal{F}, g<0 \text{ in } \bar{E}} \mathcal{J}(\Omega_g, y_g). \tag{1.5}$$

We show that for each locally optimal shape $\Omega = \Omega_g$ of (P_Ω), all associated parametrizations $g \in \mathcal{F}$ are local minimisers of (1.5) in a certain sense (Theorem 4.3). We note that the admissible set in (1.5) is not necessarily convex. To obtain that the directional derivative of the reduced objective $j : g \mapsto \mathcal{J}(\Omega_g, y_g)$ is non-negative in all feasible directions h, we only need that $\{g \in \mathcal{F} : g < 0 \text{ in } \bar{E}\}$ is open, which is indeed the case. The derivation of necessary optimality conditions in primal form for locally optimal shapes (in the context of non-smooth PDEs) constitutes a second main finding of the present manuscript (Theorem 5.1).

We end the introduction by putting into evidence the differences between our technique and other comparable works. We start with the discrepancies with other papers in the context of the FVA. Though this particular method has often been used for differentiation purposes, the novelty in the present paper is that one does not alter the original PDE; we recall that our idea is to look at the sensitivity of the solution operator of (1.3) (with $\Omega = \Omega_g$) with respect to shape functions only, cf. (1.4). The existing contributions on the topic address only smooth problems and may be divided into two categories. A large part of them resorts to extensions on the hold-all domain by means of a regularization of the Heaviside function [24–27], which yields an approximating PDE on the fixed domain. Other works [10, 21, 23] are based on the so-called Hamiltonian approach, which is more direct. Therein, an additional control variable defined on D is introduced so that the PDE is posed in the hold-all domain as well. However, the boundary condition appears as a state-control constraint and the constrained optimal control problem is in fact equivalent to the original shape optimization problem [23, Cor. 3.1]. That is, even though the extended PDE is obviously not the same as the original one (defined in Ω_g), at the minimization level, the method used in [10, 21, 23] is exact.

We now turn our attention to the rest of the literature on the sensitivity analysis of problems on perturbed domains with emphasis on non-smooth problems [1, 6, 11, 13, 28, 34] and we see that one of the most common approaches is the prominent speed/velocity method [36]. As opposed to the FVA, this has the advantage that it works in arbitrary dimensions too. The typical idea is to introduce a smooth transformation of the domain, generated by a family of vector fields, which associates to each point in Ω a unique point in the perturbed shape and viceversa. In this manner, the perturbed smooth domains are of the type $T_\lambda(\Omega)$, where T_λ is a bijection satisfying certain conditions; as a special case we mention the so-called perturbation of identity where the variations are generated via $T_\lambda(\Omega) = (\mathbb{I} + \lambda\theta)\Omega$, where θ is a smooth vector field (the velocity field). The speed method allows one to rewrite the perturbed equation in the original unperturbed domain Ω. Deriving w.r.t. λ then yields the so-called material derivative, by means of which the shape derivative of the state can be computed. This procedure is explained and applied at length in [36] and has been used by numerous authors. We refer here in particular to [12, Sec. 5.3, Thm. 5.3.1], where the shape derivative for weak formulations of PDEs is calculated by requiring Ω to be only measurable, bounded and open. However, we underline that [12] highly relies on the implicit function theorem so that the results cannot be expected to hold for non-smooth problems. Another remarkable aspect about the aforementioned contributions is that, since they involve the computation of the so-called material derivative, they require the given right-hand side, in our case f, to posses weak derivatives (mostly $f \in H^1(D)$). By contrast, we do not transfer the perturbed PDE to the original domain $\Omega = \Omega_g$, but we examine (1.4) on the common domain $\Omega_g \cap \Omega_{g+\lambda h}$, see subsection 3.2. That is, we do not compute material derivatives, so that $f \in L^p(D)$ is sufficient in the present manuscript and we want to stress this matter.

Though our functional variational concept may seem different from the classical purely geometrical one [36], we will see that, we are in fact proposing a novel way of computing the shape derivative, cf. section 6 and Remark 3.10. While this is restricted to $\mathcal{C}^2$ planar domains, it has the advantage that the involved data f and β do not require smoothness assumptions (as already pointed out). It turns out that the boundary of our perturbed domain $\Omega_{g+\lambda h}$ arises as $T_\lambda(\partial\Omega_g)$, where T_λ is the flow associated to a certain family of vector fields $\theta \in \mathcal{C}([0, \lambda_0), \mathcal{C}_c^{0,1}(D; \mathbb{R}^2))$ for which $\theta(0) = W$, see Definition 3.8. This (initial) velocity field W can be put in connection with a linearization of a Hamiltonian system which describes the boundary of the original domain, cf. (2.10).

Finally, let us state an essential matter regarding the different perspective offered by the FVA when compared with the LSM. In the LSM setting one employs the Hamilton Jacobi equation to update the level set function in numerical algorithms. This describes the evolution of a mapping g_λ with respect to λ (which there plays the role of the time variable). As a result of perturbing the zero level set of $g_0 = g$ by a known vector field $\theta : \{g = 0\} \to \mathbb{R}^2$, the Hamilton Jacobi equation reduces to finding g_λ so that

$$\frac{d}{d\lambda} g_\lambda(x) + \theta(x)\nabla g_\lambda(x) = 0 \quad \text{in } D, \quad g_0 = g. \tag{1.6}$$

The FVA theory offers an "other way around" point of view: we already know the perturbations of the functions, i.e., $g_\lambda = g + \lambda h$, and, as we will see, we can compute the velocity field $\theta = W$ associated to the variation of the zero level set of g, cf. Definition 3.8. Indeed, this satisfies (1.6) with $g_\lambda = g + \lambda h$ on $\{g = 0\}$, as shown by (3.20) below.

Outline of the paper. The present work is organized as follows. After introducing the notation, we recall in Section 2 some essential previous findings concerning the description of $\mathcal{C}^2$ planar curves via Hamiltonian systems and other fundamental properties. In section 3 we state the precise assumptions on the fixed data in (1.3) after which we examine the continuity properties of the state with respect to domain perturbations (subsection 3.1). Subsection 3.2 is entirely focused on the computation of (1.4). Here we introduce the candidate for the directional derivative of $\mathcal{S}$ and we investigate (1.4) on the common domain $\Omega_g \cap \Omega_{g+\lambda h}$ as well as on its boundary. This shall result in the first main contribution of the paper, namely Theorem 3.18. In Proposition 3.19 we gather information about (1.4) outside $\Omega_g \cap \Omega_{g+\lambda h}$, which yields the second main finding of this work, i.e., Theorem 3.20. Starting with section 4 we focus on the optimal design problem (P_Ω) and its equivalent formulation as a control problem. The main novelty here is due to subsection 4.1, where we introduce the concept of locally optimal shape and its relation with local optimality for control problems such as (1.5). Section 5 is devoted to the derivation of necessary optimality conditions for locally optimal shapes. These are stated in a general framework in Theorem 5.1. The main differentiability findings from section 3 then allow us to apply Theorem 5.1 for objectives with pointwise observation and derivatives of the state on an observation set (subsection 5.1) as well as distributed observation terms (subsection 5.2). The final section 6 of the manuscript is devoted to a comparison between the FVA and the speed/velocity method, and may be seen as a confirmation of the differentiability results obtained in section 3.

Notation. Throughout this manuscript, $\mathcal{A} \subset\subset D$ means that the closure of the set $\mathcal{A}$ is a compact subset of D and we use the symbol $|\cdot|$ for the Euclidean norm. In addition, $\mu(\mathcal{A})$ is the measure of a Lebesgue measurable set $\mathcal{A} \subset \mathbb{R}^2$ and $\mathcal{C}_c(\mathcal{A})$ is the set of all continuous functions on $\mathcal{A} \subset \mathbb{R}^2$ with compact support. We use the short-hand notation $\{g = 0\}$ for the zero level set $\{x \in D : g(x) = 0\}$ and if $\{g = 0\}$ is a union of closed curves, $\{g = 0\}^\circ$ denotes a component of this union (a single periodic curve). The mapping $\mathrm{dist}_{\mathcal{A}} : \mathbb{R}^2 \to \mathbb{R}^+$ associates to a point its distance to a compact set $\mathcal{A} \subset \mathbb{R}^2$. The Hausdorff-Pompeiu distance between two compact sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^2$ is defined as

$$d_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) := \max\{\max_{x\in\mathcal{A}} \mathrm{dist}(x, \mathcal{B}), \max_{\widetilde{x}\in\mathcal{B}} \mathrm{dist}(\widetilde{x}, \mathcal{A})\}. \tag{1.7}$$

Given a normed space X, the closed ball centered at $0 \in X$ with radius $M > 0$ is denoted by $\overline{B_X(0, M)}$.

2. Preliminaries. In this section, we collect some (meanwhile well-established) key findings about the description of planar curves via Hamiltonian systems [39], their approximating behaviour [5] and differentiability properties [40]. To this end, we recall the definition of our set of admissible shape functions:

$$\mathcal{F} = \{g \in \mathcal{C}^2(\bar{D}) : |\nabla g(x)|+|g(x)| > 0 \,\forall\, x \in D,\ g(x) > 0 \,\forall\, x \in \partial D,\ g^{-1}((-\infty, 0]) \neq \emptyset\}. \tag{2.1}$$

We begin with a fundamental result that stands at the core of the FVA:

Proposition 2.1 ([39, Prop. 2]). *For each $g \in \mathcal{F}$, the level set $\{g = 0\}$ is a finite union of disjoint closed $\mathcal{C}^2$ curves, without self intersections and not intersecting ∂D. Each component of $\{g = 0\}$ is parametrized by the solution $z : [0, \infty) \to \{g = 0\}^\circ$ of the Hamiltonian system*

$$\begin{cases} (z_1)'(t) = -\partial_2 g(z(t)), \\ (z_2)'(t) = \partial_1 g(z(t)), \\ \quad z(0) = x_0, \quad t \in [0, \infty) \end{cases} \tag{2.2}$$

when some initial point $x_0 \in \{g = 0\}^\circ$ is chosen.

Remark 2.2. *Note that $z : [0, T) \to \{g = 0\}^\circ$ is a bijection, where $T > 0$ stands for the period of the closed curve $\{g = 0\}^\circ$. If $|\nabla g(\cdot)| = 1$ on $\{g = 0\}^\circ$, its period is given by the length of the curve, see e.g. [33].*

Let $g \in \mathcal{F}$. Given a second function $h \in \mathcal{C}^2(\bar{D})$ (not necessarily in $\mathcal{F}$), we call functional variations [20] the perturbations $g + \lambda h$, $\lambda \in \mathbb{R}$ "small". The next result goes to show that the preceding Proposition 2.1 remains unaffected by small enough perturbations of the original shape function g.

Lemma 2.3 ([5, Prop. 3.4]). *For each $g \in \mathcal{F}$ and $h \in \mathcal{C}^2(\bar{D})$, there exists $\lambda_0 > 0$ so that for all $\lambda \in (0, \lambda_0)$, we have $g + \lambda h \in \mathcal{F}$ as well. That is, the assertion of*

Proposition 2.1 stays true for the level set $\{g+\lambda h=0\}$ and each of its components is parametrized by the solution of the Hamiltonian system

$$\begin{cases} z'_{\lambda,1}(t) = -\partial_2(g+\lambda h)(z_\lambda(t)), \\ z'_{\lambda,2}(t) = \partial_1(g+\lambda h)(z_\lambda(t)), \\ z_\lambda(0) = x_0^\lambda, \quad t\in[0,\infty) \end{cases} \tag{2.3}$$

when some initial point $x_0^\lambda \in D$ is chosen on each $\{g+\lambda h=0\}^\circ$.

Proof. The assertion follows by the same arguments used in the proof of [5, Lem. 3.1]. As a consequence of $g\in\mathcal{F}$ combined with the compactness of $\bar{D}$, we have by Weierstrass theorem that

$$|\nabla g| + |g| \geq \delta \quad \text{in } \bar{D} \tag{2.4}$$

for some $\delta>0$. Since $g+\lambda h \to g$ in $\mathcal{C}^2(\bar{D})$ as $\lambda \searrow 0$, it follows that there exists $\lambda_0>0$, depending only on $\delta>0$ and the given data, so that the first condition in (2.1) is satisfied by $g+\lambda h$ for all $\lambda\in(0,\lambda_0)$. A similar argument yields that the second inequality in (2.1) is fulfilled as well. Finally, we have by (2.4) and $g^{-1}((-\infty,0])\neq\emptyset$ that there exists a point $\tilde{x}\in D$ so that $g(\tilde{x})<0$. Since $g+\lambda h$ converges uniformly towards g we get that $(g+\lambda h)(\tilde{x})<0$ for $\lambda>0$ small enough dependent only on $\tilde{x}$, and thus on g. This allows us to deduce the desired statement. ☐

It turns out that the finite number of components of $\{g+\lambda h=0\}$ is independent of λ, and, in fact, equal to the number of components of $\{g=0\}$, provided that λ is small enough. This is a consequence of the following.

Proposition 2.4 ([5, Lem. 3.7, 3.8], Uniqueness of the approximating curve). *Let $g\in\mathcal{F}$ and $h\in\mathcal{C}^2(\bar{D})$. Given $\{g=0\}^\circ$, there exists $\varepsilon>0$ and $\lambda_0(\varepsilon)>0$ so that for each $\lambda\in(0,\lambda_0(\varepsilon)]$ there is exactly one closed curve $\{g+\lambda h=0\}^\circ$ satisfying*

$$\{g+\lambda h=0\}^\circ \subset \{x\in D : \operatorname{dist}(x,\{g=0\}^\circ)<\varepsilon\}.$$

Moreover, we have the convergence

$$d_{\mathcal{H}}(\{g+\lambda h=0\},\{g=0\}) \to 0 \quad \text{as } \lambda\searrow 0, \tag{2.5}$$

where $d_{\mathcal{H}}(\{g+\lambda h=0\},\{g=0\})$ denotes the Hausdorff-Pompeiu distance between the compact sets $\{g+\lambda h=0\}$ and $\{g=0\}$, i.e.,

$$\begin{aligned} &d_{\mathcal{H}}(\{g+\lambda h=0\},\{g=0\}) \\ &\quad := \max\{\max_{x\in\{g=0\}} \operatorname{dist}(x,\{g+\lambda h=0\}), \max_{\tilde{x}\in\{g+\lambda h=0\}} \operatorname{dist}(\tilde{x},\{g=0\})\}. \end{aligned} \tag{2.6}$$

From now on, whenever we write 'approximating curve' of $\{g=0\}^\circ$ we refer to the unique curve $\{g+\lambda h=0\}^\circ$ from Proposition 2.4 associated to it.

Proposition 2.5 ([21, Prop. 2.6], [5, Prop. 3.10]). *Fix $g\in\mathcal{F}$ and $h\in\mathcal{C}^2(\bar{D})$. Let $\{g=0\}^\circ$ be a closed curve with periodicity T and let $\{g+\lambda h=0\}^\circ$ be its approximating curve with periodicity T_λ. The following assertions are true:*

(i) If $x_0^\lambda \to x_0$, then, for each $B>0$, it holds $\|z_\lambda - z\|_{C^1([0,B];\mathbb{R}^2)} \to 0$ as $\lambda\searrow 0$, where z_λ and z are the trajectories of the Hamiltonian systems (2.3) and (2.2) corresponding to $\{g+\lambda h=0\}^\circ$ and $\{g=0\}^\circ$, respectively;

(ii) $T_\lambda \to T$ *as* $\lambda \searrow 0$.

To be able to discuss the differentiability properties of the Hamiltonian system (2.2) w.r.t. λ, we need to require that the zero level set of h intersects each of the components of $\{g = 0\}$ in at least one point (which we choose as initial condition in the Hamiltonian system corresponding to $\{g = 0\}^\circ$), cf. figure 2.1. That is,

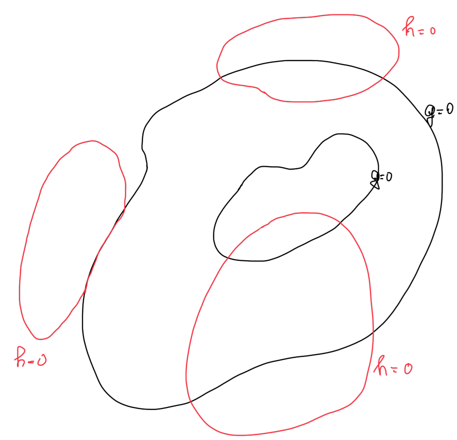

Fig. 2.1. *The zero level set of h intersects each component of $\{g = 0\}$.*

$$\{h = 0\} \cap \{g = 0\}^\circ \neq \emptyset \quad \forall \{g = 0\}^\circ \subset \{g = 0\}, \tag{2.7}$$

and in (2.2) we will choose the initial condition $z(0) = x_0 \in \{g = 0\}$, so that

$$h(x_0) = 0. \tag{2.8}$$

This mild restriction is due to the lack of differentiability of the operator $\lambda \mapsto \mathrm{proj}_{\{g+\lambda h=0\}}\, x_0$ [40, p. 9]. Note that the class of perturbed domains $\Omega_{g+\lambda h}$ is still large: it covers all non-empty $\mathcal{C}^2$ subdomains of D whose boundaries intersect the boundary of the original domain Ω_g, see also figure 3.2.

Proposition 2.6 ([40, Prop. 3.2]). *In addition to $g \in \mathcal{F}$, suppose that $h \in \mathcal{C}^2(\bar{D})$ satisfies* (2.7)*. Let z_λ and z be the trajectories of the Hamiltonian systems* (2.3) *and* (2.2) *(both with initial point x_0, cf.* (2.8)*) corresponding to $\{g + \lambda h = 0\}^\circ$ and $\{g = 0\}^\circ$, respectively. Then, it holds*

$$w_\lambda := \frac{z_\lambda - z}{\lambda} \to w \text{ in } C^1([0,T];\mathbb{R}^2) \quad \text{as } \lambda \searrow 0, \tag{2.9}$$

where $T > 0$ is the period of $\{g = 0\}^\circ$ and $w = (w_1, w_2)$ is the unique solution of

$$\begin{cases} (w_1)'(t) = -\nabla \partial_2 g(z(t)) \cdot w(t) - \partial_2 h(z(t)), \\ (w_2)'(t) = \nabla \partial_1 g(z(t)) \cdot w(t) + \partial_1 h(z(t)), \\ \quad w(0) = 0, \quad t \in [0,T]. \end{cases} \tag{2.10}$$

3. The Dirichlet problem. This section is dedicated to the study of the differentiability properties of the solution of (1.3) w.r.t. perturbations of the two-dimensional domain. These perturbations take place via functional variations as described in the introduction.

In the sequel, $g \in \mathcal{F}$ is fixed and $\Omega_g = \{x \in D : g < 0\}$ is assumed to be a (possibly multiply connected) planar *domain* (we recall that $\Omega_g \in \mathcal{C}^2$, cf. Proposition 2.1, see also [7, Thm. 4.2]). Note that, since, in general, Ω_g may have more than one component, it is reasonable to 'select' one of them in the context of solving PDEs. This will be done in section 4 via the condition $E \subset \Omega_g$ and for the moment we just assume that Ω_g is connected (however, it may have holes). We keep the whole time in mind the properties:

$$\partial\Omega_g = \{x \in D : g = 0\},$$

$$\overline{\Omega_g}^c = \{x \in D : g > 0\},$$

where $\overline{\Omega_g}^c$ is the complement of the set $\overline{\Omega}_g$. Note that these may be found for instance in [7, Thm. 4.2].

We are interested in the following non-smooth problem:

$$\begin{aligned} -\triangle y_g + \beta(y_g) &= f \quad \text{in } \Omega_g, \\ y_g &= 0 \quad \text{on } \partial\Omega_g, \end{aligned} \tag{3.1}$$

where $f \in L^p(D),\ p > 2$ and β is a not necessarily differentiable function.

The next requirement on the non-linearity β is a standiing assumption and it is thus supposed to hold throughout the manuscript (without mentioning it every time).

Assumption 3.1 (The non-smoothness).

1. *The function $\beta : \mathbb{R} \to \mathbb{R}$ is monotone increasing and locally Lipschitz continuous in the following sense: For all $M > 0$, there exists a constant $L_M > 0$ such that*

$$|\beta(y_1) - \beta(y_2)| \leq L_M |y_1 - y_2| \quad \forall\, y_1, y_2 \in [-M, M].$$

2. *The mapping β is directionally differentiable at every point, i.e.,*

$$\Big|\frac{\beta(y + \tau\, \delta y) - \beta(y)}{\tau} - \beta'(y; \delta y)\Big| \overset{\tau \searrow 0}{\longrightarrow} 0 \quad \forall\, y, \delta y \in \mathbb{R}.$$

By Assumption 3.1.1, it is straightforward to see that the Nemytskii operator $\beta : L^\infty(\mathcal{A}) \to L^\infty(\mathcal{A})$ is well-defined for each measurable set $\mathcal{A} \subset \mathbb{R}^2$. Moreover, this is Lipschitz continuous on bounded sets, i.e., for every $M > 0$, there exists $L_M > 0$ so that

$$\|\beta(y_1) - \beta(y_2)\|_{L^r(\mathcal{A})} \leq L_M \, \|y_1 - y_2\|_{L^r(\mathcal{A})} \quad \forall\, y_1, y_2 \in \overline{B_{L^\infty(\mathcal{A})}(0, M)},\ \forall\, 1 \leq r \leq \infty. \tag{3.2}$$

In addition, $\beta : L^\infty(\mathcal{A}) \to L^\varrho(\mathcal{A})$, $1 \leq \varrho < \infty$, is directionally differentiable, as a result of Assumption 3.1 and the Lebesgue's dominated convergence theorem, that is,

$$\Big\|\frac{\beta(y + \tau\, \delta y) - \beta(y)}{\tau} - \beta'(y; \delta y)\Big\|_{L^\varrho(\mathcal{A})} \overset{\tau \searrow 0}{\longrightarrow} 0 \quad \forall\, y, \delta y \in L^\infty(\mathcal{A}),\ \forall\, 1 \leq \varrho < \infty. \tag{3.3}$$

Definition 3.2 (The control-to-state map associated to (3.1)). *We define*

$$\mathcal{S} : g \in \mathcal{F} \mapsto y_g \in H_0^1(\Omega_g) \cap W^{2,p}(\Omega_g), \tag{3.4}$$

where y_g solves the equation (3.1) on the domain Ω_g.

Remark 3.3. *According to Definition 3.2, $\mathcal{S}(g)$ exists only as an element of $H_0^1(\Omega_g)$. Whenever we write $\mathcal{S}(g)$ as an element of $H_0^1(D)$ in what follows, we think of its extension by zero outside Ω_g.*

The principal aim of this section is to look at the differentiability properties of $\mathcal{S}$ w.r.t. $g \in \mathcal{F}$ (in direction $h \in \mathcal{C}^2(\bar{D})$), that is, we are interested in calculating

$$\lim_{\lambda \searrow 0} \frac{\mathcal{S}(g + \lambda h) - \mathcal{S}(g)}{\lambda}$$

in a suitable space.

For simplicity, we will use in the sequel the abbreviations $y_g := \mathcal{S}(g)$, $y_{g+\lambda h} := \mathcal{S}(g + \lambda h)$, where $\lambda > 0$ is small enough so that the unicity of the approximating curve in Proposition 2.4 is guaranteed.

3.1. Continuity properties of $\mathcal{S}$.

In this subsection we look at the convergence behaviour of $y_{g+\lambda h}$ as $\lambda \searrow 0$. We start with some essential observations.

Lemma 3.4 ([9, Thm. 8.33]). *Let $g \in \mathcal{F}$ and $h \in \mathcal{C}^2(\bar{D})$. Then, there exists $\lambda_0 > 0$ so that*

$$\|y_{g+\lambda h}\|_{\mathcal{C}^{1,\alpha}(\bar{\Omega}_{g+\lambda h})} \le c, \quad \forall\, \lambda \in (0, \lambda_0], \tag{3.5}$$

where $c > 0$ is independent of λ and $\alpha > 0$ is such that $W^{2,p}(\Omega_{g+\lambda h}) \hookrightarrow \mathcal{C}^{1,\alpha}(\bar{\Omega}_{g+\lambda h})$.

Proof. First, we observe that

$$\|y_{g+\lambda h}\|_{\mathcal{C}(\bar{\Omega}_{g+\lambda h})} \le c \quad \forall \lambda \in [0, \lambda_0), \tag{3.6}$$

where $\lambda_0 > 0$ is small (as in Lemma 2.3) and where $c > 0$ is independent of λ, see e.g. the proof of [41, Thm. 4.5]. Then, since (3.1) associated to $g + \lambda h$ can be rewritten as

$$-\triangle y_{g+\lambda h} = -\beta(y_{g+\lambda h}) + f \quad \text{in } \Omega_{g+\lambda h},\ y_{g+\lambda h}|_{\partial\Omega_{g+\lambda h}} = 0,$$

we obtain by [9, Thm. 8.33] (see also the remark at the end of Sec. 8.11 in [9]) the estimate

$$\|y_{g+\lambda h}\|_{\mathcal{C}^{1,\alpha}(\bar{\Omega}_{g+\lambda h})} \le C(\|y_{g+\lambda h}\|_{\mathcal{C}(\bar{\Omega}_{g+\lambda h})} + \|\beta(y_{g+\lambda h}) - f\|_{L^p(\Omega_{g+\lambda h})}),$$

where $\alpha > 0$ is such that $W^{2,p}(\Omega_{g+\lambda h}) \hookrightarrow \mathcal{C}^{1,\alpha}(\bar{\Omega}_{g+\lambda h})$ and $C > 0$ is independent of λ.

Note that the latter assertion is due to the fact that $C > 0$ depends only on the $\mathcal{C}^{1,\alpha}$ norms of the $\mathcal{C}^2$ mappings which define the local representation of $\partial\Omega_{g+\lambda h}$ (see the comments following [9, Thm. 8.33] and the proofs of [9, Thm. 6.1, 6.6]). These local representations can be chosen as in [7, proof of Thm. 4.2]:

$$\Psi_x : \mathcal{U}(x) \to \mathbb{R}^2, \quad \Psi_x(y) := \frac{1}{|\nabla g_\lambda(x)|}\Big((y - x)^T(-\partial_2 g_\lambda(x), \partial_1 g_\lambda(x)), g_\lambda(y)\Big),$$

where $\mathcal{U}(x)$ is a neighbourhood of $x \in \partial\Omega_{g+\lambda h}$ and we abbreviate $g_\lambda := g + \lambda h$. Hence, the fact that $C > 0$ is independent of λ reduces to the uniform boundedness of $\|g_\lambda\|_{\mathcal{C}^{1,\alpha}}$, which is true for λ small (even when we take the $\mathcal{C}^2$ norm).

Thanks to (3.6), (3.2) and $\Omega_{g+\lambda h} \subset D$ (λ small), we finally deduce (3.5) from the above estimate.□

LEMMA 3.5. *Let $g \in \mathcal{F}$ and $h \in \mathcal{C}^2(\bar{D})$. For each $v \in \mathcal{C}_c(\Omega_g)$ there exists $\lambda(v, g) > 0$ so that $v \in \mathcal{C}_c(\Omega_{g+\lambda h})$ for all $\lambda \in [0, \lambda(v, g))$.*

Proof. We show that each compact K subset of Ω_g is a compact subset of $\Omega_{g+\lambda h}$. Indeed, since g is continuous, we have

$$g(x) \leq -\delta \quad \forall\, x \in K \subset\subset \Omega_g$$

for some $\delta > 0$. Thus, there exists $\lambda > 0$, small, independent of x, so that

$$g(x) + \lambda h(x) \leq -\delta/2 < 0 \quad \forall\, x \in K.$$

Thus, $K \subset \Omega_{g+\lambda h}$. For a fixed $v \in \mathcal{C}_c(\Omega_g)$ there exists a compact subset K of Ω_g so that $v \in \mathcal{C}_c(K)$ and hence, $v \in \mathcal{C}_c(\Omega_{g+\lambda h})$ for all $\lambda \in [0, \lambda(v, g))$. □

REMARK 3.6. *An analogous argument to the one from the proof of Lemma 3.5 shows that each compact subset of $\bar{\Omega}_g^c$ is a compact subset of $\bar{\Omega}_{g+\lambda h}^c$ for λ small enough and thus, $\Omega_{g+\lambda h}$ converges to Ω_g in the sense of compact sets [12, Def. 2.2.21]. In fact,*

$$d_{\mathcal{H}}(\bar{\Omega}_{g+\lambda h}, \bar{\Omega}_g) \to 0,$$

$$d_{\mathcal{H}}(\bar{D} \setminus \Omega_{g+\lambda h}, \bar{D} \setminus \Omega_g) \to 0,$$

$$\mu(\Omega_{g+\lambda h}) \to \mu(\Omega_g), \tag{3.7}$$

see for instance [12, p. 63] and [29, Appendix 3].

One of the key characteristics of the FVA is that $\partial\Omega_g$ is divided by $\Omega_{g+\lambda h}$ into two disjoint (not necessarily connected) parts that are independent of λ (recall that, in view of (2.8), we have $\partial\Omega_g \cap \partial\Omega_{g+\lambda h} \neq \emptyset$). Here we include the case when one of these disjoint parts is void (and the other is the whole boundary), that is, $\Omega_g \subset \Omega_{g+\lambda h}$. This observation is due to the following essential (easy to check) fact

$$\partial\Omega_g \cap \partial\Omega_{g+\lambda h} = \{g = h = 0\}.$$

Hence, we have a representation of $\partial\Omega_g$ as a disjoint union of two sets that are both independent of λ, namely

$$\partial\Omega_g = \Gamma_1 \cup \Gamma_2 \tag{3.8}$$

where the relatively open set Γ_1 is defined as

$$\Gamma_1 := \partial\Omega_g \setminus \bar{\Omega}_{g+\lambda h} \tag{3.9}$$

and the relatively closed set Γ_2 is given by

$$\Gamma_2 := \partial\Omega_g \cap \bar{\Omega}_{g+\lambda h}. \tag{3.10}$$

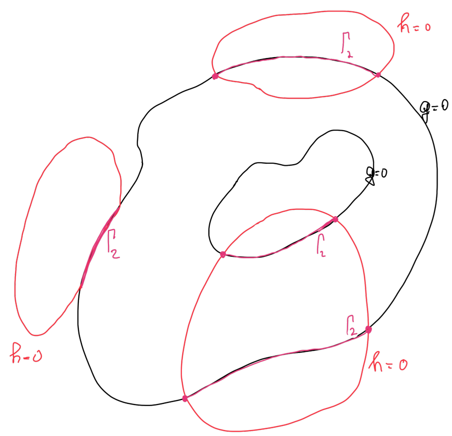

FIG. 3.1. *The set Γ_2 (the part of the boundary of $\partial\Omega_g$ included in $\bar\Omega_{g+\lambda h}$)*

This fundamental remark is true regardless of the number of components of $\partial\Omega_g$ (i.e., even when Ω_g has holes), as a consequence of Proposition 2.4. To ensure that the zero level set of h does not intersect $\{g=0\}$ an infinite number of times, we require in the remainder of the paper that $h \in \mathcal{F}$, which by Proposition 2.1, yields that $\{h=0\}$ is a finite union of disjoint closed $\mathcal{C}^2$ curves. Then, locally, $\{h=0\}^\circ$ can intersect $\{g=0\}$ in a single point or these two zero level sets may have common regions consisting of curves, as depicted on the left side in figure 3.1.

PROPOSITION 3.7. *Suppose that $g, h \in \mathcal{F}$ and (2.7) is true. Then it holds*

$$y_{g+\lambda h} \rightharpoonup y_g \ \text{in } W_0^{1,\varrho}(D) \quad \text{as } \lambda \searrow 0, \quad \forall \varrho \in [1,\infty), \tag{3.11}$$

$$\nabla y_{g+\lambda h} \to \nabla y_g \ \text{in } \mathcal{C}^{0,\alpha}(\Gamma_2) \quad \text{as } \lambda \searrow 0 \tag{3.12}$$

for some $\alpha > 0$.

Proof. (i) *Proof of* (3.11). From (3.5) we have on a subsequence

$$y_{g+\lambda h} \rightharpoonup \widetilde{y} \text{ in } W_0^{1,\varrho}(D). \tag{3.13}$$

In view of Lemma 3.5, we can test the equation associated to $y_{g+\lambda h}$ with some arbitrary $v \in \mathcal{C}_c^\infty(\Omega_g) \subset \mathcal{C}_c^\infty(\Omega_{g+\lambda h})$, where $\lambda = \lambda(v)$ is small. We have

$$\int_{\overline{\operatorname{supp} v}} \nabla y_{g+\lambda h} \nabla v \, dx + \int_{\overline{\operatorname{supp} v}} \beta(y_{g+\lambda h}) v \, dx = \int_{\overline{\operatorname{supp} v}} f v \, dx, \quad \forall \lambda \in [0, \lambda(v)). \tag{3.14}$$

Passing to the limit $\lambda \searrow 0$ in (3.14), where we rely on (3.13), the compact embedding $W_0^{1,\varrho}(D) \hookrightarrow\hookrightarrow L^\infty(D)$ and the continuity properties of β, see Assumption 3.1.1, yields

$$\int_{\overline{\operatorname{supp} v}} \nabla \widetilde{y} \nabla v \, dx + \int_{\overline{\operatorname{supp} v}} \beta(\widetilde{y}) v \, dx = \int_{\overline{\operatorname{supp} v}} f v \, dx, \tag{3.15}$$

i.e.,

$$\int_\Omega \nabla \widetilde{y} \nabla v \, dx + \int_\Omega \beta(\widetilde{y}) v \, dx = \int_\Omega f v \, dx \quad \forall v \in \mathcal{C}_c^\infty(\Omega). \tag{3.16}$$

It remains to show $\widetilde{y} = 0$ on $\partial\Omega$. Due to (3.13) we have

$$\gamma y_{g+\lambda h} \rightharpoonup \gamma\widetilde{y} \text{ in } W^{1-1/\varrho,\varrho}(\partial\Omega),$$

where $\gamma : W^{1,\varrho}(D) \to W^{1-1/\varrho,\varrho}(\partial\Omega)$ is the trace operator. Thus, $\widetilde{y} = 0$ on Γ_1, cf. (3.9), and

$$\gamma y_{g+\lambda h} \rightharpoonup \gamma\widetilde{y} \text{ in } W^{1-1/\varrho,\varrho}(\Gamma_2), \tag{3.17}$$

where we recall the definition of Γ_2 from (3.10). Fix $t \in [0, T]$ so that $z(t) \in \Omega_{g+\lambda h}$, where z is the solution of the Hamiltonian system (2.2). Since $h \in \mathcal{F}$, by assumption, we have that $\{h = 0\}$ is a finite union of disjoint closed $\mathcal{C}^2$ curves (see Proposition 2.1). Thus, t belongs to a finite union of subintervals of $[0, T]$, say I. It holds

$$|y_{g+\lambda h}(z(t)) - y_{g+\lambda h}(z_\lambda(t))| \leq c|z(t) - z_\lambda(t)|^{\hat{\alpha}}, \quad t \in I,$$

with some $\hat{\alpha} > 0$ and $c > 0$ independent of λ, where we used (3.5); recall that we take the initial point in both Hamiltonian systems (2.2) and (2.3) to be x_0, cf. (2.8). Now, in light of Proposition 2.5 we have

$$\sup_{t\in I} |y_{g+\lambda h}(z(t))| \to 0$$

which yields

$$\gamma y_{g+\lambda h} \to 0 \quad \text{in } \mathcal{C}(\Gamma_2).$$

Due to (3.17) we have $\widetilde{y} = 0$ on Γ_2 as well. In conclusion, the weak limit in (3.13) belongs to $W_0^{1,\varrho}(\Omega)$ (recall here (3.8)) and since it satisfies (3.16), we obtain $\widetilde{y} = y_g$. Taking another subsequence as in (3.13) yields the same limit and we arrive at the desired convergence result for the whole sequence $\{y_{g+\lambda h}\}$.
(ii) *Proof of* (3.12). Let K be a compact subset of $\bar{\Omega}_g$ so that $\Gamma_2 \subset K \subset \Omega_{g+\lambda h}$ (for all $\lambda \in (0, \lambda_0)$, $\lambda_0 > 0$ small) and $\operatorname{int} K$ is a $\mathcal{C}^{1,\alpha}$ domain ($\alpha > 0$ as in (3.5)) with $\overline{\operatorname{int} K} = K$. Note that such a construction is possible since $g, h \in \mathcal{F}$ and in light of (2.7). From (3.11) we know that

$$y_{g+\lambda h} \rightharpoonup y_g \text{ in } W_0^{1,\varrho}(D), \quad \forall\, \varrho \in [1, \infty).$$

Thanks to (3.5) combined with [9, Lem. 6.36], it also holds on a subsequence

$$y_{g+\lambda h} \to \widetilde{y} \text{ in } \mathcal{C}^{1,\widetilde{\alpha}}(K),$$

where $\alpha > \widetilde{\alpha} > 0$. From here we conclude $\widetilde{y} = y_g$ in K and, since $\Gamma_2 \subset K$, the convergence (3.12) follows. The proof is now complete.$\square$

3.2. Directional differentiability of $\mathcal{S}$. Now we turn our attention to our main goal in this section, which is to investigate the differentiability of $\mathcal{S}$ w.r.t. $g \in \mathcal{F}$ in a direction $h \in \mathcal{F}$ satisfying (2.7).

The vector field W and the candidate for the directional derivative. We start by introducing a vector field that will play an essential role in this paper. This associates to a point x on $\partial\Omega_g$ its velocity vector when performing the variation of the domain. As we will see in section 6, this corresponds to the initial velocity vector field in the context of the speed method.

Definition 3.8. *On the zero level set of g, we define the vector field $W : \{g = 0\} \to \mathbb{R}^2$ as follows*

$$W(x) := w((z^{-1}(x)) \quad \forall\, x \in \{g = 0\}^\circ, \quad \{g = 0\}^\circ \subset \{g = 0\} \tag{3.18}$$

where $w \in \mathcal{C}^1([0, T]; \mathbb{R}^2)$ is the unique solution of (2.10).

We note the crucial identity

$$W(z(t)) = w(t), \quad t \in [0, T_{\{g=0\}^\circ}], \quad \forall\, \{g = 0\}^\circ \subset \{g = 0\}. \tag{3.19}$$

where $T_{\{g=0\}^\circ} > 0$ is the period of $\{g = 0\}^\circ$. This will be used at length throughout the proofs of Lemmas 3.15 and 3.16 below, when we look at the behaviour of the difference quotients $\frac{y_{g+\lambda h}-y}{\lambda}$ on the boundary of the domain of interest.

In the following lemma we gather some important properties of the vector field W.

Lemma 3.9. *Suppose that $g, h \in \mathcal{F}$ and (2.7) is true. Then, the mapping W satisfies*

$$\nabla g(x)W(x) + h(x) = 0 \quad \forall\, x \in \{g = 0\}. \tag{3.20}$$

Moreover, it is Lipschitzian in the following sense: there exists $L_W > 0$ and $\delta_W > 0$ so that for each $\{g = 0\}^\circ \subset \{g = 0\}$ and for all $x_1, x_2 \in \{g = 0\}^\circ$ with $|x_1 - x_2| \le \delta_W$ we have

$$|W(x_1) - W(x_2)| \le L_W |x_1 - x_2|.$$

Proof. (i) We prove the first assertion on each component of $\{g = 0\}$. Define

$$w(\lambda, \cdot) = \lim_{\hat\lambda \to \lambda} \frac{z_{\hat\lambda} - z_\lambda}{\hat\lambda - \lambda} \in \mathcal{C}^1([0, T]; \mathbb{R}^2) \quad \forall\, \lambda \in (0, \lambda_0], \tag{3.21}$$

where $\lambda_0 > 0$ is small and $z_{\hat\lambda}, z_\lambda$ are the solutions of the Hamiltonian system (2.3) associated to the respective component $\{g = 0\}^\circ$. The above limit is computed in a similar way to w, see (2.9), cf. also the proof of [40, Prop. 3.2]. From Lemma 2.3 we know that

$$(g + \lambda h)(z_\lambda(t)) = 0, \quad t \in (0, \infty), \quad \forall\, \lambda \in (0, \lambda_0].$$

Deriving with respect to λ, where we use Proposition 2.6, cf. also (6.3), then yields

$$\nabla(g + \lambda h)(z_\lambda(t))w(\lambda, t) + h(z_\lambda(t)) = 0, \quad t \in (0, \infty), \quad \forall\, \lambda \in (0, \lambda_0].$$

Setting $\lambda := 0$ gives in turn that w satisfies

$$\nabla g(z(t))w(t) + h(z(t)) = 0, \quad t \in (0, \infty).$$

In light of (3.19), the proof of (3.20) is complete.

(ii) We only show the estimate for a fixed component $\{g = 0\}^\circ$; since their number is finite (Proposition 2.1), the desired assertion follows by building the minimum over all δ_W's and the maximum over all L_W's. Let z be the solution of the Hamiltonian system (2.2) associated to $\{g = 0\}^\circ$. We observe that, since $z_i \in \mathcal{C}^2[0, T], i = 1, 2$, it holds for all $\varepsilon > 0$

$$|z_i(t_1) - z_i(t_2) - z_i'(t_2)(t_1 - t_2)| \le c(t_1 - t_2)^2 \le \varepsilon |t_1 - t_2|, \quad \forall |t_1 - t_2| \le \varepsilon/c,$$

where $c := \frac{\|z_i''\|_{C[0,T]}}{2} > 0$ depends only on g. Then,

$$(|z_i'(t_2)| - \varepsilon)|t_1 - t_2| \leq |z_i(t_1) - z_i(t_2)|, \quad \forall |t_1 - t_2| \leq \varepsilon/c.$$

Adding the above estimates for $i = 1, 2$ yields

$$(|z_1'(t_2)| + |z_2'(t_2)| - 2\varepsilon)|t_1 - t_2| \leq \sum_{i=1}^{2} |z_i(t_1) - z_i(t_2)|, \quad \forall |t_1 - t_2| \leq \varepsilon/c. \tag{3.22}$$

Further, in light of (2.2), it holds

$$|z_1'(t_2)| + |z_2'(t_2)| \geq \sqrt{\sum_{i=1}^{2} |z_i'(t_2)|^2} = |\nabla g(z(t_2))| \geq \min_{\partial\Omega_g} |\nabla g| > 0. \tag{3.23}$$

With $\varepsilon := \frac{1}{4} \min_{\partial\Omega_g} |\nabla g| > 0$ (this is non-negative by the Weierstrass theorem, since $g \in \mathcal{F}$ and $\partial\Omega_g$ is compact), we have

$$(\frac{1}{2} \min_{\partial\Omega_g} |\nabla g|)|t_1 - t_2| \leq \sqrt{2}|z(t_1) - z(t_2)|, \quad \forall |t_1 - t_2| \leq \varepsilon/c.$$

Further we notice that $z^{-1} : \partial\Omega_g \to [0, T)$ is uniformly continuous, since it is continuous (as its inverse is continuous) on a compact set. Let $x_1, x_2 \in \partial\Omega_g$ be arbitrary but fixed and define

$$\delta_W := \delta(\varepsilon/c, z^{-1}) > 0$$

so that $|z^{-1}(x_1) - z^{-1}(x_2)| \leq \varepsilon/c$ for all $|x_1 - x_2| \leq \delta_W$. Abbreviate $t_1 = z^{-1}(x_1)$ and $t_2 = z^{-1}(x_2)$. Then, using (3.19) and the regularity of w, we get

$$|W(x_1) - W(x_2)| \leq |w(t_1) - w(t_2)| \leq L_w|t_1 - t_2| \leq \frac{2\sqrt{2}}{\min_{\partial\Omega_g} |\nabla g|}|x_1 - x_2| \tag{3.24}$$

since $|z^{-1}(x_1) - z^{-1}(x_2)| \leq \varepsilon/c$. As $x_1, x_2 \in \partial\Omega_g$ were arbitrary with $|x_1 - x_2| \leq \delta_W$, the desired Lipschitz continuity follows from (3.24), by setting $L_W := \frac{2\sqrt{2}}{\min_{\partial\Omega_g} |\nabla g|}$. ☐

Now we are in the position to introduce the candidate for the directional derivative of $\mathcal{S}$ at g in direction h. This is the unique solution $q \in W^{1,p}(\Omega_g)$ of

$$-\triangle q + \beta'(y_g; q) = 0 \quad \text{in } \Omega_g, \tag{3.25}$$

$$q + \nabla y_g W = 0 \quad \text{on } \partial\Omega_g, \tag{3.26}$$

where $W : \partial\Omega_g \to \mathbb{R}^2$ is given by Definition 3.8. The fact that (3.25)-(3.26) is indeed uniquely solvable is shown in Proposition 3.11 below. Before we proceed with its proof let us state an essential remark.

Remark 3.10. *The unique solution $q \in W^{1,p}(\Omega_g)$ of* (3.25)-(3.26) *is the shape derivative of y_{Ω_g} in direction W [36, Def. 2.85]. Indeed, when we compare with the results stated in [36, Sec. 3.1] (where, with the notations used there, we have $h(\Omega) := f - \beta(y_\Omega)$ and $z(\Gamma) := 0$), we see that* (3.25) *corresponds to [36, Eq. (3.8)] and* (3.26) *is in fact [36, Eq. (3.6)]. In the latter case, it is essential to notice that $\nabla y_g W =$*

$\nabla y_g n_{\partial\Omega_g}(W n_{\partial\Omega_g})$ on $\partial\Omega_g$, where $n_{\partial\Omega_g}:\partial\Omega_g\to\mathbb{R}^2$ is the outer normal vector field of $\partial\Omega_g$; this can be proven by differentiating $y_g(z(t))=0,\ t\in(0,\infty)$ w.r.t. variable t, which yields that, on $\partial\Omega_g$, we have $\nabla y_g = c\, n_{\partial\Omega_g}$, where c is a real-valued function defined on $\partial\Omega_g$. See also e.g. [12, Eq. (5.28)-(5.29)] for a comparison in the linear case.

Proposition 3.11. *Suppose that $g,h\in\mathcal{F}$ and (2.7) is true. The Dirichlet problem with non-homogeneous boundary conditions (3.25)-(3.26) admits a unique solution $q\in W^{1,p}(\Omega_g)$.*

Proof. Since $\beta'(y_g;\cdot):\mathbb{R}\to\mathbb{R}$ is monotone increasing (as a result of Assumption 3.1.1), we can apply for instance [41, Thm. 4.8] to obtain a first regularity result. This allows us to make use of the estimate in the standing Assumption 3.1.1 to deduce $\beta'(y_g;q)\in L^\infty(\Omega_g)\hookrightarrow W^{-1,p}(\Omega_g)$. Thanks to Lemma 3.9.(ii), we conclude that $W\in W^{1,\infty}(\partial\Omega_g)$ (see e.g. [8, Sec. 5.8.2]), whence $\nabla y_g W\in W^{1-1/p,p}(\partial\Omega_g)$ follows (recall that $y_g\in W^{2,p}(\Omega_g)$). Standard arguments from the $W^{1,p}(\Omega_g)$ regularity theory then lead to the desired assertion. $\square$

Remark 3.12. *As in the case of y_g, see Remark 3.3, q exists only as an element of $W^{1,p}(\Omega_g)$. Whenever we refer to the values of q outside Ω_g, we think of its extension by zero, which yields $q\in L^\infty(D)$.*

Analysis of the difference quotient. We continue our investigations by taking a closer look at the term

$$\frac{\mathcal{S}(g+\lambda h)-\mathcal{S}(g)}{\lambda}-q,\quad \lambda>0 \text{ small.}$$

Given $g,h\in\mathcal{F}$, we use for simplicity in what follows the abbreviations: $\omega_\lambda:=\Omega_g\cap\Omega_{g+\lambda h}$, $q_\lambda:=\frac{y_{g+\lambda h}-y}{\lambda}$ and $m_\lambda:=\|q_\lambda-q\|_{\mathcal{C}(\partial\omega_\lambda)}$, where $\lambda>0$ is small enough (so that the unicity of the approximating closed curves is guaranteed, cf. Proposition 2.4).

Lemma 3.13. *Suppose that $g,h\in\mathcal{F}$ and (2.7) is true. Then*

$$\||q_\lambda-q|-m_\lambda\|_{L^r(\widetilde{\omega}_\lambda)}\to 0,\quad \forall\, r\in[1,\infty),$$

where $\widetilde{\omega}_\lambda:=\{x\in\omega_\lambda: |q_\lambda-q|>m_\lambda\}$.

Proof. Note that

$$q_\lambda\in W^{2,p}(\omega_\lambda),$$

by Definition 3.2. In light of (3.1) (associated with g and $g+\lambda h$, respectively) and (3.25), it holds

$$-\triangle(q_\lambda-q)+[\frac{[\beta(y_{g+\lambda h})-\beta(y_g)]}{\lambda}-\beta'(y_g;q)]=0\quad \text{in } \omega_\lambda. \tag{3.27}$$

Observe that $(q_\lambda-q-m_\lambda)^+\in H_0^1(\omega_\lambda)$. This allows us to test the above equation with $(q_\lambda-q-m_\lambda)^+$. We obtain

$$\int_{\omega_\lambda}[\nabla[q_\lambda-q-m_\lambda]^+]^2\,dx+\int_{\omega_\lambda}\underbrace{[\frac{[\beta(y_{g+\lambda h})-\beta(y_g)]}{\lambda}-\beta'(y_g;q)]}_{=:A_\lambda}[q_\lambda-q-m_\lambda]^+\,dx=0. \tag{3.28}$$

Note that

$$A_\lambda = \frac{[\beta(y_{g+\lambda h}) - \beta(y_g + \lambda q)]}{\lambda} + \frac{[\beta(y_g + \lambda q) - \beta(y_g)]}{\lambda} - \beta'(y_g; q).$$

Due to the monotonicity of β (Assumption 3.1.1) and since $q_\lambda - q > m_\lambda (> 0)$ implies $y_{g+\lambda h} > y_g + \lambda q$, we can conclude that the first term in A_λ is non-negative on $\{q_\lambda - q > m_\lambda\}$. This leads to

$$\int_{\omega_\lambda} [\nabla[q_\lambda - q - m_\lambda]^+]^2 \, dx + \int_{\omega_\lambda} \Big[\frac{[\beta(y_g + \lambda q) - \beta(y_g)]}{\lambda} - \beta'(y_g; q)\Big] [q_\lambda - q - m_\lambda]^+ \, dx \leq 0. \tag{3.29}$$

Since $[q_\lambda - q - m_\lambda]^+$ can be extended by zero outside ω_λ (and it can thus be seen as an element in $H_0^1(D)$), we can apply Poincaré's inequality with a constant $c > 0$ independent of λ and obtain

$$\|[q_\lambda - q - m_\lambda]^+\|^2_{L^2(\omega_\lambda)} \leq c \Big\| \frac{[\beta(y_g + \lambda q) - \beta(y_g)]}{\lambda} - \beta'(y_g; q) \Big\|_{L^2(\omega_\lambda)} \|[q_\lambda - q - m_\lambda]^+\|_{L^2(\omega_\lambda)}. \tag{3.30}$$

This yields

$$\|[q_\lambda - q - m_\lambda]^+\|_{L^2(\omega_\lambda)} \leq c \Big\| \frac{[\beta(y_g + \lambda q) - \beta(y_g)]}{\lambda} - \beta'(y_g; q) \Big\|_{L^2(\omega_\lambda)}. \tag{3.31}$$

Adding $\|[q_\lambda - q - m_\lambda]^+\|^2_{L^2(\omega_\lambda)}$ on both sides in (3.29) then leads to

$$\begin{aligned} &\|[q_\lambda - q - m_\lambda]^+\|^2_{H^1(\omega_\lambda)} \\ &\leq \Big\| \frac{[\beta(y_g + \lambda q) - \beta(y_g)]}{\lambda} - \beta'(y_g; q) \Big\|_{L^{r'}(\omega_\lambda)} \|[q_\lambda - q - m_\lambda]^+\|_{L^r(\omega_\lambda)} \\ &\quad + \tilde{c} \|[q_\lambda - q - m_\lambda]^+\|_{L^r(\omega_\lambda)} \|[q_\lambda - q - m_\lambda]^+\|_{L^2(\omega_\lambda)}, \end{aligned} \tag{3.32}$$

where $r \in [2, \infty)$ is arbitrary and $\tilde{c} > 0$ is the embedding constant of $L^r(\omega_\lambda) \hookrightarrow L^2(\omega_\lambda)$; this is independent of λ, since $\mu(\omega_\lambda)$ is bounded by a constant independent of λ. By using the same extension argument as above and the embedding $H^1(D) \hookrightarrow L^r(D)$, as well as (3.31), we arrive at

$$\|q_\lambda - q - m_\lambda\|_{L^r(\omega_\lambda \cap \{q_\lambda - q > m_\lambda\})} \leq C \Big\| \frac{[\beta(y_g + \lambda q) - \beta(y_g)]}{\lambda} - \beta'(y_g; q) \Big\|_{L^2(\omega_\lambda)}, \tag{3.33}$$

where $C := 1 + \tilde{c}\, c > 0$ is independent of λ.

In a similar manner, we obtain

$$\|q - q_\lambda - m_\lambda\|_{L^r(\omega_\lambda \cap \{q - q_\lambda > m_\lambda\})} \leq \widetilde{C} \Big\| \frac{[\beta(y_g + \lambda q) - \beta(y_g)]}{\lambda} - \beta'(y_g; q) \Big\|_{L^2(\omega_\lambda)}, \tag{3.34}$$

where $\widetilde{C} > 0$ is independent of λ, and we only sketch the arguments.

Testing (3.27) with $(q - q_\lambda - m_\lambda)^+ \in H_0^1(\omega_\lambda)$ yields

$$\int_{\omega_\lambda} \nabla[q_\lambda - q] \nabla[q - q_\lambda - m_\lambda]^+ \, dx + \int_{\omega_\lambda} \Big[\frac{[\beta(y_{g+\lambda h}) - \beta(y_g)]}{\lambda} - \beta'(y_g; q)\Big] [q - q_\lambda - m_\lambda]^+ \, dx = 0, \tag{3.35}$$

which accounts to

$$-\int_{\omega_\lambda\cap\{q-q_\lambda>m_\lambda\}}(\nabla[q-q_\lambda-m_\lambda]^+)^2\,dx+\int_{\omega_\lambda\cap\{q-q_\lambda>m_\lambda\}}A_\lambda\,\underbrace{(q-q_\lambda-m_\lambda)}_{>0}\,dx=0. \tag{3.36}$$

Arguing as above we get that the first term in A_λ is negative on $\{q-q_\lambda>m_\lambda\}$ (since $y_{g+\lambda h}<y_g+\lambda q$ there). Therefore

$$\int_{\omega_\lambda}(\nabla[q-q_\lambda-m_\lambda]^+)^2\,dx\le\int_{\omega_\lambda}\Big[\frac{\beta(y_g+\lambda q)-\beta(y_g)}{\lambda}-\beta'(y_g;q)\Big][q-q_\lambda-m_\lambda]^+\,dx, \tag{3.37}$$

whence (3.34) follows (by the exact same arguments as above). Combining (3.33), (3.34) and (3.3) finally gives the desired assertion. □

Proposition 3.14. *Suppose that $g,h\in\mathcal{F}$ and (2.7) is true. If $m_\lambda\to 0$, then*

$$\|q_\lambda-q\|_{L^r(\omega_\lambda)}\to 0\quad\text{as }\lambda\searrow 0,\quad\forall\,r\in[1,\infty), \tag{3.38}$$

and, for each open subset $\omega\subset\subset\Omega_g$, we have

$$\|q_\lambda-q\|_{H^2(\omega)}\to 0\quad\text{as }\lambda\searrow 0. \tag{3.39}$$

Proof. Let $r\in[1,\infty)$ be fixed. In view of Lemma 3.13 and since $m_\lambda\to 0$, by assumption, we have

$$\begin{aligned}\|q_\lambda-q\|_{L^r(\omega_\lambda)}&\le\|q_\lambda-q\|_{L^r(\widetilde\omega_\lambda)}+\|q_\lambda-q\|_{L^r(\{|q_\lambda-q|<m_\lambda\})}\\&\le\||q_\lambda-q|-m_\lambda\|_{L^r(\widetilde\omega_\lambda)}+\mu(\widetilde\omega_\lambda)^{1/r}m_\lambda+\mu(\omega_\lambda)^{1/r}m_\lambda\\&\quad\to 0\quad\text{as }\lambda\searrow 0,\end{aligned}$$

where we recall $\widetilde\omega_\lambda=\{x\in\omega_\lambda:|q_\lambda-q|>m_\lambda\}$, and use that $\omega_\lambda\subset D$. This proves the first convergence.

We next show the second desired assertion. Recall that $\omega\subset\subset\Omega_g$ implies $\omega\subset\subset\Omega_{g+\lambda h}$ for $\lambda>0$ small (cf. proof of Lemma 3.5), and thus $\omega\subset\subset\omega_\lambda$. Due to [8, Sec.6.3.1, Thm. 1] applied for (3.27), we get that $q_\lambda-q\in H^2(\omega)$ and

$$\|q_\lambda-q\|_{H^2(\omega)}\le c\Big(\Big\|\frac{\beta(y_{g+\lambda h})-\beta(y_g)}{\lambda}-\beta'(y_g;q)\Big\|_{L^2(\omega_\lambda)}+\|q_\lambda-q\|_{L^2(\omega_\lambda)}\Big),$$

where $c>0$ depends only on $\frac{1}{\operatorname{dist}(\bar\omega,\partial\omega_\lambda)}$ and the given data, see [44, Ch. 3, Thm. 4.2]; to see that $\frac{1}{\operatorname{dist}(\bar\omega,\partial\omega_\lambda)}$ is bounded from above by a constant independent of λ, define $\rho:=\operatorname{dist}(\bar\omega,\partial\Omega)>0$ and use that $\partial\omega_\lambda\subset\{x\in D:\operatorname{dist}(x,\partial\Omega)\le\rho/2\}$ for λ small enough (which is due to (2.5)). Then $\operatorname{dist}(\bar\omega,\partial\omega_\lambda)\ge\rho/2$ and thus, $c>0$ is independent of λ.

Since β satisfies (3.2), the above estimate leads to

$$\begin{aligned}&\|q_\lambda-q\|_{H^2(\omega)}\\&\le c\Big(\Big\|\frac{\beta(y_{g+\lambda h})-\beta(y_g+\lambda q)}{\lambda}\Big\|_{L^2(\omega_\lambda)}+\Big\|\frac{\beta(y_g+\lambda q)-\beta(y_g)}{\lambda}-\beta'(y_g;q)\Big\|_{L^2(\omega_\lambda)}\\&\quad+\|q_\lambda-q\|_{L^2(\omega_\lambda)}\Big)\\&\le c\Big(L_M\,\|q_\lambda-q\|_{L^2(\omega_\lambda)}+\Big\|\frac{\beta(y_g+\lambda q)-\beta(y_g)}{\lambda}-\beta'(y_g;q)\Big\|_{L^2(\omega_\lambda)}\Big)\\&\quad+c\|q_\lambda-q\|_{L^2(\omega_\lambda)},\end{aligned}$$

where $M := \sup\{\|y_{g+\lambda h}\|_{L^\infty(\omega_\lambda)}, \|y_g + \lambda q\|_{L^\infty(\omega_\lambda)}\}$; the fact that $\|y_{g+\lambda h}\|_{L^\infty(\omega_\lambda)}$ is uniformly bounded is owed to (3.5) (note that $\|y_g + \lambda q\|_{L^\infty(\omega_\lambda)}$ is uniformly bounded as well since $\omega_\lambda \subset \Omega_g$ and y_g and q are both continuous on Ω_g).

Now, (3.39) follows from (3.38) and (3.3). This completes the proof. □

Given the result in Proposition 3.14, our next goal is to prove that $m_\lambda \to 0$, by looking at the convergence behaviour of the difference quotient $\frac{y_{g+\lambda h} - y}{\lambda} - q$ on the boundary of $\omega_\lambda = \Omega_g \cap \Omega_{g+\lambda h}$. To this end, we observe that

$$\partial\omega_\lambda = \Gamma_2 \cup (\partial\Omega_{g+\lambda h} \cap \Omega_g), \tag{3.40}$$

where Γ_2 is given by (3.10), that is, $\partial\omega_\lambda$ consists of a finite union of (relatively closed) curves that remain fixed with varying λ, while $(\partial\Omega_{g+\lambda h} \cap \Omega_g)$ varies with changing λ.

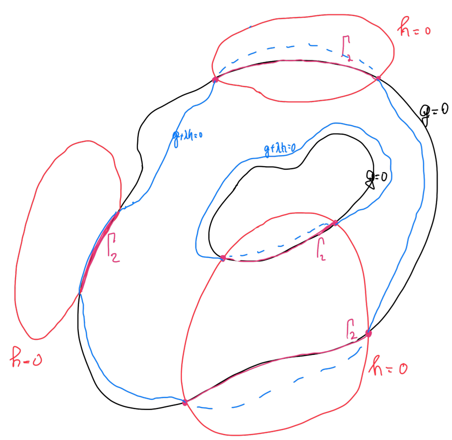

FIG. 3.2. *The blue curves describe $\partial\Omega_{g+\lambda h}$, while the black curves represent $\partial\Omega_g$. The doted blue lines are those parts of $\partial\Omega_{g+\lambda h}$ that are outside $\bar{\Omega}_g$*

LEMMA 3.15 (Behaviour of the difference quotients on Γ_2). *Suppose that $g, h \in \mathcal{F}$ and (2.7) is true. Then,*

$$\|q_\lambda - q\|_{\mathcal{C}(\Gamma_2)} \to 0 \quad \textit{as } \lambda \searrow 0.$$

Proof. Throughout this proof $\lambda > 0$ is small enough, so that the unicity of the approximating curve is guaranteed, see Proposition 2.4. Since $h \in \mathcal{F}$, by assumption, we have that $\{h = 0\}$ is a finite union of disjoint closed $\mathcal{C}^2$ curves (see Proposition 2.1). Thus, Γ_2 is a finite union of relatively closed curves and it suffices to prove the assertion for one of this curves, which we call Γ_2 as well (with a little abuse of notation). If Γ_2 is such that $\Gamma_2 \subset \partial\Omega_g \cap \partial\Omega_{g+\lambda h}$, then $y_{g+\lambda h} = 0 = y_g$ on Γ_2 and $W = 0$ on Γ_2 as well (Definition 3.8 and Proposition 2.6), whence $\frac{y_{g+\lambda h} - y_g}{\lambda} = 0 = q$ on Γ_2 follows. Otherwise, take x_0 to be the initial point of Γ_2, i.e., $h(x_0) = g(x_0) = 0$. Consider the Hamiltonian systems (2.2) and (2.3) with this initial condition and associated solutions z and z_λ, respectively; then Γ_2 is completely described by $z : I \to \{g = 0\}$, where $I = [0, \iota]$, for some $\iota > 0$ and $z(\iota) \in \{h = 0\} \cap \{g = 0\}$ (this does not exclude the case $z(\iota) = x_0$). Thus, to prove the desired assertion, it suffices to show that

$$(q_\lambda - q)(z(\cdot)) \to 0 \text{ uniformly on } I, \quad \text{as } \lambda \searrow 0.$$

To obtain a first convergence result, we make use of [43, Lem. 5.5]. We observe that $\Omega_{g+\lambda h}$ is uniformly locally quasiconvex with constants independent of λ [43, Def. 5.1]; this is due to Lemma 2.3 and since $|\nabla(g+\lambda h)|$ is uniformly bounded w.r.t. λ, for $\lambda>0$ small. In addition, thanks to (3.5), we can choose the constant $\delta>0$ in [43, Lem. 5.5] independent of λ as well (see its proof). Hence, applying [43, Lem. 5.5] for $y_{g+\lambda h}\in\mathcal{C}^1(\Omega_{g+\lambda h})$ (where we use Proposition 2.5.(i) and the fact that $z(t)\in\Omega_{g+\lambda h}$, for $t\in I$) yields : $\forall\varepsilon>0$ there exists $\lambda(\varepsilon)>0$ so that

$$\max_{t\in I}\underbrace{\Big|\frac{y_{g+\lambda h}(z(t))-y_{g+\lambda h}(z_\lambda(t))}{\lambda}+\nabla y_{g+\lambda h}(z(t))w_\lambda(t)\Big|}_{=:a_\lambda(t)}\leq\varepsilon\max_{t\in I}|w_\lambda(t)|\ \forall\lambda\in(0,\lambda(\varepsilon)).$$

Thus, as $\|w_\lambda\|_{\mathcal{C}[0,T]}$ is uniformly bounded (see (2.9)), we deduce

$$a_\lambda\to 0 \text{ uniformly on } I. \tag{3.41}$$

Since $y_g(z(t))=y_{g+\lambda h}(z_\lambda(t))(=0),\ t\in\mathbb{R}$, we have

$$\begin{aligned}
&\max_{t\in I}\Big|\frac{y_{g+\lambda h}(z(t))-y_g(z(t))}{\lambda}-q(z(t))\Big|\\
&=\max_{t\in I}\Big|\frac{y_{g+\lambda h}(z(t))-y_{g+\lambda h}(z_\lambda(t))}{\lambda}\pm\nabla y_{g+\lambda h}(z(t))w_\lambda(t)\pm\nabla y_g(z(t))w_\lambda(t)-q(z(t))\Big|\\
&\quad\leq\max_{t\in I}a_\lambda(t)+\|\nabla y_{g+\lambda h}(z(\cdot))-\nabla y_g(z(\cdot))\|_{\mathcal{C}(I)}\|w_\lambda\|_{\mathcal{C}[0,T]}\\
&\qquad+\|\nabla y_g(z(\cdot))\|_{\mathcal{C}[0,T]}\|w_\lambda-w\|_{\mathcal{C}[0,T]},
\end{aligned}$$

where we also employed (3.26) and (3.19). Due to (3.41), (2.9) and (3.12) we now arrive at the desired convergence. □

Lemma 3.16 (Behaviour of the derivative on the varying part of the boundary of ω_λ). *Suppose that $g,h\in\mathcal{F}$ and (2.7) is true. We have*

$$\|q_\lambda-q\|_{\mathcal{C}(\partial\Omega_{g+\lambda h}\cap\bar\Omega_g)}\to 0\quad as\ \lambda\searrow 0.$$

Proof. Again, fix $\lambda>0$ small so that the unicity of the approximating curve is guaranteed, see Proposition 2.4. By the same arguments as in the proof of Lemma 3.15 we have that $\partial\Omega_{g+\lambda h}\cap\bar\Omega_g$ is a finite union of (relatively closed) curves and it suffices to prove the assertion for one of this curves, which we call Γ_λ in what follows (with a little abuse of notation). The case $\Gamma_\lambda\subset\partial\Omega_g\cap\partial\Omega_{g+\lambda h}$ can be tackled as in the proof of Lemma 3.15. If $\Gamma_\lambda\subset\Omega_g$, take x_0 to be the initial point of Γ_λ, i.e., $h(x_0)=g(x_0)=0$. Consider the Hamiltonian systems (2.2) and (2.3) with this initial condition and associated solutions z and z_λ, respectively; then Γ_λ is completely described by $z_\lambda:I_\lambda\to\{g+\lambda h=0\}$, where $I_\lambda=[0,\iota_\lambda]$, for some $\iota_\lambda>0$ and $z_\lambda(\iota_\lambda)\in\{h=0\}\cap\{g=0\}$. Our goal is to show that

$$\max_{t\in I_\lambda}|q_\lambda-q|\to 0\quad\text{as }\lambda\searrow 0.$$

Arguing in the exact same way as in the proof of Lemma 3.15 we obtain

$$\max_{t\in I_\lambda}\underbrace{\Big|\frac{y_g(z(t))-y_g(z_\lambda(t))}{\lambda}+\nabla y_g(z(t))w_\lambda(t)\Big|}_{=:b_\lambda(t)}\to 0. \tag{3.42}$$

Further, we have for each $t \in I_\lambda$

$$\frac{\overbrace{y_{g+\lambda h}(z_\lambda(t))}^{=0} - y_g(z_\lambda(t))}{\lambda} - q(z_\lambda(t)) = \frac{\overbrace{y_g(z(t))}^{=0} - y_g(z_\lambda(t))}{\lambda} \\ \pm \nabla y_g(z(t))w_\lambda(t) \pm (\nabla y_g W)(z(t)) - q(z_\lambda(t)).$$

Thanks to (3.19) and (3.26), this results in

$$\begin{aligned}&\max_{t\in I_\lambda}\Big|\frac{y_{g+\lambda h}(z_\lambda(t)) - y_g(z_\lambda(t))}{\lambda} - q(z_\lambda(t))\Big| \\ &\le \max_{t\in I_\lambda} b_\lambda(t) + \|\nabla y_g(z(\cdot))\|_{\mathcal{C}[0,T]}\|w_\lambda - w\|_{\mathcal{C}[0,T]} \\ &\quad + \widehat{c}\max_{t\in I_\lambda}|z(t) - z_\lambda(t)|^\alpha,\end{aligned}$$

where we used the embedding $W^{1,p}(\Omega_g) \hookrightarrow \mathcal{C}^{0,\alpha}(\bar{\Omega}_g)$, $\alpha > 0$ and the fact that $q \in W^{1,p}(\Omega_g)$. In view of (3.42) and Propositions 2.5 and 2.6, the proof is now complete. ∎

Summarizing, from Lemmas 3.15, 3.16 and (3.40), we have

PROPOSITION 3.17. *Suppose that $g, h \in \mathcal{F}$ and (2.7) is true. It holds*

$$\|q_\lambda - q\|_{\mathcal{C}(\partial\omega_\lambda)} \to 0 \quad \text{as } \lambda \searrow 0.$$

We are now in the position to state a first differentiability result for $\mathcal{S}$. This will allow us to prove the differentiability of the objective in the presence of derivatives of the state as well as pointwise observation on a prescribed subdomain, see subsection 5.1.

THEOREM 3.18. *Assume that $g, h \in \mathcal{F}$ and (2.7) is true. Then*

$$\lim_{\lambda\searrow 0}\Big\|\frac{\mathcal{S}(g+\lambda h) - \mathcal{S}(g)}{\lambda} - q\Big\|_{H^2(\omega)} \to 0,$$

where $q \in W^{1,p}(\Omega_g)$ is the unique solution of (3.25)-(3.26) and $\omega \subset\subset \Omega_g$ is a compact subset.

Proof. The assertion is due to Propositions 3.14 and 3.17. ∎

To state a differentiability result for $\mathcal{S}$ that permits us to have distributed observation in the unknown domain in the objective as well (see subsection 5.2), we will also need to take a look at the convergence behaviour of $q_\lambda - q$ outside ω_λ (e.g. on $\Omega_{g+\lambda h} \setminus \Omega_g$ and on $\Omega_g \setminus \Omega_{g+\lambda h}$).

PROPOSITION 3.19. *Suppose that $g, h \in \mathcal{F}$ and (2.7) is true. We have*

$$\|q_\lambda - q\|_{L^r(\Omega_g\setminus\Omega_{g+\lambda h})} \to 0, \quad \forall\, r \in [1, \infty),$$

$$\|q_\lambda - q\|_{L^r(\Omega_{g+\lambda h}\setminus\Omega_g)} \to 0 \quad \forall\, r \in [1, \infty),$$

where we recall $q_\lambda = (y_{g+\lambda h} - y_g)/\lambda$ and that $y_{g+\lambda h}$ is extended by zero outside $\Omega_{g+\lambda h}$, while y_g and q are extended by zero outside Ω_g.

Proof. Let $r \in [1, \infty)$ be arbitrary but fixed. Take $\lambda > 0$ small enough and fix $\varrho \in (r, \infty)$. We first show that

$$\|q_\lambda\|_{L^\varrho(D)} \le c, \tag{3.43}$$

where $c > 0$ is independent of λ.
(i) If $x \in \Omega_{g+\lambda h} \setminus \Omega_g$, then $q_\lambda(x) = \frac{y_{g+\lambda h}(x)}{\lambda}$. By (3.5), one has

$$|y_{g+\lambda h}(x) - \underbrace{y_{g+\lambda h}(\widehat{x})}_{=0}| \leq L|x - \widehat{x}| \leq L\lambda,$$

where $\widehat{x} \in \partial\Omega_{g+\lambda h}$ and $L > 0$ is independent of λ. Hence, it holds $\|q_\lambda\|_{L^\infty(\Omega_{g+\lambda h}\setminus\Omega_g)} \leq L$.
(ii) If $x \in \Omega_g \setminus \Omega_{g+\lambda h}$, we argue as above by taking into account that $q_\lambda(x) = \frac{-y_g(x)}{\lambda}$ and $|y_g(x) - y_g(\widehat{x})| \leq L\lambda$, where $\widehat{x} \in \partial\Omega_g$; we obtain $\|q_\lambda\|_{L^\infty(\Omega_g\setminus\Omega_{g+\lambda h})} \leq L$.
(iii) If $x \notin \Omega_{g+\lambda h} \cup \Omega_g$, then $q_\lambda(x) = 0$. Due to Proposition 3.17, we also have that $\|q_\lambda\|_{L^\varrho(\omega_\lambda)}$ is uniformly bounded. From all the above, we arrive at (3.43). Further, by (3.7), one has

$$\|q_\lambda - q\|_{L^r(\Omega_g\setminus\Omega_{g+\lambda h})} \leq \mu(\Omega_g \setminus \Omega_{g+\lambda h})^{r\varrho/\varrho-r}\|q_\lambda - q\|_{L^\varrho(D)} \to 0,$$

$$\|q_\lambda - q\|_{L^r(\Omega_{g+\lambda h}\setminus\Omega_g)} \leq \mu(\Omega_{g+\lambda h} \setminus \Omega_g)^{r\varrho/\varrho-r}\|q_\lambda - q\|_{L^\varrho(D)} \to 0,$$

and the desired assertion is now proven. □

The next theorem contains the second main differentiability result for $\mathcal{S}$.

THEOREM 3.20. *Assume that $g, h \in \mathcal{F}$ and (2.7) is true. Then*

$$\lim_{\lambda\searrow 0} \Big\| \frac{\mathcal{S}(g+\lambda h) - \mathcal{S}(g)}{\lambda} - q \Big\|_{L^r(\Omega_g)} \to 0, \quad \forall\, r \in [1, \infty),$$

where $q \in W^{1,p}(\Omega_g)$ is the unique solution of (3.25)-(3.26). Moreover,

$$\lim_{\lambda\searrow 0} \Big\| \frac{\mathcal{S}(g+\lambda h) - \mathcal{S}(g)}{\lambda} - q \Big\|_{L^r(D)} \to 0, \quad \forall\, r \in [1, \infty),$$

when $q \in W^{1,p}(\Omega_g)$ is extended by zero outside Ω_g. Hence, $\mathcal{S} : \mathcal{F} \to L^r(D)$ is directionally differentiable at g in direction h with

$$\mathcal{S}'(g; h) = q.$$

Proof. The result is due to Propositions 3.17 and 3.19. □

We end this section by recalling that q is in fact the shape derivative of y_{Ω_g} in direction W [36, Def. 2.85], cf. Remark 3.10. Hence, in view of Theorem 3.20, we have obtained

$$\lim_{\lambda\searrow 0} \frac{\mathcal{S}(g+\lambda h) - \mathcal{S}(g)}{\lambda} = \lim_{\lambda\searrow 0} \frac{y_{T_\lambda^{\mathcal{W}}(\Omega_g)} - y_{\Omega_g}}{\lambda} = q \text{ in } L^r(D), \tag{3.44}$$

where $T_\lambda^{\mathcal{W}}(\Omega_g)$ is the flow generated by a family of vector fields $\mathcal{W}$ for which $\mathcal{W}(0) = W$. We stress that our perturbations happen only at the boundary level, and that, as a consequence, the whole domain gets then automatically perturbed. An additional (rather formal) confirmation of this statement is given in section 6 below, where we present the connection between the FVA and the speed method.

4. Equivalence of the shape optimization problem with a control problem with admissible set consisting of functions. In this section we turn our attention to shape optimization problems governed by (3.1), such as

$$\left.\begin{aligned} \min_{\Omega\in\mathcal{O}, E\subset\subset\Omega} \quad & \mathcal{J}(\Omega, y_\Omega) \\ \text{s.t.} \quad & -\triangle y_\Omega + \beta(y_\Omega) = f \quad \text{in } \Omega, \\ & y_\Omega = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{P_Ω}$$

where $\mathcal{J} : \mathcal{C}_2 \times H_0^1(D) \to \mathbb{R}$ is a given objective functional and $\mathcal{C}_2$ is the set of all subsets of D of class $\mathcal{C}^2$. The set of admissible shapes is given by

$$\mathcal{O} := \{\Omega \subset D : \Omega \text{ is a non-empty } \mathcal{C}^2 \text{ domain}, \ \partial\Omega \cap \partial D = \emptyset\},$$

while the observation set $E \subset\subset D$ is a non-empty subdomain of D.

By arguing exactly as in the proof of [4, Prop. 2.8], one shows that

$$\mathcal{O} = \{\Omega_g : g \in \mathcal{F}\}, \tag{4.1}$$

where we recall

$$\mathcal{F} = \{g \in \mathcal{C}^2(\bar{D}) : \ |\nabla g(x)| + |g(x)| > 0 \,\forall\, x \in D, \ g(x) > 0 \,\forall\, x \in \partial D, \ g^{-1}((-\infty, 0]) \neq \emptyset\}. \tag{4.2}$$

Note that the last condition in (4.2) ensures that $\Omega_g \neq \emptyset$. While the inclusion " $\subset$ " in (4.1) is due to arguments inspired by [7, Thm. 4.1], the opposite inclusion follows by [7, Thm. 4.2]. The latter gives that, for $g \in \mathcal{F}$, the set Ω_g is of class $\mathcal{C}^2$ (and its boundary does not intersect ∂D); but this is not necessarily a domain. However, by [4, Lem. 2.9], one can find another $\hat{g} \in \mathcal{F}$ so that $\Omega_{\hat{g}} \in \mathcal{O}$.

In the sequel we abbreviate for simplicity

$$\mathcal{O}_E := \{\Omega \in \mathcal{O} : E \subset\subset \Omega\}$$

and

$$\mathcal{F}_E := \{g \in \mathcal{F} : g < 0 \text{ in } \bar{E}\}.$$

Note that, due to the above arguments, it holds

$$\mathcal{O}_E = \{\Omega_g : g \in \mathcal{F}_E\}. \tag{4.3}$$

As in [4], we want to rewrite (P_Ω) as an optimal control problem where the admissible set consists of functions, namely

$$\left.\begin{aligned} \min_{g\in\mathcal{F}_E} \quad & \mathcal{J}(\Omega_g, y_g) \\ \text{s.t.} \quad & -\triangle y_g + \beta(y_g) = f \quad \text{in } \Omega_g, \\ & y_g = 0 \quad \text{on } \partial\Omega_g. \end{aligned}\right\} \tag{P}$$

To ensure that (P_Ω) is equivalent to (P), we need to ask that $\mathcal{J}$ satisfies for all $\Omega_1 \in \mathcal{O}, \Omega_2 \in \mathcal{C}_2$ the estimate

$$\mathcal{J}(\Omega_1, y_{\Omega_1}) \leq \mathcal{J}(\Omega_2, y_{\Omega_2}), \quad \text{if } \Omega_1 \subset \Omega_2. \tag{4.4}$$

Then, the above mentioned equivalence is to be understood as follows.

PROPOSITION 4.1. *Assume that $\mathcal{J}$ fulfills (4.4). Let $\Omega^\star \in \mathcal{O}_E$ be an optimal shape of (P_Ω). Then, each of the functions $g^\star \in \mathcal{F}_E$ that satisfy $\Omega_{g^\star} = \Omega^\star$ is a global minimizer of (P). Conversely, if $g^\star \in \mathcal{F}_E$ minimizes (P), then the component of $\Omega_{g^\star}$ that contains $\bar{E}$ is an optimal shape for (P_Ω).*

Proof. We follow the lines of the proof of [4, Prop. 2.10]. Let $\Omega^\star \in \mathcal{O}$ be an optimal shape for (P_Ω), i.e.,

$$\mathcal{J}(\Omega^\star, y_{\Omega^\star}) \le \mathcal{J}(\Omega, y_\Omega) \quad \forall\, \Omega \in \mathcal{O}_E.$$

Now, let $g^\star \in \mathcal{F}$ with $\Omega_{g^\star} = \Omega^\star$ be fixed (note that, according to [4, Lem. 2.9], there are infinitely many mappings with this property). Then, $y_{\Omega^\star} = y_{g^\star}$ and

$$\mathcal{J}(\Omega_{g^\star}, y_{g^\star}) \le \mathcal{J}(\Omega, y_\Omega) \quad \forall\, \Omega \in \mathcal{O}_E.$$

Let $g \in \mathcal{F}_E$, be arbitrary and fixed; define $\hat{g} \in \mathcal{F}$ so that $\Omega_{\hat{g}} \in \mathcal{O}$ is the component of Ω_g that contains $\bar{E}$. Testing with $\Omega_{\hat{g}}$ in the above inequality yields

$$\mathcal{J}(\Omega_{g^\star}, y_{g^\star}) \le \mathcal{J}(\Omega_{\hat{g}}, y_{\hat{g}}) \le \mathcal{J}(\Omega_g, y_g),$$

where in the second estimate we used (4.4). This proves the first statement.
To show the converse assertion, assume that $g^\star \in \mathcal{F}_E$ satisfies

$$\mathcal{J}(\Omega_{g^\star}, y_{g^\star}) \le \mathcal{J}(\Omega_g, y_g) \quad \forall\, g \in \mathcal{F}_E. \tag{4.5}$$

We denote by $\omega_{g^\star}$ the component of $\Omega_{g^\star}$ that contains $\bar{E}$. Let $\Omega \in \mathcal{O}_E$ be arbitrary but fixed. Again, we can define $\widetilde{g} \in \mathcal{F}_E$ so that $\Omega_{\widetilde{g}} = \Omega$. Then, $y_\Omega = y_{\widetilde{g}}$. In view of (4.5), where we test with $\widetilde{g}$, we have

$$\mathcal{J}(\omega_{g^\star}, y_{g^\star}) \le \mathcal{J}(\Omega_{g^\star}, y_{g^\star}) \le \mathcal{J}(\Omega_{\widetilde{g}}, y_{\widetilde{g}}) = \mathcal{J}(\Omega, y_\Omega),$$

where in the first estimate we used again (4.4). This proves the second assertion and completes the proof. ☐

With the result in Proposition 4.1 at hand, the reduced control problem we are interested in the sequel reads

$$\min_{g \in \mathcal{F}_E} \quad j(g), \tag{4.6}$$

where we abbreviate

$$j(g) := \mathcal{J}(\Omega_g, y_g)$$

in all what follows.

4.1. Local optimality. In this subsection, we go a step further and establish a notion of local optimal shape for (P_Ω) and its relation with local optimality for control problems such as (4.6).

DEFINITION 4.2. *We say that $\Omega^\star \in \mathcal{O}_E$ is locally optimal for (P_Ω) if there exists $R > 0$ so that*

$$\mathcal{J}(\Omega^\star, y_{\Omega^\star}) \le \mathcal{J}(\Omega, y_\Omega) \quad \forall\, \Omega \in \mathcal{O}_E \text{ with } d_{\mathcal{H}}(\partial\Omega^\star, \partial\Omega) \le R.$$

Theorem 4.3. *Let $\Omega^\star \in \mathcal{O}_E$ be a locally optimal shape of (P_Ω). Then, each of the functions $g^\star \in \mathcal{F}_E$ that satisfy $\Omega_{g^\star} = \Omega^\star$ is a "local minimizer" of (P) in the following (restricted) sense: There exists $r > 0$ so that*

$$\begin{aligned} j(g^\star) \leq j(g) \quad \forall\, g \in \mathcal{F}_E \text{ with } \|g - g^\star\|_{\mathcal{C}^1(\bar{D})} \leq r \text{ so that } \Omega_g \in \mathcal{O}, \\ \{g = 0\} \cap \{g^\star = 0\}^\circ \neq \emptyset \quad \forall\, \{g^\star = 0\}^\circ \subset \{g^\star = 0\}, \\ \text{and } \{g^\star = 0\} \cap \{g = 0\}^\circ \neq \emptyset \quad \forall\, \{g = 0\}^\circ \subset \{g = 0\}. \end{aligned} \tag{4.7}$$

Remark 4.4. *The concept of local optimality for (4.6) in Theorem 4.3 may appear a bit restrictive. However, the admissible set in (4.7) allows for test functions such as $g + \lambda h$, provided that λ is small enough and $h \in \mathcal{F}$ satisfies (2.7). Since this is the only type of functional variation we are interested in, Theorem 4.3 serves its purpose and we can provide first-order necessary optimality conditions for all locally optimal shapes of (P_Ω), cf. Theorem 5.1 below.*

Proof. Let $g^\star \in \mathcal{F}_E$ with $\Omega_{g^\star} = \Omega^\star$ be fixed. According to Proposition 2.1, $\{g^\star = 0\}$ constitutes of a finite number of closed curves and we denote in the sequel by $T > 0$ the largest period among these curves.

Define

$$r := \frac{R}{T \exp(2T \sup\{L_{\partial_1 g^\star}, L_{\partial_2 g^\star}\})} > 0, \tag{4.8}$$

where $L_{\partial_i g^\star} > 0, i = 1, 2$ denotes the Lipschitz constant of $\partial_i g^\star$ and $R > 0$ is the radius of local optimality of $\Omega^\star$. Let $g \in \mathcal{F}_E$ with

$$\|g - g^\star\|_{\mathcal{C}^1(\bar{D})} \leq r \tag{4.9}$$

and $\Omega_g \in \mathcal{O}$ so that

$$\{g = 0\} \cap \{g^\star = 0\}^\circ \neq \emptyset \quad \forall\, \{g^\star = 0\}^\circ \subset \{g^\star = 0\}, \tag{4.10}$$

$$\{g^\star = 0\} \cap \{g = 0\}^\circ \neq \emptyset \quad \forall\, \{g = 0\}^\circ \subset \{g = 0\}. \tag{4.11}$$

Our aim is to show that $d_{\mathcal{H}}(\partial\Omega^\star, \partial\Omega_g) \leq R$.

Take $\{g^\star = 0\}^\circ$. In light of (4.10) there is a component of $\{g = 0\}$ that intersects $\{g^\star = 0\}^\circ$, say $\{g = 0\}^\circ$. Then, we can choose in the Hamiltonian system (2.2) associated with $\{g^\star = 0\}^\circ$ the initial condition $x_0 \in \{g = 0\}^\circ$, so that

$$g^\star(x_0) = g(x_0) = 0. \tag{4.12}$$

Estimating as in the proof of [5, Prop. 3.10] we obtain

$$|z_{g^\star}(t) - z_g(t)| \leq 2 \sup\{L_{\partial_1 g^\star}, L_{\partial_2 g^\star}\} \int_0^t |z_{g^\star}(s) - z_g(s)|\, ds + t\|g^\star - g\|_{\mathcal{C}^1(\bar{D})} \quad \forall\, t \in \mathbb{R}_+,$$

and Gronwall's lemma gives in turn

$$|z_{g^\star}(t) - z_g(t)| \leq Ct\|g^\star - g\|_{\mathcal{C}^1(\bar{D})} \quad \forall\, t \in \mathbb{R}_+, \tag{4.13}$$

where

$$C := \exp(2T \sup\{L_{\partial_1 g^\star}, L_{\partial_2 g^\star}\}). \tag{4.14}$$

According to [12, Prop. 2.2.27], it holds

$$d_{\mathcal{H}}(\partial\Omega^\star, \partial\Omega_g) = \| \operatorname{dist}_{\partial\Omega^\star} - \operatorname{dist}_{\partial\Omega_g} \|_{L^\infty(\partial\Omega^\star \cup \partial\Omega_g)}.$$

Take $x \in \partial\Omega^\star \cup \partial\Omega_g$ arbitrary but fixed. If $x \in \partial\Omega^\star$, then x belongs to a closed curve $\{g^\star = 0\}^\circ$ and there is $t \in [0, T]$ so that $x = z_{g^\star}(t)$. Then

$$|\operatorname{dist}_{\partial\Omega_g} x| \le |z_{g^\star}(t) - z_g(t)| \le CT\|g^\star - g\|_{\mathcal{C}^1(\bar{D})},$$

where z_g describes a closed curve $\{g = 0\}^\circ$ intersecting $\{g^\star = 0\}^\circ$ (that is, we can choose $z_{g^\star}(0) = z_g(0) = x_0$ which allows us to make use of (4.13)). We estimate in the exact same manner for $x \in \partial\Omega_g$. We underline that this is possible due to (4.11). To summarize we obtained

$$d_{\mathcal{H}}(\partial\Omega^\star, \partial\Omega_g) \le CT\|g^\star - g\|_{\mathcal{C}^1(\bar{D})} \le \exp(2T\sup\{L_{\partial_1 g^\star}, L_{\partial_2 g^\star}\})Tr = R,$$

where we relied on (4.14), (4.9) and (4.8).

Finally we recall that $\Omega_{g^\star}$ is locally optimal for (P_Ω), cf. Definition 4.2, i.e.,

$$\mathcal{J}(\Omega_{g^\star}, y_{g^\star}) \le \mathcal{J}(\Omega, y_\Omega) \quad \forall\, \Omega \in \mathcal{O}_E \text{ with } d_{\mathcal{H}}(\partial\Omega^\star, \partial\Omega) \le R.$$

Since $\Omega_g \in \mathcal{O}_E$ (as $\Omega_g \in \mathcal{O}$ and $g \in \mathcal{F}_E$) we deduce

$$\mathcal{J}(\Omega_{g^\star}, y_{g^\star}) \le \mathcal{J}(\Omega_g, y_{\Omega_g})$$

which completes the proof. $\square$

We end this section with an essential observation concerning the opposite implication in Theorem 4.3.

Remark 4.5. *The fundamental argument standing at the core of the proof of Theorem 4.3 is that if two parametrizations are "close" to each other, then their associated shapes will be "close" too. The reverse statement is however not true. If two shapes are identical even, this does not mean that the same is true for their parametrizations, on the contrary: for each $g \in \mathcal{F}$, it holds $\Omega_g = \Omega_{n\,g},\ n \in \mathbb{N}_+$, that is, $\|g_n - g\|_{\mathcal{C}^1(\bar{D})} \to \infty$ as $n \to \infty$ for $g_n = n\,g$. Hence, it is not clear if the local optimality of some $g^\star$ for the control problem (4.6) implies that $\Omega_{g^\star}$ is a locally optimal shape for (P_Ω).*

5. Necessary optimality conditions for locally optimal shapes.

With the results from the previous section at hand, we can write a necessary optimality condition in primal form for each locally optimal shape of (P_Ω).

Theorem 5.1. *Let $\Omega^\star \in \mathcal{O}_E$ be a locally optimal shape of (P_Ω) in the sense of Definition 4.2. Then, each of the functions $g^\star \in \mathcal{F}_E$ that satisfy $\Omega_{g^\star} = \Omega^\star$ fulfill*

$$j'(g^\star; h) \ge 0 \quad \forall\, h \in \mathcal{F} \text{ with } \{h = 0\} \cap \{g^\star = 0\}^\circ \ne \emptyset \quad \forall\, \{g^\star = 0\}^\circ \subset \{g^\star = 0\},$$

provided that j is directionally differentiable at $g^\star$ in direction h.

Proof. Thanks to Theorem 4.3, we have

$$\begin{aligned} j(g^\star) \le j(g) \quad &\forall\, g \in \mathcal{F}_E \text{ with } \|g - g^\star\|_{\mathcal{C}^1(\bar{D})} \le r \text{ so that } \Omega_g \in \mathcal{O}, \\ &\{g = 0\} \cap \{g^\star = 0\}^\circ \ne \emptyset \quad \forall\, \{g^\star = 0\}^\circ \subset \{g^\star = 0\}, \\ &\text{and } \{g^\star = 0\} \cap \{g = 0\}^\circ \ne \emptyset \quad \forall\, \{g = 0\}^\circ \subset \{g = 0\}, \end{aligned} \tag{5.1}$$

with some $r > 0$. By using Weierstrass theorem, see for instance [5, Lem. 3.8], one can show that, given $h \in \mathcal{C}^2(\bar{D})$, it holds

$$g^\star \in \mathcal{F}_E \Rightarrow g^\star + \lambda h =: \widehat{g} \in \mathcal{F}_E$$

for $\lambda > 0$ small, dependent only on $g^\star$ and h and the given data. Moreover, $\|\widehat{g} - g^\star\|_{\mathcal{C}^1(\bar{D})} = \lambda \|h\|_{\mathcal{C}^1(\bar{D})} \leq r$, by choosing $\lambda > 0$ small enough. Now let $h \in \mathcal{F}$ with

$$\{h = 0\} \cap \{g^\star = 0\}^\circ \neq \emptyset \quad \forall \{g^\star = 0\}^\circ \subset \{g^\star = 0\}$$

be arbitrary but fixed and fix also $\{g^\star = 0\}^\circ \subset \{g^\star = 0\}$. Then there is $x_0 \in \{g^\star = 0\}^\circ$ so that $h(x_0) = 0$, which means that $(g^\star + \lambda h)(x_0) = 0$. That is, the second line in (5.1) is satisfied by $\hat{g} = g^\star + \lambda h$. If $\lambda > 0$ is chosen so small that $\{\widehat{g} = 0\}^\circ$ is the unique approximating curve of $\{g^\star = 0\}^\circ$ (cf. Proposition 2.4), then each of the components of $\{\widehat{g} = 0\}$ intersects $\{g^\star = 0\}$. This implies that the third line in (5.1) is satisfied by $\widehat{g} = g^\star + \lambda h$ as well.

In view of all the above, $\widehat{g}$ is admissible for (5.1) and we obtain

$$j(g^\star) \leq j(g^\star + \lambda h) \quad \forall \lambda \in (0, \lambda_0(h, g^\star)),$$

where $\lambda_0(h, g^\star) > 0$ is small enough. Hence, dividing by λ, rearranging the terms and using that j is directionally differentiable at $g^\star$ in direction h yields the desired assertion.□

In the next two subsections we investigate the directional differentiability of shape functionals $\mathcal{J}(\Omega_g, y_{\Omega_g})$ w.r.t. the shape function g for various types of instances for $\mathcal{J}$. Thanks to the above result, we can provide necessary optimality conditions not only for globally but for locally optimal shapes as well.

5.1. H^2 norms and pointwise observation on a given set. In this subsection we are concerned with shape functionals of the type

$$\mathcal{J}(y_\Omega) := \frac{1}{2}\|y_\Omega - y_d\|^2_{H^2(E)} + j_{obs}(y_\Omega), \quad E \subset\subset \Omega,$$

where $y_d \in H^2(E)$ and

$$j_{obs}(y_\Omega) := \frac{1}{2}\sum_{k=1}^{N}(y_\Omega - y_d)^2(x^k_{obs}),$$

where $N \in \mathbb{N}_+,\ x^k_{obs} \in \bar{E},\ k = 1, ..., N$. Note that performance indices such as j_{obs} are of particular interest when the optimal shape must be chosen so that the state fits a desired value y_d which can be measured only pointwise (at x^k_{obs}) by a finite number of sensors N.

The corresponding reduced objective is

$$j(g) = \frac{1}{2}\|y_g - y_d\|^2_{H^2(E)} + \frac{1}{2}\sum_{k=1}^{N}(y_g - y_d)^2(x^k_{obs}), \quad g \in \mathcal{F} \text{ with } g < 0 \text{ in } \bar{E}.$$

We now intend to apply the differentiability result from Theorem 3.18 in order to obtain a formula for the directional derivative of j.

Proposition 5.2. *Assume that $g, h \in \mathcal{F}$ and (2.7) is true. Moreover, suppose that $\Omega_g \in \mathcal{O}_E$. Then j is directionally differentiable at g in direction h with*

$$j'(g;h) = (y_g - y_d, q)_{H^2(E)} + j'_{obs}(g;h),$$

where $q = \mathcal{S}'(g;h)$ is the unique solution of (3.25)-(3.26) and

$$j'_{obs}(g;h) = \sum_{k=1}^{N} (y_g - y_d)(x^k_{obs}) q(x^k_{obs}).$$

Proof. The assumption $\Omega_g \in \mathcal{O}_E$ implies $E \subset\subset \Omega_g$. From Theorem 3.18 we deduce that $\mathcal{S} : \mathcal{F} \to H^2(E)$ is directionally differentiable with $q = \mathcal{S}'(g;h)$, and in particular

$$\lim_{\lambda \searrow 0} q_\lambda(x^k_{obs}) = q(x^k_{obs}),$$

where we use $H^2(E) \hookrightarrow \mathcal{C}(\bar{E})$ and we abbreviate again $q_\lambda := \frac{y_{g+\lambda h} - y_g}{\lambda}$. Applying the chain rule then leads to the desired statement.□

With the result in the above theorem at hand we can provide a first-order necessary optimality condition in primal form for all locally optimal shapes of (P_Ω).

Proposition 5.3. *Let $\Omega^\star \in \mathcal{O}_E$ be a locally optimal shape of (P_Ω). Then, each of the functions $g^\star \in \mathcal{F}_E$ that satisfy $\Omega_{g^\star} = \Omega^\star$ fulfill*

$$(y_{g^\star} - y_d, q)_{H^2(E)} + \sum_{k=1}^{N} (y_{g^\star} - y_d)(x^k_{obs}) q(x^k_{obs}) \geq 0$$

for all $h \in \mathcal{F}$ with

$$\{h = 0\} \cap \{g^\star = 0\}^\circ \neq \emptyset \quad \forall \{g^\star = 0\}^\circ \subset \{g^\star = 0\},$$

where $q \in W^{1,p}(\Omega_{g^\star})$ is the unique solution of (3.25)-(3.26). That is,

$$-\triangle q + \beta'(y_{g^\star}; q) = 0 \quad \text{in } \Omega_{g^\star}, \tag{5.2}$$

$$q + \nabla y_{g^\star} W = 0 \quad \text{on } \partial\Omega_{g^\star}, \tag{5.3}$$

and $W : \partial\Omega_{g^\star} \to \mathbb{R}^2$ is given by Definition 3.8.

Proof. We apply Theorem 5.1, according to which, it holds

$$j'(g^\star; h) \geq 0,$$

provided that $j'(g^\star; h)$ exists. The desired assertion is a consequence of Proposition 5.2. □

5.2. Distributed observation. In this subsection, we are interested in the case

$$\mathcal{J}(\Omega, y_\Omega) := \int_\Omega J(y_\Omega(x))\,dx + \int_\Omega \psi(x)\,dx,$$

where $\psi \in \mathcal{C}(\bar{D})$ and $J : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on bounded sets and directionally differentiable in the same sense as β. In particular, we suppose in the sequel that for every $M > 0$ the estimate holds

$$\|J(y_1) - J(y_2)\|_{L^r(\mathcal{A})} \le L_{J,M}\, \|y_1 - y_2\|_{L^r(\mathcal{A})} \quad \forall\, y_1, y_2 \in \overline{B_{L^\infty(\mathcal{A})}(0, M)},\ \forall\, 1 \le r \le \infty, \tag{5.4}$$

where $L_{J,M} > 0$. In addition,

$$\Big\| \frac{J(y + \tau\, \delta y) - J(y)}{\tau} - J'(y; \delta y) \Big\|_{L^\varrho(\mathcal{A})} \xrightarrow{\tau \searrow 0} 0 \quad \forall\, y, \delta y \in L^\infty(\mathcal{A}),\ \forall\, 1 \le \varrho < \infty. \tag{5.5}$$

Note that the above allows for tracking type functionals such as $J(y) = (y - y_d)^2, y_d \in \mathbb{R}$. Moreover, J might be non-smooth and typical examples entering this category are

$$J(y) = \max\{0, y - y_d\}, \quad J(y) = |y - y_d|, \qquad y_d \in \mathbb{R}.$$

In this subsection, the reduced objective associated to (P) is given by

$$j(g) = \int_{\{g<0\}} J(\mathcal{S}(g)(x))dx + \underbrace{\int_{\{g<0\}} \psi(x)\, dx}_{=:\Psi(g)}, \quad g \in \mathcal{F}$$

and, thanks to Theorem 3.20, we have the following result.

Proposition 5.4. *Assume that $g, h \in \mathcal{F}$ and (2.7) is true. Moreover, suppose that Ω_g is a domain. Then j is directionally differentiable at g in direction h with*

$$j'(g; h) = \int_{\{g<0\}} J'(\mathcal{S}(g)(x); q(x))dx - \int_{\{g=0\}} J(0) \frac{h}{|\nabla g|}\, d\xi + \Psi'(g; h),$$

where q is the unique solution of (3.25)-(3.26) *and*

$$\Psi'(g; h) = -\int_{\{g=0\}} \psi \frac{h}{|\nabla g|}\, d\xi. \tag{5.6}$$

Proof. By using [3, Cor. A.5], we obtain

$$\frac{1}{\lambda}\Big\{ \int_{\{g+\lambda h<0\}} \psi\, dx - \int_{\{g<0\}} \psi\, dx \Big\} \to -\int_{\{g=0\}} \psi \frac{h}{|\nabla g|}\, d\xi \quad \text{as } \lambda \searrow 0,$$

which yields (5.6).

We abbreviate for simplicity $y_{g+\lambda h} := \mathcal{S}(g + \lambda h)$, $\lambda > 0$ small, and $y_g = \mathcal{S}(g)$. Then, $g + \lambda h \in \mathcal{F}$, cf. Lemma 2.3, and we have

$$\begin{aligned}
&\frac{1}{\lambda}[j(g + \lambda h) - j(g)] - \int_{\Omega_g} J'(y_g(x); q(x))dx + \int_{g=0} J(\underbrace{y_g(\xi)}_{=0}) \frac{h}{|\nabla g|}\, d\xi \\
&= \int_{\Omega_{g+\lambda h} \setminus \Omega_g} \frac{1}{\lambda} J(y_{g+\lambda h})\, dx - \int_{\Omega_{g+\lambda h} \setminus \Omega_g} \frac{1}{\lambda} J(0)\, dx \\
&\quad + \int_{\Omega_g} \frac{1}{\lambda} (J(y_{g+\lambda h}) - J(y_g + \lambda q))dx \\
&\quad + \int_{\Omega_g} \frac{1}{\lambda} (J(y_g + \lambda q) - J(y_g)) - J'(y_g; q)dx \\
&\quad + \int_{\Omega_{g+\lambda h} \setminus \Omega_g} \frac{1}{\lambda} J(0)\, d\xi + \int_{g=0} J(0) \frac{h}{|\nabla g|}\, d\xi =: a_\lambda + b_\lambda + c_\lambda + d_\lambda.
\end{aligned}$$

Before we analize each term let us observe that, for $\lambda > 0$ small, it holds

$$\|y_{g+\lambda h}\|_{\mathcal{C}(D)}, \|y_g + \lambda q\|_{L^\infty(D)} \leq M,$$

where $M > 0$ is independent of λ (as a result of (3.5) and since $q \in W^{1,p}(\Omega)$, $p > 2$, is extended by zero outside Ω_g, see Remark 3.12). From $y_g = 0$ on $\Omega_{g+\lambda h} \setminus \Omega_g$ (see Remark 3.3) we infer

$$\begin{aligned}|a_\lambda| &\leq \frac{1}{\lambda} \int_{\Omega_{g+\lambda h}\setminus\Omega_g} |J(y_{g+\lambda h}(x)) - J(y_g(x))|\, dx \\ &\leq L_{J,M} \|q_\lambda\|_{L^1(\Omega_{g+\lambda h}\setminus\Omega_g)} \\ &\leq L_{J,M} \mu(\Omega_{g+\lambda h} \setminus \Omega_g)^{\frac{\varrho-1}{\varrho}} \|q_\lambda\|_{L^\varrho(D)}, \quad \varrho \in (1, \infty).\end{aligned}$$

where in the second estimate we used (5.4). In light of (3.7) and (3.43), we arrive at

$$a_\lambda \to 0.$$

Further, by employing again (5.4) as well as Theorem 3.20, we deduce

$$\begin{aligned}|b_\lambda| &\leq \int_{\Omega_g} \frac{1}{\lambda} |J(y_{g+\lambda h}(x)) - J(y_g + \lambda q))| dx \\ &\leq L_{J,M} \Big\| \frac{y_{g+\lambda h} - y_g}{\lambda} - q \Big\|_{L^1(\Omega_g)} \to 0.\end{aligned}$$

Due to the directional differentiability of J, cf. (5.5), we also have

$$|c_\lambda| \leq \int_{\Omega_g} \Big| \frac{J(y_g + \lambda q) - J(y_g)}{\lambda} - J'(y_g; q) \Big| dx \to 0.$$

Finally, by using for instance [3, Cor. A.5], we obtain $d_\lambda \to 0$, and by combining all the above convergences we arrive at the desired result. □

As in the previous subsection, we can now write down a first-order necessary optimality condition in primal form for all optimal shapes of (P_Ω).

Proposition 5.5. *Let $\Omega^\star \in \mathcal{O}_E$ be a locally optimal shape of (P_Ω). Then, each of the functions $g^\star \in \mathcal{F}_E$ that satisfy $\Omega_{g^\star} = \Omega^\star$ fulfill*

$$\int_{\{g^\star<0\}} J'(\mathcal{S}(g^\star); q) dx - \int_{\{g^\star=0\}} (J(0) + \psi) \frac{h}{|\nabla g^\star|}\, d\xi \geq 0$$

for all $h \in \mathcal{F}$ with

$$\{h = 0\} \cap \{g^\star = 0\}^\circ \neq \emptyset \quad \forall \{g^\star = 0\}^\circ \subset \{g^\star = 0\}.$$

Again, $q \in W^{1,p}(\Omega_{g^\star})$ solves

$$-\triangle q + \beta'(y_{g^\star}; q) = 0 \quad \text{in } \Omega_{g^\star}, \tag{5.7}$$

$$q + \nabla y_{g^\star} W = 0 \quad \text{on } \partial\Omega_{g^\star}, \tag{5.8}$$

where $W : \partial\Omega_{g^\star} \to \mathbb{R}^2$ is given by Definition 3.8.

Proof. The desired result follows by Theorem 5.1 and Proposition 5.4. □

6. The FVA and the speed method. This final section aims at putting into evidence the strong connection between the FVA and the prominent speed method (also known as the velocity method) [36]. We present a rather formal approach which will lead us to the belief that the vector field W from Definition 3.8 generates the perturbations $\Omega_{g+\lambda h}$ of the original domain Ω_g in a similar way to the speed method. This supports our previous observation (Remark 3.10) that (3.25)-(3.26) is the very same PDE that characterizes the shape derivative [36, Eqs. (3.6), (3.8)].

Throughout this section, $g, h \in \mathcal{F}$ are fixed and (2.7) is satisfied. For simplicity, we assume that Ω_g is a (possibly multiply connected) domain. In all what follows, $\lambda_0 > 0$ is fixed, small enough, so that the unicity of the approximating closed curves is guaranteed, cf. Proposition 2.4. Recall that $W : \{g = 0\} \to \mathbb{R}^2$ was defined in subsection 3.2 as follows

$$W(x) := w((z^{-1}(x)) \quad \forall\, x \in \{g = 0\}^\circ, \quad \{g = 0\}^\circ \subset \{g = 0\} \tag{6.1}$$

where $w \in \mathcal{C}^1([0, T]; \mathbb{R}^2)$ is the unique solution of (2.10) and z describes the boundary of Ω_g, that is, it solves (2.2). For $\lambda \in [0, \lambda_0)$ we can analogously define $\mathcal{W}(\lambda, \cdot) : \{g + \lambda h = 0\} \to \mathbb{R}^2$

$$\mathcal{W}(\lambda, x) := w(\lambda, (z_\lambda^{-1}(x)) \quad \forall\, x \in \{g + \lambda h = 0\}^\circ, \quad \{g + \lambda h = 0\}^\circ \subset \{g + \lambda h = 0\} \tag{6.2}$$

where

$$w(\lambda, \cdot) = \lim_{\hat{\lambda} \to \lambda} \frac{z_{\hat{\lambda}} - z_\lambda}{\hat{\lambda} - \lambda} \in \mathcal{C}^1([0, T]; \mathbb{R}^2) \tag{6.3}$$

is computed in a similar way to w for $\lambda > 0$ small enough, see the proof of [40, Prop. 3.2]. Hence, for fixed $t \in (0, \infty)$, $w(\lambda, t)$ is the velocity vector at $z_\lambda(t)$ of the trajectory $[0, \lambda_0) \ni \lambda \mapsto z_\lambda(t)$. Note that $\mathcal{W}(0) = W$.

In the case of the speed method [36, Sec. 2.9], the domain perturbations take place via $T_\lambda(\Omega_g)$, where $T_\lambda : D \to \mathbb{R}^2$ is the flow associated to a family of vector fields. Assuming that there exists $\mathcal{W}(\lambda, \cdot) \in C_c^{0,1}(D; \mathbb{R}^2)$ an extension of $\mathcal{W}(\lambda, \cdot) : \partial\Omega_{g+\lambda h} \to \mathbb{R}^2$ (denoted by the same symbol) so that $\lambda \mapsto \mathcal{W}(\lambda, \cdot) \in \mathcal{C}([0, \lambda_0), C_c^{0,1}(D; \mathbb{R}^2))$, let $T_\lambda^{\mathcal{W}} : D \to \mathbb{R}^2$ be its associated flow, i.e., $T_\lambda^{\mathcal{W}}(X) = x(\lambda, X)$ is the unique solution to

$$\frac{d}{d\lambda} x(\lambda, X) = \mathcal{W}(\lambda, x(\lambda, X)) \quad \lambda \in [0, \lambda_0), \quad x(0, X) = X \in D. \tag{6.4}$$

We are going to prove next that

$$\partial\Omega_{g+\lambda h} = T_\lambda^{\mathcal{W}}(\partial\Omega_g), \qquad \lambda \in [0, \lambda_0). \tag{6.5}$$

To this end, we introduce another vector field which associates to each point on the boundary of Ω_g a unique point on the boundary of the perturbed domain $\Omega_{g+\lambda h}$. For $\lambda \in [0, \lambda_0)$, this is given by the mapping $Z_\lambda : \{g = 0\} \to \{g + \lambda h = 0\}$ defined as

$$Z_\lambda(x) := z_\lambda(z^{-1}(x)) \quad \forall\, x \in \{g = 0\}^\circ, \quad \{g = 0\}^\circ \subset \{g = 0\}. \tag{6.6}$$

Clearly, it holds

$$Z_\lambda(z(t)) = z_\lambda(t), \quad \forall\, t \in [0, T) \tag{6.7}$$

and $Z_0(x) = x$. For a point $X \in \partial\Omega_g$, we have with $t = z^{-1}(X)$ the following

$$\frac{d}{d\lambda} Z_\lambda(X) = \lim_{\hat{\lambda}\to\lambda} \frac{Z_{\hat{\lambda}}(X) - Z_\lambda(X)}{\hat{\lambda} - \lambda} \tag{6.8}$$

$$= \lim_{\hat{\lambda}\to\lambda} \frac{z_{\hat{\lambda}}(t) - z_\lambda(t)}{\hat{\lambda} - \lambda} \tag{6.9}$$

$$= w(\lambda, t) = \mathcal{W}(\lambda, Z_\lambda(X)) \quad \lambda \in [0, \lambda_0), \tag{6.10}$$

where we use (6.7), (6.2) and (6.3). Going back to (6.4), we see that

$$x(\lambda, X) = Z_\lambda(X) \quad \text{for } X \in \partial\Omega_g,$$

since (6.4) is uniquely solvable.

Therefore, the points on the perturbed boundary $\partial\Omega_{g+\lambda h}$ appear as flows of $\mathcal{W}$ with initial condition on the original boundary. We underline that our $T_\lambda^{\mathcal{W}}$ is well-defined only on $\partial\Omega_g$ (not in the whole D as in [36]); nevertheless, the entire domain is automatically perturbed, being the interior of a finite union of disjoint planar closed curves without self intersections (Proposition 2.1).

Thus, we now have an additional (however rather formal) confirmation that the limit (1.4) we calculated in subsection 3.2 (Theorems 3.18 and 3.20) is in fact

$$\lim_{\lambda\searrow 0} \frac{y_{T_\lambda^{\mathcal{W}}(\Omega_g)} - y_{\Omega_g}}{\lambda},$$

that is, the shape derivative of y_{Ω_g} in direction $\mathcal{W}$.

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