An Objective Q-Criterion

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Classical Eulerian vortex criteria, such as the Q-, Δ-, λ_2-, λ_{ci}-, and Okubo–Weiss criterion, depend on frame-dependent quantities and therefore fail to provide objective, observer-independent diagnostics in unsteady frames. In this work, we address this longstanding limitation by introducing an objective variant of the Q-criterion, derived from a spatio-temporal variational principle that minimizes a modified time-derivative of the strain-rate tensor. Although several objective variants of the Q-criterion have been proposed before, none of these has successfully reconciled Eulerian diagnostics with the underlying Lagrangian motion of fluid particles, even in simple analytical solutions of the Navier–Stokes equations. Here, we present the first vortex criterion that consistently resolves all known pathological examples leading to false positives and false negatives in Eulerian vortex criteria applied to unsteady flows. The results establish a unified and objective framework for Eulerian vortex detection, opening new directions for the analysis and control of unsteady flow phenomena.

1. Introduction

Vortices are coherent, rotational regions in fluid flows whose identification and characterization remain central challenges in fluid mechanics. Classical Eulerian vortex criteria such as the Okubo–Weiss (OW) criterion (Okubo 1970; Weiss 1991), the Q-criterion (Hunt *et al.* 1988), the λ_2-criterion (Jeong & Hussain 1995), the Δ-criterion (Chong *et al.* 1990), and the λ_{ci}-criterion (Chakraborty *et al.* 2005) are widely used to detect vortices in two- and three-dimensional flows. While these criteria coincide in incompressible planar flows, they can yield substantially different predictions in three-dimensional (Jeong & Hussain 1995) and unsteady flows (Haller 2005), often producing false positives or false negatives.

Lagrangian approaches, based on material transport and barrier methods (Haller *et al.* 2018; Haller 2023), provide an alternative technique for vortex detection. These methods, however, require time-integration of trajectories, making them computationally expensive, sensitive to data quality, and difficult to benchmark due to the absence of classical analogues. Other approaches have attempted to construct minimally unsteady, or co-moving, velocity fields via variational principles (Bujack *et al.* 2016; Günther *et al.* 2017; Rojo & Günther 2019), which have been challenged in Haller (2021).

A systematic program to objectivize Eulerian vortex criteria was proposed by Haller (2021). In this context, Kaszás *et al.* (2023) introduced the *deformation velocity*, defined as the difference between the material velocity and its closest rigid-body approximation, while Kogelbauer & Pedergnana (2025) subsequently defined the *deformation unsteadiness* as an objective measure of the flow's temporal variation. Both approaches rely on spatio-temporal variational principles and allow the construction of objective variants

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of the Q-criterion denoted by Q_{RB} and Q_{US}, respectively. Nevertheless, as shown by Kogelbauer & Pedergnana (2025), neither of these modifications was able to consistently resolve the long-standing challenge of predicting Lagrangian dynamics from objectively modified Eulerian diagnostics in well-known pathological examples, such as the unsteady Navier–Stokes flows derived in Pedergnana *et al.* (2020). These negative results indicated that objectivity alone is not enough to define fully consistent Eulerian vortex criteria, and thus lead to the question how a consistent improvement of the Q-criterion could be achieved.

The present work addresses this challenge by introducing the *strain-rate unsteadiness velocity*, which is based on the solution to a variational principle that minimizes a modified time-derivative of the strain-rate tensor. We note that objectively modified derivatives of tensor fields are well-known in continuum mechanics, see Billington & Tate (1981), and were the motivation for the M_z criterion proposed by Haller (2005). Different from classical objective tensor derivatives, however, the formulation proposed in this work is based on a purely time-dependent tensor derived from a variational principle. This approach leads to an objective formulation of the Q-criterion, which, for the first time, resolves all previously known pathological cases causing false positives or false negatives in Eulerian vortex criteria applied to unsteady flows. Therefore, the results presented in this work provide an affirmative answer to the question posed by Haller (2021) of whether vortex criteria can be fully consistently objectivized.

2. Preliminaries

We consider the Eulerian position vector $\boldsymbol{x} \in \mathcal{D} \subset \mathbb{R}^d$, $d = 2, 3$, defined on a time-independent, connected domain $\mathcal{D}$ and a sufficiently smooth, unsteady vector field $\boldsymbol{v} : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$. For simplicity, we exclude unbounded domains or domains whose boundary changes with time although these cases could be treated in a similar way. For any function $f : \mathcal{D} \to \mathbb{R}$, we define its spatial average as

$$\overline{f} = \fint_{\mathcal{D}} f dV = \frac{1}{\mathrm{Vol}(\mathcal{D})} \int_{\mathcal{D}} f dV. \tag{2.1}$$

A general frame change in the Eulerian picture is described by the following transformations:

$$\boldsymbol{x} = \boldsymbol{Q}(t)\boldsymbol{y} + \boldsymbol{b}(t), \quad \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}, \quad \boldsymbol{Q} \in SO(\mathbb{R}^d), \quad \boldsymbol{b} \in \mathbb{R}^d. \tag{2.2}$$

A quantity is called *objective* if it transforms neutrally under a frame change of the form (2.2), see Truesdell & Noll (2004). More specifically, an Eulerian scalar α, a vector $\boldsymbol{a}$ or a matrix $\boldsymbol{A}$ is called *objective* if it transforms according to

$$\tilde{\alpha}(\boldsymbol{y}, t) = \alpha(\boldsymbol{x}, t) \quad \text{or} \quad \tilde{\boldsymbol{a}}(\boldsymbol{y}, t) = \boldsymbol{Q}^T(t)\boldsymbol{a}(\boldsymbol{x}, t) \quad \text{or} \quad \tilde{\boldsymbol{A}}(\boldsymbol{y}, t) = \boldsymbol{Q}^T(t)\boldsymbol{A}(\boldsymbol{x}, t)\boldsymbol{Q}(t), \tag{2.3}$$

in the $\boldsymbol{y}$-frame induced by (2.2). Recall that the velocity field itself is non-objective since it transforms according to

$$\tilde{\boldsymbol{v}}(\boldsymbol{y}, t) = \boldsymbol{Q}^T(t)[\boldsymbol{v}(\boldsymbol{x}, t) - \dot{\boldsymbol{Q}}(t)\boldsymbol{y} - \dot{\boldsymbol{b}}]. \tag{2.4}$$

Throughout the paper, we denote quantities in the transformed frame with a tilde and the coordinates in the transformed frame are denoted by $\boldsymbol{y}$ as defined in Eq. (2.2).
This work makes use of the *vectorization* of a matrix $\boldsymbol{A} = [A_{ij}] \in \mathbb{R}^{m \times n}$, denoted as

vec($\boldsymbol{A}$), which is the $mn \times 1$ column vector obtained by stacking the columns of $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \quad \Rightarrow \quad \mathrm{vec}(\boldsymbol{A}) = \begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \\ A_{12} \\ \vdots \\ A_{mn} \end{pmatrix}. \tag{2.5}$$

We denote the standard inner product on the space of matrices as $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{tr}(\boldsymbol{A}^T \boldsymbol{B})$ with corresponding norm $\|\boldsymbol{A}\| = \sqrt{\langle \boldsymbol{A}, \boldsymbol{A} \rangle}$ and recall the identity $\mathrm{tr}\left(\boldsymbol{A}^T \boldsymbol{B}\right) = \mathrm{vec}\left[\boldsymbol{A}\right]^T \mathrm{vec}\left[\boldsymbol{B}\right]$, see Magnus & Neudecker (2007).

3. Strain-Rate Unsteadiness Minimization

Given a velocity field $\boldsymbol{v}$, its strain rate tensor is the symmetric part of its deformation tensor,

$$\boldsymbol{S} = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T\right). \tag{3.1}$$

For incompressible velocity fields, the strain rate tensor is trace-free. Although strain rate tensors obtained from velocity fields (3.1) are the primary application of our analysis, we stress that the subsequent calculations apply to any symmetric tensor field $\boldsymbol{S}$.
Consider the following modified time derivative of the symmetric strain rate tensor $\boldsymbol{S}$,

$$\overset{\circ}{\boldsymbol{S}}(\boldsymbol{x}, t) = \partial_t \boldsymbol{S}(\boldsymbol{x}, t) - \boldsymbol{T}(t)\boldsymbol{S}(\boldsymbol{x}, t) - \boldsymbol{S}(\boldsymbol{x}, t)\boldsymbol{T}^T(t), \tag{3.2}$$

for any real three-by-three matrix $\boldsymbol{T}$. We define the space-time averaged modified time derivative of the strain rate, regarded as a functional acting on $t \mapsto \boldsymbol{T}(t)$,

$$\mathcal{S}[\boldsymbol{T}] = \frac{1}{2}\int_{t_0}^{t_1} \fint_{\mathcal{D}} \|\overset{\circ}{\boldsymbol{S}}(\boldsymbol{x}, t; \boldsymbol{T}(t))\|^2 dV dt. \tag{3.3}$$

The functional (3.3) is convex with respect to $\boldsymbol{T}$, thus guaranteeing that extremizers will be minimizers. To minimize the action (3.3) we set the first variation, calculated in Appendix A, equal to zero:

$$\frac{\delta \mathcal{S}}{\delta \boldsymbol{T}} = \overline{\boldsymbol{M}^T \left(\boldsymbol{M}\ \mathrm{vec}[\boldsymbol{T}] - \mathrm{vec}[\partial_t \boldsymbol{S}]\right)} = \boldsymbol{0}, \tag{3.4}$$

where $\boldsymbol{M} = \boldsymbol{S} \otimes \boldsymbol{I} + \left(\boldsymbol{I} \otimes \boldsymbol{S}\right)\boldsymbol{K}$, $\otimes$ is the Kronecker product, and $\boldsymbol{K}$ is the commutation matrix satisfying $\mathrm{vec}[\boldsymbol{A}^T] = \boldsymbol{K}\ \mathrm{vec}[\boldsymbol{A}]$. The spectrum of the matrix $\boldsymbol{M}(\boldsymbol{x}, t) \in \mathbb{R}^{9\times 9}$ is given by

$$\mathrm{Spec}(\boldsymbol{M}) = \{0\} \cup \{\lambda_i + \lambda_j \,:\, i, j = 1, 2, 3\}, \tag{3.5}$$

where λ_i, $i = 1, 2, 3$ are the eigenvalues of the rate-of-strain tensor $\boldsymbol{S}(\boldsymbol{x}, t)$, see Magnus & Neudecker (2007). Despite $\boldsymbol{M}$ being pointwise singular at all $(\boldsymbol{x}, t)$, because the eigenvectors of $\boldsymbol{M}^T\boldsymbol{M}$ vary in space, its spatial average $\overline{\boldsymbol{M}^T\boldsymbol{M}}$ is generally nonsingular. An exception occurs when the velocity field $\boldsymbol{v}$ is linear in $\boldsymbol{x}$. In vectorized form, assuming $\overline{\boldsymbol{M}^T\boldsymbol{M}}$ is nonsingular, the solution of (3.4) is then given by

$$\mathrm{vec}[\boldsymbol{T}] = \overline{\boldsymbol{M}^T\boldsymbol{M}}^{-1}\, \overline{\boldsymbol{M}^T \mathrm{vec}[\partial_t \boldsymbol{S}]}. \tag{3.6}$$

In the following, we denote the matrix (3.6) that minimizes the action (3.3) by $\boldsymbol{T}$. We discuss below a spatially linear flow field where $\overline{\boldsymbol{M}^T\boldsymbol{M}}$ is singular but the exact global minimizer of (3.3) can be directly obtained by replacing the inverse $\overline{\boldsymbol{M}^T\boldsymbol{M}}^{-1}$ in Eq. (3.6) with the Moore–Penrose pseudoinverse $\overline{\boldsymbol{M}^T\boldsymbol{M}}^{+}$, see Magnus & Neudecker (2007). In Appendix B, we show that the matrix $\boldsymbol{T}$ transforms like a spin tensor under a general frame change:

$$\widetilde{\boldsymbol{T}} = \boldsymbol{Q}^T\boldsymbol{T}\boldsymbol{Q} - \boldsymbol{Q}^T\dot{\boldsymbol{Q}}. \tag{3.7}$$

We emphasize that this transformation property only holds for the minimizer of (3.3), and is not an *a priori* invariance of the action. As a consequence of (3.7), the modified time-derivative (3.2) is objective for the minimizers $\boldsymbol{T}$:

$$\begin{aligned}\widetilde{\overset{\circ}{\boldsymbol{S}}} &= \boldsymbol{Q}^T\left(\partial_t\boldsymbol{S} - \dot{\boldsymbol{Q}}\boldsymbol{Q}^T\boldsymbol{S} - \boldsymbol{S}\boldsymbol{Q}\dot{\boldsymbol{Q}}^T\right)\boldsymbol{Q} - \widetilde{\boldsymbol{T}}\boldsymbol{Q}^T\boldsymbol{S}\boldsymbol{Q} - \boldsymbol{Q}^T\boldsymbol{S}\boldsymbol{Q}\widetilde{\boldsymbol{T}}^T \\ &= \boldsymbol{Q}^T\left(\partial_t\boldsymbol{S} - \boldsymbol{T}\boldsymbol{S} - \boldsymbol{S}\boldsymbol{T}^T\right)\boldsymbol{Q}.\end{aligned} \tag{3.8}$$

Furthermore, it implies that the *strain-rate unsteadiness velocity* (SUV), defined as

$$\boldsymbol{v}_{\mathrm{s}}(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{x},t) - \boldsymbol{T}(t)\left[\boldsymbol{x} - \overline{\boldsymbol{x}}(t)\right] - \overline{\boldsymbol{v}}(t), \tag{3.9}$$

is objective. In Eq. (3.9), $\overline{\boldsymbol{x}}(t)$ and $\overline{\boldsymbol{v}}(t)$ are the location and velocity of the center of gravity of the domain. We thus introduce the following modified Q-criterion for $\boldsymbol{v}_{\mathrm{s}}$,

$$Q_{\mathrm{s}} = \frac{1}{2}\left(\|\boldsymbol{W}(\boldsymbol{x},t) - \mathrm{skew}\left[\boldsymbol{T}\right](t)\|^2 - \|\boldsymbol{S}(\boldsymbol{x},t) - \mathrm{symm}\left[\boldsymbol{T}(t)\right]\|^2\right), \tag{3.10}$$

where $\mathrm{skew}\,[\boldsymbol{T}]$ and $\mathrm{symm}\,[\boldsymbol{T}]$ are the skew-symmetric and symmetric parts of $\boldsymbol{T}$. The quantity Q_{s} is an objective Eulerian scalar which can be used to define vortical regions in space in analogy to the Q-criterion and its objective analogues Q_{RB} and Q_{US} defined in Kaszás *et al.* (2023) and Kogelbauer & Pedergnana (2025), respectively.

4. Examples

4.1. *Spatially linear unsteady velocity field*

Consider the spatially linear family of unsteady Navier–Stokes solutions given by

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{A}(t)\boldsymbol{x}, \quad \boldsymbol{A}(t) = \begin{pmatrix} -\sin(Ct) & \cos(Ct) - \frac{\omega}{2} \\ \cos(Ct) + \frac{\omega}{2} & \sin(Ct)\end{pmatrix}. \tag{4.1}$$

over the square domain $[-L/2, L/2]^2$. In this example, $\boldsymbol{M}$ does not depend on $\boldsymbol{x}$ and $\overline{\boldsymbol{M}^T\boldsymbol{M}}$ is singular with rank three. We therefore apply the modified solution formula

$$\mathrm{vec}[\boldsymbol{T}] = \overline{\boldsymbol{M}^T\boldsymbol{M}}^{+}\,\overline{\boldsymbol{M}^T\,\mathrm{vec}[\partial_t\boldsymbol{S}]}. \tag{4.2}$$

The matrix $\boldsymbol{T}$ defined by (4.2) transforms like a spin tensor if

$$(\boldsymbol{Q}^T \otimes \boldsymbol{Q}^T)\boldsymbol{M}^+\boldsymbol{M}\;\mathrm{vec}[\dot{\boldsymbol{Q}}\boldsymbol{Q}^T] = \mathrm{vec}[\boldsymbol{Q}^T\dot{\boldsymbol{Q}}], \tag{4.3}$$

see Appendix B. This condition is satisfied in this example for general $\boldsymbol{Q} \in SO(2)$. From Eq. (4.2), we obtain

$$\boldsymbol{T} = \begin{pmatrix} 0 & -\frac{C}{2} \\ \frac{C}{2} & 0\end{pmatrix}. \tag{4.4}$$

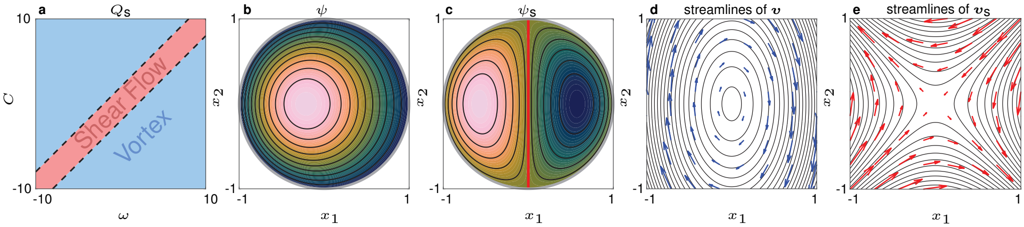

FIGURE 1. Vortex detection in analytical flow fields. **a** The objective Q_s-criterion predicts the correct fluid particle motion for the linear family of flow fields (4.1) as a function of its parameters. See Fig. 4 of Kogelbauer & Pedergnana (2025) for corresponding predictions of the classical Q-criterion and its other objective analogues. **b**,**c** Analysis of a two-dimensional flow defined by the streamfunction (4.6). **b** Instantaneous streamlines of the velocity field at time $t = \pi/4$. **c** Instantaneous streamlines of the objective field $\boldsymbol{v}_s$ defined by (4.8) at the same time instant correctly identify the flow topology of the fluid particle motion. Brighter values correspond to larger values of the streamfunction. The zero isocontour, indicating the separation line of the flow, is colored in red. **d**,**e** Analysis of the spatially nonlinear unsteady Navier–Stokes field (4.9). **d** Instantaneous streamlines of the velocity field $\boldsymbol{v}$, shown here for $t_0 = \pi/4$, indicate a vortical flow. Streamlines at different times show analogous flow patterns. **e** The streamlines of the corresponding objective field $\boldsymbol{v}_s$ correctly indicate a mixing region near the origin.

The modified rate $\overset{\circ}{\boldsymbol{S}}$ vanishes identically in this example, showing that $\boldsymbol{T}$ is the exact global minimizer of (3.3). The objective SUV field $\boldsymbol{v}_s$ defined in (3.9) is given by

$$\boldsymbol{v}_s(\boldsymbol{x}, t) = [\boldsymbol{A}(t) - \boldsymbol{T}]\,\boldsymbol{x} = \begin{pmatrix} -\sin(Ct) & \cos(Ct) - \frac{\omega - C}{2} \\ \cos(Ct) + \frac{\omega - C}{2} & \sin(Ct) \end{pmatrix} \boldsymbol{x}. \tag{4.5}$$

The velocity gradient of $\boldsymbol{v}_s$ is given by $\boldsymbol{\nabla}\boldsymbol{v}_s = \boldsymbol{A}(t) - \boldsymbol{T}$ with eigenvalues $\pm\sqrt{1 - (\omega - C)^2/4}$, predicting a vortical flow around the origin for $|\omega - C| > 2$ and a shear flow for $|\omega - C| < 2$. Equivalent predictions are obtained by the objective Q_s-criterion defined in (3.10), see Fig. 1**a**. As discussed in Pedergnana *et al.* (2020), these parameter ranges correspond exactly to the fluid particle motion defined by the field (4.1), which is, in general, judged incorrectly by the classical Q-criterion and its objective analogues Q_{RB} and Q_{US}.

4.2. *Separation and reattachment flow*

We now consider the kinematic example of Lekien & Haller (2008) given by the unsteady stream function defined on the unit circle, see also Kaszás *et al.* (2023):

$$\psi(\boldsymbol{x}, t) = \left(|\boldsymbol{x}|^2 - 1\right)[x_1 \sin(\omega_s t) + x_2 \cos(\omega_s t)] - \tfrac{1}{2}\,\omega_s |\boldsymbol{x}|^2. \tag{4.6}$$

The matrix $\overline{\boldsymbol{M}^T\boldsymbol{M}}$ is nonsingular for all t. Computing the solution (3.6) yields

$$\boldsymbol{T} = \begin{pmatrix} 0 & \frac{\omega_s}{2} \\ -\frac{\omega_s}{2} & 0 \end{pmatrix}. \tag{4.7}$$

In this example, the modified rate (3.2) vanishes identically as well, showing that $\boldsymbol{T}$ is the exact global minimizer. The SUV field $\boldsymbol{v}_s = \boldsymbol{v} - \boldsymbol{T}\boldsymbol{x}$ corresponds to the streamfunction

$$\psi_s(\boldsymbol{x}, t) = \left(|\boldsymbol{x}|^2 - 1\right)[x_1 \sin(\omega_s t) + x_2 \cos(\omega_s t)]\,. \tag{4.8}$$

In contrast to the original stream function, ψ_s correctly predicts the topology of the fluid particle motion defined by (4.6), which consists of two diametrically opposed vortex cells separated by a rotating separation line, see Fig. 1**b**,**c**.

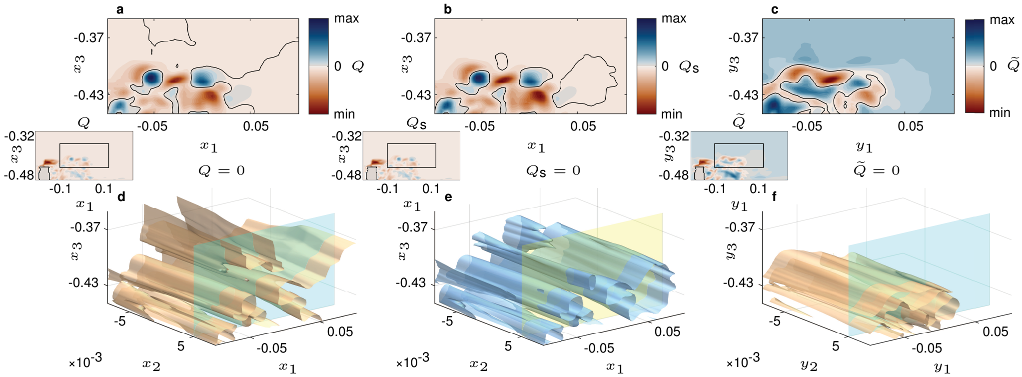

FIGURE 2. Vortex detection in the wind wake behind the research vessel Tangaroa. **a**, **b** Comparison of the Q- and Q_s-criteria and **c** Q-criterion computed in a rotating frame described by (4.11), all in in the plane $y = 0.005$. The black curves show the zero isocontours. The small insets show the position of the larger insets and all criteria in a global view of the wind wake. **d**, **e**, **f** Zero isocontours of the three vortex criteria. The out-of-plane axes are scaled for visualization purposes. The transparent plane shows the location where the iscontours in **a**, **b**, **c** were extracted.

4.3. *Cubic unsteady Navier–Stokes flow with hidden mixing region*

We use the results derived in this work to examine the spatially cubic Navier–Stokes field presented in Pedergnana *et al.* (2020) given by

$$\boldsymbol{u}(\boldsymbol{x},t) = \begin{pmatrix} \sin(4t) & \cos(4t)+2 \\ \cos(4t)-2 & -\sin(4t) \end{pmatrix} \boldsymbol{x} \;+\; \alpha(t) \begin{pmatrix} x\,(x^2-3y^2) \\ -\,y\,(3x^2-y^2), \end{pmatrix} \tag{4.9}$$

over the square domain $[-1,1]^2$. The matrix $\overline{\boldsymbol{M}^T\boldsymbol{M}}$ is nonsingular for all t. For the choice $\alpha = 0.005$, at time $t_0 = \pi/4$, and the solution (3.6) takes the form

$$\boldsymbol{T}(t_0) = \begin{pmatrix} 0 & 1.99972 \\ -1.99972 & 0 \end{pmatrix}. \tag{4.10}$$

While the instantaneous streamlines of $\boldsymbol{v}$ erroneously predict a vortical flow near the origin, those of the objective field $\boldsymbol{v}_\mathrm{s} = \boldsymbol{v} - \boldsymbol{T}\boldsymbol{x}$ correctly indicate a mixing region near the origin, see Fig. 1**d**,**e**.

4.4. *Unsteady three-dimensional flow in a wind wake*

As our final example, we analyze vortices in a simulated, unsteady three-dimensional flow data set describing the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ of a wind wake behind the Tangaroa research vessel operated by the National Institute of Water and Atmospheric Research of New Zealand, see Popinet *et al.* (2004). Details on the simulations, which were conducted in a frame in which the ship is at rest, can be found in the reference. In Fig. 2, we compare the classical Q-criterion to the objective Q_s-criterion and the Q-criterion in a transformed frame corresponding to an Euclidean frame change (2.2) with

$$\boldsymbol{Q}(t) = \begin{pmatrix} \cos\mathit{\Omega}(t-t_0) & 0 & \sin\mathit{\Omega}(t-t_0) \\ 0 & 0 & 0 \\ -\sin\mathit{\Omega}(t-t_0) & 0 & \cos\mathit{\Omega}(t-t_0) \end{pmatrix}, \tag{4.11}$$

$\boldsymbol{b}(t)$ arbitrary and $\mathit{\Omega} = 35$. At the time $t_0 = 0.49$, the matrix $\overline{\boldsymbol{M}^T\boldsymbol{M}}$ is nonsingular and we obtain

$$\boldsymbol{T}(t_0) = \begin{pmatrix} -2.9331 & -1.7045 & -1.8468 \\ 0.9799 & 1.8019 & 1.4561 \\ 3.1358 & 4.2781 & 3.1321 \end{pmatrix} \tag{4.12}$$

for the domain shown in Fig. 2**b**, **e**. We observe that, in the ship's rest frame, Q and Q_{s} yield qualitatively similar predictions, one can observe that the latter criterion removes spurious features in the field. Under a frame change, the Q-criterion yields widely differing predictions, while Q_{s} is frame-invariant by definition.

5. Discussion

Kaszás *et al.* (2023) resolved the kinematic flow given discussed in Section (4.2) by subtracting the closest rigid-body approximation from the velocity field. However, their Eulerian analysis was unable to consistently detect the Lagrangian particle motion in the nonlinear Navier–Stokes fields discussed in Pedergnana *et al.* (2020), which were constructed as weakly nonlinear extensions of the original example of Haller (2005). Kogelbauer & Pedergnana (2025) obtained an improved, objective Q-criterion by minimizing unsteadiness of the velocity. While their work dealt with the unsteadiness of *absolute* motion and achieved only a partial solution, the present work minimizes the unsteadiness of *relative*, infinitesimal motion described by the rate-of-strain tensor. This appears to be the correct approach, leading to the complete solution of the problem posed by Haller (2021).

6. Conclusion

We have introduced a new spatio-temporal variational principle for an observer-independent measure of flow unsteadiness through a modified time-derivative of the strain rate tensor. Through this principle, we derived an optimal tensor field $\boldsymbol{T}$, whose transformation properties allow us to construct an objective co-moving velocity field $\boldsymbol{v}_{\mathrm{s}}$ and an associated objective Q-criterion. These results constitute a notable advance to a longstanding problem of objective vortex detection in unsteady flows using Eulerian criteria, demonstrating the complete resolution of all known pathological examples of the classical Q-criterion in the literature. The present approach therefore constitutes a significant improvement not only compared to classical Eulerian vortex criteria, but also with regard to recently proposed objective analogues of the Q-criterion, which, despite their objectivity, have not fully reconciled Eulerian diagnostics with the underlying Lagrangian dynamics. Due to their Eulerian nature, the methods developed in this work can be readily applied to complex, three-dimensional flows from simulations or experiments at low computational cost. These results also allow the construction of objective analogues of other well-known quantities derived from v_{s}, such as vorticity, potential vorticity and helicity, opening a multitude of promising avenues for future studies of transport of dynamically active quantities in unsteady three-dimensional flows.

Acknowledgments. The flow data set used in Section 4.4 is part of the open-source flow database provided by the Computer Graphics Laboratory at ETH Zürich. The colormaps used in Fig. 1**b**,**c** and Fig. 2 are taken from Crameri *et al.* (2020).

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Declaration of Interests. The authors report no conflict of interest.

Appendix A. First variation of $\mathcal{S}$

We compute the first variation of the action defined in (3.3) as

$$\frac{\delta\mathcal{S}}{\delta\boldsymbol{T}}\hat{\boldsymbol{T}} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{S}[\boldsymbol{T}+\varepsilon\hat{\boldsymbol{T}}] \tag{A 1}$$

$$= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{1}{2}\int_{t_0}^{t_1} \fint_{\mathcal{D}} \|\partial_t\boldsymbol{S} - (\boldsymbol{T}+\varepsilon\hat{\boldsymbol{T}})\boldsymbol{S} - \boldsymbol{S}(\boldsymbol{T}+\varepsilon\hat{\boldsymbol{T}})^T\|^2\, dV dt \tag{A 2}$$

$$= \int_{t_0}^{t_1} \fint_{\mathcal{D}} \mathrm{tr}\left[(\hat{\boldsymbol{T}}\boldsymbol{S}+\boldsymbol{S}\hat{\boldsymbol{T}}^T)^T(\boldsymbol{T}\boldsymbol{S}+\boldsymbol{S}\boldsymbol{T}^T-\partial_t\boldsymbol{S})\right] dV dt. \tag{A 3}$$

$$= \int_{t_0}^{t_1} \fint_{\mathcal{D}} \mathrm{vec}[\hat{\boldsymbol{T}}]^T\, \boldsymbol{M}^T\, (\boldsymbol{M}\ \mathrm{vec}[\boldsymbol{T}] - \mathrm{vec}[\partial_t\boldsymbol{S}])\ dV\, dt, \tag{A 4}$$

where $\boldsymbol{M} = \boldsymbol{S}^T\otimes\boldsymbol{I} + (\boldsymbol{I}\otimes\boldsymbol{S})\,\boldsymbol{K}$. Using $\boldsymbol{S}^T = \boldsymbol{S}$ yields the expression given in Section 3.

Appendix B. Transformation properties of $\boldsymbol{T}$

In this appendix, we derive the transformation properties of the matrix $\boldsymbol{T}$ which appears in the modified rate of the rate-of-strain tensor $\boldsymbol{S}$ defined in (3.2). This derivation makes use of the transformation properties of $\boldsymbol{S}$, which is itself an objective tensor, see Truesdell & Noll (2004), and its time derivative:

$$\widetilde{\boldsymbol{S}} = \boldsymbol{Q}^T\boldsymbol{S}\boldsymbol{Q} \quad\Rightarrow\quad \widetilde{\boldsymbol{S}}^T = \boldsymbol{Q}^T\boldsymbol{S}^T\boldsymbol{Q}, \tag{B 1}$$

$$\begin{aligned}\widetilde{\partial_t\boldsymbol{S}} &= \boldsymbol{Q}^T\partial_t\boldsymbol{S}\boldsymbol{Q} + \dot{\boldsymbol{Q}}^T\boldsymbol{S}\boldsymbol{Q} + \boldsymbol{Q}^T\boldsymbol{S}\dot{\boldsymbol{Q}}\\ &= \boldsymbol{Q}^T\left(\partial_t\boldsymbol{S} - \dot{\boldsymbol{Q}}\boldsymbol{Q}^T\boldsymbol{S} - \boldsymbol{S}\boldsymbol{Q}\dot{\boldsymbol{Q}}^T\right)\boldsymbol{Q},\end{aligned} \tag{B 2}$$

where we used $\widetilde{\boldsymbol{v}}_\mathrm{d} = \boldsymbol{Q}^T\boldsymbol{v}_\mathrm{d}$ and the property that $\dot{\boldsymbol{Q}} = -\boldsymbol{Q}\dot{\boldsymbol{Q}}^T\boldsymbol{Q}$ for the time-derivative of rotation matrices. We also recall the following three identities for the Kronecker product and the vectorization operator. For square matrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{M},\boldsymbol{N},\boldsymbol{P},\boldsymbol{Q}\in\mathbb{R}^{n\times n}$, we have

$$(\boldsymbol{M}\boldsymbol{A}\boldsymbol{P})\otimes(\boldsymbol{N}\boldsymbol{B}\boldsymbol{Q}) = (\boldsymbol{M}\otimes\boldsymbol{N})\,(\boldsymbol{A}\otimes\boldsymbol{B})\,(\boldsymbol{P}\otimes\boldsymbol{Q}), \tag{B 3}$$

$$\mathrm{vec}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{C}^T\otimes\boldsymbol{A})\ \mathrm{vec}(\boldsymbol{B}) = (\boldsymbol{I}_n\otimes\boldsymbol{A})(\boldsymbol{C}^T\otimes\boldsymbol{I}_n)\ \mathrm{vec}(\boldsymbol{B}), \tag{B 4}$$

$$(\boldsymbol{A}\otimes\boldsymbol{B})^T = \boldsymbol{A}^T\otimes\boldsymbol{B}^T. \tag{B 5}$$

The transformation property of an objective matrix $\boldsymbol{A}$ under a frame change (2.2) is

$$\mathrm{vec}[\widetilde{\boldsymbol{A}}] = \mathrm{vec}[\boldsymbol{Q}^T\boldsymbol{A}\boldsymbol{Q}] = (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\ \mathrm{vec}[\boldsymbol{A}]. \tag{B 6}$$

B.1. *Transformation property of* $\boldsymbol{M}$

Using the fact that the commutation matrix $\boldsymbol{K}$ satisfies $\boldsymbol{K}\,(\boldsymbol{A}\otimes\boldsymbol{B}) = (\boldsymbol{B}\otimes\boldsymbol{A})\,\boldsymbol{K}$, we calculate the transformation property of $\widetilde{\boldsymbol{M}}$ under an Euclidean frame change:

$$\begin{aligned}\widetilde{\boldsymbol{M}} &= \widetilde{\boldsymbol{S}}^T\otimes\boldsymbol{I} + (\boldsymbol{I}\otimes\widetilde{\boldsymbol{S}})\boldsymbol{K}\\
&= (\boldsymbol{Q}^T\boldsymbol{S}^T\boldsymbol{Q})\otimes\boldsymbol{I} + (\boldsymbol{I}\otimes\boldsymbol{Q}^T\boldsymbol{S}\boldsymbol{Q})\boldsymbol{K}\\
&= (\boldsymbol{Q}^T\otimes\boldsymbol{I})(\boldsymbol{S}^T\otimes\boldsymbol{I})(\boldsymbol{Q}\otimes\boldsymbol{I}) + (\boldsymbol{I}\otimes\boldsymbol{Q}^T)(\boldsymbol{I}\otimes\boldsymbol{S})(\boldsymbol{I}\otimes\boldsymbol{Q})\boldsymbol{K}\\
&= (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)(\boldsymbol{S}^T\otimes\boldsymbol{I})(\boldsymbol{Q}\otimes\boldsymbol{Q}) + (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)(\boldsymbol{I}\otimes\boldsymbol{S})(\boldsymbol{Q}\otimes\boldsymbol{Q})\boldsymbol{K}\\
&= (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)(\boldsymbol{S}^T\otimes\boldsymbol{I})(\boldsymbol{Q}\otimes\boldsymbol{Q}) + (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)(\boldsymbol{I}\otimes\boldsymbol{S})\boldsymbol{K}(\boldsymbol{Q}\otimes\boldsymbol{Q})\\
&= (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\boldsymbol{S}^T\otimes\boldsymbol{I} + (\boldsymbol{I}\otimes\boldsymbol{S})\boldsymbol{K}(\boldsymbol{Q}\otimes\boldsymbol{Q})\\
&= (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\boldsymbol{M}\,(\boldsymbol{Q}\otimes\boldsymbol{Q}).\end{aligned} \tag{B 7}$$

and, consequently,

$$\overline{\widetilde{\boldsymbol{M}}^T\widetilde{\boldsymbol{M}}} = (\boldsymbol{Q}^T \otimes \boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\boldsymbol{M}}\,(\boldsymbol{Q}\otimes\boldsymbol{Q}). \tag{B 8}$$

B.2. *Transformation property of* $\overline{\boldsymbol{M}^T\,\mathrm{vec}[\partial_t \boldsymbol{S}]}$

The transformation property (B 7) implies that we can write the transformed spatial average as

$$\begin{aligned}\overline{\widetilde{\boldsymbol{M}}^T\ \mathrm{vec}[\widetilde{\partial_t \boldsymbol{S}}]} &= \overline{(\boldsymbol{Q}\otimes\boldsymbol{Q})^T\,\boldsymbol{M}^T\,(\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)^T\ \mathrm{vec}\left[\boldsymbol{Q}^T\partial_t\boldsymbol{S}\boldsymbol{Q}+\dot{\boldsymbol{Q}}^T\boldsymbol{S}\boldsymbol{Q}+\boldsymbol{Q}^T\boldsymbol{S}\dot{\boldsymbol{Q}}\right]}\\ &= \overline{(\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\boldsymbol{M}^T\,(\boldsymbol{Q}\otimes\boldsymbol{Q})\ \mathrm{vec}\left[\boldsymbol{Q}^T\partial_t\boldsymbol{S}\boldsymbol{Q}+\dot{\boldsymbol{Q}}^T\boldsymbol{S}\boldsymbol{Q}+\boldsymbol{Q}^T\boldsymbol{S}\dot{\boldsymbol{Q}}\right]}.\end{aligned} \tag{B 9}$$

Treating each term in the square bracket separately, we obtain

$$(\boldsymbol{Q}\otimes\boldsymbol{Q})\ \mathrm{vec}[\boldsymbol{Q}^T\partial_t\boldsymbol{S}\boldsymbol{Q}] = (\boldsymbol{Q}\boldsymbol{Q}^T\otimes\boldsymbol{Q}\boldsymbol{Q}^T)\ \mathrm{vec}[\partial_t\boldsymbol{S}] = \mathrm{vec}[\partial_t\boldsymbol{S}], \tag{B 10}$$

$$(\boldsymbol{Q}\otimes\boldsymbol{Q})\ \mathrm{vec}[\dot{\boldsymbol{Q}}^T\boldsymbol{S}\boldsymbol{Q}] = (\boldsymbol{Q}\dot{\boldsymbol{Q}}^T\otimes\boldsymbol{Q}\boldsymbol{Q}^T)\ \mathrm{vec}[\boldsymbol{S}], \tag{B 11}$$

$$(\boldsymbol{Q}\otimes\boldsymbol{Q})\ \mathrm{vec}[\boldsymbol{Q}^T\boldsymbol{S}\dot{\boldsymbol{Q}}] = (\dot{\boldsymbol{Q}}\boldsymbol{Q}^T\otimes\boldsymbol{Q}\boldsymbol{Q}^T)\ \mathrm{vec}[\boldsymbol{S}]. \tag{B 12}$$

Combining the above results gives

$$(\boldsymbol{Q}\otimes\boldsymbol{Q})\ \mathrm{vec}[\boldsymbol{Q}^T\partial_t\boldsymbol{S}\boldsymbol{Q}+\dot{\boldsymbol{Q}}^T\boldsymbol{S}\boldsymbol{Q}+\boldsymbol{Q}^T\boldsymbol{S}\dot{\boldsymbol{Q}}] = \mathrm{vec}[\partial_t\boldsymbol{S}+\dot{\boldsymbol{Q}}\boldsymbol{Q}^T\boldsymbol{S}+\boldsymbol{S}\boldsymbol{Q}\dot{\boldsymbol{Q}}^T]. \tag{B 13}$$

Substituting back into (B 9) and using $\dot{\boldsymbol{Q}}\boldsymbol{Q}^T = -\boldsymbol{Q}\dot{\boldsymbol{Q}}^T$, we obtain

$$\overline{\widetilde{\boldsymbol{M}}^T\ \mathrm{vec}[\widetilde{\partial_t\boldsymbol{S}}]} = (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\ \mathrm{vec}\left[\partial_t\boldsymbol{S}-\dot{\boldsymbol{Q}}^T\boldsymbol{Q}\boldsymbol{S}-\boldsymbol{S}\dot{\boldsymbol{Q}}\boldsymbol{Q}^T\right]}. \tag{B 14}$$

B.3. *Transformation property of* vec$[\boldsymbol{T}]$

By combining the above results, we obtain

$$\begin{aligned}\mathrm{vec}[\widetilde{\boldsymbol{T}}] &= \overline{\widetilde{\boldsymbol{M}}^T\widetilde{\boldsymbol{M}}}^{-1}\ \overline{\widetilde{\boldsymbol{M}}^T\,\mathrm{vec}[\widetilde{\partial_t\boldsymbol{S}}]}\\ &= \left[(\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\boldsymbol{M}}\,(\boldsymbol{Q}\otimes\boldsymbol{Q})\right]^{-1}(\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\ \mathrm{vec}[\partial_t\boldsymbol{S}-\dot{\boldsymbol{Q}}^T\boldsymbol{Q}\boldsymbol{S}-\boldsymbol{S}\dot{\boldsymbol{Q}}\boldsymbol{Q}^T]}\\ &= (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\boldsymbol{M}}^{-1}\,\overline{\boldsymbol{M}^T\ \mathrm{vec}[\partial_t\boldsymbol{S}-\dot{\boldsymbol{Q}}^T\boldsymbol{Q}\boldsymbol{S}-\boldsymbol{S}\dot{\boldsymbol{Q}}\boldsymbol{Q}^T]}\\ &= (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\boldsymbol{M}}^{-1}\,\overline{\boldsymbol{M}^T\ \mathrm{vec}[\partial_t\boldsymbol{S}]}\\ &\quad-(\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\boldsymbol{M}}^{-1}\,\overline{\boldsymbol{M}^T\ \mathrm{vec}[\dot{\boldsymbol{Q}}^T\boldsymbol{Q}\boldsymbol{S}+\boldsymbol{S}\dot{\boldsymbol{Q}}\boldsymbol{Q}^T]}.\end{aligned} \tag{B 15}$$

Using $\mathrm{vec}[\dot{\boldsymbol{Q}}^T\boldsymbol{Q}\boldsymbol{S}+\boldsymbol{S}\dot{\boldsymbol{Q}}\boldsymbol{Q}^T] = \boldsymbol{M}\ \mathrm{vec}[\boldsymbol{Q}^T\dot{\boldsymbol{Q}}]$, we can further simplify:

$$\begin{aligned}(\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\boldsymbol{M}}^{-1}\,\overline{\boldsymbol{M}^T\ \mathrm{vec}[\dot{\boldsymbol{Q}}^T\boldsymbol{Q}\boldsymbol{S}+\boldsymbol{S}\dot{\boldsymbol{Q}}\boldsymbol{Q}^T]} &= (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\,\overline{\boldsymbol{M}^T\boldsymbol{M}}^{-1}\,\overline{\boldsymbol{M}^T\boldsymbol{M}}\ \mathrm{vec}[\dot{\boldsymbol{Q}}\boldsymbol{Q}^T]\\ &= (\boldsymbol{Q}^T\otimes\boldsymbol{Q}^T)\ \mathrm{vec}[\dot{\boldsymbol{Q}}\boldsymbol{Q}^T]\\ &= \mathrm{vec}[\boldsymbol{Q}^T(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T)\boldsymbol{Q}]\\ &= \mathrm{vec}[\boldsymbol{Q}^T\dot{\boldsymbol{Q}}],\end{aligned} \tag{B 16}$$

Therefore, the matrix $\boldsymbol{T}$ transforms like a spin tensor under a general frame change. The condition (4.3) for spatially linear fields follows from these results if the inverse of $\overline{\boldsymbol{M}^T\boldsymbol{M}}$ is replaced by its Moore–Penrose pseudoinverse.

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