POLYNOMIALLY EFFECTIVE EQUIDISTRIBUTION FOR CERTAIN UNIPOTENT SUBGROUPS IN QUOTIENTS OF SEMISIMPLE LIE GROUPS

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Abstract. We prove an effective equidistribution theorem for orbits of certain unipotent subgroups in arithmetic quotients of semisimple Lie groups with a polynomial error term. This provides the first infinite family of examples where effective equidistribution, with polynomial error rate, of non-horospherical unipotent subgroups in semisimple quotients is obtained.

As applications, we obtain effective estimate on distribution of lattice orbits on homogeneous spaces, as well as an effective version of the Oppenheim conjecture for indefinite quadratic forms with a polynomial error rate in all dimension $d \geq 3$. We also prove a sub-modularity inequality for irreducible representation of connected algebraic group, which is crucial to our proof and is of independent interest.

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Part 0. Introduction

1. Introduction

An important theme in homogeneous dynamics is the behavior of orbits of Ad-unipotent subgroups from *any* initial point. More precisely, let G be a Lie group, $\Gamma < G$ be a lattice and $U \leq G$ be an Ad-unipotent subgroup. Raghunathan conjectured that for *any* initial point $x \in X = G/\Gamma$, the orbit closure $\overline{U.x}$ is a periodic orbit $L.x$ of some subgroup $U \leq L \leq G$. We say $L.x$ is periodic if $\mathrm{Stab}(x) \cap L$ is a lattice in L. In the literature the conjecture was first stated in the paper [Dan81] and in a more general form in [Mar90] where the subgroup U is not necessarily Ad-unipotent but generated by Ad-unipotent elements.

Raghunathan's conjecture was proved in full generality by Ratner in [Rat90a, Rat90b, Rat91a, Rat91b]. In her landmark work, Ratner also classified all ergodic invariant probability measures under the action of U and proved an equidistribution theorem for orbits of U. These remarkable theorems have been highly influential and have led to a lot of important applications.

Prior to Ratner's proof, the conjecture was known in certain cases. In particular, the following important special case relates to Oppenheim conjecture on distribution of values of indefinite quadratic forms on integer points. In his seminal work [Mar89], Margulis proved Oppenheim conjecture by showing every $\mathrm{SO}(2,1)$-orbit in $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ is either periodic or unbounded. Later, Dani and Margulis [DM89, DM90] showed that any $\mathrm{SO}(2,1)$-orbit is either periodic or dense. They also classified possible orbit closures of a one-parameter unipotent subgroup of $\mathrm{SO}(2,1)$ in $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$. We refer to the book by Morris [Mor05] for a detailed historical background.

Based on equidistribution results for unipotent subgroups, information on asymptotics of distribution of values of indefinite quadratic forms with signature (p,q) on integer points when $p \geq 3$ or $(p,q) = (2,2)$ is provided by Eskin, Margulis and Mozes in [EMM98, EMM05]. Recently, Kim extended the ideas by Eskin, Margulis and Mozes to indefinite quadratic forms with signature $(2,1)$ in [Kim24].

Because of its intrinsic interest and in view of the applications, *effective* results on distribution of the orbits of unipotent groups have been sought after for some time.

An effective proof of the Oppenheim conjecture with a poly-logarithmic rate was obtained by Lindenstrauss and Margulis in [LM14]. In the landmark works by Lindenstrauss and Mohammadi [LM23] and later with Wang and Yang and by Yang [LMW22, Yan25, LMWY25], effective density and equidistribution theorems with polynomial rate for orbits of unipotent subgroups are established in quotients of quasi-split, almost simple linear algebraic groups of absolute rank 2. In [LMWY25], they established a proof of effective Oppenheim conjecture with a polynomial rate when the dimension $d = 3$ building on their effective equidistribution theorem. We refer to [LMW22] and [Moh23] for a throughout survey on both historical background and recent progress.

In [Lin25], the author established an effective equidistribution for some 2-dimensional unipotent subgroups in quotients of $\mathrm{SL}_4(\mathbb{R})$ aiming at establishing a proof of effective Oppenheim conjecture with a polynomial rate when the dimension $d = 4$.

The main motivation for this paper is to extend these results to cover a wide class of unipotent subgroups in a maximal semisimple subgroup H of a semisimple group G. Before we state the main theorem, let us introduce the following notions.

Let $\mathbf{G} \leq \mathrm{SL}_N$ be a connected semisimple $\mathbb{Q}$-group, let $G = \mathbf{G}(\mathbb{R})$ be its group of $\mathbb{R}$-points, and let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. Let $\Gamma = G \cap \mathrm{SL}_N(\mathbb{Z})$. It is a lattice of G by the theorem of Borel–Harish-Chandra. Let $X = G/\Gamma$ and let μ_X be the unique probability Haar measure on X.

We fixed an inner product on $\mathfrak{g}$ induced by the standard Euclidean metric on $\mathfrak{sl}_N(\mathbb{R})$ and we use $\|\cdot\|$ to denote the induced norm. It induces a right G-invariant Riemannian metric on G and hence a Riemannian metric d_X on $X = G/\Gamma$.

Let $\mathbf{H}$ be a connected semisimple $\mathbb{R}$-subgroup of $\mathbf{G}$ and let $H = \mathbf{H}(\mathbb{R})$ be its group of $\mathbb{R}$-points. Suppose H has no compact factor. Let $\mathfrak{h}$ be its Lie algebra and let $\mathfrak{h} = \bigoplus_{i=1}^{k} \mathfrak{h}_i$ be the unique decomposition of $\mathfrak{h}$ into its simple ideals. Since $\mathbf{H}$ is semisimple, there exists a decomposition of $\mathfrak{g}$ into H-invariant complement as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$.

Let $\mathbf{a} \in \mathfrak{h}$ be a semisimple element with $\|\mathbf{a}\| = 1$ whose projection to each simple factor is non-zero. Suppose $\mathrm{ad}\,\mathbf{a}$ is diagonalizable over $\mathbb{R}$. Let $a_t = \exp(t\mathbf{a})$ and let U be the expanding horospherical subgroup of H as the following:

$$U = \{u \in H : a_{-t} u a_t \to e \text{ as } t \to +\infty\}.$$

Let $\mathfrak{u}$ be the Lie algebra of U. Let $B_r^{\mathfrak{u}}$ be the ball of radius r under the norm $\|\cdot\|$ centered at the origin and let $B_r^U = \exp(B_r^{\mathfrak{u}})$. We fix a Haar measure $\mathrm{d}u$ on U so that the measure of B_1^U is of measure 1.

Throughout this paper, we fixed the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ as above. By *absolute constant* we mean constant depending explicitly on the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$. We will discuss more details on this dependence in Section 2.

The following are standing assumption throughout this paper.

Irreducible. *The subspace $\mathfrak{r}$ is an irreducible representation of H.*

Proximal. *The action of $\mathrm{Ad}(a_t)|_{\mathfrak{r}}$ on $\mathfrak{r}$ is proximal, that is, the largest eigenvalue of $\mathrm{Ad}(a_1)|_{\mathfrak{r}}$ on $\mathfrak{r}$ has multiplicity 1.*

Theorem 1.1. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible and Proximal, then there exist absolute constants $A_1 > A_2 \geq 1$ and $\kappa > 0$ so that the following holds. For all $x_0 \in X$ and large enough R depending explicitly on x_0, for any $T \geq R^{A_1}$, at least one of the following is true.*

(1) For all $\phi \in \mathrm{C}_c^\infty(X)$,

$$\left| \int_{B_1^U} \phi(a_{\log T} u.x_0)\,\mathrm{d}u - \int_X \phi\,\mathrm{d}\mu_X \right| \leq \mathcal{S}(\phi) R^{-\kappa}$$

where $\mathcal{S}(\phi)$ is a certain Sobolev norm.

(2) There exists $x \in X$ so that $H.x$ is periodic with $\mathrm{vol}(H.x) \leq R$ and

$$d_X(x_0, x) \leq T^{-\frac{1}{A_2}}.$$

Remark 1.2. We remark that the dependence of R on x_0 is of the form $R \gg \mathrm{inj}(x_0)^{-\star}$. See Section 2 for the precise definition of $\mathrm{inj}(x_0)$ and the convention on $\star$-notations.

1.1. **Examples.** We provide examples so that the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible and Proximal. Note that these two conditions depend only on $(G, H, \mathbf{a})$, we will only describe them in this subsection.

Example 1.1. Let $\mathbf{G}$ be a quasi-split semisimple group with absolute rank 2 and let $\mathbf{H}$ be a principal SL_2 in $\mathbf{G}$. Under this condition, $\mathbf{G}$ is locally isomorphic to one of the following:

$$\mathrm{SL}_2(\mathbb{C}), \quad \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}), \quad \mathrm{SL}_3(\mathbb{R}), \quad \mathrm{SU}(2,1), \quad \mathrm{Sp}_4(\mathbb{R}), \quad \mathrm{G}_2(\mathbb{R}).$$

In this case, the complement $\mathfrak{r}$ is an irreducible representation of SL_2 and Irreducible and Proximal are automatically satisfied. This is the case studied by Lindenstrauss–Mohammadi–Wang–Yang in [LMWY25].

Example 1.2. Let $\mathbf{G}_0$ be a $\mathbb{Q}$-group that is $\mathbb{R}$-simple and $\mathbb{R}$-split, e.g., SL_d, $\mathrm{SO}(n,n)$, Sp_{2m}. Let $\mathbf{G} = \mathbf{G}_0 \times \mathbf{G}_0$ and let

$$\mathbf{H} = \{(h,h) : h \in \mathbf{G}_0\} < \mathbf{G}.$$

Let $\mathbf{a}$ be a *regular* element in $\mathfrak{h}$, that is, it lies in the interior of a Weyl chamber. In this case, Irreducible follows from the fact that $\mathbf{G}_0$ is $\mathbb{R}$-simple and Proximal follows from the fact that $\mathbf{G}_0$ is $\mathbb{R}$-split and $\mathbf{a}$ is regular.

Example 1.3. Let $\mathbf{G} = \mathrm{SL}_{2n}$ and let $\mathbf{H} = \mathrm{Sp}_{2n}$. Let $\mathbf{a}$ be a *regular* element in $\mathfrak{h}$. A direct computation shows that the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfying Irreducible and Proximal.

Example 1.4. Let $d = p + q$ and let Q_0 be the quadratic form of signature (p, q) defined as the following:

$$Q_0(x_1, \dots, x_{p+q}) = 2x_1 x_{p+q} + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q-1}^2.$$

Let $\mathbf{G} = \mathrm{SL}_d$ and let $\mathbf{H} = \mathrm{SO}(Q_0)$. Let $e_1, \dots, e_{p+q}$ be the standard basis of $\mathbb{R}^d$. Let a_t be the following diagonal flow in H defined as the following:

$$a_t e_1 = e^t e_1, \qquad a_t e_{p+q} = e^{-t} e_{p+q}, \qquad a_t e_i = e_i \text{ for } i = 2, \dots, p+q-1.$$

A direct computation shows that the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfying Irreducible and Proximal. This example is a generalization of the cases in [LMWY23] and [Lin25].

We remark that these data has been used by Eskin–Margulis–Mozes in [EMM98, EMM05] to prove an asymptotic formula for values of distribution of irrational quadratic form on integer points.

We also remark that e.g. if $p = q \geq 3$, this choice of $\{a_t\}_{t \in \mathbb{R}}$ lies on a wall of Weyl chamber and it is *not* regular.

Example 1.5. We also provide a family of examples where $\mathbf{H}$ is semisimple but not simple. We identify $\mathbb{R}^{nm} \cong \mathbb{R}^n \otimes \mathbb{R}^m$. Let $\mathbf{G} = \mathrm{SL}_{mn}$ and let $\mathbf{H} = \mathrm{SL}_n \times \mathrm{SL}_m$. The group $\mathbf{H} = \mathrm{SL}_n \times \mathrm{SL}_m$ is embedded into $\mathbf{G}$ via the tensor product of irreducible representations $\mathbb{R}^n$ and $\mathbb{R}^m$ of each of its factor. Let $\mathbf{a}$ be a regular element in $\mathfrak{sl}_n \oplus \mathfrak{sl}_m$. In this case, the complement $\mathfrak{r}$ is isomorphic to $\mathfrak{sl}_n \otimes \mathfrak{sl}_m$ as a representation of $\mathrm{SL}_n \times \mathrm{SL}_m$.

We remark that in the previous example when Q_0 has signature $(2,2)$, $\mathbf{H}$ is locally isomorphic to $\mathrm{SL}_2 \times \mathrm{SL}_2$ and $\mathfrak{r} \cong \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ as in this example.

1.2. **Applications.** As mentioned before, homogeneous dynamics has had several applications in number theory and geometry. In this brief section, we mention two applications of Theorem 1.1.

1.2.1. *Distribution of lattice orbit in homogeneous spaces.* Motivated by central problems in arithmetic geometry, homogeneous dynamics has been successful in providing sharp estimates in orbital counting problems on homogeneous varieties. In the above notation, this concerns Γ-orbits in $H\backslash G$.

There has been extensive research on this problem especially on the whole group G and the symmetric space $K\backslash G$ where all Γ-orbits are discrete (see [BO12] and references therein). There has also been research, albeit to a lesser extent, on non-discrete Γ-orbits, where one expects a limiting distribution of Γ-orbits on $H\backslash G$ according to some natural limiting process.

A major work in this direction was done by Gorodnik–Weiss [GW07]. For any closed subgroup $S \leq G$, we denote by $B_r^S(x) \subset S$ the open ball of radius $r > 0$ centered at $x \in X$ with respect to the metric induced by the Riemannian metric on G. We write

$$H_T := H \cap KB_T^G(e)K \quad \text{and} \quad \Gamma_T := \Gamma \cap KB_T^G(e)K$$

for all $T > 0$. For all $y_0 \in H\backslash G$, there is a canonical measure $\nu_{y_0} \ll \mu_{H\backslash G}$ as defined in [GW07, Eq. (12) and Proposition 5.1] which we normalize as $\hat{\nu}_{y_0} := \mathrm{vol}(X)^{-1}\nu_{y_0}$. In [GW07], they proved that $\hat{\nu}_{y_0}$ is the limiting density of the orbit $\Gamma.y_0 \subset H\backslash G$ whenever it is dense in the following sense.

Theorem 1.3 ([GW07, Theorem 1.1])**.** *Let $y_0 \in H\backslash G$ such that $\Gamma.y_0 \subset H\backslash G$ is dense. Then, for all $\psi \in C_c(H\backslash G)$, we have*

$$\lim_{T\to+\infty} \frac{1}{\mu_H(H_T)} \sum_{\gamma\in\Gamma_T} \psi(\gamma.y_0) = \int_{H\backslash G} \psi \, d\hat{\nu}_{y_0}.$$

In a previous joint paper by Sarkar and the author, we establish an estimate of error term under an effective equidistribution hypothesis, see [LS24, Theorem 1.12]. Combining it with Theorem 1.1, we have the following unconditional result.

Before we state it, let us introduce the following notation. For all Lipschitz function ψ on $H\backslash G$ with compact support, we define a corresponding constant

$$D_\psi := \inf\{r > 0 : \mathrm{supp}(\psi) \subset B_r^G(e) \cdot H \subset H\backslash G\} > 0.$$

Theorem 1.4. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible and Proximal. There exists $\varsigma_H > 0$ depending only on H and $\kappa > 0$ such that the following holds. Let ψ be a Lipschitz function on $H\backslash G$ with compact support, $g_0 \in G$, $x_0 = g_0\Gamma \in X$, and $y_0 = Hg_0 \in H\backslash G$. There exists $M_{\psi,g_0} > 0$ such that for all $R \gg_{G,H,\Gamma,g_0,\chi} 1$ and $T \gg_{G,H,\Gamma} \log(R) + M_{\psi,g_0}$, at least one of the following holds.*

(1) We have:

(a) if $\mathrm{rank}(G) = 1$, then

$$\left| \frac{1}{\mu_H(H_T)} \sum_{\gamma\in\Gamma_T} \psi(\gamma.y_0) - \int_{H\backslash G} \psi \, d\hat{\nu}_{y_0} \right| \ll_{D_\psi} \|\psi\|_{\mathrm{Lip}} R^{-\kappa};$$

(b) if $\mathrm{rank}(G) \geq 2$*, then*

$$\left|\frac{1}{\mu_H(H_T)} \sum_{\gamma \in \Gamma_T} \psi(\gamma . y_0) - \int_{H \backslash G} \psi \, d\hat{\nu}_{y_0}\right| \ll_{D_\psi} \|\psi\|_{\mathrm{Lip}} \big(T^{-\frac{1}{2\dim(G)+5}} + R^{-\kappa}\big).$$

(2) There exists $x \in X$ *with*

$$d(x_0, x) \leq e^{-\frac{\varsigma_H}{2} T}$$

such that Hx *is periodic with* $\mathrm{vol}(Hx) \leq R$*.*

1.2.2. *Effective version of the Oppenheim conjecture.* In a forthcoming paper, we will investigate the applications of Theorem 1.1 to distribution of value of indefinite quadratic forms on integer points in all dimension $d \geq 3$ and further counting results. In particular, we will obtain an effective version of the Oppenheim conjecture for quadratic forms with at least three variables.

Theorem 1.5. *For all integer* $d \geq 3$*, and all algebraic irrational indefinite quadratic form* Q*, there exists* $\kappa = \kappa_Q > 0$ *so that the following holds. For all* T *large enough depending explicitly on* Q *and all*

$$s \in [-T^\kappa, T^\kappa],$$

there exists a primitive vector $v \in \mathbb{Z}^d$ *with* $0 < \|v\| \leq T$ *so that*

$$|Q(v) - s| \leq T^{-\kappa}.$$

This theorem is obtained in [Lin25] by combining the work by Lindenstrauss, Mohammadi, Wang and Yang for quadratic forms of signature $(2,1)$ [LMWY25, Theorem 2.5], the work by the author for quadratic forms of signature $(2,2)$ and $(3,1)$ [Lin25, Theorem 1.4], and the work by Buterus, Götze, Hille and Margulis for indefinite quadratic forms with at least 5 variables [BGHM22, Corollary 1.4]. We remark that the work by Buterus, Götze, Hille and Margulis utilizes analytic number theory and a quantitative version of Meyer's theorem.

In contrast, the proof here will be purely based on homogeneous dynamics and we deal with all dimension $d \geq 3$ at once.

1.3. **On the proof of Theorem 1.1.** We now discuss the proof of Theorem 1.1.

The strategy of the proof of Theorem 1.1 is similar to the general strategy in [LMW22, LMWY25, Lin25]. However, due to the complication from H and the $\mathrm{Ad}(H)$-invariant complement $\mathfrak{r}$, the achievement to higher dimension is harder. Before we point out the difficulties and the solutions, let us recall the general strategy developed in [LMW22, LMWY25].

In [LMWY25], the proof can be roughly divided into three phases:

(1) Initial dimension from effective closing lemma;

(2) Improving dimension using ingredients from projection theorems;

(3) From large dimension to equidistribution.

The major difficulties in our setting come from phase (1) and (2). Due to the complexity of H, especially the case where H is semisimple but not simple, phase (1) cannot be proved directly as in [LMWY25, Section 4]. However, thanks to the effective closing lemma for long unipotent orbits proved by Lindenstrauss, Margulis, Mohammadi, Shah and Wieser in [LMM+24], we obtain a similar initial phase. The proof is similar to [Lin25, Part 1] but the proof needs modifications to be applicable to the general setting at hand. One of the crucial ingredient is an effective version of Łojasiewiecz's inequality. This phase is carried out in Part 1.

The difficulty from phase (2) is more severe. In [LMWY25] and [Lin25], phase (2) can be very roughly speaking further divided into three steps. First, one establishes a dimension improving result for the linear $\mathrm{Ad}(H)$-action on $\mathfrak{r}$, see [LMWY25, Theorem 6.1] or [Lin25, Theorem 7.10]. Building on this, one establishes a Margulis function estimate which provides dimension improvement in the transverse direction in X, see [LMWY25, Lemma 7.2] or [Lin25, Proposition 16.2]. With the Margulis function estimate, one runs a bootstrap process to get a high dimension (close to $\dim(\mathfrak{r})$) in the transverse direction to H, see [LMWY25, Section 8] or [Lin25, Part 3]. The major difficulty comes from the first step.

The $\mathrm{Ad}(H)$-invariant complement $\mathfrak{r}$ can be decomposed into eigenspaces $\mathfrak{r}_\lambda$ for $\mathbf{a}$. Let $\mathfrak{r}^{(\mu)} = \bigoplus_{\lambda\geq\mu}\mathfrak{r}_\lambda$ and let $\pi^{(\mu)}$ be the orthogonal projection to $\mathfrak{r}^{(\mu)}$. Let $\pi_u^{(\mu)} = \pi^{(\mu)} \circ \mathrm{Ad}(u)$. Roughly speaking, we say the family of maps $\pi_u^{(\mu)} : \mathfrak{r} \to \mathfrak{r}^{(\mu)}$ is optimal if for all set A and almost all parameter $u \in \mathsf{B}_1^U$ one has $\dim \pi_u^{(\mu)}(A) = \min\{\dim A, \dim \mathfrak{r}^{(\mu)}\}$. We say the family of maps $\pi_u^{(\mu)} : \mathfrak{r} \to \mathfrak{r}^{(\mu)}$ is (sub)-critical if for all set A and almost all parameter $u \in \mathsf{B}_1^U$ one has $\dim \pi_u^{(\mu)}(A) = \frac{\dim \mathfrak{r}^{(\mu)}}{\dim \mathfrak{r}} \dim A$. The dim here stands for a suitable dimension notion for fractal-like set. In [Lin25], the first step is obtained by combining an optimal projection theorem to the space $\mathfrak{r}^{(\mu)}$ with largest μ and a (sub)-critical projection theorem for all other $\{\pi_u^{(\mu)}\}_{u\in\mathsf{B}_1^U}$.

As noted in [LMWY25, Section 8] and [Lin25, Section 16.2], an ϵ-improvement in dimension is enough for phase (2). In this paper, we replace the optimal projection theorem by Theorem 7.1, which is a consequence of Bourgain's discretized projection theorem. This is the only place in this paper which uses Proximal and it will be interesting to prove a stronger projection theorem without this condition. It will also be interesting to utilize the optimal projection theorem for moment curve by Gan–Guo–Wang [GGW24] to prove an optimal projection theorem under the condition Proximal.

We now discuss the (sub)-critical bound for $\{\pi_u^{(\mu)}\}$ where μ might not be the largest eigenvalue. In [Lin25], the author obtained such bound when $(G, H) = (\mathrm{SL}_4(\mathbb{R}), \mathrm{SO}(2, 2))$ and $(G, H) = (\mathrm{SL}_4(\mathbb{R}), \mathrm{SO}(3, 1))$. A similar strategy is used in the work by Bénard–He–Zhang in [BHZ25, Section 6]. In this paper, we extend the argument in [Lin25, Section 12] and obtain a (sub)-critical bound for projections in general irreducible representations. We refer to the introductory part in Part 2 and Section 8 for a more detailed exposition.

As an interesting bi-product of the proof of (sub)-critical estimate, we obtain the following inequality. Let F be a field of characteristic 0 and let $\tilde{H}$ be a connected linear algebraic group over F. Let V be a regular irreducible representation of $\tilde{H}$ defined over F.

Theorem 1.6. *For all subspaces $W, W' \subseteq V$, there exists $\tilde{h} \in \tilde{H}$ so that*

$$\dim[(\tilde{h}.W) \cap W'] \leq \frac{\dim W}{\dim V} \dim W'. \tag{1}$$

Note that the set of such $\tilde{h}$ forms a Zariski open subset of $\tilde{H}$.

Remark 1.7. Although we state the above theorem as irreducible representation for general connected algebraic groups, it is essentially a theorem for reductive groups.

Any irreducible representation of connected algebraic group factors through its reductive quotient since the fixed point of the unipotent radical is a sub-representation.

Remark 1.8. The dimension bound in Theorem 1.6 is optimal in general. Suppose $\tilde{H} = \tilde{H}_1 \times \tilde{H}_2$ and let V_1 and V_2 be two irreducible representation of $\tilde{H}_1$ and $\tilde{H}_2$ respectively. Let $V = V_1 \otimes V_2$ and let $v_i \in V_i$ be nonzero vectors. Then if we set $W = V_1 \otimes Fv_2$ and $W' = Fv_1 \otimes V_2$, for all $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)$

$$\dim[(\tilde{h}.W) \cap W'] = \dim F(v_1 \otimes h_2.v_2) = 1 = \frac{\dim W}{\dim V} \dim W'.$$

The case of Theorem 1.6 where $\tilde{H}$ is a real reductive group defined over $\mathbb{Q}$ and both the representation and *the subspace* W are defined over $\mathbb{Q}$ was proved by Eskin–Mozes–Shah using non-divergence estimate for translate of algebraic measure in homogeneous space, see [EMS97, Corollary 1.4]. John Stalker has a proof using unitary trick for irreducible representation over $\mathbb{R}$ for reductive groups. The proof we give in Section 9 is purely combinatorial and is inspired by the submodularity inequality of entropy. It utilizes only irreducibility and Zariski connectedness.

Very recently, Bénard–He independently established similar subcritical estimate in adjoint representation of simple Lie groups, see [BH25b, Section 5]. Their proof is based on structure of root system of simple Lie algebra and a case-by-case study. In contrast, our proof for the above (sub)-critical estimate is combinatorial and utilizes only the irreducibility of V. We refer to Section 8 for a more detailed exposition of our method.

Part 1 and Part 2 are independent. Part 1 is devoted to phase (1). Its main theorem is Theorem 2.6. Part 2 is devoted to the linear dimension improvement result in phase (2). Its main theorem is Theorem 6.1. We refer to Subsection 2.9 for the proof of Theorem 1.1 assuming Theorem 2.6 and Theorem 6.1.

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2. Notations and Preliminaries

As indicated in the introduction, Parts 1 and 2 can be read independently. In this section, we introduce the notations and preliminaries used throughout this whole paper. New notations and preliminaries needed inside Parts 1 and 2 will be introduced in those 'preparation' sections. We remark that *inside* Parts 1 and 2, we might slightly change the conventions for simplicity. We will always clarify the changes at the beginning of each section.

2.1. **Lie groups and Lie algebras.** We use corresponding Fraktur letters for the Lie algebras of Lie groups throughout the paper. For example, $\mathfrak{s}$ is the Lie algebra of Lie group S. For a Lie group S, we use S° to denote its identity component under the *Hausdorff topology*. For an algebraic group $\mathbf{S}$, we use $\mathbf{S}^0$ to denote its identity component under the *Zariski topology*. For a group G acting on a space X, we use $g.x$ to denote this action. Sometimes the action is clear from the context and we will use $g.x$ without introducing it explicitly. For example, for $v \in \mathfrak{g}$ and $g \in G$, we write $g.v = \mathrm{Ad}(g)v$.

2.2. **The data** $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$**.** We first recall that from the introduction the definition and condition on the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$.

Let $\mathbf{G} \leq \mathrm{SL}_N$ be a connected semisimple $\mathbb{Q}$-group, let $G = \mathbf{G}(\mathbb{R})$ be its group of $\mathbb{R}$-points, and let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. Let $\Gamma = G \cap \mathrm{SL}_N(\mathbb{Z})$. It is a lattice of G by the theorem of Borel–Harish-Chandra. Let $X = G/\Gamma$ and let μ_X be the unique probability Haar measure on X.

We fixed an inner product on $\mathfrak{g}$ induced by the standard Euclidean metric on $\mathfrak{sl}_N(\mathbb{R})$ and we use $\|\cdot\|$ to denote the induced norm. It induces a right G-invariant Riemannian metric on G and hence a Riemannian metric d_X on $X = G/\Gamma$.

Let $\mathbf{H}$ be a connected semisimple $\mathbb{R}$-subgroup of $\mathbf{G}$ and let $H = \mathbf{H}(\mathbb{R})$ be its group of $\mathbb{R}$-points. Suppose H has no compact factor. Let $\mathfrak{h}$ be its Lie algebra and let $\mathfrak{h} = \bigoplus_{i=1}^k \mathfrak{h}_i$ be the unique decomposition of $\mathfrak{h}$ into its simple ideals. Since $\mathbf{H}$ is semisimple, there exists a decomposition of $\mathfrak{g}$ into H-invariant complement as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$.

Let $\mathbf{a} \in \mathfrak{h}$ be a semisimple element with $\|\mathbf{a}\| = 1$ whose projection to each simple factor is non-zero. Suppose $\mathrm{ad}\,\mathbf{a}$ is diagonalizable over $\mathbb{R}$. Let $a_t = \exp(t\mathbf{a})$ and let U be the expanding horospherical subgroup of H as the following:

$$U = \{u \in H : a_{-t} u a_t \to e \text{ as } t \to +\infty\}.$$

Let $\mathfrak{u}$ be the Lie algebra of U. Let $B_r^{\mathfrak{u}}$ be the ball of radius r under the norm $\|\cdot\|$ centered at the origin and let $B_r^U = \exp(B_r^{\mathfrak{u}})$. We fix a Haar measure $\mathrm{d}u$ on U so that the measure of B_1^U is of measure 1.

We fix a Cartan involution of $\mathfrak{h}$ so that $\theta(\mathbf{a}) = -\mathbf{a}$. We have the Cartan decomposition $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$. We remark that by Mostow's theorem [Mos55], one can choose θ to be also a Cartan involution of $\mathfrak{g}$ though we don't need it in this paper. Let

$$\mathfrak{h} = \mathfrak{u}^- \oplus \mathfrak{m}_0 \oplus \mathfrak{a} \oplus \mathfrak{u}$$

be the corresponding restricted root decomposition. Let $A_H = \exp(\mathfrak{a}) \leq H$, $M_0 = \exp(\mathfrak{m}_0)$ and $U^- = \exp(\mathfrak{u}^-)$. Let $U^+ = U$ and $\mathfrak{u}^+ = \mathfrak{u}$. We remark that since the projection of $\mathbf{a}$ to each simple ideal factor in $\mathfrak{h}$ is nontrivial, so is $\mathfrak{u}$ and we can write $\mathfrak{u} = \oplus_i \mathfrak{u}_i$ where $\mathfrak{u}_i < \mathfrak{h}_i$ and $\mathfrak{u}_i \neq \{0\}$.

We set $\lambda_{\min}$ to be the least positive eigenvalue of $\mathbf{a}$ and $\lambda_{\max}$ to be the largest positive eigenvalue of $\mathbf{a}$ on $\mathfrak{g}$.

In Part 1, we need a $\mathbb{Q}$-structure on $\mathfrak{h}$. Let us recall the following result in [Kam14]. See also [Mor04, Mor15]. We remark that when $\mathbf{H}$ is $\mathbb{R}$-split, the existence of such basis is a classical result by Chevalley and Serre.

Proposition 2.1 ([Kam14, Theorem 4.1])**.** *There exists a basis* $\{x_i\}_{i=1}^h$ *of* $\mathfrak{h}$ *with structure constants* $c_{i,j}^k \in \frac{1}{2}\mathbb{Z}$*.*

$$[x_i, x_j] = \sum_{k=1}^{h} c_{ij}^k x_k.$$

Throughout this paper, by dependence on the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$, we allow dependence on the size of the above basis of $\mathfrak{h}$.

2.3. **Constants and $\star$-notations.** For $A \ll B^\star$, we mean there exist constants $C > 0$ and $\kappa > 0$ depending at most on the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ such that $A \leq CB^\kappa$. For $A \asymp B$, we mean $A \ll B$ and $B \ll A$. We also use the notion of $O(\cdot)$ where $f = O(g)$ is the same as $|f| \ll g$. For $A \ll_D B$, we mean there exist constant $C_D > 0$ depending on D and at most on the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ so that $A \leq C_D B$. For example, in Part 2, we will heavily use the notation $A \ll_\epsilon B^{O(\sqrt{\epsilon})}$. This is equivalent to the following. There exists a constant C_ϵ depending on ϵ and at most also on the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ and a constant E depending at most on the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ so that $A \leq C_\epsilon B^{E\sqrt{\epsilon}}$.

2.4. **Norms and balls.** Let $\|\cdot\|_\infty$ be the maximum norm from $\mathfrak{sl}_N(\mathbb{R})$. For any subspace $V \subseteq \mathfrak{g} \subset \mathfrak{sl}_N(\mathbb{R})$ and $v \in V$, we define

$$B_r^{V,\infty}(v) = \{w \in V : \|w - v\| \leq r\}.$$

If $v = 0$, we often omit it and denote the ball by $B_r^{V,\infty}$.

We define

$$\mathsf{B}_r^U = \exp(B_r^{\mathfrak{u},\infty}), \mathsf{B}_r^A = \exp(B_r^{\mathfrak{a},\infty}), \mathsf{B}_r^{M_0} = \exp(B_r^{\mathfrak{m}_0,\infty}), \mathsf{B}_r^{U^-} = \exp(B_r^{\mathfrak{u}^-,\infty})$$

and

$$\mathsf{B}_r^{M_0A} = \mathsf{B}_r^{M_0}\mathsf{B}_r^A = \exp(B_r^{\mathfrak{a}\oplus\mathfrak{m}_0,\infty}).$$

We set

$$\mathsf{B}_r^H = \mathsf{B}_r^{U^-}\mathsf{B}_r^{M_0A}\mathsf{B}_r^U,$$
$$\mathsf{B}_r^{s,H} = \mathsf{B}_r^{U^-}\mathsf{B}_r^{M_0A},$$

and $\mathsf{B}_r^G = \mathsf{B}_r^H \exp(B_r^{\mathfrak{r},\infty})$.

2.5. **Natural measures.** Note that $U = \exp(\mathfrak{u})$. Since U is abelian, the exponential map exp is an isomorphism between Lie groups if we identify $\mathfrak{u}$ with $\mathbb{R}^2$ using the standard coordinate in $\mathfrak{sl}_N(\mathbb{R})$. Let $\tilde{m}_U$ be the push-forward of the standard Lebesgue measure Leb under the exponential map. Let m_U be the rescaling of $\tilde{m}_U$ so that it assign B_1^U with measure 1. This is a U-invariant measure on U. For the ball $\mathsf{B}_1^U \subset U$, we use $m_{\mathsf{B}_1^U}$ to denote the restriction of m_U to B_1^U.

Similarly, we can define m_A, m_{M_0}, m_{U^-} via the push-forward of the standard Lebesgue measure on subspaces in $\mathfrak{sl}_N(\mathbb{R})$. They are Haar measures on the corresponding groups. Let m_H be the corresponding Haar measure on H. It is proportional to the measure defined by the volume form induced by the Riemannian metric from the Cartan involution θ.

Recall that since Γ is a lattice in G, there is a unique probability G invariant measure μ_X on $X = G/\Gamma$. This measure is proportional to the measure defined by the volume form induced by the Riemannian metric d_X.

2.6. **Commutation relations.** We record the following consequences of Baker–Campbell–Hausdorff formula.

Lemma 2.2. *There exists $\eta_0 > 0$ and $C_0 > 0$ so that the following holds for all $0 < \eta \le \eta_0$. For all $w_1, w_2 \in B_\eta^{\mathfrak{r}}(0)$, there exists $h \in H$ and $\bar{w} \in \mathfrak{r}$ with*

$$\|h - \mathrm{Id}\| \le C_0\eta, \text{ and } \quad \|\bar{w} - (w_1 - w_2)\| \le C_0\eta\|w_1 - w_2\|$$

so that

$$\exp(w_1)\exp(-w_2) = h\exp(\bar{w}).$$

In particular,

$$\frac{1}{2}\|w_1 - w_2\| \le \|\bar{w}\| \le \frac{3}{2}\|w_1 - w_2\|.$$

Proof. This is a direct application of Baker–Campbell–Hausdorff formula. See [LM23, Lemma 2.1]. ∎

We take a further minimum so that for all $\eta \le \eta_0$ the following holds.

(1) The exponential map restrict to $B_\eta^{\mathfrak{g},\infty}$ is a bi-analytic map.
(2) The maps

$$\begin{aligned}&B_\eta^{\mathfrak{u}^+,\infty} \times B_\eta^{\mathfrak{m}_0,\infty} \times B_\eta^{\mathfrak{a},\infty} \times B_\eta^{\mathfrak{u}^-,\infty} \to H\\ &(X_{\mathfrak{u}^+}, X_{\mathfrak{m}_0}, X_{\mathfrak{a}}, X_{\mathfrak{u}^-}) \mapsto \exp(X_{\mathfrak{u}^+})\exp(X_{\mathfrak{m}_0})\exp(X_{\mathfrak{a}})\exp(X_{\mathfrak{u}^-}),\end{aligned} \tag{2}$$

$$\begin{aligned}&B_\eta^{\mathfrak{u}^-,\infty} \times B_\eta^{\mathfrak{m}_0,\infty} \times B_\eta^{\mathfrak{a},\infty} \times B_\eta^{\mathfrak{u}^+,\infty} \to H\\ &(X_{\mathfrak{u}^-}, X_{\mathfrak{m}_0}, X_{\mathfrak{a}}, X_{\mathfrak{u}^+}) \mapsto \exp(X_{\mathfrak{u}^-})\exp(X_{\mathfrak{m}_0})\exp(X_{\mathfrak{a}})\exp(X_{\mathfrak{u}^+}),\end{aligned} \tag{3}$$

$$\begin{aligned}&B_\eta^{\mathfrak{u}^+,\infty} \times B_\eta^{\mathfrak{u}^-,\infty} \times B_\eta^{\mathfrak{m}_0,\infty} \times B_\eta^{\mathfrak{a},\infty} \to H\\ &(X_{\mathfrak{u}^+}, X_{\mathfrak{u}^-}, X_{\mathfrak{a}}, X_{\mathfrak{m}_0}) \mapsto \exp(X_{\mathfrak{u}^+})\exp(X_{\mathfrak{u}^-})\exp(X_{\mathfrak{m}_0})\exp(X_{\mathfrak{a}})\end{aligned} \tag{4}$$

are bi-analytic map to their images.
(3) The map

$$\begin{aligned}&\mathsf{B}_\eta^H \times B_\eta^{\mathfrak{r},\infty} \to G\\ &(\mathsf{h}, X_{\mathfrak{r}}) \mapsto \mathsf{h}\exp(X_{\mathfrak{r}})\end{aligned}$$

is a bi-analytic map to its image.
(4) Lemma 2.2 holds.

For a parameter $\eta \le \eta_0$ and $\beta = \eta^{\dim G}$, we set

$$\mathsf{E} = \mathsf{B}_\beta^{U^-}\mathsf{B}_\beta^{M_0A}\mathsf{B}_\eta^{U^+}.$$

The choice of the parameter η will always be clear from the context.

2.7. **Injectivity radius.** For all $x \in X = G/\Gamma$, we set

$$\mathrm{inj}(x) = \sup\{\eta : \mathsf{B}_{100C_0\eta}^G \to \mathsf{B}_{100C_0\eta}^G.x \text{ is a diffeomorphism}\}.$$

The constant C_0 comes from Theorem 2.2. Taking a further minimum if necessary, we always assume that the injectivity radius of x defined using the Riemannian metric d_X dominates $\mathrm{inj}(x)$.

For all $\eta > 0$, let

$$X_\eta = \{x \in X : \mathrm{inj}(x) \ge \eta\}.$$

2.8. **A different formulation for Theorem 1.1.** Recall that we set $\mathsf{B}_1^U = \exp(B_1^{\mathfrak{u}})$ and assign m_U to be the Haar measure on U so that $m_U(\mathsf{B}_1^U) = 1$. The following theorem is a slightly different formulation of Theorem 1.1.

Theorem 2.3. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible and Proximal, then there exist absolute constants $A_3 > A_4 \geq 1$ and $\kappa > 0$ so that the following holds. For all $x_0 \in X$ and large enough R depending explicitly on x_0, for any $T \geq R^{A_3}$, at least one of the following is true.*

(1) For all $\phi \in \mathrm{C}_c^\infty(X)$,

$$\left| \int_{\mathsf{B}_1^U} \phi(a_{\log T} u.x_0) \,\mathrm{d}m_U(u) - \int_X \phi \,\mathrm{d}\mu_X \right| \leq \mathcal{S}(\phi) R^{-\kappa}$$

where $\mathcal{S}(\phi)$ is a certain Sobolev norm.

(2) There exists $x \in X$ so that $H.x$ is periodic with $\mathrm{vol}(H.x) \leq R$ and

$$d_X(x_0, x) \leq T^{-\frac{1}{A_4}}.$$

Theorem 2.3 is equivalent to Theorem 1.1. Therefore we will focus on the study of the orbit of $a_{\log T}\mathsf{B}_1^U$ in this paper.

2.9. **Proof of Theorem 2.3 assuming Theorem 2.6 and Theorem 6.1.**

2.9.1. *From large dimension to effective equidistribution.* The goal of this subsection is prove the following lemma.

Lemma 2.4. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible. Then there exists absolute $\varrho_0 \in (0, 1)$ so that the following holds. Let $\delta_0 \in (0, 1)$. Let $\ell_1, \ell_2 > 0$ with $\kappa_1 \ell_2 \geq \max\{\ell_1, |\log \eta|\}$ and $\ell_2 \ll |\log \delta_0|$, and let $\varrho \in (0, \varrho_0]$. Let μ be a probability measure on $B_\varrho^{\mathfrak{r}}(0)$ satisfying*

$$\mu(B_\delta^{\mathfrak{r}}(w)) \leq \Upsilon \delta^{\dim \mathfrak{r}} \quad \forall w \in \mathfrak{r}, \delta \geq \delta_0.$$

Then for all $\phi \in \mathrm{C}_c^\infty(X) + \mathbb{C}\mathbb{1}_X$ and all $x \in X_\eta$, we have

$$\begin{aligned} &\int_{\mathsf{B}_1^U} \int_{\mathsf{B}_1^U} \int_{\mathfrak{r}} \phi(a_{\ell_1} u_1 a_{\ell_2} u_2 \exp(w).x) \,\mathrm{d}\mu(w) \,\mathrm{d}m_U(u_2) \,\mathrm{d}m_U(u_1) \\ &= \int_X \phi \,\mathrm{d}\mu_X + O(\mathcal{S}(\phi)(\varrho^\star + \eta + \Upsilon^\star \varrho^{-\star} e^{-\kappa_1 \ell_1})). \end{aligned}$$

Proof. The statement can be proved by following the proof of [LMWY25, Lemma 5.2] step-by-step, see also [OS25, Theorem 11.1] for a variant when G is simple. Note that we need to establish an estimate of decay of correlation for the flow $a_t = \exp(t\mathbf{a})$, which in turns relies on the following lemma. ∎

Lemma 2.5. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible. Then for all simple ideal factor $\mathfrak{g}_i$ of $\mathfrak{g}$, the projection of $\mathbf{a}$ on $\mathfrak{g}_i$ is non-trivial.*

Proof. If $\mathfrak{g}_i \subseteq \mathfrak{h}$, then it is an ideal of $\mathfrak{h}$ and the lemma follows from the condition that the projection of $\mathbf{a}$ onto every simple ideal factor of $\mathfrak{h}$ is non-trivial.

If not, note that $\mathfrak{g}_i$ is also an $\mathfrak{h}$-module and $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_i$ due to irreducibility of $\mathfrak{r}$. Therefore, there exists $\mathfrak{r}' \subseteq \mathfrak{g}_i$ so that $\mathfrak{r}' \cong \mathfrak{r}$ as $\mathfrak{h}$-module. If the projection of $\mathbf{a}$ onto $\mathfrak{g}_i$ is 0, then $[\mathbf{a}, \mathfrak{g}_i] = \{0\}$. To get a contradiction, it suffices to show that $\mathfrak{r}$ is a non-trivial $\mathfrak{h}$-module.

Suppose not, by irreducibility of $\mathfrak{r}$, it has to be one dimensional and this implies that the center $\mathfrak{Z}(\mathfrak{g})$ is non-trivial, contradict to the assumption $\mathfrak{g}$ is semisimple. ∎

2.9.2. *Proof of Theorem 2.3.*

Proof of Theorem 2.3. The statement can be proved by following [Lin25, Part 3] step-by-step. In particular, replace [Lin25, Theorem 2.3] by Theorem 2.6, [Lin25, Theorem 7.10] by Theorem 6.1 and [Lin25, Lemma 15.2] by Lemma 2.4, the rest of the arguments are mutatis mutandis. ∎

Part 1. Closing lemma and initial dimension

The main result of this part is Theorem 2.6. We remark that all results in this part utilize only Irreducible. Before we state the result, let us fix some parameters.

Let $0 < \epsilon' < 0.001$ be a small constant. Let $\beta = e^{-\epsilon' t}$ and $\eta = \beta^{1/\dim G}$. We assume that t is large enough so that $t^{100} \leq e^{\epsilon' t}$ and $100 C_0 \eta \leq \eta_0$ where η_0 is defined in Section 2 and C_0 is from Lemma 2.2. Recall we set

$$\mathsf{E} = \mathsf{B}^{U^-}_{\beta} \mathsf{B}^{M_0 A}_{\beta} \mathsf{B}^{U^+}_{\eta}.$$

Let us introduce the notions of *sheeted sets* and a *Margulis function* to state the main result. A subset $\mathcal{E} \subseteq X$ is called a *sheeted set* if there exists a base point $y \in X_\eta$ and a finite set of transverse cross-section $F \subset B^{\mathfrak{r}}_{\eta}$ so that the map $(\mathsf{h}, w) \mapsto \mathsf{h} \exp(w).y$ is bi-analytic on $\mathsf{E} \times B^{\mathfrak{r}}_{\eta}$ and

$$\mathcal{E} = \bigsqcup_{w \in F} \mathsf{E} \exp(w).y.$$

For all $z \in \mathcal{E}$, let

$$I_{\mathcal{E}}(z) = \{ w \in \mathfrak{r} : \|w\| < \mathrm{inj}(z), \exp(w).z \in \mathcal{E} \}.$$

Let us recall the (modified) Margulis function defined in [LMWY25]. For every $0 < \delta < 1$ and $0 < \alpha < \dim \mathfrak{r}$, we define the (modified) Margulis function of a sheeted set $\mathcal{E}$:

$$f^{(\alpha)}_{\mathcal{E},\delta}(z) = \sum_{w \in I_{\mathcal{E}}(z) \setminus \{0\}} \max\{\|w\|, \delta\}^{-\alpha}.$$

Roughly speaking, the Margulis function provides a measurement on the dimension in the transverse direction of the sheeted set $\mathcal{E}$ for scales at least δ. We refer to Section 4 for discussion on its connection with Frostman-type condition and α-energy.

The statement of following theorem also needs the notion of *admissible measures* introduced in [LMW22]. We refer to Subsection 4.1 for its precise definition, see also [LMWY25, Appendix D] or [LMW22, Section 7]. Informally, an admissible measure $\mu_{\mathcal{E}}$ associated to a sheeted set $\mathcal{E}$ is a probability measure on $\mathcal{E}$ that is equivalent to Haar measure of H on each sheet. Moreover, each sheet is assigned with roughly equal weight.

Let λ be the normalized Haar measure on

$$\mathsf{B}^{s,H}_{\beta + 100\beta^{\dim G}} = \mathsf{B}^{U^-}_{\beta + 100\beta^{\dim G}} \mathsf{B}^{M_0 A}_{\beta + 100\beta^{\dim G}},$$

and

$$\nu_t = (a_t)_* m_{\mathsf{B}^U_1}.$$

The following theorem is the main result of this part.

Theorem 2.6. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible. Then there exist absolute constants $A_5 > 1$, $C_1 > 1$, $D_1 > 1$, $E_1, E_2 > 1$, $M_1 > 1$, $\epsilon_0 > 0$, and L so that the following holds. For all $x_1 \in X_\eta$ and $R \gg \eta^{-E_1}$, let $\delta_0 = R^{-\frac{1}{A_5}}$. For all $D \geq D_1 + 1$, let $t = M \log R$ where $M = M_1 + C_1 D$ and $\mu_t = \nu_t * \delta_{x_1}$.*

Suppose that for all periodic orbit $H.x'$ with $\mathrm{vol}(H.x') \leq R$, we have

$$d_X(x_1, x') > R^{-D}.$$

Then there exists a family of sheeted sets $\mathcal{F} = \{\mathcal{E}\}$ with associated L-admissible measures $\{\mu_{\mathcal{E}} : \mathcal{E} \in \mathcal{F}\}$ so that the following holds.

(1) There exists $\{c_{\mathcal{E}}\}$ with $c_{\mathcal{E}} > 0$ and $\sum_{\mathcal{E}} c_{\mathcal{E}} = 1$ so that for all $u' \in \mathsf{B}_1^U$, $d \geq 0$ and all $\phi \in \mathrm{C}_c^\infty(X)$

$$\int_X \phi(a_d u' x) \,\mathrm{d}(\lambda * \mu_t)(x) = \sum_{\mathcal{E}} c_{\mathcal{E}} \int_X \phi(a_d u'.x) \,\mathrm{d}\mu_{\mathcal{E}}(x) + O(\mathcal{S}(\phi)(\beta^\star)). \tag{5}$$

(2) For all sheeted set $\mathcal{E} \in \mathcal{F}$ with cross-section $F \subset B_\eta^{\mathfrak{r}}$. The number of sheets satisfies

$$\beta^\star \delta_0^{-2\epsilon_0} \leq \#F \leq \beta^{-\star} e^{\lambda_{\max} t}. \tag{6}$$

Moreover, we have the Margulis function estimate

$$f_{\mathcal{E},\delta_0}^{(\epsilon_0)}(z) \leq \beta^{-E_2} \#F \quad \forall z \in \mathcal{E}. \tag{7}$$

The proof of Theorem 2.6 relies on the following lemma. It is a generalization of [Lin25, Lemma 2.4].

Lemma 2.7. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible. Then there exist constants $A_6 > 1$, $C_2 > 1$, $D_2 > 1$, $M_2 > 1$, and $\epsilon_1 > 0$ so that the following holds. For all $D \geq D_2 + 1$, $x_1 \in X_\eta$ and $R \gg \eta^{-\star}$, let $M = M_2 + C_2 D$, $t = M \log R$, $\mu_t = \nu_t * \delta_{x_1}$ and $\delta_0 = R^{-\frac{1}{A_6}}$.*

Suppose that for all periodic orbit $H.x'$ with $\mathrm{vol}(H.x') \leq R$, we have

$$d(x_1, x') > R^{-D}.$$

Then for all $y \in X_{3\eta}$, $r_H \leq \frac{1}{4}\min\{\mathrm{inj}(y), \eta_0\}$, $r \in [\delta_0, \eta]$, we have

$$(\lambda * \mu_t)((\mathsf{B}_{r_H}^H)^{\pm 1} \exp(B_r^{\mathfrak{r}}).y) \ll \eta^{-\star} r^{\epsilon_1}.$$

As in [Lin25, Part 1], the above lemma is deduced from a single-scale estimate (Lemma 2.8) and an avoidance principle (Proposition 4.2).

Lemma 2.8. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible. Then there exist constants $A_7, C_3, E_3, M_3 > 1$ and $\epsilon_2 > 0$ so that the following holds. For all $D > 0$, $x_1 \in X_\eta$ and $R \gg \eta^{-E_3}$, let $M = M_3 + C_3 D$, $t = M \log R$, $\mu_t = \nu_t * \delta_{x_1}$ and $\delta = R^{-\frac{1}{A_7}}$.*

Suppose that for all periodic orbit $H.x'$ with $\mathrm{vol}(H.x') \leq R$, we have

$$d(x_1, x') > R^{-D}.$$

Then for all $y \in X_{3\eta}$ and all $r_H \leq \frac{1}{2}\min\{\mathrm{inj}(y), \eta_0\}$, we have

$$\mu_t((\mathsf{B}_{r_H}^H)^{\pm 1} \exp(B_\delta^{\mathfrak{r}}).y) \leq \delta^{\epsilon_2}.$$

The proof of Lemma 2.8 heavily relies on the following effective closing lemma for large balls in unipotent groups proved by Lindenstrauss–Margulis–Mohammadi–Shah–Wieser. Before we state it, let us introduce the following notions.

For any $\mathbb{Q}$-subgroup $\mathbf{M}$ of $\mathbf{G}$, let $\mathfrak{m}$ be its Lie algebra. It is a $\mathbb{Q}$-subspace of $\mathfrak{g}$. We define $v_M \in \wedge^{\dim \mathfrak{m}}\mathfrak{g}$ to be one of the primitive integral vector in the line $\wedge^{\dim \mathfrak{m}}\mathfrak{m}$.

For any subspace (not necessarily $\mathbb{Q}$-subspace) $\mathfrak{s} \subseteq \mathfrak{g}$, we define $\hat{v}_{\mathfrak{s}}$ to be the corresponding point in $\mathbb{P}(\wedge^{\dim \mathfrak{s}}\mathfrak{g})$. For any $0 < r \leq \dim \mathfrak{g}$, we equip $\mathbb{P}(\wedge^r\mathfrak{g})$ with the Fubini-Study metric d where $\mathrm{d}(\hat{v}, \hat{w})$ is the angle between the corresponding lines in $\mathbb{P}(\wedge^r\mathfrak{g})$.

Theorem 2.9 (Lindenstrauss–Margulis–Mohammadi–Shah–Wieser)**.** *There exist constants $A_8, A_9 > 1$ and $E_4 > 1$ depending on $\mathbf{G}$ and $\mathbf{a}$ so that the following holds. Let $\tau \in (0,1)$ and $e^t > S \geq E_4\tau^{-A_8}$. Let $x = g\Gamma \in X_\tau$ be a point.*

Suppose there exists $\mathcal{E} \subseteq \mathsf{B}_1^U$ with the following properties.

(1) $|\mathcal{E}| > S^{-\frac{1}{A_8}}$.

(2) For any $u, u' \in \mathcal{E}$, there exists $\gamma \in \Gamma$ with

$$\|a_t u a_{-t} g\gamma g^{-1} a_t (u')^{-1} a_{-t}\| \leq S^{\frac{1}{A_8}},$$
$$\mathrm{d}(a_t u a_{-t} g\gamma g^{-1} a_t (u')^{-1} a_{-t}.\hat{v}_{\mathfrak{h}}, \hat{v}_{\mathfrak{h}}) \leq S^{-1}.$$

Then one of the following is true.

(1) There exists a non-trivial proper $\mathbb{Q}$-subgroup $\mathbf{M}$ with radical to be unipotent so that

$$\sup_{u\in \mathsf{B}_1^U} \|a_t u a_{-t} g.v_M\| \leq S^{A_9},$$
$$\sup_{z\in B_1^{\mathfrak{u}}, u \in \mathsf{B}_1^U} \|z \wedge (a_t u a_{-t} g.v_M)\| \leq e^{-\frac{t}{A_9}} S^{A_9}.$$

(2) There exists a non-trivial proper normal $\mathbb{Q}$-subgroup $\mathbf{M}$ with

$$\sup_{z\in B_1^{\mathfrak{u}}} \|z \wedge v_M\| \leq S^{-\frac{1}{A_9}}.$$

Remark 2.10. We remark that in [LMM$^+$24] the notion X_τ is defined via the *heights* of points in X instead of the notion inj defined in this paper. However, the transition between them is well-known, see for example [SS24, Proposition 26].

We now sketch an outline for Part 1. In Section 3, we recall the relations between different measurements for complexity of a periodic orbit. In Section 4, we recall the notion of sheeted sets and Margulis function and deduce Lemma 2.7 and Theorem 2.6 from Lemma 2.8. The rest of this part is devoted to the proof of Lemma 2.8. We prove a linear algebra lemma in Section 5 and combine it with the effective closing lemma for large balls in unipotent orbits (Theorem 2.9) to prove Lemma 2.8 in Section 6.

3. Preparation I: Measurement for complexity of periodic orbits

For a periodic orbit, there are various ways to measure its complexity. We briefly recall their relations in this section. We refer to [Lin25, Section 3] and [EMV09, Section 17] for a detailed exposition. For a periodic orbit $Hg\Gamma$ inside $X = G/\Gamma$, one can attach the following quantities.

First, from the Riemannian metric on $X = G/\Gamma$, there is a natural volume form on all its embedded submanifolds. Therefore, we can define the volume of a periodic orbit $Hg\Gamma$. We use $\mathrm{vol}(Hg\Gamma)$ to denote this quantity.

Second, if $Hg\Gamma$ is periodic, $g^{-1}Hg \cap \Gamma$ is a lattice in $g^{-1}Hg$ and therefore it is Zariski dense in $g^{-1}Hg$. There exists a $\mathbb{Q}$-subgroup $\mathbf{M} \leq \mathbf{G}$ so that $g^{-1}Hg = \mathbf{M}(\mathbb{R})$. Recall that v_M is defined as one of the primitive integer vector of the line $\wedge^{\dim \mathbf{M}}\mathfrak{m}$ inside $\wedge^{\dim \mathbf{M}}\mathfrak{g}$. The height of $\mathbf{M}$ is defined to be $\mathrm{ht}(\mathbf{M}) = \|v_M\|$. However, the group $\mathbf{M}$ *does* depends on the choice of representative g: if we change g to $g\gamma$, then we need to change $\mathbf{M}$ to $\gamma^{-1}\mathbf{M}\gamma$. The length $\|\mathrm{Ad}(\gamma^{-1})v_M\|$ can be significantly different from $\|v_M\|$.

We have the following proposition (see [Lin25, Corollary 3.2]).

Proposition 3.1. *There exists an absolute constant $c > 1$ so that the following holds. For all $\mathbb{Q}$-subgroup $\mathbf{M}$ with $\mathbf{M}(\mathbb{R}) = g^{-1}Hg$, we have*

$$\mathrm{vol}(Hg\Gamma) \ll \mathrm{ht}(\mathbf{M})^c.$$

We also need the following lemma relates the height of a $\mathbb{Q}$-subgroup and the height of its normalizer.

Proposition 3.2. *There exists an absolute constant $c_0 > 1$ so that the following holds. For all semisimple $\mathbb{Q}$-subgroup $\mathbf{M}$ with $N_{\mathbf{G}}(\mathbf{M}) = g^{-1}\mathbf{H}g$, we have*

$$\mathrm{vol}(Hg\Gamma) \ll \mathrm{ht}(\mathbf{M})^{c_0}.$$

Proof. It follows from [LMMS24, Lemma 4.8] and the fact that the normalizer of a semisimple subgroup in a semisimple group is reductive, cf. [Bou05, Chapter VII, §1, no.3, Lemma 2]. ∎

4. Preparation II: sheeted set, admissible measures, and Margulis function

We recall the notions of sheeted set, admissible measures and Margulis function in [LMWY25]. After this preparation, we prove Lemma 2.7 and Theorem 2.6 assuming Lemma 2.8.

4.1. **Sheeted set and admissible measure.** Recall that $\eta \leq \frac{1}{100C_0}\eta_0$ be a small parameter where η_0 and C_0 are from Lemma 2.2. Recall

$$\mathsf{E} = \mathsf{B}_\beta^{U^-}\mathsf{B}_\beta^{M_0A}\mathsf{B}_\eta^{U^+}.$$

Recall that a subset $\mathcal{E} \subseteq X$ is called a sheeted set if there exists a base point $y \in X_\eta$ and a finite set of transverse cross-section $F \subset B_\eta^{\mathfrak{r}}$ so that the map $(h, w) \mapsto h\exp(w).y$ is injective on $\mathsf{E} \times B_\eta^{\mathfrak{r}}$ and

$$\mathcal{E} = \bigsqcup_{w\in F} \mathsf{E}\exp(w).y.$$

We now recall the definition of Λ-admissible measure in [LMWY25, Appendix D]. A probability measure $\mu_{\mathcal{E}}$ on $\mathcal{E}$ is called Λ-admissible if

$$\mu_{\mathcal{E}} = \frac{1}{\sum_{w\in F}\mu_w(X)}\sum_{w\in F}\mu_w$$

where μ_w are measures on $\mathsf{E}\exp(w).y$ with the following properties. For all $w \in F$, there exists a function ϱ_w defined on E with $\frac{1}{\Lambda} \le \varrho_w \le \Lambda$ so that for all $\mathsf{E}' \subseteq \mathsf{E}$, we have

$$\mu_w(\mathsf{E}'\exp(w).y) = \int_{\mathsf{E}'} \varrho_w(\mathsf{h})\,\mathrm{d}m_H(\mathsf{h}).$$

Moreover, there exists $\mathsf{E}_w = \cup_{i=1}^{\Lambda} \mathsf{E}_{w,i} \subseteq \mathsf{E}$ so that

(1) $\mu_w\big((\mathsf{E} \setminus \mathsf{E}_{w,i})\exp(w).y\big) \le \Lambda\eta\mu_w(X)$,
(2) the complexity of $\mathsf{E}_{w,i}$ is bounded by Λ for all i, and
(3) $\mathrm{Lip}(\varrho_w|_{\mathsf{E}_{w,i}}) \le \Lambda$.

4.2. **Dimension, energy and Margulis function.** For a finite set $F \subset \mathfrak{r}$, we set μ_F be the normalized counting measure on F. It is a probability measure. We say that the set F has dimension $\ge \alpha$ for scales larger than δ if there exists $C > 1$ so that

$$\mu_F(B(x,r)) \le Cr^\alpha \quad \forall x \in \mathfrak{r} \text{ and } r \ge \delta.$$

In literatures, this is always denoted by (C,α)-Frostman-type condition or (C,α)-nonconcentration condition. We also define the (modified) α-energy of F as follows.

$$\mathcal{G}^{(\alpha)}_{F,\delta}(w) = \sum_{w' \in F : w' \neq w} \max\{\|w' - w\|, \delta\}^{-\alpha}.$$

We recall the notion of (modified) Margulis function in [LMWY25, Section 7]. Suppose $\mathcal{E}$ is a sheeted set. For all $z \in \mathcal{E}$, let

$$I_{\mathcal{E}}(z) = \{w \in \mathfrak{r} : \|w\| < \mathrm{inj}(z), \exp(w).z \in \mathcal{E}\}.$$

For every $0 < \delta < 1$ and $0 < \alpha < \dim \mathfrak{r}$, we define the (modified) Margulis function as follows.

$$f^{(\alpha)}_{\mathcal{E},\delta}(z) = \sum_{w \in I_{\mathcal{E}}(z) \setminus \{0\}} \max\{\|w\|, \delta\}^{-\alpha}.$$

We have the following connection between those notions, see [Lin25, Proposition 7.1].

Proposition 4.1. *Suppose $F \subset B_1^{\mathfrak{r}}$ is a finite set and suppose $\mathcal{E} = \mathsf{E}\exp(F).y$ is a sheeted set. We have the following properties.*

(1) Suppose F is a set of dimension $\ge \alpha$ for scales larger than δ, then for all $w \in F$ and $0 < \beta < \alpha$,

$$\mathcal{G}^{(\beta)}_{F,\delta}(w) \le 2^{\dim \mathfrak{r}} C\Big(1 + \frac{1}{1 - 2^{\beta - \alpha}}\Big)\#F.$$

(2) Suppose for all $w \in F$ we have

$$\mathcal{G}^{(\alpha)}_{F,\delta}(w) \le C\#F,$$

then for all $z \in \mathcal{E}$, we have

$$f^{(\alpha)}_{\mathcal{E},\delta}(z) \ll C\#F.$$

(3) Let $\hat{\mathcal{E}} = (\overline{\mathsf{E} \setminus \partial_{5\beta^{\dim G}} \mathsf{E}}) \exp(F).y$. *Suppose for all* $z \in \mathcal{E}$, *we have*

$$f^{(\alpha)}_{\mathcal{E},\delta}(z) \leq \Upsilon.$$

Then for all $z \in \hat{\mathcal{E}}$ *and all* $w \in I_{\mathcal{E}}(z)$, *we have*

$$\mathcal{G}^{(\alpha)}_{I_{\mathcal{E}}(z),\delta}(w) \ll \Upsilon.$$

4.3. **Proof of Lemma 2.7.** As in the introduction, the proof of Lemma 2.7 relies on following avoidance principle obtained in [SS24, Theorem 2].

Proposition 4.2. *There exist* m, s_0, A_{10}, C_4, *and* D_3 *depending only on* (G, H, Γ), *so that the following holds. Let* $R_1, R_2 \geq 1$. *Suppose* $x_0 \in X$ *is so that*

$$d_X(x_0, x) \geq (\log R_2)^{D_3} R_2^{-1}$$

for all x *with* $\mathrm{vol}(H.x) \leq R_1$. *Then for all* $s \geq A_{10} \max\{\log R_2, |\log \mathrm{inj}(x_0)|\} + s_0$ *and all* $\eta \in (0,1]$, *we have*

$$m_U\left(\left\{u \in \mathsf{B}^U_1 : \begin{array}{l} \mathrm{inj}(a_s u.x_0) \leq \eta \textit{ or } \exists x \textit{ with } \mathrm{vol}(H.x) \leq R_1 \\ \textit{and } d_X(a_s u.x_0, x) \leq {C_4}^{-1} R_1^{-D_3} \end{array}\right\}\right) \leq C_4(R_1^{-1} + \eta^{\frac{1}{\mathsf{m}}}).$$

Proof of Lemma 2.7. The proof is the same as [Lin25, Section 5.3] by applying the above avoidance principle. Note that the Følner property [Lin25, Lemma 5.1] holds for general $\{a_t\}_{t\in\mathbb{R}}$ and U. ■

4.4. **Proof of Theorem 2.6.**

Proof of Theorem 2.6. The proof is identical to [Lin25, Section 7]. We record a sketch and needed change here. First, we decompose μ_t into local pieces of size $\beta^{\dim G}$ in $U^- M_0 A$-direction and then smear it using λ which is of size $\beta + 100\beta^{\dim G}$ in $U^- M_0 A$-direction. This ensures that in size β ball near the origin, it looks roughly like Haar measure and the boundary contributes only small error. Next, we decompose the measure once again according to the weight on each H-sheets. This ensures that we get admissible measures at the end. In [Lin25], it is claimed that the uniform expandion of $\mathrm{Ad}(a_t)$ on $\mathfrak{u}$ is used. However, it is only used in [Lin25, Lemma 7.3, 7.4] and one can replace the term e^{2t} by $\frac{m_U(\mathrm{Ad}(a_t)\mathsf{B}^U_1)}{m_U(\mathsf{B}^U_1)}$. The rest of the arguments follows. ■

5. Linear algebra

This section is devoted to prove the following lemma.

Lemma 5.1. *Suppose the data* $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ *satisfy Irreducible. Then there exists a constant* $C_5 > 0$ *so that the following holds for all* $\tilde{\eta} \in (0,1)$, $\tilde{R} \gg \tilde{\eta}^{-2}$, *and* $\tilde{t} \geq C_5(C_5 + 1) \log R$.

Suppose there exist a non-trivial connected proper $\mathbb{R}$*-subgroup* $\mathbf{M}$ *of* $\mathbf{G}$ *with the following properties. Suppose the radical of* $\mathbf{M}$ *is unipotent. Let* $M = \mathbf{M}(\mathbb{R})$ *and let* v_M *be a non-zero vector in the line corresponding to* $\mathfrak{m} = \mathrm{Lie}(M)$ *in* $\wedge^{\dim M}\mathfrak{g}$. *Suppose*

$$\|v_M\| \geq \tilde{\eta}, \tag{8}$$

$$\sup_{u \in \mathsf{B}^U_1} \|a_{\tilde{t}} u v_M\| \leq \tilde{R}. \tag{9}$$

Then we have

$$\|v_M\| \ll \tilde{R}.$$

Moreover, for all $A \geq 1$, if $\tilde{t} \geq AC_5(C_5+1)\log \tilde{R}$ and

$$\sup_{z\in B_1^{\mathfrak{u}}, u\in \mathsf{B}_1^U} \|z \wedge (a_{\tilde{t}} u v_M)\| \leq e^{-\frac{\tilde{t}}{A}} \tilde{R}, \tag{10}$$

then there exists $g' \in G$ with $\|g' - I\| \leq \tilde{R}^{C_5} e^{-\frac{1}{C_5}\tilde{t}}$ so that

$$g' N_G(M)(g')^{-1} = H.$$

5.1. **Isolation of $\mathfrak{u}$ from proper ideals of $\mathfrak{g}$.** We record the following lemma in this subsection.

Lemma 5.2. *There exists $c_1 > 0$ so that the following holds. Suppose $\mathfrak{I}$ is a proper ideal of $\mathfrak{h}$. Let $v_{\mathfrak{I}}$ be a vector in the line $\wedge^{\dim \mathfrak{I}}\mathfrak{I} \subseteq \wedge^{\dim \mathfrak{I}}\mathfrak{g}$. Then we have*

$$\sup_{z\in\mathfrak{u}, \|z\|=1} \|z \wedge v_{\mathfrak{I}}\| \geq c_1 \|v_{\mathfrak{I}}\|.$$

Proof. Since there are only finitely many ideals of $\mathfrak{g}$ containing the radical, it suffices to show that $\mathfrak{h} \cap \mathfrak{I}$ is a proper ideal in $\mathfrak{h}$. If not, since $\mathfrak{h}$ is not an ideal of $\mathfrak{g}$, we have $\mathfrak{h} \subsetneq \mathfrak{I} \subsetneq \mathfrak{g}$ and contradict to the fact that $\mathfrak{r}$ is irreducible. ∎

5.2. **An equivariant projection and its consequence.** We record an equivariant projection from [EMV09]. In this section, S is a semisimple subgroup of G with Lie algebra $\mathfrak{s}$. Let $\bar{v}_S$ be a *unit vector* in the line corresponding to $\mathfrak{s}$ in $\wedge^{\dim \mathfrak{s}}\mathfrak{g}$.

Lemma 5.3. *There exists a neighborhood $\mathcal{N}_S$ of $\bar{v}_S$ and a projection map $\Pi : \mathcal{N}_S \to G.\bar{v}_S$ so that the following holds. For all $v \in \mathcal{N}_S$ with $g.v = v$ for some $g \in B_1^G$, we have $g.\Pi(v) = \Pi(v)$.*

Proof. See [EMV09, Lemma 13.2]. ∎

The following consequence is useful in later proofs.

Proposition 5.4. *Let $\mathbf{S}$ be a connected semisimple $\mathbb{R}$-subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{s}$ and let $\mathbf{S}'$ be a connected $\mathbb{R}$-subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{s}'$. Let $S = \mathbf{S}(\mathbb{R})$ and $S' = \mathbf{S}'(\mathbb{R})$. Suppose the radical of $\mathbf{S}'$ is unipotent and $\dim \mathfrak{s} = \dim \mathfrak{s}'$. Let $\bar{v}_{S'}$ be a unit vector in the line corresponding to $\mathfrak{s}'$ in $\wedge^{\dim \mathfrak{s}}\mathfrak{g}$.*

If $\bar{v}_{S'} \in \mathcal{N}_S$ as in Lemma 5.3, then there exists $g' \in G$ so that $\|g' - \mathrm{Id}\| \ll \|\bar{v}_{S'} - \bar{v}_S\|$ and

$$\mathbf{S}' \subseteq g' N_{\mathbf{G}}(\mathbf{S})(g')^{-1}.$$

Proof. Note that since the radical of $\mathbf{S}'$ is unipotent, for all $s' \in S'$, $s'.v_{S'} = v_{S'}$. By Lemma 5.3, for all $s' \in B_1^{S'}$ we have $s'.\Pi(v_{S'}) = \Pi(v_{S'}) = g'.v_S$. Since $(S')^\circ$ is generated by $B_1^{S'}$, for all $s' \in (S')^\circ$ we have

$$s' \in N_G(g'S(g')^{-1}) = g'N_G(S)(g')^{-1}.$$

Therefore, $\mathbf{S}' \cap N_{\mathbf{G}}(\mathbf{S})$ is a variety of same dimension as $\mathbf{S}'$. By Zariski connectedness of $\mathbf{S}'$, $\mathbf{S}' \leq N_{\mathbf{G}}(\mathbf{S})$. ∎

5.3. **Effective Łojasiewiecz's inequality and its consequence.** The proof of Lemma 5.1 uses an effective version of Łojasiewiecz's inequality [Ło59]. It asserts that the distance between a point to zero locus of an analytic function can be controlled by the value of that function. We use an effective version of this statement for polynomials proved in [Sol91], see also [LMM+24, Theorem 3.2]. The height of a polynomial in $\mathbb{Z}[x_1, \dots, x_n]$ is defined to be the maximum of its coefficients in absolute value.

Theorem 5.5 (Solernó [Sol91])**.** *For any $d \in \mathbb{N}$, there exists $C(d) > 1$ with the following property.*

Let $h > 1$ and let $f_1, \dots, f_r \in \mathbb{Z}[x_1, \dots, x_n]$ have degree at most d and height at most h. Let $\mathcal{V} \subseteq \mathbb{R}^n$ be the zero locus of $f_1, \dots, f_r$. Then for $w \in \mathbb{R}^n$

$$\min\{1, d(w, \mathcal{V})\} \ll_d (1 + \|w\|)^{C(d)} h^{C(d)} \max_{1 \le i \le r} |f_i(w)|^{\frac{1}{C(d)}}$$

where $d(w, \mathcal{V}) = \inf_{v \in \mathcal{V}} d(w, v)$.

The effective Łojasiewiecz's inequality allow us to approximate $\mathfrak{m}$ by a subalgebra $\mathfrak{w}$ which is also a $\mathfrak{h}$-module. We record this consequence in the following lemma.

Lemma 5.6. *Suppose the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible. Then there exists an absolute constant $C_6 > 1$ so that the following holds for all $\tilde{\eta} \in (0, 0.1)$, $\tilde{R} \gg \tilde{\eta}^{-2}$, and $\tilde{t} \ge \frac{C_6}{\lambda_{\min}}(C_6 + 1) \log R$.*

Suppose there exist a non-trivial connected proper $\mathbb{R}$-subgroup $\mathbf{M}$ of $\mathbf{G}$. Let $M = \mathbf{M}(\mathbb{R})$ and let v_M be a non-zero vector in the line corresponding to $\mathfrak{m} = \mathrm{Lie}(M)$ in $\wedge^{\dim M}\mathfrak{g}$. Suppose

$$\|v_M\| \ge \tilde{\eta}, \tag{11}$$

$$\sup_{u \in \mathsf{B}_1^U} \|a_{\tilde{t}} u v_M\| \le \tilde{R}. \tag{12}$$

Then we have

$$\|v_M\| \ll \tilde{R}.$$

Moreover, there exists a subalgebra $\mathfrak{w} < \mathfrak{g}$ with $\dim \mathfrak{w} = \dim \mathfrak{m}$ satisfies the following.

1. *The subalgebra $\mathfrak{w}$ is a $\mathfrak{h}$-module.*
2. *Let $v_{\mathfrak{w}}$ be a vector in the line $\wedge^m \mathfrak{w}$. We have*

$$\|x \wedge v_{\mathfrak{w}}\| \ll e^{-\frac{\lambda_{\min}}{C_6}\tilde{t}} \tilde{R}^{C_6} \|x\| \|v_{\mathfrak{w}}\|, \quad \forall x \in \mathfrak{m}.$$

To apply Theorem 5.5, we need a $\mathbb{Q}$-structure on $\mathfrak{h}$. Let us recall the following result in [Kam14].

Proposition ([Kam14, Theorem 4.1])**.** *There exists a basis $\{x_i\}_{i=1}^h$ of $\mathfrak{h}$ with structure constants $c_{i,j}^k \in \frac{1}{2}\mathbb{Z}$.*

$$[x_i, x_j] = \sum_{k=1}^h c_{ij}^k x_k.$$

Proof of Lemma 5.6. Let $m = \dim M$. Let $V = \wedge^m \mathfrak{g}$. This is a representation of H with the following decomposition

$$V = V^H \oplus V^{\mathrm{non}}$$

where V^H consists of fixed vectors of H and V^{non} is the direct sum of non-trivial sub-representations. This decomposition is unique. We write $v_M = v_0 + v^{\mathrm{non}}$ according to this decomposition. Since

$$\sup_{u\in\mathsf{B}_1^U} \|a_{\tilde{t}} u v_M\| \asymp \|v_0\| + \sup_{u\in\mathsf{B}_1^U} \|a_{\tilde{t}} u.v^{\mathrm{non}}\| \le \tilde{R},$$

we have

$$\|v^{\mathrm{non}}\| \ll \tilde{R}e^{-\lambda_{\min}\tilde{t}}, \qquad \|v_0\| \le \tilde{R}.$$

where the implied constants depend only on the representation V, cf. [Sha96, Section 5]. Therefore, we have $\|v_M\| \ll \tilde{R}$, which proves the first assertion.

Let $\mathcal{V}$ be the variety in $\mathfrak{g}^{h+m}$ consisting of tuples $(x_1, \dots, x_h, w_1, \dots, w_m)$ satisfying the following polynomial equations:

$$\begin{aligned} &[x_i, x_j] = \sum_{k=1}^{h} c_{ij}^k x_k, \quad i, j = 1, \dots, h, \\ &\sum_{j=1}^{m} w_1 \wedge [x_i, w_j] \wedge \cdots \wedge w_m = 0, \quad i = 1, \dots, h, \\ &[w_k, w_l] \wedge w_1 \wedge \cdots \wedge w_m = 0, \quad k, l = 1, \dots, m. \end{aligned} \tag{13}$$

Note that since $\mathfrak{g}$ is a $\mathbb{Q}$-Lie algebra in $\mathfrak{sl}_N$ and $c_{ij}^k \in \mathbb{Q}$, the above polynomials have rational coefficients with height bounded by constant depending explicitly on $\mathfrak{g}$ and $\mathfrak{h}$. Let C be the constant from Theorem 5.5 applying to the above polynomials.

Since $\|v_M\| \ll \tilde{R}$, there exists basis $\{w_j^{(0)}\}_{j=1}^m$ of $\mathfrak{m}$ with size $\|w_j^{(0)}\| \ll \tilde{R}$ for all $j = 1, \dots, m$. Let $x_i^{(0)}$ be a basis of $\mathfrak{h}$ from Proposition 2.1. We have

$$d(((x_i^{(0)})_{i=1}^h, (w_j^{(0)})_{j=1}^m), \mathcal{V}) \ll \tilde{R}^C e^{-\frac{\lambda_{\min}}{C}\tilde{t}}.$$

Let $((x_i^{(1)})_{i=1}^h, (w_j^{(1)})_{j=1}^m) \in \mathcal{V}$ so that

$$\|x_i^{(0)} - x_i^{(1)}\| \ll \tilde{R}^C e^{-\frac{\lambda_{\min}}{C}\tilde{t}}, \|w_j^{(0)} - w_j^{(1)}\| \ll \tilde{R}^C e^{-\frac{\lambda_{\min}}{C}\tilde{t}}.$$

If $\tilde{t} \gg_{\mathfrak{g}} \log \eta$ and $\tilde{R} \gg \tilde{\eta}^{-2}$, we have

$$\|x_1^{(1)} \wedge \cdots \wedge x_h^{(1)}\| \gg \|x_1^{(0)} \wedge \cdots \wedge x_h^{(0)}\| > 0, \|w_1^{(1)} \wedge \cdots \wedge w_m^{(1)}\| \gg \tilde{\eta} > 0$$

and hence

$$\dim\{x_1^{(1)}, \dots, x_h^{(1)}\} = h, \quad \dim\{w_1^{(1)}, \dots, w_m^{(1)}\} = m.$$

Let $\mathfrak{h}^{(1)} = \mathrm{span}\{x_1^{(1)}, \dots, x_h^{(1)}\}$. It is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{h}$. By Proposition 5.4, there exists $\|g^{(1)} - \mathrm{Id}\| \ll \tilde{R}^C e^{-\frac{\lambda_{\min}}{C}\tilde{t}}$ so that $\mathfrak{h}^{(1)} \subseteq \mathrm{Ad}(g^{(1)})\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. Since $\mathfrak{r}$ is a non-trivial simple $\mathfrak{h}$-module, we have $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Therefore, $\mathfrak{h}^{(1)} = \mathrm{Ad}(g^{(1)})\mathfrak{h}$.

Let $\mathfrak{w}^{(1)} = \mathrm{span}\{w_1^{(1)}, \dots, w_m^{(1)}\}$. It is both a subalgebra of $\mathfrak{g}$ with $\dim \mathfrak{w} = \dim \mathfrak{m}$ and a $\mathfrak{h}^{(1)}$-module. Let $\mathfrak{w} = \mathrm{Ad}(g^{(1)})^{-1}\mathfrak{w}^{(1)}$. It is both a subalgebra of $\mathfrak{g}$ and a

$\mathfrak{h}$-module. It suffices to show property (2). Since $\|w_i^{(0)} - w_i^{(1)}\| \ll \tilde{R}^C e^{-\frac{\lambda_{\min}}{C}\tilde{t}}$, we have

$$\left\|x \wedge \left(\mathrm{Ad}(g^{(1)})^{-1} w_1^{(1)}\right) \wedge \cdots \wedge \left(\mathrm{Ad}(g^{(1)})^{-1} w_m^{(1)}\right)\right\| \ll \tilde{R}^C e^{-\frac{\lambda_{\min}}{C}\tilde{t}}.$$

Since

$$\left\|\left(\mathrm{Ad}(g^{(1)})^{-1} w_1^{(1)}\right) \wedge \cdots \wedge \left(\mathrm{Ad}(g^{(1)})^{-1} w_m^{(1)}\right)\right\| \gg \tilde{\eta}$$

and $\tilde{\eta} \gg R^{-\frac{1}{2}}$, we have

$$\|x \wedge v_{\mathfrak{w}}\| \ll e^{-\frac{\lambda_{\min}}{2C}\tilde{t}} \tilde{R}^{2C} \|x\| \|v_{\mathfrak{w}}\|, \quad \forall x \in \mathfrak{m}.$$

Let $C_6 = 2C$, the proof of the lemma is complete. ∎

5.4. **Almost invariant subalgebras of semisimple Lie algebra.** We recall the following lemma from [LMM+24]. It states that in a *semisimple* Lie algebra, a subalgebra cannot be too close to an ideal unless it is that ideal itself.

Lemma 5.7 ([LMM+24, Lemma 3.3])**.** *Let $\mathfrak{s}$ be a semisimple Lie algebra and let $\|\cdot\|$ be a norm on it. There exists c_2 depending only on $\mathfrak{s}$ and $\|\cdot\|$ so that the following holds. Suppose $\mathfrak{h}_1$ is a Lie subalgebra of $\mathfrak{s}$ for which there exists a Lie ideal $\mathfrak{h}_2 \lhd \mathfrak{s}$ with $\dim \mathfrak{h}_1 = \dim \mathfrak{h}_2$ and*

$$\mathrm{d}(\hat{v}_{\mathfrak{h}_1}, \hat{v}_{\mathfrak{h}_2}) \le c_2.$$

Then $\mathfrak{h}_1 = \mathfrak{h}_2$. In particular, $\mathfrak{h}_1$ is a Lie ideal.

Remark 5.8. We remark that in [LMM+24] they assume $\mathfrak{s}$ to be a $\mathbb{Q}$-Lie algebra so that c_2 depends only on the height of $\mathfrak{s}$. In this paper, we will apply this lemma only to the semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$. The Lie algebra $\mathfrak{g}$ has a natural $\mathbb{Q}$-structure from the setting. For the Lie algebra $\mathfrak{h}$, we utilize the $\mathbb{Q}$-structure from Proposition 2.1.

5.5. **Proof of Lemma 5.1.** For reader's convenience, we recall Lemma 5.1 here.

Lemma. *There exists a constant $C_5 > 0$ so that the following holds for all $\tilde{\eta} \in (0,1)$, $\tilde{R} \gg \tilde{\eta}^{-2}$, and $\tilde{t} \ge C_5(C_5+1)\log R$.*

Suppose there exist a non-trivial connected proper $\mathbb{R}$-subgroup $\mathbf{M}$ of $\mathbf{G}$. Suppose the radical of $\mathbf{M}$ is unipotent. Let $M = \mathbf{M}(\mathbb{R})$ and let v_M be a non-zero vector in the line corresponding to $\mathfrak{m} = \mathrm{Lie}(M)$ in $\wedge^{\dim M}\mathfrak{g}$. Suppose

$$\|v_M\| \ge \tilde{\eta},$$

$$\sup_{u \in \mathsf{B}_1^U} \|a_{\tilde{t}} u v_M\| \le \tilde{R}.$$

Then we have

$$\|v_M\| \ll \tilde{R}.$$

Moreover, for all $A \ge 1$, if $\tilde{t} \ge AC_5(C_5+1)\log \tilde{R}$ and

$$\sup_{z \in B_1^{\mathfrak{u}}, u \in \mathsf{B}_1^U} \|z \wedge (a_{\tilde{t}} u v_M)\| \le e^{-\frac{\tilde{t}}{A}} \tilde{R}, \tag{10}$$

then there exists $g' \in G$ with $\|g' - I\| \le \tilde{R}^{C_5} e^{-\frac{1}{C_5}\tilde{t}}$ so that

$$g' N_G(M)(g')^{-1} = H.$$

Proof of Lemma 5.1. We start by showing that $\mathfrak{m}$ is not a proper ideal of $\mathfrak{g}$. Indeed, if so, by Eq. (10), we have

$$\sup_{z\in B_1^{\mathfrak{u}}}\|z\wedge v_M\| = \sup_{z\in B_1^{\mathfrak{u}}, u\in \mathsf{B}_1^U}\|z\wedge (a_{\tilde{t}}uv_M)\| \leq e^{-\frac{\tilde{t}}{A}}\tilde{R}.$$

We get a contradiction if $\tilde{R}\gg\tilde{\eta}^{-2}$ and the implied constant is large enough depending only on c_1 in Lemma 5.2.

Applying Lemma 5.6, we have that $\|v_M\|\ll\tilde{R}$. This proves the first assertion. Moreover, there exists a subalgebra $\mathfrak{w}<\mathfrak{g}$ with $\dim\mathfrak{w}=\dim\mathfrak{m}$ so that it is also a $\mathfrak{h}$-module and

$$\|x\wedge v_{\mathfrak{w}}\|\ll e^{-\frac{\lambda_{\min}}{C_6}\tilde{t}}\tilde{R}^{C_6}\|x\|\|v_{\mathfrak{w}}\|,\quad \forall x\in\mathfrak{m}$$

where $v_{\mathfrak{w}}$ is any vector in the line $\wedge^m\mathfrak{w}$.

We claim that $\mathfrak{w}$ is *not* an ideal of $\mathfrak{g}$. Suppose it is, then by Lemma 5.7, if R is large enough, we have $\mathfrak{m}=\mathfrak{w}$ and

$$\|z\wedge v_{\mathfrak{w}}\| = \|z\wedge v_M\| = \|z\wedge a_t u.v_M\|\ll e^{-\frac{\tilde{t}}{A}}\tilde{R},$$

which contradict to Lemma 5.2.

If $\mathfrak{w}\not\subseteq\mathfrak{h}$, then we have $\mathfrak{h}+\mathfrak{w}=\mathfrak{g}$ due to irreducibility of $\mathfrak{r}$. Since $\mathfrak{w}$ is both a subalgebra and a $\mathfrak{h}$-module, it has to be an ideal of $\mathfrak{g}$ and we get a contradiction. Therefore, $\mathfrak{w}\subseteq\mathfrak{h}$. Since it is $\mathfrak{h}$-module, it is an ideal of $\mathfrak{h}$, i.e., there exists $\mathcal{I}\subseteq\{1,\dots,k\}$ so that $\mathfrak{w}=\oplus_{i\in\mathcal{I}}\mathfrak{h}_i$. Moreover, since $\mathfrak{w}$ is not an ideal of $\mathfrak{g}$ and $\mathfrak{r}$ is simple $\mathfrak{h}$-module, we have $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{w})=\mathfrak{h}$.

Applying Lemma 5.4 to $\mathfrak{s}=\mathfrak{w}$ and $\mathfrak{s}'=\mathfrak{m}$, there exists g' with $\|g'-\mathrm{Id}\|\ll e^{-\frac{\lambda_{\min}}{C_6}\tilde{t}}\tilde{R}^{C_6}$ so that

$$\mathfrak{m}\subseteq \mathrm{Ad}(g')^{-1}\mathfrak{n}_{\mathfrak{g}}(\mathfrak{w}) = \mathrm{Ad}(g')^{-1}\mathfrak{h}.$$

Applying Lemma 5.7 to $\mathrm{Ad}(g')\mathfrak{m}$ and $\mathfrak{w}$ in $\mathfrak{h}$, we have $\mathrm{Ad}(g')^{-1}\mathfrak{m}=\mathfrak{w}$ and

$$g'N_G(M)(g')^{-1}=H.$$

∎

6. Applying effective closing lemma for large balls in unipotent orbits

In this subsection, we combine Theorem 2.9 and Lemma 5.1 to prove Lemma 2.8. For reader's convenience, we restate Lemma 2.8 as the following.

Lemma. *Suppose the data $(G,\Gamma,H,\mathfrak{r},\mathbf{a},U)$ satisfy Irreducible. Then there exist constants $A_7, C_3, E_3, M_3>1$ and $\epsilon_2>0$ so that the following holds. For all $D>0$, $x_1\in X_\eta$ and $R\gg\eta^{-E_3}$, let $M=M_3+C_3D$, $t=M\log R$, $\mu_t=\nu_t*\delta_{x_1}$ and $\delta=R^{-\frac{1}{A_7}}$.*

Suppose that for all periodic orbit $H.x'$ with $\mathrm{vol}(H.x')\leq R$, we have

$$d(x_1,x')>R^{-D}.$$

Then for all $y\in X_{3\eta}$ and all $r_H\leq\frac{1}{2}\min\{\mathrm{inj}(y),\eta_0\}$, we have

$$\mu_t((\mathsf{B}^H_{r_H})^{\pm1}\exp(B^{\mathfrak{r}}_{\delta}).y)\leq\delta^{\epsilon_2}.$$

Proof of Lemma 2.8. We prove the lemma for $\mathsf{B}^H_{r_H}$. The proof for $(\mathsf{B}^H_{r_H})^{-1}$ is exactly the same.

Let C_5 be the constant coming from Lemma 5.1 and c_0 be the constant coming from the comparison between volume and height in Proposition 3.2. Let $A_8 > 1$, $A_9 > 1$ and E_4 be as in Theorem 2.9. Let $M_3 = A_9(C_5(C_5+1)+1)/c_0$, $A_7 = cA_9$, and $\epsilon_2 = \frac{1}{2A_8}$.

For initial point $x_1 \in X_\eta$, by reduction theory, we can write $x_1 = g_1\Gamma$ where

$$\max\{\|g_1\|, \|g_1^{-1}\|\} \le \eta^{-A}. \tag{14}$$

The constant A depends only on (G,Γ). Let $R \ge E_4{}^{c_0A_8A_9}\eta^{-2cA_8A_9A}$. Let $\delta = R^{-\frac{1}{A_7}} \le \eta$ and let $D > 0$, $M = M_3 + C_3D$, and $t = M\log R$. Let $\tilde{R} = \eta^A R^{\frac{1}{c_0}}$ and $S = \tilde{R}^{\frac{1}{A_9}}$. Note that $S \ge E_4\eta^{-A_8}$.

Suppose there exists $y \in X_{3\eta}$ and $r_H \le \frac{1}{2}\min\{\mathrm{inj}(y), \eta_0\}$ so that

$$\mu_t(\mathsf{B}^H_{r_H}\exp(B^{\mathfrak{r}}_\delta).y) = \nu_t * \delta_{x_1}(\mathsf{B}^H_{r_H}\exp(B^{\mathfrak{r}}_\delta).y) > \delta^{\epsilon_2}.$$

Let

$$\mathcal{E}_y = \{u \in \mathsf{B}^U_1 : a_tu.x_1 \in \mathsf{B}^H_{r_H}\exp(B^{\mathfrak{r}}_\delta).y\},$$

we have

$$|\mathcal{E}_y| > \delta^{\epsilon_2} = R^{-\frac{1}{2c_0A_8A_9}} \ge (\eta^{-A/A_9}R^{-\frac{1}{c_0A_9}})^{\frac{1}{A_8}} = S^{\frac{1}{A_8}}.$$

Fix a $u_1 \in \mathcal{E}$ and let $\mathcal{E} = \mathcal{E}_yu_1^{-1} \subseteq \mathsf{B}^U_2$. For all $u, u' \in \mathcal{E}$, we have

$$a_tuu_1.x_1 = h\exp(w).a_tu'u_1.x_1$$

where $h \in \mathsf{B}^H_{2r_H}$ and $w \in B^{\mathfrak{r}}_{2\delta}$. Re-centering at $x_2 = a_tu_1.x_1$, we have

$$a_tua_{-t}x_2 = h\exp(w).a_tu'a_{-t}x_2.$$

Note that $x_2 \in \mathsf{B}^H_{r_H}\exp(B^{\mathfrak{r}}_\delta).y \subseteq X_{2\eta}$. Write $x_2 = g_2\Gamma$ where

$$\max\{\|g_2\|, \|g_2\|\} \le \eta^{-A}.$$

There exists $\gamma_{u,u'} \in \Gamma$ so that

$$a_tua_{-t}g_2\gamma_{u,u'} = h\exp(w)a_tu'a_{-t}g_2.$$

Therefore,

$$a_tua_{-t}g_2\gamma_{u,u'}g_2^{-1}a_t(u')^{-1}a_{-t} = h\exp(w).$$

In summary, we have

(1) $|\mathcal{E}| > \delta^{\epsilon_2} \ge S^{\frac{1}{A_8}}$.
(2) For all $u, u' \in \mathcal{E}$, there exists $\gamma \in \Gamma$ so that

$$\|a_tua_{-t}g_2\gamma_{u,u'}g_2^{-1}a_t(u')^{-1}a_{-t}\| = \|h\exp(w)\| \ll 1,$$
$$\mathrm{d}(a_tua_{-t}g_2\gamma_{u,u'}g_2^{-1}a_t(u')^{-1}a_{-t}.\hat{v}_{\mathfrak{h}}, \hat{v}_{\mathfrak{h}}) \le \delta = R^{-\frac{1}{c_0A_9}} \le S^{-1}.$$

We apply Theorem 2.9 with $e^t \ge S = \eta^{A/A_9}R^{\frac{1}{c_0A_9}} \ge E_4\eta^{-A_8}$ and the base point $x_2 = g_2\Gamma \in X_{2\eta}$. Note that by Lemma 5.2, case (2) in Theorem 2.9 cannot hold if R is large enough so that $S^{-\frac{1}{A_9}} = R^{-\frac{1}{c_0A_9{}^2}} \le c_1$ where c_1 is from Lemma 5.2. Therefore, there exists a non-trivial proper $\mathbb{Q}$-subgroup $\mathbf{M}$ so that

$$\sup_{u\in\mathsf{B}^U_2}\|a_tua_{-t}g_2.v_M\| \le S^{A_9} = \eta^AR^{\frac{1}{c_0}},$$

$$\sup_{z\in B_1^{\mathfrak{u}},u\in \mathsf{B}_2^U}\|z\wedge(a_tua_{-t}g_2.v_M)\|\leq e^{-\frac{t}{A_9}}S^{A_9}=e^{-\frac{t}{A_9}}\eta^A R^{\frac{1}{c_0}}.$$

Since $x_2=g_2\Gamma=a_tu_1g_1\Gamma$, there exists $\gamma\in\Gamma$ so that

$$g_2\gamma=a_tu_1g_1.$$

Therefore, we have

$$\sup_{u\in \mathsf{B}_2^U}\|a_tuu_1g_1\gamma.v_M\|\leq S^{A_9}=\eta^A R^{\frac{1}{c_0}}=\tilde{R}.$$

Since $u_1\in\mathsf{B}_1^U$, we have

$$\sup_{u\in \mathsf{B}_1^U}\|a_tug_1\gamma.v_M\|\leq\tilde{R}.$$

Similarly, we have

$$\sup_{z\in B_1^{\mathfrak{u}},u\in \mathsf{B}_1^U}\|z\wedge(a_tug_1\gamma.v_M)\|\leq e^{-\frac{t}{A_9}}S^{A_9}=e^{-\frac{t}{A_9}}\tilde{R}.$$

Note that $\mathbf{M}$ is a $\mathbb{Q}$-group and v_M is a primitive non-zero integer vector in $\mathfrak{g}_{\mathbb{Z}}$, we have $\|v_M\|\geq 1$. Combine it with Eq. (14), we have

$$\|g_1\gamma.v_M\|\geq\eta^A.$$

We now apply Lemma 5.1 with $\tilde{t}=t$, $\tilde{R}=\eta^A R^{\frac{1}{c_0}}$, A_9, and $g_1\gamma.v_M\in\wedge^m\mathfrak{g}$. Note that

$$\begin{aligned}&\tilde{R}=\eta^A R^{1/c_0}\geq\eta^{-A},\\&\tilde{t}=M\log R\geq M_3\log R\geq A_9(C_5(C_5+1)+1)\log\tilde{R},\end{aligned}$$

the condition of the lemma is satisfied. There exists $g'\in G$ with $\|g'-\mathrm{Id}\,\|\leq\tilde{R}^{C_5}e^{-\frac{1}{C_5}\tilde{t}}$ so that

$$\mathbf{H}=g'g_1\gamma N_{\mathbf{G}}(\mathbf{M})\gamma^{-1}g_1^{-1}(g')^{-1}.$$

Since $\mathbf{M}$ is a $\mathbb{Q}$-subgroup of $\mathbf{G}$, its normalizer $N_{\mathbf{G}}(\mathbf{M})$ is also a $\mathbb{Q}$-subgroup of $\mathbf{G}$. Therefore, the orbit $Hg'g_1\Gamma$ is periodic. Moreover,

$$\|g'-\mathrm{Id}\,\|\leq\tilde{R}^{C_5}e^{-\frac{1}{C_5}t}\leq R^{-D}.$$

Note that $g_1\gamma v_M$ satisfies

$$\|g_1\gamma v_M\|\ll\tilde{R}=\eta^A R^{\frac{1}{c_0}}.$$

Combining with Eq. (14), we have

$$\|\gamma v_M\|\ll R^{\frac{1}{c_0}}.$$

By Proposition 3.2, We have

$$\mathrm{vol}(Hg'g_1\Gamma)=\mathrm{vol}(Hg'g_1\gamma\Gamma)\ll\mathrm{ht}(\gamma\mathbf{M}\gamma^{-1})^{c_0}\leq R.$$

We get a contradiction to the initial Diophantine condition. This proves the lemma. ∎

Part 2. Dimension improvement in the transverse complement

The main result of this part is Theorem 6.1. It is a linear dimension improvement result in the representation $\mathfrak{r}$ of H. It is an analog of [LMWY25, Theorem 6.1] and [Lin25, Theorem 7.10]. Let us first fix some notations.

As in the introduction, $\mathbf{H}$ is a connected semisimple $\mathbb{R}$-group and $H = \mathbf{H}(\mathbb{R})$ is the group of its $\mathbb{R}$-points. Let $\mathfrak{h}$ be the Lie algebra of H and let $\mathfrak{h} = \oplus_{i=1}^{k}\mathfrak{h}_i$ be the unique (up to permutation) decomposition of $\mathfrak{h}$ into simple ideals. In this part, we use V to denote a regular representation of $\mathbf{H}$. Let $\mathbf{a} \in \mathfrak{h}$ be a semisimple element so that its projection to each $\mathfrak{h}_i$ is non-trivial. Suppose $\operatorname{ad}\mathbf{a}$ is diagonalizable over $\mathbb{R}$. Let $a_t = \exp(t\mathbf{a})$ and let

$$U = \{u \in H : a_{-t}ua_t \to e \text{ as } t \to +\infty\}.$$

Recall that

$$\mathsf{B}_1^U = \exp(B_1^{\mathfrak{u},\infty}(0))$$

and m_U is the Haar measure on U so that $m_U(\mathsf{B}_1^U) = 1$.

Let V be a finite dimensional non-trivial representation of H.

GIrrep. *The representation V is an irreducible representation of H.*

GProx. *The action of a_t on V is proximal, in other word, the largest eigenvalue $\mu_{\max} > 0$ of $\mathbf{a}$ on V has multiplicity 1.*

Note that if the data $(G, \Gamma, H, \mathfrak{r}, \mathbf{a}, U)$ satisfy Irreducible and Proximal, the complement $\mathfrak{r}$ satisfies the two conditions.

By Mostow's simultaneous Cartan decomposition theorem [Mos55], there exist an inner product on V and an associated orthonormal basis so that matrix representations of elements in A_H are diagonal. Moreover, the matrix transpose is the adjoint operation under this inner product. We identify $V \cong \mathbb{R}^n$ under this basis.

Let $|\cdot|_\delta$ denote the δ-covering number according to the inner product fixed by Mostow's theorem. We remark that for the results in this part, changing to a different norm will only affect the estimate by a constant factor.

For a finite set F, let μ_F be the uniform probability measure on F. For all $\alpha \in (0, \dim V)$ and scale $\delta \in (0,1)$, recall we defined the following (modified) α-energy of the set F in Subsection 4.2:

$$\mathcal{G}_{F,\delta}^{(\alpha)}(w) = \sum_{w' \in F, w' \neq w} \max\{\|w' - w\|, \delta\}^{-\alpha}.$$

The following is the main result of this part.

Theorem 6.1. *Suppose the representation V of H satisfies GIrrep and GProx. Let $\alpha \in (0, n)$. There exists $\epsilon_1 = \epsilon_1(\alpha) > 0$ and $\delta_1 = \delta_1(\alpha) > 0$ so that the following holds for all $\epsilon \in (0, \epsilon_1)$ and $\delta \in (0, \delta_1)$. Suppose there exists a finite set $F \subset B_1^V(0)$ with $\#F \gg_\epsilon 1$ satisfying*

$$\mathcal{G}_{F,\delta}^{(\alpha)}(w) \leq \Upsilon \quad \forall w \in F.$$

Then for all $\ell \gg_\epsilon 1$, there exists $J \subset \mathsf{B}_1^U$ with $m_U(\mathsf{B}_1^U \setminus J) \ll_\epsilon |\log\delta| e^{-\epsilon\mu_{\max}\ell}$ so that the following holds. For all $u \in J$ there exists $F_u \subseteq F$ with $\#(F \setminus F_u) \ll_\epsilon |\log\delta| e^{-\epsilon\mu_{\max}\ell} \#F$ so that for all $w \in F_u$

$$\mathcal{G}_{F_u(w),\delta'}^{(\alpha)}(a_\ell u.w) \ll_\epsilon e^{-\epsilon\mu_{\max}\ell}\Upsilon$$

where the new scale $\delta' = e^{\mu_{\max}\ell}\max\{\delta, \#F^{-\frac{1}{\alpha}}\}$ *and the set*

$$F_u(w) = \{a_\ell u.w' : w' \in F_u, \|a_\ell u.w' - a_\ell u.w\| \leq e^{-\mu_{\max}\ell}\}.$$

Moreover, the function ϵ_1 *and* δ_1 *has a uniform lower bound when* α *varies in a compact subset of* $(0, n)$*.*

As in [LMWY25] and [Lin25], Theorem 6.1 follows from the following theorem.

Theorem 6.2. *Suppose the representation* V *of* H *satisfies GIrrep and GProx. Let* $\alpha \in (0, n)$*. There exists* $\epsilon_1 = \epsilon_1(\alpha) > 0$ *and* $\delta_1 = \delta_1(\alpha) > 0$ *so that the following holds for all* $\epsilon \in (0, \epsilon_1)$ *and* $\delta_0 \in (0, \delta_1)$*.*

Let $F \subset B_1^V(0)$ *be a finite set satisfying*

$$\mu_F(B_\delta^V(x)) \leq C\delta^\alpha \quad \forall x \in V$$

for some $C \geq 1$ *and all* $\delta \geq \delta_0$*.*

For all $\ell \gg_\epsilon 1$ *and* $\delta \in [e^{\mu_{\max}\ell}\delta_0, e^{-\mu_{\max}\ell}]$*, there exists* $J_{\ell,\delta} \subseteq \mathsf{B}_1^U$ *with* $m_U(\mathsf{B}_1^U \setminus J_{\ell,\delta}) \ll_\epsilon e^{-\epsilon\mu_{\max}\ell}$ *so that the following holds. Let* $u \in J_{\ell,\delta}$*, there exists* $F_{\ell,\delta,u} \subseteq F$ *with*

$$\mu_F(F \setminus F_{\ell,\delta,u}) \ll_\epsilon e^{-\epsilon\mu_{\max}\ell}$$

such that for all $w \in F_{\ell,\delta,u}$ *we have*

$$\mu_F(\{w' \in F_{\ell,\delta,u} : \|a_\ell u.w' - a_\ell u.w\| \leq \delta\}) \ll_\epsilon Ce^{-\epsilon\mu_{\max}\ell}\delta^\alpha.$$

Theorem 6.2 in turns follows from the following theorem on covering numbers.

Theorem 6.3. *Suppose the representation* V *of* H *satisfies GIrrep and GProx. Let* $\alpha \in (0, n)$*. There exists* $\epsilon_1 = \epsilon_1(\alpha) > 0$ *and* $\delta_1 = \delta_1(\alpha) > 0$ *so that the following holds for all* $\epsilon \in (0, \epsilon_1)$ *and* $\delta_0 \in (0, \delta_1)$*.*

Let $F \subset B_1^V(0)$ *be a finite set satisfying*

$$\mu_F(B_\delta^V(x)) \leq C\delta^\alpha \quad \forall x \in V$$

for some $C \geq 1$ *and all* $\delta \geq \delta_0$*.*

There exists $C_\epsilon > 1$ *so that the following holds. For all* $\ell \gg_\epsilon 1$ *and* $\delta \in [e^{\mu_{\max}\ell}\delta_0, e^{-\mu_{\max}\ell}]$*, we define the exceptional set* $\mathcal{E}(F)$ *to be*

$$\mathcal{E}(F) = \{u \in \mathsf{B}_1^U : \exists F' \subseteq F \text{ with } \mu_F(F') \geq e^{-\epsilon\mu_{\max}\ell} \\ \text{and } |a_\ell u.F'|_\delta < C_\epsilon^{-1}C^{-1}e^{\epsilon\mu_{\max}\ell}\delta^{-\alpha}\}.$$

We have

$$m_U(\mathcal{E}(F)) \leq C_\epsilon e^{-\epsilon\mu_{\max}\ell}.$$

A key step in the proof of the above theorem is an estimate of covering number using certain anisotropic tubes explicated later, see Theorem 6.4. Before we state it, let us introduce some notations.

Note that since V is a regular representation of $\mathbf{H}$ and $\mathbf{a}$ is a semisimple element in $\mathfrak{h}$, V can be decomposed into eigenspaces of $\mathbf{a}$. Let

$$V = \bigoplus_\lambda V_\lambda$$

be such decomposition. Note that there is a *total order* on $\{\lambda\}$ (since we only focus on the element $\mathbf{a}$) and we denote

$$V^{(\mu)} = \bigoplus_{\lambda \geq \mu} V_\lambda.$$

Let $\mu_1 < \cdots < \mu_m = \mu_{\max}$ be all the eigenvalues of $\mathbf{a}$ and let $k_1, \ldots, k_m$ be the dimension of the corresponding eigenspaces. Under the Proximal, $k_m = 1$.

For a subspace $W \subseteq V$, we always use π_W to denote the orthogonal projection to W. For $V^{(\mu)}$, we simplify the notation by setting $\pi^{(\mu)} = \pi_{V^{(\mu)}}$. We use $\pi_u^{(\mu)}$ to denote the projections $\pi^{(\mu)} \circ u$.

We adapt the notations in [BH24] for partitions using anisotropic tubes associated to the flag $\{V^{(\mu)}\}_\mu$. Let $\mathcal{D}_\delta(\mu)$ be the partition of $V^{(\mu)}$ via δ-cubes. For a m-tuple $\mathbf{r} = (\mathsf{r}_1, \ldots, \mathsf{r}_\mathsf{m})$ satisfying $0 \le \mathsf{r}_1 \le \cdots \le \mathsf{r}_\mathsf{m} = 1$, we define

$$\mathcal{D}_\delta^{\mathbf{r}} = \bigvee_{i=1}^{m} (\pi^{(\mu_i)})^{-1} \mathcal{D}_{\delta^{\mathsf{r}_i}}(\mu_i)$$

to be the partition consisting of (possibly anisotropic) tubes. We will use T to denote an atom in $\mathcal{D}_\delta^{\mathbf{r}}$. Roughly, T is a tube of size

$$\prod_{j=1}^{k_1} \delta^{\mathsf{r}_1} \times \cdots \times \prod_{j=1}^{k_m} \delta^{\mathsf{r}_\mathsf{m}}$$

with edges parallel to an orthogonal basis compatible with the weight space decomposition $V = \bigoplus_i V_{\mu_i}$. Its volume satisfies

$$\mathrm{vol}(T) \sim \delta^{\sum_{i=1}^m \mathsf{k}_\mathsf{i} \mathsf{r}_\mathsf{i}}.$$

Theorem 6.4. *Suppose the representation V of H satisfies GIrrep and GProx. Let $\alpha \in (0, n)$. There exists $\epsilon_1 = \epsilon_1(\alpha) > 0$ and $\delta_1 = \delta_1(\alpha) > 0$ so that the following holds for all $\epsilon \in (0, \epsilon_1)$ and $\delta_0 \in (0, \delta_1)$.*

Let $F \subset B_1^V(0)$ be a finite set satisfying

$$\mu_F(B_\delta^V(x)) \le C\delta^\alpha \quad \forall x \in V$$

for some $C \ge 1$ and all $\delta \ge \delta_0$.

Fix a m-tuple $\mathbf{r}$. Then there exists $C_{\epsilon,\mathbf{r}}$ so that the following holds. For all $\delta_0 \le \delta < \delta_1$, we define the exceptional set $\mathcal{E}(F)$ to be

$$\begin{aligned}\mathcal{E}(F) = \{u \in \mathsf{B}_1^U : &\exists F' \subseteq F \text{ with } \mu_F(F') \ge \delta^\epsilon \text{ and} \\ &|u.F'|_{\mathcal{D}_\delta^{\mathbf{r}}} < C_{\epsilon,\mathbf{r}}^{-1} C^{-1} \mathrm{vol}(T)^{-\frac{1}{n}\alpha - (\mathsf{r}_\mathsf{m} - \mathsf{r}_{\mathsf{m}-1})\epsilon}\}.\end{aligned}$$

We have

$$m_U(\mathcal{E}(F)) \le C_{\epsilon,\mathbf{r}} \delta^\epsilon.$$

The proof of Theorem 6.3 assuming Theorem 6.4 is similar to [Lin25, Section 8]. We now discuss the proof of Theorem 6.4. As indicated in the introduction, the proof is based on Bourgain's discretized projection theorem to the highest weight space and subcritical bound for other $V^{(\mu)}$'s. This argument utilizes a submodularity inequality, similar arguments have appeared in special cases in [BHZ25] and [Lin25, Section 14]. We adapted it to the general case.

Very recently, Bénard–He established a general multislicing machinery, see [BH25b, Section 3]. It is plausible that the arguments in this paper can be simplified if one applies their general multislicing machinery.

In the rest of this introductory part, we discuss the following subcritical estimate for projections in irreducible representation of H.

Theorem 6.5. *Suppose the representation V of H satisfies only GIrrep. There exists $M > 1$ depending only on U and V so that for all $0 < \epsilon < \frac{1}{100}$ and $0 < \delta \ll_{\epsilon,V} 1$ the following holds for all eigenvalue μ of $\mathbf{a}$.*

Let $A \subseteq B_1^V$, we set the exceptional set of projection to be

$$\mathcal{E}(A) = \{u \in B_1^U : |\pi_u^{(\mu)}(A)|_\delta < \delta^{M\epsilon} |A|_\delta^{\frac{\dim V^{(\mu)}}{\dim V}}\}.$$

Then we have

$$m_U(\mathcal{E}(A)) \leq \delta^\epsilon.$$

The above subcritical estimate has been established in two special cases in [Lin25, Section 12]. The argument in [Lin25, Section 12] utilizes property of the action of U on V and explicit computations, see [Lin25, Lemma 11.7, 12.4, 12.6]. In contrast, the proof of Theorem 6.5 utilizes the action of H on V and is inspired by the combinatorial argument in [Lin25, Section 12].

As a bi-product, we obtain the following result (Theorem 1.6) regarding intersection of orbit of a subspace in irreducible representations with certain Schubert variety. Let F be a field of characteristic 0 and let $\tilde{H}$ be a connected linear algebraic group over F. Let V be a regular irreducible representation of $\tilde{H}$ defined over F.

Theorem. *For all subspaces $W, W' \subseteq V$, there exists $\tilde{h} \in \tilde{H}$ so that*

$$\dim[(\tilde{h}.W) \cap W'] \leq \frac{\dim W}{\dim V} \dim W'.$$

Note that the set of such $\tilde{h}$ forms a Zariski open subset of $\tilde{H}$.

Part 2 is organized as the following. We first sketch how to deduce Theorems 6.1–6.4 from Theorem 6.5 and and Theorem 7.1 in Section 7. The rest of this part is devoted to the proof of Theorem 6.5 and we refer to Section 8 for a more detailed outline.

7. Proof of Theorems 6.1–6.4 assuming Theorem 6.5 and Theorem 7.1

We sketch the proof of Theorems 6.1–6.4 assuming Theorem 6.5 and Theorem 7.1. It is very similar to [Lin25, Section 14]. The main difference is that in [Lin25], we established an optimal projection [Lin25, Theorem 13.1]. In this paper we replace it with a generalization of Bourgain's discretized projection theorem [Bou10], see Theorem 7.1.

7.1. Supercritical projection theorem.

We record the following theorem. This is the only place in this paper where we need GProx.

Theorem 7.1. *Suppose V is a representation of H satisfying GIrrep and GProx. Let $\mu_{\max}$ be the largest eigenvalue of $\mathbf{a}$ on V.*

For all $0 < \alpha < n$ and $\kappa > 0$, there exists $\epsilon = \epsilon(\alpha, \kappa) > 0$ so that the following holds for all $0 < \delta \ll 1$.

Suppose $A \subseteq B_1^V(0)$ be a set satisfying the following conditions:

$$|A|_\delta \geq \delta^{-\alpha+\epsilon},$$

$$|A \cap B_r^V(x)|_\delta \leq \delta^{-\epsilon} r^\kappa |A|_\delta, \quad \forall r \geq \delta, x \in V.$$

Then there exists $\mathcal{E} \subseteq \mathsf{B}_1^U$ with $m_U(\mathcal{E}) < \delta^\epsilon$ so that for all $u \notin \mathcal{E}$ and all $A' \subseteq A$ with $|A'|_\delta \geq \delta^\epsilon |A|_\delta$, we have

$$|\pi_u^{(\mu_{\max})}(A')|_\delta \geq \delta^{-\frac{\alpha}{n}-\epsilon}.$$

Proof. By [He20, Theorem 1], it suffices to establish a non-degenerate condition for $U^-.V_{\mu_{\max}}$ in $\mathbb{P}(V)$. This is well-known for representation of H satisfying GIrrep and GProx, see e.g. [Sha96, Section 5] or [Lin25, Theorem 11.4] for a proof. ∎

7.2. **Proof of Theorems 6.1–6.3 assuming Theorem 6.4 and Theorem 7.1.**

Proof of Theorem 6.1 assuming Theorem 6.2. The statement can be proved by following the proof of [LMWY25, Theorem 6.1] step-by-step and replacing [LMWY25, Lemma 6.2] by Theorem 6.2. ∎

We now deduce Theorem 6.2 from Theorem 6.3. This procedure is well-known. We reproduce it here for completeness.

Proof of Theorem 6.2 assuming Theorem 6.3. Applying Theorem 6.3, there exists $\mathcal{E} \subset \mathsf{B}_1^U$ with $m_U(\mathcal{E}) \ll_\epsilon e^{-\epsilon\mu_{\max}\ell}$ such that for all $u \notin \mathcal{E}$ and all F' with $\mu_F(F') \geq e^{-\epsilon\mu_{\max}\ell}$, we have

$$|a_\ell u.F'|_\delta \geq C_\epsilon^{-1} C^{-1} e^{\epsilon\mu_{\max}\ell} \delta^{-\alpha}.$$

We define

$$\mathcal{D}_{\delta,\mathrm{bad}}^{\ell,u} = \{Q \in \mathcal{D}_\delta : (a_\ell u)_*\mu_F(Q) > C_\epsilon C e^{-\epsilon\mu_{\max}\ell}\delta^\alpha\}.$$

Let

$$F'_{\ell,\delta,u} = (a_\ell u)^{-1} \bigcup_{Q \in \mathcal{D}_{\delta,\mathrm{bad}}^{\ell,u}} ((a_\ell u.F) \cap Q).$$

Since $(a_\ell u)_*\mu_F$ is a probability measure, we have

$$\#\mathcal{D}_{\delta,\mathrm{bad}}^{\ell,u} < C_\epsilon^{-1} C^{-1} e^{\epsilon\mu_{\max}\ell}\delta^{-\alpha},$$

which is equivalent to

$$|a_\ell u F'_{\ell,\delta,u}|_\delta < C_\epsilon^{-1} C^{-1} e^{\epsilon\mu_{\max}\ell}\delta^{-\alpha}.$$

Therefore, we have

$$\mu_F(F'_{\ell,\delta,u}) \leq e^{-\epsilon\mu_{\max}\ell}.$$

Let $F_{\ell,\delta,u} = F \backslash F'_{\ell,\delta,u}$. For all δ-(dyadic) cube Q, we have

$$(a_\ell u)_*(\mu_F|_{F_{\ell,\delta,u}})(Q) \leq C_\epsilon C e^{-\epsilon\mu_{\max}\ell}\delta^\alpha$$

which proves the theorem. ∎

Proof of Theorem 6.3 assuming Theorem 6.4. The statement can be proved by following the proof of [Lin25, Theorem 7.12] step-by-step. ∎

Proof of Theorem 6.4 assuming Theorem 6.5 and Theorem 7.1. The statement can be proved by following [Lin25, Section 14] and replacing [Lin25, Theorem 13.1] by Theorem 7.1. ∎

8. Subcritical estimates for projections in irreducible representations: an outline

The goal of the rest of the paper is to prove Theorem 6.5. For reader's convenience, we record it in the following theorem.

Theorem. *Suppose the representation V of H satisfies only GIrrep. There exists $M > 1$ depending only on U and V so that for all $0 < \epsilon < \frac{1}{100}$ and $0 < \delta \ll_{\epsilon,V} 1$ the following holds for all eigenvalue μ of $\mathbf{a}$.*

Let $A \subseteq B_1^V$, we set the exceptional set of projection to be

$$\mathcal{E}(A) = \{u \in B_1^U : |\pi_u^{(\mu)}(A)|_\delta < \delta^{M\epsilon}|A|_\delta^{\frac{\dim V^{(\mu)}}{\dim V}}\}.$$

Then we have

$$m_U(\mathcal{E}(A)) \leq \delta^\epsilon.$$

An important step in the proof of Theorem 6.5 is to establish the following estimate for dimension of projections of linear subspaces.

Theorem 8.1. *For any subspace $W \subseteq V$ of dimension k, there exists $m \ll_{n,k} 1$ and a Zariski open dense subset $\mathcal{O}_W \subseteq H^m$ so that for all $(h_1, \dots, h_m) \in \mathcal{O}_W$ and all subspace $W' \subseteq V$*

$$\dim \pi_{h_1.W}(W') + \cdots + \dim \pi_{h_m.W}(W') \geq \frac{mk}{n} \dim W'. \tag{15}$$

The integer m and the subset $\mathcal{O}_W \subseteq H^m$ does not *depend on W'.*

Theorem 8.1 is equivalent to the following.

Theorem 8.2. *For all subspace $W \subseteq V$ of dimension k, there exist positive integer $m \ll_{n,k} 1$ and a Zariski open dense subset $\mathcal{O}_W \subseteq H^m$ so that for all $(h_1, \dots, h_m) \in \mathcal{O}_W$ and all subspace $W' \subseteq V$*

$$\dim[h_1.W \cap W'] + \cdots + \dim[h_m.W \cap W'] \leq \frac{mk}{n} \dim W'. \tag{16}$$

The integer m and the subset $\mathcal{O}_W \subseteq H^m$ does not *depend on W'.*

In fact, we will prove the Theorem 8.2 in a more general setting. Let F be a field of characteristic 0 and let $\tilde{H}$ be a connected linear algebraic group over F. Let V be a regular irreducible representation of $\tilde{H}$ defined over F. By irreducible we mean there is no non-trivial sub-representation. For simplicity, we still use n to denote the dimension of V.

Theorem 8.3. *For all subspace $W \subseteq V$ of dimension k, there exists $m \ll_{n,k} 1$ and a Zariski open dense subset $\mathcal{O}_W \subseteq \tilde{H}^m$ so that for all $(\tilde{h}_1, \dots, \tilde{h}_m) \in \mathcal{O}_W$ and all subspace $W' \subseteq V$*

$$\dim[\tilde{h}_1.W \cap W'] + \cdots + \dim[\tilde{h}_m.W \cap W'] \leq \frac{mk}{n} \dim W'. \tag{17}$$

The integer m and the subset $\mathcal{O}_W \subseteq \tilde{H}^m$ does not *depend on W'.*

Remark 8.4. We remark that any irreducible representation of connected algebraic group factors through its reductive quotient since the fixed point of the unipotent radical is a sub-representation.

As a consequence, we have the following. This is Theorem 1.6.

Corollary 8.5. *For all subspaces $W, W' \subseteq V$, there exists $\tilde{h} \in \tilde{H}$ so that*

$$\dim[(\tilde{h}.W) \cap W'] \leq \frac{\dim W}{\dim V} \dim W'. \tag{18}$$

Note that the set of such $\tilde{h}$ forms a Zariski open subset of $\tilde{H}$.

The case of Corollary 8.5 where $\tilde{H}$ is a real reductive group defined over $\mathbb{Q}$ and the representation *and the subspace* W are both defined over $\mathbb{Q}$ was proved by Eskin–Mozes–Shah [EMS97, Corollary 1.4]. Their proof is based on a non-divergence estimate for translate of algebraic measure on homogeneous space. The proof we give here is purely combinatorial and is based on the idea of entropy. It utilizes only irreducibility and Zariski connectedness.

Before we sketch the idea of the proof of Theorem 6.5, let us recall the following sub-modularity inequality for entropy.

Lemma 8.6. *For all probability measure μ and all partition $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ and $\mathcal{S}$ satisfying*

$$\mathcal{P} \vee \mathcal{Q} = \mathcal{R},$$
$$\mathcal{S} \prec \mathcal{P}, \mathcal{S} \prec \mathcal{Q},$$

we have

$$H(\mu, \mathcal{P}) + H(\mu, \mathcal{Q}) \geq H(\mu, \mathcal{R}) + H(\mu, \mathcal{S}).$$

Proof. This is the convexity of the conditional entropy $H(\mu, \cdot|\mathcal{S})$. ∎

For simplicity, we get back to the setting where H is semisimple and assume that the representation V is faithful in the following exposition. Under this condition, for all $h \in H$, there exists a unique element of H so that the matrix representative of it on V is the transpose of the one of h. We denote this element by h^t. The following linear algebra lemma allows us to roughly identify $\pi_W \circ h$ with $\pi_{h^t.W}$.

Lemma 8.7. *For all $h \in H$ and subspace $W \subseteq V$, there exists a linear isomorphism*

$$\varphi_h : W \to h^t.W$$

so that

$$\varphi_h \circ \pi_W \circ h = \pi_{h^t.W}$$

and

$$\max\{\mathrm{Lip}(\varphi_h), \mathrm{Lip}(\varphi_h^{-1})\} \ll \|h\|^\star.$$

We outline the proof of Theorem 6.5. Without loss of generality, δ is a dyadic scale. Let $\mu = \mu_{A^{(\delta)}}$ be the re-normalized Lebesgue measure on the δ-neighborhood $A^{(\delta)}$ of A. For all subspace $W \subseteq V$, let $\mathcal{D}_\delta(W)$ be the partition of W using δ-dyadic cubes. For simplicity, we will denote it by $\mathcal{D}_\delta$ if the subspace W is clear from the context. The covering number $|A|_\delta$ is closely related to $H(\mu, \mathcal{D}_\delta)$ and for simplicity we will focus on the latter in this outline. In the rest of this section, W is always a non-trivial subspace of V.

We will outline a proof of $H(\mu, \pi_{h.W}^{-1}\mathcal{D}_\delta) \geq \frac{\dim W}{\dim V} H(\mu, \mathcal{D}_\delta) - O(1)$ for generic h in the following. The transition between H-action and U-action follows directly from the restricted root space decomposition.

By irreducibility of V, there exists a minimal positive integer $q \leq n$ so that for generic $(h_1, \dots, h_q)$,

$$h_1.W + \dots + h_q.W = V.$$

Therefore, iterate the sub-modularity inequality for entropy, we have

$$\begin{aligned}
&H(\mu, \pi^{-1}_{h_1.W}\mathcal{D}_\delta) + \dots + H(\mu, \pi^{-1}_{h_q.W}\mathcal{D}_\delta)\\
&\geq H(\mu, \pi^{-1}_{\sum_{j=1}^m h_j.W}\mathcal{D}_\delta) + \sum_{k=1}^{q-1} H(\mu, \pi^{-1}_{(\sum_{j=1}^k h_j.W)\cap h_{k+1}.W}\mathcal{D}_\delta) - O(1)\\
&= H(\mu, \mathcal{D}_\delta) + \sum_{k=1}^{q-1} H(\mu, \pi^{-1}_{(\sum_{j=1}^k h_j.W)\cap h_{k+1}.W}\mathcal{D}_\delta) - O(1)
\end{aligned}$$

where the implied constant depends on $(h_1, \dots, h_q)$. Note that by minimality of q, we have

$$\dim \left(\sum_{j=1}^k h_j.W\right) \cap h_{k+1}.W < \dim W$$

for all $k = 1, \dots, q-1$. Suppose we can run certain induction argument, then we have

$$\begin{aligned}
&H(\mu, \pi^{-1}_{h_1.W}\mathcal{D}_\delta) + \dots + H(\mu, \pi^{-1}_{h_q.W}\mathcal{D}_\delta)\\
&\geq H(\mu, \mathcal{D}_\delta) + \sum_{k=1}^{q-1} \frac{\dim(\sum_{j=1}^k h_j.W) \cap h_{k+1}.W}{\dim V} H(\mu, \mathcal{D}_\delta) - O(1)\\
&= \frac{H(\mu, \mathcal{D}_\delta)}{\dim V}\left(\dim V + \sum_{k=1}^{q-1}\left[\dim\left(\sum_{j=1}^k h_j.W\right) + \dim h_{k+1}.W - \dim\left(\sum_{j=1}^{k+1} h_j.W\right)\right]\right) - O(1)\\
&= \frac{\dim W}{\dim V} H(\mu, \mathcal{D}_\delta) - O(1).
\end{aligned}$$

This shows the entropy estimate in Theorem 6.5 in average.

To run the above induction argument, we need to study the families of subspaces of the form $\left(\sum_{j=1}^k h_j.W\right) \cap h_{k+1}.W$. They are no longer in the same Grassmannian nor form a single H-orbit. However, for generic choice of $(h_1, \dots, h_q)$, it lies in a same Grassmannian. This resolve the first problem. Note that the families of subspaces of the form $\left(\sum_{j=1}^k h_j.W\right) \cap h_{k+1}.W$ are stratified by H-orbit. A Fubini-type argument will allow us to run an induction argument sketched as above.

The above outline can be rigorously translated into a proof of Theorem 6.5. However, the details are quite tedious, especially in keeping track on certain explicit dependence coming from the comparison between $\pi^{-1}_{W_1+W_2}\mathcal{D}_\delta$ and $\pi^{-1}_{w_1}\mathcal{D}_\delta \vee \pi^{-1}_{W_2}\mathcal{D}_\delta$.

The proof given here follows a slightly different outline and make use of recent developments of Hölder–Brascamp–Lieb inequality. Roughly speaking, the Hölder–Brascamp–Lieb inequality provides us a linear criterion to the entropy estimate in Theorem 6.5, see e.g. [BCCT10, Theorem 2.1]. Therefore, we can prove the linear version of Theorem 6.5, Theorem 8.1 using the above outline without tedious tracking constant procedure and apply recent development of Hölder–Brascamp–Lieb inequality in [Gre21] to prove Theorem 6.5.

Remark 8.8. We remark the connection between the above proof and the proof in the special case in [Lin25]. In fact, the main ingredient in the above proof is that the H-action can always move 1-dimension out. In [Lin25], it is done by the U^--action via variants of [Sha96, Lemma 5.1], see [Lin25, Lemma 11.7, 12.6, 12.7]. In an earlier draft of this paper, we prove Theorem 6.5 under both GIrrep and GProx using the U^--action but it is unclear how to achieve Theorem 6.5 under only GIrrep purely using the U^--action. The H-action, however, has the property of moving 1-dimension out directly.

We remark that very recently, Bénard–He established an effective Brascamp–Lieb inequality [BH25a]. It will be interesting to compare the estimate of Brascamp–Lieb constant there with the estimate in this paper which builds on Gressman's work [Gre21]. The latter in turns, utilizes geometric invariant theory.

The rest of the paper is organized as the following. Section 9 is devoted to the proof of Theorem 8.3. Then we collect results from recent developments of Hölder–Brascamp–Lieb inequality from [Gre21] in Section 10. In Section 11, we combine all the ingredients to prove Theorem 6.5.

9. Subcritical estimates for subspaces in irreducible representations

The goal of this section is to prove Theorem 8.3. In fact, we will prove a more general version, Theorem 9.2. Before stating the result, let us introduce the following notion.

For simplicity, in this section, F is a field of characteristic 0 and H is a connected algebraic group defined over F. Let V be an irreducible representation of H of dimension n. For all $0 \leq k \leq n$, we use $\mathrm{Gr}_k(V)$ to denote the Grassmannian of k-dimensional subspaces in V. The actions of $h \in H$ on H^ℓ and $\prod_{i=1}^q \mathrm{Gr}_{k_i}(V)$ in this section are referred to the left actions by diagonally embedded h in H^ℓ and H^q respectively.

Definition 9.1. We say $\mathcal{W} \subseteq \mathrm{Gr}_k(V)$ is H-stratified with data $(\ell, \mathcal{O}, \Psi)$ if there exists $\ell \geq 1$, a H-left invariant Zariski open dense subset $\mathcal{O} \subseteq H^\ell$, and an left H-equivariant regular map $\Psi : \mathcal{O} \to \mathrm{Gr}_k(V)$ so that $\Psi(\mathcal{O}) = \mathcal{W}$.

Theorem 9.2. *Suppose $\mathcal{W}$ is a H-stratified subset of $\mathrm{Gr}_k(V)$ with data $(\ell, \mathcal{O}, \Psi)$, then there exist a positive integer $m \ll_n 1$ and a Zariski open dense subset $\mathcal{O}_{\mathcal{W}} \subseteq \mathcal{O}^m \subseteq (H^\ell)^m$ so that for all subspace $W' \subseteq V$ and all $(\mathbf{h}_1, \ldots, \mathbf{h}_m) \in \mathcal{O}$, we have*

$$\dim(\Psi(\mathbf{h}_1) \cap W') + \cdots + \dim(\Psi(\mathbf{h}_m) \cap W') \leq \frac{mk}{n} \dim W'. \tag{19}$$

The bound of integer m does not *depend on $\mathcal{W}$ and its corresponding data $(\ell, \mathcal{O}, \Psi)$. The subset $\mathcal{O}_{\mathcal{W}}$ does* not *depend on W'.*

Proof of Theorem 8.3 assuming Theorem 9.2. Let $\mathcal{W} = H.W$, then it follows directly from Theorem 9.2. ■

To study families of subspaces of form $\left(\sum_{j=1}^k h_j.W\right) \cap h_{k+1}.W$, we introduce the following notation.

Definition 9.3. Fix $h \geq 1$ and a connected finite (rooted) tree $\mathcal{T}$ with height h satisfying the following conditions.

(1) Let o be the root of $\mathcal{T}$. If $h \geq 2$, $\deg(o) \geq 2$.

(2) Any vertex either has ≥ 2 descendants or has no descendant. The set of the vertices with no descendant is denoted by $\mathcal{L}(\mathcal{T})$, the leaves of $\mathcal{T}$.

Fix a function

$$\omega : \{v \in \mathcal{T} : v \text{ has} \geq 2 \text{ descendants}\} \to \left\{\sum, \bigcap\right\},$$

for all k, we define a map

$$\Phi_\omega^{(\mathcal{T})} : \mathrm{Gr}_k(V)^{\mathcal{L}(\mathcal{T})} \to \bigsqcup_{i=0}^{n} \mathrm{Gr}_i(V)$$

in the following inductive way.

(1) If $\mathcal{T}$ has height 1, then the function ω is a function with domain being empty set, we define $\Phi_\omega^{(\mathcal{T})}$ to be the identity map on $\mathrm{Gr}_k(V)$.

(2) If $\mathcal{T}$ has height $h \geq 2$, its root o has ≥ 2 descendant. Let $\{\mathcal{T}_i\}_{i=1}^d$ be the sub-trees of $\mathcal{T}$ rooted at descendants of o and let ω_i be the restriction of ω to $\mathcal{T}_i$. We can identify

$$\mathrm{Gr}_k(V)^{\mathcal{L}(\mathcal{T})} \cong \prod_{i=1}^{d} \mathrm{Gr}_k(V)^{\mathcal{L}(\mathcal{T}_i)}$$

By inductive hypothesis, $\Phi_{\omega_i}^{(\mathcal{T}_i)}$ has been defined.

(a) If $\omega(o) = \sum$, we define

$$\Phi_\omega^{(\mathcal{T})} = \sum_{i=1}^{d} \Phi_{\omega_i}^{(\mathcal{T}_i)}.$$

(b) If $\omega(o) = \bigcap$, we define

$$\Phi_\omega^{(\mathcal{T})} = \bigcap_{i=1}^{d} \Phi_{\omega_i}^{(\mathcal{T}_i)}.$$

We have the following direct lemma.

Lemma 9.4. *For all H-invariant subset $\mathcal{W} \subseteq \mathrm{Gr}_k(V)$, all finite connected rooted tree $\mathcal{T}$, certain function ω and $i = 0, \ldots, n$, the set*

$$\Phi_\omega^{(\mathcal{T})}(\mathcal{W}) \cap \mathrm{Gr}_i(V)$$

is H-invariant.

Proof. Since H is a group, its element acts on V with an inverse. Also note that the map $\Phi_\omega^{(\mathcal{T})}$ is H-equivariant. ∎

We have the following lemma. It shows that from the operation $\Phi_\omega^{(\mathcal{T})}$, we can form an H-stratified subset of certain Grassmannian.

Lemma 9.5. *Let $\mathcal{W}$ be a H-stratified set with data $(\ell, \mathcal{O}, \Psi)$. Then for all finite connected rooted tree $\mathcal{T}$ and function ω satisfying the conditions in Definition 9.3, there exist an integer $k_{\mathcal{T},\omega}$ and an H-invariant Zariski open dense subset $\mathcal{O}_{\mathcal{T},\omega} \subseteq \mathcal{O}^{\mathcal{L}(\mathcal{T})} \subseteq (H^\ell)^{\mathcal{L}(\mathcal{T})}$ so that the following holds.*

(1) The image $\Phi_\omega^{(\mathcal{T})}[(\Psi(\mathcal{O}_{\mathcal{T},\omega}))^{\mathcal{L}(\mathcal{T})}]$ lies in $\mathrm{Gr}_{k_{\mathcal{T},\omega}}(V)$.

(2) The map $\Phi_\omega^{(\mathcal{T})} \circ (\Psi^{\mathcal{L}(\mathcal{T})})$ is regular on $\mathcal{O}_{\mathcal{T},\omega}$.

Proof. it suffices to show there exists $k_{\mathcal{T},\omega}$ and a Zariski open dense subset $\tilde{\mathcal{O}}_{\mathcal{T},\omega} \subseteq \mathcal{O}^{\mathcal{L}(\mathcal{T})} \subseteq (H^\ell)^{\mathcal{L}(\mathcal{T})}$ so that for all $(\mathbf{h}_v)_{v\in\mathcal{L}(\mathcal{T})} \in \tilde{\mathcal{O}}_{\mathcal{T},\omega}$, we have

$$\dim \Phi_\omega^{(\mathcal{T})}\left((\Psi(\mathbf{h}_v))_{v\in\mathcal{L}(\mathcal{T})}\right) = k_{\mathcal{T},\omega}.$$

Let $\mathcal{O}_{\mathcal{T},\omega} = \cup_{h\in H}\Delta(h).\tilde{\mathcal{O}}_{\mathcal{T},\omega}$ and note that

$$\dim \Phi_\omega^{(\mathcal{T})}\left((\Psi(h\mathbf{h}_v))_{v\in\mathcal{L}(\mathcal{T})}\right) = \dim h.\Phi_\omega^{(\mathcal{T})}\left((\Psi(\mathbf{h}_v))_{v\in\mathcal{L}(\mathcal{T})}\right) = k_{\mathcal{T},\omega}.$$

The set $\mathcal{O}_{\mathcal{T},\omega}$ is a $\Delta(H)$-invariant Zariski open dense subset of $H^{\mathcal{L}(\mathcal{T})}$ satisfies the requirement of the lemma. The regularity of $\Phi_\omega^{(\mathcal{T})}$ on $\mathcal{O}_{\mathcal{T},\omega}$ follows from the regularity of $\Phi_\omega^{(\mathcal{T})}$ on $\tilde{\mathcal{O}}_{\mathcal{T},\omega}$ and the fact that regularity of a map is a local condition.

We identify V with F^n by fixing a basis of V. Under such basis, every vector in V is identified with a column vector in F^n.

We prove it by induction of the height of the tree $\mathcal{T}$. If the height is 1, the conclusion follows from the data $(\ell, \mathcal{O}, \Psi)$.

Now suppose the result is proved for all trees with height $\leq h-1$. Let $\mathcal{T}$ be a tree with height h and let ω be an associated function. Let $\{\mathcal{T}_i\}_{i=1}^d$ be the sub-trees of $\mathcal{T}$ rooted at descendants of o and let ω_i be the restriction of ω to $\mathcal{T}_i$. We identify

$$\mathrm{Gr}_k(V)^{\mathcal{L}(\mathcal{T})} \cong \prod_{i=1}^d \mathrm{Gr}_k(V)^{\mathcal{L}(\mathcal{T}_i)}.$$

By induction hypothesis, there exists integers $\{k_i : i = 1,\dots,d\}$ and Zariski open dense subsets $\{\mathcal{O}_i : i = 1,\dots,d, \mathcal{O}_i \subseteq \mathcal{O}^{\mathcal{L}(\mathcal{T}_i)}\}$ so that for all $(\mathbf{h}_v)_{v\in\mathcal{L}(\mathcal{T}_i)} \in \mathcal{O}_i \subseteq \mathcal{O}^{\mathcal{L}(\mathcal{T}_i)}$,

$$\dim \Phi_{\omega_i}^{(\mathcal{T}_i)}\left((\Psi(\mathbf{h}_v))_{v\in\mathcal{L}(\mathcal{T}_i)}\right) = k_i.$$

Moreover, the map $\Phi_{\omega_i}^{(\mathcal{T}_i)} \circ (\Psi^{\mathcal{L}(\mathcal{T}_i)})$ is regular. In particular, if $k_i \neq 0$, then there exist Zariski open dense subsets $\mathcal{O}_i' \subseteq \mathcal{O}_i$ and regular maps $w_1^{(i)},\dots,w_{k_i}^{(i)}$

$$w_j^{(i)} : \mathcal{O}_i' \to V, \quad j = 1,\dots,k_i$$

so that the k_i-dimensional subspace $\Phi_{\omega_i}^{(\mathcal{T}_i)}\left((\Psi(\mathbf{h}_v))_{v\in\mathcal{L}(\mathcal{T}_i)}\right)$ is spanned by $\{w_j^{(i)}((\mathbf{h}_v)_{v\in\mathcal{L}(\mathcal{T}_i)}) : j = 1,\dots,k_i\}$ if $(\mathbf{h}_v)_{v\in\mathcal{L}(\mathcal{T}_i)} \in \mathcal{O}_i'$.

Case 1. $\omega(o) = \sum$. Without loss of generality, $k_i \neq 0$ for all $i = 1,\dots,d$. Let A be the $n \times (\sum_i k_i)$ matrix with column vectors $w_j^{(i)}((\mathbf{h}_v)_{v\in\mathcal{L}(\mathcal{T}_i)})$. Let $k_{\mathcal{T},\omega}$ be the maximal integer so that there exists $k_{\mathcal{T},\omega} \times k_{\mathcal{T},\omega}$ minor that is a nonzero element in the field of rational functions of $(\mathbf{h}_v)_{v\in\mathcal{L}(\mathcal{T})}$. Let $\tilde{\mathcal{O}}_{\mathcal{T},\omega}$ be the subset of $\prod_{i=1}^d \mathcal{O}_i'$ so that there exists a $k_{\mathcal{T},\omega} \times k_{\mathcal{T},\omega}$ minor of A which is non-zero. This is a Zariski open dense subset of $\mathcal{O}^{\mathcal{L}(\mathcal{T})}$. The regularity of $\Phi_\omega^{(\mathcal{T})} \circ (\Psi)^{\mathcal{L}(\mathcal{T})}$ follows directly from taking the $k_{\mathcal{T},\omega}$ columns of A where certain $k_{\mathcal{T},\omega} \times k_{\mathcal{T},\omega}$ minor is non-zero.

Case 2. $\omega(o) = \bigcap$. Without loss of generality, $k_i \neq 0$ for all $i = 1,\dots,d$. Otherwise the intersection is $\{0\}$ and the proof is complete. Note that for all $m \geq 1$ and subspaces $W_1,\dots,W_m$, the map

$$\begin{aligned} W_1 \times \cdots \times W_m &\to V^{m-1} \\ (w_1, w_2, \dots, w_m) &\mapsto (w_1 + w_2, w_2 + w_3, \dots, w_{m-1} + w_m) \end{aligned}$$

has kernel $W_1 \cap \cdots \cap W_m$ (diagonally embedded). In the form of matrices, let A' be the $(dn) \times (\sum_{i=1}^d k_i)$ matrix constructed as the following:

$$A' = \begin{pmatrix} (w_i^{(1)})_{i=1}^{k_1} & (w_i^{(2)})_{i=1}^{k_2} & 0 & \cdots & 0 \\ 0 & (w_i^{(2)})_{i=1}^{k_2} & (w_i^{(3)})_{i=1}^{k_3} & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & 0 & (w_i^{(d-1)})_{i=1}^{k_{d-1}} & (w_i^{(d)})_{i=1}^{k_d} \end{pmatrix}.$$

Let r be the maximal integer so that there exists $r \times r$ minor that is a nonzero element in the field of rational functions of $(\mathbf{h}_v)_{v \in \mathcal{L}(\mathcal{T})}$. Let $k_{\mathcal{T},\omega} = \sum_i k_i - r = \dim \ker A'$. Let $\tilde{\mathcal{O}}_{\mathcal{T},\omega}$ be the subset of $\prod_{i=1}^d \mathcal{O}_i'$ so that there exists a $r \times r$ minor of A which is non-zero. This is a Zariski open dense subset of $\mathcal{O}^{\mathcal{L}(\mathcal{T})}$. The regularity of $\Phi_\omega^{(\mathcal{T})} \circ (\Psi)^{\mathcal{L}(\mathcal{T})}$ follows directly from the fact that there is a basis of $\ker A'$ with entries are polynomial on entries of A' and inverse of a $r \times r$-minor.

The proof of the inductive step is complete and the lemma follows. ∎

The following lemma utilizes the irreducibility of V.

Lemma 9.6. *For all non-trivial subspaces $W, W' \subseteq V$, there exists a Zariski open dense subset $\mathcal{O}_{W,U} \subseteq H$ so that for all $h \in \mathcal{O}_{W,W'}$, we have*

$$\dim[(h.W) \cap W'] < \min\{\dim W, \dim W'\}.$$

Proof. Note that the condition $\dim[(h.W) \cap W'] < \min\{\dim W, \dim W'\}$ is an open condition. Suppose this does not hold, then for all $h \in H$, we have

$$\dim W \geq \dim[(h.W) \cap W'] \geq \dim W.$$

This implies that $h.W \subseteq W'$ for all $h \in H$ and hence W' contains a nontrivial sub-representation, contradicting to the fact that V is irreducible. ∎

Lemma 9.7. *Let $\mathcal{W}$ be a H-stratified set in $\mathrm{Gr}_k(V)$ with data $(\ell, \mathcal{O}, \Psi)$ where $0 < k < n$. For all non-trivial subspace $W' \subseteq V$, there exists a Zariski open dense subset $\mathcal{O}_{\mathcal{W},W'} \subseteq \mathcal{O}$ so that for all $\mathbf{h} \in \mathcal{O}_{\mathcal{W},W'}$, we have*

$$\dim[\Psi(\mathbf{h}) \cap W'] < \min\{k, \dim W'\}.$$

Proof. Note that by regularity of Ψ, the condition $\dim[\Psi(\mathbf{h}) \cap W'] < \min\{k, \dim W'\}$ is an open condition. It suffices to show that there exists $\mathbf{h} \in \mathcal{O}$ so that the inequality holds. The existence of such $\mathbf{h}$ follows from Lemma 9.6, H-invariance of $\mathcal{O}$ and the condition that Ψ is H-equivariant. ∎

The following lemma is a consequence of Lemma 9.5 and Lemma 9.7.

Lemma 9.8. *Let $\mathcal{W}$ be a H-stratified set in $\mathrm{Gr}_k(V)$ with data $(\ell, \mathcal{O}, \Psi)$ where $0 < k < n$. There exist a minimal positive integer $1 \leq q \leq n$ and a H-left invariant Zariski open dense subset $\tilde{\mathcal{O}} \subseteq \mathcal{O}^q \subseteq (H^\ell)^q$ so that for all $(\mathbf{h}_i)_{i=1,\ldots,q} \in \tilde{\mathcal{O}}$, we have*

$$\Psi(\mathbf{h}_1) + \cdots + \Psi(\mathbf{h}_q) = V.$$

Moreover, for all $2 \leq q' \leq q$, there exists $k_{q'} < k$ so that the following hold. For all $(\mathbf{h}_i)_{i=1}^q$, we have $\dim \left(\sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i)\right) \cap \Psi(\mathbf{h}_{q'}) = k_{q'}$ and the map

$$\Psi^{(q')} : \tilde{\mathcal{O}} \to \mathrm{Gr}_{k_{q'}}(V)$$

$$(\mathbf{h}_i)_{i=1}^q \mapsto \left(\sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i)\right) \cap \Psi(\mathbf{h}_{q'}).$$

is regular. The $(\ell q, \tilde{\mathcal{O}}, \Psi^{(q')})$ *forms a data of a* H*-stratified set in* $\mathrm{Gr}_{k_{q'}}(V)$*. In particular, for all* $(\mathbf{h}_i)_{i=1}^q \in \tilde{\mathcal{O}}$ *and all* $1 < q' \leq q$*,*

$$\dim \left(\sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i)\right) \cap \Psi(\mathbf{h}_{q'}) < k.$$

Proof. For all integer $q' \geq 2$, let $\mathcal{T}_{q'}$ and $\omega_{q'}$ be the finite connected rooted tree and the function as in Definition 9.3 corresponding to the operation $\left(\sum_{i=1}^{q'-1} W_i\right) \cap W_{q'}$. Let $k_{q'}$ and $\mathcal{O}_{q'} \subseteq \mathcal{O}^{q'}$ be the integer and H-invariant Zariski open dense subset obtained by applying Lemma 9.5 to the tree $\mathcal{T}_{q'}$ and $\omega_{q'}$.

We construct Zariski open dense subset of $\mathcal{O}^{q'}$ inductively. Let $\tilde{\mathcal{O}}_2 = \mathcal{O}_2 \subseteq \mathcal{O}^2$. Suppose $\tilde{\mathcal{O}}_{q'}$ has been constructed, let $\tilde{\mathcal{O}}_{q'+1} = \left(\tilde{\mathcal{O}}_{q'} \times \mathcal{O}\right) \cap \mathcal{O}_{q'+1}$. By Zariski connectedness of H, $\tilde{\mathcal{O}}_{q'+1}$ is a Zariski open dense subset of $\mathcal{O}^{q'}$. Since each $\mathcal{O}_{q'}$ is H-invariant, $\tilde{\mathcal{O}}_{q'}$ is H-invariant in $\mathcal{O}^{q'}$ for all q'. We claim that for all $q' \geq 2$ and $(\mathbf{h}_1, \ldots, \mathbf{h}_{q'}) \in \tilde{\mathcal{O}}_{q'}$, we have

$$\dim \sum_{i=1}^{q'} \Psi(\mathbf{h}_i) \geq \min\{k + q' - 1, n\} \tag{20}$$

and $k_{q'} < k$ if

$$\dim \sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i) < n.$$

We prove the claim by induction on q'. If $q' = 2$, for all $(\mathbf{h}_1, \mathbf{h}_2) \in \tilde{\mathcal{O}}_2 = \mathcal{O}_2$, we have

$$\begin{aligned} \dim[\Psi(\mathbf{h}_1) + \Psi(\mathbf{h}_2)] &= \dim \Psi(\mathbf{h}_1) + \dim \Psi(\mathbf{h}_2) - \dim \Psi(\mathbf{h}_1) \cap \Psi(\mathbf{h}_2) \\ &= 2k - k_2. \end{aligned}$$

Since $0 < k < n$, it suffices to show that $k_2 < k$. Fix $(\mathbf{h}_1, \mathbf{h}_2) \in \tilde{\mathcal{O}}_2 = \mathcal{O}_2$. Since H is Zariski connected, there exists a Zariski open dense subset $\mathcal{U}_{\mathbf{h}_1} \subseteq \mathcal{O}$ so that for all $\mathbf{h} \in \mathcal{U}_{\mathbf{h}_1}$, $(\mathbf{h}_1, \mathbf{h}) \in \mathcal{O}_2$. Applying Lemma 9.7 with $W' = \Psi(\mathbf{h}_1)$, we have $\dim \Psi(\mathbf{h}_1) \cap \Psi(\mathbf{h}) < k$ for $\mathbf{h} \in \mathcal{U}'_{\mathbf{h}_1}$ where $\mathcal{U}'_{\mathbf{h}_1}$ is another Zariski open dense subset of $\mathcal{O}$. By Zariski connectedness of H, the intersection $\mathcal{U}'_{\mathbf{h}_1} \cap \mathcal{U}_{\mathbf{h}_1}$ is non-empty which shows that $k_2 < k$.

Now suppose Eq. (20) holds for all integers $\leq q' - 1$. If we have

$$\dim \sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i) \geq n \quad \forall (\mathbf{h}_i)_{i=1}^{q'-1} \in \tilde{\mathcal{O}}_{q'-1},$$

then the proof is complete. Therefore we assume that for all $(\mathbf{h}_i)_{i=1}^{q'-1} \in \tilde{\mathcal{O}}_{q'-1}$, we have

$$k + q' - 2 \leq \dim \sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i) < n.$$

As in the base case, fix $(\mathbf{h}_i)_{i=1}^{q'-1} \in \tilde{\mathcal{O}}_{q'-1}$. Since H is Zariski connected, there exists a Zariski open dense subset $\mathcal{U}_{(\mathbf{h}_i)_{i=1}^{q'-1}} \subseteq \mathcal{O}$ so that for all $\mathbf{h} \in \mathcal{U}_{(\mathbf{h}_i)_{i=1}^{q'-1}}$, we have $(\mathbf{h}_1, \dots, \mathbf{h}_{q'-1}, \mathbf{h}) \in \tilde{\mathcal{O}}_{q'}$. Apply Lemma 9.7 to the non-trivial subspace, there exists a Zariski open dense subset $\mathcal{U}'_{(\mathbf{h}_i)_{i=1}^{q'-1}} \subseteq \mathcal{O}$ so that for all $\mathbf{h} \in \mathcal{U}'_{(\mathbf{h}_i)_{i=1}^{q'-1}}$, we have

$$\dim \left(\sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i) \right) \cap \Psi(\mathbf{h}) < k.$$

Since H is Zariski connected, there exists $\mathbf{h} \in \mathcal{U}_{(\mathbf{h}_i)_{i=1}^{q'-1}} \cap \mathcal{U}'_{(\mathbf{h}_i)_{i=1}^{q'-1}}$. For such $\mathbf{h}$, we have

$$k_{q'} = \dim \left(\sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i) \right) \cap \Psi(\mathbf{h}) < k.$$

Therefore, for all $(\mathbf{h}_i)_{i=1}^{q'}$, we have

$$\begin{aligned} \dim \sum_{i=1}^{q'} \Psi(\mathbf{h}_i) =& \dim \left(\sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i) \right) + \dim \Psi(\mathbf{h}_{q'}) - k_q \\ &\geq k + q' - 2 + 1 = k + q' - 1. \end{aligned}$$

This complete the proof of the inductive step and hence the claim.

By equation Eq. (20) and $k > 0$, there exists a minimal integer $q \leq n$ so that for all $(\mathbf{h}_1, \dots, \mathbf{h}_q) \in \tilde{\mathcal{O}}_q$, we have

$$\sum_{i=1}^{q} \Psi(\mathbf{h}_i) = V.$$

Let $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_q$. The minimality of q shows that $k_{q'} < k$ for all $q' < q$. Let $\pi_{\leq i}$ be the projection of $\mathcal{O}^q$ to the first i coordinates, we have

$$\Psi^{(q')} = \Phi_{\omega_{q'}}^{(\mathcal{T}_{q'})} \circ \Psi^{q'} \circ \pi_{\leq q'}.$$

The regularity of $\Psi^{(q')}$ follows directly from the construction of $\tilde{\mathcal{O}}_q$ and Lemma 9.5. The proof of the lemma is complete. ∎

Recall the following linear algebra lemma. It can be viewed as a (linear) dual version of sub-modularity inequality for entropy if we set $\mathcal{P} = \pi_{W_1}^{-1}\mathcal{D}_\delta$, $\mathcal{Q} = \pi_{W_2}^{-1}\mathcal{D}_\delta$, $\mathcal{R} = \pi_{W_1+W_2}^{-1}\mathcal{D}_\delta$ and $\mathcal{S} = \pi_{W_1 \cap W_2}^{-1}\mathcal{D}_\delta$.

Lemma 9.9. *For all subspaces $W', W_1, W_2 \subseteq V$, we have*

$$\dim W' \cap W_1 \cap W_2 + \dim W' \cap (W_1 + W_2) \geq \dim W' \cap W_1 + \dim W' \cap W_2.$$

Proof. Note that $\dim W' \cap (W_1 + W_2) \geq \dim[(W' \cap W_1) + (W' \cap W_2)]$ and

$$\begin{aligned} &\dim W' \cap W_1 \cap W_2 + \dim[(W' \cap W_1) + (W' \cap W_2)] \\ =& \dim W' \cap W_1 + \dim W' \cap W_2. \end{aligned}$$

∎

Proof of Theorem 9.2. Without loss of generality, we assume $0 < k < n$ and $W' \neq \{0\}$ and $W' \neq V'$.

We prove the theorem by induction on k. For $k = 1$, let

$$\mathcal{U} = \{(\mathbf{h}_1, \ldots, \mathbf{h}_n) \in \mathcal{O}^n : \Psi(\mathbf{h}_1) + \cdots + \Psi(\mathbf{h}_n) = V\}.$$

By Lemma 9.7 and the fact that $k = 1$, there exists $(\mathbf{h}_1, \ldots, \mathbf{h}_n) \in \mathcal{O}^n$ so that

$$\Psi(\mathbf{h}_1) + \cdots + \Psi(\mathbf{h}_n) = \Psi(\mathbf{h}_1) \oplus \cdots \oplus \Psi(\mathbf{h}_n) = V.$$

Since $\dim\left(\sum_{i=1}^n \Psi(\mathbf{h}_i)\right) = n$ is an open condition, the set $\mathcal{U}$ is a Zariski open dense subset. Let $m = n$ and $\mathcal{O}_{\mathcal{W}} = \mathcal{U}$, for all $(\mathbf{h}_i)_{i=1}^n \in \mathcal{O}_{\mathcal{W}} = \mathcal{U}$ and subspace $W' \subseteq V$, we have

$$\sum_{i=1}^n \dim \Psi(\mathbf{h}_i) \cap W' \leq \dim\left(\sum_{i=1}^n \Psi(\mathbf{h}_i)\right) \cap W' = \dim W' = n\frac{1}{n}\dim W'.$$

The proof of the base case is complete.

We now prove the inductive step. Suppose the theorem holds for all H-stratified subsets in $\mathrm{Gr}_{k'}(V)$ for all $k' \leq k-1$. By Lemma 9.8, there exist integer $1 \leq q \leq n$ and H-left invariant Zariski open dense subset $\tilde{\mathcal{O}} \subseteq \mathcal{O}^q \subseteq (H^\ell)^q$ so that for all $(\mathbf{h}_i)_{i=1,\ldots,q} \in \tilde{\mathcal{O}}$, we have

$$\Psi(\mathbf{h}_1) + \cdots + \Psi(\mathbf{h}_q) = V.$$

Moreover, for all $2 \leq q' \leq q$ there exists $k_{q'} < k$ so that the map

$$\begin{aligned} \Psi^{(q')} : \tilde{\mathcal{O}} &\to \mathrm{Gr}_{k_{q'}}(V) \\ (\mathbf{h}_i)_{i=1}^q &\mapsto \left(\sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i)\right) \cap \Psi(\mathbf{h}_{q'}). \end{aligned}$$

is well-defined and regular.

Applying the inductive hypothesis to the H-stratified data $(\ell q, \tilde{\mathcal{O}}, \Psi^{(q')})$ in $\mathrm{Gr}_{k_{q'}}(V)$ for all $2 \leq q' \leq q$, there exists $m_{q'} \ll_n 1$ and $\tilde{\mathcal{O}}^{(q')} \subseteq \tilde{\mathcal{O}}^{m_{q'}}$ so that for all $[(\mathbf{h}_i^{(j)})_{i=1}^q]_{j=1}^{m_{q'}} \in \tilde{\mathcal{O}}^{(q')}$ and all subspace $W' \subseteq V$, we have

$$\begin{aligned} &\sum_{j=1}^{m_{q'}} \dim\left(\left(\sum_{i=1}^{q'-1} \Psi(\mathbf{h}_i^{(j)})\right) \cap \Psi(\mathbf{h}_{q'}^{(j)}) \cap W'\right) \\ =& \sum_{j=1}^{m_{q'}} \dim\left(\Psi^{(q')}((\mathbf{h}_i^{(j)})_{i=1}^q) \cap W'\right) \\ \leq& \, m_{q'}\frac{k_{q'}}{n}\dim W'. \end{aligned} \tag{21}$$

Let $m = n!\prod_{i=1}^{q-1} m_i$. Since $q \leq n$ and by inductive hypothesis $m_i \ll_n 1$, we have $m \ll_n 1$. Let

$$\mathcal{O}_{\mathcal{W}} = \tilde{\mathcal{O}}^{\frac{m}{q}} \cap \bigcap_{2 \leq q' \leq q} \left(\tilde{\mathcal{O}}^{(q')}\right)^{\frac{m}{m_{q'}}} \subseteq \mathcal{O}^m.$$

Since H is Zariski connected, $\mathcal{O}_{\mathcal{W}}$ is a Zariski open dense subset. For all $[(\mathbf{h}_i^{(j)})_{i=1}^q]_{j=1}^{\frac{m}{q}} \in \tilde{\mathcal{O}}_{\mathcal{W}}$, by sub-modularity inequality we have

$$\begin{aligned}
&\sum_{j=1}^{\frac{m}{q}}\sum_{i=1}^{q} \dim\left(W' \cap \Psi(\mathbf{h}_i^{(j)})\right) \\
\leq &\sum_{j=1}^{\frac{m}{q}}\left[\dim W' \cap \left(\sum_{i=1}^{q}\Psi(\mathbf{h}_i^{(j)})\right) + \sum_{q'=2}^{q}\dim W' \cap \left(\sum_{i=1}^{q'-1}\Psi(\mathbf{h}_i^{(j)})\right)\cap \Psi(\mathbf{h}_{q'}^{(j)})\right] \\
= &\sum_{j=1}^{\frac{m}{q}}\left[\dim W' + \sum_{q'=2}^{q}\dim W' \cap \Psi^{(q')}((\mathbf{h}_i^{(j)})_{i=1}^q)\right] \\
= &\frac{m}{q}\dim W' + \sum_{q'=2}^{q}\sum_{p=1}^{\frac{m}{qm_{q'}}}\sum_{j=1}^{m_{q'}}\dim W' \cap \Psi^{(q')}((\mathbf{h}_i^{(j+(p-1)m_{q'})})_{i=1}^q).
\end{aligned}$$

By Eq. (21), we have

$$\begin{aligned}
&\sum_{j=1}^{\frac{m}{q}}\sum_{i=1}^{q} \dim\left(W' \cap \Psi(\mathbf{h}_i^{(j)})\right) \\
\leq &\frac{m}{q}\dim W' + \sum_{q'=2}^{q}\sum_{p=1}^{\frac{m}{qm_{q'}}}\sum_{j=1}^{m_{q'}}\dim W' \cap \Psi^{(q')}((\mathbf{h}_i^{(j+(p-1)m_{q'})})_{i=1}^q) \\
\leq &\frac{m}{q}\dim W' + \sum_{q'=2}^{q}\sum_{p=1}^{\frac{m}{qm_{q'}}} m_{q'}\frac{k_{q'}}{n}\dim W' \\
= &\frac{m}{q}\left(1+\sum_{q'=2}^{q}\frac{k_{q'}}{n}\right)\dim W'.
\end{aligned} \tag{22}$$

We now calculate $\sum_{q'=2}^q k_{q'}$. Note that for all $(\mathbf{h}_i)_{i=1}^q \in \tilde{\mathcal{O}}$, we have

$$\begin{aligned}
k_{q'} &= \dim\left(\sum_{i=1}^{q'-1}\Psi(\mathbf{h}_i)\right)\cap\Psi(\mathbf{h}_{q'}) \\
&= \dim\left(\sum_{i=1}^{q'-1}\Psi(\mathbf{h}_i)\right) + \dim\Psi(\mathbf{h}_{q'}) - \dim\left(\sum_{i=1}^{q'-1}\Psi(\mathbf{h}_i)\right).
\end{aligned}$$

Therefore,

$$\sum_{q'=2}^{q} k_{q'} = \dim\Psi(\mathbf{h}_1) + (q-1)k - \dim\sum_{i=1}^{q}\Psi(\mathbf{h}_i) = qk-n.$$

Combine this with Eq. (22), we have

$$\sum_{j=1}^{\frac{m}{q}}\sum_{i=1}^{q}\dim\left(W'\cap\Psi(\mathbf{h}_i^{(j)})\right) \leq m\frac{k}{n}\dim W'$$

for all $[(\mathbf{h}_i^{(j)})_{i=1}^q]_{j=1}^{\frac{m}{q}} \in \tilde{\mathcal{O}}_{\mathcal{W}}$. This complete the proof of the inductive step and hence the theorem. ∎

10. The Hölder–Brascamp–Lieb inequality

The Hölder–Brascamp–Lieb inequality allow us to translate the linear dimension estimate in Theorem 8.1 to the entropy estimate in Theorem 6.5. For reader's convenience, we recall related notions and results in this section. We will mainly follow the notations in [Gre21, Section 3].

In this section, we get back to the original setting where H be a semisimple real algebraic group with Lie algebra $\mathfrak{h}$ and V is a faithful irreducible representation of H. Recall that we fixed an inner product and an orthonormal basis from Mostow's theorem [Mos55] so that the matrix transpose is the adjoint operation under this inner product and $h^t \in H$ for all $h \in H$. We identify $V \cong \mathbb{R}^n$ using this basis.

For all $j = 1, \ldots, m$, $\{n_j\}_{j=1}^m \subseteq \mathbb{N}$, linear maps $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ and $p_j \in [0, 1]$, we define the Brascamp–Lieb constant $\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)$ to be the smallest non-negative real number so that

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\pi_j(x))^{p_j} \,\mathrm{d}x \leq \mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m) \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j} \tag{23}$$

for all non-negative measurable functions $f_j \in L^1(\mathbb{R}^{n_j})$. If such real number does not exist, we say $\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m) = \infty$ or equivalently $[\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)]^{-1} = 0$.

For simplicity, we identify π_j with its $n_j \times n$ matrix representation under the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^{n_j}$ in this section. Following [Gre21], we use $\{\pi_j\}_{j=1}^m$ to denote the *m-tuple* $(\pi_1, \ldots, \pi_m)$ for simplicity.

The following theorem is proved by Bennett–Carbery–Christ–Tao. It provides a linear criterion for finiteness of the Brascamp–Lieb constant $\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)$.

Theorem 10.1 ([BCCT10, Theorem 2.1])**.** *Suppose π_j's are surjective. Then the Brascamp–Lieb constant $\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)$ is finite if and only if*

$$n = \sum_{j=1}^m p_j n_j \tag{24}$$

and

$$\dim(U) \leq \sum_{j=1}^m p_j \dim(\pi_j(U)) \textit{ for all subspace } U \subseteq V. \tag{25}$$

Proof. This is exactly [BCCT10, Theorem 2.1]. We remark that p_j's here are the reciprocals of the exponents in [BCCT10]. ∎

The following propositions are direct consequences.

Proposition 10.2. *Let W be a subspace of V of dimension k and let m and $\mathcal{O}_W$ be as in Theorem 8.1. Then for all $(h_1, \ldots, h_m) \in \mathcal{O}_W$, the Brascamp–Lieb constant*

$$\mathrm{BL}\left(\left\{ \pi_{h_j.W}, \frac{n}{km} \right\}_{j=1}^m \right) < +\infty.$$

Proof. Let $p_j = \frac{n}{km}$ for all $j = 1, \dots, m$. It suffices to check the conditions Eqs. (24) and (25) in Theorem 10.1. Note that $n_j = k$ for all $j = 1, \dots, m$, we have

$$\sum_{j=1}^{m} p_j n_j = \sum_{j=1}^{m} \frac{n}{km} \cdot k = n.$$

This verifies condition Eq. (24).

By Theorem 8.1, for any subspace $W' \subseteq V$ we have

$$\dim W' \leq \sum_{j=1}^{m} \frac{n}{mk} \dim \pi_{h_j.W}(W') = \sum_{j=1}^{m} p_j \dim \pi_{h_j.W}(W').$$

This verifies condition Eq. (25). ∎

Proposition 10.3. *Let W be a subspace of V of dimension k and let m and $\mathcal{O}_W$ be as in Theorem 8.1. Then for all $(h_1, \dots, h_m) \in (\mathcal{O}_W)^t$, the Brascamp–Lieb constant*

$$\mathrm{BL}\left(\left\{\pi_W \circ h_j, \frac{n}{km}\right\}_{j=1}^{m}\right) < +\infty.$$

Proof. As in the previous proposition, let $p_j = \frac{n}{km}$ for all $j = 1, \dots, m$. Condition Eq. (24) follows exactly in the same computation. For condition Eq. (25), by Theorem 8.1, for all subspace $W' \subseteq V$ we have

$$\begin{aligned}\dim W' \leq \sum_{j=1}^{m} \frac{n}{mk} \dim \pi_{h_j^t.W}(W') &= \sum_{j=1}^{m} p_j \dim \pi_{h_j^t.W}(W') \\ &= \sum_{j=1}^{m} p_j \dim \pi_W(h_j.W').\end{aligned}$$

The last inequality follows from Lemma 8.7. This verifies the condition Eq. (25). ∎

Using the Bruhat decomposition, we have the following proposition.

Proposition 10.4. *For all μ, there exist positive integer $m \ll_n 1$ and a Zariski open dense subset $\mathcal{O}'_\mu \subseteq (U^+)^m$ so that for all $(u_1, \dots, u_m) \in \mathcal{O}'_\mu$, the Brascamp–Lieb constant*

$$\mathrm{BL}\left(\left\{\pi_{u_j^t.V^{(\mu)}}, \frac{n}{m \dim V^{(\mu)}}\right\}_{j=1}^{m}\right) < +\infty.$$

Proof. Let $P = P(a_t) = \{h \in H : a_{-t} h a_t \text{ is bounded when } t \to +\infty\}$. This is a parabolic subgroup of H. Let U^- be the opposite horospherical subgroup of $\{a_t\}_{t>0}$. By dimension counting, there exists a Zariski open dense subset of H in $U^- P$. Note that by the basis from Mostow's theorem, $U^- = U^t$.

Applying Theorem 8.1 to subspace $V^{(\mu)}$, there exists a positive integer $m \ll_n 1$ and a Zariski open dense subset $\tilde{\mathcal{O}}_\mu \subseteq H^m$ so that for all $(h_1, \dots, h_m) \in \tilde{\mathcal{O}}_\mu$ and all subspaces $W' \subseteq V$ we have

$$\dim W' \leq \sum_{j=1}^{m} \frac{n}{m \dim V^{(\mu)}} \dim \pi_{h_j.V^{(\mu)}}(W').$$

Therefore, there exists a Zariski open dense subset $\tilde{\mathcal{O}}'_\mu \subseteq (U^- P)^m$ so that for all $(u_j^t p_j)_{j=1}^m \in \tilde{\mathcal{O}}_\mu$ and all subspaces $W' \subseteq V$ we have

$$\dim W' \leq \sum_{j=1}^m \frac{n}{m \dim V^{(\mu)}} \dim \pi_{u_j^t \cdot V^{(\mu)}}(W').$$

Pullback to $U \times P$ and project to the first factor, we obtain the Zariski open dense subset $\mathcal{O}'_\mu \subseteq (U^+)^m$ satisfying the conditions in the proposition. ■

By Lemma 8.7, we have the following proposition.

Proposition 10.5. *For all μ, there exist positive integer $m \ll_n 1$ and a Zariski open dense subset $\mathcal{O}'_\mu \subseteq (U^+)^m$ so that for all $(u_1, \ldots, u_m) \in \mathcal{O}'_W$, the Brascamp–Lieb constant*

$$\mathrm{BL}\left(\left\{\pi_{u_j}^{(\mu)}, \frac{n}{m \dim V^{(\mu)}}\right\}_{j=1}^m\right) < +\infty.$$

Proof. The proof is exactly the same as the proof of Proposition 10.3 using Lemma 8.7 and the previous proposition. ■

We also need an explicit estimate on the the Brascamp–Lieb constant $\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)$. It was shown by Lieb [Lie90] (see also [BL76]) that any Brascamp–Lieb inequality has an extremizing sequence of Gaussian. Using this, Gressman obtained the following explicit expression of Brascamp–Lieb constant in [Gre21].

Lemma 10.6 ([Gre21, Lemma 1])**.** *Suppose the exponents $\{p_j\}_{j=1}^m$ and dimensions $\{n_j\}_{j=1}^m$ satisfies*

$$\sum_{j=1}^m \frac{p_j n_j}{n} = 1. \tag{26}$$

Then the Brascamp–Lieb constant $\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)$ satisfies

$$[\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)]^{-1} = \inf_{\substack{A \in \mathrm{SL}_n(\mathbb{R}), \\ A_j \in \mathrm{SL}_{n_j}(\mathbb{R}), j=1,\ldots,m}} \prod_{j=1}^m n_j^{-\frac{p_j n_j}{2}} \|A_j \pi_j A^t\|_{\mathrm{HS}}^{p_j n_j}, \tag{27}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm computed with respect to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^{n_j}$'s and A^t is the transpose of A.

We now follow [Gre21, Section 3] to show a polynomial bound (on entries of $\{\pi_j\}_{j=1}^m$) for Brascamp–Lieb constant $\mathrm{BL}(\{\pi_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$.

From now on in this section, we use $\{\tilde{\pi}_j\}_{j=1}^m$ to denote matrices of sizes $n_j \times n$ with entries to be distinct indeterminants. Since the ring of polynomials over an infinite field is canonically isomorphic to the ring of polynomial functions, we will also use $\{\tilde{\pi}_j\}_{j=1}^m$ to denote matrices of sizes $n_j \times n$ with arbitrary entries. We say that a polynomial Φ is polynomial on $\{\tilde{\pi}_j\}_{j=1}^m$ if it is a polynomials on those indeterminants with coefficients in $\mathbb{C}$. For such polynomials, we define its Hilbert–Schmidt norm $\|\Phi\|_{\mathrm{HS}}$ to be the maximum of $|\Phi|$ on all m-tuples of linear maps $\{\pi_j\}_{j=1}^m$ so that $\|\pi_j\|_{\mathrm{HS}} \leq 1$ for all $j = 1, \ldots, m$. It is comparable to the maximum of absolute value of coefficient of Φ.

Let IP be the collection of *non-zero* polynomials on $\{\tilde{\pi}_j\}_{j=1}^m$ satisfying the following.

The polynomials Φ is homogeneous with degree $d_j > 0$ in each $\tilde{\pi}_j$ and is $\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_m} \times \mathrm{SL}_n$-invariant, i.e.,

$$\Phi(\{t_j \pi_j\}_{j=1}^m) = t_1^{d_1} \cdots t_m^{d_m} \Phi(\{\pi_j\}_{j=1}^m) \text{ for all } t_1, \dots, t_m \in \mathbb{R} \tag{28}$$

and

$$\Phi(\{A_j \pi_j A^t\}_{j=1}^m) = \Phi(\{\pi_j\}_{j=1}^m) \tag{29}$$

for all $A \in \mathrm{SL}_n(\mathbb{R}), A_j \in \mathrm{SL}_{n_j}(\mathbb{R}), j = 1, \dots, m$. Moreover, the degrees need to satisfies

$$\frac{p_1 n_1}{d_1} = \cdots = \frac{p_m n_m}{d_m} = \frac{1}{s_\Phi} \tag{30}$$

for some real number s_Φ.

The arguments in [Gre21, Section 3] provide the following lemma.

Lemma 10.7. *Suppose p_j's are non-zero rational numbers and satisfies*

$$\sum_{j=1}^m \frac{p_j n_j}{n} = 1. \tag{31}$$

If there exists $\{\pi_j^{(0)}\}_{j=1}^m$ so that $\mathrm{BL}(\{\pi_j^{(0)}, p_j\}_{j=1}^m) < +\infty$, then $\mathrm{IP} \neq \emptyset$.

Moreover, if $\mathrm{IP} \neq \emptyset$, there exists a non-empty finite set $\mathrm{IP}_0 \subseteq \mathrm{IP}$ of polynomials not *depending on the previous $\{\pi_j^{(0)}\}_{j=1}^m$ satisfying the following.*

(1) *The degree of polynomials in IP_0 satisfy*

$$\sup_{\Phi \in \mathrm{IP}_0} \deg \Phi \ll_{n, \{n_j\}_{j=1}^m, \{p_j\}_{j=1}^m} 1.$$

Also, the numbers s_Φ's are positive integers and satifies $s_\Phi \ll_{n, \{n_j\}_{j=1}^m, \{p_j\}_{j=1}^m} 1$. The dependence on $\{p_j\}_{j=1}^m$ is explicit dependence on the size of their numerators and denominators.

(2) *For all $\{\pi_j\}_{j=1}^m$, we have*

$$[\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)]^{-1} \geq \left(\prod_{j=1}^m n_j^{-\frac{p_j n_j}{2}} \right) \sup_{\Phi \in \mathrm{IP}_0} \|\Phi\|_{\mathrm{HS}}^{-\frac{1}{s_\Phi}} |\Phi(\{\pi_j\}_{j=1}^m)|^{\frac{1}{s_\Phi}}. \tag{32}$$

(3) *If $\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m) < +\infty$, then*

$$\sup_{\Phi \in \mathrm{IP}_0} \|\Phi\|_{\mathrm{HS}}^{-\frac{1}{s_\Phi}} |\Phi(\{\pi_j\}_{j=1}^m)|^{\frac{1}{s_\Phi}} > 0. \tag{33}$$

Proof. This is essentially a combination of parts of proofs of [Gre21, Lemma 2, Proposition 3, Lemma 3]. We explicate the proof here for reader's convenience.

We first show that for all $\Phi \in \mathrm{IP}$, we have

$$[\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)]^{-1} \geq \left(\prod_{j=1}^m n_j^{-\frac{p_j n_j}{2}} \right) \|\Phi\|_{\mathrm{HS}}^{-\frac{1}{s_\Phi}} |\Phi(\{\pi_j\}_{j=1}^m)|^{\frac{1}{s_\Phi}}. \tag{34}$$

Indeed, by the invariance of Φ under the action of $\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_m} \times \mathrm{SL}_n$ and homogeneity of Φ, we have

$$|\Phi(\{\pi_j\}_{j=1}^m)| = |\Phi(\{A_j \pi_j A^t\}_{j=1}^m)| \leq \|\Phi\|_{\mathrm{HS}} \prod_{j=1}^m \|A_j \pi_j A^t\|_{\mathrm{HS}}^{d_j}.$$

By the degree condition Eq. (30), we have

$$\|\Phi\|_{\mathrm{HS}}^{-\frac{1}{s_\Phi}} |\Phi(\{\pi_j\}_{j=1}^m)|^{\frac{1}{s_\Phi}} \le \prod_{j=1}^m \|A_j\pi_j A^t\|_{\mathrm{HS}}^{\frac{d_j}{s_\Phi}} = \prod_{j=1}^m \|A_j\pi_j A^t\|_{\mathrm{HS}}^{p_j n_j}$$

for all $A \in \mathrm{SL}_n(\mathbb{R}), A_j \in \mathrm{SL}_{n_j}(\mathbb{R}), j = 1, \ldots, m$.

Applying Lemma 10.6, we have

$$\begin{aligned}\|\Phi\|_{\mathrm{HS}}^{-\frac{1}{s_\Phi}} |\Phi(\{\pi_j\}_{j=1}^m)|^{\frac{1}{s_\Phi}} &\le \inf_{\substack{A\in\mathrm{SL}_n(\mathbb{R}),\\ A_j\in\mathrm{SL}_{n_j}(\mathbb{R}), j=1,\ldots,m}} \|A_j\pi_j A^t\|_{\mathrm{HS}}^{p_j n_j} \\ &\le \left(\prod_{j=1}^m n_j^{\frac{p_j n_j}{2}}\right) [\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m)]^{-1}\end{aligned}$$

This completes the proof of Eq. (34).

Now we construct polynomials in IP_0 explicitly and show that $\mathrm{IP} \neq \emptyset$ in the pathway. Since all p_j's are rational and non-zero, there exist positive integers $q, q_1, \ldots, q_n$ so that $p_j n_j = \frac{q_j}{q}$. Note that we can choose q_j's and q bounded above by constants depending only on the size of numerator and denominator of p_j's. Let

$$V = \bigotimes_{j=1}^m \bigotimes_{i=1}^q \left(\mathbb{R}^{n_j} \otimes (\mathbb{R}^n)^*\right).$$

It a $\mathbb{R}$-linear space with a natural linear action of the real reductive group $G = \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_m} \times \mathrm{SL}_n$ as the following.

For all matrices $A \in \mathrm{SL}_n$, $A_j \in \mathrm{SL}_{n_j}$ for $j = 1, \ldots, m$ and all element $\otimes_{j=1}^m \otimes_{i=1}^q (x_j^{(i)} \otimes \psi_j^{(i)}) \in V$, we define

$$(A_1, \ldots, A_m, A).\left(\otimes_{j=1}^m \otimes_{i=1}^q (x_j^{(i)} \otimes \psi_j^{(i)})\right) = \otimes_{j=1}^m \otimes_{i=1}^q ((A_j.x_j^{(i)}) \otimes (A.\psi_j^{(i)}))$$

where $A.\psi_j^{(i)}$'s are all the natural dual action of the standard action of SL_n on $\mathbb{R}^n$.

Note that under the natural isomorphism $\mathbb{R}^{n_j} \otimes (\mathbb{R}^n)^* \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n_j})$, we can identify any element of the form $\tilde{\pi}_1^{\otimes q_1} \otimes \cdots \otimes \tilde{\pi}_m^{\otimes q_m}$ as an element in V. The Euclidean inner products on $\mathbb{R}^n$ and $\mathbb{R}^{n_j}$'s naturally induce an inner product and norm on V. Under the above identification, we have

$$\|\tilde{\pi}_1^{\otimes q_1} \otimes \cdots \otimes \tilde{\pi}_m^{\otimes q_m}\| = \prod_{j=1}^m \prod_{i=1}^q \|\tilde{\pi}_j\|_{\mathrm{HS}}.$$

Let $\mathcal{O}(V)^G$ be the ring of invariant polynomials of the above representation. By Hilbert's theorem, it is a finitely generated $\mathbb{C}$-algebra. Let $\{P_i\}_{i\in\mathcal{I}}$ be a set of polynomial so that $\{1, P_i\}_{i\in\mathcal{I}}$ generates $\mathcal{O}(V)^G$. By [Der04, Section 1.4], there is an algorithm to construct such generators using Buchberger's algorithm and Reynold operator. In this case, the Reynold operator can be explicitly constructed via Cayley's Ω-process (see e.g. [Stu93, Section 4.3] or [Gre21, Section 3.2]). Therefore, we can assume $\deg P_i \ll_{n,\{n_j\}_{j=1}^m} 1$ for all $i \in \mathcal{I}$ where the implied constants are computable and depend only on n and $\{n_j\}_{j=1}^m$. Moreover, we can assume without loss of generality that those P_i's are homogeneous polynomials.

We now construct IP_0 from $\{P_i\}_{i\in\mathcal{I}}$. In fact, we will construct Φ_P satisfying conditions Eqs. (28)–(30) for all homogeneous polynomials in $\mathcal{O}(V)^G$. Note that up to this step, $\mathcal{O}(V)^G$ might be isomorphic to $\mathbb{C}$ and all such P_i's (and hence Φ_{P_i}'s) might be zero.

Suppose $P \in \mathcal{O}(V)^G$ is a homogeneous polynomial with degree d, we define Φ_P to be the polynomial so that its value at any m-tuple of matrices $\{\tilde{\pi}_j\}_{j=1}^m$ is of the following:

$$\Phi_P(\{\tilde{\pi}_j\}_{j=1}^m) = P(\tilde{\pi}_1^{\otimes q_1} \otimes \cdots \otimes \tilde{\pi}_m^{\otimes q_m}), \tag{35}$$

where again we naturally identify $\tilde{\pi}_1^{\otimes q_1} \otimes \cdots \otimes \tilde{\pi}_m^{\otimes q_m}$ as an element in V using the natural isomorphism $\mathbb{R}^{n_j} \otimes (\mathbb{R}^n)^* \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n_j})$.

We now show that if $\mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m) < +\infty$, we have

$$\sup_i \|\Phi_{P_i}\|_{\mathrm{HS}}^{-\frac{1}{s_{\Phi_{P_i}}}} |\Phi_{P_i}(\{\pi_j\}_{j=1}^m)|^{\frac{1}{s_\Phi}} > 0. \tag{36}$$

Note that this shows that on V, there is an invariant homogeneous polynomial $P \in \mathrm{IP}_0$ separates 0 and the vector $\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}$. In particular, this implies that $\mathcal{O}(V)^G$ is nontrivial. Therefore, Eq. (36) implies that there exists Φ_{P_i} which is non-zero and $\mathrm{IP} \neq \emptyset$.

To proof Eq. (36), we make use of geometric invariant theory and the fact that $\mathrm{BL}(\{\pi_j, p_j\}) < +\infty$. We have

$$\begin{aligned}
&\inf_{\substack{A \in \mathrm{SL}_n(\mathbb{R}),\\ A_j \in \mathrm{SL}_{n_j}(\mathbb{R}), j=1,\dots,m}} \|(A_1, \dots, A_m, A).\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}\| \\
= &\inf_{\substack{A \in \mathrm{SL}_n(\mathbb{R}),\\ A_j \in \mathrm{SL}_{n_j}(\mathbb{R}), j=1,\dots,m}} \prod_{j=1}^m \|A_j \pi_j^{(0)} A^t\|_{\mathrm{HS}}^{q_j} \\
= &\inf_{\substack{A \in \mathrm{SL}_n(\mathbb{R}),\\ A_j \in \mathrm{SL}_{n_j}(\mathbb{R}), j=1,\dots,m}} \left(\prod_{j=1}^m \|A_j \pi_j A^t\|_{\mathrm{HS}}^{p_j n_j} \right)^q > 0.
\end{aligned}$$

The last inequality follows from Lemma 10.6 and the assumption that $\mathrm{BL}(\{\pi_j, p_j\}) < +\infty$.

If the orbit $G.\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}$ is closed in Hausdorff topology, then by [Bir71, Corollary 5.3], it is also closed in Zariski topology and there exists i so that the homogeneous polynomial separates 0 and $\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}$.

If not, then by [RS90, Lemma 3.3], there exists a $v_0 \in V$ so that $G.v_0$ is closed and a semisimple element $X \in \mathfrak{g}$ so that

$$v_0 = \lim_{t \to -\infty} \exp(tX).\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}.$$

Note that for all $g \in G$, we have

$$\begin{aligned}
\|g.v_0\| &= \lim_{t \to -\infty} \|g \exp(tX).\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}\| \\
&\geq \inf_{h \in G} \|h.\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}\| > 0.
\end{aligned}$$

Note that by [Bir71, Corollary 5.3], $G.v_0$ is also closed in Zariski topology. Since $0 \notin G.v_0$, there exists i so that the homogeneous polynomial $P_i(v_0) \neq 0$. Since $G.v_0$ lies in the closure of $G.\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}$ under Hausdorff topology, it also lies in the Zariski closure of $G.\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}$. It is the unique closed orbit in this closure. Therefore,

$$\Phi_{P_i}(\{\pi_j\}_{j=1}^m) = P_i(\pi_1^{\otimes q_1} \otimes \cdots \otimes \pi_m^{\otimes q_m}) = P_i(v_0) \neq 0,$$

which proves Eq. (36).

We now show that Φ_P satisfies conditions Eqs. (28)–(30).

Indeed, since P is invariant under the group $G = \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_m} \times \mathrm{SL}_n$, we have

$$\begin{aligned}\Phi_P(\{A_j \tilde{\pi}_j A^t\}_{j=1}^m) &= P(A_1^{\otimes q_1} \tilde{\pi}_1^{\otimes q_1} (A^{\otimes q_1})^t \otimes \cdots \otimes A_m^{\otimes q_m} \tilde{\pi}_m^{\otimes q_m} (A^{\otimes q_1})^t) \\ &= P((A_1, \ldots, A_m, A).\tilde{\pi}_1^{\otimes q_1} \otimes \cdots \otimes \tilde{\pi}_m^{\otimes q_m}) \\ &= P(\tilde{\pi}_1^{\otimes q_1} \otimes \cdots \otimes \tilde{\pi}_m^{\otimes q_m}) = \Phi_P(\{\tilde{\pi}_j\}_{j=1}^m).\end{aligned}$$

This shows that Φ_P is invariant as a polynomial function and hence is an invariant polynomial. This verifies condition Eq. (29).

For the homogeneity, note that we have

$$\begin{aligned}\Phi_P(\{t_j \tilde{\pi}_j\}_{j=1}^m) &= P(t_1^{q_1} \cdots t_m^{q_m} \tilde{\pi}_1^{\otimes q_1} \otimes \cdots \otimes \tilde{\pi}_m^{\otimes q_m}) \\ &= (t_1^{q_1} \cdots t_m^{q_m})^d P(\tilde{\pi}_1^{\otimes q_1} \otimes \cdots \otimes \tilde{\pi}_m^{\otimes q_m}).\end{aligned}$$

Therefore, Φ_P is homogeneous with degree $d_j = q_j d$ for $j = 1, \ldots, m$. Moreover,

$$\frac{p_j n_j}{d_j} = \frac{p_j n_j}{q_j d} = \frac{1}{qd}.$$

This shows that $s_{\Phi_P} = qd$ is an integer and satisfies the bound $s_\Phi \ll_{n,\{n_j\}_{j=1}^m,\{p_j\}_{j=1}^m} 1$. The conditions Eqs. (28) and (30) are verified. This completes the proof of the lemma. ∎

As consequences, we have the following propositions.

Proposition 10.8. *For all subspace $W \subseteq V$ of dimension k, there exist a positive integer $m \ll_n 1$, a Zariski open dense subset $\mathcal{O}'_W \subseteq (H)^m$, a non-zero polynomial Φ on H^m with degree $\ll_V 1$ and a positive integer $s_\Phi \ll_V 1$ so that for all $(h_1, \ldots, h_m) \in \mathcal{O}'_W$, the Brascamp–Lieb constant*

$$\mathrm{BL}\left(\left\{\pi_W \circ h_j, \frac{n}{km}\right\}_{j=1}^m\right) \ll_n \|\Phi\|_\infty^{\frac{1}{s_\Phi}} |\Phi(h_1, \ldots, h_m)|^{-\frac{1}{s_\Phi}}.$$

The norm $\|\Phi\|_\infty$ is the maximum of absolute value of coefficient in Φ.

Proof. Let m be the positive integer and $\mathcal{O}_W \subseteq H^m$ be the Zariski open dense subset given by Proposition 10.3. Let $\mathcal{O}'_W = (\mathcal{O}_W)^t$. For all $(h_1, \ldots, h_m) \in \mathcal{O}'_W$, Proposition 10.3 implies that the Brascamp–Lieb constant $\mathrm{BL}\left(\left\{\pi_W \circ h_j, \frac{n}{km}\right\}_{j=1}^m\right)$ is finite. By Lemma 10.7, there exists a nonzero homogeneous polynomial $\tilde{\Phi} \in \mathrm{IP}_0$, we have

$$[\mathrm{BL}(\{\pi_W \circ h_j, \frac{n}{km}\}_{j=1}^m)]^{-1} \tag{37}$$

$$\geq \left(\prod_{j=1}^m k^{-\frac{n}{2m}}\right) \|\tilde{\Phi}\|_{\mathrm{HS}}^{-\frac{1}{s_{\tilde{\Phi}}}} |\tilde{\Phi}(\{\pi_W \circ h_j\}_{j=1}^m)|^{\frac{1}{s_{\tilde{\Phi}}}} > 0. \tag{38}$$

Let

$$\Phi(h_1, \ldots, h_m) := \tilde{\Phi}(\{\pi_W \circ h_j\}_{j=1}^m)$$

and $s_\Phi = s_{\tilde{\Phi}}$. We have $\deg \Phi \ll_V 1$ where the implied constant depends on n and the degree of this polynomial representation. We also have $s_\Phi = s_{\tilde{\Phi}}$ is a positive integer and $s_\Phi = s_{\tilde{\Phi}} \ll_n 1$. In fact, in this case, we have

$$p_j n_j = \frac{n}{km} \cdot k = \frac{n}{m} = \frac{q_j}{q}.$$

Therefore we can take $q = m \ll_n 1$ and $q_j = n$. As in the proof of the previous lemma, we set

$$V = \bigotimes_{j=1}^{m} \bigotimes_{i=1}^{q} (\mathbb{R}^{n_j} \otimes (\mathbb{R}^n)^*) = \bigotimes_{j=1}^{m} \bigotimes_{i=1}^{m} \left(\mathbb{R}^k \otimes (\mathbb{R}^n)^*\right).$$

Since $\tilde{\Phi} = \tilde{\Phi}_P$ for some homogeneous polynomial $P \in \mathcal{O}(V)^G$ with degree $d \ll_{n,m} 1$, we have

$$s_\Phi = s_{\tilde{\Phi}} = qd = md \ll_n 1.$$

The last inequality follows from $m \ll_n 1$. By Eq. (37), we have

$$\mathrm{BL}\left(\left\{\pi_W \circ h_j, \frac{n}{km}\right\}_{j=1}^{m}\right) \ll_n \|\Phi\|_\infty^{\frac{1}{s_\Phi}} |\Phi(h_1, \ldots, h_m)|^{-\frac{1}{s_\Phi}},$$

which completes the proof of the proposition. ■

Proposition 10.9. *For all μ, there exist positive integer $m \ll_n 1$, a Zariski open dense subset $\mathcal{O}'_\mu \subseteq (U^+)^m$, a non-zero polynomial Φ on $(U^+)^m$ with degree $\ll_V 1$ and a positive integer $s_\Phi \ll_n 1$ so that for all $(u_1, \ldots, u_m) \in \mathcal{O}'_\mu$, the Brascamp–Lieb constant*

$$\mathrm{BL}\left(\left\{\pi_{u_j}^{(\mu)}, \frac{n}{m \dim V^{(\mu)}}\right\}_{j=1}^{m}\right) \ll_n \|\Phi\|_\infty^{\frac{1}{s_\Phi}} |\Phi(u_1, \ldots, u_m)|^{-\frac{1}{s_\Phi}}.$$

The norm $\|\Phi\|_\infty$ is the maximum of absolute value of coefficient in Φ.

Proof. The proof is identical to the proof of Proposition 10.8 if we replace Proposition 10.3 by Proposition 10.5. ■

11. Proof of Theorem 6.5

With all the ingredients developed in previous sections, we now proceed the detail of proof of Theorem 6.5 in this section. Due to the polynomial nature of actions of unipotent groups, we need an estimate on the size of the set where certain polynomial function is small. This is known as Remez's inequality and is used by Kleinbock and Margulis and later Kleinbock and Tomanov in [KM98, KT07] to verify the '(C, α)-good' property. We record the form we need in the following lemma.

Lemma 11.1 ([KT07, Lemma 3.4])**.** *For all $d, k \in \mathbb{N}$, there exists a constant $0 < C \ll_{d,k} 1$ so that the following holds. Let $P \in \mathbb{R}[x_1, \ldots, x_d]$ be a polynomial with degree at most k. For all ball $B \subset \mathbb{R}^d$ and $\epsilon > 0$, we have*

$$\mathrm{Leb}\{x \in B : |P(x)| < \epsilon\} \leq C\left(\frac{\epsilon}{\|P\|_{L^\infty(B)}}\right)^{\frac{1}{dk}} \mathrm{Leb}(B).$$

Proof of Theorem 6.5. For simplicity, let $k = \dim V^{(\mu)}$. Recall that we use $A^{(\delta)}$ to denote the δ-neighborhood of A.

Let $m \ll_n 1$, $\mathcal{O}'_\lambda \subseteq (U^+)^m$, Φ and positive integer $s_\Phi \ll_n 1$ be as in Proposition 10.9. Recall that we have $\deg \Phi \ll_V 1$. Let $p_j = \frac{n}{mk}$ for all $j = 1, \ldots, m$.

Let M_1 be a large positive integer so that $M_1 > m(m+2)\deg \Phi$. Since $m \ll_n 1$ and $\deg \Phi \ll_V 1$, the integer M_1 can be chosen to depend only on V. Let $M = 2M_1$. Let $\epsilon \in (0, \frac{1}{100})$ and $0 < \delta \ll_{\epsilon, V} 1$ be a small enough number depending only on ϵ and the representation V. We will explicate this dependence later in the proof.

Suppose the theorem does not hold for M, ϵ and δ as above. Then there exists $A \subseteq B_1^V$ so that $m_U(\mathcal{E}(A)) > \delta^\epsilon$. Recall that from the definition of exceptional set, for all $u \in \mathcal{E}(A)$, we have

$$|\pi_u^{(\lambda)}(A)|_\delta < \delta^{M\epsilon}|A|_\delta^{\frac{k}{n}}.$$

Let

$$\mathcal{E}_{\mathrm{BL}} = \{(u_1, \ldots, u_m) \in (B_1^U)^m : |\Phi(u_1, \ldots, u_m)| < \|\Phi\|_\infty \delta^{M_1\epsilon}\}.$$

Since $m \ll_n 1$ and $\deg \Phi \ll_n 1$, by Remez's inequality (Lemma 11.1), there exists constant C depending only on $\dim U$ and n so that

$$m_U^{\otimes m}(\mathcal{E}_{\mathrm{BL}}) \le C\delta^{\frac{M_1\epsilon}{m\deg\Phi}} \le C\delta^{(m+2)\epsilon} \tag{39}$$

If δ is small enough depending on ϵ so that $\delta^{-\epsilon} \ge C$, we have

$$m_U^{\otimes m}(\mathcal{E}_{\mathrm{BL}}) \le \delta^{(m+1)\epsilon} \tag{40}$$

Let $\tilde{\mathcal{E}} = \mathcal{E}(A)^m \setminus \mathcal{E}_{\mathrm{BL}}$, it has measure

$$m_U^{\otimes m}(\tilde{\mathcal{E}}) \ge \delta^{m\epsilon} - \delta^{(m+1)\epsilon} > \delta^{(m+1)\epsilon}$$

if δ is small enough depending only on ϵ so that $\delta^\epsilon < \frac{1}{2}$. In particular, we have $\tilde{\mathcal{E}} \neq \emptyset$. For all $(u_1, \ldots, u_m) \in \tilde{\mathcal{E}}$, we have

$$\mathrm{BL}(\{\pi_{u_j}^{(\lambda)}, p_j\}_{j=1}^m) \ll_n \left(\|\Phi\|_\infty |\Phi(u_1, \ldots, u_m)|^{-1}\right)^{\frac{1}{s_\Phi}} \le \delta^{-\frac{M_1}{s_\Phi}\epsilon} \le \delta^{-M_1\epsilon}.$$

Let $f_j = \mathbb{1}_{\pi_{u_j}^{(\lambda)}(A^{(\delta)})} \in L^1(V^{(\mu)})$, note that

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\pi_{u_j}^{(\lambda)}(x))^{p_j}\,\mathrm{d}x \ge \mathrm{Leb}(A^{(\delta)}) \gg_n \delta^n |A|_\delta.$$

By the definition of the Brascamp–Lieb constant, we have

$$\begin{aligned}
\delta^n|A|_\delta \ll \int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\pi_{u_j}^{(\lambda)}(x))^{p_j}\,\mathrm{d}x &\le \mathrm{BL}(\{\pi_j, p_j\}_{j=1}^m) \prod_{j=1}^m \left(\int_{V^{(\mu)}} f_j\right)^{p_j} \\
&\ll_n \delta^{-M_1\epsilon} \prod_{j=1}^m \mathrm{Leb}(\pi_{u_j}^{(\lambda)}(A^{(\delta)}))^{\frac{n}{km}} \\
&\ll_n \delta^{-M_1\epsilon} \prod_{j=1}^m \left(\delta^k |\pi_{u_j}^{(\lambda)}(A)|_\delta\right)^{\frac{n}{km}}.
\end{aligned}$$

Therefore,

$$\left(\prod_{j=1}^{m} |\pi_{u_j}^{(\lambda)}(A)|_{\delta}^{\frac{n}{k}}\right)^{\frac{1}{m}} \gg_n \delta^{M_1\epsilon}|A|_{\delta}. \tag{41}$$

For all $(u_1, \dots, u_m) \in \tilde{\mathcal{E}} = \mathcal{E}(A)^m \setminus \mathcal{E}_{\mathrm{BL}}$, we have

$$|\pi_{u_j}^{(\lambda)}(A)|_{\delta} < \delta^{M\epsilon}|A|_{\delta}^{\frac{k}{n}} \tag{42}$$

for all $j = 1, \dots, m$. Combining Eq. (41) and Eq. (42), we have

$$\delta^{M_1\epsilon} \ll_n \delta^{Mm\epsilon} = \delta^{2M_1\epsilon},$$

which leads to a contradiction if δ is small enough depending on ϵ and n. The proof of the theorem is complete. ■

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