A GREEN'S FUNCTION APPROACH TO LINEARIZED MONGE–AMPÈRE EQUATIONS IN DIVERGENCE FORM AND APPLICATION TO SINGULAR ABREU TYPE EQUATIONS

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Abstract. In this paper, we establish local and global regularity estimates for linearized Monge–Ampère equations in divergence form via critical Lorentz space estimates for the Green's function of the linearized Monge–Ampère operator and its gradient. These estimates hold under suitable conditions on the data and the convex Monge–Ampère potential is assumed to have Hessian determinant bounded between two positive constants. As an application, we obtain the solvability in all dimensions of the second boundary value problem for a class of singular fourth-order Abreu type equations that arise from the approximation analysis of variational problems subject to convexity constraints.

1. Introduction

In this paper, we establish local and global regularity estimates for solutions to linearized Monge–Ampère equations in divergence form

$$L_u v := D_i(U^{ij}D_j v) = \operatorname{div}\mathbf{F} + \mu, \tag{1.1}$$

where $\mathbf{F}$ is a vector field and μ is a signed Radon measure, via critical Lorentz space estimates for the Green's function, also known as the fundamental solution, of the linearized Monge–Ampère operator

$$L_u := D_i(U^{ij}D_j).$$

Throughout,

$$U = (U^{ij})_{1\le i,j\le n} := (\det D^2u)(D^2u)^{-1}$$

is the cofactor matrix of the Hessian matrix D^2u of a convex Monge–Ampère potential $u \in C^3(\Omega)$, where the function u satisfies

$$0 < \lambda \le \det D^2 u \le \Lambda \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.2}$$

where λ and Λ are constants. We always assume $n \ge 2$ and repeated indices are summed.

Due to the divergence-free property of U, that is, $D_iU^{ij} = 0$ for all j, the operator L_u can be rewritten in nondivergence form:

$$L_u = D_j(U^{ij}D_i) = U^{ij}D_{ij}.$$

The coefficient matrix U of L_u arises from linearizing the Monge-Ampère operator $\det D^2u$. One can also note that $L_u v$ is the coefficient of t in the expansion

$$\det D^2(u+tv) = \det D^2u + t\,U^{ij}D_{ij}v + \cdots + t^n \det D^2 v.$$

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Under the assumption (1.2), the cofactor matrix U is positive definite, so (1.1) is an elliptic equation. However, it is in general degenerate and/or singular since the eigenvalues of U could tend to zero and/or infinity. This makes its analysis challenging.

Caffarelli and Gutiérrez [6] established a fundamental interior Harnack inequality for the homogeneous linearized Monge–Ampère equation $U^{ij}D_{ij}v = 0$. Their result is an affine invariant version of the classical Harnack inequality of Moser, and Krylov and Safonov for elliptic equations in divergence form and nondivergence form, respectively. Interior Hölder estimates consequently follow. Central in their analysis is the geometry of sections of the Monge–Ampère potential function u which replace Euclidean balls. Sections are defined as sublevel sets of convex functions after subtracting their supporting hyperplanes. Since then, many developments and applications have been obtained by many authors including [8, 15–19, 21–23, 25, 27–29, 32–34, 37, 41–43, 45]. We refer the reader to [27] for an overview and will restrict ourselves to works closely related to the subject of this paper.

Equations of the type (1.1) arise in several contexts such as the semigeostrophic equations in meteorology [2, 3, 9, 10, 22, 31] and singular Abreu equations in the calculus of variations with a convexity constraint [7, 18, 24, 26, 30].

Let us mention some works related to the case of $\mu = 0$ under (1.2). Loeper [31] established Hölder estimates for (1.1) using integral information on v under the assumption that $\det D^2u$ is close to a constant. When $n = 2$ and (1.2) is satisfied, the second author [22, 25] obtained interior and global Hölder estimates for (1.1) assuming $\mathbf{F} \in L^\infty(\Omega; \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. With the additional assumption $D^2u \in L^{1+\varepsilon_n}(\Omega)$ for some $\varepsilon_n > (n+1)(n-2)/2$, these estimates were extended to higher dimensions in [27, Chapter 15]. This assumption holds in dimension two in view of the $W^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation established by De Philippis–Figalli–Savin [11] and Schmidt [40]. In view of Caffarelli's $W^{2,p}$ estimates [4] and Wang's counterexamples [46] for the Monge–Ampère equation, the above integrability condition may be thought of as an assumption on Λ/λ in (1.2). Such higher integrability of D^2u was required for the Moser iteration. Later, by assuming $(D^2u)^{1/2}\mathbf{F} \in L^q(\Omega; \mathbb{R}^n)$ for some $q > n$ and using De Giorgi's iteration, Wang [45] established interior Hölder estimates for (1.1) relying on the L^∞ norm of the solution v, while the second author's results only rely on the L^p norm of v for $p \in (1, +\infty)$. For all bounded vector field $\mathbf{F}$, Wang's condition essentially requires D^2u to be in L^r where $r > n/2$. In this case, Wang's Hölder estimates were established via Moser's iteration technique by Kim [17] who also studied equations with drifts.

Very recently, when $\mathbf{F} = 0$, Cui–Wang–Zhou [8] use potential theory and Campanato type estimates to study equation (1.1) for signed Radon measures μ whose total variation on each compactly contained section of u of height h grows like $h^{\frac{n-2}{2}+\varepsilon}$ for some $\varepsilon > 0$. In particular, they obtained interior Hölder estimates for (1.1) under (1.2) when $\mu = \operatorname{div}\mathbf{F}_0$ is a negative Radon measure and $\mathbf{F}_0$ is a bounded vector field. Remarkably, these results give new interior higher-order estimates for singular Abreu equations. In all these works [8, 17, 45], the Monge–Ampère Sobolev inequality [22, 41] plays an important role.

We will give here a unified, different proof of the interior Hölder estimates in [8, 45]. Moreover, we also obtain a global version. Our approach uses fine properties of the Green's function of the linearized Monge–Ampère operator L_u and avoids the Monge–Ampère Sobolev inequality. In the context of linearized Monge–Ampère equations, our approach has its root in the previous work of Nguyen and the second author [28] where global Hölder estimates were obtained using L^q norm of the right-hand side where $q > n/2$. Crucial to our approach are uniform estimates for solutions to (1.1) with zero boundary value. Our approach gives uniform estimates assuming only $(D^2u)^{1/2}\mathbf{F}$ being in the Lorentz space $L^{n,1}$ when $n \geq 3$.

As an application, we obtain the solvability of the second boundary value problem for singular Abreu type equations, some of which in dimensions at least three are not accessible by previous approaches.

The analysis of the Green's function of the linearized Monge–Ampère operator starts with the work of Tian–Wang [41] and then Maldonado [32–34] and the second author [19, 21, 22, 25, 27]. Let $g_S(\cdot, y)$ be the Green's function of L_u in $S \subset \Omega$ with pole $y \in S$. Building on properties of the Green's function put together in [27], we will establish interior and global estimates for $(D^2u)^{-1/2}D_x g_S(\cdot, y)$ in the weak $L^{n/(n-1)}$ space, also known as the Lorentz space $L^{n/(n-1),\infty}$, when $n \geq 3$. This space is critical as can be seen for the case of $u(x) = |x|^2/2$ where L_u now becomes the Laplace operator, and its Green's function does not belong to smaller Lorentz spaces. Our results are affine analogues of those in Grüter–Widman [14] for uniformly elliptic equations in dimensions $n \geq 3$. We have an extra logarithmic factor for Lorentz space type estimates in dimension two. These estimates build upon Lorentz space type estimates for the Green's function g_S itself; in this case, we have more refined analysis as we have a good control on the shapes of the superlevel sets of g_S.

The quantity $(D^2u)^{-1/2}D_x g_S(\cdot, y)$ was first studied by Maldonado [32, 34] as a natural Monge–Ampère gradient in the sense of the Monge–Ampère Sobolev inequality obtained in [22, 41]. Our estimates for $(D^2u)^{-1/2}D_x g_S(\cdot, y)$ in strong L^p spaces where $p \in (1, n/(n-1))$ follow from Lorentz space estimates and they recover and extend results established in [34] in the case of sections compactly contained in the domain. As applications, we will use these estimates to obtain local and global uniform and Hölder estimates for solutions to (1.1) under suitable assumptions.

Before stating our main results, we introduce some relevant concepts.

Definition 1.1 (Sections)**.** Let $u \in C^1(\Omega) \cap C(\overline{\Omega})$ be a convex function and $h > 0$. If $x_0 \in \overline{\Omega}$, then the *section of u centered at x_0 with height h* is defined by

$$S_u(x_0, h) := \big\{x \in \overline{\Omega} : u(x) < u(x_0) + Du(x_0) \cdot (x - x_0) + h\big\}.$$

In the case of $x_0 \in \partial\Omega$, we require that u is differentiable at x_0.

Definition 1.2 (Green's function of the linearized Monge–Ampère operator)**.** Let Ω be a bounded convex domain in $\mathbb{R}^n$ and $u \in C^3(\Omega)$ be a convex function satisfying (1.2). Assume $V \subset \Omega$ is open. Let δ_y be the Dirac measure giving the unit mass to y. Then, for each $y \in V$, there exists a unique function $g_V(\cdot, y) : V \to [0, \infty]$ with the following properties:

(a) $g_V(\cdot, y) \in W_0^{1,q}(V) \cap W^{1,2}(V \backslash B_r(y))$ for all $q < \frac{n}{n-1}$ and all $r > 0$.

(b) $g_V(\cdot, y)$ is a weak solution of

$$-D_i\left(U^{ij}D_j g_V(\cdot, y)\right) = \delta_y \quad \text{in } V, \quad g_V(\cdot, y) = 0 \quad \text{on } \partial V,$$

that is, denoting $D_j g_V(x, y) = D_{x_j} g_V(x, y) = \frac{\partial}{\partial x_j} g_V(x, y)$, we have

$$\int_V U^{ij} D_j g_V(x, y) D_i \psi(x)\, dx = \psi(y) \quad \text{for all } \psi \in C_c^\infty(V). \tag{1.3}$$

We call $g_V(\cdot, y)$ the *Green's function* of the linearized Monge–Ampère operator $L_u = D_i\left(U^{ij}D_j\right)$ in V with *pole* y. We set $g_V(y,y) = +\infty$.

Definition 1.3 (Proper sets)**.** We call a nonempty open set $V \subset \mathbb{R}^n$ *proper* if V satisfies an exterior cone condition at every boundary point. Examples of such proper sets include intersections of a bounded convex domain $\Omega \subset \mathbb{R}^n$ with sections of a convex function $u \in C^1(\overline{\Omega})$ or open balls.

Definition 1.4 (Lorentz spaces)**.** Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and $f : \Omega \longrightarrow \mathbb{R}$ be a Lebesgue-measurable function. For $1 \leq p < \infty$ and $0 < q \leq \infty$, we define

$$\|f\|_{L^{p,q}(\Omega)} := \begin{cases} p^{\frac{1}{q}}\Big(\int_0^\infty t^q|\{x \in \Omega : |f(x)| > t\}|^{\frac{q}{p}}\frac{dt}{t}\Big)^{\frac{1}{q}} & \text{if } q < \infty,\\ \sup_{t>0} t|\{x \in \Omega : |f(x)| > t\}|^{1/p} & \text{if } q = \infty. \end{cases}$$

The *Lorentz space* $L^{p,q}(\Omega)$ consists of all Lebesgue-measurable functions f defined on Ω such that $\|f\|_{L^{p,q}(\Omega)} < \infty$. The Lorentz space $L^{p,\infty}$ coincides with the *weak L^p space*, and the Lorentz space $L^{p,p}$ is the usual L^p space.

Notation. We use $c = (*, \ldots, \star)$ and $C = C(*, \ldots, \star)$ to denote positive constants c, C depending on the quantities appearing in the parentheses; they may change from line to line. We use $D_i = \partial/\partial_{x_i}$ and $D_x f(x,y)$ to denote the gradient of f in the x-variable. For a Lebesgue measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes its n-dimensional Lebesgue measure. We use $\mathcal{H}^{n-1}$ to denote the $(n-1)$-dimensional Hausdorff measure. For a locally integrable function $\mu : \mathbb{R}^n \to \mathbb{R}$, we can view it as a signed Radon measure and denote $|\mu|(A) = \int_A |\mu|\,dx$ for any Lebesgue measurable set $A \subset \mathbb{R}^n$.

The rest of this paper will be organized as follows. We state our main results in Section 2. In Section 3, we recall some background materials for our analysis and state a key estimate for the Monge–Ampère gradient of the Green's function. In Section 4, we will prove Theorems 2.1 and 2.2 on Lorentz space estimates for the Green's function. In Section 5, we establish maximum principles for linearized Monge–Ampère equation (1.1). We prove an interior Harnack inequality in Theorem 2.3 and interior Hölder estimates for solutions to (1.1) in Section 6. The proof of Theorem 2.4 on global Hölder estimates will be given in Section 7. We present an application to the solvability of singular Abreu equations in Section 8.

2. Statement of the main results

In this section, we state our main results and give some brief comments on them.

2.1. **Lorentz space estimates for the Green's function.** Our first main result establishes Lorentz space estimates for the Green's function of the linearized Monge–Ampère operator in compactly contained sections.

Theorem 2.1. *Let $u \in C^3(\Omega)$ be a convex function satisfying* (1.2)*, where $\Omega \subset \mathbb{R}^n$. Assume $S_u(x_0, 2h) \Subset \Omega$ where $x_0 \in \Omega$ and $h > 0$. Let $g_{S_h}(\cdot, y)$ be the Green's function of the linearized Monge–Ampère operator $D_i(U^{ij}D_j)$ in $S_u(x_0,h)$ with pole $y \in S_u(x_0,h)$. Then, for $t > 0$,*

$$\begin{aligned} &|\{x \in S_u(x_0,h) : |(D^2u(x))^{-1/2}D_x g_{S_u(x_0,h)}(x,y)| > t\}| \\ &\leq \begin{cases} Ct^{-2}\big(h + \log\max\{t,1\}\big) & \text{if } n = 2,\\ Ct^{-\frac{n}{n-1}} & \text{if } n \geq 3, \end{cases} \end{aligned} \tag{2.1}$$

where $C = C(n, \lambda, \Lambda) > 0$.

Consequently, for all $y \in S_u(x_0, h)$,

- *if* $n \geq 3$, *then*

$$\|(D^2u)^{-1/2} D_x g_{S_u(x_0,h)}(\cdot, y)\|_{L^{\frac{n}{n-1},\infty}(S_u(x_0,h))} \leq C(n, \lambda, \Lambda); \tag{2.2}$$

- *if* $n \geq 2$ *and* $p \in \left(1, \frac{n}{n-1}\right)$, *then*

$$\int_{S_u(x_0,h)} |(D^2u(x))^{-1/2} D_x g_{S_u(x_0,h)}(x, y)|^p \, dx \leq C(n, \lambda, \Lambda, p) h^{\frac{n}{2} - \frac{n-1}{2} p}. \tag{2.3}$$

Next, we state global analogues of estimates in Theorem 2.1 under suitable assumptions for which Savin's boundary localization theorem for the Monge–Ampère equation [38, Theorem 3.1] is applicable.

Global structural assumptions. Let $\Omega \subset \mathbb{R}^n$ be a convex domain and assume that there exists a constant $\rho > 0$ such that

$$\Omega \subset B_{1/\rho}(0) \subset \mathbb{R}^n \tag{2.4}$$

and for each $y \in \partial\Omega$,

$$\text{there is an interior ball } B_\rho(z) \subset \Omega \text{ such that } y \in \partial B_\rho(z). \tag{2.5}$$

Let $u \in C^{1,1}(\overline{\Omega}) \cap C^3(\Omega)$ be a convex function satisfying

$$0 < \lambda \leq \det D^2 u \leq \Lambda \quad \text{ in } \Omega. \tag{2.6}$$

Assume further that on $\partial\Omega, u$ separates quadratically from its tangent hyperplanes; namely, for all $x_0, x \in \partial\Omega$, we have

$$\rho \left|x - x_0\right|^2 \leq u(x) - u\left(x_0\right) - Du\left(x_0\right) \cdot \left(x - x_0\right) \leq \rho^{-1} \left|x - x_0\right|^2. \tag{2.7}$$

Theorem 2.2. *Assume that* u *and* Ω *satisfy* (2.4)–(2.7). *Let* $V \subset \Omega$ *be open, proper, and contained in a section* $S_u(x_0, h)$ *of* u *of height* $h > 0$. *Let* $g_V(\cdot, y)$ *be the Green's function of the linearized Monge–Ampère operator* $D_i(U^{ij} D_j)$ *in* V *with pole* $y \in V$. *Then, for* $t > 0$,

$$|\{x \in V : |(D^2u(x))^{-1/2} D_x g_V(x, y)| > t\}| \leq \begin{cases} Ct^{-2}(h + \log \max\{t, 1\}) & \text{if } n = 2, \\ Ct^{-\frac{n}{n-1}} & \text{if } n \geq 3, \end{cases} \tag{2.8}$$

where $C = C(n, \lambda, \Lambda, \rho) > 0$.

Consequently, for all $y \in V$,

- *if* $n \geq 3$, *then*

$$\|(D^2u)^{-1/2} D_x g_V(\cdot, y)\|_{L^{\frac{n}{n-1},\infty}(V)} \leq C(n, \lambda, \Lambda, \rho). \tag{2.9}$$

- *if* $n \geq 2$ *and* $p \in \left(1, \frac{n}{n-1}\right)$, *then*

$$\int_V |(D^2u(x))^{-1/2} D_x g_V(x, y)|^p \, dx \leq C(n, \lambda, \Lambda, \rho, p) h^{\frac{n}{2} - \frac{n-1}{2} p}. \tag{2.10}$$

Under the assumptions (2.4)–(2.7), by [27, Lemma 9.7], there exists $M = M(n, \lambda, \Lambda, \rho) > 0$ such that

$$\overline{\Omega} \subset S_u(x_0, M) \quad \text{for all} \quad x_0 \in \overline{\Omega}.$$

Therefore, Ω is a section of u with height comparable to 1 and Theorem 2.2 is applicable to $V = \Omega$.

2.2. **Applications of Lorentz space estimates to the linearized Monge–Ampère equations.** Applying Theorem 2.1 and the Caffarelli–Gutiérrez Harnack inequality (Theorem 6.1), we obtain a Harnack inequality for nonnegative solutions to (1.1).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ and $u \in C^3(\Omega)$ be a convex function satisfying* (1.2). *Let $\mathbf{F} \in W^{1,n}_{loc}(\Omega;\mathbb{R}^n)$ and $\mu \in L^n_{loc}(\Omega)$. Suppose that $S_u(x_0, 2h) \Subset \Omega$ where $x_0 \in \Omega$ and $h > 0$. Let $v \in W^{2,n}_{loc}(S_u(x_0,h)) \cap C(\overline{S_u(x_0,h)})$ be a nonnegative solution to*

$$U^{ij}D_{ij}v = div\,\mathbf{F} + \mu \quad in \quad S_u(x_0,h).$$

(i) *Assume $n \geq 3$, $\mu = 0$, and $(D^2u)^{1/2}\mathbf{F} \in L^{n,1}(S_u(x_0,h);\mathbb{R}^n)$. Then*

$$\sup_{S_u(x_0,h/2)} v \leq C(n,\lambda,\Lambda)\Big(\inf_{S_u(x_0,h/2)} v + \|(D^2u)^{1/2}\mathbf{F}\|_{L^{n,1}(S_u(x_0,h))}\Big). \tag{2.11}$$

(ii) *Assume that $(D^2u)^{1/2}\mathbf{F} \in L^q(S_u(x_0,h);\mathbb{R}^n)$ for some $q > n$ and there exist $M_0 \geq 0$ and $\varepsilon > 0$ such that*

$$|\mu|(S_u(z,s)) \leq M_0 s^{\frac{n-2}{2}+\varepsilon} \quad \text{for all sections } S_u(z,s) \Subset S_u(x_0,2h). \tag{2.12}$$

Then

$$\sup_{S_u(x_0,h/2)} v \leq C(n,\lambda,\Lambda)\Big(\inf_{S_u(x_0,h/2)} v + C_*\|(D^2u)^{1/2}\mathbf{F}\|_{L^q(S_u(x_0,h))}h^{\frac{q-n}{2q}} + C_\star M_0 h^{\varepsilon}\Big), \tag{2.13}$$

where $C_* = C_*(n,\lambda,\Lambda,q) > 0$ and $C_\star = C_\star(n,\lambda,\Lambda,\varepsilon) > 0$.

Next, we discuss Hölder continuity of solutions to (1.1). Theorem 2.3 allows us to give a new proof of the interior Hölder estimates for (1.1) when $(D^2u)^{1/2}\mathbf{F} \in L^q(\Omega;\mathbb{R}^n)$ for some $q > n$ and $\mu \in L^n(\Omega)$ satisfying (2.12). These estimates are stated in Theorem 6.2. They were first proved by Wang [45, Theorem 1.5] and Cui–Wang–Zhou [8, Theorem 1.2]. Moreover, we are able to obtain the following global Hölder estimates.

Theorem 2.4. *Let Ω be a uniformly convex domain in $\mathbb{R}^n$, that is, for all $z \in \partial\Omega$, there is a ball $B_R(z_0)$ such that $\Omega \subset B_R(z_0)$ and $\partial\Omega \cap \partial B_R(z_0) = \{z\}$ for some uniform convexity radius $R > 0$. Let $\partial\Omega \in C^3$ and $u \in C^{1,1}(\overline{\Omega}) \cap C^3(\Omega)$ be a convex function satisfying* (1.2) *and $u|_{\partial\Omega} \in C^3$. Assume $\mathbf{F} \in W^{1,n}(\Omega;\mathbb{R}^n)$, $\mu \in L^n(\Omega)$, and $\phi \in C^\alpha(\partial\Omega)$ for some $\alpha \in (0,1)$. Let $v \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ be the solution to*

$$U^{ij}D_{ij}v = div\,\mathbf{F} + \mu \quad in\ \Omega, \quad v = \phi \quad on\ \partial\Omega.$$

Assume $(D^2u)^{1/2}\mathbf{F} \in L^q(\Omega;\mathbb{R}^n)$ for some $q > n$, and there exist $M_0 \geq 0$ and $\varepsilon > 0$ such that

$$|\mu|(S_u(z,s)) \leq M_0 s^{\frac{n-2}{2}+\varepsilon} \quad \text{for all sections } S_u(z,s) \subset \overline{\Omega}. \tag{2.14}$$

Then, there exists $\beta = \beta(n,\lambda,\Lambda,q,\varepsilon,\alpha) \in (0,1)$ such that

$$\|v\|_{C^\beta(\overline{\Omega})} \leq C\big(\|\phi\|_{C^\alpha(\partial\Omega)} + \|(D^2u)^{1/2}\mathbf{F}\|_{L^q(\Omega)} + M_0\big),$$

where $C > 0$ depends only on $n, \lambda, \Lambda, q, \varepsilon, \alpha, \|u\|_{C^3(\partial\Omega)}$, R, and the C^3 regularity of $\partial\Omega$.

As an application of Theorem 2.4, we obtain in Section 8 the solvability of the second boundary value problem for singular fourth-order Abreu type equations, some of which in dimensions at least three are not accessible by previous approaches.

2.3. **Comments.** We briefly comment on our results, assumptions, and methods of the proofs.

Remark 2.5. Some remarks on the results are in order.

(1) Under assumption (1.2), when $y = x_0$ and the integral over the whole section $S_u(x_0, h)$ in inequality (2.3) is replaced by one over $S_u(x_0, h/2)$, Maldonado [34, Theorem 1.1] established the following estimate by a different method:
$$\int_{S_u(x_0,h/2)} |(D^2u(x))^{-1/2} D_x g_{S_u(x_0,h)}(x, x_0)|^p \, dx \le C(n, \lambda, \Lambda, p) h^{\frac{n}{2} - \frac{n-1}{2}p}.$$

(2) When $n \ge 3$, the exponent $\frac{n}{2} - \frac{n-1}{2}p$ in (2.3) comes from applying (2.2) and the Hölder inequality in Lorentz spaces together with the volume estimates for sections. For the same reason, the exponent $\frac{q-n}{2q}$ in (2.13) comes from (2.11).

(3) It would be interesting to remove the logarithmic terms in Theorems 2.1 and 2.2 in dimension two.

(4) By applying Theorem 2.2 and the boundary Harnack inequality for the linearized Monge–Ampère equation [21], we can obtain a boundary analogue of Theorem 2.3.

Remark 2.6. We comment briefly on some assumptions.

(1) In Theorems 2.3 and 2.4, we assume the vector field $\mathbf{F}$ to be in $W^{1,n}$ and the measure μ to be an L^n function only to use the representation formula (3.1) involving the Green's function, $\operatorname{div} \mathbf{F}$, and μ. However, our estimates will not depend on this regularity of $\mathbf{F}$ and μ. In particular, they do not depend on $\|\mu\|_{L^n}$ in Theorem 2.4. Note that
$$|\mu|(S_u(z, s)) \le \|\mu\|_{L^n(\Omega)} |S_u(z, s)|^{\frac{n-1}{n}} \le C(n, \lambda) \|\mu\|_{L^n(\Omega)} s^{\frac{n-1}{2}}.$$

(2) As will be seen in applications (see Lemma 3.5 and Theorem 8.1), when $\mu = \operatorname{div} \mathbf{F}_0$ for a vector field $\mathbf{F}_0 \in W^{1,n}(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ such that μ has a definite sign, M_0 will be chosen to depend on $\|\mathbf{F}_0\|_{L^\infty(\Omega)}$ rather than on $\|\operatorname{div} \mathbf{F}_0\|_{L^n(\Omega)}$. This is the case of $\mathbf{F}_0$ being the gradient of a convex or concave function. We do not pursue the issue of finding optimal regularity conditions on $\mathbf{F}$ and μ in this paper.

Remark 2.7. We comment briefly on the methods of the proofs.

(1) The key idea in the proofs of Theorems 2.1 and 2.2 is to estimate the distribution function for the Monge–Ampère gradient $(D^2u)^{-1/2}Dg$ of the Green's function g of the linearized Monge–Ampère operator via estimates of the distribution function of g (see Proposition 3.10) and then optimize. This allows us to take advantage of results for the distribution function of g that have been developed so far. Though simple, this idea can be applicable to more general elliptic operators in divergence form.

(2) In the proofs of Theorems 2.3 and 2.4, we follow the framework in [27, Chapter 15] for linearized Monge–Ampère equations in divergence form and use the Caffarelli–Gutiérrez Harnack inequality, Savin's boundary localization theorem, and maximum principles (such as Propositions 5.1 and 5.3) to estimate solutions with prescribed boundary values.

(3) In the course of proving the maximum principles such as (5.3), instead of working with $D_x g_{S_h}(\cdot, y) \cdot \mathbf{F}$, we work with $(D^2u)^{-1/2} D_x g_{S_h}(\cdot, y) \cdot (D^2u)^{1/2}\mathbf{F}$. The integrability of the Monge–Ampère gradient of the Green's function in Theorems 2.1 and 2.2 explains the assumed integrability of $(D^2u)^{1/2}\mathbf{F}$ and this seems to be optimal without further restrictions. On the other hand, when μ is a Radon measure with certain measure growth of the total variation $|\mu|$ on sections, the layer-cake formula can be used to

obtain uniform estimates; see (5.7). For this, our refined control on the shape of the superlevel sets of the Green's function proves to be crucial.

3. Preliminaries and a key estimate

In this section, we recall some properties of sections of solutions to the Monge–Ampère equation and the Green's functions of the linearized Monge–Ampère operator and state a key estimate for the Monge–Ampère gradient of the Green's function.

We will frequently use the following volume estimates; see [27, Lemmas 5.6 and 5.8].

Lemma 3.1 (Volume estimates for sections)**.** *Let $u \in C^2(\Omega)$ be a convex function satisfying* (1.2) *on a bounded convex domain $\Omega \subset \mathbb{R}^n$. Let $x \in \Omega$ and $h > 0$. Then*

$$|S_u(x,h)| \leq C(n)\lambda^{-1/2}h^{n/2}.$$

If $S_u(x,h) \Subset \Omega$, then

$$c(n)\Lambda^{-1/2}h^{n/2} \leq |S_u(x,h)|,$$

where c, C are positive constants depending only on n.

Note that if $S_u(x_0,h) \Subset \Omega$, then the convexity of u implies that $S_u(x_0,h)$ is a bounded convex domain and $u(x) = u(x_0) + Du(x_0)\cdot(x-x_0) + h$ on $\partial S_u(x_0,h)$.

We also need the following geometric property of sections; see [27, Theorem 5.30].

Theorem 3.2 (Inclusion property of interior sections)**.** *Let $u \in C^2(\Omega)$ be a convex function satisfying* (1.2)*. Then, there exist constants $c_0(n,\lambda,\Lambda) > 0$ and $p_1(n,\lambda,\Lambda) \geq 1$ with the following property. Assume $S_u(x_0, 2t) \Subset \Omega$ and $0 < r < s \leq 1$. If $x_1 \in S_u(x_0, rt)$, then*

$$S_u\left(x_1, c_0(s-r)^{p_1}t\right) \subset S_u\left(x_0, st\right).$$

We collect here some facts on the Green's function of the linearized Monge–Ampère operator; see [27, Chapter 14] for more details.

Remark 3.3. We will use the following properties of the Green's functions in the setting of Definition 1.2.

(1) By approximation arguments, we can use $\psi \in W_0^{1,2}(V) \cap C(\overline{V})$ as test functions to (1.3).
(2) (Representation formula) Assume that V is proper. If $\varphi \in L^n(V)$, then there exists a unique solution $\psi \in W^{2,n}_{\mathrm{loc}}(V) \cap W_0^{1,2}(V) \cap C(\overline{V})$ to

$$-U^{ij}D_{ij}\psi = \varphi \quad \text{in } V \quad \text{and } \psi = 0 \quad \text{on } \partial V.$$

Use ψ as a test function to (1.3), then the following holds:

$$\psi(y) = \int_V g_V(\cdot, y)\varphi\, dx. \tag{3.1}$$

We also need the following regularity of the Green's function away from the pole.

Proposition 3.4. *Let u, Ω, and V satisfy one of the following sets of conditions:*

(1) *Ω is bounded convex domain in $\mathbb{R}^n$, $u \in C^3(\Omega)$ is a convex function satisfying* (1.2)*, and $V \Subset \Omega$ is open.*
(2) *u and Ω satisfy* (2.4)*–*(2.7)*, and $V \subset \Omega$ is open.*

Let $g_V(\cdot, x_0)$ be the Green's function of $L_u = D_i\left(U^{ij}D_j\right)$ in V with pole $x_0 \in V$.

- *If* $E \Subset V \setminus \{x_0\}$, *then* $g_V(\cdot, x_0) \in W^{2,n}_{\rm loc}(E) \cap W^{1,2}(E) \cap C(\overline{E})$ *and* $L_u g_V(\cdot, x_0) = 0$ *in* E.
- *Assume* V *is proper. Then* $g_V(\cdot, x_0) \in C\left(\overline{V} \setminus B_r(x_0)\right)$ *for all* $r > 0$.

We state a global Hölder gradient estimate for the Monge–Ampère equation and verify (2.14) under natural conditions that are suitable for applications.

Lemma 3.5. *Assume that* u *and* Ω *satisfy* (2.4)–(2.7). *There exists* $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$ *such that the following statements hold.*

(i) *There exists* $C_* = C_*(n, \lambda, \Lambda, \rho) > 0$ *such that*

$$[Du]_{C^\alpha(\overline{\Omega})} := \sup_{x \neq y \in \overline{\Omega}} |Du(x) - Du(y)|/|x - y|^\alpha \leq C_*. \tag{3.2}$$

(ii) *Let* $S_u(x_0, t_0)$ *be a section of* u *with* $x_0 \in \overline{\Omega}$ *and* $t_0 > 0$. *Then*

$$\mathcal{H}^{n-1}(\partial S_u(x_0, t_0)) \leq C(n, \lambda, \Lambda, \rho) t_0^{\frac{n-2}{2} + \frac{\alpha}{\alpha+1}}. \tag{3.3}$$

Consequently, if $\mathbf{F} \in W^{1,n}(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ *and* $\mu := div\, \mathbf{F}$ *has a definite sign, then*

$$|\mu|(S_u(x_0, t_0)) \leq C(n, \lambda, \Lambda, \rho) \|\mathbf{F}\|_{L^\infty(S_u(x_0, t_0))} t_0^{\frac{n-2}{2} + \frac{\alpha}{\alpha+1}}. \tag{3.4}$$

For the proof of Lemma 3.5 (ii), we will use the following observation in Cui–Wang–Zhou [8, Lemma 4.4]:

Lemma 3.6. *Let* $X \subset \mathbb{R}^n$ *be a bounded convex domain containing an open ball* $B_r(x_0)$. *Then*

$$\mathcal{H}^{n-1}(\partial X) \leq \frac{n|X|}{r}.$$

In [8, Lemma 4.4], the lemma was proved for ∂X smooth. However, it is still valid for general bounded convex domain X. To see this, let $\{X_m\}_{m=1}^\infty$ be a sequence of uniformly convex domains with C^∞ boundaries such that $\overline{X_m}$ converges to $\overline{X}$ in the Hausdorff distance; see [27, Theorem 2.51]. Then applying [8, Lemma 4.4] to X_m and letting $m \to \infty$, we obtain the stated estimate.

Proof of Lemma 3.5. By the global $C^{1,\alpha}$ estimates for u (see [27, Theorem 9.5]), there exist $\alpha(n, \lambda, \Lambda) \in (0, 1)$ and $C_*(n, \lambda, \Lambda, \rho)$ such that (3.2) holds.

From the mean value theorem, (3.2) easily implies that

$$B_{C_*^{-1/(1+\alpha)} h^{1/(1+\alpha)}}(x) \subset S_u(x, h) \quad \text{whenever } S_u(x, h) \Subset \Omega. \tag{3.5}$$

By the dichotomy of sections in [27, Proposition 9.8], one of the following is true:

(a) $S_u(x_0, 2t_0) \subset \Omega$.

(b) There exist $z \in \partial\Omega$ and a constant $\bar{c}(n, \lambda, \Lambda, \rho) > 1$ such that $S_u(x_0, 2t_0) \subset S_u(z, \bar{c}t_0)$.

Suppose (a) is true. Then, by Lemma 3.6 and the volume estimates in Lemma 3.1, we have

$$\mathcal{H}^{n-1}(\partial S_u(x_0, t_0)) \leq \frac{n|S_u(x_0, t_0)|}{C_*^{-1/(1+\alpha)} t_0^{1/(1+\alpha)}} \leq \frac{C(n, \lambda) t_0^{n/2}}{C_*^{-1/(1+\alpha)} t_0^{1/(1+\alpha)}} = C(n, \lambda, \Lambda, \rho) t_0^{\frac{n-2}{2} + \frac{\alpha}{\alpha+1}}.$$

Suppose (b) is true. If $\bar{c}t_0 < c$ where $c = c(n, \lambda, \Lambda, \rho) > 0$ is small, then, [27, (9.11)] gives

$$S_u(z, \bar{c}t_0) \subset B_{Ct_0^{1/2}|\log t_0|}(z).$$

By the monotonicity of the surface measure with respect to inclusion of convex sets (see [27, Lemma 2.71]), we have

$$\mathcal{H}^{n-1}(\partial S_u(x_0,t_0)) \leq \mathcal{H}^{n-1}(\partial B_{Ct_0^{1/2}|\log t_0|}(z)) \leq C(n,\lambda,\Lambda,\rho)t_0^{\frac{n-2}{2}+\frac{\alpha}{\alpha+1}}.$$

If $\bar{c}t_0 > c$, then, again by [27, Lemma 2.71], we have

$$\mathcal{H}^{n-1}(\partial S_u(x_0,t_0)) \leq \mathcal{H}^{n-1}(\partial\Omega) \leq \mathcal{H}^{n-1}(\partial B_\rho(0)) \leq C(n)\rho^{n-1} \leq C(n,\lambda,\Lambda,\rho)t_0^{\frac{n-2}{2}+\frac{\alpha}{\alpha+1}}.$$

We have established (3.3) in all cases.

To prove (3.4), we can assume without loss of generality that $\mu \geq 0$. Let ν be the outer unit normal vector field on $\partial S_u(x_0,t_0)$. Then, the divergence theorem gives

$$\begin{aligned}|\mu|(S_u(x_0,t_0)) = \int_{S_u(x_0,t_0)} \operatorname{div}\mathbf{F}\,dx &= \int_{\partial S_u(x_0,t_0)} \mathbf{F}\cdot\nu\,d\mathcal{H}^{n-1}\\ &\leq \|\mathbf{F}\|_{L^\infty(S_u(x_0,t_0))}\mathcal{H}^{n-1}(\partial S_u(x_0,t_0))\\ &\leq C(n,\lambda,\Lambda,\rho)\|\mathbf{F}\|_{L^\infty(S_u(x_0,t_0))}t_0^{\frac{n-2}{2}+\frac{\alpha}{\alpha+1}}.\end{aligned}$$

The lemma is proved. □

For later references, we record here an easy consequence of the Hölder inequality and the volume estimates in Lemma 3.1.

Remark 3.7. Assume that u and Ω satisfy (2.4)–(2.7) and $\mu \in L^q(\Omega)$ where $q > n/2$. Then, for all $x_0 \in \overline{\Omega}$ and $t_0 > 0$, we have

$$|\mu|(S_u(x_0,t_0)) \leq C(n,q,\lambda,\Lambda,\rho)\|\mu\|_{L^q(S_u(x_0,t_0))}t_0^{\frac{n}{2}-\frac{n}{2q}}.$$

We will use the following Hölder inequality in Lorentz spaces, due to O'Neil [35].

Theorem 3.8 (Hölder inequality in Lorentz spaces)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. Let $f, g : \Omega \longrightarrow \mathbb{R}$ be two Lebesgue-measurable functions. Suppose $1 \leq p_1, p_2, p < \infty$ and $0 < q_1, q_2, q \leq \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. If $\|f\|_{L^{p_1,q_1}(\Omega)} < \infty$ and $\|g\|_{L^{p_2,q_2}(\Omega)} < \infty$, then*

$$\|fg\|_{L^{p,q}(\Omega)} \leq C(p_1,p_2,q_1,q_2)\|f\|_{L^{p_1,q_1}(\Omega)}\|g\|_{L^{p_2,q_2}(\Omega)}.$$

We recall the layer-cake formula (see [27, Lemma 2.75]).

Lemma 3.9 (The layer-cake formula)**.** *Let $E \subset \mathbb{R}^n$ be a Borel set and let $v : E \to \mathbb{R}$ be a measurable function. Assume that $\mu : E \to \mathbb{R}$ is an integrable, nonnegative function. Let us also use μ for the measure with density $\mu(x)$. Then*

$$\int_E |v(x)|\mu(x)\,dx = \int_0^\infty \mu(E \cap \{|v| > t\})\,dt,$$

and for $p \in (1,\infty)$, we have

$$\int_E |v(x)|^p\,dx = p\int_0^\infty t^{p-1}|E \cap \{|v| > t\}|\,dt.$$

The following proposition is our key estimate. It allows us to estimate the distribution function for the Monge–Ampère gradient of the Green's function g via estimates of the distribution function of g.

Proposition 3.10. *Let $\Omega \subset \mathbb{R}^n$ and let $V \subset \Omega$ be open and proper. Let $u \in C^3(\Omega)$ be a convex function satisfying $\det D^2 u \geq \lambda > 0$. Let $g_V(\cdot, y)$ be the Green's function of the linearized Monge–Ampère operator $D_i(U^{ij}D_j)$ in V with pole $y \in V$. Then for all $y \in V$ and constants $k, t > 0$, we have*

$$|\{x \in V : |(D^2u(x))^{-1/2}D_x g_V(x,y)| > t\}| \leq \frac{k}{\lambda t^2} + |\{x \in V : g_V(x,y) > k\}|. \tag{3.6}$$

Proof. Fix $y \in V$. Let

$$\xi := g_V(\cdot, y) \quad \text{and } \phi(x) := (D^2u(x))^{-1/2}D_x g_V(x,y).$$

To simplify, we use this notation

$$\{|\phi| > s\} := \{x \in V : |\phi(x)| > s\}, \quad \{\xi > s\} := \{x \in V : \xi(x) > s\}.$$

For constants $t, k > 0$, we have

$$\{|\phi| > t\} \subset \{\xi > k\} \cup \Big(\{|\phi| > t\} \cap \{\xi \leq k\}\Big). \tag{3.7}$$

We claim that

$$\int_{\{\xi \leq k\}} U^{ij}D_i\xi D_j\xi \, dx = k. \tag{3.8}$$

This identity appears in various forms in [27] such as equations (14.23) and (14.71) there. For the reader's convenience, we include its proof. Indeed, let

$$\xi_k := \min\{\xi, k\}.$$

Then $\xi_k = \xi$ in $\{\xi < k\}$ and $D\xi_k = 0$ in $\{\xi > k\}$. From Proposition 3.4, $\xi_k \in W_0^{1,2}(V) \cap C(\overline{V})$. Remark 3.3 tells that we can use ξ_k as a test function in (1.3), so

$$k = \xi_k(y) = \int_V U^{ij}D_i\xi D_j\xi_k \, dx = \int_{\{\xi \leq k\}} U^{ij}D_i\xi D_j\xi \, dx.$$

Thus, (3.8) is proved as claimed.

Using (3.8) and recalling $|\phi|^2 = (D^2u)^{-1}D\xi \cdot D\xi$, we have

$$\begin{aligned}\big|\{|\phi| > t\} \cap \{\xi \leq k\}\big| &\leq \int_{\{\xi \leq k\}} \frac{|\phi|^2}{t^2}\, dx = \int_{\{\xi \leq k\}} \frac{(D^2u)^{-1}D\xi \cdot D\xi}{t^2}\, dx \\ &= t^{-2}\int_{\{\xi \leq k\}} (\det D^2u)^{-1}U^{ij}D_i\xi D_j\xi \, dx \\ &\leq \frac{k}{\lambda t^2}.\end{aligned} \tag{3.9}$$

The proposition follows from (3.7) and (3.9). □

4. Lorentz space estimates for the Green's function

In this section, we prove Theorems 2.1 and 2.2.

For Theorem 2.1, we first establish geometric controls and measure estimates for the superlevel sets of the Green's function $g_{S_u(x_0,h)}(\cdot, y)$ for compactly contained sections $S_u(x_0, h)$. This lemma extends [27, Theorem 14.11] into the weak L^p space at the end point $p = n/(n-2)$ when $n \geq 3$.

Lemma 4.1. *Let $u \in C^3(\Omega)$ be a convex function satisfying (1.2), where $\Omega \subset \mathbb{R}^n$. Assume $S_u(x_0, 2h) \Subset \Omega$ where $x_0 \in \Omega$ and $h > 0$. Let $g_{S_u(x_0,h)}(\cdot, y)$ be the Green's function of the linearized Monge–Ampère operator $D_i(U^{ij}D_j)$ in $S_u(x_0, h)$ with pole $y \in S_u(x_0, h)$. There exist constants $\eta(n, \lambda, \Lambda) \in (0, 1)$ and $\tau_0(n, \lambda, \Lambda) > 1$ such that the following statements hold.*

(i) *For all $y \in S_u(x_0, h)$, we have $S_u(y, 2\eta h) \Subset S_u(x_0, 2h)$ and for all $t > \tau_0 h^{-\frac{n-2}{2}}$,*

$$\{x \in S_u(x_0, h) : g_{S_u(x_0,h)}(x, y) > t\} \subset \begin{cases} S_u(y, 2\eta h 2^{-t/\tau_0}) & \text{if } n = 2,\\ S_u(y, (4\tau_0)^{\frac{2}{n-2}} t^{-\frac{2}{n-2}}) & \text{if } n \geq 3.\end{cases} \tag{4.1}$$

(ii) *For all $y \in S_u(x_0, h)$ and $t > 0$, we have*

$$|\{x \in S_u(x_0, h) : g_{S_u(x_0,h)}(x, y) > t\}| \leq \begin{cases} C(\lambda, \Lambda) h 2^{-t/\tau_0} & \text{if } n = 2,\\ C(n, \lambda, \Lambda) t^{-\frac{n}{n-2}} & \text{if } n \geq 3.\end{cases} \tag{4.2}$$

As a consequence, if $n \geq 3$, then

$$\|g_{S_u(x_0,h)}(\cdot, y)\|_{L^{\frac{n}{n-2},\infty}(S_u(x_0,h))} \leq C(n, \lambda, \Lambda).$$

Proof. The inclusion property of interior sections (Theorem 3.2) shows that there is a constant $\eta = \eta(n, \lambda, \Lambda) \in (0, 1)$ such that $S_u(y, 2\eta h) \Subset S_u(x_0, 2h)$ for any $y \in S_u(x_0, h)$. Let

$$\tilde{g}(x) := g_{S_u(x_0,2h)}(x, y) - g_{S_u(x_0,h)}(x, y).$$

As observed at the beginning of the proof of [27, Lemma 14.7], we have $\tilde{g} \in W^{2,n}_{\text{loc}}(S_u(x_0, h)) \cap C(\overline{S_u(x_0, h)})$ and it satisfies

$$U^{ij} D_{ij} \tilde{g} = 0 \quad \text{in } S_u(x_0, h) \quad \text{and } \tilde{g} \geq 0 \quad \text{on } \partial S_u(x_0, h).$$

The Aleksandrov–Bakelman–Pucci (ABP) maximum principle (see [13, Theorem 9.1]) shows that $\tilde{g} \geq 0$ in $S_u(x_0, h)$. Thus,

$$g_{S_u(x_0,2h)}(x, y) \geq g_{S_u(x_0,h)}(x, y) \geq 0 \quad \text{for all } x \in S_u(x_0, h).$$

Consequently, we have for all $y \in S_u(x_0, h)$ and $t > 0$,

$$\{x \in S_u(x_0, h) : g_{S_u(x_0,h)}(x, y) > t\} \subset \{x \in S_u(x_0, 2h) : g_{S_u(x_0,2h)}(x, y) > t\}. \tag{4.3}$$

To simplify, we denote $S := S_u(x_0, 2h)$. Fix $y \in S_u(x_0, h)$. By [27, Lemma 14.10], we have

$$\{x \in S : g_S(x, y) > \tau(2h)^{-\frac{n-2}{2}}\} \subset \begin{cases} S_u(y, 2\eta h 2^{-\tau/\tau_0}) & \text{if } n = 2,\\ S_u(y, 2(4\tau_0 \tau^{-1})^{\frac{2}{n-2}} h) & \text{if } n \geq 3,\end{cases}$$

for all $\tau > \tau_0 = C_1 \eta^{-n/2} > 1$, where $C_1 = C_1(n, \lambda, \Lambda) > 0$ is large.

It follows that

$$\{x \in S : g_S(x, y) > t\} \subset \begin{cases} S_u(y, 2\eta h 2^{-t/\tau_0}) & \text{if } n = 2,\\ S_u(y, (4\tau_0)^{\frac{2}{n-2}} t^{-\frac{2}{n-2}}) & \text{if } n \geq 3\end{cases} \quad \text{if } t > \tau_0 h^{-\frac{n-2}{2}}. \tag{4.4}$$

From (4.3) and (4.4), we obtain (4.1). Part (i) is proved.

From the volume estimates of sections in Lemma 3.1, we obtain for some $C_2(n, \lambda, \Lambda) > 0$ and all $t > \tau_0 h^{-\frac{n-2}{2}}$,

$$|\{x \in S : g_S(x, y) > t\}| \leq \begin{cases} |S_u(y, 2\eta h 2^{-t/\tau_0})| \leq C_2 h 2^{-t/\tau_0} & \text{if } n = 2,\\ |S_u(y, (4\tau_0)^{\frac{2}{n-2}} t^{-\frac{2}{n-2}})| \leq C_2 t^{-\frac{n}{n-2}} & \text{if } n \geq 3.\end{cases} \tag{4.5}$$

Consider $n=2$. If $0<t\le\tau_0$, then using Lemma 3.1, we have

$$|\{x\in S: g_S(x,y)>t\}|\le |S|\le C_3(\lambda)h\le 2C_3h2^{-t/\tau_0}. \tag{4.6}$$

From (4.3), (4.5) and (4.6), by choosing $C=\max\{C_2,2C_3\}$, we obtain (4.2) for $n=2$.

Consider now $n\ge 3$. If $0<t\le \tau_0 h^{-\frac{n-2}{2}}$, then using Lemma 3.1, we have

$$|\{x\in S: g_S(x,y)>t\}|\le |S|\le C_4(n,\lambda)h^{n/2}\le C_4(n,\lambda,\Lambda)t^{-\frac{n}{n-2}}. \tag{4.7}$$

From (4.3), (4.5) and (4.7), by choosing $C=\max\{C_2,C_4\}$, we obtain (4.2) for $n\ge 3$. Part (ii) is proved. The proof of the lemma is complete. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Denote $S_h:=S_u(x_0,h)$. Fix $y\in S_h$. Let

$$\xi(x):=g_{S_h}(x,y)\quad \text{and } \phi(x):=(D^2u(x))^{-1/2}D_x g_{S_h}(x,y).$$

Step 1: We first prove (2.1) from which (2.2) follows.

We will use Proposition 3.10 so it remains to estimate the measure $|\{x\in S_h:\xi(x)>k\}|$ from above and then optimize over k.

Combining Proposition 3.10 and Lemma 4.1, we obtain for all $k>0$ and $t>0$

$$\big|\{x\in S_h: |\phi(x)|>t\}\big|\le \begin{cases}\frac{k}{\lambda t^2}+C_1(\lambda,\Lambda)h2^{-k/\tau_0} & \text{if } n=2,\\ \frac{k}{\lambda t^2}+C_1(n,\lambda,\Lambda)k^{-\frac{n}{n-2}} & \text{if } n\ge 3,\end{cases} \tag{4.8}$$

where $\tau_0(\lambda,\Lambda)>1$.

Letting in (4.8)

$$k=\begin{cases}h+2\log_{2^{1/\tau_0}}\max\{t,1\} & \text{if } n=2,\\ t^{\frac{n-2}{n-1}} & \text{if } n\ge 3,\end{cases}$$

we obtain (2.1).

Step 2: We prove (2.3) for $p\in\big(1,\frac{n}{n-1}\big)$.

Consider $n\ge 3$. Let χ_{S_h} be the characteristic function of S_h. By the Hölder inequality in Lorentz spaces (Theorem 3.8) and (2.2), we have

$$\begin{aligned}\int_{S_h}|\phi|^p\,dx=\|\phi\|^p_{L^{p,p}(S_h)}&\le C_2(n,p)\|\phi\|^p_{L^{\frac{n}{n-1},\infty}(S_h)}\|\chi_{S_h}\|^p_{L^{\frac{np}{n+p-np},p}(S_h)}\\ &\le C_3(n,\lambda,\Lambda,p)\int_0^\infty t^{p-1}|\{x\in S_h:|\chi_{S_h}(x)|>t\}|^{\frac{n+p-np}{n}}\,dt\\ &=C_3\int_0^1 t^{p-1}|S_h|^{\frac{n+p-np}{n}}\,dt\\ &\le C_4(n,\lambda,\Lambda,p)h^{\frac{n}{2}-\frac{n-1}{2}p},\end{aligned} \tag{4.9}$$

where the last inequality comes from Lemma 3.1. This proves (2.3) for $n\ge 3$.

Consider now $n=2$. Then $p\in(1,2)$. From (4.8), and for $k>0$ and $q\in(1,\infty)$ to be chosen, we have

$$|\{x\in S_h:|\phi(x)|>t\}|\le \frac{k}{\lambda t^2}+C_1(\lambda,\Lambda)2^{-k/\tau_0}h\le\frac{k}{\lambda t^2}+C_5(\lambda,\Lambda,q)k^{-q}h.$$

Let $k = t^{\frac{2}{1+q}} h^{\frac{1}{1+q}}$ so that $kt^{-2} = k^{-q} h$. Then

$$|\{x \in S_h : |\phi(x)| > t\}| \leq C_6(\lambda, \Lambda, q) h^{\frac{1}{1+q}} t^{\frac{-2q}{1+q}}.$$

By the layer-cake formula (Lemma 3.9), we have for every $\tau > 0$

$$\begin{aligned}
\int_{S_h} |\phi|^p \, dx &= \int_{\{x \in S_h : |\phi(x)| \leq \tau\}} |\phi|^p \, dx + \int_{\{x \in S_h : |\phi(x)| > \tau\}} |\phi|^p \, dx \\
&\leq \tau^p |S_h| + p \int_\tau^\infty t^{p-1} \big|\{x \in S_h : |\phi(x)| > t\}\big| \, dt \\
&\leq C_7(\lambda, \Lambda) \tau^p h + C_8(\lambda, \Lambda, p, q) h^{\frac{1}{1+q}} \int_\tau^\infty t^{p-1-\frac{2q}{1+q}} \, dt.
\end{aligned}$$

We may choose $q = \frac{p+1}{2-p}$ so that $p - \frac{2q}{1+q} < 0$ (due to $p \in (1, 2)$) and the integral above converges. Then,

$$\int_{S_h} |\phi|^p \, dx \leq C_7 \tau^p h + C_9(\lambda, \Lambda, p) h^{\frac{1}{1+q}} \tau^{p-\frac{2q}{1+q}}.$$

Setting $\tau = h^{-1/2}$, we obtain (2.3) for $n = 2$. This completes the proof. $\square$

For Theorem 2.2, extending [27, Theorem 14.22] into the weak L^p space at the end point $p = \frac{n}{n-2}$, we establish the following geometric controls on the superlevel sets and distribution function estimates for the Green's function g_V. Here, V is no longer required to be compactly contained in the domain but we need suitable global conditions on the domain and the Monge–Ampère potential.

Lemma 4.2. *Let u and Ω satisfy (2.4)–(2.7). Let $V \subset \Omega$ be open, proper, and contained in a section $S_u(x_0, h)$ of u of height $h > 0$. Let $g_V(\cdot, y)$ be the Green's function of the linearized Monge–Ampère operator $D_i(U^{ij} D_j)$ in V with pole $y \in V$.*

(i) *There exist positive constants $C_*(n, \lambda, \Lambda, \rho)$, $\bar{C}(\lambda, \Lambda, \rho)$ such that for all $y \in V$, we have*

$$\{x \in V : g_V(x, y) > t\} \subset \begin{cases} S_u(y, h2^{-t/\bar{C}}) & \text{if } n = 2, \\ S_u(y, C_* t^{-\frac{2}{n-2}}) & \text{if } n \geq 3 \end{cases} \quad \text{if } t > C_*(n, \lambda, \Lambda, \rho). \tag{4.10}$$

(ii) *For all $y \in V$ and all $t > 0$, we have*

$$|\{x \in V : g_V(x, y) > t\}| \leq \begin{cases} C(\lambda, \Lambda, \rho) h 2^{-t/C_1} & \text{if } n = 2, \\ C(n, \lambda, \Lambda, \rho) t^{-\frac{n}{n-2}} & \text{if } n \geq 3, \end{cases} \tag{4.11}$$

where $C_1 = C_1(\lambda, \Lambda, \rho) > 1$. Consequently,

$$\|g_V(\cdot, y)\|_{L^{\frac{n}{n-2}, \infty}(V)} \leq C(n, \lambda, \Lambda, \rho) \quad \text{if } n \geq 3.$$

Proof. From the bound on the Green's function [27, Theorem 14.21], we have

$$\max_{x \in \partial S_u(y, t_0)} g_\Omega(x, y) \leq \begin{cases} C_1(n, \lambda, \Lambda, \rho) \log_2(1/t_0) & \text{if } n = 2, \\ C_1(n, \lambda, \Lambda, \rho) t_0^{-\frac{n-2}{2}} & \text{if } n \geq 3, \end{cases}$$

for all $y \in \Omega$ and $0 < t_0 < c = c(n, \lambda, \Lambda, \rho) < 1$. Therefore, for $0 < s < c$, we have

$$S_u(y, s) \supset \begin{cases} \{x \in \Omega : g_\Omega(x, y) > C_1(\lambda, \Lambda, \rho) \log_2(1/s)\} & \text{if } n = 2, \\ \{x \in \Omega : g_\Omega(x, y) > C_1(n, \lambda, \Lambda, \rho) s^{-\frac{n-2}{2}}\} & \text{if } n \geq 3. \end{cases} \tag{4.12}$$

We first prove (4.11) for the case $n \geq 3$. Fix $y \in V$. As in (4.3), we have

$$\{x \in V : g_V(x,y) > t\} \subset \{x \in \Omega : g_\Omega(x,y) > t\} \quad \text{for all } t > 0.$$

From (4.12), we see that for $t > C_1 c^{-\frac{n-2}{2}}$,

$$\{x \in V : g_V(x,y) > t\} \subset \{x \in \Omega : g_\Omega(x,y) > t\} \subset S_u\big(y, \big(t/C_1\big)^{-\frac{2}{n-2}}\big). \tag{4.13}$$

By Lemma 3.1, we have for $t > C_1 c^{-\frac{n-2}{2}}$,

$$|\{x \in V : g_V(x,y) > t\}| \leq \big|S_u\big(y, \big(t/C_1\big)^{-\frac{2}{n-2}}\big)\big| \leq C_2(n,\lambda,\Lambda,\rho) t^{-\frac{n}{n-2}}. \tag{4.14}$$

For $0 < t \leq C_1 c^{-\frac{n-2}{2}}$, recalling $\Omega \subset B_{1/\rho}(0) \subset \mathbb{R}^n$, we have

$$|\{x \in V : g_V(x,y) > t\}| \leq |\Omega| \leq C_3(n,\lambda,\Lambda,\rho) t^{-\frac{n}{n-2}}. \tag{4.15}$$

Combining (4.14) and (4.15), we obtain (4.11) for all $t > 0$ when $n \geq 3$.

We prove (4.11) for the case $n = 2$. As remarked after the statement of Theorem 2.2, Ω is itself a section of u with height comparable to 1. Fix $y \in V$. Note that

$$\{x \in V : g_V(x,y) > t\} \subset \{x \in S_u(x_0,h) \cap \Omega : g_{S_u(x_0,h)\cap\Omega}(x,y) > t\} \quad \text{for all } t > 0.$$

By repeating the proof of [27, Theorem 14.21] for $g_{S_u(x_0,h)\cap\Omega}(x,y)$ instead of $g_\Omega(x,y)$ (for example, in [27, (14.48)], we can replace c by h and c_2 by $c_2 h$), we obtain

$$\max_{x \in \partial S_u(y,t_0)} g_{S_u(x_0,h)\cap\Omega}(x,y) \leq C_4(\lambda,\Lambda,\rho)\log_2(h/t_0)$$

for all $y \in S_u(x_0,h) \cap \Omega$ and $0 < t_0 < ch$.

It follows that for all $y \in V$ and all $t > C_4 \log_2(1/c)$,

$$\{x \in V : g_V(x,y) > t\} \subset \{x \in S_u(x_0,h) \cap \Omega : g_{S_u(x_0,h)\cap\Omega}(x,y) > t\} \subset S_u(y, h2^{-t/C_4}). \tag{4.16}$$

From (4.13) and (4.16), we obtain (4.10), which is a global version of (4.1). Part (i) is proved.

Using the volume estimates for sections (Lemma 3.1), we obtain for $t > C_4 \log_2(1/c)$,

$$|\{x \in V : g_V(x,y) > t\}| \leq |S_u(y, h2^{-t/C_4})| \leq C(\lambda) h 2^{-t/C_4}. \tag{4.17}$$

On the other hand, if $0 < t \leq C_4 \log_2(1/c)$, then

$$|\{x \in V : g_V(x,y) > t\}| \leq |S_u(x_0,h)| \leq C(\lambda) h \leq C_5(\lambda,\Lambda,\rho) h 2^{-t/C_4}. \tag{4.18}$$

Combining (4.17) and (4.18), we obtain (4.11) for all $t > 0$ when $n = 2$. Part (ii) is completely proved and so is the lemma. □

Proof of Theorem 2.2. The proof is similar to that of Theorem 2.1. It uses Proposition 3.10 and Lemma 4.2 instead of Lemma 4.1, so we skip it. □

5. Maximum principles for linearized Monge–Ampère equations

In this section and the next, we present some applications of Theorems 2.1 and 2.2 to the regularity of solutions to linearized Monge–Ampère equations in divergence form. This section focuses on maximum principles for subsolutions to (1.1). They imply uniform estimates for solutions to (1.1) with zero boundary value.

We begin with the case of compactly contained sections.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain and $u \in C^3(\Omega)$ be a convex function satisfying (1.2). Assume $\mathbf{F} \in W^{1,n}_{loc}(\Omega;\mathbb{R}^n)$, $\mu \in L^n_{loc}(\Omega)$, and $S_u(x_0, 2h) \Subset \Omega$ where $x_0 \in \Omega$ and $h > 0$. Assume $v \in W^{2,n}_{loc}(S_u(x_0,h)) \cap C(\overline{S_u(x_0,h)})$ satisfies*

$$-U^{ij}D_{ij}v \leq div\,\mathbf{F} + \mu \qquad in \qquad S_u(x_0,h).$$

(i) *Assume $n \geq 3$, $\mu = 0$, and $(D^2u)^{1/2}\mathbf{F} \in L^{n,1}(S_u(x_0,h);\mathbb{R}^n)$. Then*

$$\sup_{S_u(x_0,h)} v \leq \sup_{\partial S_u(x_0,h)} v + C(n,\lambda,\Lambda)\|(D^2u)^{1/2}\mathbf{F}\|_{L^{n,1}(S_u(x_0,h))}.$$

(ii) *Assume $(D^2u)^{1/2}\mathbf{F} \in L^q(S_u(x_0,h);\mathbb{R}^n)$ for some $q > n$ and there exist $M_0 \geq 0$ and $\varepsilon > 0$ such that for $\mu^+ = \max\{\mu, 0\}$, we have*

$$\mu^+(S_u(z,s)) \leq M_0 s^{\frac{n-2}{2}+\varepsilon} \quad \text{for all sections } S_u(z,s) \Subset S_u(x_0,2h). \tag{5.1}$$

Then

$$\sup_{S_u(x_0,h)} v \leq \sup_{\partial S_u(x_0,h)} v + C_*(n,\lambda,\Lambda,q)\|(D^2u)^{1/2}\mathbf{F}\|_{L^q(S_u(x_0,h))} h^{\frac{q-n}{2q}} + C_\star(n,\lambda,\Lambda,\varepsilon)M_0 h^{\varepsilon}.$$

Proof. Let us denote $S_h := S_u(x_0,h)$. Let $\psi \in W^{2,n}_{\mathrm{loc}}(S_h) \cap W^{1,2}_0(S_h) \cap C(\overline{S_h})$ solve

$$-U^{ij}D_{ij}\psi = \operatorname{div}\mathbf{F} + \mu \quad \text{in } S_h \quad \text{and} \quad \psi = 0 \quad \text{on } \partial S_h;$$

see [13, Theorem 9. 30]. From the assumption, we have

$$-U^{ij}D_{ij}(v-\psi) \leq 0 \quad \text{in } S_h.$$

Since $v - \psi \in W^{2,n}_{\mathrm{loc}}(S_h) \cap C(\overline{S_h})$, the ABP maximum principle gives

$$\sup_{S_h} v - \sup_{S_h}\psi \leq \sup_{S_h}(v-\psi) \leq \sup_{\partial S_h}(v-\psi) = \sup_{\partial S_h} v.$$

Hence,

$$\sup_{S_h} v \leq \sup_{\partial S_h} v + \sup_{S_h}\psi. \tag{5.2}$$

For $y \in S_h$, let $g_{S_h}(x,y)$ be the Green's function of the linearized Monge–Ampère operator $D_i(U^{ij}D_j)$ in S_h with pole y. Using the representation formula (3.1) and integration by parts, we find

$$\begin{aligned}\psi(y) = \int_{S_h} g_{S_h}(\cdot,y)(\operatorname{div}\mathbf{F} + \mu)\,dx &= -\int_{S_h} D_x g_{S_h}(x,y)\cdot\mathbf{F}(x)\,dx + \int_{S_h} g_{S_h}(\cdot,y)\mu\,dx \\ &\leq \int_{S_h} |(D^2u)^{-1/2}D_x g_{S_h}(x,y)\cdot(D^2u)^{1/2}\mathbf{F}|\,dx \\ &\quad + \int_{S_h} g_{S_h}(\cdot,y)\mu\,dx := \psi_1(y) + \psi_2(y).\end{aligned} \tag{5.3}$$

We prove part (i) when $n \geq 3$ and $\mu = 0$. By Theorem 2.1 and the Hölder inequality in Lorentz spaces (Theorem 3.8), we have

$$\begin{aligned}\psi(y) \leq \psi_1(y) &\leq C_0(n)\|(D^2u)^{-1/2}D_x g_{S_h}(\cdot, y)\|_{L^{\frac{n}{n-1},\infty}(S_h)}\|(D^2u)^{1/2}\mathbf{F}\|_{L^{n,1}(S_h)} \\ &\leq C_1(n,\lambda,\Lambda)\|(D^2u)^{1/2}\mathbf{F}\|_{L^{n,1}(S_h)}.\end{aligned}\tag{5.4}$$

Since $y \in S_h$ is arbitrary, (5.2)–(5.4) establish part (i).

We prove part (ii) by estimating $\psi_1(y)$ and $\psi_2(y)$. Recall from (5.3) that

$$\psi_1(y) = \int_{S_h} |(D^2u)^{-1/2}D_x g_{S_h}(x,y)\cdot (D^2u)^{1/2}\mathbf{F}|\,dx.$$

Since $q > n$, we have $\frac{q}{q-1} < \frac{n}{n-1}$. Applying the Hölder inequality and Theorem 2.1, we have

$$\begin{aligned}\psi_1(y) &\leq \|(D^2u)^{-1/2}D_x g_{S_h}(\cdot, y)\|_{L^{\frac{q}{q-1}}(S_h)}\|(D^2u)^{1/2}\mathbf{F}\|_{L^q(S_h)} \\ &\leq C_2(n,\lambda,\Lambda,q)\|(D^2u)^{1/2}\mathbf{F}\|_{L^q(S_h)}h^{\frac{q-n}{2q}}.\end{aligned}\tag{5.5}$$

We estimate

$$\psi_2(y) = \int_{S_h} g_{S_h}(\cdot, y)\mu\,dx.$$

By Lemma 4.1, there are constants $\eta(n,\lambda,\Lambda) \in (0,1)$ and $\tau_0(n,\lambda,\Lambda) > 1$ such that for any $y \in S_h$, we have $S_u(y, 2\eta h) \Subset S_u(x_0, 2h)$ and

$$\{x \in S_h : g_{S_h}(x,y) > t\} \subset \begin{cases} S_u(y, 2\eta h 2^{-t/\tau_0}) & \text{if } n = 2, \\ S_u(y, (4\tau_0)^{\frac{2}{n-2}}t^{-\frac{2}{n-2}}) & \text{if } n \geq 3 \end{cases} \quad \text{if } t > \tau_0 h^{-\frac{n-2}{2}}.\tag{5.6}$$

Thus, by choosing $\tau_1 = \tau_1(n,\lambda,\Lambda)$ large enough, we see that all sections in (5.6) are contained in $S_u(y, \eta h) \Subset S_u(x_0, 2h)$ when $t > \tau_1 h^{-\frac{n-2}{2}} := T$.

Consider $n \geq 3$. Using the layer-cake formula (Lemma 3.9) and recalling (5.1) and that $g_{S_h}(x,y) \geq 0$, we can estimate

$$\begin{aligned}\psi_2(y) &\leq \int_{S_h} g_{S_h}(x,y)\mu^+(x)\,dx \\ &= \int_0^\infty \mu^+\big(\{x \in S_h : g_{S_h}(x,y) > t\}\big)\,dt \\ &= \int_0^T \mu^+\big(\{x \in S_h : g_{S_h}(x,y) > t\}\big)\,dt + \int_T^\infty \mu^+\big(\{x \in S_h : g_{S_h}(x,y) > t\}\big)\,dt \\ &\leq T\mu^+(S_h) + \int_T^\infty \mu^+\big(S_u(y, (4\tau_0)^{\frac{2}{n-2}}t^{-\frac{2}{n-2}})\big)\,dt \\ &\leq M_0 T h^{\frac{n-2}{2}+\varepsilon} + C_3 M_0 \int_T^\infty t^{-\frac{2}{n-2}(\frac{n-2}{2}+\varepsilon)}\,dt \\ &\leq M_0 T h^{\frac{n-2}{2}+\varepsilon} + C_4 M_0 T^{-\frac{2\varepsilon}{n-2}} \\ &\leq C_5(n,\lambda,\Lambda,\varepsilon)M_0 h^{\varepsilon}.\end{aligned}\tag{5.7}$$

Consider $n = 2$. Then $T = \tau_1$, and as above, we have

$$\begin{aligned}\psi_2(y) &\le T\mu^+(S_h) + \int_T^\infty \mu^+\big(S_u(y, 2\eta h 2^{-t/\tau_0})\big)\,dt \\ &\le M_0 T h^\varepsilon + C_6 h^\varepsilon M_0 \int_T^\infty 2^{-t\varepsilon/\tau_0}\,dt \\ &\le C_7(\lambda, \Lambda, \varepsilon) M_0 h^\varepsilon.\end{aligned} \tag{5.8}$$

Combining (5.3) with (5.5), (5.7), and (5.8), we obtain the asserted estimate in part (ii). The proposition is proved. □

Remark 5.2. For a nonnegative Radon measure μ, we define the truncated Riesz potential I_u^μ with respect to u by

$$I_u^\mu(y, h) = \int_0^h \frac{\mu(S_u(y,s))}{s^{\frac{n}{2}}} ds. \tag{5.9}$$

Then, the estimates for $\psi_2(y)$ in the proof of Proposition 5.1 (ii) can be expressed using $I_u^{\mu^+}$. Indeed, by the change of variables

$$s = \begin{cases} (4\tau_0)^{\frac{2}{n-2}} t^{-\frac{2}{n-2}} & \text{when } n \ge 3, \\ 2\eta h 2^{-t/\tau_0} & \text{when } n = 2, \end{cases}$$

we have in (5.7) and (5.8)

$$\psi_2(y) \le C h^{-\frac{n-2}{2}} \mu^+(S_u(x_0, h)) + C I_u^{\mu^+}(y, \eta h), \tag{5.10}$$

where $C = C(n, \lambda, \Lambda) > 0$ and $\eta = \eta(n, \lambda, \Lambda) \in (0, 1)$ is such that $S_u(y, 2\eta h) \Subset S_u(x_0, 2h)$.

Next, we state a boundary version of Proposition 5.1.

Proposition 5.3. *Assume u and Ω satisfy* (2.4)–(2.7). *Let $V \subset \Omega$ be open, proper, and contained in a section $S_u(x_0, h)$ of u of height $h > 0$. Assume $\mathbf{F} \in W^{1,n}(V; \mathbb{R}^n)$, $\mu \in L^n(\Omega)$, and $v \in W^{2,n}_{loc}(V) \cap C(\overline{V})$ satisfies*

$$-U^{ij} D_{ij} v \le div\, \mathbf{F} + \mu \quad \text{in } V.$$

(i) *Assume $n \ge 3$, $\mu = 0$, and $(D^2u)^{1/2}\mathbf{F} \in L^{n,1}(V; \mathbb{R}^n)$. Then*

$$\sup_V v \le \sup_{\partial V} v + C(n, \lambda, \Lambda, \rho) \big\|(D^2u)^{1/2}\mathbf{F}\big\|_{L^{n,1}(V)}.$$

(ii) *Assume $(D^2u)^{1/2}\mathbf{F} \in L^q(V; \mathbb{R}^n)$ for some $q > n$ and there exist $M_0 \ge 0$ and $\varepsilon > 0$ such that for $\mu^+ = \max\{\mu, 0\}$, we have*

$$\mu^+(S_u(z,s)) \le M_0 s^{\frac{n-2}{2}+\varepsilon} \quad \text{for all sections } S_u(z,s) \subset \overline{\Omega}.$$

Then,

$$\sup_V v \le \sup_{\partial V} v + C_*(n, \lambda, \Lambda, q, \rho) \big\|(D^2u)^{1/2}\mathbf{F}\big\|_{L^q(V)} h^{\frac{q-n}{2q}} + C_\star(n, \lambda, \Lambda, \rho, \varepsilon) M_0 h^\varepsilon.$$

Proof. Our proof is similar to that of Proposition 5.1. It uses the representation formula (3.1), Lemma 4.2 instead of Lemma 4.1 for part (ii), and Theorem 2.2 instead of Theorem 2.1, so we skip it. □

As discussed after the statement of Theorem 2.2, the above maximum principle is applicable when $V = \Omega$ where the height is bounded by a constant $M(n, \lambda, \Lambda, \rho)$.

6. Interior regularity for linearized Monge–Ampère equations

In this section, we prove an interior Harnack inequality and interior Hölder estimates for solutions to (1.1).

6.1. Interior Harnack inequality.

In this subsection, we prove Theorem 2.3. The main tools are uniform bounds for solutions to (1.1) with zero boundary value and the Caffarelli–Gutiérrez Harnack inequality for the linearized Monge–Ampère equation [6, Theorem 5].

Theorem 6.1 (Caffarelli–Gutiérrez Harnack inequality)**.** *Let $u \in C^2(\Omega)$ be a convex function satisfying* (1.2) *in a domain Ω in $\mathbb{R}^n$. Let $v \in W^{2,n}_{loc}(\Omega)$ be a nonnegative solution of the linearized Monge–Ampère equation $U^{ij}D_{ij}v = 0$ in a section $S_u(x_0, 2h) \Subset \Omega$. Then*

$$\sup_{S_u(x_0,h)} v \leq C(n, \lambda, \Lambda) \inf_{S_u(x_0,h)} v.$$

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. Denote $S_h := S_u(x_0, h)$ and $S_{h/2} := S_u(x_0, h/2)$. Let $w \in W^{2,n}_{\mathrm{loc}}(S_h) \cap C(\overline{S_h})$ satisfy

$$U^{ij}D_{ij}w = \operatorname{div} \mathbf{F} + \mu \quad \text{ in } S_h \quad \text{ and } \quad w = 0 \quad \text{ on } \partial S_h;$$

see [13, Theorem 9.30]. Notice that $v - w \in W^{2,n}_{\mathrm{loc}}(S_h) \cap C(\overline{S_h})$ satisfies

$$U^{ij}D_{ij}(v - w) = 0 \quad \text{ in } S_h \quad \text{ and } v - w = v \geq 0 \text{ on } \partial S_h.$$

Hence, the ABP maximum principle shows that $v-w \geq 0$ in $\overline{S_h}$. Then, applying the Caffarelli–Gutiérrez Harnack inequality (Theorem 6.1) to $v - w$, we find

$$\sup_{S_{h/2}}(v - w) \leq C_1(n, \lambda, \Lambda) \inf_{S_{h/2}}(v - w).$$

Thus

$$\sup_{S_{h/2}} v \leq (1 + C_1)\big(\inf_{S_{h/2}} v + \|w\|_{L^\infty(S_h)}\big). \tag{6.1}$$

From Proposition 5.1 applied to w and $-w$, we have

$$\|w\|_{L^\infty(S_h)} \leq \begin{cases} C_2(n, \lambda, \Lambda)\|(D^2u)^{1/2}\mathbf{F}\|_{L^{n,1}(S_h)} & \text{in part (i)}, \\ C_3\|(D^2u)^{1/2}\mathbf{F}\|_{L^q(S_h)}h^{\frac{q-n}{2q}} + C_4M_0h^\varepsilon & \text{in part (ii)}, \end{cases}$$

where $C_2 = C_2(n, \lambda, \Lambda) > 0$, $C_3 = C_3(n, \lambda, \Lambda, q) > 0$, and $C_4 = C_4(n, \lambda, \Lambda, \varepsilon) > 0$. Combining this with (6.1) completes the proof of the theorem. □

6.2. Interior Hölder estimates.

In this subsection, using Theorem 2.3, we give a new proof of the following theorem, first obtained by Wang [45, Theorem 1.5] and Cui–Wang–Zhou [8, Theorem 1.2], on interior Hölder estimates for (1.1).

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain and $u \in C^3(\Omega)$ be a convex function satisfying* (1.2)*. Let $S_u(x_0, 2h_0) \Subset \Omega$ where $x_0 \in \Omega$ and $h_0 > 0$ and $v \in W^{2,n}_{loc}(S_u(x_0, 2h_0)) \cap C(\overline{S_u(x_0, 2h_0)})$ be a solution to*

$$U^{ij}D_{ij}v = div\mathbf{F} + \mu \quad in \quad S_u(x_0, 2h_0),$$

where $\mathbf{F} \in W^{1,n}_{loc}(\Omega;\mathbb{R}^n)$ *with* $(D^2u)^{1/2}\mathbf{F} \in L^q(S_u(x_0,2h_0);\mathbb{R}^n)$ *for some* $q > n$, *and* $\mu \in L^n(S_u(x_0,2h_0))$ *and there exist* $M_0 \geq 0$ *and* $\varepsilon > 0$ *such that*

$$|\mu|(S_u(z,s)) \leq M_0 s^{\frac{n-2}{2}+\varepsilon} \quad \textit{for all sections } S_u(z,s) \Subset S_u(x_0,2h_0).$$

Letting

$$M := \sup_{x,y\in S_u(x_0,h_0)} |Du(x)-Du(y)| \quad \textit{and} \quad \tilde{M} := h_0^{\frac{q-n}{2q}} \|(D^2u)^{1/2}\mathbf{F}\|_{L^q(S_u(x_0,h_0))} + M_0 h_0^{\varepsilon}.$$

Then

$$|v(y)-v(z)| \leq C\big(\|v\|_{L^\infty(S_u(x_0,h_0))} + \tilde{M}\big) h_0^{-\gamma} M^\gamma |y-z|^\gamma \qquad \textit{for all } y,z \in S_u(x_0,h_0/2), \tag{6.2}$$

where $C = C(n,\lambda,\Lambda,q,\varepsilon) > 0$ *and* $\gamma = \gamma(n,\lambda,\Lambda,q,\varepsilon) \in (0,1)$.

Remark 6.3. Some remarks on Theorem 6.2 are in order.

(1) If $\mathbf{F} = 0$ in Theorem 6.2, the term $\|v\|_{L^\infty(S_u(x_0,h_0))}$ in (6.2) can be replaced by $\|v\|_{L^p(S_u(x_0,h_0))}$ for any $p > 0$ in [8].

(2) From Caffarelli's $C^{1,\alpha}$ estimates [5] (see also [27, Corollary 5.22]), we have

$$M := \sup_{x,y\in S_u(x_0,h_0)} |Du(x)-Du(y)| \leq C_1(n,\lambda,\Lambda,\mathrm{diam}(S_u(x_0,2h_0)))h_0^{-\alpha}, \tag{6.3}$$

where $\alpha = \alpha(n,\lambda,\Lambda) > 0$.

Proof of Theorem 6.2. We divide the proof into two steps.

Step 1: Oscillation decay of solutions to (1.1) in small sections. We claim that there exist $C > 0$ and $\gamma \in (0,1)$ depending only on $n,\lambda,\Lambda,q,\varepsilon$ such that for any $0 < h \leq h_0$, we have

$$\operatorname*{osc}_{S_u(x_0,h)} v \leq C\big(\frac{h}{h_0}\big)^\gamma\big(\operatorname*{osc}_{S_u(x_0,h_0)} v + h_0^{\frac{q-n}{2q}} \|(D^2u)^{1/2}\mathbf{F}\|_{L^q(S_u(x_0,h_0))} + M_0 h_0^{\varepsilon}\big). \tag{6.4}$$

Denote $S_h := S_u(x_0,h)$. Since $\bar{v} := v - \inf_{S_h} v$ is a nonnegative solution to

$$U^{ij}D_{ij}\bar{v} = \operatorname{div}\mathbf{F} + \mu \quad \text{in} \quad S_h,$$

we can apply Theorem 2.3 to obtain for some $C(n,\lambda,\Lambda) > 3$

$$C^{-1}\sup_{S_{h/2}} \bar{v} \leq \inf_{S_{h/2}} \bar{v} + C_1(n,\lambda,\Lambda,q)\big\|(D^2u)^{1/2}\mathbf{F}\big\|_{L^q(S_h)} h^{\frac{q-n}{2q}} + C_2(n,\lambda,\Lambda,\varepsilon)M_0 h^{\varepsilon}.$$

Thus,

$$\begin{aligned}\operatorname*{osc}_{S_{h/2}} v = \sup_{S_{h/2}} \bar{v} - \inf_{S_{h/2}} \bar{v} &\leq (1-C^{-1})\sup_{S_{h/2}} \bar{v} + C_1 h^{\frac{q-n}{2q}} \big\|(D^2u)^{1/2}\mathbf{F}\big\|_{L^q(S_h)} + C_2 M_0 h^{\varepsilon} \\ &\leq \beta \operatorname*{osc}_{S_h} v + C_1 h^{\frac{q-n}{2q}} \big\|(D^2u)^{1/2}\mathbf{F}\big\|_{L^q(S_h)} h^{\frac{q-n}{2q}} + C_2 M_0 h^{\varepsilon},\end{aligned} \tag{6.5}$$

where we can choose

$$\beta = \beta(n,\lambda,\Lambda,q,\varepsilon) \in (1-C^{-1},1) \quad \text{such that} \quad \min\big\{\beta 2^{\frac{q-n}{2q}}, \beta 2^{\varepsilon}\big\} > 1.$$

A standard iteration argument (see, for example, the proof of [27, Theorem 12.13]) gives (6.4) with $\gamma = \log(1/\beta)\log 2 \in (0,1)$ as claimed.

Step 2: Interior Hölder estimates. Let $y,z \in S_{h_0/2}$. By the inclusion property of sections (Theorem 3.2), there exists $\tau = \tau(n,\lambda,\Lambda) > 0$, such that $S_u(y,2\tau h_0) \subset S_{h_0} \Subset \Omega$. Then, we obtain the same estimates as (6.4) in $S_u(y,h)$ for $h \leq \tau h_0$. We consider two cases:

Case 1: $z \in S_u(y, \tau h_0)$. Choose $0 < r \le \tau h_0$, such that $z \in S_u(y,r) \setminus S_u(y, r/2)$. By the definition of sections and the mean value theorem, we have
$$\frac{r}{2} \le u(z) - u(y) - Du(y)\cdot(z-y) \le \sup_{S_{h_0}} |Du(\cdot) - Du(y)| \cdot |y-z| \le M|y-z|.$$
Thus $2M|y-z| \ge r$. Then, using (6.4),
$$\begin{aligned} |v(y)-v(z)| &\le \underset{S_u(y,r)}{\operatorname{osc}} v \le C\big(\|v\|_{L^\infty(S_u(y,\tau h_0))} + \tilde{M}\big)\tau^{-\gamma} h_0^{-\gamma} r^\gamma \\ &\le C_3\big(\|v\|_{L^\infty(S_{h_0})} + \tilde{M}\big) h_0^{-\gamma} M^\gamma |y-z|^\gamma, \end{aligned} \tag{6.6}$$
where $C_3 = C_3(n, \lambda, \Lambda, q, \varepsilon)$ can be chosen such that $C_3 > 2\tau^{-\gamma}$.

Case 2: $z \notin S_u(y, \tau h_0)$. As in Case 1, we have $\tau h_0 \le M|y-z|$. Then, clearly (6.6) holds from $|v(y) - v(z)| \le 2\|v\|_{L^\infty(S_{h_0})}$ and the choice of C_3.

Combining these two cases, we obtain (6.2) as asserted. ☐

7. Global Hölder estimates for linearized Monge–Ampère equations

In this section, we give a proof of Theorem 2.4.

First, we prove Hölder estimates for solutions to (1.1) at the boundary.

Proposition 7.1. *Assume that u and Ω satisfy (2.4)–(2.7). Let $\mathbf{F} \in W^{1,n}(\Omega; \mathbb{R}^n)$, $\mu \in L^n(\Omega)$, and $v \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ be the solution to*
$$U^{ij} D_{ij} v = div\,\mathbf{F} + \mu \quad \text{in } \Omega, \quad v = \phi \quad \text{on } \partial\Omega,$$
where $\phi \in C^\alpha(\partial\Omega)$ for some $\alpha \in (0,1)$. Assume $(D^2u)^{1/2}\mathbf{F} \in L^q(\Omega;\mathbb{R}^n)$ for some $q > n$, and there exist $M_0 \ge 0$ and $\varepsilon_0 > 0$ such that
$$|\mu|(S_u(z,s)) \le M_0 s^{\frac{n-2}{2}+\varepsilon_0} \quad \text{for all sections } S_u(z,s) \subset \overline{\Omega}.$$
Let
$$\alpha_0 := \min\Big\{\alpha, \frac{3(q-n)}{8q}, \frac{3\varepsilon_0}{4}\Big\}, \quad \alpha_1 := \frac{\alpha_0}{\alpha_0 + 3n}.$$
Then, there exist constants $\delta, C > 0$ depending only on $n, \lambda, \Lambda, \alpha, \rho$, q and ε_0, such that
$$|v(x) - v(x_0)| \le C\Big(\|\phi\|_{C^\alpha(\partial\Omega)} + \|(D^2u)^{1/2}\mathbf{F}\|_{L^q(\Omega)} + M_0\Big)|x-x_0|^{\alpha_1} \text{ if } x_0 \in \partial\Omega,\ x \in \Omega \cap B_\delta(x_0).$$

Proof. The proof follows closely that of [27, Proposition 14.32] using maximum principles and barriers. Since our setting is different, we include the details for the reader's convenience.

Since $\alpha_0 \le \alpha$, we have $\|\phi\|_{C^{\alpha_0}(\partial\Omega)} \le C_0(\alpha, \alpha_0, \rho)\|\phi\|_{C^\alpha(\partial\Omega)}$. Thus, when proving the desired result, we can replace $\|\phi\|_{C^\alpha(\partial\Omega)}$ by $\|\phi\|_{C^{\alpha_0}(\partial\Omega)}$.

Let $L_u := U^{ij} D_{ij}$. By homogeneity, we can assume that
$$\mathbf{K} := \|v\|_{L^\infty(\Omega)} + \|\phi\|_{C^{\alpha_0}(\partial\Omega)} + \|(D^2u)^{1/2}\mathbf{F}\|_{L^q(\Omega)} + M_0 = 1.$$
By Proposition 5.3 (ii),
$$\|v\|_{L^\infty(\Omega)} \le C(n,\lambda,\Lambda,\rho,q,\varepsilon)\big(\|\phi\|_{C^\alpha(\partial\Omega)} + \|(D^2u)^{1/2}\mathbf{F}\|_{L^q(\Omega)} + M_0\big).$$
Without loss of generality, we assume $x_0 = 0$, $u(0) = 0$, and $Du(0) = 0$. It suffices to show, using barriers, that for all $x \in \Omega \cap B_\delta(0)$,
$$|v(x) - v(0)| \le C|x|^{\alpha_1},$$

for some structural constants δ and C depending only on $n, \lambda, \Lambda, \alpha, \rho$, q and ε_0.

For any $\varepsilon \in (0,1)$, we let

$$h_{\pm} := v - v(0) \pm \varepsilon \pm \frac{6}{\delta_2^3} w_{\delta_2} \quad \text{ in } A := \Omega \cap B_{\delta_2}(0),$$

where $\delta_2 \in (0,1)$ small is to be chosen later and the function w_{δ_2} is defined by

$$w_{\delta_2}(x) = w_{\delta_2}(x', x_n) := M_{\delta_2} x_n + u(x) - \tilde{\delta_2}|x'|^2 - \Lambda^n(\lambda\tilde{\delta_2})^{1-n} x_n^2 \quad \text{ for } x = (x', x_n) \in \overline{\Omega},$$

where

$$\tilde{\delta_2} := \delta_2^3/2 \quad \text{and} \quad M_{\delta_2} := \Lambda^n(\lambda\tilde{\delta_2})^{1-n}.$$

From [27, Lemma 13.7], w_{δ_2} has the following properties for δ_2 small:

$$\begin{cases} U^{ij} D_{ij} w_{\delta_2} \leq -\max\{\tilde{\delta_2} \operatorname{trace}(U), n\Lambda\} \quad \text{ in } \Omega, \\ w_{\delta_2} \geq 0 \quad \text{ in } \overline{\Omega \cap B_{\delta_2}(0)}, \quad \text{ and } \quad w_{\delta_2} \geq \tilde{\delta_2} \quad \text{ on } \Omega \cap \partial B_{\delta_2}(0). \end{cases} \tag{7.1}$$

Note that if $x \in \partial\Omega$ and $|x| \leq \delta_1(\varepsilon) := \varepsilon^{1/\alpha_0}$, then

$$|v(x) - v(0)| = |\phi(x) - \phi(0)| \leq |x|^{\alpha_0} \leq \varepsilon.$$

If $x \in \Omega \cap \partial B_{\delta_2}(0)$, we have $\frac{6}{\delta_2^3} w_{\delta_2}(x) \geq 3$ from (7.1). Moreover, we have

$$|v(x) - v(0) \pm \varepsilon| \leq 2\|v\|_{L^\infty(A)} + \varepsilon \leq 2\mathbf{K} + \varepsilon \leq 3.$$

It follows that if $\delta_2 \leq \delta_1$, then

$$h_- \leq 0 \quad \text{ and } h_+ \geq 0 \quad \text{ on } \partial A.$$

Also from (7.1), we have

$$L_u h_+ \leq \operatorname{div} \mathbf{F} + \mu, \quad L_u h_- \geq \operatorname{div} \mathbf{F} + \mu \quad \text{ in } A.$$

By [27, inclusion (14.59)], we have $A \subset S_u(0, \delta_2^{3/2})$ when δ_2 is small.

Now, applying the maximum principle in Proposition 5.3 (ii) to h_+ and h_-, we obtain

$$\max\{-h_+, h_-\} \leq C_0\big(\|(D^2u)^{1/2}\mathbf{F}\|_{L^q(A)} \delta_2^{\frac{3}{2}\cdot\frac{q-n}{2q}} + M\delta_2^{\frac{3}{2}\cdot\varepsilon_0}\big) \leq C_1 \delta_2^{n\tilde{\beta}} \text{ in } A, \tag{7.2}$$

where $C_1 > 1$ depends only on $n, \lambda, \Lambda, q, \varepsilon_0$, and ρ, and

$$\tilde{\beta} = \min\big\{\frac{3(q-n)}{4nq}, \frac{3\varepsilon_0}{2n}\big\}.$$

Next, from the boundary estimates of u (see [27, Proposition 8.23]), we have for δ_2 small

$$\begin{aligned} w_{\delta_2}(x) \leq M_{\delta_2} x_n + u(x) &\leq M_{\delta_2}|x| + C_2(n, \lambda, \Lambda, \rho)|x|^2|\log|x||^2 \\ &\leq 2M_{\delta_2}|x| \qquad \text{for all } x \in A. \end{aligned} \tag{7.3}$$

Let $\delta_2 = \delta_1$. Combining (7.2) and (7.3) and recalling the definition of h_+ and h_-, we obtain

$$\begin{aligned} |v(x) - v(0)| \leq \varepsilon + \frac{6}{\delta_2^3} w_{\delta_2}(x) + C_1\delta_2^{n\tilde{\beta}} &\leq \varepsilon + \frac{12}{\delta_2^3}\Lambda^n(\lambda\frac{\delta_2^3}{2})^{1-n}|x| + C_1\delta_2^{n\tilde{\beta}} \\ &= \varepsilon + C_3(n, \lambda, \Lambda)\delta_2^{-3n}|x| + C_1\delta_2^{n\tilde{\beta}} \\ &= \varepsilon + C_3\varepsilon^{-3n/\alpha_0}|x| + C_1\delta_2^{n\tilde{\beta}}. \end{aligned}$$

If we further restrict $\varepsilon \le C_1^{-1}$, then

$$C_1\delta_2^{n\tilde{\beta}} = C_1\delta_1^{n\tilde{\beta}} = C_1\varepsilon^{\frac{1}{\alpha_0}n\tilde{\beta}} \le C_1\varepsilon^2 \le \varepsilon.$$

From the previous two estimates, we have for all $\varepsilon \le C_1^{-1}$ and $|x| \le \delta_1 = \varepsilon^{1/\alpha_0}$ that

$$|v(x) - v(0)| \le 2\varepsilon + C_3\varepsilon^{-3n/\alpha_0}|x|. \tag{7.4}$$

We may choose $\varepsilon = |x|^{\frac{\alpha_0}{\alpha_0+3n}}$ where $|x| \le \delta := C_1^{-(\alpha_0+3n)/\alpha_0}$ so that the conditions on ε and x are satisfied. Then, from (7.4), we have

$$|v(x) - v(0)| \le (2 + C_3)|x|^{\frac{\alpha_0}{\alpha_0+3n}} = (2 + C_3)|x|^{\alpha_1} \quad \text{in} \quad \Omega \cap B_\delta(0).$$

The proposition is proved. $\square$

If $x_0 \in \Omega$, denote by

$$\bar{h}(x_0) := \sup\left\{h > 0 : S_u(x_0, h) \subset \Omega\right\}$$

the maximal height of all sections of u centered at x_0 and contained in Ω. $S_u(x_0, \bar{h}(x_0))$ is called the *maximal interior section of u centered at x_0*, and it is tangent to the boundary.

Thanks to Savin's boundary localization theorem [38, Theorem 3.1], we have the following useful properties of the maximal interior sections; see [39] and [27, Proposition 9.2].

Proposition 7.2 (Shape of maximal interior sections)**.** *Let u and Ω satisfy* (2.4)–(2.7). *Assume that for some $y \in \Omega$, the maximal interior section $S_u(y, \bar{h}(y)) \subset \Omega$ is tangent to $\partial\Omega$ at 0; that is, $\partial S_u(y, \bar{h}(y)) \cap \partial\Omega = \{0\}$. If $h := \bar{h}(y) \le c$ where $c = c(n, \lambda, \Lambda, \rho) > 0$ is small, then there exists a small positive constant $\kappa_0(n, \lambda, \Lambda, \rho)$ such that*

$$\begin{cases} Du(y) = ae_n \quad \text{for some } a \in \left[\kappa_0 h^{1/2}, \kappa_0^{-1}h^{1/2}\right], \\ \kappa_0 E_h \subset S_u(y, h) - y \subset \kappa_0^{-1}E_h, \quad \text{and} \quad \kappa_0 h^{1/2} \le \operatorname{dist}(y, \partial\Omega) \le \kappa_0^{-1}h^{1/2}, \end{cases}$$

where $e_n := (0, 0, \cdots, 1) \in \mathbb{R}^n$ and the ellipsoid E_h is given by $E_h = A_h^{-1}B_{h^{1/2}}(0)$, with A_h being a linear transformation on $\mathbb{R}^n$ with the following properties:

$$\det A_h = 1, \quad A_h x = x - \tau_h x_n, \quad \tau_h \cdot e_n = 0, \quad \|A_h^{-1}\| + \|A_h\| \le \kappa_0^{-1}|\log h|.$$

Finally, we are able to prove Theorem 2.4.

Proof of Theorem 2.4. Note that our assumption implies (2.4)–(2.7), where ρ now depends only on $n, \lambda, \Lambda, \|u\|_{C^3(\partial\Omega)}$, R, and the C^3 regularity of $\partial\Omega$; see [27, Proposition 4.7]. Therefore, Proposition 7.1 can be applied to all $x_0 \in \partial\Omega$. Our proof is similar to that of [27, Theorem 13.2]. Since our setting is slightly different, we include the details for the reader's convenience.

By Proposition 5.3 (ii),

$$\|v\|_{L^\infty(\Omega)} \le C(n, \lambda, \Lambda, \rho, q, \varepsilon)\big(\|\phi\|_{C^\alpha(\partial\Omega)} + \|(D^2u)^{1/2}\mathbf{F}\|_{L^q(\Omega)} + M_0\big). \tag{7.5}$$

By homogeneity, we can assume that

$$\mathbf{K} := \|\phi\|_{C^\alpha(\partial\Omega)} + \|(D^2u)^{1/2}\mathbf{F}\|_{L^q(\Omega)} + M_0 = 1.$$

By the global Hölder gradient estimate in Lemma 3.5, we have

$$M := \sup_{x,y\in\overline{\Omega}} |Du(x) - Du(y)| \le C_\star(n, \lambda, \Lambda, \rho). \tag{7.6}$$

Let $c, \kappa_0 \in (0, 1)$ be as in Proposition 7.2 and let $\Omega_s := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > s\}$ for $s > 0$.

Step 1: Hölder estimates in the interior of Ω.

By Proposition 7.2, the maximal interior section $S_u(y, \overline{h}(y))$ of u centered at $y \in \overline{\Omega}_c$ satisfies

$$\bar{h}(y) \geq (c\kappa_0)^2 =: c_1. \tag{7.7}$$

Therefore, from (7.6) (see also (3.5)), we can find $c_2 = c_2(n, \lambda, \Lambda, \rho) \in (0, c_1)$ such that

$$B_{c_2}(y) \subset S_u(y, c_1/8) \subset S_u(y, c_1/2) \Subset \Omega \quad \text{whenever} \quad y \in \overline{\Omega}_c. \tag{7.8}$$

From Theorem 6.2, (7.5), (7.6), and (7.8), we can find $\gamma = \gamma(n, \lambda, \Lambda, q, \varepsilon) \in (0, 1)$ such that

$$|v(z_1) - v(z_2)| \leq C_1(n, \lambda, \Lambda, \rho, q, \varepsilon)|z_1 - z_2|^\gamma \quad \text{for all } z_1, z_2 \in B_{c_2}(y).$$

Repeating the above argument, we can find $c_3(n, \lambda, \Lambda, \rho) \in (0, c_2)$ such that

$$|v(z_1) - v(z_2)| \leq C_2(n, \lambda, \Lambda, \rho, q, \varepsilon)|z_1 - z_2|^\gamma \quad \text{for all } z_1, z_2 \in B_{c_3}(y) \quad \text{where} \quad y \in \Omega_{c_2}.$$

Step 2: Hölder estimates in a maximal interior section whose center is in $\overline{\Omega} \setminus \Omega_{c_2}$.

Let $y \in \Omega$ be such that $\operatorname{dist}(y, \partial\Omega) = r \leq c_2$. Let $S_u(y, \overline{h})$ be the maximal interior section of u centered at y and $y_0 \in \partial\Omega \cap \partial S_u(y, \bar{h})$ where $\bar{h} = \bar{h}(y)$. Then, the arguments in (7.7) and (7.8) give $\bar{h} \leq c$. By Proposition 7.2,

$$\kappa_0 \overline{h}^{1/2} \leq r \leq \kappa_0^{-1} \overline{h}^{1/2}, \text{ and } \kappa_0 E \subset S_u(y, \overline{h}) - y \subset \kappa_0^{-1} E, \tag{7.9}$$

where $E := \overline{h}^{1/2} A_{\overline{h}}^{-1} B_1(0)$ with $A_{\overline{h}}$ being a linear transformation on $\mathbb{R}^n$ such that

$$\|A_{\overline{h}}\| + \|A_{\overline{h}}^{-1}\| \leq \kappa_0^{-1} |\log \overline{h}| \quad \text{ and } \det A_{\overline{h}} = 1. \tag{7.10}$$

From (7.9), (7.10), and

$$(1/8)(S_u(y, \bar{h}) - y) \subset S_u(y, \bar{h}/8) - y,$$

we obtain constants $C_3, \overline{c_2} > 0$ depending only on n, λ, Λ and ρ such that

$$B_{C_3 r |\log r|}(y) \supset S_u(y, \overline{h}) \supset S_u(y, \overline{h}/8) \supset B_{\overline{c_2} \frac{r}{|\log r|}}(y) \supset B_{\overline{c_2} r^2}(y). \tag{7.11}$$

Let $\bar{v} = v - v(y_0)$. Then

$$U^{ij} D_{ij} \bar{v} = \operatorname{div} \mathbf{F} + \mu \quad \text{in} \quad \Omega.$$

By Theorem 6.2, there exists $C_4 = C_4(n, \lambda, \Lambda, q, \varepsilon, \rho)$ such that for all $z_1, z_2 \in S_u(y, \bar{h}/8)$

$$\begin{aligned} |v(z_1) - v(z_2)| &= |\bar{v}(z_1) - \bar{v}(z_2)| \\ &\leq C_4 \big(\|\bar{v}\|_{L^\infty(S_u(y, \bar{h}))} + \|(D^2 u)^{1/2} \mathbf{F}\|_{L^q(\Omega)} \bar{h}^{\frac{q-n}{2q}} + M_0 \bar{h}^\varepsilon \big) \bar{h}^{-\gamma} |z_1 - z_2|^\gamma. \end{aligned} \tag{7.12}$$

We need to estimate $\|\bar{v}\|_{L^\infty(S_u(y, \bar{h}))}$. By (7.11) and Proposition 7.1 applied at $y_0 \in \partial\Omega$,

$$\|\bar{v}\|_{L^\infty(S_u(y, \bar{h}))} \leq C_5(n, \lambda, \Lambda, \rho, \alpha, q, \varepsilon) [\operatorname{diam}(S_u(y, \bar{h}))]^{\alpha_1} \leq C_6 (r |\log r|)^{\alpha_1}, \tag{7.13}$$

where $C_6 = C_6(n, \lambda, \Lambda, \rho, \alpha, q, \varepsilon) > 0$ and $\alpha_1 = \alpha_1(n, \alpha, q, \varepsilon) \in (0, \alpha)$.

From (7.9), (7.11)–(7.13), $q > n$, and $\mathbf{K} \leq 1$, we can find $\beta = \beta(n, \lambda, \Lambda, q, \varepsilon, \alpha) \in (0, \min\{\alpha_1, \gamma\})$ such that

$$|v(z_1) - v(z_2)| \leq C_7(n, \lambda, \Lambda, \rho, q, \varepsilon, \alpha)|z_1 - z_2|^\beta \qquad \text{for all } z_1, z_2 \in S_u(y, \overline{h}/8). \tag{7.14}$$

Step 3: Completion of the proof. Let $x, y \in \Omega$. We show that

$$|v(x) - v(y)| \leq C_8(n, \lambda, \Lambda, \alpha, q, \varepsilon, \rho)|x - y|^\beta. \tag{7.15}$$

By (7.5), it suffices to consider

$$|x - y| < c_3.$$

Let $x_0, y_0 \in \partial\Omega$ be such that $r_x := \mathrm{dist}(x, \partial\Omega) = |x - x_0|$ and $r_y := \mathrm{dist}(y, \partial\Omega) = |y - y_0|$. We may assume $r_x \leq r_y$. Let $\bar{h} = \bar{h}(y)$ be as in Step 2.

Case 1: $r_y \leq c_2$.

If $x \in B_{\overline{c_2} r_y^2}(y)$, then we obtain (7.15) directly from (7.11) and (7.14).

Assume now $x \notin B_{\overline{c_2} r_y^2}(y)$. Then, $|x - y| \geq \overline{c_2} r_y^2$, and

$$|x_0 - y_0| \leq |x - x_0| + |y - y_0| + |x - y| \leq 2r_y + |x - y| \leq \tilde{c_2}|x - y|^{1/2}, \tag{7.16}$$

where $\tilde{c_2} = \tilde{c_2}(n, \lambda, \Lambda, \rho)$. Note that

$$|v(x) - v(y)| \leq |v(x) - v(x_0)| + |v(y) - v(y_0)| + |v(x_0) - v(y_0)|. \tag{7.17}$$

We can apply Proposition 7.1 to get

$$|v(x) - v(x_0)| \leq C_6|x - x_0|^{\alpha_1} \quad \text{and} \quad |v(y) - v(y_0)| \leq C_6|y - y_0|^{\alpha_1}. \tag{7.18}$$

Combining (7.16)–(7.18), and recalling the C^α regularity of v on $\partial\Omega$, we obtain

$$|v(x) - v(y)| \leq 2C_6 r_y^{\alpha_1} + (\tilde{c_2})^\alpha |x - y|^{\alpha/2} \leq C_8(n, \lambda, \Lambda, \alpha, q, \varepsilon, \rho)|x - y|^\beta.$$

Case 2: $r_y \geq c_2$. Since $|x - y| < c_3$, (7.15) follows from Step 1.

The proof of the theorem is complete. □

8. Application to singular Abreu type equations

In this section, we will use Theorem 2.4 to establish the Sobolev solvability in all dimensions for a class of singular fourth-order Abreu type equations. These equations arise from the approximation analysis of convex functionals subject to convexity constraints such as the Rochet–Choné model [36] with a quadratic cost in the monopolist's problem in economics. We refer the readers to [7, 18, 24, 26, 30] for more details.

Theorem 2.4 provides us a tool to directly establish, in all dimensions, the global higher-order estimates for singular Abreu equations that were first obtained in the work of Kim, Wang, Zhou, and the second author [18]. Moreover, it allows us to obtain new solvability results in dimensions at least three that were not accessible by previous methods.

Our solvability results state as follows.

Theorem 8.1 (Solvability of the second boundary value problem for singular Abreu type equations)**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth, uniformly convex domain. Let $G(t) : (0, \infty) \to \mathbb{R}$ be*

$$\textit{either} \quad G(t) = \log t \quad \textit{or} \quad G(t) = \log t / \log\log(t + e^{e^{4n}}).$$

Assume that $f \in L^p(\Omega)$ with $f \leq 0$ and $p > n$, $\varphi \in W^{4,p}(\Omega)$ and $\psi \in W^{2,p}(\Omega)$ with $\min_{\partial\Omega} \psi > 0$. Consider the following second boundary value problem for a uniformly convex function u:

$$\begin{cases} U^{ij} D_{ij} w &= -\Delta u + f \quad \textit{in} \quad \Omega, \\ \qquad w &= G'(\det D^2 u) \quad \textit{in} \quad \Omega, \\ \qquad u &= \varphi, \quad w = \psi \quad \textit{on} \quad \partial\Omega. \end{cases} \tag{8.1}$$

Here $(U^{ij}) = (\det D^2 u)(D^2 u)^{-1}$. Then, there exists a uniformly convex solution $u \in W^{4,p}(\Omega)$ to (8.1) *with*

$$\|u\|_{W^{4,p}(\Omega)} \leq C = C(n, p, G, \varphi, \psi, \Omega). \tag{8.2}$$

Before sketching the proof of Theorem 8.1, we make some pertinent remarks and related analysis.

When $G(t) = \log t$, the expression $U^{ij}D_{ij}w = U^{ij}D_{ij}[(\det D^2u)^{-1}]$ appears in the Abreu's equation [1] in the constant scalar curvature problem in Kähler geometry [12]. For a convex function u without further regularity, Δu can be just a nonnegative Radon measure so it is a very singular term. For this reason, we call the first equation of (8.1) singular Abreu equation. This fourth-order equation in u can be rewritten as a system of two equations: one is a Monge–Ampère equation for u in the form of

$$\det D^2u = G'^{-1}(w),$$

and the other is a linearized Monge–Ampère equation for w in the form of

$$U^{ij}D_{ij}w = -\Delta u + f.$$

Since we prescribe the boundary values of both u and its Hessian determinant $\det D^2u$ via w, we call (8.1) a second boundary value problem.

Without the term $-\Delta u$ on the right-hand side of (8.1), Theorem 8.1 was proved in [20] (see Theorems 1.1 and 1.2 and Remark 1.4 there) without the sign restriction on f. However, when $-\Delta u$ appears on the right-hand side of (8.1), it creates several difficulties in establishing solvability results for (8.1) using degree theory and a priori estimates.

The first difficulty lies in obtaining the *a priori* positive lower and upper bounds for det D^2u, which is a critical step in applying the regularity results of the linearized Monge-Ampère equation. For example, it is not known if one can obtain a lower bound for $\det D^2u$ in dimensions $n \geq 3$ for $G(t) = t^{\frac{1}{n+2}}$ which arises from the affine maximal surface equation [42].

The second difficulty, granted that the bounds $0 < \lambda \leq \det D^2u \leq \Lambda < \infty$ have been established, consists in obtaining Hölder estimates for w in the linearized Monge-Ampère equation $U^{ij}D_{ij}w = -\Delta u + f$. If one applies the global Hölder estimates in [28], this requires Δu to be in L^q with $q > n/2$. However, Δu is a priori at most $L^{1+\varepsilon}$ for some small constant $\varepsilon(\lambda, \Lambda, n) > 0$ (see [11, 40, 46]) so it does not have enough integrability in dimensions $n \geq 3$. New ideas are required.

When $G(t) = \log t$, Theorem 8.1 was proved in [18, Theorem 1.1]. The key idea in [18] is to transform the equation $U^{ij}D_{ij}w = -\Delta u + f$ with singular term $-\Delta u$ into *a family of linearized Monge–Ampère equations* with bounded drifts and L^p right-hand sides for which the interior Harnack inequality and Hölder estimates in [23] are applicable and at each boundary point x_0 of interest, one can make the drift vanish at that point for the purpose of obtaining pointwise Hölder estimates at x_0. Indeed, let

$$\eta(x) := w(x)e^{|Du(x)-Du(x_0)|^2/2}. \tag{8.3}$$

Then η solves

$$U^{ij}D_{ij}\eta - (\det D^2u)(Du - Du(x_0)) \cdot D\eta = e^{|Du-Du(x_0)|^2/2}f. \tag{8.4}$$

Note that once the Hessian determinant bounds have been established, the potential method in [8] gives new interior higher-order estimates for (8.1); see [8, Theorem 5.1]. Here, we are able to handle estimates up to the boundary. This is crucial for the solvability using the degree theory.

Interestingly, the transformations (8.3) and (8.4) are very specific to the case $G(t) = \log t$. They do not transform in any helpful ways for other G, including $G(t) = \log t/\log\log(t+e^{e^{4n}})$. This case is not accessible by the methods in [18, 24] in dimensions at least three. For this G, we can, however, use Theorem 2.4 to obtain solvability.

Sketch of proof of Theorem 8.1. It suffices to establish the a priori bound (8.2) for uniformly convex solution $u \in W^{4,p}(\Omega)$ to (8.1) since the existence then follows from the degree theory.

Step 1: Hessian determinant bounds. We assert that

$$C_1^{-1} \leq \det D^2 u \leq C_1 \quad \text{in } \Omega, \tag{8.5}$$

where $C_1 = C_1(n, p, G, \varphi, \psi, \Omega) > 0$.

Indeed, from the convexity of u, we have $\Delta u \geq 0$. Hence, recalling $f \leq 0$, we find

$$U^{ij} D_{ij} w = -\Delta u + f \leq 0 \quad \text{in } \Omega.$$

By the maximum principle, w attains its minimum value in $\overline{\Omega}$ on the boundary. Thus

$$w \geq \min_{\partial\Omega} w = \min_{\partial\Omega} \psi > 0 \quad \text{in } \Omega.$$

This together with $\det D^2 u = (G')^{-1}(w)$ gives the upper bound for the Hessian determinant:

$$\det D^2 u \leq C_2 := (G')^{-1}(\min_{\partial\Omega} \psi) \quad \text{in } \Omega. \tag{8.6}$$

Using $u = \varphi$ on $\partial\Omega$ together with Ω being smooth and uniformly convex, we can construct suitable barrier functions (see [27, Theorem 3.30]) to obtain

$$\sup_{\Omega} |u| + \|Du\|_{L^\infty(\Omega)} \leq C_3 = C_3(n, G, \varphi, \psi, \Omega). \tag{8.7}$$

Denote the Legendre transform u^* of u by

$$u^*(y) = x \cdot Du(x) - u(x), \quad \text{where } y = Du(x) \in \Omega^* := Du(\Omega).$$

Then

$$x = Du^*(y) \quad \text{and} \quad D^2 u(x) = (D^2 u^*(y))^{-1}.$$

Let $(u^{*ij})_{1\leq i,j\leq n}$ be the inverse matrix of $D^2 u^*$ and

$$\begin{aligned} w^* &:= G\big((\det D^2 u^*)^{-1}\big) - (\det D^2 u^*)^{-1} G'\big(\det D^2 u^*)^{-1}\big) \\ &= G(\det D^2 u) - (\det D^2 u) G'(\det D^2 u). \end{aligned} \tag{8.8}$$

Computing as in [20, Lemma 2.7], we have

$$U^{ij} D_{ij} w = -u^{*ij} D_{ij} w^*,$$

and (see also [18, (2.9)])

$$\Delta u = u^{*ii}.$$

Here, we use the notation

$$D_{ij} w^* = \frac{\partial^2 w^*}{\partial y_i \partial y_j}.$$

It follows that

$$u^{*ij} D_{ij}\big(w^* - |y|^2/2\big)\big) = -U^{ij} D_{ij} w - \Delta u = -f(x) = -f(Du^*(y)) \quad \text{in } \Omega^*. \tag{8.9}$$

Note that (8.7) implies

$$\operatorname{diam}(\Omega^*) + \|u^*\|_{L^\infty(\Omega^*)} \leq C_4(n, G, \varphi, \psi, \Omega). \tag{8.10}$$

Moreover, for $y = Du(x) \in \partial\Omega^*$ where $x \in \partial\Omega$, we have from (8.8)

$$\begin{aligned} w^*(y) &= G(G'^{-1}(w(x))) - G'^{-1}(w(x))w(x) \\ &= G(G'^{-1}(\psi(x))) - G'^{-1}(\psi(x))\psi(x). \end{aligned} \tag{8.11}$$

From (8.10) and (8.11), by applying the ABP estimate [13, Theorem 9.1] to $w^* - |y|^2/2$ on Ω^* in equation (8.9), and then changing of variables $y = Du(x)$ with $dy = \det D^2u\, dx$, we obtain

$$\begin{aligned} \|w^* - |y|^2/2\|_{L^\infty(\Omega^*)} &\leq \|w^* - |y|^2/2\|_{L^\infty(\partial\Omega^*)} + C(n)\mathrm{diam}(\Omega^*)\Big\|\frac{f(Du^*)}{(\det u^{*ij})^{1/n}}\Big\|_{L^n(\Omega^*)} \\ &\leq C + C\Big(\int_{\Omega^*} \frac{|f|^n(Du^*)}{(\det D^2u^*)^{-1}}\, dy\Big)^{1/n} \\ &= C + C\Big(\int_{\Omega} \frac{|f|^n(x)}{\det D^2u} \det D^2u\, dx\Big)^{1/n} \\ &\leq C_5 + C_5\|f\|_{L^p(\Omega)}, \end{aligned}$$

where $C_5 = C_5(n, p, G, \varphi, \psi, \Omega)$.

In particular, w^* is bounded from below by a negative structural constant. Due to (8.8), the formula for G gives a positive structural lower bound for the Hessian determinant:

$$\det D^2u \geq C_6(n, p, G, \varphi, \psi, \Omega) > 0 \quad \text{in } \Omega. \tag{8.12}$$

Combining (8.6) and (8.12), we obtain (8.5) as asserted.

Step 2: Higher-order derivative estimates.

Now, by (8.5), (8.7), Lemma 3.5 and Remark 3.7, we can apply Theorem 2.4 with

$$\mathbf{F} = 0 \quad \text{and} \quad \mu = -\operatorname{div}(Du) + f.$$

We obtain some $\alpha = \alpha(n, p, G, \varphi, \psi, \Omega) \in (0, 1)$ such that $w \in C^\alpha(\overline{\Omega})$ with

$$\|w\|_{C^\alpha(\Omega)} \leq C_7(n, p, G, \varphi, \psi, \Omega).$$

Rewriting the equation for w, we find

$$\det D^2u = (G')^{-1}(w) \quad \text{in} \quad \Omega, \quad u = \varphi \quad \text{on} \quad \partial\Omega, \tag{8.13}$$

with $(G')^{-1}(w) \in C^\alpha(\overline{\Omega})$, and $\varphi \in C^3(\overline{\Omega})$ because $\varphi \in W^{4,p}(\Omega)$ and $p > n$. We obtain from the global Schauder estimates for the Monge–Ampère equation [44, Theorem 1.1] that

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C_8(n, p, G, \varphi, \psi, \Omega).$$

Thus, the first equation of (8.1) is a uniformly elliptic, second order partial differential equations in w with $C^\alpha(\overline{\Omega})$ coefficients and L^p right-hand side. Hence, from the standard $W^{2,p}$ theory for uniformly elliptic equations (see [13, Chapter 9]), we obtain the following $W^{2,p}(\Omega)$ estimates

$$\|w\|_{W^{2,p}(\Omega)} \leq C(n, p, G, \varphi, \psi, \Omega).$$

Now, we can differentiate and apply the standard Schauder and Calderón–Zygmund theories to (8.13) to obtain the following global $W^{4,p}(\Omega)$ estimate for u:

$$\|u\|_{W^{4,p}(\Omega)} \leq C = C(n, p, G, \varphi, \psi, \Omega).$$

The theorem is proved. $\square$

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