On the largest degrees in intersecting hypergraphs

Peter Frankl[1], Jian Wang[2]

[1]Rényi Institute, Budapest, Hungary

[2]Department of Mathematics
Taiyuan University of Technology
Taiyuan 030024, P. R. China

E-mail: [1]frankl.peter@renyi.hu, [2]wangjian01@tyut.edu.cn

Abstract

Let $\binom{[n]}{k}$ denote the collection of all k-subsets of the standard n-set $[n]=\{1,2,\ldots,n\}$. Let $n>2k$ and let $\mathcal{F}\subset\binom{[n]}{k}$ be an *intersecting* k-graph, i.e., $F\cap F'\neq\emptyset$ for all $F,F'\in\mathcal{F}$. The number of edges $F\in\mathcal{F}$ containing $x\in[n]$ is called the *degree* of x. Assume that $d_1\geq d_2\geq\ldots\geq d_n$ are the degrees of $\mathcal{F}$ in decreasing order. An important result of Huang and Zhao states that for $n>2k$ the minimum degree d_n is at most $\binom{n-2}{k-2}$. For $n\geq 6k-9$ we strengthen this result by showing $d_{2k+1}\leq\binom{n-2}{k-2}$. As to the second and third largest degrees we prove the best possible bound $d_3\leq d_2\leq\binom{n-2}{k-2}+\binom{n-3}{k-2}$ for $n>2k$. Several more best possible results of a similar nature are established.

AMS classification: 05D05.

Key words: extremal set theory, intersecting hypergraphs, ith largest degree.

1 Introduction

Let $[n]$ denote the standard n-element set $\{1,2,\ldots,n\}$ and $2^{[n]}$ its power set. Let $\binom{[n]}{k}$ be the family of all k-subsets of $[n]$. A family $\mathcal{F}\subset\binom{[n]}{k}$ is called *t-intersecting* if $|F\cap F'|\geq t$ for all $F,F'\in\mathcal{F}$ (if $t=1$ simply *intersecting*). To avoid trivialities, without further mention we are going to assume $n\geq 2k$ in the case of intersecting families and $n>2k-t$ in the case of t-intersecting families. For $x\in[n]$, define

$$\mathcal{F}(x)=\{F\setminus\{x\}\colon x\in F\in\mathcal{F}\}\text{ and }\mathcal{F}(\bar{x})=\{F\colon x\notin F\in\mathcal{F}\}.$$

Note that $|\mathcal{F}(x)|+|\mathcal{F}(\bar{x})|=|\mathcal{F}|$.

Definition 1.1. Suppose that $x_1,\ldots,x_n$ is a permutation of $1,2,\ldots,n$ satisfying $|\mathcal{F}(x_1)|\geq\ldots\geq|\mathcal{F}(x_n)|$. Set $d_i(\mathcal{F})=|\mathcal{F}(x_i)|$, i.e., $d_i(\mathcal{F})$ is the ith largest degree of $\mathcal{F}$. When $\mathcal{F}$ is clear from the context we sometimes write d_i for short.

In the literature $d_1(\mathcal{F})$ the largest degree is often denoted by $\Delta(\mathcal{F})$ and $d_n(\mathcal{F})$ the minimum degree by $\delta(\mathcal{F})$.

Let us recall some results concerning these quantities in intersecting k-graphs, $\mathcal{F}\subset\binom{[n]}{k}$.

Exact Erdős-Ko-Rado Theorem ([2], [3], [16]). Suppose that $n \geq (t+1)(k-t+1)$ and $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting. Then

$$|\mathcal{F}| \leq \binom{n-t}{k-t}. \tag{1.1}$$

The Hilton-Milner Theorem is a strong stability result for the $t=1$ case of the Erdős-Ko-Rado Theorem. We say that $\mathcal{F} \subset \binom{[n]}{k}$ is *non-trivial* intersecting if $\cap\{F\colon F \in \mathcal{F}\} = \emptyset$.

Hilton-Milner Theorem ([11]). If $n > 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ is non-trivial intersecting, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{1.2}$$

Corollary 1.2. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $d_1(\mathcal{F}) < |\mathcal{F}|$. Then*

$$d_1(\mathcal{F}) \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1}.$$

Note that for the full star $\mathcal{S} = \{S \in \binom{[n]}{k}\colon 1 \in S\}$, $d_i(\mathcal{S}) = \binom{n-2}{k-2}$ for $2 \leq i \leq n$. Huang and Zhao gave a beautiful algebraic proof showing that $d_n(\mathcal{F}) \leq \binom{n-2}{k-2}$ is always true.

Theorem 1.3 ([12]). *For $n > 2k$, if $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, then*

$$d_n(\mathcal{F}) \leq \binom{n-2}{k-2}.$$

There is a natural sequence of intersecting families containing the Hilton-Milner Family, $\mathcal{H}_k$.

Definition 1.4. For $2 \leq \ell \leq k$, define

$$\mathcal{H}_\ell = \mathcal{H}_\ell(n,k) = \left\{H \in \binom{[n]}{k} : 1 \in H,\ H \cap [2, \ell+1] \neq \emptyset\right\} \cup \left\{H \in \binom{[n]}{k} : [2, \ell+1] \subset H\right\}.$$

It is easy to verify that

$$d_i(\mathcal{H}_\ell) = \binom{n-2}{k-2} + \binom{n-\ell-1}{k-\ell} \text{ for } 2 \leq i \leq \ell+1, \tag{1.3}$$

$$|\mathcal{H}_\ell| = \binom{n-1}{k-1} - \binom{n-\ell-1}{k-1} + \binom{n-\ell-1}{k-\ell}. \tag{1.4}$$

A by now classical result of the first author relates the maximum degree and the size of an intersecting family.

Theorem 1.5 ([6]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, $n > 2k$, $2 \leq \ell \leq k$ and $d_1(\mathcal{F}) \leq d_1(\mathcal{H}_\ell) = \binom{n-1}{k-1} - \binom{n-\ell-1}{k-1}$. Then*

$$|\mathcal{F}| \leq |\mathcal{H}_\ell|.$$

Moreover, in case of equality either $\mathcal{F}$ is isomorphic to $\mathcal{H}_\ell$ or $\ell = 3$ and $\mathcal{F} \cong \mathcal{H}_2$.

Since $\mathcal{H}_2 = \{H \in \binom{[n]}{k} : |H \cap [3]| \geq 2\}$, it is often called the *triangle family*.

In this paper, we are mainly concerned with the ith degree of an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$. For $k = 2$, there are only two types of intersecting families, the stars and the triangle. Therefore we always assume $k \geq 3$.

Our first result shows that the maximum of d_2 and d_3 are given by the triangle family. The surprisingly short proof relies on the idea of Daykin [1].

Theorem 1.6. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, $n > 2k$. Then*

$$d_2(\mathcal{F}) \leq \binom{n-2}{k-2} + \binom{n-3}{k-2}. \tag{1.5}$$

Moreover, if equality holds for $d_2(\mathcal{F})$ then $\mathcal{F}$ is isomorphic to $\mathcal{H}_2$.

For $n \geq 6k - 9$, we determine $d_{2k+1}(\mathcal{F})$ for an arbitrary intersecting family $\mathcal{F} \subset \binom{[n]}{k}$.

Theorem 1.7. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Then for $n \geq 6k - 9$,*

$$d_{2k+1}(\mathcal{F}) \leq \binom{n-2}{k-2}.$$

Note that in the above range Theorem 1.7 is a considerable strengthening of the Huang-Zhao Theorem (Theorem 1.3).

For $n \geq \lceil \frac{8k}{3} \rceil$, we prove that $d_{\lceil \frac{8k}{3} \rceil}(\mathcal{F}) \leq \binom{n-2}{k-2}$.

Theorem 1.8. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Then for $n \geq \lceil \frac{8k}{3} \rceil$,*

$$d_{\lceil \frac{8k}{3} \rceil}(\mathcal{F}) \leq \binom{n-2}{k-2}.$$

For $n \geq \binom{t+2}{2} k^2$, we can prove the stronger bound $d_{k+2}(\mathcal{F}) \leq \binom{n-t-1}{k-t-1}$ for any t-intersecting family $\mathcal{F} \subset \binom{[n]}{k}$.

Theorem 1.9. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and $n \geq \binom{t+2}{2} k^2$, $k > t \geq 1$. Then*

$$d_{k+2}(\mathcal{F}) \leq \binom{n-t-1}{k-t-1}. \tag{1.6}$$

Define

$$\mathcal{H}(n,k,t) = \left\{ H \in \binom{[n]}{k} : [t] \subset H, H \cap [t+1, k+1] \neq \emptyset \right\} \cup \{[k+1] \setminus \{j\} : 1 \leq j \leq t\}.$$

It is easy to check that

$$d_{t+1}(\mathcal{H}(n,k,t)) = \ldots = d_{k+1}(\mathcal{H}(n,k,t)) = \binom{n-t-1}{k-t-1} + t.$$

This shows that in a certain sense (1.6) is best possible.

Concerning the general case d_ℓ for $4 \leq \ell \leq k+1$ we have the following.

Problem 1.10. *Determine or estimate the minimum value $n_0(k, \ell)$ such that for $n > n_0(k, \ell)$ the $(\ell+1)$th largest degree in an intersecting k-graph $\mathcal{F}$ on n vertices satisfies*

$$d_{\ell+1}(\mathcal{F}) \leq \binom{n-2}{k-2} + \binom{n-\ell-1}{k-\ell}. \tag{1.7}$$

The next two results establish (1.7) under certain constraints.

Theorem 1.11. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, $n \geq 6k$. Then*

$$d_4(\mathcal{F}) \leq \binom{n-2}{k-2} + \binom{n-4}{k-3}.$$

For the case $\ell \geq 4$ let us prove a slightly weaker bound.

Theorem 1.12. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, $4 \leq \ell \leq k$ and $n > 2\ell^2 k$. Then*

$$d_{\ell+1}(\mathcal{F}) \leq \binom{n-2}{k-2} + \binom{n-\ell-1}{k-\ell}. \tag{1.8}$$

Example 1.13. *For $n \geq k \geq r \geq 1$ define*

$$\mathcal{L}_r = \left\{F \in \binom{[n]}{k} : |F \cap [2r-1]| \geq r\right\}.$$

Clearly $\mathcal{L}_r$ is intersecting and

$$d_i(\mathcal{L}_r) = \sum_{r-1 \leq j \leq 2r-2} \binom{2r-2}{j}\binom{n-2r+1}{k-j-1} \text{ for } i = 1, 2, \ldots, 2r-1.$$

Recall that

$$d_4(\mathcal{H}_3) = \binom{n-2}{k-2} + \binom{n-4}{k-3} = \binom{n-5}{k-2} + 4\binom{n-5}{k-3} + 4\binom{n-5}{k-4} + \binom{n-5}{k-5}.$$

Then for $n < 3k-2$,

$$d_4(\mathcal{L}_3) - d_4(\mathcal{H}_3) = 2\binom{n-5}{k-3} - \binom{n-5}{k-2} = \left(\frac{2(k-2)}{n-k-2} - 1\right)\binom{n-5}{k-2} > 0.$$

Thus Theorem 1.11 does not hold for $n < 3k-2$.

Note also that

$$d_5(\mathcal{H}_4) = \binom{n-2}{k-2} + \binom{n-5}{k-4} = \binom{n-5}{k-2} + 3\binom{n-5}{k-3} + 4\binom{n-5}{k-4} + \binom{n-5}{k-5}.$$

Then for $n < 4k-4$,

$$d_5(\mathcal{L}_3) - d_5(\mathcal{H}_4) = 3\binom{n-5}{k-3} - \binom{n-5}{k-2} = \left(\frac{3(k-2)}{n-k-2} - 1\right)\binom{n-5}{k-2} > 0.$$

Thus $d_5(\mathcal{L}_3) > d_5(\mathcal{H}_4)$ for $n < 4k-4$.

2 Preliminaries and the proof of Theorem 1.6

For $\mathcal{F} \subset \binom{[n]}{k}$, let us define the *$\ell$th shadow of $\mathcal{F}$* as

$$\partial^{(\ell)}\mathcal{F} = \left\{E \in \binom{[n]}{k-\ell} : \text{there exists } F \in \mathcal{F} \text{ such that } E \subset F\right\}.$$

The quintessential Kruskal-Katona Theorem [14, 13] gives the best possible lower bounds on $|\partial^{(\ell)}\mathcal{F}|$ for given size of $\mathcal{F}$.

Recall the *co-lexicographic order* $A <_C B$ for $A, B \in \binom{[n]}{k}$ defined by, $A <_C B$ if and only if $\max\{i\colon i \in A \setminus B\} < \max\{i\colon i \in B \setminus A\}$. Let $\mathcal{C}(n,k,m)$ be the collection of the first m sets $A \in \binom{[n]}{k}$ in the co-lexicographic order.

Kruskal–Katona Theorem ([14, 13]). If $\mathcal{F} \subset \binom{[n]}{k}$, $|\mathcal{F}| = m$, then

$$|\partial^{(\ell)}\mathcal{F}| \geq |\partial^{(\ell)}\mathcal{C}(n,k,m)|. \tag{2.1}$$

Recall the *lexicographic order* $A <_L B$ for $A, B \in \binom{[n]}{k}$ defined by, $A <_L B$ if and only if $\min\{i\colon i \in A \setminus B\} < \min\{i\colon i \in B \setminus A\}$. For $n > k > 0$ and $\binom{n}{k} \geq m > 0$, let $\mathcal{L}(n,k,m)$ denote the collection of the first m sets $A \in \binom{[n]}{k}$ in the lexicographic order.

We also need the following reformulation of the Kruskal–Katona Theorem, which was first appeared in [10]. Two families $\mathcal{A}$, $\mathcal{B}$ are called *cross-intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Kruskal–Katona Theorem ([14, 13, 10]). Let n, a, b be positive integers, $n \geq a + b$. Suppose that $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting. Then $\mathcal{L}(n,a,|\mathcal{A}|)$ and $\mathcal{L}(n,b,|\mathcal{B}|)$ are cross-intersecting as well.

For $A \subset B \subset [n]$, define

$$\mathcal{F}(A,B) = \{F \setminus B\colon F \cap B = A\}.$$

We use $\mathcal{F}(A)$ to denote $\mathcal{F}(A,A)$. We also use $\mathcal{F}(i,\bar{j})$ to denote $\mathcal{F}(\{i\},\{i,j\})$, $\mathcal{F}(i,j)$ to denote $\mathcal{F}(\{i,j\},\{i,j\})$ and $\mathcal{F}(\bar{i},\bar{j})$ to denote $\mathcal{F}(\emptyset,\{i,j\})$.

Let us recall the following corollaries of the Kruskal–Katona Theorem.

Corollary 2.1. *Suppose that $n \geq 2k+2$. Let $\mathcal{A} \subset \binom{[n]}{k}$ and $\mathcal{B} \subset \binom{[n]}{k+2}$ be cross-intersecting. If $|\mathcal{A}| \geq \binom{n-1}{k-1} + \binom{n-2}{k-2}$ then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k}.$$

Proof. By the Kruskal–Katona Theorem, we may assume that $\mathcal{A} = \mathcal{L}(n,k,|\mathcal{A}|)$ and $\mathcal{B} = \mathcal{L}(n,k+2,|\mathcal{B}|)$. By $|\mathcal{A}| \geq \binom{n-1}{k-1} + \binom{n-2}{k-2}$, we infer that $|\mathcal{A}| = \binom{n-1}{k-1} + \binom{n-2}{k-1} + |\mathcal{A}(\bar{1},\bar{2})|$ and $|\mathcal{B}| = |\mathcal{B}(1,2)|$. Since $\mathcal{A}(\bar{1},\bar{2})$ and $\mathcal{B}(1,2)$ are cross-intersecting,

$$|\mathcal{A}(\bar{1},\bar{2})| + |\mathcal{B}(1,2)| \leq \binom{n-2}{k}.$$

Thus $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k}$ follows. $\square$

Corollary 2.2. *Suppose $n \geq a+b$. Let $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ be cross-intersecting families. If $|\mathcal{A}| \geq \binom{n-2}{a-2} + \binom{n-3}{a-2} + \ldots + \binom{n-d}{a-2}$ for some $2 \leq d \leq b+1$, then*

$$|\mathcal{B}| \leq \binom{n-1}{b-1} + \binom{n-d}{b-d+1}.$$

Moreover, if $|\mathcal{A}| \geq \binom{n-\ell}{a-\ell}$ for some $1 \leq \ell \leq a$, then

$$|\mathcal{B}| \leq \binom{n}{b} - \binom{n-\ell}{b}.$$

Proof. By the Kruskal–Katona Theorem, we may assume $\mathcal{A} = \mathcal{L}(n, a, |\mathcal{A}|)$ and $\mathcal{B} = \mathcal{L}(n, b, |\mathcal{B}|)$. If $|\mathcal{A}| \geq \binom{n-2}{a-2} + \binom{n-3}{a-2} + \ldots + \binom{n-d}{a-2}$, then

$$\left\{F \in \binom{[n]}{a} : \{1, i\} \subset F \text{ for some } i \in [2, d]\right\} \subset \mathcal{A}.$$

As $\mathcal{A}$, $\mathcal{B}$ are cross-intersecting,

$$\mathcal{B} \subset \left\{F \in \binom{[n]}{b} : 1 \in F \text{ or } [2, d] \subset F\right\}.$$

Consequently,

$$|\mathcal{B}| \leq \binom{n-1}{b-1} + \binom{n-d}{b-d+1}.$$

If $|\mathcal{A}| \geq \binom{n-\ell}{a-\ell}$, then

$$\left\{F \in \binom{[n]}{a} : [\ell] \subset F\right\} \subset \mathcal{A}.$$

As $\mathcal{A}$, $\mathcal{B}$ are cross-intersecting,

$$\mathcal{B} \subset \left\{F \in \binom{[n]}{b} : F \cap [\ell] \neq \emptyset\right\}.$$

Thus,

$$|\mathcal{B}| \leq \binom{n}{b} - \binom{n-\ell}{b}.$$

□

Daykin [1] showed that the $t = 1$ case of Erdős-Ko-Rado follows from the Kruskal-Katona Theorem. The following result can be extracted from his proof.

Daykin Theorem ([1]). Let $n > 2k$, $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$. If $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting then

$$\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq \binom{n-1}{k-1}.$$

In case of equality $\mathcal{A} = \mathcal{B} = \{F \in \binom{[n]}{k} : x \in F\}$ for some $x \in [n]$.

Proof. Suppose that $|\mathcal{A}| \geq |\mathcal{B}| \geq \binom{n-1}{k-1}$. We need to show $|\mathcal{A}| = \binom{n-1}{k-1}$. Assume indirectly $|\mathcal{A}| > \binom{n-1}{k-1}$ and fix $\mathcal{A}_0 \subset \mathcal{A}$, $|\mathcal{A}_0| = \binom{n-1}{k-1} + 1$. The initial segment $\mathcal{L}(n, k, \binom{n-1}{k-1} + 1)$ of the lexicographic order consists of the full star of 1 and the set $[2, k+1]$. By the Kruskal-Katona Theorem every member of $\mathcal{L}(n, k, |\mathcal{B}|)$ intersects each of these sets. Hence $|\mathcal{B}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} < \binom{n-1}{k-1}$, the desired contradiction.

Using the uniqueness part of the Kruskal-Katona Theorem in the case $|\mathcal{A}| = \binom{n-1}{k-1}$ (cf. e.g. [4] or [9]), it follows that $|\mathcal{A}| = |\mathcal{B}| = \binom{n-1}{k-1}$ holds iff $\mathcal{A}$ and $\mathcal{B}$ are identical full stars. □

Proof of Theorem 1.6. Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family with $d_1(\mathcal{F}) = |\mathcal{F}(1)|$, $d_2(\mathcal{F}) = |\mathcal{F}(2)|$. Note that

$$d_1(\mathcal{F}) = |\mathcal{F}(1, 2)| + |\mathcal{F}(1, \bar{2})|, \ d_2(\mathcal{F}) = |\mathcal{F}(1, 2)| + |\mathcal{F}(\bar{1}, 2)|.$$

Hence $|\mathcal{F}(\bar{1}, 2)| \leq |\mathcal{F}(1, \bar{2})|$. Since $\mathcal{F}$ is intersecting, $\mathcal{F}(\bar{1}, 2)$ and $\mathcal{F}(1, \bar{2})$ are cross-intersecting. By the Daykin Theorem, we get

$$|\mathcal{F}(\bar{1}, 2)| = \min\{|\mathcal{F}(\bar{1}, 2)|, |\mathcal{F}(1, \bar{2})|\} \leq \binom{(n-2)-1}{(k-1)-1}.$$

Thus $d_2(\mathcal{F}) \leq \binom{n-2}{k-2} + \binom{n-3}{k-2}$ follows. In case of equality, $|\mathcal{F}(1, 2)| = \binom{n-2}{k-2}$ and $\mathcal{F}(\bar{1}, 2)$, $\mathcal{F}(1, \bar{2})$ are identical full stars. Without loss of generality assume that $\mathcal{F}(\bar{1}, 2)$, $\mathcal{F}(1, \bar{2})$ are both full stars of center 3. Then $\mathcal{F}$ is isomorphic to $\mathcal{H}_2$ and $d_1(\mathcal{F}) = d_2(\mathcal{F}) = d_3(\mathcal{F})$ follows. □

3 Saturation, minimal transversals and the shifted case

When considering the maximum of the ith largest degree of a t-intersecting family, we may always assume that $\mathcal{F}$ is saturated. That is, the addition of any new k-sets to $\mathcal{F}$ would destroy the t-intersecting property.

Let $\binom{[n]}{\leq k}$ denote the collection of all subsets of $[n]$ with size at most k. For $\mathcal{G} \subset \binom{[n]}{\leq k}$, define

$$\langle \mathcal{G} \rangle = \left\{ F \in \binom{[n]}{k} \colon \text{ there exists } G \in \mathcal{G} \text{ such that } G \subset F \right\}.$$

For a t-intersecting family $\mathcal{F}$, define the *t-transversal family*

$$\mathcal{T}_t(\mathcal{F}) := \{T \subset [n] \colon |T| \leq k,\ |T \cap F| \geq t \text{ for all } F \in \mathcal{F}\}.$$

Define the *basis* $\mathcal{B}_t(\mathcal{F})$ as the family of all minimal members (for containment) of $\mathcal{T}_t(\mathcal{F})$, that is , the collection of all *minimal t-transversals*. The *t-covering number* $\tau_t(\mathcal{F})$ is defined by $\tau_t(\mathcal{F}) = \min\{|T| \colon T \in \mathcal{T}_t(\mathcal{F})\}$. For $t = 1$, we often write $\mathcal{T}(\mathcal{F})$, $\mathcal{B}(\mathcal{F})$ and $\tau(\mathcal{F})$ for $\mathcal{T}_1(\mathcal{F})$, $\mathcal{B}_1(\mathcal{F})$ and $\tau_1(\mathcal{F})$, respectively. The 1-covering number is simply called the *covering number.*

Lemma 3.1 ([7]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a saturated t-intersecting family and $\mathcal{B} = \mathcal{B}_t(\mathcal{F})$. Then $\mathcal{B}$ is t-intersecting with $\langle \mathcal{B} \rangle = \mathcal{F}$.*

Let us recall the shifting method, which was invented by Erdős, Ko and Rado [2]. For $\mathcal{F} \subset \binom{[n]}{k}$ and $1 \leq i < j \leq n$, define the shift

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) \colon F \in \mathcal{F}\},$$

where

$$S_{ij}(F) = \begin{cases} F' := (F \setminus \{j\}) \cup \{i\}, & \text{if } j \in F, i \notin F \text{ and } F' \notin \mathcal{F}; \\ F, & \text{otherwise.} \end{cases}$$

It is well known (cf. [5]) that shifting preserves the t-intersecting property.

Let us define the *shifting partial order* $\prec$. For two k-sets A and B where $A = \{a_1, \ldots, a_k\}$, $a_1 < \ldots < a_k$ and $B = \{b_1, \ldots, b_k\}$, $b_1 < \ldots < b_k$ we say that A precedes B and denote it by $A \prec B$ if $a_i \leq b_i$ for all $1 \leq i \leq k$.

A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *shifted* if $A \prec B$ and $B \in \mathcal{F}$ always imply $A \in \mathcal{F}$. By repeated shifting one can transform an arbitrary k-graph into a shifted k-graph with the same number of edges.

We say that a family $\mathcal{F} \subset \binom{[n]}{k}$ is *saturated t-intersecting* if $\mathcal{F}$ is t-intersecting and any addition of an extra k-set would destroy the t-intersecting property.

Lemma 3.2 ([3]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is shifted, saturated and t-intersecting. Let $\mathcal{B} = \mathcal{B}_t(\mathcal{F})$. Then $\mathcal{B} \subset 2^{[2k-t]}$.*

Proof. Suppose the contrary and let $B \in \mathcal{B}$ with $B \not\subset [2k-t]$. Choose $j \in B$ with $j > 2k - t$. By the minimality of $\mathcal{B}$, there exists $F \in \mathcal{F}$ with $|F \cap (B \setminus \{j\})| \leq t - 1$.

Then $|F \cap B| \geq t$ implies $j \in F$ and $|F \cap (B \setminus \{j\})| = |(F \setminus \{j\}) \cap (B \setminus \{j\})| = t - 1$. Thus,

$$|(F \setminus \{j\}) \cup (B \setminus \{j\})| = |F \setminus \{j\}| + |B \setminus \{j\}| - |(F \setminus \{j\}) \cap (B \setminus \{j\})| = 2(k-1) - (t-1) = 2k-1-t.$$

Hence we can fix $i \leq 2k - t$ with $i \notin F \cup B$. Consequently by shiftedness $F' = (F \setminus \{j\}) \cup \{i\} \in \mathcal{F}$ and $|F' \cap B| = t - 1$, a contradiction. $\square$

Lemma 3.3 ([5]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted t-intersecting family. Then $\mathcal{F}(i)$ is t-intersecting for all $i > 2k - t$.*

For a shifted t-intersecting family, we can prove the following result.

Theorem 3.4. *Let $n > (t+1)(k-t)$. If $\mathcal{F} \subset \binom{[n]}{k}$ is a shifted t-intersecting family, then*

$$d_{2k-t+1}(\mathcal{F}) \leq \binom{n-t-1}{k-t-1}.$$

Proof. Let $\mathcal{F} \subset \binom{[n]}{k}$ be shifted, saturated and t-intersecting and let $\mathcal{B} = \mathcal{B}_t(\mathcal{F})$ be its basis. Then by Lemma 3.2 we infer that $\mathcal{B} \subset 2^{[2k-t]}$. By saturatedness for every $i \in [2k-t+1, n]$, the link $\mathcal{F}(i)$ is isomorphic, that is,

$$\mathcal{F}(i) = \left\{ E \subset \binom{[n] \setminus \{i\}}{k-1} \colon \text{ there exists } B \in \mathcal{B} \text{ such that } B \subset E \right\}.$$

By Lemma 3.3, $\mathcal{F}(i)$ is t-intersecting for $i > 2k - t$. Then by the Exact Erdős-Ko-Rado Theorem, for $n - 1 \geq (t+1)(k-t)$ we have

$$d_i(\mathcal{F}) = |\mathcal{F}(i)| \leq \binom{(n-1)-t}{(k-1)-t} = \binom{n-t-1}{k-t-1}.$$

□

4 Sharpening the Huang–Zhao Theorem

By the Huang–Zhao Theorem we know that $d_n(\mathcal{F}) \leq \binom{n-2}{k-2}$ for all intersecting families $\mathcal{F} \subset \binom{[n]}{k}$, $n > 2k$. As we have shown in Theorem 3.4, in case of shifted families $d_{2k}(\mathcal{F}) \leq \binom{n-2}{k-2}$, that is, already the $2k$th degree is quit small. In this section we prove that for $n \geq 6k - 9$, $d_{2k+1}(\mathcal{F}) \leq \binom{n-2}{k-2}$ holds and for $n \geq \lceil \frac{8k}{3} \rceil$, $d_{\lceil \frac{8k}{3} \rceil}(\mathcal{F}) \leq \binom{n-2}{k-2}$ holds for general (non-shifted) intersecting families.

We need the following lemma.

Lemma 4.1 ([8]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family and let $|\mathcal{F}(u)| = d_1(\mathcal{F})$. If $n \geq 2k \geq 6$ and $|\mathcal{F}(\bar{u}, v)| \geq 5\binom{n-4}{k-3}$ for some $v \in [n] \setminus \{u\}$, then there exists $w \in [n] \setminus \{u, v\}$ such that*

$$|\mathcal{F}(\emptyset, \{u,v,w\})| \leq \binom{n-7}{k-4} \text{ and } |\mathcal{F}(\{x\}, \{u,v,w\})| \leq \binom{n-4}{k-3}, \ x = u, v, w.$$

Corollary 4.2. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family and let $|\mathcal{F}(u)| = d_1(\mathcal{F})$. If $n \geq 2k \geq 6$ and $|\mathcal{F}(\bar{u}, v)| \geq 5\binom{n-4}{k-3}$ for some $v \in [n] \setminus \{u\}$, then*

$$d_4(\mathcal{F}) \leq 6\binom{n-3}{k-3}.$$

Proof. Indeed, by Lemma 4.1 there exists $w \in [n] \setminus \{u, v\}$ such that

$$|\mathcal{F}(\emptyset, \{u,v,w\})| \leq \binom{n-7}{k-4} \text{ and } |\mathcal{F}(\{x\}, \{u,v,w\})| \leq \binom{n-4}{k-3}, \ x = u, v, w.$$

It follows that for any $x \in [n] \setminus \{u, v, w\}$,

$$|\mathcal{F}(x)| \leq 3\binom{n-3}{k-3} + \binom{n-7}{k-4} + 3\binom{n-4}{k-3} \leq 6\binom{n-3}{k-3}.$$

□

Proposition 4.3. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family and let $|\mathcal{F}(u)| = d_1(\mathcal{F})$. If $|\mathcal{F}(v)| > \binom{n-2}{k-2}$ and $|\mathcal{F}(\bar{u}, v)| \le \binom{n-4}{k-2}$ for some $v \in [n] \setminus \{u\}$, then*

$$|\mathcal{F}(\bar{u}, v)| > \frac{1}{2}|\mathcal{F}(\bar{u})|.$$

Proof. Note that

$$|\mathcal{F}(v)| = |\mathcal{F}(u, v)| + |\mathcal{F}(\bar{u}, v)| > \binom{n-2}{k-2}. \tag{4.1}$$

By $|\mathcal{F}(\bar{u}, v)| \le \binom{n-4}{k-2}$,

$$|\mathcal{F}(u, v)| > \binom{n-2}{k-2} - |\mathcal{F}(\bar{u}, v)| \ge \binom{n-2}{k-2} - \binom{n-4}{k-2}. \tag{4.2}$$

Applying Corollary 2.1 to the pair of cross-intersecting families $\mathcal{F}(u, v)$, $\mathcal{F}(\bar{u}, \overline{v})$ yields

$$|\mathcal{F}(u, v)| + |\mathcal{F}(\bar{u}, \overline{v})| \le \binom{n-2}{k-2}.$$

Comparing with (4.1), we get $|\mathcal{F}(\bar{u}, v)| > |\mathcal{F}(\bar{u}, \overline{v})|$. Since $|\mathcal{F}(\bar{u})| = |\mathcal{F}(\bar{u}, v)| + |\mathcal{F}(\bar{u}, \overline{v})|$, $|\mathcal{F}(\bar{u}, v)| > \frac{1}{2}|\mathcal{F}(\bar{u})|$ follows. □

Proof of Theorem 1.7. Suppose that $|\mathcal{F}(1)| = d_1(\mathcal{F})$. By Corollary 4.2, if $|\mathcal{F}(\bar{1}, x)| \ge 5\binom{n-4}{k-3}$ for some $x \in [2, n]$, then by $n \ge 6k - 10$,

$$d_{2k+1}(\mathcal{F}) \le d_4(\mathcal{F}) \le 6\binom{n-3}{k-3} \le \binom{n-2}{k-2}.$$

Thus we may assume $|\mathcal{F}(\bar{1}, x)| < 5\binom{n-4}{k-3}$ for all $x \in [2, n]$.

Note that $n \ge 6k - 9$ implies that

$$|\mathcal{F}(\bar{1}, x)| < 5\binom{n-4}{k-3} \le \frac{5(k-2)}{n-k-1}\binom{n-4}{k-2} \le \binom{n-4}{k-2}. \tag{4.3}$$

Arguing indirectly we may assume that there exist $2k$ distinct elements $x_1, \dots, x_{2k} \in [2, n]$ satisfying $|\mathcal{F}(x_i)| > \binom{n-2}{k-2}$. Applying Proposition 4.3 with $u = 1$ and $v = x_i$, we obtain that

$$|\mathcal{F}(\bar{1}, x_i)| > \frac{1}{2}|\mathcal{F}(\bar{1})|,\ 1 \le i \le 2k.$$

Summing these $2k$ inequalities,

$$k|\mathcal{F}(\bar{1})| = \sum_{x \in [2,n]} |\mathcal{F}(\bar{1}, x)| \ge \sum_{1 \le i \le 2k} |\mathcal{F}(\bar{1}, x_i)| > k|\mathcal{F}(\bar{1})|,$$

a contradiction. Thus $d_{2k+1} \le \binom{n-2}{k-2}$. □

Lemma 4.4. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family with $n > 2k$. Then either $d_{2k+1}(\mathcal{F}) \le \binom{n-2}{k-2}$ or $|\mathcal{F}| \le \binom{n-2}{k-2} + 2\binom{n-3}{k-2}$ holds.*

Proof. Suppose that $|\mathcal{F}(1)| = d_1(\mathcal{F})$. Then the *diversity*, $\gamma(\mathcal{F})$ is simply $|\mathcal{F}(\bar{1})| = |\mathcal{F}| - d_1(\mathcal{F})$.

Let us show that $\gamma(\mathcal{F}) \le \binom{n-4}{k-2}$ implies $d_{2k+1}(\mathcal{F}) \le \binom{n-2}{k-2}$. Arguing indirectly assume that there exist $2k$ distinct elements $x_1, \dots, x_{2k} \in [2, n]$ satisfying $|\mathcal{F}(x_i)| > \binom{n-2}{k-2}$. Note that

$$|\mathcal{F}(\bar{1}, x_i)| \le |\mathcal{F}(\bar{1})| = \gamma(\mathcal{F}) \le \binom{n-4}{k-2}.$$

Applying Proposition 4.3 with $u = 1$ and $v = x_i$, we obtain that

$$|\mathcal{F}(\bar{1}, x_i)| > \frac{1}{2}|\mathcal{F}(\bar{1})|,\ 1 \le i \le 2k.$$

Summing these $2k$ inequalities,

$$k|\mathcal{F}(\bar{1})| = \sum_{x\in[2,n]} |\mathcal{F}(\bar{1}, x)| \ge \sum_{1\le i\le 2k} |\mathcal{F}(\bar{1}, x_i)| > k|\mathcal{F}(\bar{1})|,$$

a contradiction. Thus we may assume that $\gamma(\mathcal{F}) > \binom{n-4}{k-2}$.

For $n > 2k$, $|\mathcal{F}(\bar{1})| = \gamma(\mathcal{F}) > \binom{n-4}{k-2} \ge \binom{n-4}{k-3}$. By applying Corollary 2.2 with $\mathcal{A} = \mathcal{F}(\bar{1})$ and $\mathcal{B} = \mathcal{F}(1)$,

$$|\mathcal{F}(1)| \le \binom{n-1}{k-1} - \binom{n-4}{k-1}.$$

By Theorem 1.5, we conclude that

$$|\mathcal{F}| \le |\mathcal{H}_3| = \binom{n-2}{k-2} + 2\binom{n-3}{k-2}.$$

Note that this can be deduced directly from Theorem 2 in [15] as well. $\square$

Proof of Theorem 1.8. By Lemma 4.4 we may assume that $|\mathcal{F}| \le \binom{n-2}{k-2} + 2\binom{n-3}{k-2}$. If $n \ge 6k - 9$, then by Theorem 1.7,

$$d_{\lceil \frac{8k}{3} \rceil}(\mathcal{F}) \le d_{2k+1}(\mathcal{F}) \le \binom{n-2}{k-2}.$$

Thus we may assume $n \le 6k - 10$.

Note that $n \le 6k - 10$ is equivalent to $\frac{k-2}{n-2} \ge \frac{1}{6}$. Then

$$|\mathcal{F}| \le \binom{n-2}{k-2} + 2\binom{n-3}{k-2} = \left(3 - \frac{2(k-2)}{n-2}\right)\binom{n-2}{k-2} \le \frac{8}{3}\binom{n-2}{k-2}. \tag{4.4}$$

If $d_{\lceil \frac{8k}{3} \rceil} > \binom{n-2}{k-2}$, then

$$\left\lceil \frac{8k}{3} \right\rceil \binom{n-2}{k-2} < \sum_{1\le i\le \lceil \frac{8k}{3} \rceil} d_i \le \sum_{x\in[n]} |\mathcal{F}(x)| = k|\mathcal{F}| < \frac{8}{3}k\binom{n-2}{k-2},$$

a contradiction. Thus $d_{\lceil \frac{8k}{3} \rceil} \le \binom{n-2}{k-2}$ holds. $\square$

Let us prove a statement that assumes $\tau(\mathcal{F}) = 2$.

Proposition 4.5. *Let $\mathcal{F} \subset \binom{n}{k}$ be an intersecting family with $\tau(\mathcal{F}) = 2$. Then for $n > 2k$,*

$$d_{2k+1}(\mathcal{F}) < \binom{n-2}{k-2}.$$

Proof. Without loss of generality we assume that $\mathcal{F}$ is saturated. By Lemma 4.4 we may assume that $|\mathcal{F}| \le \binom{n-2}{k-2} + 2\binom{n-3}{k-2}$. Argue by contradiction, let $\{x_1, x_2\}$ be a transversal and suppose that $z_1, \ldots, z_{2k-2} \in [n]\setminus\{x_1, x_2\}$ have degree at least $\binom{n-2}{k-2}$. By saturatedness, we have $|\mathcal{F}(\{x_1, x_2\})| = \binom{n-2}{k-2}$. Let

$$\tilde{\mathcal{F}} = \mathcal{F} \setminus \{F \in \mathcal{F} \colon \{x_1, x_2\} \subset F\}.$$

Then $|\tilde{\mathcal{F}}| < 2\binom{n-3}{k-2}$ and $|\tilde{\mathcal{F}}(z_i)| \ge \binom{n-2}{k-2} - \binom{n-3}{k-3} = \binom{n-3}{k-2}$, $i = 1, 2, \ldots, 2k-2$. Since $F \cap \{x_1, x_2\} \ne \emptyset$ for each $F \in \tilde{\mathcal{F}}$,

$$2(k-1)\binom{n-3}{k-2} \le \sum_{1 \le i \le 2k-2} |\tilde{\mathcal{F}}(z_i)| \le \sum_{x \in [n]\setminus\{x_1,x_2\}} |\tilde{\mathcal{F}}(x)| < (k-1)|\tilde{\mathcal{F}}| < 2(k-1)\binom{n-3}{k-2},$$

a contradiction. Thus at most $2k-3$ vertices in $[n] \setminus \{x_1, x_2\}$ have degree at least $\binom{n-2}{k-2}$ and the proposition follows. □

5 Stronger bounds for large n

We continue assuming that $\mathcal{F} \subset \binom{[n]}{k}$ is a saturated t-intersecting family and $n > 2k - t$. Recall that $\mathcal{T} = \mathcal{T}_t(\mathcal{F})$ is the family of t-transversals of size not exceeding k and the base $\mathcal{B} = \mathcal{B}_t(\mathcal{F})$ is the family of members of $\mathcal{T}$ that are minimal for containment.

Let us first prove a statement for $\tau_t(\mathcal{F}) \ge t + 2$.

Proposition 5.1. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be saturated and t-intersecting. If $n \ge \binom{t+2}{2}k^2$ and $\tau_t(\mathcal{F}) \ge t + 2$ then*

$$d_{1+\tau_t(\mathcal{F})} < \binom{n-t-1}{k-t-1} \tag{5.1}$$

Proof. Set $r = \tau_t(\mathcal{F})$ and fix $T_1 \in \mathcal{T}$ satisfying $|T_1| = r \ge t + 2$. To prove (5.1) it is sufficient to show

$$|\mathcal{F}(y)| < \binom{n-t-1}{k-t-1} \text{ for all } y \in [n] \setminus T_1. \tag{5.2}$$

To prove (5.2) for a fixed $y \in [n] \setminus T_1$ let us construct rk^{r-t} sequences $(z_1, z_2, \ldots, z_{r-1})$ in the following way.

Choose a t-subset $\{z_1, z_2, \ldots, z_t\} \subset T_1$, define a sequence $(z_1, z_2, \ldots, z_t)$. If $z_1, \ldots, z_\ell$ are chosen and $\ell < r - 1$ then choose $F_{\ell+1} \in \mathcal{F}$ satisfying $|\{y, z_1, z_2, \ldots, z_\ell\} \cap F_{\ell+1}| < t$. This is possible by $\tau_t(\mathcal{F}) = r$. Now extend the sequence to $(z_1, z_2, \ldots, z_{\ell+1})$ in all possible ways with $z_{\ell+1} \in F_{\ell+1} \setminus \{y, z_1, z_2, \ldots, z_\ell\}$.

It should be clear from the construction that every member $G \in \mathcal{F}(y)$ contains at least one of the sets $\{z_1, z_2, \ldots, z_\ell\}$ at each stage, in particular for $\ell = r - 1$. Consequently,

$$|\mathcal{F}(y)| \le \binom{r}{t} \cdot k^{r-1-t}\binom{n-r}{k-r}.$$

To conclude the proof of (5.2) we show that for $r \ge t + 2$ and $n \ge \binom{t+2}{2}k^2$,

$$\binom{r}{t} \cdot k^{r-1-t}\binom{n-r}{k-r} < \binom{n-t-1}{k-t-1}. \tag{5.3}$$

For $r = t+2$, $\binom{t+2}{2}k\binom{n-t-2}{k-t-2} = \binom{t+2}{2}\frac{k(k-t-1)}{n-t-1}\binom{n-t-1}{k-t-1}$ and the desired inequality follows from $n \geq \binom{t+2}{2}k^2 > \binom{t+2}{2}k(k-t-1)+t+1$. For $r \geq t+3$ just note that the LHS of (5.3) is a monotone decreasing function of t:

$$\frac{\binom{r+1}{t}k^{r-t}\binom{n-r-1}{k-r-1}}{\binom{r}{t}k^{r-1-t}\binom{n-r}{k-r}} = \frac{r+1}{r+1-t}\frac{k(k-r)}{n-r} < \frac{t+4}{4}\frac{k^2}{n} < 1. \qquad \square$$

A collection of sets $F_0, \dots, F_k$ is called a *sunflower of size* $k+1$ *with center* C if $F_i \cap F_j = C$ for all distinct $i, j \in \{0, 1, \dots, k\}$.

Proof of Theorem 1.9. Let $\mathcal{F} \subset \binom{[n]}{k}$ be saturated and t-intersecting. By Proposition 5.1 we may assume $\tau_t(\mathcal{F}) \leq t+1$. Let $\mathcal{T} = \mathcal{T}_t(\mathcal{F})$, $\mathcal{B} = \mathcal{B}_t(\mathcal{F})$ and $\mathcal{B}_0 = \{B \in \mathcal{B} \colon |B| = t+1\}$. If there exists $B \in \mathcal{B}$ with $|B| = t$, then $B \subset F$ for all $F \in \mathcal{F}$. By saturatedness $d_{k+2}(\mathcal{F}) = \binom{n-t-1}{k-t-1}$ and we are done. Thus we may assume $\tau_t(\mathcal{F}) = t+1$ and $\mathcal{B}_0 \neq \emptyset$.

Claim 5.2. $|\cup \mathcal{B}_0| \leq k+1$.

Proof. Suppose for contradiction that $|\cup \mathcal{B}_0| \geq k+2$. Choose $B_{t+1}, B_{t+2} \in \mathcal{B}_0$. Then by symmetry we may assume $B_{t+i} = [t] \cup \{t+i\}$, $i = 1, 2$. For $t+2 < j \leq k+2$ fix $B_j \in \mathcal{B}_0$ with $j \in B_j$. Now $|B_j| = t+1$ and $|B_{t+i} \cap B_j| \geq t$ imply $B_j = [t] \cup \{j\}$. Hence $B_{t+1}, B_{t+2}, \dots, B_{k+2}$ form a sunflower with center $[t]$. Now $[t] \subset F$ follows for all $F \in \mathcal{F}$, contradicting our assumption $\tau_t(\mathcal{F}) = t+1$. $\square$

By Claim 5.2 and symmetry we may assume $\cup \mathcal{B}_0 \subset [k+1]$. To prove the theorem it is sufficient to show

$$|\mathcal{F}(x)| < \binom{n-t-1}{k-t-1} \text{ for } k+2 < x \leq u. \tag{5.4}$$

We distinguish two cases.

Case 1. $\mathcal{B}_0$ is not a sunflower and $|\mathcal{B}_0| \geq 3$.

Without loss of generality, assume $[t+1], [t] \cup \{t+2\} \in \mathcal{B}_0$. Since $\mathcal{B}_0$ is not a sunflower, there is some $B_0 \in \mathcal{B}_0$ such that $|B_0 \cap [t]| < t$. Since $|B_0 \cap [t+1]| \geq t$, it follows that $|B_0 \cap [t]| = t-1$. Since $|B_0 \cap ([t] \cup \{t+2\})| \geq t$, we have $\{t+1, t+2\} \subset B_0$ whence $B_0 \subset [t+2]$. Now the t-intersecting property implies $|F \cap [t+2]| \geq t+1$ for all $F \in \mathcal{F}$. Indeed, $F \cap [t+1]$, $F \cap ([t] \cup \{t+2\})$ and $F \cap B_0$, all three have size at least t. This implies $|F \cap [t+2]| \geq t+1$ for all $F \in \mathcal{F}$. Thus for any $x \in [t+3, n]$,

$$|\mathcal{F}(x)| \leq (t+2)\binom{n-t-2}{k-t-2} < \binom{n-t-1}{k-t-1} \text{ for } n \geq (t+2)k.$$

Case 2. $\mathcal{B}_0$ is a sunflower.

Without loss of generality, assume that $\mathcal{B}_0 = \{[t] \cup \{q\} \colon t+1 \leq q \leq t+s\}$ for some s, $1 \leq s \leq k-t+1$. Since $[t] \notin \mathcal{B}_0$, there exists $F_0 \in \mathcal{F}$ such that $|F_0 \cap [t]| \leq t-1$. Since $|F_0 \cap ([t] \cup \{q\})| \geq t$ for all $t+1 \leq q \leq t+s$, it follows that $|F_0 \cap [t]| = t-1$ and $\{t+1, t+2, \dots, t+s\} \subset F_0$. Then $|[t] \cup F_0| = k+1$. Let $x \in [n] \setminus ([t] \cup F_0)$. For any $F \in \mathcal{F}$ with $x \in F$, we infer that either $[t] \subset F$ and $F \cap (F_0 \setminus [t]) \neq \emptyset$ or $|F \cap [t]| = t-1$ and $\{t+1, t+2, \dots, t+s\} \subset F$. Then

$$|\mathcal{F}(x)| \leq (k-t+1)\binom{n-t-2}{k-t-2} + t\binom{n-t-s-1}{k-t-s}. \tag{5.5}$$

For $s \geq 2$ and $n \geq k^2$,

$$|\mathcal{F}(x)| \leq (k-t+1)\binom{n-t-2}{k-t-2} + t\binom{n-t-3}{k-t-2} < \binom{n-t-1}{k-t-1}.$$

For $s = 1$, assume $\mathcal{B}_0 = \{[t+1]\}$. Let $x \in [t+2, n]$. For any $T \in \binom{[t+1]}{t}$, since $T \cup \{x\} \notin \mathcal{B}_0$, there exists F_T such that $|F_T \cap (T \cap \{x\})| \leq t-1$. It follows that for $n \geq (t+1)(k-t)^2$,

$$|\mathcal{F}(x)| \leq \binom{t+1}{t}(k-t+1)\binom{n-t-2}{k-t-2} < \binom{n-t-1}{k-t-1}. \qquad \square$$

6 Proofs of Theorems 1.11 and 1.12

Lemma 6.1. *Let $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ be cross-intersecting, $|\mathcal{A}| \leq |\mathcal{B}|$ and $n \geq (r+1)k$. Then*

$$r|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k}. \tag{6.1}$$

Proof. By the Kruskal–Katona Theorem we may assume that $\mathcal{A}, \mathcal{B}$ are initial segments in the lexicographic order. Hence $\mathcal{A} \subset \mathcal{B}$ and $1 \in A$ for all $A \in \mathcal{A}$. Define a bipartite graph on $X = \mathcal{A}(1)$, $Y = \binom{[2,n]}{k}$ by putting an edge iff the two sets are disjoint. The degree of $A \setminus \{1\}$ for $A \in \mathcal{A}$ is $\binom{n-k}{k}$. On the other hand for $B \in Y$ its degree is at most $\binom{n-k-1}{k-1} \leq \frac{1}{r}\binom{n-k}{k}$. It follows that

$$|\mathcal{A}|\binom{n-k}{k} \leq \frac{1}{r}\binom{n-k}{k}|N(\mathcal{A})|,$$

where $N(\mathcal{A})$ is the set of neighbors of $\mathcal{A}$. Thus the neighborhood of $\mathcal{A}$ in Y has size at least $r|\mathcal{A}|$ and these k-sets are not in $\mathcal{B}$. $\square$

Lemma 6.2 ([8]). *Let $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ be cross-intersecting families with $n \geq a+b$. If $|\mathcal{A}| \geq \binom{n-1}{a-1} + \binom{n-2}{a-1} + \ldots + \binom{n-d}{a-1}$, $d < b$, then*

$$|\mathcal{A}| + \frac{\binom{n-d}{a}}{\binom{n-d}{b-d}}|\mathcal{B}| \leq \binom{n}{a}.$$

Proof of Theorem 1.11. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a saturated intersecting family and let $|\mathcal{F}(1)|$, $|\mathcal{F}(2)|$, $|\mathcal{F}(3)|$, $|\mathcal{F}(4)|$ be the largest four degrees. To prove the theorem it suffices to show that there exists $i \in [4]$ such that

$$f(i) := |\mathcal{F}(i)| \leq \binom{n-2}{k-2} + \binom{n-4}{k-3}.$$

Set $f(Q) = |\mathcal{F}(Q, [4])|$ for $Q \subset [4]$. Then

$$f(i) = |\mathcal{F}(i)| = \sum_{i \in Q \subset [4]} f(Q). \tag{6.2}$$

For the proof it is useful to break up (6.2) into two parts:

$$f_{small}(i) = f(\{i\}) + \sum_{Q \in \binom{[4]}{2}, i \in Q} f(Q) \text{ and } f_{big}(i) = \sum_{i \in Q \subset [4],\ |Q| \geq 3} f(Q).$$

Claim 6.3. If $f_{small}(i) \le \binom{n-4}{k-2}$ then $f(i) \le \binom{n-2}{k-2} + \binom{n-4}{k-3}$.

Proof. Evidently, $f_{big}(i) \le 3\binom{n-4}{k-3} + \binom{n-4}{k-4}$. Hence,

$$f(i) = f_{small}(i) + f_{big}(i) \le \binom{n-2}{k-2} + \binom{n-4}{k-3}.$$

□

By Claim 6.3, it suffices to show that for some $i \in [4]$,

$$f_{small}(i) \le \binom{n-4}{k-2}. \tag{6.3}$$

Consider $P, P' \in \binom{[4]}{2}$ where $P = [4] \setminus P'$. Then $\mathcal{F}(P, [4])$ and $\mathcal{F}(P', [4])$ are cross-intersecting. Let us make a graph $\mathcal{G}$ on the vertex set $[4]$ by drawing an edge P iff $f(P) \ge \binom{n-5}{k-3} + \binom{n-6}{k-3}$. By the Daykin Theorem there are no two disjoint edges. Hence either $\mathcal{G}$ is a triangle or a star. We distinguish four cases according to the structure of $\mathcal{G}$.

Case 1. $\mathcal{G} \neq \emptyset$ and there exists an isolated vertex in $\mathcal{G}$.

Without loss of generality, let $\{1, 2\} \in \mathcal{G}$ and let 4 be isolated in $\mathcal{G}$. Since $\mathcal{F}(\{4\}, [4])$, $\mathcal{F}(\{1, 2\}, [4])$ are cross-intersecting and $f(\{1, 2\}) \ge \binom{n-5}{k-3} + \binom{n-6}{k-3}$, by the Kruskal–Katona Theorem we infer that

$$f(\{4\}) \le \binom{n-6}{k-3}.$$

Since $\mathcal{F}(\{1, 2\}, [4])$, $\mathcal{F}(\{3, 4\}, [4])$ are cross-intersecting and $f(\{1, 2\}) \ge \binom{n-5}{k-3} + \binom{n-6}{k-3}$, by the Kruskal–Katona Theorem

$$f(\{3, 4\}) \le \binom{n-6}{k-4}.$$

Since 4 is isolated, by the definition of $\mathcal{G}$ we have $f(\{j, 4\}) < \binom{n-5}{k-3} + \binom{n-6}{k-3}$ for $j = 1, 2$. Then for $(n-4) \ge 5(k-2)$,

$$\begin{aligned} f_{small}(4) = f(\{4\}) + \sum_{1 \le j \le 3} f(\{j, 4\}) &< \binom{n-6}{k-3} + \binom{n-6}{k-4} + 2\left(\binom{n-5}{k-3} + \binom{n-6}{k-3}\right) \\ &< 5\binom{n-5}{k-3} \\ &\le \binom{n-4}{k-2}. \end{aligned}$$

Case 2. $\mathcal{G} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$.

By symmetry assume that $f(\{1, 2\}) \ge f(\{1, 3\}) \ge f(\{1, 4\})$. Note that $n \ge 4k - 3$ implies

$$\frac{\binom{n-6}{k-2}}{\binom{n-6}{k-4}} \ge \frac{\binom{n-6}{k-2}}{\binom{n-6}{k-3}} = \frac{n-k-3}{k-2} \ge 3.$$

Since $\mathcal{F}(\{1, 2\}, [4])$, $\mathcal{F}(\{4\}, [4])$ are cross-intersecting and $|\mathcal{F}(\{1, 2\}, [4])| \ge \binom{n-5}{k-3} + \binom{n-6}{k-3}$, by Lemma 6.2 we get

$$f(\{1, 4\}) + 3f(\{4\}) \le f(\{1, 2\}) + \frac{\binom{n-6}{k-2}}{\binom{n-6}{k-3}} f(\{4\}) \le \binom{n-4}{k-2}. \tag{6.4}$$

Since $\mathcal{F}(\{1,2\},[4])$, $\mathcal{F}(\{3,4\},[4])$ are cross-intersecting, by Lemma 6.2 we get

$$f(\{1,4\})+3f(\{3,4\})\leq f(\{1,2\})+\frac{\binom{n-6}{k-2}}{\binom{n-6}{k-4}}f(\{3,4\})\leq\binom{n-4}{k-2}. \tag{6.5}$$

Similarly,

$$f(\{1,4\})+3f(\{2,4\})\leq f(\{1,3\})+3f(\{2,4\})\leq\binom{n-4}{k-2}. \tag{6.6}$$

Adding (6.4), (6.5), (6.6) and dividing by 3, we obtain that

$$f_{small}(4)=f(\{1,4\})+f(\{4\})+f(\{2,4\})+f(\{3,4\})\leq\binom{n-4}{k-2}.$$

Case 3. $\mathcal{G}=\emptyset$ and $f(\{i\})\geq\binom{n-6}{k-3}+\binom{n-7}{k-3}$ for $i=1,2,3,4$.

Note that $\mathcal{F}(\{i\},[4])$, $\mathcal{F}(\{j\},[4])$ are cross-intersecting for $1\leq i<j\leq 4$. By the Daykin Theorem, one of $\mathcal{F}(\{i\},[4])$ and $\mathcal{F}(\{j\},[4])$ has size at most $\binom{n-5}{k-2}$. Without loss of generality assume $|\mathcal{F}(\{4\},[4])|\leq\binom{n-5}{k-2}$. Since $|\mathcal{F}(\{i\},[4])|=f(\{i\})\geq\binom{n-6}{k-3}+\binom{n-7}{k-3}$ for all $i=1,2,3,4$, by Corollary 2.2,

$$|\mathcal{F}(\{j,4\})|\leq\binom{n-5}{k-3}+\binom{n-7}{k-4}\text{ for }j=1,2,3 \tag{6.7}$$

and

$$|\mathcal{F}([4]\setminus\{j\})|\leq\binom{n-5}{k-4}+\binom{n-7}{k-5}\text{ for }j=1,2,3. \tag{6.8}$$

Adding (6.7), (6.8) and $|\mathcal{F}([4],[4])|\leq\binom{n-4}{k-4}$,

$$\begin{aligned}f(4)&=f(\{4\})+\sum_{1\leq j\leq 3}f(\{j,4\})+\sum_{1\leq j\leq 3}f([4]\setminus\{j\})+f([4])\\&\leq\binom{n-5}{k-2}+3\left(\binom{n-5}{k-3}+\binom{n-7}{k-4}\right)+3\left(\binom{n-5}{k-4}+\binom{n-7}{k-5}\right)+\binom{n-4}{k-4}.\end{aligned}$$

Using $\binom{n-2}{k-2}=\binom{n-5}{k-2}+3\binom{n-5}{k-3}+3\binom{n-5}{k-4}+\binom{n-5}{k-5}$, $\binom{n-7}{k-4}+\binom{n-7}{k-5}=\binom{n-6}{k-4}$ and $n\geq 4k$, we obtain that

$$\begin{aligned}f(4)&\leq\binom{n-2}{k-2}+3\binom{n-6}{k-4}+\binom{n-4}{k-4}-\binom{n-5}{k-5}\\&\leq\binom{n-2}{k-2}+4\binom{n-5}{k-4}\\&\leq\binom{n-2}{k-2}+\binom{n-4}{k-3}.\end{aligned}$$

Case 4. $\mathcal{G}=\emptyset$ and $f(\{i\})<\binom{n-6}{k-3}+\binom{n-7}{k-3}$ for some $i\in[4]$.

Without loss of generality assume that

$$f(\{4\})<\binom{n-6}{k-3}+\binom{n-7}{k-3}. \tag{6.9}$$

Note that $\mathcal{G}=\emptyset$ implies

$$f(\{j,4\}) \le \binom{n-5}{k-3} + \binom{n-6}{k-3}. \tag{6.10}$$

If $f(\{i\}) \ge \binom{n-6}{k-3}+\binom{n-7}{k-3}$ for some $i\in[3]$, by Corollary 2.2 we infer that

$$f(\{j,4\}) \le \binom{n-5}{k-3} + \binom{n-7}{k-4} \text{ for } j\neq i. \tag{6.11}$$

Let

$$I=\left\{i\in[3]\colon f(\{i\}) \ge \binom{n-6}{k-3}+\binom{n-7}{k-3}\right\}.$$

Subcase 4.1. $|I|\ge 2$.
By (6.9), (6.11) and $n\ge 5k$,

$$\begin{aligned} f_{small}(4) = \sum_{1\le j\le 3} f(\{j,4\}) + f(\{4\}) &\le 3\left(\binom{n-5}{k-3}+\binom{n-7}{k-4}\right)+\binom{n-6}{k-3}+\binom{n-7}{k-3} \\ &< 5\binom{n-5}{k-3} \le \binom{n-4}{k-2}. \end{aligned}$$

Subcase 4.2. $|I|=1$.
By (6.9), (6.10), (6.11) and $n\ge 6k$,

$$\begin{aligned} f_{small}(4) &= \sum_{1\le j\le 3} f(\{j,4\}) + f(\{4\}) \\ &\le 2\left(\binom{n-5}{k-3}+\binom{n-7}{k-4}\right)+\binom{n-5}{k-3}+\binom{n-6}{k-3}+\binom{n-6}{k-3}+\binom{n-7}{k-3} \\ &< 6\binom{n-5}{k-3} \le \binom{n-4}{k-2}. \end{aligned}$$

Subcase 4.3. $I=\emptyset$.
By the definition of I,

$$f(i) < \binom{n-6}{k-3}+\binom{n-7}{k-3},\ i=1,2,3,4. \tag{6.12}$$

Let $\mathcal{G}'$ be a graph on the vertex set $[4]$ with $P\in\binom{[4]}{2}$ being an edge iff $f(P)>\binom{n-5}{k-3}$. By the Daykin Theorem there are no two disjoint edges.

If there is an isolated vertex of $\mathcal{G}'$, without loss of generality assume 4 is isolated, then $\{j,4\}\notin\mathcal{G}'$ for $j=1,2,3$, implying $f(\{j,4\})\le\binom{n-5}{k-3}$. By (6.12) and $n\ge 5k$,

$$\begin{aligned} f_{small}(4) &= \sum_{1\le j\le 3} f(\{j,4\}) + f(\{4\}) \\ &\le 3\binom{n-5}{k-3}+\binom{n-6}{k-3}+\binom{n-7}{k-3} \\ &< 5\binom{n-5}{k-3} \le \binom{n-4}{k-2}. \end{aligned}$$

Thus we may assume $\mathcal{G}' = \{\{1,2\},\{1,3\},\{1,4\}\}$. Then $\{1,4\},\{2,4\} \notin \mathcal{G}'$ implies $f(\{1,4\}) \leq \binom{n-5}{k-3}$ and $f(\{2,4\}) \leq \binom{n-5}{k-3}$. Thus by (6.10), (6.12) and $n \geq 6k$

$$\begin{aligned} f_{small}(4) &= \sum_{1\leq j\leq 3} f(\{j,4\}) + f(\{4\}) \\ &\leq 2\binom{n-5}{k-3} + \binom{n-5}{k-3} + \binom{n-6}{k-3} + \binom{n-6}{k-3} + \binom{n-7}{k-3} \\ &< 6\binom{n-5}{k-3} < \binom{n-4}{k-2}. \end{aligned}$$

□

Proof of Theorem 1.12. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a saturated intersecting family. Let $|\mathcal{F}(1)|$, $|\mathcal{F}(2)|, \ldots, |\mathcal{F}(\ell+1)|$ be the largest $\ell+1$ degrees. We are going to show that for some $i \in [\ell+1]$,

$$f(i) := |\mathcal{F}(i)| \leq \binom{n-2}{k-2} + \binom{n-\ell-1}{k-\ell}.$$

Set $f(Q) = |\mathcal{F}(Q,[\ell+1])|$ for $Q \subset [\ell+1]$. Then

$$f(i) = \sum_{i\in Q\subset[\ell+1]} f(Q). \tag{6.13}$$

For $p \in [\ell+1]\setminus\{i\}$, we define

$$\begin{aligned} f_1(i) &= \sum_{i\in Q\subset[\ell+1],\ 1\leq|Q|\leq 2} f(Q), \\ f_2(i,p) &= \sum_{i\in Q\subset[\ell+1],\ p\notin Q,\ 3\leq|Q|\leq\ell-1} f(Q), \\ f_3(i,p) &= \sum_{i\in Q\subset[\ell+1],\ |Q|\geq\ell} f(Q) + \sum_{i\in Q\subset[\ell+1],\ p\in Q,\ 3\leq|Q|\leq\ell-1} f(Q). \end{aligned}$$

Clearly $f(i) = f_1(i) + f_2(i,p) + f_3(i,p)$ holds for every $p \in [\ell+1]\setminus\{i\}$.

Claim 6.4. If $f_1(i) \leq \frac{1}{2}\binom{n-\ell-1}{k-2}$ or $f_1(i)+f_2(i,p) \leq \binom{n-\ell-1}{k-2}$ for some $p \in [\ell+1]\setminus\{i\}$, then $f(i) \leq \binom{n-2}{k-2} + \binom{n-\ell-1}{k-\ell}$.

Proof. Evidently,

$$\begin{aligned} f_3(i,p) &= \sum_{i\in Q\subset[\ell+1],\ |Q|\geq\ell} f(Q) + \sum_{i\in Q\subset[\ell+1],\ p\in Q,\ 3\leq|Q|\leq\ell-1} f(Q) \\ &\leq \ell\binom{n-\ell-1}{k-\ell} + \binom{n-\ell-1}{k-\ell-1} + \sum_{1\leq j\leq\ell-3}\binom{\ell-1}{j}\binom{n-\ell-1}{k-2-j} \\ &= \binom{n-\ell-1}{k-\ell} + \sum_{1\leq j\leq\ell-1}\binom{\ell-1}{j}\binom{n-\ell-1}{k-2-j} \\ &= \binom{n-\ell-1}{k-\ell} + \binom{n-2}{k-2} - \binom{n-\ell-1}{k-2}. \end{aligned}$$

If $f_1(i)+f_2(i,p) \leq \binom{n-\ell-1}{k-2}$ for some $p \in [\ell+1]\setminus\{i\}$, then $f(i) \leq \binom{n-2}{k-2} + \binom{n-\ell-1}{k-\ell}$ follows. If $f_1(i) \leq \frac{1}{2}\binom{n-\ell-1}{k-2}$, we are left to show that

$$f_2(i,p) < \frac{1}{2}\binom{n-\ell-1}{k-2} \text{ for some } p \in [\ell+1]\setminus\{i\}. \tag{6.14}$$

Let us fix some $p \in [\ell+1] \setminus \{i\}$. Since

$$f_2(i,p) = \sum_{i\in Q\subset[\ell+1],\ p\notin Q,\ 3\leq|Q|\leq\ell-1} f(Q) \leq \sum_{3\leq j\leq \ell-1} \binom{\ell-1}{j-1}\binom{n-\ell-1}{k-j}$$

and $n \geq 2\ell k$ implies

$$\frac{\binom{\ell-1}{j-1}\binom{n-\ell-1}{k-j}}{\binom{\ell-1}{j}\binom{n-\ell-1}{k-j-1}} = \frac{j}{\ell-j}\cdot\frac{n-k-\ell+j}{k-j} > 2,$$

it follows that for $n \geq 2\ell^2 k$,

$$f_2(i,p) \leq 2\binom{\ell-1}{2}\binom{n-\ell-1}{k-3} < (\ell-1)(\ell-2)\binom{n-\ell-1}{k-3} < \frac{1}{2}\binom{n-\ell-1}{k-2}$$

and (6.14) holds. $\square$

By Claim 6.4, it suffices to show that for some $i \in [\ell+1]$,

$$f_1(i) \leq \frac{1}{2}\binom{n-\ell-1}{k-2} \tag{6.15}$$

or for some $p \in [\ell+1]\setminus\{i\}$,

$$f_1(i) + f_2(i,p) \leq \binom{n-\ell-1}{k-2}. \tag{6.16}$$

Consider $P, P' \in \binom{[\ell+1]}{2}$ with $P \cap P' = \emptyset$. Then $\mathcal{F}(P,[\ell+1])$ and $\mathcal{F}(P',[\ell+1])$ are cross-intersecting. Let us make a graph $\mathcal{G}$ on vertex set $[\ell+1]$, drawing an edge P iff $f(P) \geq \binom{n-\ell-2}{k-3} + \binom{n-\ell-3}{k-3}$. By the Daykin Theorem there are no two disjoint edges. Hence either $\mathcal{G}$ is a triangle or a star. We distinguish four cases according to the structure of $\mathcal{G}$.

Case 1. $\mathcal{G} \neq \emptyset$ and there exists an isolated vertex in $\mathcal{G}$.

Without loss of generality, let $\{1,2\} \in \mathcal{G}$ and $\ell+1$ be isolated. Since $\mathcal{F}(\{\ell+1\},[\ell+1])$, $\mathcal{F}(\{1,2\},[\ell+1])$ are cross-intersecting and $f(\{1,2\}) \geq \binom{n-\ell-2}{k-3} + \binom{n-\ell-3}{k-3}$, by the Kruskal–Katona Theorem we have

$$f(\{\ell+1\}) \leq \binom{n-\ell-3}{k-3}.$$

Since $\ell+1$ is isolated, by the definition of $\mathcal{G}$ we infer $f(\{j,\ell+1\}) < \binom{n-\ell-2}{k-3} + \binom{n-\ell-3}{k-3}$ for $j = 1,2,\ldots,\ell$. It follows that for $n \geq (4\ell+2)k$,

$$\begin{aligned}
f_1(\ell+1) &= f(\{\ell+1\}) + \sum_{1\leq j\leq\ell} f(\{j,\ell+1\})\\
&< \binom{n-\ell-3}{k-3} + \ell\left(\binom{n-\ell-2}{k-3} + \binom{n-\ell-3}{k-3}\right)\\
&< (2\ell+1)\binom{n-\ell-2}{k-3}\\
&< \frac{1}{2}\binom{n-\ell-1}{k-2},
\end{aligned}$$

which proves (6.15).

Case 2. $\mathcal{G}$ is a star of size ℓ.

Without loss of generality, assume that $\mathcal{G} = \{\{1,2\},\{1,3\},\dots,\{1,\ell+1\}\}$ and $f(\{1,2\}) \geq f(\{1,3\}) \geq \dots \geq f(\{1,\ell+1\})$. Note the $n \geq (2\ell+1)k$ implies

$$\frac{\binom{n-\ell-3}{k-2}}{\binom{n-\ell-3}{k-4}} \geq \frac{\binom{n-\ell-3}{k-2}}{\binom{n-\ell-3}{k-3}} = \frac{n-\ell-k}{k-2} > 2\ell.$$

Since $\mathcal{F}(\{1,2\},[\ell+1])$, $\mathcal{F}(\{\ell+1\},[\ell+1])$ are cross-intersecting and $|\mathcal{F}(\{1,2\},[\ell+1])| \geq \binom{n-\ell-2}{k-3} + \binom{n-\ell-3}{k-3}$, by Lemma 6.2 we get

$$f(\{1,\ell+1\}) + 2\ell f(\{\ell+1\}) \leq f(\{1,2\}) + \frac{\binom{n-\ell-3}{k-2}}{\binom{n-\ell-3}{k-3}} f(\{\ell+1\}) \leq \binom{n-\ell-1}{k-2}. \tag{6.17}$$

Since $\mathcal{F}(\{1,2\},[\ell+1])$ and $\mathcal{F}(\{3,\ell+1\},[\ell+1])$ are cross-intersecting, by Lemma 6.2 we get

$$f(\{1,\ell+1\}) + 2\ell f(\{3,\ell+1\}) \leq f(\{1,2\}) + \frac{\binom{n-\ell-3}{k-2}}{\binom{n-\ell-3}{k-4}} f(\{3,\ell+1\}) \leq \binom{n-\ell-1}{k-2}.$$

Similarly,

$$f(\{1,\ell+1\}) + 2\ell f(\{2,\ell+1\}) \leq f(\{1,3\}) + 2\ell f(\{2,\ell+1\}) \leq \binom{n-\ell-1}{k-2}.$$

In general, we have

$$f(\{1,\ell+1\}) + 2\ell f(\{i,\ell+1\}) \leq \binom{n-\ell-1}{k-2},\ i = 2,3,\dots,\ell. \tag{6.18}$$

Adding (6.17), (6.18) over $i = 2,3,\dots,\ell$ and dividing 2ℓ, we obtain that

$$f_1(\ell+1) = f(\{\ell+1\}) + \sum_{1\leq i\leq \ell} f(\{i,\ell+1\}) \leq \frac{1}{2}\binom{n-\ell-1}{k-2} + \frac{1}{2} f(\{1,\ell+1\}). \tag{6.19}$$

For any $Q \subset [2,\ell+1]$ with $3 \leq |Q| \leq \ell-1$, there exists some $x \in [2,\ell+1] \setminus Q$. Note that $n \geq 2(\ell+1)k$ implies

$$\begin{aligned}
\frac{\binom{n-\ell-3}{k-2}}{\binom{n-\ell-3}{k-|Q|-2}} &= \frac{(n-\ell-k+|Q|-1)\dots(n-\ell-k-2)}{(k-2)\dots(k-|Q|-1)} \\
&> \left(\frac{n-\ell-k+|Q|-1}{k-2}\right)^{|Q|} \\
&\geq (2\ell)^{|Q|} > 2^{|Q|-1}\binom{\ell-1}{|Q|-1}.
\end{aligned}$$

Since $\mathcal{F}(\{1,x\},[\ell+1])$ and $\mathcal{F}(Q,[\ell+1])$ are cross-intersecting, by Lemma 6.2 we get

$$f(\{1,\ell+1\}) + 2^{|Q|-1}\binom{\ell-1}{|Q|-1} f(Q) \leq f(\{1,x\}) + \frac{\binom{n-\ell-3}{k-2}}{\binom{n-\ell-3}{k-|Q|-2}} f(Q) \leq \binom{n-\ell-1}{k-2}.$$

It follows that

$$f(Q) \leq 2^{-|Q|+1}\binom{\ell-1}{|Q|-1}^{-1}\left(\binom{n-\ell-1}{k-2} - f(\{1,\ell+1\})\right).$$

Then

$$
\begin{aligned}
f_2(\ell+1,1) &= \sum_{i\in Q\subset[\ell+1],\ 1\notin Q,\ 3\le|Q|\le\ell-1} f(Q)\\
&= \sum_{i\in Q\subset[\ell+1],\ 1\notin Q,\ 3\le|Q|\le\ell-1} 2^{-|Q|+1}\binom{\ell-1}{|Q|-1}^{-1}\left(\binom{n-\ell-1}{k-2}-f(\{1,\ell+1\})\right)\\
&= \left(\binom{n-\ell-1}{k-2}-f(\{1,\ell+1\})\right)\sum_{3\le j\le\ell-1}2^{-j+1}\\
&< \frac{1}{2}\binom{n-\ell-1}{k-2}-\frac{1}{2}f(\{1,\ell+1\}). \qquad (6.20)
\end{aligned}
$$

Adding (6.19) and (6.20), we conclude that

$$
f_1(\ell+1)+f_2(\ell+1,1)<\binom{n-\ell-1}{k-2},
$$

proving (6.16).

Case 3. $\mathcal{G}=\emptyset$ and $f(\{i\})\ge(\ell-1)\binom{n-\ell-3}{k-3}$ for $i=1,2,\ldots,\ell+1$.

Note that $\mathcal{F}(\{i\},[\ell+1])$, $\mathcal{F}(\{j\},[\ell+1])$ are cross-intersecting for $1\le i<j\le\ell+1$. By the Daykin Theorem, one of $\mathcal{F}(\{i\},[\ell+1])$ and $\mathcal{F}(\{j\},[\ell+1])$ has size at most $\binom{n-\ell-2}{k-2}$. Without loss of generality assume

$$
|\mathcal{F}(\{\ell+1\},[\ell+1])|\le\binom{n-\ell-2}{k-2}. \qquad (6.21)
$$

Note that for any $Q\subsetneq[\ell+1]$ and $i\in[\ell+1]\setminus Q$, $\mathcal{F}(\{i\},[\ell+1])$, $\mathcal{F}(Q,[\ell+1])$ are cross-intersecting. Since $|\mathcal{F}(\{i\},[\ell+1])|\ge(\ell-1)\binom{n-\ell-3}{k-3}$ for all $i=1,2,\ldots,\ell+1$, by Corollary 2.2

$$
|f(Q)|\le\binom{n-\ell-2}{k-|Q|-1}+\binom{n-2\ell-1}{k-|Q|-\ell+1}.
$$

It follows that

$$
\sum_{\ell+1\in Q\subset[\ell+1],\ |Q|\ge2} f(Q)\le\sum_{2\le j\le\ell}\binom{\ell}{j-1}\left(\binom{n-\ell-2}{k-j-1}+\binom{n-2\ell-1}{k-j-\ell+1}\right) \qquad (6.22)
$$

Adding (6.21), (6.22) and $|\mathcal{F}([\ell+1],[\ell+1])|\le\binom{n-\ell-1}{k-\ell-1}$, for $n\ge4k$ we obtain that

$$
f(\ell+1)\le\binom{n-\ell-2}{k-2}+\sum_{2\le j\le\ell}\binom{\ell}{j-1}\left(\binom{n-\ell-2}{k-j-1}+\binom{n-2\ell-1}{k-j-\ell+1}\right)+\binom{n-\ell-1}{k-\ell-1}.
$$

Using $\binom{n-2}{k-2}=\sum_{1\le j\le\ell+1}\binom{\ell}{j-1}\binom{n-\ell-2}{k-j-1}$, it follows that

$$
f(\ell+1)\le\binom{n-2}{k-2}+\sum_{2\le j\le\ell}\binom{\ell}{j-1}\binom{n-2\ell-1}{k-j-\ell+1}+\binom{n-\ell-2}{k-\ell-1}.
$$

Since for $2\le j\le\ell-1$ and $n\ge2(\ell+1)k$

$$
\frac{\binom{\ell}{j}\binom{n-2\ell-1}{k-j-\ell}}{\binom{\ell}{j-1}\binom{n-2\ell-1}{k-j-\ell+1}}=\frac{(\ell-j+1)(k-j-\ell+1)}{j(n-\ell-k+j-1)}<\frac{\ell(k-\ell)}{n-\ell-k}<\frac{1}{2},
$$

we have

$$\sum_{2\leq j\leq \ell}\binom{\ell}{j-1}\binom{n-2\ell-1}{k-j-\ell+1}<2\ell\binom{n-2\ell-1}{k-\ell-1}<2\ell\binom{n-\ell-2}{k-\ell-1}.$$

Thus for $n\geq (2\ell+1)k$,

$$f(\ell+1)<\binom{n-2}{k-2}+(2\ell+1)\binom{n-\ell-2}{k-\ell-1}<\binom{n-2}{k-2}+\binom{n-\ell-1}{k-\ell}$$

and the theorem follows.

Case 4. $\mathcal{G}=\emptyset$ and $f(\{i\})<(\ell-1)\binom{n-\ell-3}{k-3}$ for some $i\in[\ell+1]$.

Without loss of generality assume that $f(\{\ell+1\})<(\ell-1)\binom{n-\ell-3}{k-3}$. By $\mathcal{G}=\emptyset$, we infer that

$$f(\{j,\ell+1\})\leq\binom{n-\ell-2}{k-3}+\binom{n-\ell-3}{k-3},\ j=1,2,\ldots,\ell+1.$$

Then for $n\geq 6\ell k$ we obtain that

$$f_1(\ell+1)=\sum_{1\leq j\leq \ell}f(\{j,\ell+1\})+f(\{\ell+1\})<3\ell\binom{n-\ell-2}{k-3}\leq\frac{1}{2}\binom{n-\ell-1}{k-2},$$

proving (6.15). $\square$

7 Concluding remarks

In this paper, we mainly considered the maximum ith degree of an intersecting family for $2\leq i\leq k+1$ and $i=2k+1$.

One of the main results states that every intersecting k-graph on n vertices has at least $n-2k$ vertices of degree at most $\binom{n-2}{k-2}$ for $n\geq 6k$. Recall that the Huang-Zhao Theorem showed that every intersecting k-graph on n vertices has a vertex of degree at most $\binom{n-2}{k-2}$ for $n>2k$. An intriguing problem is whether our result holds for the full range.

Conjecture 7.1. Let $\mathcal{F}\subset\binom{[n]}{k}$ be an intersecting family with $n>2k$. Then there are at least $n-2k$ vertices of degree at most $\binom{n-2}{k-2}$.

In Theorem 1.9 we proved that every t-intersecting k-graph on n vertices has at least $n-k-1$ vertices of degree at most $\binom{n-t-1}{k-t-1}$ for $n\geq\binom{t+2}{2}k^2$. Let us close this paper by a stronger conjecture.

Conjecture 7.2. Let $\mathcal{F}\subset\binom{[n]}{k}$ be a t-intersecting family. Then for some absolute constant c and $n\geq ckt$, $d_{k+2}(\mathcal{F})\leq\binom{n-t-1}{k-t-1}$.

Acknowledgement. The second author was supported by National Natural Science Foundation of China Grant no. 12471316 and Natural Science Foundation of Shanxi Province Grant no. RD2500002993.

References

[1] D.E. Daykin, Erdős-Ko-Rado from Kruskal-Katona, J. Combin. Theory, Ser. A 17 (1972), 254–255.

[2] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. 12 (1961), 313–320.

[3] P. Frankl, The Erdős-Ko-Rado theorem is true for $n = ckt$, Coll. Math. Soc. J. Bolyai 18 (1978), 365–375.

[4] P. Frankl, A new short proof for the Kruskal-Katona theorem, Discrete Math. 48 (1984), 327–329.

[5] P. Frankl, The shifting technique in extremal set theory, Surveys in Combinatorics 123 (1987), 81–110.

[6] P. Frankl, Erdős-Ko-Rado theorem with conditions on the maximal degree, J. Comb. Theory, Ser. A 46(2) (1987), 252–263.

[7] P. Frankl, J. Wang, Intersections and distinct intersections in cross-intersecting families, Europ. J. Combin. 110 (2022), 103665.

[8] P. Frankl, J. Wang, On the C-diversity of intersecting hypergraphs, Eur. J. Comb. 130 (2025), 104199.

[9] Z. Füredi, J. R. Griggs, Families of finite sets with minimum shadows, Combinatorica 6 (1986), 355–363.

[10] A.J.W. Hilton, The Erdős-Ko-Rado Theorem with valency conditions, unpublished manuscript, 1976.

[11] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets, Quart.J. Math. Oxford Ser. 18 (1967), 369–384.

[12] H. Huang, Y. Zhao, Degree versions of the Erdős-Ko-Rado Theorem and Erdős hypergraph matching conjecture, Journal of Combinatorial Theory, Ser. A, 150 (2017), 233–247.

[13] G.O.H. Katona, A theorem of finite sets, Theory of Graphs. Proc. Colloq. Tihany, Akad. Kiadó (1966), 187–207.

[14] J.B. Kruskal, The number of simplices in a complex, Mathematical Optimization Techniques 251 (1963), 251–278.

[15] A. Kupavskii, D. Zakharov, Regular bipartite graphs and intersecting families, J. Comb. Theory Ser. A 155 (2018), 180-189.

[16] R. M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, Combinatorica 4 (1984), 247–257.