Odd Induced Subgraphs in Graphs of Maximum Degree Four

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Abstract

A graph is called *odd* if all of its vertex degrees are odd. A long-standing conjecture asked whether there exists a positive constant c such that every n-vertex graph without isolated vertices contains an odd induced subgraph on at least cn vertices. In 2022, Ferber and Krivelevich resolved this conjecture affirmatively with $c = 10^{-4}$. A natural question is to determine the largest possible constant c. In 1994, Caro remarked that if $2/7$ is a valid value for c, then it is the largest possible one. To the best of our knowledge, the bound $c \geq 2/7$ has not been improved. Previous research has established tight bounds for specific graph classes—for instance, $c = 2/5$ for graphs with maximum degree at most 3 and without isolated vertices. In this paper, we prove that $c = 2/7$ is the tight bound for graphs with maximum degree at most 4 and without isolated vertices. Our result provides some support for $2/7$ being the largest value of c.

Keywords: odd induced subgraphs, maximum degree, extremal graph theory.

1 Introduction

A graph is called *even* (respectively, *odd*) if every vertex has even (respectively, odd) degree. By a classical theorem of Gallai, the vertex set of every graph can be partitioned into two parts, each inducing an even subgraph. In particular, every n-vertex graph has an induced even subgraph on at least $n/2$ vertices. No such statement holds for odd induced subgraphs in general, due to the possible presence of isolated vertices.

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A long-standing conjecture [2, 8] stated that there is a constant $c > 0$ such that every n-vertex graph without isolated vertices has an odd induced subgraph with at least cn vertices. This conjecture attracted considerable attention. Caro [2] proved that every n-vertex graph without isolated vertices has an odd induced subgraph on at least $\Omega(\sqrt{n})$ vertices. This result was improved to $\Omega(n/\log n)$ by Scott [8]. Recently, Ferber and Krivelevich [3] confirmed the conjecture, proving that one can take $c = 10^{-4}$. This raises a natural question: what is the largest value of c? (Formally, if $f_o(G)$ denotes the size of a largest induced odd subgraph in G, then we can define c as the infimum of $f_o(G)/|V(G)|$ over all graphs G without isolated vertices.) Caro [2] mentioned, as a known fact, that $c \leq \frac{2}{7}$. Are there n-vertex graphs without isolated vertices in which all odd induced subgraphs have less than $2n/7$ vertices? Theorem 1.1 gives some support for the negative answer.

For several graph classes, the optimal constant c is already known. Radcliffe and Scott [6] proved that every n-vertex tree has an odd induced subgraph of order at least $2\lfloor(n+1)/3\rfloor$. Berman et al. [1] established the optimal value $c = 2/5$ for graphs of maximum degree 3, and Hou et al. [4] showed that the same holds for graphs of treewidth at most two. Rao et al. [7] proved that $c \geq \frac{2}{5}$ for planar graphs with girth at least 7, and $c \geq \frac{1}{3}$ for planar graphs with girth at least 6, and that both bounds are tight.

In this paper, we extend the degree-bound result of Berman et al. [1]. We determine the maximum constant c such that every graph with maximum degree at most 4 and no isolated vertices contains an odd induced subgraph on at least cn vertices.

Theorem 1.1. *Every graph of order n without isolated vertices and with maximum degree at most four has an odd induced subgraph of order at least $2n/7$. Furthermore, this bound is sharp.*

A related line of research concerns a conjecture of Scott [8] who proposed that every n-vertex graph with chromatic number χ and no isolated vertices contains an odd induced subgraph with at least $n/(2\chi)$ vertices. Recent work of Wang and Wu [9] disproved this conjecture for bipartite graphs, while confirming it for all line graphs; the conjecture remains open for graphs with $\chi \geq 3$.

In the next section, we prove Theorem 1.1. We conclude the paper in Section 3.

2 Proof of Theorem 1.1

Assume that the theorem is false and let G be a counterexample to the theorem of the smallest possible order. Note that G is connected. We will prove several claims concerning G, ending in a contradiction, which will complete the proof. Let G have order n, $\Delta(G) \leq 4$, and have no isolated vertices. Note that when $n \leq 7$, every edge forms an odd induced subgraph of order at least $2n/7$, so we assume $n > 7$.

Claim 1. $\delta(G) > 1$.

Proof. For the sake of contradiction, assume that $\delta(G) = 1$. Let $p \in V(G)$ be a vertex of degree one, and $N_G(p) = \{x\}$. Let $G' = G - N_G[x]$ and $N_G(x) = \{p, y_1, \ldots, y_{d_G(x)-1}\}$. Let I be the set of isolated vertices in G' and let $G'' = G' - I$. Then G'' contains an odd induced subgraph H of order at least $2|V(G'')|/7$ by the minimality of n.

Note that $|I \cup N_G[x]| \geq 8$, since otherwise H together with the edge px is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. Therefore, $d_G(x) \geq 3$ and $9 \geq |I| \geq 3$.

Note that every y_i which has two neighbors in I, must have one neighbor in $V(G'')$, since otherwise suppose that $\{z_1, z_2\} \subseteq N_G(y_1) \cap I$ and y_1 has no neighbors in $V(G'')$, and H together with $G[\{x, y_1, z_1, z_2\}]$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

If $d_G(x) = 3$, then $|I| = 4$, y_1 is not adjacent to y_2 and $|N_G(y_1) \cap I| = |N_G(y_1) \cap I| = 2$. Let $N_G(y_1) \cap I = \{z_1, z_2\}$. Let $G^* = G - \{p, x, y_2\} - I$. Then G^* contains an odd induced subgraph H^* of order at least $2|V(G^*)|/7 = 2(|V(G)| - 7)/7$ and H^* together with the edge px if $y_1 \notin V(H^*)$ or H^* together with the edge set $\{y_1z_1, y_1z_2\}$ if $y_1 \in V(H^*)$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

Now we have that $d_G(x) = 4$ and $3 \leq |I| \leq 6$.

Case 1: $5 \leq |I| \leq 6$.

Without loss of generality, assume $|N_G(y_1) \cap I| = |N_G(y_2) \cap I| = 2$ and $(N_G(y_1) \cap I) \cap (N_G(y_2) \cap I) = \emptyset$. Then y_1 is not adjacent to y_2. Let $W = N_G(x) \cup N_G(y_1) \cup N_G(y_2)$ and let G^* be the graph obtained from G by deleting W and all isolated vertices in the resulting graph. Then G^* contains an odd induced subgraph H^* of order at least $2|V(G^*)|/7 \geq 2(|V(G)| - 19)/7$. Let $N_G(y_1) \cap I = \{z_1, z_2\}$ and $N_G(y_2) \cap I = \{z_3, z_4\}$. Then H^* together with $G[\{p, x, y_1, y_2, z_1, z_2, z_3, z_4\}]$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

Case 2: $3 \leq |I| \leq 4$.

If $|N_G(y_{i_0}) \cap I| = 2$ for some $i_0 \in \{1, 2, 3\}$, then let $W = N_G(x) \cup N_G(y_{i_0})$ and let G^* be the graph obtained from G by deleting W and all isolated vertices in the resulting graph. Then G^* contains an odd induced subgraph H^* of order at least $2|V(G^*)|/7 \geq 2(|V(G)| - 13)/7$. Let $N_G(y_{i_0}) \cap I = \{z_1, z_2\}$. H^* together with $G[\{x, y_{i_0}, z_1, z_2\}]$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

Thus $|N_G(y_1) \cap I| = |N_G(y_2) \cap I| = |N_G(y_3) \cap I| = 1$ and $|I| = 3$. Let $N_G(y_i) \cap I = \{z_i\}$ for $i \in \{1, 2, 3\}$.

If each y_i has no neighbors in $V(G'')$, then $V(G) = N_G[x] \cup I$ and $G[\{x, y_1, y_2, y_3\}]$ (if y_i is not adjacent to y_j for every $i, j \in \{1, 2, 3\}$), or $G[\{p, x, y_{i_1}, y_{i_2}, z_{i_1}, z_{i_2}\}]$ (if y_{i_1} is adjacent to y_{i_2} for some $i_1, i_2 \in \{1, 2, 3\}$) is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

So, without loss of generality, assume that y_1 has a neighbor in $V(G'')$. Let $G^* = G - \{p, x, y_2, y_3\} - I$. Then G^* has no isolated vertices and it contains an odd induced subgraph H^* of order at least $2|V(G^*)|/7 = 2(|V(G)| - 7)/7$. Then H^* together with the edge px if $y_1 \notin V(H^*)$ or H^* together with the edge set $\{xy_1, y_1z_1\}$ if $y_1 \in V(H^*)$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. $\square$

Claim 2. $\delta(G) > 2$.

Proof. For the sake of contradiction, assume that $\delta(G) = 2$.

Let $p \in V(G)$ such that $N_G(p) = \{x_1, x_2\}$. Let $W = N_G(p) \cup N_G(x_1)$. Then $|W| \leq 6$. Let $G' = G - W$ and let I be the set of isolated vertices in G'. Since $\delta(G) = 2$, $|I| \leq 6$. If $|W \cup I| \leq 7$, then the odd induced subgraph of order at least $2(|V(G)| - 7)/7$ in $G' - I$, together with the edge px_1 is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. Thus, $|W \cup I| \geq 8$ and $2 \leq |I| \leq 6$.

Subclaim 2.1. *Each vertex in W has at most one neighbor in I.*

Proof. Suppose that $u \in W$ has two neighbors in I.

Note that $N_G(u) \setminus (W \cup I) \neq \emptyset$, since otherwise, let $\{z_1, z_2\} \subseteq N_G(u) \cap I$, and without loss of generality, assume that $u \in N_G(x_1) \setminus \{p\}$, and the odd induced subgraph of order at least $2(|V(G)| - 12)/7$ in $G' - I$, together with the edge set $\{x_1u, uz_1, uz_2\}$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

Then every vertex in W which has two neighbors in I, has one neighbor outside $W \cup I$. Then $|I| \leq 4$, since $\delta(G) = 2$. Without loss of generality, assume that $u \in N_G(x_1) \setminus \{p\}$ and $N_G(u) \cap I = \{z_1, z_2\}$ and $N_G(u) \setminus (W \cup I) = \{k\}$. Then the odd induced subgraph of order at least $2(|V(G)| - 14)/7$ in the graph obtained from $G' - I - \{k\}$ by deleting all isolated vertices, together with the edge set $\{x_1u, uz_1, uz_2\}$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. $\square$

Subclaim 2.1 implies that $|I| = 2$ and $|W| = 6$. Then each vertex in $W \setminus \{p, x_1\}$ has exactly one neighbor in I. Let $N_G(x_1) = \{p, y_1, y_2, y_3\}$ and $I = \{z_1, z_2\}$. Without loss of generality, let $N_G(z_1) = \{y_1, y_2\}$ and $N_G(z_2) = \{y_3, x_2\}$. If each vertex in $\{x_2, y_1, y_2, y_3\}$ has no neighbors outside $W \cup I$, then $V(G) = W \cup I$ and $G[\{p, x_1, y_1, y_2\}]$, or $G[\{p, x_1, y_1, y_3\}]$, or $G[\{p, x_1, y_2, y_3\}]$, or $G[y_1, z_1, x_2, z_2]$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. Without loss of generality, suppose that $N_G(y_1) \setminus (W \cup I) \neq \emptyset$. Let $W^* = W \setminus \{y_1\}$ and $G^* = G - W^* - I$. Then the odd induced subgraph H^* of order at least $2(|V(G)| - 7)/7$ in G^*, together with the edge px_1 (if $y_1 \notin V(H^*)$) or together with the edge set $\{x_1y_1, y_1z_1\}$ (if $y_1 \in V(H^*)$), is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. $\square$

Claim 3. *G contains no triangles.*

Proof. For the sake of contradiction, assume that G contains a triangle. Let ab be an edge of this triangle and $W = N_G(a) \cup N_G(b)$. Then $|W| \leq 7$. Let $G' = G - W$. Let I be the set of isolated vertices in G'.

Note that $|W \cup I| \geq 8$, since otherwise the odd induced subgraph of order at least $2(|V(G)| - 7)/7$ in $G' - I$, together with the edge ab is an odd induced subgraph of order at least $2|V(G)|/7$. Therefore, $1 \leq |I| \leq 4$.

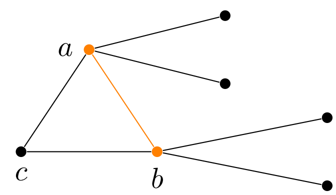

Figure 1: The graph $G[W]$ when G contains a triangle.

Subclaim 3.1. *Each vertex in W has at most one neighbor in I.*

Proof. Suppose that $u \in W$ has two neighbors in I.

Note that $N_G(u) \setminus (W \cup I) \neq \emptyset$, since otherwise, let $\{z_1, z_2\} \subseteq N_G(u) \cap I$, and without loss of generality, assume that $u \in N_G(a) \setminus \{b\}$ and the odd induced subgraph of order at least $2(|V(G)| - 11)/7$ in $G' - I$, together with the edge set $\{au, uz_1, uz_2\}$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

So, every vertex in W which has two neighbors in I, has one neighbor outside $W \cup I$. Then $|I| \leq 3$ since $\delta(G) \geq 3$. Without loss of generality, assume that $u \in N_G(a) \setminus \{b\}$ and $N_G(u) \cap I = \{z_1, z_2\}$ and $N_G(u) \setminus (W \cup I) = \{k\}$. Then the odd induced subgraph of order at least $2(|V(G)| - 14)/7$ in the graph obtained from $G' - I - \{k\}$ by deleting all isolated vertices, together with the edge set $\{au, uz_1, uz_2\}$ is an odd induced subgraph of order at least $2|V(G)|/7$. □

Subclaim 3.1 implies that $|I| = 1$ and $|W| = 7$. Let $I = \{z\}$, $N_G(a) = \{b, c, y_1, y_2\}$ and $N_G(b) = \{a, c, y_3, y_4\}$. Let $W' = \{w \in W : N_G(w) \setminus (W \cup \{z\}) \neq \emptyset\}$.

Case 1: $W' = \emptyset$.

In this case, we have that $V(G) = W \cup \{z\}$. Note that y_1 is adjacent to y_2, and y_3 is adjacent to y_4, since otherwise $G[\{a, b, y_1, y_2\}]$ or $G[\{a, b, y_3, y_4\}]$ is an odd induced subgraph of order at least $2|V(G)|/7$. Moreover, if neither y_1 nor y_2 is adjacent to c, then the edge set $\{bc, y_1y_2\}$ forms an odd induced subgraph of order at least $2|V(G)|/7$. Thus, without loss of generality, we can assume that y_1 is adjacent to c and by symmetry, assume that y_3 is adjacent to c. Then $d_G(c) = 4$ and $N_G(c) = \{a, b, y_1, y_3\}$. Since $\delta(G) \geq 3$, y_2 or y_4 is adjacent to z. Then $G[\{b, c, y_2, z\}]$ or $G[\{a, c, y_4, z\}]$ is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

Case 2: There is a neighbor u of z such that $u \in W'$.

Let $W^* = (W \cup \{z\}) \setminus \{u\}$ and $G^* = G \setminus W^*$. Then G^* has no isolated vertices, and it has an odd induced subgraph H of order at least $2(|V(G)| - 7)/7$. If H contains u, then $G[V(H) \cup \{z, a\}]$ (if $u \in \{c, y_1, y_2\}$) or $G[V(H) \cup \{z, b\}]$ (if $u \in \{c, y_3, y_4\}$) is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. If not, then H together with the edge ab is an odd induced subgraph of order at least $2|V(G)|/7$, which is also a contradiction.

Case 3: $W' \neq \emptyset$ and $N_G(z) \cap W' = \emptyset$.

Note that there are some $w \in W'$ and some $u \in N_G(z)$ such that w is not adjacent to u since $\delta(G) \geq 3$ and $\Delta(G) \leq 4$. Let $W^* = (W \cup \{z\}) \setminus \{w\}$ and $G^* = G \setminus W^*$. Then G^* has no isolated vertices and it has an odd induced subgraph H of order at least $2(|V(G)| - 7)/7$. H together with the edge zu is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. $\square$

Claim 4. *G is 4-regular.*

Proof. For the sake of contradiction, assume that there is a vertex, say p, such that $N_G(p) = \{x_1, x_2, x_3\}$. Then let $N_G(x_1) = \{p, y_1, \ldots, y_{d_G(x_1)-1}\}$, and $W = N_G(p) \cup N_G(x_1)$ and $G' = G - W$. Let I be the set of isolated vertices in G'.

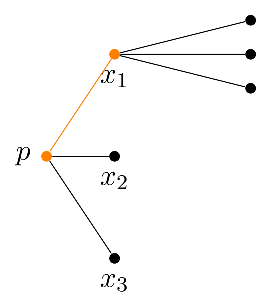

Figure 2: The graph $G[W]$ when G contains no triangles and $\delta(G) = 3$.

Note that $|W \cup I| \geq 8$, since otherwise the odd induced subgraph of order at least $2(|V(G)| - 7)/7$ in $G' - I$, together with the edge px_1 is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. Therefore, $1 \leq |I| \leq 5$.

If $|I| = 4$ or 5, then $|N_G(y_{i_0}) \cap I| = 3$ for some $i_0 \in [d_G(x_1) - 1]$ or $|N_G(x_{j_0}) \cap I| = 3$ for some $j_0 \in \{2, 3\}$. Without loss of generality, let $N_G(y_1) \cap I = \{z_1, z_2, z_3\}$. Then the odd induced subgraph of order at least $2(|V(G)| - 12)/7$ in $G' - I$, together with the edge set $\{y_1z_1, y_1z_2, y_1z_3\}$ is an odd induced subgraph of order at least $2|V(G)|/7$.

If $|I|=2$ or 3, then $|N_G(y_{i_0})\cap I|\geq 2$ for some $i_0\in[d_G(x_1)-1]$ or $|N_G(x_{j_0})\cap I|\geq 2$ for some $j_0\in\{2,3\}$. Without loss of generality, let $\{z_1,z_2\}\subseteq N_G(y_1)\cap I$. Let $G^*=G-W-I-N_G(y_1)$. Then the odd induced subgraph of order at least $2(|V(G)|-14)/7$ in the graph obtained from G^* by deleting all isolated vertices, together with the edge set $\{x_1y_1,y_1z_1,y_1z_2\}$ is an odd induced subgraph of order at least $2|V(G)|/7$.

If $|I|=1$, then $|W|=7$. Let $I=\{z_1\}$ and $W'=\{w\in W \mid N_G(w)\setminus(W\cup\{z_1\})\neq\emptyset\}$. Note that $W'\neq\emptyset$, since otherwise $V(G)=W\cup\{z_1\}$ and $G[\{p,x_1,x_2,x_3\}]$ is an odd induced subgraph of order at least $2|V(G)|/7$. If there is a neighbor of z_1, without loss of generality, say y_1, such that $y_1\in W'$, then let $W^*=(W\cup\{z_1\})\setminus\{y_1\}$ and $G^*=G-W^*$. The odd induced subgraph H^* of order at least $2(|V(G)|-7)/7$ in G^*, together with the edge px_1 (if $y_1\notin V(H^*)$) or together with the edge set $\{x_1y_1,y_1z_1\}$ (if $y_1\in V(H^*)$), is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. Therefore, $N_G(z_1)\cap W'=\emptyset$. Since $d_G(z_1)\geq 3$, we can always find $w\in W'$ and $u\in N_G(z_1)$ such that w is not adjacent to u. Let $W^*=(W\cup\{z_1\})\setminus\{w\}$ and $G^*=G-W^*$. Then the odd induced subgraph H^* of order at least $2(|V(G)|-7)/7$ in G^*, together with the edge z_1u is an odd induced subgraph of order at least $2|V(G)|/7$. $\square$

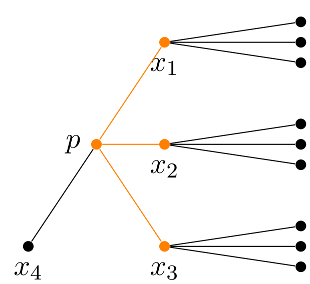

Figure 3: The graph $G[W]$ when G contains no triangles and is 4-regular.

Let p be an arbitrary vertex of G. Let $N_G(p)=\{x_1,x_2,x_3,x_4\}$, $W=N_G(p)\cup N_G(x_1)\cup N_G(x_2)\cup N_G(x_3)$ and $G'=G-W$. Then $|W|\leq 14$. Let I be the set of isolated vertices in G'.

Note that $|W\cup I|\geq 15$, since otherwise $G'-I$ contains an odd induced subgraph H' of order at least $2(|V(G)|-14)/7$, and H' together with the edge set $\{px_1,px_2,px_3\}$ is an odd induced subgraph of order at least $2|V(G)|/7$. Therefore, $1\leq|I|\leq 7$.

Firstly, we give three useful observations to describe the neighbors of the vertex in W.

Observation 1. *Each vertex in W has at most two neighbors in I.*

Proof. Suppose that there is a vertex in the set W that has three neighbors in I. Since the neighborhoods of p, x_1, x_2 and x_3 are all contained in W, without loss of generality, we may assume that x_4, or one vertex in $N_G(x_1)\setminus\{p\}$, say y_1, has three neighbors in I.

Firstly, if x_4 has three neighbors in I, say z_1, z_2 and z_3, then the odd induced subgraph of order at least $2(|V(G)|-|W|-|I|)/7 \geq 2(|V(G)|-21)/7$ in $G'-I$, together with the edge set $\{z_1x_4, z_2x_4, x_4p, px_1, px_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction.

If y_1 has three neighbors in I, also say z_1, z_2 and z_3, then the odd induced subgraph of order at least $2(|V(G)|-|W|-|I|)/7 \geq 2(|V(G)|-21)/7$ in $G'-I$, together with the edge set $\{px_2, y_1z_1, y_1z_2, y_1z_3\}$, is an odd induced subgraph of order at least $2|V(G)|/7$, which is a contradiction. □

Observation 2. *If $|I| \leq 5$, then x_4 has at most one neighbor in I.*

Proof. For the sake of contradiction, assume that $N_G(x_4) \cap I = \{z_1, z_2\}$.

Firstly, if $N_G(x_4) \subseteq W \cup I$, then the odd induced subgraph of order at least $2(|V(G)|-|W|-|I|)/7 \geq 2(|V(G)|-19)/7$ in G', together with the edge set $\{z_1x_4, z_2x_4, x_4p, px_1, px_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$.

So, let $N_G(x_4) \setminus (W \cup I) = \{k\}$. Let $G^* = G - W - I - \{k\}$. If G^* contains at most one isolated vertex, the odd induced subgraph of order at least

$$2(|V(G)|-|W|-|I|-1-1)/7 \geq 2(|V(G)|-21)/7$$

in G^*, together with the edge set $\{z_1x_4, z_2x_4, x_4p, px_1, px_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$. Therefore, G^* contains at least two isolated vertices, say t_1 and t_2. Then let $G_2^* = G^* - N_G(k)$. Then the odd induced subgraph of order at least

$$2(|V(G)|-|W|-|I|-1-6)/7 \geq 2(|V(G)|-26)/7$$

in the graph obtained from G_2^* by deleting all isolated vertices, together with the edge set $\{px_1, px_2, px_4, x_4z_1, x_4k, kt_1, kt_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$. □

Observation 3. *If $|W \cup I| = 15$, then x_4 has no neighbors in I or has no neighbors outside $W \cup I$.*

Proof. For the sake of contradiction, assume that $z \in N_G(x_4) \cap I$ and $N_G(x_4) \setminus (W \cup I) \neq \emptyset$. Let $W^* = W \setminus \{x_4\}$ and $G^* = G - W^* - I$. Then the odd induced subgraph H^* of order at least $2(|V(G)|-14)/7$ in G^*, together with the edge set $\{px_1, px_2, px_3\}$ (if $x_4 \notin V(H^*)$), or together with the edge set $\{x_4z, x_4p, px_1, px_2\}$ (if $x_4 \in V(H^*)$), is an odd induced subgraph of order at least $2|V(G)|/7$. □

Then $|I| \leq 4$, and moreover $12 \leq |W| \leq 14$, and if $|W| = 12$ then $|I| = 3$, by Observation 1 and Observation 2.

Let $W_0 = \{w \in W : |N_G(w) \cap I| = 0\}$, $W_1 = \{w \in W : |N_G(w) \cap I| = 1\}$ and $W_2 = \{w \in W : |N_G(w) \cap I| = 2\}$. Observation 1 implies that $W = W_0 \cup W_1 \cup W_2$. Observation 2 implies that $x_4 \notin W_2$.

If $|W| = 12$ and $|I| = 3$, then $4 \le |W_2| \le 6$. If $|W_2| = 4$, then $|W_1| = 4$, $|W_0| = 0$, and by Observation 2 and 3, $|N_G(x_4) \cap I| = 1$ and $N_G(x_4) \setminus (W \cup I) = \emptyset$. Without loss of generality, assume that $y_1 \in N_G(x_1) \setminus (N_G(x_2) \cup N_G(x_3))$ such that $y_1 \in W_2$. Let $z \in (N_G(y_1) \cap I) \setminus N_G(x_4)$. The odd induced subgraph of order at least $2(|V(G)| - 12 - 3 - 4)/7 \ge 2(|V(G)| - 19)/7$ in the graph obtained from $G' - I - N_G(y_1)$ by deleting all isolated vertices, together with the edge set $\{px_2, px_3, px_4, y_1z\}$ (if $x_4 \notin N_G(y_1)$) or together with $G[\{p, x_4, y_1\} \cup I]$ (if $x_4 \in N_G(y_1)$), is an odd induced subgraph of order at least $2|V(G)|/7$. If $|W_2| \ge 5$, then $|W_1| = 2$ or $|W_2| = 6$. Therefore, without loss of generality, assume that $y_1, y_2 \in N_G(x_1) \setminus (N_G(x_2) \cup N_G(x_3))$ such that $y_1 \in W_2$ and $y_2 \in W_1 \cup W_2$. Let $K = N_G(y_1) \cup N_G(y_2) \setminus (W \cup I)$. Then $|K| \le 3$. Let $G^* = G - W - I - K$ and let I^* be the set of isolated vertices in G^*. Then we have that

$$|I^*| \le \left\lfloor \frac{1}{4}\left(\sum_{v \in W \setminus \{p, x_1, x_2, x_3\}} |N_G(v) \cap V(G')| + \sum_{u \in K} (d_G(u) - 1) - 4|I| - |K| \right) \right\rfloor .$$

If $|K| \le 2$, we have that

$$|I^*| \le \max\{ \left\lfloor \frac{1}{4}(22 + 6 - 12 - 2) \right\rfloor, \left\lfloor \frac{1}{4}(22 + 3 - 12 - 1) \right\rfloor, \left\lfloor \frac{1}{4}(22 - 12) \right\rfloor \} = 3,$$

and the odd induced subgraph of order at least $2(|V(G^*)| - |I^*|)/7 \ge 2(|V(G)| - 20)/7$ in $G^* - I^*$, together with the edge set $\{px_1, px_2, px_3, x_1y_1, x_1y_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$. Thus $|K| = 3$ and $y_2 \in W_1$, and $|I^*| \le \lfloor \frac{1}{4}(22 + 9 - 12 - 3) \rfloor = 4$. If $|I^*| \le 3$, similarly, the odd induced subgraph of order at least $2(|V(G^*)| - |I^*|)/7 \ge 2(|V(G)| - 21)/7$ in $G^* - I^*$, together with the edge set $\{px_1, px_2, px_3, x_1y_1, x_1y_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$. If $|I^*| = 4$, one can find that $V(G) = W \cup I \cup K \cup I^*$ and x_4 has at least two neighbors in I^*, say t_1 and t_2. Then $G[\{p, x_1, x_2, x_4, y_1, y_2, t_1, t_2\}]$ is an odd induced subgraph of order $8 > 2|V(G)|/7$.

Claim 5. *G does not contain two different cycles of length four that have common edges.*

Proof. If G contains two different cycles of length four that have common edges, we can let p be the vertex in the two cycles of length four which is adjacent to three vertices in the two cycles, and let these three vertices be x_1, x_2 and x_3, and let the other neighbor of p in G be x_4. The claim follows by doing the same discussions all above. □

If $|W| = 13$, we have that $|I| = 2$, 3 or 4.

(a) One common edge   (b) Two common edges

Figure 4: Two different cycles of length four with common edges.

If $|W| = 13$ and $|I| = 2$, let $N_G(x_1) = \{p, y_1, y_2, y_3\}$, $N_G(x_2) = \{p, y_3, y_4, y_5\}$, $N_G(x_3) = \{p, y_6, y_7, y_8\}$ and $I = \{z_1, z_2\}$. Note that $N_G(x_4) \setminus (W \cup I) \neq \emptyset$ by Claim 5. Then by Observation 3, $x_4 \in W_0$. If $y_3 \in W_1 \cup W_2$, let $z_1 \in N_G(y_3) \cap I$, and by Claim 5, $N_G(z_1) \subseteq \{y_3, y_6, y_7, y_8\}$ but $\{y_6, y_7, y_8\} \not\subseteq N_G(z_1)$, which contradicts to $d_G(z_1) = 4$. Thus $y_3 \in W_0$. Moreover, by Claim 5, every two $y_i^{\prime}s$ which are adjacent to the same x_j, cannot belong to W_2 at the same time, and $1 \leq |W_2| \leq 3$. Without loss of generality, assume that $y_1 \in W_2$ and $y_2 \in W_1$ and $N_G(y_2) \cap I = \{z_1\}$. Let $K = N_G(y_1) \cup N_G(y_2) \setminus (W \cup I)$. Then $|K| \leq 3$. Let $G^* = G - W - I - K$ and let I^* be the set of isolated vertices in G^*. Similarly, if $|K| \leq 2$, we have that

$$|I^*| \leq \max\{\left\lfloor \frac{1}{4}(26+6-8-2) \right\rfloor, \left\lfloor \frac{1}{4}(26+3-8-1) \right\rfloor, \left\lfloor \frac{1}{4}(26-8) \right\rfloor\} = 5.$$

If $|I^*| \leq 4$, then the odd induced subgraph of order at least $2(|V(G^*)| - |I^*|)/7 \geq 2(|V(G)| - 21)/7$ in $G^* - I^*$, together with the edge set $\{px_1, px_2, px_3, x_1y_1, x_1y_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$. If $|I^*| = 5$, then $|K| = 2$ and one vertex in K, say k, has three neighbors in I^*, say t_1, t_2 and t_3. Then the odd induced subgraph of order at least $2(|V(G^*)| - |I^*|)/7 \geq 2(|V(G)| - 22)/7$ in $G^* - I^*$, together with the edge set $\{px_1, px_2, px_3, kt_1, kt_2, kt_3\}$, is an odd induced subgraph of order at least $2|V(G)|/7$.

Therefore, $|K| = 3$ and $|I^*| \leq \left\lfloor \frac{1}{4}(26+9-8-3) \right\rfloor = 6$. If $|I^*| \leq 3$, the odd induced subgraph of order at least $2(|V(G^*)| - |I^*|)/7 \geq 2(|V(G)| - 21)/7$ in $G^* - I^*$, together with the edge set $\{px_1, px_2, px_3, x_1y_1, x_1y_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$. If $4 \leq |I^*| \leq 6$, then some vertex in K, say k, has two neighbors in I^*, say t_1 and t_2. Let $G_1 = G - W - I - K - I^* - N_G(k)$. Then the graph obtained from G_1 by deleting all isolated vertices, contains an odd induced subgraph H_1 of order at least $2(|V(G)| - 13 - 2 - 3 - 6 - 1 - 3)/7 \geq 2(|V(G)| - 28)/7$. If k is adjacent to y_1, then k is not adjacent to y_2 and H_1 together with the edge set $\{px_3, y_1z_1, y_1z_2, y_1k, kt_1, kt_2\}$ is an odd induced subgraph of order at least $2|V(G)|/7$. If k is adjacent to y_2, then k is not adjacent to y_1 and H_1 together with the edge set $\{px_3, y_1z_2, y_2k, kt_1, kt_2\}$ is an odd induced subgraph of order at least $2|V(G)|/7$.

If $|W| = 13$ and $|I| = 3$, let $N_G(x_1) = \{p, y_1, y_2, y_3\}$, $N_G(x_2) = \{p, y_3, y_4, y_5\}$ and $N_G(x_3) = \{p, y_6, y_7, y_8\}$. Note that $|W_2| \geq 3$, and by Claim 5, if $y_3 \in W_2$, we have $\{y_1, y_2, y_4, y_5\} \cap W_2 = \emptyset$. Without loss of generality, assume that $y_1 \in W_2$ and $y_2 \in W_1 \cup W_2$.

If $y_2 \in W_2$, then $|(N_G(y_1) \cap I) \cap (N_G(y_2) \cap I)| = 1$ by Claim 5. Let $K = N_G(y_1) \cup N_G(y_2) \setminus (W \cup I)$. Then $|K| \leq 2$. Let $G^* = G - W - I - K$ and let I^* be the set of isolated vertices in G^*. Similarly, we have that

$$|I^*| \leq \max\{\left\lfloor \frac{1}{4}(26+6-12-2) \right\rfloor, \left\lfloor \frac{1}{4}(26+3-12-1) \right\rfloor, \left\lfloor \frac{1}{4}(26-12) \right\rfloor\} = 4.$$

If $|I^*| \leq 3$, then the odd induced subgraph of order at least $2(|V(G^*)| - |I^*|)/7 \geq 2(|V(G)| - 21)/7$ in $G^* - I^*$, together with the edge set $\{px_1, px_2, px_3, x_1y_1, x_1y_2\}$, is an odd induced subgraph of order at least $2|V(G)|/7$. Therefore, $|I^*| = 4$. If one vertex in K, say k, has three neighbors in I^*, say t_1, t_2 and t_3, then the odd induced subgraph of order at least $2(|V(G^*)| - |I^*|)/7 \geq 2(|V(G)| - 22)/7$ in $G^* - I^*$, together with the edge set $\{px_1, px_2, px_3, kt_1, kt_2, kt_3\}$, is an odd induced subgraph of order at least $2|V(G)|/7$. If every vertex in K has at most two neighbors in I^*, then $|K| = 2$. Note that

$$\begin{aligned}
\sum_{v \in W \setminus \{p, x_1, x_2, x_3\}} |N_G(v) \cap V(G')| &\geq \sum_{t \in I^*} |N_G(t) \cap W| + \textstyle\sum_{z \in I} |N_G(z) \cap W| + |K| \\
&\geq 16 - 4 + 12 + 2 \\
&= 26 \\
&\geq \sum_{v \in W \setminus \{p, x_1, x_2, x_3\}} |N_G(v) \cap V(G')|.
\end{aligned}$$

Then $\bigcup_{v \in W \setminus \{p, x_1, x_2, x_3\}} N_G(v) \subseteq \{p, x_1, x_2, x_3\} \cup I \cup K \cup I^*$, which means that y_7, y_8 have no neighbors in $G^* - I^*$ and y_1, y_2 are not adjacent to y_7, y_8. Now the odd induced subgraph of order at least $2(|V(G^*)| - |I^*|)/7 \geq 2(|V(G)| - 22)/7$ in $G^* - I^*$, together with the edge set $\{px_1, px_2, px_3, x_1y_1, x_1y_2, x_3y_7, x_3y_8\}$, is an odd induced subgraph of order at least $2|V(G)|/7$.

Therefore, $y_2 \in W_1$, and we may assume that $\{y_4, y_5\} \not\subset W_2$, $\{y_6, y_7\} \not\subset W_2$, $\{y_7, y_8\} \not\subset W_2$ and $\{y_6, y_8\} \not\subset W_2$ otherwise we can do the same discussion above. Recall that $|W_2| \geq 3$ and if $y_3 \in W_2$, then $\{y_1, y_2, y_4, y_5\} \cap W_2 = \emptyset$. So $|W_2| = 3$ and $|W_1| = 6$. Without loss of generality, let $y_6 \in W_2$ and $y_7, y_8 \in W_1$. Let $N_G(y_6) \cap I = \{z_1, z_2\}$. By Claim 5, $N_G(y_7) \cap \{z_1, z_2\} = \emptyset$ or $N_G(y_8) \cap \{z_1, z_2\} = \emptyset$. Without loss of generality, let $N_G(y_7) \cap \{z_1, z_2\} = \emptyset$. Then $(N_G(y_6) \cap I) \cup (N_G(y_7) \cap I) = I$. Let $G_2^* = G - W - N_G(y_6) - N_G(y_7)$. Then the graph obtained from G_2^* by deleting all isolated vertices, contains an odd induced subgraph H_2^* of order at least $2(|V(G)| - 13 - 3 - 3 - 9)/7 \geq 2(|V(G)| - 28)/7$. H_2^* together with the edge set $\{px_1, px_2, px_3, x_3y_6, x_3y_7, y_6z_1, y_6z_2\}$ is an odd induced subgraph of order at least $2|V(G)|/7$.

If $|W| = 13$ and $|I| = 4$, then $|W_2| \geq 7$. But by Claim 5, we have that $|W_2| \leq 6$, which is a contradiction.

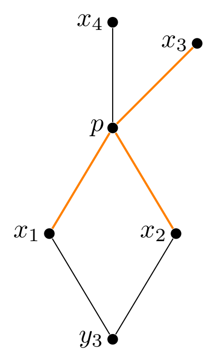

Figure 5: A cycle of length four.

Claim 6. *G contains no cycles of length four.*

Proof. If G contains a cycle of length four, we can let p be the vertex in this cycle which is adjacent to two vertices in this cycle, and let these two vertices be x_1, x_2, and let the other neighbors of p in G be x_3, x_4. The claim follows by doing the same discussions all above. $\square$

By Observation 2, all the above discussions and Claim 6, we have $|W| = 14$ and $|I| = 1$. Moreover, let $I = \{z\}$. Then z is adjacent to x_4 and Observation 3 implies that $N_G(x_4) \subseteq W \cup I$, but at this moment G will contain a cycle of length four since $d_G(x_4) = 4$, which is a contradiction.

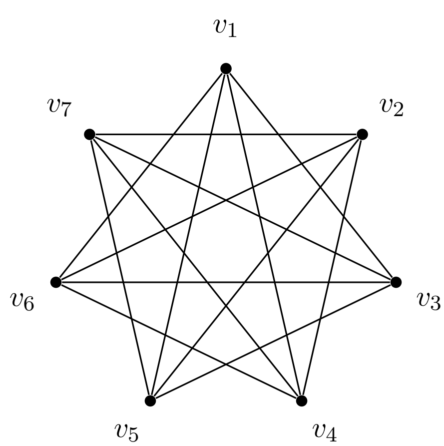

Figure 6: K_7 without a Hamilton cycle

To illustrate the tightness of the coefficient, we consider the following example: Let G be a graph obtained by deleting a Hamiltonian cycle from a K_7, see Fig. 6. We claim that the maximum order of an odd induced subgraph, say H, of G is exactly 2 (any edge is an odd induced subgraph). Note that $|V(H)|$ is even by the hand-shaking lemma. If $|V(H)| = 4$, there is a vertex of degree 2 in H, a contradiction. If $|V(H)| = 6$, there is a vertex of degree 4 in H, a contradiction. So, H is an edge.

3 Conclusion

As we only consider the graphs without isolated vertices, so we omit saying this in the following. In this paper, we determined the optimal constant c such that every n-vertex graph with maximum degree at most four contains an odd induced subgraph of order at least cn. We proved that $c = \frac{2}{7}$ and this value cannot be improved. Together with the result of Berman, Wang and Wargo [1], which established the sharp bound $c = \frac{2}{5}$ for graphs of maximum degree at most three, our result completes the exact determination of the largest possible induced odd subgraph in graphs of maximum degree at most four. What is the largest value of c for graphs with maximum degree five? If it is no less than $\frac{2}{7}$ then it would further support the possibility that $\frac{2}{7}$ is the optimal constant for all graphs.

Our work fits into a broader line of research investigating the existence of large odd induced subgraphs under structural restrictions. In particular, it complements the recent positive resolution by Ferber and Krivelevich [3] of a conjecture that there is a constant $c > 0$ that every graph contains an odd induced subgraph of order at least cn. Ferber and Krivelevich's proof shows that $c \geq 10^{-4}$. Our work parallels several optimal results known for classes such as trees [6], graphs of treewidth at most two [4], and planar graphs with large girth [7]. It remains a compelling direction to understand to what extent similar sharp bounds can be obtained for other sparse families of graphs, including graphs of bounded average degree or bounded degeneracy.

Another natural direction is related to the conjecture of Scott [8], which states that every n-vertex graph with chromatic number χ contains an odd induced subgraph of order at least $n/(2\chi)$. This conjecture has recently been disproved for bipartite graphs but remains open for graphs with $\chi \geq 3$; see Wang and Wu [9]. It would be interesting to investigate whether the extremal constructions arising in our proof shed further light on this problem.

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