Frustration indices of signed subcubic graphs

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Abstract

The frustration index of a signed graph is defined as the minimum number of negative edges among all switching-equivalent signatures. This can be regarded as a generalization of the classical Max-Cut problem in graphs, as the Max-Cut problem is equivalent to determining the frustration index of signed graphs with all edges being negative signs. In this paper, we prove that the frustration index of an n-vertex signed connected simple subcubic graph, other than $(K_4, -)$, is at most $\frac{3n+2}{8}$, and we characterize the family of signed graphs for which this bound is attained. This bound can be further improved to $\frac{n}{3}$ for signed 2-edge-connected simple subcubic graphs, with the exceptional signed graphs being characterized. As a corollary, every signed 2-edge-connected simple cubic graph on at least 10 vertices and with m edges has its frustration index at most $\frac{2}{9}m$, where the upper bound is tight as it is achieved by an infinite family of signed cubic graphs.

Keywords: Frustration index, Signed graphs, Cubic graphs, Max-cut

1 Introduction

The Max-Cut problem seeks a vertex partition of a graph that maximizes the number of edges between the two parts. For a graph G, let $b(G)$ denote the size of its maximum cut, and let $v(G)$ and $e(G)$ denote the number of vertices and edges, respectively. Although the Max-Cut problem is shown to be NP-hard [12], several positive algorithmic results exist under specific constraints. Hadlock [10] proved that a maximum cut can be found in polynomial time for planar graphs. Barahona [2] further showed that the problem is polynomially solvable for graphs not contractible to K_5. Goemans and Williamson [9] proposed a polynomial-time algorithm that finds a cut in any graph G of size at least approximately $0.878b(G)$, which is best possible.

Edwards [7, 8] established a fundamental lower bound of $b(G)$ based on its number of vertices and edges: $b(G) \geq \frac{e(G)}{2} + \frac{v(G)-1}{4}$. In 1997, J. Komlós [13] studied simple graphs G of girth g, and provided a sharp upper bound of $\min\{v(G), c\frac{v(G)}{g} \log \frac{2v(G)}{g}\}$, where c is a constant. In 1982, Locke [15] showed that for any k-regular graph G, the number of edges of a maximal $(k-1)$-colorable subgraph is greater than $\frac{k^2-2}{k^2}e(G)$, which implies, in particular, that $b(G) \geq \frac{7}{9}e(G)$ for cubic graphs G. Further refinements were obtained in subcubic settings. Hopkins and Staton [11] proved that $b(G) \geq \frac{4}{5}e(G)$ for triangle-free cubic graphs. Bondy and Locke [3] extended the result of Hopkins and Staton to triangle-free subcubic graphs, and Xu and Yu [18] characterized the seven extremal graphs achieving the equality.

Zhu [20] gave a more concise proof of Bondy and Locke's result, and later [21] improved the bound to a tight one: $b(G) \geq \frac{17}{21}e(G)$. For cubic graphs with large girths, Zýka [22] showed that $b(G) \geq \frac{6}{7}e(G)$ when the girth g of G tends to infinity.

An equivalent formulation of the MAX-CUT problem can be given in terms of signed graphs and their *frustration indices*. A *signed graph* (G, σ) consists of an underlying graph G together with a *signature* $\sigma : E(G) \to \{+, -\}$. The notation $\widehat{G}$ is also used to refer to a signed graph when the signature is clear from the context. The set of negative edges in (G, σ) is denoted by $E^-_{(G,\sigma)}$. The *switching* operation at an edge-cut $[X, X^c]$ is to change the signs of all edges in the cut. Two signatures σ and σ' on G are *switching equivalent* if one can be obtained from the other by switching at some edge-cut. The *frustration index* of (G, σ), denoted by $F(G, \sigma)$, is defined to be the minimum number of negative edges in any switching-equivalent signature:

$$F(G, \sigma) = \min\{|E^-_{(G,\sigma')}| : \sigma' \text{ is switching equivalent to } \sigma\},$$

where the minimum is taken over all switching-equivalent signatures σ' of σ. Let $(G, -)$ denote the signed graph with all edges being negative. Noting that in $(G, -)$ all odd cycles have an odd number of negative edges and all even cycles have an even number of negative edges, we may apply a switching at an edge-cut of G with the maximum size, thereby turning all those edges into positive signs. The fundamental relationship between the frustration index and the max-cut follows:

$$b(G) = e(G) - F(G, -) \geq e(G) - \max_{\sigma} F(G, \sigma),$$

where the maximum is taken over all arbitrary signatures σ of G.

Since computing the frustration index can be reduced from the MAX-CUT problem where all edges are negative, it is known to be NP-hard. In 1981, J. Akiyama, D. Avis, V. Chvátal, and H. Era [1] established a non-trivial lower bound for the frustration index of a signed simple graph (G, σ); this bound was later refined by P. Solé and T. Zaslavsky in 1994 [17]. The trivial upper bound $\frac{e(G)}{2}$ was improved by S. Poljak and D. Turzík [16] in 1982. Those two results together provide the following:

$$\frac{1}{2}e(G) - \sqrt{\frac{1}{2}\ln 2}\sqrt{e(G)(v(G)-1)} \leq F(G, \sigma) \leq \frac{1}{2}e(G) - \frac{1}{2}\lceil \frac{1}{2}(v(G)-1) \rceil.$$

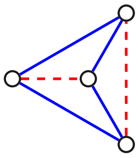
In 2004, D. Král and H. J. Voss [14] proved that for a signed planar graph (G, σ), it holds that $F(G, \sigma) \leq 2\nu(G, \sigma)$, where $\nu(G, \sigma)$ denotes the maximum number of edge-disjoint negative cycles in (G, σ). For further questions and recent developments in the theory of signed graphs, we refer the reader to the survey [19].

In this paper, we study the frustration indices of signed subcubic graphs and improve upon the existing results. In all figures, negative edges are depicted as dashed red lines, positive edges as solid blue lines, and edges with unspecified signs are shown as solid gray lines. Our first main result is as follows.

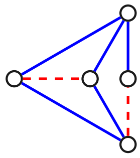
Theorem 1.1. *Every signed 2-edge-connected simple subcubic graph (G, σ), except for the five specific signed graphs $\widehat{\Gamma}_1, \ldots, \widehat{\Gamma}_5$ shown in Figure 1, satisfies that*

$$F(G, \sigma) \leq \frac{1}{3}v(G).$$

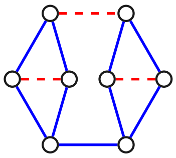
Moreover, there exists an infinite family of signed 2-edge-connected simple subcubic graphs with the frustration index equal to one-third of the number of vertices.



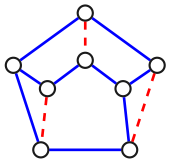
(a) $F(\widehat{\Gamma}_1) = \frac{1}{2}v(\widehat{\Gamma}_1)$



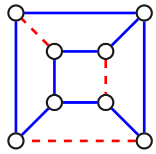
(b) $F(\widehat{\Gamma}_2) = \frac{2}{5}v(\widehat{\Gamma}_2)$



(c) $F(\widehat{\Gamma}_3) = \frac{3}{8}v(\widehat{\Gamma}_3)$



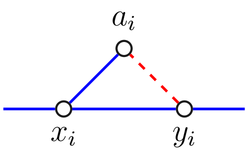
(d) $F(\widehat{\Gamma}_4) = \frac{3}{8}v(\widehat{\Gamma}_4)$



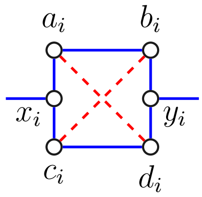
(e) $F(\widehat{\Gamma}_5) = \frac{3}{8}v(\widehat{\Gamma}_5)$

Figure 1: Five exceptional signed subcubic graphs with $F(G, \sigma) > \frac{1}{3}v(G)$

We now present an infinite family of signed 2-edge-connected simple subcubic graphs (G_n, σ_n) with $F(G_n, \sigma_n) = \frac{1}{3}v(G_n)$. Each of (G_n, σ_n) is formed by connecting the components $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_n$ by adding positive edges $x_i y_{i-1}$ with indices taken modulo n. Here $\widehat{F}_i$ is one of the following two kinds of signed gadgets: (1) a negative triangle with exactly one vertex of degree 2; (2) any 2-subdivision of $\widehat{\Gamma}_1$, where we subdivide a negative edge into a negative edge and a positive edge, and subdivide a positive edge into two positive edges. See Figure 2a and Figure 2b. Note that $|E^-_{\widehat{F}_i}| = \frac{1}{3}v(F_i)$ and thus $F(G_n, \sigma_n) = \frac{1}{3}v(G_n)$.



(a) The first kind of signed gadget



(b) The second kind of signed gadget: a 2-subdivision of $\widehat{\Gamma}_1$

Figure 2: A subgraph $\widehat{G_i}$

Theorem 1.1 yields the following corollary for signed cubic graphs, which particularly generalizes the result of Locke [15] and shows that every 2-edge-connected simple cubic graph G on at least 10 vertices contains an edge-cut of size at least $\frac{7}{9}e(G)$.

Corollary 1.2. *Every signed 2-edge-connected simple cubic graph (G, σ) on at least 10 vertices satisfies that $F(G, \sigma) \leq \frac{2}{9}e(G)$. Moreover, the upper bound is tight.*

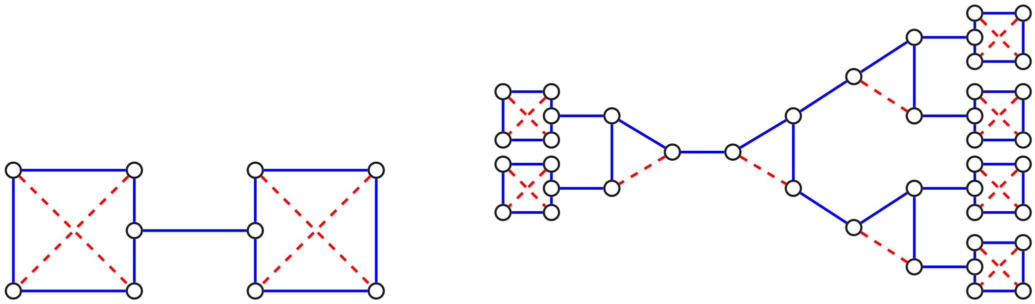
For signed subcubic graphs which are not 2-edge-connected and may contain some cut-edges, we have the following result with the assistance of Theorem 1.1. Each maximal 2-edge-connected component of a connected graph obtained by removing all cut-edges is called a *block*. A block of a connected graph is called a *leaf-block* if it is connected to the rest of the graph via a single cut-edge.

Theorem 1.3. *Every signed connected simple subcubic graph (G, σ) other than $\widehat{\Gamma}_1$ satisfies that*

$$F(G, \sigma) \leq \frac{3v(G) + 2}{8}.$$

Moreover, the equality holds if and only if (G, σ) is cubic and contains at least one cut-edge, each leaf-block is isomorphic to $\widehat{\Gamma}_2$, and each non-leaf-block is isomorphic to a negative triangle.

Two examples are illustrated in Figure 3, showing that the upper bound of Theorem 1.3 is tight.



(a) $F(G, \sigma) = \frac{3v(G)+2}{8}$

(b) $\frac{F(G,\sigma)}{v(G)} = \frac{3k+4}{8k+10}$ where k is the number of non-leaf-blocks

Figure 3: Examples illustrating the tightness of Theorem 1.3

Note that all the above extremal examples contain certain small cycles. We conjecture that the bounds established above can be further improved by forbidding specific configurations. For example, we propose the following conjecture.

Conjecture 1.4. *Every signed connected simple subcubic graph (G, σ) with girth at least 5 satisfies that $F(G, \sigma) \leq \frac{3}{10}v(G)$.*

The condition of girth 5 in Conjecture 1.4 is necessary due to the signed graph $\widehat{\Gamma}_5$ in Figure 1e. Note that the bound $\frac{3}{10}v(G)$ is tight if it is true, as evidenced by a signed Petersen graph with all edges negative, which attains this bound. We leave this conjecture for future research.

We now introduce some terminologies in the end of this section. Let G be a graph. Let X be a vertex subset of $V(G)$ and u be a vertex of $V(G)$. We use $N(u)$ to denote the set of neighbors of u in G, and use $d_X(u)$ to denote the number of neighbors of u that are in X. For a positive integer k, a vertex u is called a *k-vertex* if its degree is equal to k, that is, $|N(u)| = k$. For any $X \subset V(G)$, let $[X, X^c]$ denote the *edge-cut* between X and X^c. An edge-cut $[X, X^c]$ is called a *k-edge-cut* if it consists of k edges. A graph G is *k-edge-connected* if every edge-cut of G contains at least k edges. In this paper, to simplify the notation, we write $[x_1, x_2, ..., x_n]$ for the edge-cut $[\{x_1, x_2, ..., x_n\}, \{x_1, x_2, ..., x_n\}^c]$.

Let (G, σ) be a signed graph. A vertex x of (G, σ) is called a *negative* (or *positive*) neighbor of y if the edge xy is negative (or positive) in (G, σ). A *digon* is a signed graph on 2 vertices with two parallel edges, one positive and the other negative.

We say an edge-cut (or simply, a cut) of a signed graph is *unequilibrated* if it contains more negative edges than positive ones, and *equilibrated* if it contains the same number of negative edges and positive edges.

In this paper, we define the *subdivision* operation of any signed graph as follows: to subdivide a negative edge xy, delete xy, add a new vertex z, then add a positive edge xz and a negative edge yz; to subdivide a positive edge xy, delete xy, add a new vertex z, then add two positive edges xz and yz.

In the sequel, we adopt the following convention: when constructing a signed graph (G', σ') from the original signed graph (G, σ), we assume that every unchanged edge inherits its sign from the original signature.

The remainder of the paper is organized as follows: In the next section, we first prove Theorem 1.1 under the assumption that a key lemma (Lemma 2.3) holds, and then proceed to establish Theorem 1.3. In Section 3, we characterize all exceptional signed subcubic graphs of Theorem 1.1 on at most 9 vertices. Specifically, we explain how we discover the five exceptional signed graphs. In Section 4, we first establish some properties of the minimum counterexample to Theorem 1.1, and finally provide a proof of Lemma 2.3.

2 Frustration indices of signed subcubic graphs

We start with the following well-known observation in signed graphs.

Observation 2.1. *Let (G, σ) be a signed graph with $|E^-_{(G,\sigma)}| = F(G, \sigma)$. For any k-edge-cut $[X, X^c]$ of (G, σ), we have $|E^-_{(G,\sigma)} \cap [X, X^c]| \leq \lfloor \frac{k}{2} \rfloor$.*

It follows that a signed graph (G, σ) such that $|E^-_{(G,\sigma)}| = F(G, \sigma)$ cannot have any unequilibrated edge-cut. In particular, at each vertex of a signed subcubic graph (G, σ) with $|E^-_{(G,\sigma)}| = F(G, \sigma)$, there is at most one negative edge incident to it. A direct consequence is that for all subcubic graphs (G, σ), $F(G, \sigma) \leq \frac{1}{2}v(G)$.

Let k be a positive integer. Let $\mathcal{T}$ denote the set of signed trees T where $v(T) = 2k + 1$, and $[V(T), V(T)^c]$ containing at least $k + 2$ negative edges. Similarly, let $\mathcal{C}$ denote the set of signed odd cycles C with $v(C) = 2k + 1$ such that $[V(C), V(C)^c]$ contains at least $k + 1$ negative edges. A signed graph is called $\mathcal{T}$-free ($\mathcal{C}$-free, respectively) if it does not contain a tree $T \in \mathcal{T}$ (a cycle $C \in \mathcal{C}$, respectively) as an induced subgraph. Note that in any subcubic graph containing a tree T or a cycle C as an induced subgraph, the cut $[V(T), V(T)^c]$ contains at most $v(T) + 2$ edges, while the cut $[V(C), V(C)^c]$ contains at most $v(C)$ edges. This observation leads to the following lemma.

Lemma 2.2. *Every signed subcubic graph (G, σ) such that $|E^-_{(G,\sigma)}| = F(G, \sigma)$ is $\mathcal{T}$-free and $\mathcal{C}$-free.*

Next we start to prove Theorem 1.1. Let (G, σ) be a minimum counterexample to Theorem 1.1 with $v(G)$ being minimized. Thus (G, σ) is subcubic, simple, 2-edge-connected, not isomorphic to any of signed graphs in Figure 1, and satisfies that $F(G, \sigma) > \frac{1}{3}v(G)$. Furthermore, among all switching-equivalent signatures, we choose the signature σ such that $|E^-_{(G,\sigma)}| = F(G, \sigma)$.

We define two subsets X and Y of $V(G)$ with respect to the signature σ:

$$X := \{u \mid u \text{ is not incident to any negative edge in } (G, \sigma)\} \text{ and } \; Y = V(G) - X.$$

Thus (X, Y) is a bipartition of $V(G)$. Since $E^-_{(G,\sigma)} \subset E(G[Y])$, every vertex of Y has at least one neighbor in Y, i.e., $d_X(u) \leq 2$ for every vertex $u \in Y$. Moreover, for each $i \in \{0, 1, 2, 3\}$ and $j \in \{0, 1, 2\}$, we define that

$$X_i := \{u \in X \mid d_Y(u) = i\}, \quad \text{and} \quad Y_j := \{v \in Y \mid d_X(v) = j\}.$$

Note that $X = \bigcup\limits_{k=0}^{3} X_k$ and $Y = \bigcup\limits_{\ell=0}^{2} Y_\ell$. We shall establish the following key lemma, with the proof postponed to Section 4.

Lemma 2.3. *In the minimum counterexample (G,σ), $|X_3|+|Y_0|\le|Y_2|$.*

We first prove Theorem 1.1 assuming that Lemma 2.3 holds.

Proof of Theorem 1.1. Assume that (G,σ) is a minimum counterexample with $v(G)$ being minimized and $|E^-_{(G,\sigma)}|=F(G,\sigma)$. Let X,Y, X_i (for $i\in\{0,1,2,3\}$), and Y_j (for $j\in\{0,1,2\}$) be defined as above. By Lemma 2.3, we have $|X_3|+|Y_0|\le|Y_2|$, i.e., $|X_3|\le|Y_2|-|Y_0|$. Note that $|X|+|Y|=v(G)$, $|X_2|=v(G)-|Y|-|X_0|-|X_1|-|X_3|$, and $|Y_1|=|Y|-|Y_0|-|Y_2|$. Note that

$$\begin{aligned}\sum_{u\in X}d_Y(u)&=\sum_{i=0}^{3}\sum_{u\in X_i}d_Y(u)=0|X_0|+|X_1|+2|X_2|+3|X_3|\\&=|X_1|+2(v(G)-|Y|-|X_0|-|X_1|-|X_3|)+3|X_3|,\end{aligned}$$

and

$$\sum_{v\in Y}d_X(v)=\sum_{i=0}^{2}\sum_{v\in Y_i}d_X(v)=0|Y_0|+|Y_1|+2|Y_2|=|Y|-|Y_0|+|Y_2|\ge|Y|+|X_3|.$$

Since $\sum\limits_{u\in X}d_Y(u)=\sum\limits_{v\in Y}d_X(v)$, combining the above two inequalities,

$$|Y|+|X_3|\le|X_1|+2(v(G)-|Y|-|X_0|-|X_1|-|X_3|)+3|X_3|,$$

solving which we get $|Y|\le\frac{2}{3}v(G)-\frac{1}{3}(2|X_0|+|X_1|)$. Since $|E^-_{(G,\sigma)}|=F(G,\sigma)$, by Observation 2.1 (G,σ) contains no adjacent negative edges. Thus the number of negative edges of (G,σ) is at most half the size of the set Y, i.e., $|E^-_{(G,\sigma)}|\le\frac{1}{2}|Y|$. Hence,

$$F(G,\sigma)\le\frac{1}{3}v(G)-\frac{1}{3}|X_0|-\frac{1}{6}|X_1|,$$

a contradiction. This completes the proof of Theorem 1.1. $\square$

Next we shall apply Theorem 1.1 to establish Theorem 1.3. We first derive a result based on Theorem 1.1 that provides a sufficient condition for a signed connected simple subcubic graph (G,σ) to have its frustration index at most $\frac{1}{3}v(G)$.

In the sequel, for every leaf-block H, let e_H denote the cut-edge connecting to H, and let $v_H\in e_H$ denote the vertex in H, and $v_{H^c}\in e_H$ denote the vertex not in H.

Proposition 2.4. *Let (G,σ) be a signed connected simple subcubic graph not isomorphic to any of $\widehat{\Gamma}_1,\dots,\widehat{\Gamma}_5$. If (G,σ) has no leaf-block isomorphic to $\widehat{\Gamma}_2$, then $F(G,\sigma)\le\frac{1}{3}v(G)$.*

Proof. Assume that (G,σ) is a minimum counterexample with $v(G)$ being minimized. So $F(G,\sigma)>\frac{1}{3}v(G)$. By Theorem 1.1, G must have a cut-edge. Let H be a leaf-block of (G,σ) and let e_H be the cut-edge connecting H with $G-H$. Note that $d_H(v_H)\le 2$ and $d_{G-H}(v_{H^c})\le 2$. By the assumption, H is not isomorphic to $\widehat{\Gamma}_2$ and, moreover, as $d_H(v_H)\le 2$, H is not isomorphic to any of the exceptional signed graphs in Figure 1. Hence, applying Theorem 1.1 to $(H,\sigma|_H)$, we have $F(H,\sigma|_H)\le\frac{1}{3}v(H)$. Now we consider the signed subgraph $(G-H,\sigma|_{G-H})$. If $G-H$ is also a block (which is a leaf-block), then as $d_{G-H}(v_{H^c})\le 2$, by symmetry we have $F(G-H,\sigma|_{G-H})\le\frac{1}{3}v(G-H)$. Since e_H is a cut-edge,

$$F(G,\sigma)=F(H,\sigma|_H)+F(G-H,\sigma|_{G-H})\le\frac{1}{3}v(H)+\frac{1}{3}v(G-H)=\frac{1}{3}v(G),$$

a contradiction. Otherwise, $G-H$ contains at least one cut-edge. Since $d_{G-H}(v_{H^c}) \le 2$, the newly-built leaf-block of $G-H$ cannot be isomorphic to $\widehat{\Gamma}_2$. Noting that $F(G-H,\sigma|_{G-H}) = F(G,\sigma) - F(H,\sigma|_H) > \frac{1}{3}v(G) - \frac{1}{3}v(H) = \frac{1}{3}v(G-H)$, $G-H$ is a smaller counterexample, a contradiction. □

Now we provide the proof of Theorem 1.3.

Proof of the first part of Theorem 1.3. Assume to the contrary that (G,σ) is a signed connected simple subcubic graph, not isomorphic to $\widehat{\Gamma}_1$, satisfying $F(G,\sigma) > \frac{3v(G)+2}{8}$, and with $v(G)$ being minimized. As $F(\widehat{\Gamma}_i,\sigma) < \frac{3v(G)+2}{8}$ for $i \in \{2,3,4,5\}$, by Theorem 1.1, (G,σ) contains at least one cut-edge.

We begin by establishing that every leaf-block of (G,σ) is isomorphic to $\widehat{\Gamma}_2$. Assume not and let H be a leaf-block that is not isomorphic to $\widehat{\Gamma}_2$. Note that since $d_H(v_H) \le 2$, H is not isomorphic to any of the exceptional signed graphs $\widehat{\Gamma}_i$'s. By Theorem 1.1, $F(H,\sigma|_H) \le \frac{v(H)}{3} < \frac{3v(H)}{8}$. Since e_H is a cut-edge, we have

$$F(G-H,\sigma|_{G-H}) = F(G,\sigma) - F(H,\sigma|_H) > \frac{3v(G)+2}{8} - \frac{3v(H)}{8} = \frac{3v(G-H)+2}{8}.$$

Observing that $d_{G-H}(v_{H^c}) \le 2$, $(G-H,\sigma|_{G-H})$ cannot be isomorphic to $\widehat{\Gamma}_1$, contradicting the minimality of (G,σ).

Next, we show that for each leaf-block H of (G,σ), $d_G(v_{H^c}) = 3$. It is easy to see that $d_G(v_{H^c}) \ge 2$. If $d_G(v_{H^c}) = 2$, then we denote the other neighbor of v_{H^c} by v, and form a new signed graph by deleting v_{H^c} and adding a positive edge $v_H v$. The resulting signed graph has the same frustration index as (G,σ) but fewer vertices, and not isomorphic to $\widehat{\Gamma}_1$, contradicting the minimality of (G,σ).

We then claim that for each leaf-block H of (G,σ), the vertex $v_{H^c} \in e_H$ is in a triangle. Let x and y be the two neighbors of v_{H^c} in $G-H$, and assume to the contrary that $xy \notin E(G)$. We form a new signed graph (G',σ') from (G,σ) by deleting H and the vertex v_{H^c}, and adding an edge xy with $\sigma'(xy) = \sigma(xv_{H^c})\sigma(yv_{H^c})$. The signed graph (G',σ') is simple and connected. Moreover, we have that

$$F(G',\sigma') = F(G,\sigma) - F(H,\sigma|_H) > \frac{3v(G)+2}{8} - 2 = \frac{3(v(G')+6)+2}{8} - 2 > \frac{3v(G')+2}{8}.$$

Note that (G',σ') is not isomorphic to $\widehat{\Gamma}_1$, as otherwise (G,σ) contains 10 vertices with $F(G,\sigma) = 4$, contradicting the assumption of $F(G,\sigma) > \frac{3v(G)+2}{8}$. So (G',σ') is a smaller counterexample, a contradiction.

Moreover, we prove that each leaf-block is connected to a triangle via a cut-edge, and that each such triangle is associated with exactly one leaf-block. Let H be a leaf-block. By the previous claims, it is connected to a triangle $v_{H^c}xy$. Assume to the contrary that at least one of x and y coincides with some $v_{H_0^c}$, say $x = v_{H_0^c}$. If y is a 2-vertex, then (G,σ) is a signed graph on 13 vertices with $F(G,\sigma) \le 5$, contradicting the assumption of $F(G,\sigma) > \frac{3v(G)+2}{8}$. So y is a 3-vertex and let y' be the third neighbor of y. In this case, we form a signed graph (G'',σ'') by deleting $H_0, v_{H_0^c}, x$ and y, and adding an edge $v_{H_0^c}y'$ with $\sigma'(v_{H_0^c}y') = \sigma(yy')$. Noting that (G'',σ'') is simple and not isomorphic to $\widehat{\Gamma}_1$, we have that

$$F(G'',\sigma'') \ge F(G,\sigma) - 3 > \frac{3v(G)+2}{8} - 3 = \frac{3(v(G'')+8)+2}{8} - 3 = \frac{3v(G'')+2}{8},$$

that is to say, (G'',σ'') is a smaller counterexample, a contradiction.

Let $H_1, \ldots, H_m$ be the leaf-blocks of G, and let (G^*, σ^*) denote the signed graph $(G - \cup H_i, \sigma|_{G-\cup H_i})$. Note that the triangles corresponding to distinct vertices v_{H_i} are vertex-disjoint in G. Assume not, and if there exist two triangles $v_{H_i}xy$ and $v_{H_j}xy$ sharing the same edge xy, then (G, σ) is a signed graph on 14 vertices with $F(G, \sigma) \leq 5$, contradicting the assumption of $F(G, \sigma) > \frac{3v(G)+2}{8}$. Since $d_{G^*}(v^c_{H_i}) = 2$ for each $i \in \{1, 2, \ldots, m\}$, (G^*, σ^*) contains at least m vertices of degree 2, each lying in a distinct triangle and all of which are pairwise vertex-disjoint. Then $v(G^*) \geq 3m$. Moreover, (G^*, σ^*) is not isomorphic to any of $\widehat{\Gamma}_i$'s for $i \in \{1, 2, \ldots, 5\}$. Therefore, as (G^*, σ^*) has no leaf-block isomorphic to $\widehat{\Gamma}_2$, by Proposition 2.4, $F(G^*, \sigma^*) \leq \frac{1}{3}v(G^*)$. Thus we have

$$\frac{F(G, \sigma)}{v(G)} = \frac{F(G^*, \sigma^*) + 2m}{v(G^*) + 5m} \leq \frac{\frac{1}{3}v(G^*) + 2m}{v(G^*) + 5m} \leq \frac{3}{8},$$

where the last inequality follows from the fact of $v(G^*) \geq 3m$, a contradiction. □

The final part of our analysis is devoted to characterizing the family of signed graphs that achieve the bound $\frac{3v(G)+2}{8}$.

Proof of the moreover part of Theorem 1.3. One direction is straightforward. If a signed connected simple graph (G, σ) is cubic and contains at least one cut-edge, with each leaf-block isomorphic to $\widehat{\Gamma}_2$ and each non-leaf-block isomorphic to a negative triangle, then (G, σ) can be constructed from a cubic tree T (in which all non-leaf vertices have degree 3) by replacing each 3-vertex with a negative triangle, and attaching each leaf vertex to the unique 2-vertex of a distinct copy of $\widehat{\Gamma}_2$. It is easy to see that if $|E^-_{(G,\sigma)}| = F(G, \sigma)$, then (G, σ) has $8k + 10$ vertices with exactly $3k + 4$ negative edges, where k is the number of 3-vertices in T. Hence, $F(G, \sigma) = \frac{3v(G)+2}{8}$.

We now show the other direction.

Claim 2.5. *If a signed connected simple subcubic graph (G, σ) not isomorphic to $\widehat{\Gamma}_1$ satisfies that $F(G, \sigma) = \frac{3v(G)+2}{8}$, then (G, σ) is cubic and contains at least one cut-edge, each leaf-block is isomorphic to $\widehat{\Gamma}_2$, and each other block is isomorphic to a negative triangle.*

Proof of the claim. Let (G, σ) be a signed connected simple subcubic graph not isomorphic to $\widehat{\Gamma}_1$ such that $F(G, \sigma) = \frac{3v(G)+2}{8}$.

Since $F(\widehat{\Gamma}_i, \sigma) < \frac{3v(G)+2}{8}$ for $i \in \{2, 3, 4, 5\}$ and (G, σ) is not isomorphic to $\widehat{\Gamma}_1$, by Theorem 1.1, (G, σ) contains at least one cut-edge.

Next we claim that every leaf-block of (G, σ) is isomorphic to $\widehat{\Gamma}_2$. Otherwise, let H be a leaf-block not isomorphic to $\widehat{\Gamma}_2$. Observing that $d_H(v_H) \leq 2$, $(H, \sigma|_H)$ is not isomorphic to any of the exceptional signed graphs $\widehat{\Gamma}_i$'s. So by Theorem 1.1, $F(H, \sigma|_H) \leq \frac{v(H)}{3} < \frac{3v(H)}{8}$. Then $F(G - H, \sigma|_{G-H}) = F(G, \sigma) - F(H, \sigma|_H) > \frac{3v(G)+2}{8} - \frac{3v(H)}{8} = \frac{3v(G-H)+2}{8}$. By the first part of Theorem 1.3, $G - H$ must be isomorphic to $\widehat{\Gamma}_1$, which is not possible as v^c_H is a 2-vertex in $G - H$.

Moreover, we show that for each leaf-block H of (G, σ), $d_G(v_{H^c}) = 3$ and v_{H^c} is contained in either a triangle or another leaf-block. Observe that $d_G(v_{H^c}) \geq 2$. If $d_G(v_{H^c}) = 2$, let v be the other neighbor of v_{H^c}. The new signed graph formed from (G, σ) by deleting v_{H^c} and adding a positive edge $v_H v$, is simple and clearly not isomorphic to $\widehat{\Gamma}_1$. It has the same frustration index as (G, σ), but fewer vertices, contradicting the first part of Theorem 1.3. If v_{H^c} is in another leaf-block isomorphic to $\widehat{\Gamma}_2$, then (G', σ') is isomorphic to $\widehat{\Gamma}_1$ and thus (G, σ) is isomorphic to the signed graph in Figure 3a. So we may assume to the contrary that x and y are two neighbors of v_{H^c} in $G-H$ and $xy \notin E(G)$. The new signed graph (G',σ'), formed from (G,σ) by deleting H and the vertex v_{H^c}, and adding an edge xy with $\sigma'(xy)=\sigma(xv_{H^c})\sigma(yv_{H^c})$, is simple, connected, and not isomorphic to $\widehat{\Gamma}_1$. We have that $F(G',\sigma')=F(G,\sigma)-F(H,\sigma|_H) > \frac{3v(G)+2}{8}-2=\frac{3(v(G')+6)+2}{8}-2>\frac{3v(G')+2}{8}$, a contradiction to the first part of Theorem 1.3.

Now we prove that each non-leaf-block of (G,σ) is isomorphic to a negative triangle. Since the signed graph in Figure 3a has no non-leaf-block, we can assume in the following that for each leaf-block H of (G,σ), v_{H^c} is contained in a triangle. Assume to the contrary that (G,σ) contains a non-leaf-block Q that is not isomorphic to a negative triangle. Among all such signed graphs, we select one with the smallest number of vertices. Let H be a leaf-block of G and assume that v_{H^c} is in a triangle $v_{H^c}xy$.

We next show that each of x and y is of degree 3. Assume not and at least one of them is a 2-vertex, say x. Note that y cannot be of degree 2, as otherwise, $v_{H^c}xy$ is a leaf-block not isomorphic to $\widehat{\Gamma}_2$, a contradiction. Let y' denote the third neighbor of y. We form a new signed graph (G',σ') from (G,σ) by deleting H, v_{H^c}, x, and y. Since yy' is a cut-edge and $F(G[\{v_{H^c},x,y\},\sigma]) \le 1$, we have that $F(G',\sigma') \ge F(G,\sigma)-3=\frac{3v(G)+2}{8}-3=\frac{3(v(G')+8)+2}{8}-3=\frac{3v(G')+2}{8}$. Moreover, since $d_{G'}(y') \le 2$, (G',σ') is not isomorphic to $\widehat{\Gamma}_1$. Thus by the first part of Theorem 1.3, $F(G',\sigma')=\frac{3v(G')+2}{8}$. It implies that $F(G[\{v_{H^c},x,y\},\sigma])=1$, i.e., $v_{H^c}xy$ is a negative triangle. Moreover, since yy' is a cut-edge, $G[V(Q)]=G'[V(Q)]$. If $G'[V(Q)]$ is a leaf-block, then $y' \in V(Q)$. As $d_{G'}(y') \le 2$, $G'[V(Q)]$ is not isomorphic to $\widehat{\Gamma}_2$. Noting that (G',σ') satisfies that $F(G',\sigma')=\frac{3v(G')+2}{8}$, it is a contradiction to the previous claim. So Q is still a non-leaf-block of (G',σ') and thus (G',σ') is a smaller counterexample, a contradiction.

Let x' and y' be the third neighbors of x and y, respectively. We claim that $x' \neq y'$. Suppose to the contrary that $x'=y'$. Observe that $F(G[\{v_{H^c},x,y,x'\}],\sigma) \le 1$. If x' is a 2-vertex, then $F(G,\sigma) < \frac{3v(G)+2}{8}$, a contradiction. Hence, we may assume that x' is a 3-vertex. Let x'' denote the other neighbor of x'. Let (G'',σ'') be the signed graph obtained from (G,σ) by deleting the leaf-block H and the vertices v_{H^c}, x, y, and x'. Since $x'x''$ is a cut-edge, we have $F(G'',\sigma'') \ge F(G,\sigma)-3$. Moreover, (G'',σ'') is not isomorphic to $\widehat{\Gamma}_1$, for otherwise (G,σ) would have 14 vertices satisfying that $F(G,\sigma) \le 5$, contradicting the assumption that $F(G,\sigma)=\frac{3v(G)+2}{8}$. Now, $F(G'',\sigma'') \ge F(G,\sigma)-3=\frac{3v(G)+2}{8}-3=\frac{3(v(G'')+9)+2}{8}-3>\frac{3v(G'')+2}{8}$, which contradicts the first part of Theorem 1.3.

We consider the following two cases based on whether $x'y'$ is an edge or not. From now on, we choose σ to be the signature such that $|E^-_{(G,\sigma)}|=F(G,\sigma)$.

Case 1 $x'y' \in E(G)$. In this case, at least one of x' and y' is of degree 3, as otherwise, $G[\{v_H,x,y,x',y'\}]$ is a leaf-block that is not isomorphic to $\widehat{\Gamma}_2$, a contradiction. Without loss of generality, we assume that $d_G(y')=3$. Let x'' be the third neighbor of x' if it exists. We form (G_1,σ_1) from (G,σ) by deleting H, v_{H^c}, x, y, and x', and adding an edge $x''y'$ with $\sigma_1(x''y')=\sigma(x'x'')\sigma(x'y')$ if x'' exists. Since $d_{G_1}(y') \le 2$, (G_1,σ_1) cannot be isomorphic to $\widehat{\Gamma}_1$. Noting that in (G,σ) there is at most one negative edge among the edges $v_{H^c}x, v_{H^c}y, xy, xx'$, and yy', we have

$$F(G_1,\sigma_1) \ge F(G,\sigma)-3=\frac{3v(G)+2}{8}-3=\frac{3(v(G_1)+9)+2}{8}-3>\frac{3v(G_1)+2}{8},$$

a contradiction to the first part of Theorem 1.3.

Case 2 $x'y' \notin E(G)$. We form (G_2,σ_2) from (G,σ) by deleting H, v_{H^c}, x, and y, and adding an edge $x'y'$ with $\sigma_2(x'y') = \sigma(xx')\sigma(yy')$. Noting that $F(G[\{v_{H^c}, x, y\}, \sigma]) \leq 1$, we have

$$F(G_2, \sigma_2) \geq F(G, \sigma) - 3 = \frac{3v(G)+2}{8} - 3 = \frac{3(v(G_2)+8)+2}{8} - 3 = \frac{3v(G_2)+2}{8}.$$

Now (G_2, σ_2) is not isomorphic to $\widehat{\Gamma}_1$, as otherwise, (G, σ) contains 12 vertices satisfying that $F(G, \sigma) \leq 4$, a contradiction to the assumption of $F(G, \sigma) = \frac{3v(G)+2}{8}$. Hence, by the first part of Theorem 1.3, $F(G_2, \sigma_2) = \frac{3v(G_2)+2}{8}$. So $v_{H^c}xy$ is a negative triangle.

If (G_2, σ_2) contains a non-leaf-block not isomorphic to a negative triangle, then (G_2, σ_2) is a smaller counterexample, contradicting the minimality. So we assume that every non-leaf-blocks of (G_2, σ_2) is isomorphic to a negative triangle. Since $Q \subset G$, $\{x', y'\} \subset V(Q)$ and thus x' and y' are in the same block, say Q', of (G_2, σ_2). By the minimality of (G, σ), either Q' is a leaf-block of (G_2, σ_2), or Q' is a negative triangle.

- In the former case, since $F(G_2, \sigma_2) = \frac{3v(G_2)+2}{8}$, if Q' is a leaf-block of (G_2, σ_2), then $Q' = \widehat{\Gamma}_2$. Thus $e_{Q'}$ is a cut-edge in (G, σ), and neither x' nor y' is an endpoint of $e_{Q'}$. We denote two connected components of $G - e_{Q'}$ by H_1 and H_2, where $v_H \in H_1$. As $v_{Q'^c}$ is a 2-vertex in H_2, $(H_2, \sigma|_{H_2})$ is not isomorphic to $\widehat{\Gamma}_1$. Noting that H_1 has 13 vertices with $F(H_1, \sigma|_{H_1}) \leq 4$, we have that

$$F(H_2, \sigma|_{H_2}) = F(G, \sigma) - F(H_1, \sigma|_{H_1}) \geq \frac{3v(G)+2}{8} - 4 = \frac{3(v(H_2)+13)+2}{8} - 4 > \frac{3v(H_2)+2}{8},$$

  a contradiction to the first part of Theorem 1.3.
- In the latter case, if Q' is a negative triangle, then the block Q consists of Q' and a 5-cycle that share the common edge xy. Note that $F(Q, \sigma) \leq 1$. Since $v_{H^c}xy$ forms a negative triangle, the 5-cycle cannot contain any negative edge other than xy; in particular, the edges xx' and yy' are both positive. Now, since Q' is a negative triangle, the only edge in (G_2, σ_2) that can be negative is $x'y'$. However, recalling that $\sigma_2(x'y') = \sigma(xx')\sigma(yy')$, it follows that one of xx' or yy' must be negative in (G, σ), which leads to a contradiction.

Hence, we conclude that each non-leaf-block of (G, σ) is isomorphic to a negative triangle.

Finally, we show that (G, σ) is cubic. Assume to the contrary that x is a 2-vertex of G. The above discussion implies that x is in a non-leaf-block isomorphic to a negative triangle, denoted xyz. Note that if one of y and z is also 2-vertex, then it is a leaf-block not isomorphic to $\widehat{\Gamma}_2$, a contradiction. So we let y' and z' be the third neighbor of y and z, respectively. As xyz is a block, $y'z' \notin E(G)$. We form a new signed graph (G''', σ''') from (G, σ) by deleting x, y and z, and adding a positive edge $y'z'$. We have that

$$F(G''', \sigma''') = F(G, \sigma) - 1 = \frac{3v(G)+2}{8} - 1 = \frac{3(v(G''')+3)}{8} - 1 > \frac{3v(G''')+2}{8}.$$

Note that (G''', σ''') is simple and connected. Moreover, (G''', σ''') is not isomorphic to $\widehat{\Gamma}_1$, as otherwise, (G, σ) contains 7 vertices satisfying that $F(G, \sigma) = 3$, a contradiction to the assumption that $F(G, \sigma) = \frac{3v(G)+2}{8}$. Now (G''', σ''') is a smaller counterexample, contradicting the first part of Theorem 1.3. $\Diamond$

We complete the proof of Theorem 1.3. $\square$

3 Signed 2-edge-connected simple subcubic graphs on at most 9 vertices

In this section, we characterize 2-edge-connected simple signed subcubic graphs (G, σ) on at most 9 vertices. Recall that it follows from Observation 2.1 that for every signed subcubic graph (G, σ), $F(G, \sigma) \leq \frac{1}{2}v(G)$. In the following, we first characterize all signed 2-edge-connected subcubic graphs for which this bound is attained; we emphasize that parallel edges are allowed in this result.

Lemma 3.1. *Let (G, σ) be a signed 2-edge-connected subcubic graph such that $|E^-_{(G,\sigma)}| = F(G, \sigma)$. If $F(G, \sigma) = \frac{1}{2}v(G)$, then either (G, σ) is isomorphic to $\widehat{\Gamma}_1$ up to switching, or each vertex of (G, σ) is in a digon.*

Proof. Let (G, σ) be a signed 2-edge-connected subcubic graph such that $F(G, \sigma) = \frac{1}{2}v(G)$. Moreover, assume that $|E^-_{(G,\sigma)}| = F(G, \sigma)$. Now $E^-_{(G,\sigma)} = \{u_1v_1, \cdots, u_kv_k\}$ is a perfect matching of G. Note that every vertex x is adjacent to exactly one negative edge of $E^-_{(G,\sigma)}$, and we denote the negative edge by e_x. Moreover, we use x' and x'' to denote the other two positive neighbors of x. We first claim that $e_x, e_{x'}$, and $e_{x''}$ cannot be distinct. Otherwise, $G[\{x, x', x''\}] \in \mathcal{T} \cup \mathcal{C}$ is an induced subgraph of G (noting that in the latter case $G[\{x, x', x''\}]$ is a positive triangle), a contradiction to Lemma 2.2. If $x' = x''$, then $G[\{x, x'\}] \in \mathcal{C}$ is an induced subgraph of G, again a contradiction to Lemma 2.2. Hence, either x is in a digon (i.e., either $e_x = e_{x'}$ or $e_x = e_{x''}$), or there is a negative triangle in $G[\{x, x', x''\}]$ (i.e., $e_{x'} = e_{x''}$).

We shall prove that if the latter case occurs ($e_{x'} = e_{x''}$), then (G, σ) is isomorphic to $\widehat{\Gamma}_1$ up to switching. We denote the other positive neighbor of x' and x'' by y' and y'' respectively, and denote the negative neighbor of x by y. As both x and y' are positive neighbors of x', the negative edges $e_x, e_{x'}$, and $e_{y'}$ are not distinct, and thus $e_{y'} \in \{e_x, e_{x'}\}$, that is to say, $y' \in \{x, x'', y\}$. Note that $y' \neq x$, as otherwise x is adjacent to three positive edges, a contradiction. If $y' = x''$, then between x' and x'' there is a digon, and thus xy is a cut-edge, a contradiction. So $y' = y$. Now as both x and y'' are positive neighbors of x'', the negative edges $e_x, e_{x''}$, and $e_{y''}$ are not distinct. Similarly as $y'' \notin \{x', x\}$, $y'' = y$. Now $\{x, x', x'', y\}$ forms a K_4, and as xy and $x'x''$ are both negative, (G, σ) is isomorphic to $\widehat{\Gamma}_1$. $\square$

Note that for each $\widehat{\Gamma}_i$ in Figure 1, its signature is the one that achieves its frustration index. That is because all of them are 2-edge-connected planar graphs, so each edge is contained in exactly two facial cycles. Thus the number of negative facial cycles should be at most twice of the frustration index. Moreover, by Lemma 3.1, for signed simple subcubic graphs (G, σ), either $F(G, \sigma) < \frac{1}{2}v(G)$ or (G, σ) is isomorphic to $\widehat{\Gamma}_1$, so any signature π on Γ_i would imply that $F(\Gamma_i, \pi) \leq \frac{1}{3}v(\Gamma_i)$ for $i \in \{2, 3, 4, 5\}$.

Now we claim that all the other signed 2-edge-connected simple subcubic graphs (G, σ) on at most 9 vertices satisfy $F(G, \sigma) \leq \frac{1}{3}v(G)$. We begin by considering cubic graphs, noting that such graphs must have an even number of vertices.

For $v(G) = 4$, the complete graph K_4 is the unique cubic graph on 4 vertices. Here, $\frac{1}{3}v(K_4) < F(K_4, \sigma) \leq \frac{1}{2}v(K_4)$ holds if and only if the signature σ satisfies that $|E^-_{(G,\sigma)}| = 2$ and thus $E^-_{(G,\sigma)}$ forms a perfect matching, as shown in Figure 1a.

For $v(G) = 6$, by Lemma 3.1 we have $F(G, \sigma) < \frac{1}{2}v(G) = 3$, i.e., $F(G, \sigma) \leq \frac{1}{3}v(G) = 2$.

For $v(G) = 8$, it is proved in [4] that there are only five such cubic graphs. We have demonstrated that three of them (see Figures 1c to 1e) admit signatures that serve as exceptional signed graphs. It remains to show that the remaining two graphs, W_1 and W_2, depicted in Figure 4, satisfy $F(W_i, \sigma) \leq \frac{1}{3}v(W_i)$ for any arbitrary signature σ.

Lemma 3.2. *Let W_1 and W_2 be two graphs as depicted in Figure 4a and Figure 4b. For each $i \in \{1, 2\}$ and any signature σ, $F(W_i, \sigma) \leq \frac{1}{3}v(W_i)$.*

(a) W_1 (b) W_2

Figure 4: Two 8-vertex cubic graphs W_1 and W_2

Proof. Assume to the contrary that for each $i \in \{1, 2\}$, there is a signature σ_i such that $F(W_i, \sigma_i) > \frac{1}{3}v(W_i)$. Moreover, we may assume that $|E^-_{(W_i,\sigma_i)}| = F(W_i, \sigma_i)$. Since W_1 and W_2 has no parallel edges, and neither of them is isomorphic to K_4, Lemma 3.1 implies that $F(W_i, \sigma_i) < \frac{1}{2}v(W_i)$, and thus $3 \leq |E^-_{(W_i,\sigma_i)}| < 4$, i.e., $|E^-_{(W_i,\sigma_i)}| = 3$. We consider two signed graphs case by case:

Case 1. For (W_1, σ_1), we first claim that the cycle $aa'c'c$ cannot be all-positive under σ_1.

Suppose for contradiction that the cycle $aa'c'c$ is all-positive in (W_1, σ_1). We consider two possibilities based on the sign of bd.

Assume that bd is negative under σ_1. In this case, by Observation 2.1, $ab, bd', b'd$, and cd are all positive edges. We claim that $a'b'$ is positive. Assume not, and both $b'd$ and $b'd'$ are positive. As $|E^-_{(W_1,\sigma_1)}| = 3$, $c'd'$ must be negative. However, it implies that $W_1[\{b', d', b\}] \in \mathcal{T}$ is an induced subgraph of W_1, a contradiction to Lemma 2.2. Since $a'b'$ is positive and $|E^-_{(W_1,\sigma_1)}| = 3$, both $c'd'$ and $d'b'$ are negative, a contradiction to Observation 2.1.

Assume that bd is positive under σ_1. By symmetry, $b'd'$ is also a positive edge. Now consider the remaining six edges, by Observation 2.1 there are only two possibilities under symmetry: either $E^-_{(W_1,\sigma_1)} = \{ab, b'd, c'd'\}$, or $E^-_{(W_1,\sigma_1)} = \{ab, b'd', cd\}$. But in the previous case, $W_1[\{b', d', b\}] \in \mathcal{T}$ is an induced subgraph of W_1 and in the latter case, $W_1[\{d', b', d\}] \in \mathcal{T}$ is an induced subgraph of W_1, both contradicting Lemma 2.2.

Hence, we conclude that the cycle $aa'c'c$ cannot be all-positive under σ_1. By symmetry, we may assume that either ac or aa' is negative in (W_1, σ_1).

- If ac is negative, then by Observation 2.1, edges ab, aa', cc', and cd are all positive. We first claim that $a'c'$ is also positive. Assume not, as $|E^-_{(W_1,\sigma_1)}| = 3$, there exists a negative edge e that is adjacent to neither ac nor $a'c'$, thus one of $\{b, b'\}$ is in e. Now either $W_1[\{a, a', b'\}]$ or $W_1[\{a', a, b\}] \in \mathcal{T}$ is an induced subgraph of W_1, a contradiction to Lemma 2.2. We then claim that $a'b'$ is also positive. Assume not, as $|E^-_{(W_1,\sigma_1)}| = 3$, there exists a negative edge e that is adjacent to neither ac nor $a'b'$, thus one of $\{b, c'\}$ is in e. Now either $W_1[\{a, a', c'\}] \in \mathcal{T}$ or $W_1[\{b, a, a'\}] \in \mathcal{T}$ is an induced subgraph of W_1, a contradiction to Lemma 2.2. So $c'd'$ is positive by symmetry. Now we consider the remaining edges, which are precisely those in the 4-cycle $bd'b'd$. As $|E^-_{(W_1,\sigma_1)}| = 3$, either $\{bd, b'd'\} \subset E^-_{(W_1,\sigma_1)}$ or $\{bd', b'd\} \subset E^-_{(W_1,\sigma_1)}$, and thus either $W_1[\{a, b, d'\}] \in \mathcal{T}$ or $W_1[\{a, b, d\}] \in \mathcal{T}$ is an induced subgraph of G, a contradiction Lemma 2.2.

- If aa' is negative, then by Observation 2.1, edges $ac, ab, a'c'$, and $a'b'$ are all positive. We claim that the edge-cut $[a,b]$ cannot contain two negative edges. Assume not, we may switch at $[a,b]$ and denote the resulting signature by σ_1'. Note that $|E^-_{(W_1,\sigma_1)}| = |E^-_{(W_1,\sigma_1')}|$. Now ac is negative in (W_1,σ_1'), which has been discussed above. By symmetry, the edge-cut $[a',b']$ cannot contain two negative edges. Thus edges $bd, b'd', b'd$, and bd' are all positive. Among the remaining edges $\{cd, cc', c'd'\}$, as $|E^-_{(W_1,\sigma_1)}| = 3$, cd and $c'd'$ are negative, and thus $W_1[\{a',c',c\}] \in \mathcal{T}$ is an induced subgraph of W_1, a contradiction to Lemma 2.2.

Case 2. For (W_2,σ_2), We first claim that the triangle abc of W_2 cannot be all-positive.

Assume to the contrary that abc is all-positive in (W_2,σ_2). Since $\{au, bv, cw\}$ is an edge-cut, at most one of them can be negative under σ_2. If au is negative, then by Observation 2.1 ux and uy are positive. We consider edges in the cycle $xwyv$, as the other edges are positive except ua. As $|E^-_{(W_2,\sigma_2)}| = 3$, either $\{xw, yv\} \subset E^-_{(W_2,\sigma_2)}$ or $\{xv, wy\} \subset E^-_{(W_2,\sigma_2)}$, and thus either $W_2[\{u,x,w\}] \in \mathcal{T}$ or $W_2[\{u,y,w\}] \in \mathcal{T}$ is an induced subgraph of W_2, a contradiction to Lemma 2.2. So au is positive and by symmetry, bv and cw are both positive. But now the subgraph induced on $\{u,v,x,y,w\}$ is simple and has five vertices, and by Observation 2.1 it contains at most 2 negative edges. So it contradicts the fact that $|E^-_{(W_2,\sigma_2)}| = 3$.

Hence, the triangle abc of W_2 cannot be all-positive. Note that x and y are both adjacent to u, v, and w. By symmetry, we may assume that ab is negative in (W_2,σ_2). Now by Observation 2.1, edges au, ac, bv, and bc are all positive. We claim that cw is also positive. Assume not, then by Observation 2.1 edges xw and yw are all positive. As $|E^-_{(W_2,\sigma_2)}| = 3$, there exists a negative edge e such that either u or v is an endpoint of e, thus either $W_2[\{u,a,c\}] \in \mathcal{T}$ or $W_2[\{v,b,c\}] \in \mathcal{T}$ is an induced subgraph of W_2, a contradiction to Lemma 2.2. We then claim that ux is positive. Assume not, by Observation 2.1 uy, xw, and xv are all positive. As $|E^-_{(W_2,\sigma_2)}| = 3$, either wy or vy is negative, and thus either $W_2[\{a,u,y\}] \in \mathcal{T}$ or $W_2[\{b,v,x\}] \in \mathcal{T}$ is an induced subgraph of W_2, a contradiction to Lemma 2.2. Now by symmetry, uy, vx, and vy are all positive. Since $|E^-_{(W_2,\sigma_2)}| = 3$, xw and yw must be both negative, a contradiction to Observation 2.1. □

Next, we consider signed 2-edge-connected simple subcubic graphs containing 2-vertices, where the total number of vertices may be odd (or even).

Observation 3.3. *Let (G,σ) be a signed graph and let (G',σ') be formed from (G,σ) by subdividing some edges. Then $F(G',\sigma') = F(G,\sigma)$.*

Observe that any signed tree has frustration index 0. So we only consider signed subcubic graphs with at least one cycle.

If $v(G) = 3$, then G is isomorphic to 3-cycle. It is easy to see that there are only two signed triangles $(C_3,+)$ and $(C_3,-)$ up to switching, where $F(C_3,+) = 0$ and $F(C_3,-) = 1$. Thus for any signature σ, $F(C_3,\sigma) \leq \frac{1}{3}v(C_3)$.

If $v(G) \in \{5,7,9\}$, then G is a subcubic graph, and can be obtained from subdividing k edges of a cubic graph G', where k is an odd number. For any signature σ on G, let π denote the corresponding signature on G' that is induced by σ before the subdivision is performed. If $k \geq 3$, then $v(G') \leq v(G) - 3 \leq 6$. As $F(G',\pi) \leq \frac{1}{2}v(G')$ for any signature π and by Observation 3.3, we have $F(G,\sigma) = F(G',\pi) \leq \frac{1}{2}v(G') \leq \frac{1}{3}v(G)$. If $k = 1$, then (G',π) has at most one digon. As $v(G') \geq 4$, there exist some vertices that are not in any digon. So by Lemma 3.1, either (G',π) is isomorphic to $\widehat{\Gamma}_1$ or $F(G',\pi) < \frac{1}{2}v(G')$. Assume that there is a signature σ such that $F(G,\sigma) > \frac{1}{3}v(G)$. By Observation 3.3, if $v(G) = 9$, then $F(G,\sigma) = F(G',\pi) < \frac{1}{2}v(G') = 4$, a contradiction; if $v(G) = 7$, then $F(G,\sigma) = F(G',\pi) <$

$\frac{1}{2}v(G') = 3$, a contradiction. So $v(G) = 5$, and $F(G,\sigma) = F(G',\pi) = \frac{1}{2}v(G') = 2$. This implies that G is a subdivision of $\widehat{\Gamma}_1$, as shown in Figure 1b.

For $v(G) \in \{4, 6, 8\}$, it can be formed from G' where $v(G') \in \{3, 5, 7\}$ by subdividing some edges. By Observation 3.3, $F(G,\sigma) = F(G',\pi) \le \frac{1}{3}v(G') < \frac{1}{3}v(G)$.

4 Proof of Lemma 2.3

In this section, we shall first provide some reducible configurations for the minimum counterexample (G,σ) to Theorem 1.1 and then prove Lemma 2.3. Based on the discussion in Section 3, in the sequel, we may assume that $v(G) \ge 10$.

We first give some lemmas related to the exceptional signed graphs $\widehat{\Gamma}_i$. A signed graph (H,π) is said to be *critically k-frustrated* if $F(H,\pi) = k$ and, for any edge $e \in E(H)$, $F(H - e, \sigma|_{H-e}) < k$. It follows from Lemma 4.3 of [6] that $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are critically 2-frustrated, and from Lemma 3.5 of [5] that $\widehat{\Gamma}_3, \widehat{\Gamma}_4$ and $\widehat{\Gamma}_5$ are critically 3-frustrated.

Lemma 4.1. [6] *Let k be a positive integer. A signed graph (H,π) is critically k-frustrated if and only if every positive edge of (H,π) is contained in an equilibrated cut.*

By Lemma 4.1, we immediately have the following result.

Corollary 4.2. *For $i \in \{1,2,3,4,5\}$, every positive edge of $\widehat{\Gamma}_i$ is contained in an equilibrated cut.*

4.1 Properties of the minimum counterexample (G,σ) to Theorem 1.1

Lemma 4.3. *The minimum counterexample (G,σ) is cubic.*

Proof. Since (G,σ) is 2-edge-connected, there is no 1-vertex in (G,σ). Assume that z is a 2-vertex with two neighbors x and y. We first claim that $xy \in E(G)$. Suppose for contradiction that $xy \notin E(G)$, we then form a signed graph (G',σ') from (G,σ) by deleting the vertex z and adding an edge xy such that $\sigma'(xy) = \sigma(xz)\sigma(zy)$. Note that (G',σ') is 2-edge-connected and simple (i.e., no parallel edges is created). Moreover, $|E^-_{(G',\sigma')}| = F(G',\sigma')$ and thus

$$F(G',\sigma') = F(G,\sigma) > \frac{1}{3}v(G) = \frac{1}{3}(v(G')+1) > \frac{1}{3}v(G').$$

Since $v(G') = v(G) - 1 \ge 9$, (G',σ') cannot be isomorphic to any of the five exceptional signed graphs $\widehat{\Gamma}_i$'s, and thus (G',σ') is a smaller counterexample, contradicting the minimality of (G,σ). So $xy \in E(G)$. Furthermore, since (G,σ) is 2-edge-connected, each of x and y is a 3-vertex. Let x' and y' denote the third neighbor of x and y, respectively. We then consider the following two cases.

Case 1. $xy \in E(G)$ and the triangle xyz is positive.

In this case, we form a signed graph (G'',σ'') from (G,σ) by deleting vertices y and z, and adding an edge xy' with $\sigma''(xy') = \sigma(yy')$. Now no parallel edge is created and G'' is 2-edge-connected.

Since for any unequilibrated cut of (G'',σ'') containing the edge xy', replacing xy' with yy' results in an unequilibrated cut in (G,σ), it follows that (G'',σ'') contains no unequilibrated cut, and hence $F(G'',\sigma'') = |E^-_{(G'',\sigma'')}|$. Now we have that

$$F(G'',\sigma'') = F(G,\sigma) > \frac{1}{3}v(G) = \frac{1}{3}(v(G'')+2) > \frac{1}{3}v(G'').$$

Note that $v(G'') = v(G) - 2 \geq 8$. If (G'', σ'') is isomorphic to one of the five exceptional signed graphs, then (G'', σ'') can only be one of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$, as shown in Figures 1c to 1e. Observe that none of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$ has any 2-vertex. However, $d_{G''}(x) = 2$, which leads to a contradiction. Thus (G'', σ'') is a smaller counterexample, contradicting the minimality of (G, σ).

Case 2. $xy \in E(G)$ and the triangle xyz is negative.

In this case, without loss of generality, exactly one of xy and xz is negative. By Observation 2.1, both xx' and yy' are positive. We consider the following two subcases: If $x'y' \notin E(G)$, then let (G_1, σ_1) be formed from (G, σ) by deleting vertices x, y, and z, and adding a positive edge $x'y'$; If $x'y' \in E(G)$, then let (G_2, σ_2) be formed from (G, σ) by deleting vertices x, y, and z. Note that no parallel edge is created, and in each case the resulting signed graph (G_i, σ_i) remains 2-edge-connected.

We next verify that $F(G_i, \sigma_i) = |E^-_{(G_i,\sigma_i)}|$ for $i \in \{1, 2\}$. Assume not and it implies that σ_i is not the signature that achieves the frustration index of (G_i, σ_i). So (G_i, σ_i) contains an unequilibrated cut C_i. If C_i does not contain the edge $x'y'$, then itself is an unequilibrated cut in (G, σ), a contradiction. Thus we assume that $x'y' \in C_i$ for each $i \in \{1, 2\}$. Since exactly one of xy and xz is negative, for $i = 1$, replacing $x'y'$ with $\{xy, xz\}$ results in an unequilibrated cut of (G, σ); for $i = 2$, adding $\{xy, xz\}$ to C_2 results in an unequilibrated cut of (G, σ); both contradicting Observation 2.1. Thus we have that

$$F(G_i, \sigma_i) = F(G, \sigma) - 1 > \frac{1}{3}v(G) - 1 = \frac{1}{3}(v(G_i) + 3) - 1 = \frac{1}{3}v(G_i).$$

Now $v(G_1) = v(G_2) = v(G) - 3 \geq 7$. If (G_i, σ_i) is not isomorphic to any of $\widehat{\Gamma}_i$'s, then it is a smaller counterexample, contradicting the minimality of (G, σ). So we know that $v(G_i) = 8$ and (G_i, σ_i) can only be one of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$. Since $d_{G_2}(x') \leq 2$ but none of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$ has any vertex of degree less than 2, we know that $i = 1$. By Corollary 4.2, the newly-added positive edge $x'y'$ must be in an equilibrated cut of (G_1, σ_1). Since exactly one of xy and xz is negative, and neither xy nor xz is in (G_1, σ_1), replacing $x'y'$ with $\{xy, xz\}$ will imply an unequilibrated cut of (G, σ), a contradiction Observation 2.1. □

Lemma 4.4. *The minimum counterexample (G, σ) contains no adjacent triangles.*

Proof. Assume to the contrary that there are two adjacent triangles acd and bcd in G. Let x and y denote the third neighbor of a and b respectively. Note that $x \neq y$, as otherwise the edge incident to x which is different from $\{ax, bx\}$ is a cut-edge, a contradiction. Let (H, σ) denote the signed subgraph of (G, σ) induced by the vertex set $\{a, b, c, d\}$.

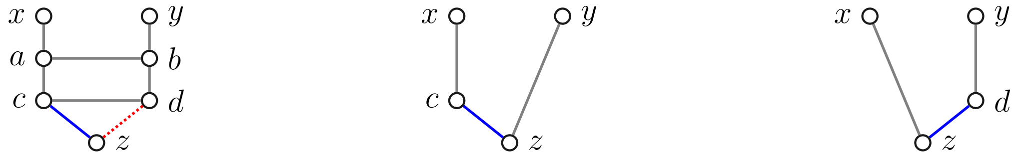
We claim that there exists a signature π (of G) switching equivalent to σ such that $|E^-_{(G,\pi)}| = F(G, \sigma)$ and $|E^-_{(H,\pi|_H)}| \leq 1$. If $|E^-_{(H,\sigma)}| \leq 1$, then we let $\pi = \sigma$; if $|E^-_{(H,\sigma)}| > 1$, then by Observation 2.1 either $E^-_{(H,\sigma)} = \{ac, bd\}$ or $E^-_{(H,\sigma)} = \{ad, bc\}$. Without loss of generality, assume that $E^-_{(H,\sigma)} = \{ac, bd\}$. We form (G, π) by switching at the equilibrated cut $[a, d]$. Now $E^-_{(H,\pi|_H)} = \{cd\}$, and as we form π from σ by switching at an equilibrated cut, $|E^-_{(G,\pi)}| = F(G, \sigma)$, thus the claim is proved.

Now we form a signed graph (G', π') from (G, π) by deleting vertices b, c, and d, and adding an edge ay with $\pi'(ay) = \pi(by)$. Note that the resulting signed graph (G', π') is still 2-edge-connected. Furthermore, for any unequilibrated cut of (G', π') containing the edge ay, replacing it with by results in an unequilibrated cut in (G, π). Therefore, (G', π') contains no unequilibrated cut, and hence $F(G', \pi') = |E^-_{(G',\pi')}|$. Since $|E^-_{(H,\pi|_H)}| \leq 1$, we have that

$$F(G',\pi') \geq F(G,\pi) - 1 > \frac{1}{3}v(G) - 1 = \frac{1}{3}(v(G')+3) - 1 = \frac{1}{3}v(G').$$

Note that $v(G') = v(G) - 3 \geq 7$. If (G',π') is not isomorphic to any of $\widehat{\Gamma}_i$'s, then it is a smaller counterexample, contradicting the minimality of (G,π). So we know that $v(G') = 8$ and (G',π') can only be one of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$. However, none of these three signed graphs contain any 2-vertex, contradicting the fact that a is a 2-vertex in (G',π'). □

Lemma 4.5. *The minimum counterexample (G,σ) contains no triangle adjacent to a 4-cycle, where the triangle has a negative edge that is not the common edge shared with the 4-cycle.*



(a) (G,σ) (b) (G_1,π_1) (c) (G_2,π_2)

Figure 5: Configurations in Lemma 4.5

Proof. Assume not, and the underlying graph G contains a triangle cdz and a 4-cycle $abdc$. Let x and y be the third neighbor of a and b, respectively. Note that neither x nor y is coincide with z, as otherwise by or ax is a cut-edge, a contradiction. By symmetry we may assume that dz is negative and thus cz is positive. See Figure 5a. Let (H,σ) denote the signed subgraph of (G,σ) induced by the vertex set $\{a,b,c,d,z\}$.

We claim that there exists a signature π (of G) switching equivalent to σ such that $|E^-_{(G,\pi)}| = F(G,\sigma)$ and $|E^-_{(H,\pi|_H)}| \leq 1$. If $|E^-_{(H,\sigma)}| \leq 1$, then we let $\pi = \sigma$; if $|E^-_{(H,\sigma)}| > 1$, then by Observation 2.1, either $E^-_{(H,\sigma)} = \{ab, dz\}$ or $E^-_{(H,\sigma)} = \{ac, dz\}$. If $E^-_{(H,\sigma)} = \{ab, dz\}$, we form (G,π) by switching at the equilibrated cut $[b,d]$; if $E^-_{(H,\sigma)} = \{ac, dz\}$, we form (G,π) by switching at the equilibrated cut $[c,z]$. Now $E^-_{(H,\pi|_H)} = \{cd\}$, and as we form π from σ by switching at an equilibrated cut, $|E^-_{(G,\pi)}| = F(G,\sigma)$, thus the claim is proved.

Since z is a 3-vertex, if both x and y are adjacent to z, then $x = y$. However, now we have that $v(G) = 6$, contradicting the fact that $v(G) \geq 10$. So we consider the following two cases.

- If y is not adjacent to z, then let (G_1,π_1) be formed from (G,π) by deleting vertices a, b, and d, and adding edges xc and yz such that $\pi_1(xc) = \pi(xa)$ and $\pi_1(yz) = \pi(yb)$, as shown in Figure 5b;
- If x is not adjacent to z, then let (G_2,π_2) be formed from (G,π) by deleting vertices a, b, and c, adding edges yd and xz such that $\pi_2(yd) = \pi(yb)$ and $\pi_2(xz) = \pi(xa)$, and changing the sign of dz to be positive under π_2, as shown in Figure 5c.

It is easy to see that both G_1 and G_2 are 2-edge-connected.

Next we shall prove that $|E^-_{(G_i,\pi_i)}| = F(G_i,\pi_i)$ for each $i \in \{1,2\}$. Assume to the contrary that $|E^-_{(G_i,\pi_i)}| > F(G_i,\pi_i)$. It implies that (G_i,π_i) contains an unequilibrated cut. We claim that there exists an unequilibrated cut of (G_1,π_1) not containing cz. For a cut containing

cz, if it does not contain cx, then replacing cz with cx will result in an unequilibrated cut of (G_1, π_1); if it contains cx, then deleting cx and cz will result in an unequilibrated cut (G_1, π_1). Thus we shall choose the one not containing cz and denote it by $C_1 := [A_1, B_1]$. Now based on C_1, we consider the following cut C_1' of (G, π) as follows:

$$C_1' = \begin{cases} [A_1, B_1 \cup \{a, b, d\}] & \text{if } x \in A_1, \{y, z\} \subset B_1, \\ [A_1, B_1 \cup \{a, b, d\}] & \text{if } \{x, y\} \subset A_1, z \in B_1, \\ [A_1 \cup \{a, b, d\}, B_1] & \text{if } \{x, z\} \subset A_1, y \in B_1, \\ [A_1 \cup \{a, b, d\}, B_1] & \text{if } \{x, y, z\} \subset A_1, \end{cases}$$

where $C_1' = C_1 \setminus \{cx\} \cup \{ax\}$ in the first case, $C_1' = C_1 \setminus \{cx, zy\} \cup \{ax, by\}$ in the second case, $C_1' = C_1 \setminus \{zy\} \cup \{by\}$ in the third case, and $C_1' = C_1$ in the last case. Now C_1' is an unequilibrated cut in (G, π), a contradiction to Observation 2.1. By symmetry, there exists an unequilibrated cut C_2 of (G_2, π_2) not containing dz, and based on C_2 we shall again find an unequilibrated cut in (G, π). So $|E^-_{(G_i,\pi_i)}| = F(G_i, \pi_i)$ for each $i \in \{1, 2\}$, and in both cases, as $|E^-_{(H,\pi|_H)}| \leq 1$, we have that

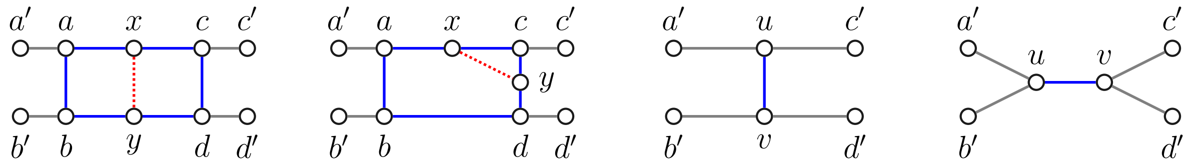
$$F(G_i, \pi_i) \geq F(G, \pi) - 1 > \frac{1}{3}v(G) - 1 = \frac{1}{3}(v(G_i) + 3) - 1 = \frac{1}{3}v(G_i).$$

Note that $v(G_i) = v(G) - 3 \geq 7$. If (G_i, π_i) is not isomorphic to any of $\widehat{\Gamma}_i$'s, then it is a smaller counterexample, contradicting the minimality of (G, π). So we know that $v(G_i) = 8$ and (G_i, π_i) can only be one of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$. However, G_1 has a 2-vertex c and G_2 has a 2-vertex d, contradicting the fact that none of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$ has any 2-vertex. $\square$

Lemma 4.6. *The minimum counterexample (G, σ) does not contain the following:*

(1) Adjacent negative 4-cycles sharing a common negative edge, such that the resulting 6-cycle has exactly one chord;

(2) Adjacent negative 3-cycle and 5-cycle sharing a common negative edge, such that the resulting 6-cycle has exactly one chord.



(a) Adjacent 4-cycles (b) Adjacent 3-cycle and 5-cycle (c) (G_1, σ_1) (d) (G_2, σ_2)

Figure 6: Configurations in Lemma 4.6

Proof. Assume to the contrary that (G, σ) contains two negative 4-cycles $abyx$ and $cdyx$ sharing one common negative edge xy, as shown in Figure 6a; or contains a 3-cycle cxy and a 5-cycle $abdyx$ sharing one common negative edge xy, as shown in Figure 6b. For both cases, let a', b', c', and d' be the third neighbors of vertices a, b, c, and d, respectively. Since the 6-cycle $axcdyb$ or $axcydb$ has only one chord xy, $\{a, b\} \cap (N(c) \cup N(d)) = \emptyset$. For both cases, we construct two signed graphs (G_1, σ_1) and (G_2, σ_2) as follows: delete vertices $\{a, b, c, d, x, y\}$ and replace the part connecting a', b', c', and d' by the two signed subgraphs depicted in Figure 6c and Figure 6d. In particular, we define that $\sigma_1(a'u) = \sigma_2(a'u) = \sigma(a'a)$, $\sigma_1(b'v) = \sigma_2(b'u) = \sigma(b'b)$, $\sigma_1(c'u) = \sigma_2(c'v) = \sigma(c'c)$, and $\sigma_1(d'v) = \sigma_2(d'v) = \sigma(d'd)$. The resulting signed graphs are denoted by (G_1, σ_1) and (G_2, σ_2), respectively.

We first claim that at least one of (G_1, σ_1) and (G_2, σ_2) is 2-edge-connected and has no parallel edges, and denote it by (G', σ'). We consider the vertex set $\{a', b', c', d'\}$. If three of them coincide, say $a' = b' = c'$, then dd' is a cut-edge of G, a contradiction; If two of them coincide, then for cases that $a' = d'$ or $b' = c'$, both graphs are 2-edge-connected with no parallel edges, and for other cases, by symmetry we only consider the cases that $a' = c'$ or $a' = b'$. For the case $a' = c'$, (G_2, σ_2) is 2-edge-connected with no parallel edges, and for the case $a' = b'$, (G_1, σ_1) is 2-edge-connected with no parallel edges. If all a', b', c', and d' are distinct, then both (G_1, σ_1) and (G_2, σ_2) has no parallel edges, and as G is 2-edge-connected, at least one of them is 2-edge-connected. Thus the claim is proved.

Next we claim that $F(G_i, \sigma_i) = F(G, \sigma) - 1$ for each $i \in \{1, 2\}$. Assume not and thus (G', σ') contains an unequilibrated cut C. We form a new cut C' of (G, σ) based on the following edge replacement operations.

(i) If C contains uv, then: for the case of two adjacent negative 4-cycles, we replace uv by $\{ab, xy, cd\}$ when $i = 1$ and replace uv by $\{cx, xy, by\}$ when $i = 2$; for the case of adjacent negative 3-cycle and 5-cycle, we replace uv by $\{ab, xy, cy\}$ when $i = 1$ and replace uv by $\{cx, xy, bd\}$ when $i = 2$.

(ii) If C contains wz for $i = 1, 2$, where $w \in \{a', b', c', d'\}$ and $z \in \{u, v\}$, then for both cases, we replace wz by ww_0, where w_0 is the vertex that satisfies the following conditions: $w_0 \in N(w)$, w_0 is the vertex which is contained in the 6-cycle, and $\sigma(ww_0) = \sigma(wz)$;

(iii) Otherwise, let C'=C.

Note that the conditions (i) and (ii) can occur simultaneously. If so, then we apply both operations. It is easy to see that C' is an unequilibrated cut in (G, σ), a contradiction to Observation 2.1. Hence,

$$F(G', \sigma') = F(G, \sigma) - 1 > \frac{1}{3}v(G) - 1 = \frac{1}{3}(v(G') + 4) - 1 > \frac{1}{3}v(G').$$

Note that $v(G') = v(G) - 4 \geq 6$. If (G', σ') is not isomorphic to any of $\widehat{\Gamma}_i$'s, then it is a smaller counterexample, contradicting the minimality of (G, σ). So it must hold that $v(G') = 8$ and (G', σ') can only be one of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$. Note that as $v(G) = v(G')+4 = 12$, (G, σ) is a signed graph on 12 vertices with 4 negative edges and thus $F(G, \sigma) = \frac{1}{3}v(G)$, a contradiction. $\square$

Lemma 4.7. *Assume that $\{v_1u_1, v_2u_2\}$ is a 2-edge-cut of the minimum counterexample (G, σ). Let H be a 2-edge-connected component of $G - \{v_1u_1, v_2u_2\}$, where $\{u_1, u_2\} \subset V(H)$. If both v_1u_1 and v_2u_2 are positive, and v_1 is not adjacent to v_2, then neither v_1 nor v_2 is incident with a negative edge.*

(a) (G, σ) (b) (G', σ')

Figure 7: Configurations in Lemma 4.7

Proof. Let $\{u_1v_1, u_2v_2\} := [V(H), V(G) \setminus V(H)]$ be a 2-edge-cut of (G, σ) where $\{u_1, u_2\} \subset V(H)$. Without loss of generality, assume to the contrary that v_1 is incident to a negative edge v_1x, see Figure 7a. Note that $v_1 \neq v_2$, since otherwise G would contain either a cut-edge that connects v_1, or a 2-vertex that is v_1 itself, either of which leads to a contradiction. Note that H is 2-edge-connected and has no parallel edges.

We first claim that $|E^-_{(H,\sigma|_H)}| = F(H, \sigma|_H)$. Assume to the contrary that $|E^-_{(H,\sigma|_H)}| > F(H, \sigma|_H)$, so there exists an unequilibrated cut $C := [H_1, H_2]$ where $V(H) = H_1 \cup H_2$ in $(H, \sigma|_H)$. Note that C must separate u_1 and u_2, as otherwise C itself is an unequilibrated cut of (G, σ), a contradiction. Without loss of generality, we assume that $u_1 \in H_1$ and $u_2 \in H_2$. Now since $v_1 \notin H$, the cut $[H_1 \cup \{v_1\}, V(G) \setminus (H_1 \cup \{v_1\})]$, which is indeed $C \cup \{xv_1, yv_1\}$, is an unequilibrated cut of (G, σ), a contradiction to Observation 2.1. We next claim that $F(H, \sigma|_H) \leq \frac{1}{3}v(H)$. Assume not and since $(H, \sigma|_H)$ is 2-edge-connected, and has no parallel edges, by the minimality of the choice of (G, σ), $(H, \sigma|_H)$ is isomorphic to one of $\widehat{\Gamma}_i$'s. But it is not possible since u_1 and u_2 are two 2-vertices in $(H, \sigma|_H)$ but none of the exceptional graphs has two 2-vertices.

Now we form a signed graph (G', σ') from (G, σ) as follows: delete all vertices of H, and add a positive edge v_1v_2. Note that (G', σ') is 2-edge-connected and has no parallel edges. Furthermore, for any unequilibrated cut of (G', σ') containing the edge v_1v_2, replacing it with u_2v_2 results in an unequilibrated cut in (G, σ). Therefore, (G', σ') contains no unequilibrated cut, and hence $F(G', \sigma') = |E^-_{(G',\sigma')}|$ where $|E^-_{(G',\sigma')}| = F(G, \sigma) - F(H, \sigma|_H)$. Since $F(H, \sigma|_H) \leq \frac{1}{3}v(H)$, we have that

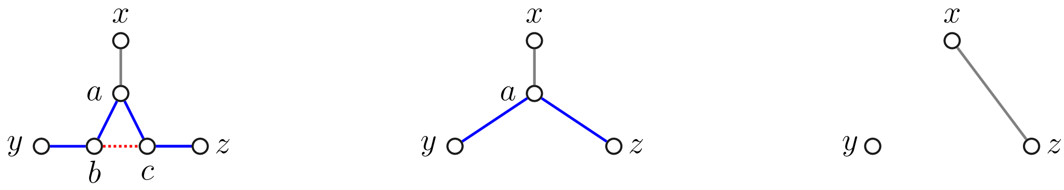
$$F(G', \sigma') = F(G, \sigma) - F(H, \sigma|_H) > \frac{1}{3}v(G) - \frac{1}{3}v(H) = \frac{1}{3}v(G').$$

If (G', σ') is not isomorphic to any of $\widehat{\Gamma}_i$'s, then it is a smaller counterexample, contradicting the minimality of (G, σ). So (G', σ') must be one of $\widehat{\Gamma}_i$'s.

We discuss these five signed graphs separately. Note that $(G', \sigma') - v_1v_2$ is also a subgraph of (G, σ). If (G', σ') is isomorphic to either $\widehat{\Gamma}_1$ or $\widehat{\Gamma}_5$, then as both $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_5$ are edge-transitive, there exists σ_1 switching-equivalent to σ with $|E^-_{(G,\sigma_1)}| = F(G, \sigma)$ satisfying the following conditions: (1) either $(G', \sigma_1) - v_1v_2$ contains a pair of adjacent triangles sharing a negative edge, which contradicts Lemma 4.4, (2) or $(G', \sigma_1) - v_1v_2$ contains adjacent 4-cycles where the common edge is negative, and the 6-cycle they formed has only one chord, which contradicts Lemma 4.6(1). If (G', σ') is isomorphic to $\widehat{\Gamma}_2$, then G has a 2-vertex, a contradiction to Lemma 4.3. If (G', σ') is isomorphic to $\widehat{\Gamma}_3$, then $(G', \sigma') - v_1v_2$ must contain two adjacent triangles, a contradiction to Lemma 4.4. If (G', σ') is isomorphic to $\widehat{\Gamma}_4$, then as $\widehat{\Gamma}_4$ is a signed planar graph and every facial cycle is negative, there exists σ^* switching-equivalent to σ' with $|E^-_{(G,\sigma^*)}| = F(G', \sigma')$ satisfying the following: (1) $(G', \sigma^*) - v_1v_2$ contains a triangle adjacent to a 4-cycle, where one edge of the 3-cycle that is not the common edge shared with the 4-cycle is negative, which contradicts Lemma 4.5; or (2) $(G', \sigma^*) - v_1v_2$ contains a triangle adjacent to a 5-cycle, where the common shared edge is negative, and the 6-cycle they formed has only one chord, which contradicts Lemma 4.6(2). □

Lemma 4.8. *The minimum counterexample (G, σ) contains no negative triangle.*

Proof. Assume to the contrary that (G, σ) contains a negative triangle abc where bc is negative. Let x, y, and z be the third neighbors of a, b, and c, respectively. By Lemma 4.4 we know that vertices x, y, z are distinct, and furthermore, by Lemma 4.5 x is not the neighbor of y or z, as shown in Figure 8a. We form a signed graph (G', σ') from (G, σ) by



(a) (G,σ) (b) (G',σ') (c) (G'',σ'')

Figure 8: Configurations in Lemma 4.8

deleting vertices b and c, and adding two positive edges ay and az, as shown in Figure 8b. Note that (G',σ') is also 2-edge-connected. Now we consider the following two cases based on the situation of the edge ya:

Case 1. *The edge ya is in a 2-edge-cut of (G',σ').*

In this case, we may assume that $\{ya, uv\} := [V(H), V(G')\setminus V(H)]$ is a 2-edge-cut in (G',σ') where $\{y,u\}\subset H$ and H is 2-edge-connected. It is easy to see that $[V(H), V(G)\setminus V(H)](=\{yb,uv\})$ is an edge-cut in (G,σ) where $\{y,u\}\subset H$ and H is 2-edge-connected. We first claim that $\{a,b,c\}\cap V(H)=\emptyset$. As yb is in the 2-edge-cut, $b\notin V(H)$. Assume to the contrary that either: $a\in V(H)$ and $c\notin V(H)$, or $a\notin V(H)$ and $c\in V(H)$, or $\{a,c\}\subset V(H)$. Then $[V(H),V(G)\setminus V(H)]$ contains: $\{yb,ab,ac\}$, $\{yb,ac,bc\}$, or $\{yb,ab,bc\}$ respectively, each contradicting the fact that $[V(H),V(G)\setminus V(H)]$ is a 2-edge-cut. We then claim that uv is a positive edge, as otherwise $[V(H)\cup\{b\}, V(G)\setminus(V(H)\cup\{b\})] := \{uv,ab,bc\}$ is an unequilibrated cut of (G,σ), a contradiction to Observation 2.1. We finally claim that v is not adjacent to b. Assume not, and since $v\notin V(H)$ and $y\in V(H)$, we have $v\in\{a,c\}$. As $\{a,b,c\}\cap V(H)=\emptyset$, if $v=a$, then $uv=ax$, and $[V(H)\cup\{a,b,c\}, V(G)\setminus(V(H)\cup\{a,b,c\})] = \{cz\}$; if $v=c$, then $uv=cz$, and $[V(H)\cup\{a,b,c\}, V(G)\setminus(V(H)\cup\{a,b,c\})]=\{ax\}$. In both cases G has a cut-edge, a contradiction. So now there exists an all-positive edge-cut $\{yb,uv\}$ such b is incident to a negative edge bc, and not adjacent to v, a contradiction to Lemma 4.7.

Case 2. *The edge ya is not in any 2-edge-cut of (G',σ').*

In this case, we form (G'',σ'') from (G',σ') by deleting the vertex a and adding an edge xz where $\sigma''(xz)=\sigma'(xa)$, as shown in Figure 8c. Since x is not a neighbor of z, (G'',σ'') has no parallel edges; and as ya is not in any 2-edge-cut of (G',σ'), (G'',σ'') is 2-edge-connected. Next we shall prove that $F(G'',\sigma'')=|E^-_{(G'',\sigma'')}|$. Assume not and it implies that (G'',σ'') contains an unequilibrated cut $C:=[A,B]$. We consider the following cut C' of (G,σ):

$$C' = \begin{cases} [A, B\cup\{a,b,c\}] & \text{if } x\in A, \{y,z\}\subset B \\ [A\cup\{b\}, B\cup\{a,c\}] & \text{if } \{x,y\}\subset A, z\in B, \\ [A\cup\{a,c\}, B\cup\{b\}] & \text{if } \{x,z\}\subset A, y\in B, \\ [A\cup\{a,b,c\}, B] & \text{if } \{x,y,z\}\subset A, \end{cases}$$

where $C'=C\setminus\{xz\}\cup\{xa\}$ in the first case, $C'=C\setminus\{xz\}\cup\{xa,ab,bc\}$ in the second case, $C'=C\cup\{ab,bc\}$ in the third case, and $C'=C$ in the last case. In each case, C' is an unequilibrated cut in (G,σ), a contradiction to Observation 2.1. Thus we have that

$$F(G'',\sigma'') = F(G,\sigma)-1 > \frac{1}{3}v(G)-1 = \frac{1}{3}(v(G'')+3)-1 = \frac{1}{3}v(G'').$$

Note that $v(G'')=v(G)-3\geq 7$. If (G'',σ'') is not isomorphic to any of $\widehat{\Gamma}_i$'s, then it is a smaller counterexample, contradicting the minimality of (G,σ). So we know that

$v(G'') = 8$ and (G'', σ'') can only be one of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$. However, (G'', σ'') has a 2-vertex y, contradicting the fact that $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$ are all cubic. □

Lemma 4.9. *The minimum counterexample (G, σ) contain no $\widehat{H}_1$ as a subgraph, where $\widehat{H}_1$ is the graph depicted in Figure 2b.*

Proof. Assume for the contradiction that (G, σ) contains $\widehat{H}_1$ as a subgraph. We form (G', σ') from (G, σ) by replacing $\widehat{H}_1$ with the subgraph in Figure 2a. it is easy to see that (G', σ') is 2-edge-connected and simple. Moreover, we claim that $F(G', \sigma') = F(G, \sigma) - 1$. Assume not and it implies that (G', σ') contains an unequilibrated cut $C := [A, B]$. We construct a new cut C' (of (G, σ)) as follows:

$$C' = \begin{cases} [A, B \cup \{a_i, b_i, c_i, d_i\}] & \text{if } x_i \in A, \{a_i, y_i\} \subset B, \\ [A \cup \{c_i\}, B \cup \{b_i, d_i\}] & \text{if } \{x_i, a_i\} \subset A, y_i \in B, \\ [A \cup \{a_i, b_i, c_i, d_i\}, B \setminus \{a_i\}] & \text{if } \{x_i, y_i\} \subset A, a_i \in B, \\ [A \cup \{b_i, c_i, d_i\}, B] & \text{if } \{x_i, a_i, y_i\} \subset A, \end{cases}$$

where $C' = C \setminus \{x_i y_i\} \cup \{x_i c_i\}$ in the first case, $C' = C \setminus \{a_i y_i, x_i y_i\} \cup \{a_i b_i, a_i d_i, c_i b_i, c_i d_i\}$ in the second case, $C' = C \setminus \{x_i a_i, a_i y_i\}$ in the third case, and $C' = C$ in the last case. It is easy to see that C' is an unequilibrated cut of (G, σ), a contradiction to Observation 2.1. Hence, we have that

$$F(G', \sigma') = F(G, \sigma) - 1 > \frac{1}{3} v(G) - 1 = \frac{1}{3}(v(G') + 3) - 1 = \frac{1}{3} v(G').$$

Note that $v(G') = v(G) - 3 \geq 7$. If (G', σ') is not isomorphic to any of $\widehat{\Gamma}_i$'s, then it is a smaller counterexample, contradicting the minimality of (G, σ). So we know that $v(G') = 8$ and (G', σ') can only be one of $\widehat{\Gamma}_3, \widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$. It is impossible since (G', σ') has a 2-vertex a_i. □

Lemma 4.10. *The minimum counterexample (G, σ) contains no $\widehat{H}_2$ as a signed subgraph, where $\widehat{H}_2$ is the graph depicted in Figure 9a and $x \notin \{b', c'\}$.*

(a) A signed subgraph $\widehat{H}_2$ (b) (G', σ')

Figure 9: Configurations in Lemma 4.10

Proof. Assume to the contrary that (G, σ) contains $\widehat{H}_2$ as a subgraph, labeled as in Figure 9a. Note that $b' \neq c'$, as otherwise (G, σ) contains $\widehat{H}_1$ as a subgraph, contradicting Lemma 4.9. We form a signed graph (G', σ') from (G, σ) by deleting vertices $\{y, z, b, c\}$, and adding a negative edge ub' and a positive edge uc', as shown in Figure 9b. It is easy to see that (G', σ') is 2-edge-connected. Since $x \notin \{b', c'\}$, (G', σ') contains no parallel edges. Moreover, we claim that $F(G',\sigma') = F(G,\sigma) - 1$. Assume not and it implies that (G',σ') contains an unequilibrated cut $C := [A, B]$. We consider the following cut C' of (G,σ):

$$C' = \begin{cases} [A\cup\{y,z\}, B\cup\{b,c\}] & \text{if } u\in A, \{b',c'\}\subset B, \\ [A\cup\{y,b,z,c\}, B] & \text{if } \{u,b'\}\subset A, c'\in B, \\ [A\cup\{y,c\}, B\cup\{b,z\}] & \text{if } \{u,c'\}\subset A, b'\in B, \\ [A\cup\{y,b,z,c\}, B] & \text{if } \{u,b',c'\}\subset A, \end{cases}$$

where $C' = C\setminus\{b'u, c'u\}\cup\{by, bz, cy, cz\}$ in the first case, $C' = C\setminus\{c'u\}\cup\{c'c\}$ in the second case, $C' = C\setminus\{b'u\}\cup\{by, uz, cz\}$ in the third case, and $C' = C$ in the last case. The edge-cut C' is an unequilibrated cut of (G,σ), a contradiction to Observation 2.1. Hence, we have that

$$F(G',\sigma') = F(G,\sigma) - 1 > \frac{1}{3}v(G) - 1 = \frac{1}{3}(v(G') + 4) - 1 > \frac{1}{3}v(G').$$

Note that $v(G') = v(G) - 4 \geq 6$. If (G',σ') is not isomorphic to any of $\widehat{\Gamma}_i$'s, then it is a smaller counterexample, contradicting the minimality of (G,σ). So we know that $v(G') = 8$ and (G',σ') can only be one of $\widehat{\Gamma}_3$, $\widehat{\Gamma}_4$, and $\widehat{\Gamma}_5$. Thus (G,σ) is a signed subcubic graph on 12 vertices with 3 negative edges, i.e., $F(G,\sigma) = \frac{1}{3}v(G)$, a contradiction. □

4.2 Proof of Lemma 2.3

Recall that for $i\in\{0,1,2,3\}$, $X_i = \{u\in X \mid d_Y(u) = i\}$, and for $j\in\{0,1,2\}$, $Y_j = \{v\in Y \mid d_X(v) = j\}$. We aim to show that

$$|X_3| + |Y_0| \leq |Y_2|.$$

It follows from Lemma 4.3 that the minimum counterexample (G,σ) is cubic, and from Lemma 4.8 that it contains no negative triangle.

Lemma 4.11. *$Y_0 = \emptyset$.*

Proof. Assume to the contrary that there is a vertex $u\in Y_0$. Let x and y be two positive neighbors of u. Since $d_X(u) = 0$, both x and y are in Y. If xy is not a negative edge, then $G[\{u,x,y\}]\in\mathcal{T}$ is an induced subgraph of G, a contradiction. So xy must be a negative edge. It implies that u is in a negative triangle, a contradiction to Lemma 4.8. □

Lemma 4.12. *Every vertex of X_3 has at least two neighbors that are vertices in Y_2.*

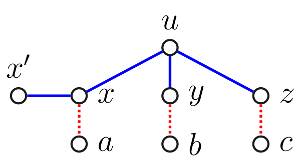


Figure 10: A vertex $u\in X_3$ has three neighbors $x,y,z\in Y$

Proof. Let u be an vertex of X_3 in (G,σ) with its three neighbors x, y, and z. Since $u\in X_3$, $\{x,y,z\}\subset Y$. We first claim that $\{x,y,z\}$ is an independent set, as shown in Figure 10. If not, then without loss of generality we may assume that x is adjacent to y. By Lemma 4.8,

xy must be a positive edge (i.e., uxy is a positive triangle). Now $G[\{u,x,y\}] \in \mathcal{C}$ is an induced subgraph of G, a contradiction.

Assume to the contrary that at least two of x, y, and z are not in Y_2. By Lemma 4.11, we know that $Y_0 = \emptyset$. So without loss of generality, we may assume $\{x,y\} \subset Y_1$, i.e., $d_X(x) = 1$ and $d_X(y) = 1$. Let a, b, c be the other neighbor of x, y, z where ax, by, cz are all negative edges in (G, σ). By definition, each of a, b, and c is in Y. Let x' be the third neighbor of x. As $d_X(x) = 1$ and $u \in X \cap N(x)$, we have that $x' \in Y$, i.e., x' is incident with a negative edge. Thus $G[\{x, x', u, y, z\}] \in \mathcal{T}$ is an induced subgraph, unless x' coincides with exactly one of b and c. Without loss of generality, we assume that $x' = b$, i.e., xb is a positive edge. Similar arguments apply to y, and hence y is connected to exactly one of a and c via a positive edge. We consider the following two cases:

- Assume that $yc \in E(G)$. In this case, b is not the neighbor of z or c, as otherwise either $G[\{x,b,z\}] \in \mathcal{T}$ or $G[\{x,b,c\}] \in \mathcal{T}$ is an induced subgraph of (G, σ), a contradiction to Lemma 2.2. Moreover, since x is neither a neighbor of c nor z, the 6-cycle $xbyczu$ has only one chord yu. By switching at an equilibrated cut $[y, c]$, the resulting signed graph (G, σ') satisfies that $|E^-_{(G,\sigma')}| = F(G, \sigma)$. Furthermore, (G, σ') contains two adjacent negative 4-cycles sharing a common negative edge yu (where the 6-cycle is exactly $xbyczu$), a contradiction to Lemma 4.6(1).

- Assume that $ya \in E(G)$. In this case, if z is adjacent to exactly one of a and b via a positive edge, by symmetry, say $zb \in E(G)$, then $G[\{x,b,z\}] \in \mathcal{T}$ is an induced subgraph of G, a contradiction. Thus it contains $\widehat{H}_2$ as a subgraph, which contradicts Lemma 4.10.

We complete the proof of the lemma. $\square$

Now consider the bipartite subgraph of G between X_3 and Y_2. By double counting on the number of edges between X_3 and Y_2, we have that

$$\sum_{v \in X_3} d_{Y_2}(v) = \sum_{u \in Y_2} d_{X_3}(u).$$

For $v \in X_3$, by Lemma 4.12, we have that $d_{Y_2}(v) \geq 2$; and for $u \in Y_2$, by the definition of Y_2, we have that $d_{X_3}(u) \leq d_X(u) = 2$. Therefore, since $Y_0 = \emptyset$ by Lemma 4.11,

$$|X_3| + |Y_0| = |X_3| \leq \frac{1}{2} \sum_{v \in X_3} d_{Y_2}(v) = \frac{1}{2} \sum_{u \in Y_2} d_{X_3}(u) \leq |Y_2|.$$

This completes the proof of Lemma 2.3.

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