Power-law Bianchi type I inflation with multiple vector fields

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We seek power-law Bianchi type I solutions for an inflationary universe in a model with one scalar field non-minimally coupled to three vector fields aligned along the three axes. As a result, we find four types of power-law solutions that are classified according to the number of non-vanishing vector fields. Moreover, we find that all these solutions are stable for certain conditions on the parameters (stability regions), which are described both quantitatively and qualitatively. The attractor property of these solutions is also confirmed by numerical calculations.

I. INTRODUCTION

In modern cosmology, the cosmological principle, which is a hypothesis stating that the universe is homogeneous and isotropic on large scales, has served as an underlying assumption. A universe obeying the cosmological principle would be uniquely described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric [1]. Moreover, the cosmological principle is supported by the cosmic no-hair conjecture proposed by Hawking et al [2, 3]. According to the conjecture, all homogeneities and anisotropies (a.k.a. classical spatial hairs) would vanish as the universe expands. Therefore, if the conjecture holds, the initial conditions become unimportant and the cosmological principle is realized in later time. Remarkably, the validity of the cosmic no-hair conjecture has been examined extensively by many people, e.g., see Refs. [4–21] for an incomplete list of literature. Many of them have focused on the so-called inflationary phase of the early universe, which was proposed as a resolution to some fundamental issues in modern cosmology such as the flatness, horizon and magnetic monopole problems [22–24], to see if the cosmic no-hair conjecture is valid or not.

Recently, the cosmological principle and cosmic no-hair conjecture have been challenged by the leading observations of the Cosmic Microwave Background (CMB). In particular, the Wilkinson Microwave Anisotropy Probe (WMAP) detected two anomalies on the CMB including the cold spot and hemispheric asymmetry [25], which were also confirmed by the Planck satellite [26]. Therefore, it is reasonable to seek an exotic inflationary model that violates the cosmological principle and cosmic no-hair conjecture [27] because the mentioned CMB anomalies have been believed to not show up in the standard inflationary models, in which the cosmological principle holds as well as the cosmic no-hair should never be broken down. One of the candidate for the origin of the observed anomalies is vector field(s). However, vector fields are usually ignored in studies of inflation since they are believed to decay very quickly as the universe expands, which is consistent with the cosmic no-hair conjecture.

However, an anisotropic inflationary model proposed by Kanno, Soda, and Watanabe (KSW) showed the opposite [28, 29]. Motivated by the supergravity, they introduced a non-minimal coupling between one scalar field ϕ and one vector field A_μ such as $f^2(\phi)F^{\mu\nu}F_{\mu\nu} \propto \exp(2\rho\phi/M_p)F^{\mu\nu}F_{\mu\nu}$. As a result, they were success to derive an anisotropic power-law inflationary solution, which has been shown to be stable against perturbations under some specific condition of parameters. In fact, the gauge kinetic function, i.e., $f(\phi)$, plays a crucial role in keeping the anisotropies from being diluted. The KSW model therefore becomes a true counterexample of the cosmic no-hair conjecture, which has received a lot of attention [30, 31]. Interestingly, many other counterexamples, which are nothing but non-trivial extensions of the KSW model, have been proposed in recent years [32–65]. It is noticeable that in most of such these models, the spacetime is chosen to be a locally rotationally symmetric (LRS) Bianchi type I metric, which has the following form,

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)(dy^2 + dz^2). \tag{1.1}$$

In this paper, we would like to propose an extension of KSW model for multiple vector fields, i.e., three vector fields along three axes. We also impose the Bianchi type I metric in the most general form,

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2, \tag{1.2}$$

instead of the LRS one shown above. It is worth noting that anisotropic inflation with multiple vector fields have been studied in Refs. [38, 39, 42, 56, 57, 60, 63]. However, the background considered in these papers is still that

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used in the original KSW model, i.e., the LRS Bianchi type I metric. In our model, we are mainly concerned with the evolution and fate of the vector fields and anisotropies with different values of parameters. In particular, we will find all isotropic and anisotropic solutions of the model and investigate their stability. As a result, we will see that all solutions are stable and attractive for some specific conditions of parameters (stability regions). We will also give some interesting qualitative descriptions for these stability regions.

Our paper will be organized as follows: (i) A brief introduction of our study has been written in Sect. I. (ii) In Sect. II, we will present the general setup of our model and derive the corresponding field equations. (iii) In Sect. III, we will solve the field equations to find the anisotropic power-law solutions. (iv) In Sect. IV, we will investigate the stability of solutions by transforming the field equations into autonomous equations of dynamical variables. Numerical calculations to support the above stability analysis will be conducted in this section. (v) Conclusions and further discussions will be given in Sect. V. (vi) Finally, Appendices A and B are devoted to some detailed calculations.

II. MODEL SETUP

We consider the Bianchi type I (BI) metric, which describes a homogeneous but anisotropic universe, as

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2, \tag{2.1}$$

where $a(t)$, $b(t)$ and $c(t)$ are the scale factors corresponding to the x-, y- and z- axes respectively. It is convenient to parameterize the metric as

$$ds^2 = -dt^2 + e^{2\alpha(t)-2\sigma_b(t)-2\sigma_c(t)}dx^2 + e^{2\alpha(t)+2\sigma_b(t)}dy^2 + e^{2\alpha(t)+2\sigma_c(t)}dz^2. \tag{2.2}$$

Another way to parameterize the metric (2.1) can be found in Ref. [30]. The averaged Hubble parameter defined as

$$H \equiv \frac{1}{3}\left(H_a + H_b + H_c\right) = \dot{\alpha}, \tag{2.3}$$

where $H_a \equiv \dot{a}/a$, $H_b \equiv \dot{b}/b$ and $H_c \equiv \dot{c}/c$. We also define the spatial anisotropies as

$$X_b \equiv \frac{H_b - H}{H} = \frac{\dot{\sigma}_b}{\dot{\alpha}}, \quad X_c \equiv \frac{H_c - H}{H} = \frac{\dot{\sigma}_c}{\dot{\alpha}}, \quad X_a \equiv \frac{H_a - H}{H} = \frac{-\dot{\sigma}_b - \dot{\sigma}_c}{\dot{\alpha}} = -X_b - X_c, \tag{2.4}$$

which measure the corresponding spatial deviations from an isotropy along y-, z-, and x-axes, respectively. Note that when two of the anisotropies are equal, say $X_b = X_c$, we can use the coordinate transformation $z \to c(t)z/b(t)$ to make the scale factors $b(t)$ and $c(t)$ equal to each other and then recover the Bianchi type I metric that is locally rotationally symmetric about x-axis (denoted by the x-rsBI metric for short) and has the following form,

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)dx^2 + b^2(t)(dy^2 + dz^2) \\ &= -dt^2 + e^{2\alpha(t)-4\sigma(t)}dx^2 + e^{2\alpha(t)+2\sigma(t)}(dy^2 + dz^2). \end{aligned} \tag{2.5}$$

where $\sigma(t) \equiv \sigma_b(t) = \sigma_c(t)$. Similarly, when $X_a = X_b = X_c$ (and consequently they are all equal to zero), we obtain the well-known FLRW metric as

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \\ &= -dt^2 + e^{2\alpha(t)}(dx^2 + dy^2 + dz^2). \end{aligned} \tag{2.6}$$

In many studies on anisotropic inflation, the metric has been taken to be (2.5). A typical example can be named is the KSW model [28, 29], in which the authors have considered a supergravity-motivated action,

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu}\right], \tag{2.7}$$

where the reduced Planck mass has been set to be one, i.e., $M_p = 1$, for convenience. In addition, R is the Ricci scalar, ϕ is a scalar field, $V(\phi)$ is the potential and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the vector field A_μ. In harmony with the metric (2.5), a homogeneous scalar field and a vector field of the form,

$$\phi = \phi(t), \quad A_\mu = (0, A_a(t), 0, 0), \tag{2.8}$$

are introduced. The metric (2.5) have also appeared in other studies, e.g., Refs. [37, 44, 46, 52, 55, 59], with different actions and field configurations that are also required to be in harmony with the metric.

In our model, we would like to consider a more general action

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2}-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)-\frac{1}{4}\sum_i f_i^2(\phi)F_{(i)\mu\nu}F_{(i)}{}^{\mu\nu}\right], \tag{2.9}$$

which is a straightforward generalization of (2.7) for multiple vector fields. From the principle of least action, we derive the corresponding Einstein field equations to be

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\left[\frac{1}{2}\partial_\rho\phi\partial^\rho\phi+V(\phi)+\frac{1}{4}\sum_i f_i^2(\phi)F_{(i)\rho\sigma}F_{(i)}{}^{\rho\sigma}\right]+\sum_i f_i^2(\phi)F_{(i)\mu\sigma}F_{(i)\nu}{}^{\sigma} \tag{2.10}$$

along with the equation of motion of scalar field ϕ given by

$$\Box\phi-V'(\phi)-\frac{1}{2}\sum_i f_i(\phi)f_i'(\phi)F_{(i)\rho\sigma}F_{(i)}{}^{\rho\sigma}=0, \tag{2.11}$$

where the prime stands for derivative with respect to ϕ and $\Box\equiv\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu)$ being the d'Alembert operator. Lastly, the equation of motion of vector field is defined to be

$$\partial_\mu\left[\sqrt{-g}f_i^2(\phi)F_{(i)}{}^{\mu\nu}\right]=0. \tag{2.12}$$

Instead of the metric (2.5), we consider the Bianchi type I metric in the most general form (2.2). In harmony with the chosen metric, we introduce a homogeneous scalar field along with three vector fields of the following forms,

$$\phi=\phi(t),\quad A_\mu=(0,A_a(t),0,0),\quad B_\mu=(0,0,B_b(t),0),\quad C_\mu=(0,0,0,C_c(t)), \tag{2.13}$$

where we have renamed the vector fields as follows: $A_\mu\equiv A_{(1)\mu}$, $B_\mu\equiv A_{(2)\mu}$, and $C_\mu\equiv A_{(3)\mu}$, just for convenience. If we instead choose the KSW's field configuration (2.8), the difference between anisotropies in y- and z-axes decays quickly as the universe expands and therefore it is more convenient to consider the metric (2.5) instead of (2.2) [30]. However as we will see later, it is not the case for the field configuration (2.13). It is noted that the action (2.9) as well as the field configuration (2.13) have also been considered in Refs. [38, 39, 42]. Especially in Ref. [38], anisotropic power-law solutions are confirmed to exist by numerical calculations. In our paper, we try to derive the exact solutions and investigate their stability and attractor properties.

For the assumed metric (2.2) and field configuration, non-trivial solutions for three vector fields (2.12) are solved from their equation of motion to be

$$\dot{A}_a=p_af_a^{-2}e^{-\alpha-2\sigma_b-2\sigma_c},\quad \dot{B}_b=p_bf_b^{-2}e^{-\alpha+2\sigma_b},\quad \dot{C}_c=p_cf_c^{-2}e^{-\alpha+2\sigma_c}, \tag{2.14}$$

where the dot stands for the derivative with respect to cosmic time t. In addition, p_a, p_b, and p_c are constants of integration. We can therefore write explicitly the Einstein field equations (2.10) as

$$3\dot\alpha^2-\dot\sigma_b^2-\dot\sigma_c^2-\dot\sigma_b\dot\sigma_c=\frac{1}{2}\dot\phi^2+V(\phi)+\frac{p_a^2}{2}f_a^{-2}(\phi)e^{-4\alpha-2\sigma_b-2\sigma_c}+\frac{p_b^2}{2}f_b^{-2}(\phi)e^{-4\alpha+2\sigma_b}+\frac{p_c^2}{2}f_c^{-2}(\phi)e^{-4\alpha+2\sigma_c}, \tag{2.15}$$

$$\ddot\alpha+3\dot\alpha^2=V(\phi)+\frac{p_a^2}{6}f_a^{-2}(\phi)e^{-4\alpha-2\sigma_b-2\sigma_c}+\frac{p_b^2}{6}f_b^{-2}(\phi)e^{-4\alpha+2\sigma_b}+\frac{p_c^2}{6}f_c^{-2}(\phi)e^{-4\alpha+2\sigma_c}, \tag{2.16}$$

$$\ddot\sigma_b+3\dot\alpha\dot\sigma_b=\frac{p_a^2}{3}f_a^{-2}(\phi)e^{-4\alpha-2\sigma_b-2\sigma_c}-\frac{2p_b^2}{3}f_b^{-2}(\phi)e^{-4\alpha+2\sigma_b}+\frac{p_c^2}{3}f_c^{-2}(\phi)e^{-4\alpha+2\sigma_c}, \tag{2.17}$$

$$\ddot\sigma_c+3\dot\alpha\dot\sigma_c=\frac{p_a^2}{3}f_a^{-2}(\phi)e^{-4\alpha-2\sigma_b-2\sigma_c}+\frac{p_b^2}{3}f_b^{-2}(\phi)e^{-4\alpha+2\sigma_b}-\frac{2p_c^2}{3}f_c^{-2}(\phi)e^{-4\alpha+2\sigma_c}. \tag{2.18}$$

On the other hand, the field equation of scalar field (2.11) apparently reads

$$\ddot\phi+3\dot\alpha\dot\phi+V'(\phi)-p_a^2f_a^{-3}(\phi)f_a'(\phi)e^{-4\alpha-2\sigma_b-2\sigma_c}-p_b^2f_b^{-3}(\phi)f_b'(\phi)e^{-4\alpha+2\sigma_b}-p_c^2f_c^{-3}(\phi)f_c'(\phi)e^{-4\alpha+2\sigma_c}=0, \tag{2.19}$$

thanks to the solutions shown in Eq. (2.14). Apparently, Eqs. (2.15), (2.16), (2.17), (2.18), and (2.19) will reduce to field equations in Ref. [29] if $p_b=p_c=0$ and $\sigma_b=\sigma_c=\sigma$.

III. ANISOTROPIC POWER-LAW SOLUTIONS

In order to realize power-law inflationary solutions, the potential and the coupling functions should take the following forms [29],

$$V(\phi)=V_0e^{\lambda\phi},\quad f_a(\phi)=f_{a0}e^{\rho_a\phi},\quad f_b(\phi)=f_{b0}e^{\rho_b\phi},\quad f_c(\phi)=f_{c0}e^{\rho_c\phi},\tag{3.1}$$

where V_0, f_{a0}, f_{b0}, f_{c0}, λ, ρ_a, ρ_b, and ρ_c are constants. We investigate the power-law solutions in this model by considering the compatible ansatz [29],

$$\alpha=\zeta\log t,\quad \sigma_b=\eta_b\log t,\quad \sigma_c=\eta_c\log t,\quad \phi=\xi\log t+\phi_0.\tag{3.2}$$

Consequently, the metric (2.2) now becomes a power-law metric given by

$$ds^2=-dt^2+t^{2\zeta-2\eta_b-2\eta_c}dx^2+t^{2\zeta+2\eta_b}dy^2+t^{2\zeta+2\eta_c}dz^2,\tag{3.3}$$

and the corresponding anisotropies (2.4) turn out to be

$$X_a=\frac{\eta_a}{\zeta},\quad X_b=\frac{\eta_b}{\zeta},\quad X_c=\frac{\eta_b}{\zeta}.\tag{3.4}$$

where $\eta_a\equiv-\eta_b-\eta_c$. Apparently, the metric (3.3) will reduce to the rsBI one if two of three η_a, η_b, and η_c are equal to each other, and will become the FLRW one if $\eta_a=\eta_b=\eta_c=0$.

For convenience, we define the following variables,

$$u=V_0e^{\lambda\phi_0},\quad \omega_a=p_a^2f_{a0}^{-2}e^{-2\rho_a\phi_0},\quad \omega_b=p_b^2f_{b0}^{-2}e^{-2\rho_b\phi_0},\quad \omega_c=p_c^2f_{c0}^{-2}e^{-2\rho_c\phi_0}.\tag{3.5}$$

As a result, we can rewrite the field equations (2.15), (2.16), (2.17), (2.18), and (2.19) as follows

$$\begin{aligned}\frac{-3\zeta^2+\eta_b^2+\eta_c^2+\eta_b\eta_c+\xi^2/2}{t^2}+ut^{\lambda\xi}+\frac{\omega_a}{2t^{2(\rho_a\xi+2\zeta+\eta_b+\eta_c)}}+\frac{\omega_b}{2t^{2(\rho_b\xi+2\zeta-\eta_b)}}+\frac{\omega_c}{2t^{2(\rho_c\xi+2\zeta-\eta_c)}}&=0,\\ \frac{-\zeta+3\zeta^2}{t^2}-ut^{\lambda\xi}-\frac{\omega_a}{6t^{2(\rho_a\xi+2\zeta+\eta_b+\eta_c)}}-\frac{\omega_b}{6t^{2(\rho_b\xi+2\zeta-\eta_b)}}-\frac{\omega_c}{6t^{2(\rho_c\xi+2\zeta-\eta_c)}}&=0,\\ \frac{-\eta_b+3\zeta\eta_b}{t^2}-\frac{\omega_a}{3t^{2(\rho_a\xi+2\zeta+\eta_b+\eta_c)}}+\frac{2\omega_b}{3t^{2(\rho_b\xi+2\zeta-\eta_b)}}-\frac{\omega_c}{3t^{2(\rho_c\xi+2\zeta-\eta_c)}}&=0,\\ \frac{-\eta_c+3\zeta\eta_c}{t^2}-\frac{\omega_a}{3t^{2(\rho_a\xi+2\zeta+\eta_b+\eta_c)}}-\frac{\omega_b}{3t^{2(\rho_b\xi+2\zeta-\eta_b)}}+\frac{2\omega_c}{3t^{2(\rho_c\xi+2\zeta-\eta_c)}}&=0,\\ \frac{-\xi+3\zeta\xi}{t^2}+\lambda ut^{\lambda\xi}-\frac{\rho_a\omega_a}{t^{2(\rho_a\xi+2\zeta+\eta_b+\eta_c)}}-\frac{\rho_b\omega_b}{t^{2(\rho_b\xi+2\zeta-\eta_b)}}-\frac{\rho_c\omega_c}{t^{2(\rho_c\xi+2\zeta-\eta_c)}}&=0,\end{aligned}\tag{3.6}$$

respectively. Interestingly, these equations are no longer ordinary differential equations. Furthermore, they can be reduced to a set of algebraic equations if suitable constraints are imposed. We will show how to do this in the next subsections.

A. Solution type 0

In a case where all the vector fields are vanishing, i.e., $p_a=p_b=p_c=0$, the system (3.6) becomes

$$\begin{aligned}-\lambda\xi&=2,\\ 3\zeta^2-\eta_b^2-\eta_c^2-\eta_b\eta_c-\frac{\xi^2}{2}-u&=0,\\ \zeta-3\zeta^2+u&=0,\\ \eta_b-3\zeta\eta_b&=0,\\ \eta_c-3\zeta\eta_c&=0,\\ \xi-3\zeta\xi-\lambda u&=0,\end{aligned}\tag{3.7}$$

where the first equation is given to ensure that all terms in each equation of the system (3.6) have the same time dependence. The system (3.7) has the solution

$$\zeta=\frac{2}{\lambda^2},\quad \eta_a=\eta_b=\eta_c=0,\quad \xi=-\frac{2}{\lambda},\quad u=\frac{2(6-\lambda^2)}{\lambda^4}.\tag{3.8}$$

Since (3.8) is the solution where there is no non-vanishing vector field, it is convenient to name it the solution type 0. We notice that $\eta_a = \eta_b = \eta_c = 0$ for this solution. This means that it corresponds to a power-law FLRW metric,

$$ds^2 = -dt^2 + t^{4/\lambda^2}(dx^2 + dy^2 + dz^2), \tag{3.9}$$

where the scale factors $a(t) = b(t) = c(t) = t^{2/\lambda^2}$. This type of solution can be found in Ref. [29].

B. Solutions type I

In a case where A_μ is the only non-vanishing vector field, i.e., $p_a \neq 0$ and $p_b = p_c = 0$, the system (3.6) becomes

$$\begin{aligned}
-\lambda\xi = 2, \quad \rho_a\xi + 2\zeta + \eta_b + \eta_c &= 1,\\
3\zeta^2 - \eta_b^2 - \eta_c^2 - \eta_b\eta_c - \frac{\xi^2}{2} - u - \frac{\omega_a}{2} &= 0,\\
\zeta - 3\zeta^2 + u + \frac{\omega_a}{6} &= 0,\\
\eta_b - 3\zeta\eta_b + \frac{\omega_a}{3} &= 0,\\
\eta_c - 3\zeta\eta_c + \frac{\omega_a}{3} &= 0,\\
\xi - 3\zeta\xi - \lambda u + \rho_a\omega_a &= 0,
\end{aligned} \tag{3.10}$$

where the first two equations are given to ensure that all terms in each equation of the system (3.6) have the same time dependence. The system (3.10) has the following solution,

$$\begin{aligned}
&\zeta = \frac{8+\lambda^2+8\lambda\rho_a+12\rho_a^2}{6\lambda(\lambda+2\rho_a)}, \quad \eta_a = \frac{2(4-\lambda^2-2\lambda\rho_a)}{3\lambda(\lambda+2\rho_a)}, \quad \eta_b = \eta_c = \frac{-4+\lambda^2+2\lambda\rho_a}{3\lambda(\lambda+2\rho_a)}, \quad \xi = -\frac{2}{\lambda},\\
&u = \frac{(2+\lambda\rho_a+2\rho_a^2)(8-\lambda^2+4\lambda\rho_a+12\rho_a^2)}{2\lambda^2(\lambda+2\rho_a)^2}, \quad \omega_a = \frac{(-4+\lambda^2+2\lambda\rho_a)(8-\lambda^2+4\lambda\rho_a+12\rho_a^2)}{2\lambda^2(\lambda+2\rho_a)^2}.
\end{aligned} \tag{3.11}$$

Since Eq. (3.11) is the solution for the scenario, where A_μ is the only non-vanishing vector field ($p_a \neq 0$ and $p_b = p_c = 0$), we name it the solution type I_a. We also notice that $\eta_b = \eta_c$ for this solution, so it corresponds to the x-rsBI metric mentioned above. In fact, this solution is nothing but the anisotropic solution found in Ref. [29]. Similarly, in a case where B_μ is the only non-vanishing vector field, i.e., $p_b \neq 0$ and $p_a = p_c = 0$, it is straightforward to obtain the corresponding solution type I_b as

$$\begin{aligned}
&\zeta = \frac{8+\lambda^2+8\lambda\rho_b+12\rho_b^2}{6\lambda(\lambda+2\rho_b)}, \quad \eta_a = \eta_c = \frac{-4+\lambda^2+2\lambda\rho_b}{3\lambda(\lambda+2\rho_b)}, \quad \eta_b = \frac{2(4-\lambda^2-2\lambda\rho_b)}{3\lambda(\lambda+2\rho_b)}, \quad \xi = -\frac{2}{\lambda},\\
&u = \frac{(2+\lambda\rho_b+2\rho_b^2)(8-\lambda^2+4\lambda\rho_b+12\rho_b^2)}{2\lambda^2(\lambda+2\rho_b)^2}, \quad \omega_b = \frac{(-4+\lambda^2+2\lambda\rho_b)(8-\lambda^2+4\lambda\rho_b+12\rho_b^2)}{2\lambda^2(\lambda+2\rho_b)^2},
\end{aligned} \tag{3.12}$$

which corresponds to the y-rsBI metric, i.e., locally rotationally symmetric about y-axis. Finally, in a case where C_μ is the only non-vanishing vector field, i.e., $p_c \neq 0$ and $p_a = p_b = 0$, it is also straightforward to have the corresponding solution type I_c as

$$\begin{aligned}
&\zeta = \frac{8+\lambda^2+8\lambda\rho_c+12\rho_c^2}{6\lambda(\lambda+2\rho_c)}, \quad \eta_a = \eta_b = \frac{-4+\lambda^2+2\lambda\rho_c}{3\lambda(\lambda+2\rho_c)}, \quad \eta_c = \frac{2(4-\lambda^2-2\lambda\rho_c)}{3\lambda(\lambda+2\rho_c)}, \quad \xi = -\frac{2}{\lambda},\\
&u = \frac{(2+\lambda\rho_c+2\rho_c^2)(8-\lambda^2+4\lambda\rho_c+12\rho_c^2)}{2\lambda^2(\lambda+2\rho_c)^2}, \quad \omega_b = \frac{(-4+\lambda^2+2\lambda\rho_c)(8-\lambda^2+4\lambda\rho_c+12\rho_c^2)}{2\lambda^2(\lambda+2\rho_c)^2},
\end{aligned} \tag{3.13}$$

which corresponds to a z-rsBI metric, i.e., locally rotationally symmetric about z-axis.

C. Solutions type II

In a case where B_μ and C_μ are the only two non-vanishing vector fields, i.e., $p_a = 0$, $p_b \neq 0$, and $p_c \neq 0$, the system (3.6) becomes

$$\begin{aligned} -\lambda\xi = 2, \quad \rho_b\xi + 2\zeta - \eta_b = 1, \quad \rho_c\xi + 2\zeta - \eta_c &= 1, \\ 3\zeta^2 - \eta_b^2 - \eta_c^2 - \eta_b\eta_c - \frac{\xi^2}{2} - u - \frac{\omega_b}{2} - \frac{\omega_c}{2} &= 0, \\ \zeta - 3\zeta^2 + u + \frac{\omega_b}{6} + \frac{\omega_c}{6} &= 0, \\ \eta_b - 3\zeta\eta_b - \frac{2\omega_b}{3} + \frac{\omega_c}{3} &= 0, \\ \eta_c - 3\zeta\eta_c + \frac{\omega_b}{3} - \frac{2\omega_c}{3} &= 0, \\ \xi - 3\zeta\xi - \lambda u + \rho_b\omega_b + \rho_c\omega_c &= 0, \end{aligned} \tag{3.14}$$

where the first three equations are given to ensure that all terms in each equation of the system (3.6) have the same time dependence. The system (3.14) has the following solution,

$$\begin{aligned} \zeta &= \frac{2 + \lambda^2 + 4\lambda(\rho_b + \rho_c) + 4(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)}{3\lambda\left[\lambda + 2(\rho_b + \rho_c)\right]}, \\ \eta_b &= \frac{4 - \lambda^2 - 2\lambda(2\rho_b - \rho_c) - 4(\rho_b^2 - 2\rho_c^2 + \rho_b\rho_c)}{3\lambda\left[\lambda + 2(\rho_b + \rho_c)\right]}, \\ \eta_c &= \frac{4 - \lambda^2 - 2\lambda(2\rho_c - \rho_b) - 4(\rho_c^2 - 2\rho_b^2 + \rho_b\rho_c)}{3\lambda\left[\lambda + 2(\rho_b + \rho_c)\right]}, \quad \xi = -2/\lambda, \\ u &= \frac{2\left[2 + \lambda(\rho_b + \rho_c) + 2(\rho_b^2 + \rho_c^2)\right]\left[1 + \lambda(\rho_b + \rho_c) + 2(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right]}{\lambda^2\left[\lambda + 2(\rho_b + \rho_c)\right]^2}, \\ \omega_b &= \frac{2\left[-4 + \lambda^2 + 2\lambda\rho_b - 4\rho_c(\rho_c - \rho_b)\right]\left[1 + \lambda(\rho_b + \rho_c) + 2(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right]}{\lambda^2\left[\lambda + 2(\rho_b + \rho_c)\right]^2}, \\ \omega_c &= \frac{2\left[-4 + \lambda^2 + 2\lambda\rho_c - 4\rho_b(\rho_b - \rho_c)\right]\left[1 + \lambda(\rho_b + \rho_c) + 2(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right]}{\lambda^2\left[\lambda + 2(\rho_b + \rho_c)\right]^2}. \end{aligned} \tag{3.15}$$

Since Eq. (3.15) is the solution where B_μ and C_μ are the only two non-vanishing vector fields, we name it the solution of type II$_{bc}$. In general, η_a, η_b, and η_c are different, so the solution type II$_{bc}$ corresponds to a general Bianchi type I metric with three different scale factors. Note that if $\rho_b = \rho_c$, then $\eta_b = \eta_c$ and the metric becomes the x-rsBI metric.

Similarly, in the case where A_μ and C_μ are the only two non-vanishing vector fields, i.e., $p_a \neq 0$, $p_b = 0$, and $p_c \neq 0$, we are able obtain the corresponding solution type II$_{ac}$ given by

$$\begin{aligned} \zeta &= \frac{2 + \lambda^2 + 4\lambda(\rho_a + \rho_c) + 4(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)}{3\lambda\left[\lambda + 2(\rho_a + \rho_c)\right]}, \\ \eta_b &= \frac{2\left[-4 + \lambda^2 + \lambda(\rho_a + \rho_c) - 2(\rho_a^2 + \rho_c^2 - 2\rho_a\rho_c)\right]}{3\lambda\left[\lambda + 2(\rho_a + \rho_c)\right]}, \\ \eta_c &= \frac{4 - \lambda^2 - 2\lambda(2\rho_c - \rho_a) - 4(\rho_c^2 - 2\rho_a^2 + \rho_a\rho_c)}{3\lambda\left[\lambda + 2(\rho_a + \rho_c)\right]}, \quad \xi = -2/\lambda, \\ u &= \frac{2\left[2 + \lambda(\rho_a + \rho_c) + 2(\rho_a^2 + \rho_c^2)\right]\left[1 + \lambda(\rho_a + \rho_c) + 2(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right]}{\lambda^2\left[\lambda + 2(\rho_a + \rho_c)\right]^2}, \\ \omega_a &= \frac{2\left[-4 + \lambda^2 + 2\lambda\rho_a - 4\rho_c(\rho_c - \rho_a)\right]\left[1 + \lambda(\rho_a + \rho_c) + 2(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right]}{\lambda^2\left[\lambda + 2(\rho_a + \rho_c)\right]^2}, \\ \omega_c &= \frac{2\left[-4 + \lambda^2 + 2\lambda\rho_c - 4\rho_a(\rho_a - \rho_c)\right]\left[1 + \lambda(\rho_a + \rho_c) + 2(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right]}{\lambda^2\left[\lambda + 2(\rho_a + \rho_c)\right]^2}, \end{aligned} \tag{3.16}$$

which corresponds to another general Bianchi type I metric, which will become the y-rsBI metric if $\rho_a = \rho_c$.

Finally, in the case where A_μ and B_μ are the only two non-vanishing vector fields, i.e., $p_a \neq 0$, $p_b \neq 0$, and $p_c = 0$, we obtain the corresponding solution type II_{ab} given by

$$\begin{aligned}
\zeta &= \frac{2+\lambda^2+4\lambda(\rho_a+\rho_b)+4(\rho_a^2+\rho_b^2+\rho_a\rho_b)}{3\lambda\left[\lambda+2(\rho_a+\rho_b)\right]},\\
\eta_b &= \frac{4-\lambda^2-2\lambda(2\rho_b-\rho_a)-4(\rho_b^2-2\rho_a^2+\rho_a\rho_b)}{3\lambda\left[\lambda+2(\rho_a+\rho_b)\right]},\\
\eta_c &= \frac{2\left[-4+\lambda^2+\lambda(\rho_a+\rho_b)-2(\rho_a^2+\rho_b^2-2\rho_a\rho_b)\right]}{3\lambda\left[\lambda+2(\rho_a+\rho_b)\right]}, \quad \xi = -2/\lambda,\\
u &= \frac{2\left[2+\lambda(\rho_a+\rho_b)+2(\rho_a^2+\rho_b^2)\right]\left[1+\lambda(\rho_a+\rho_b)+2(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right]}{\lambda^2\left[\lambda+2(\rho_a+\rho_b)\right]^2},\\
\omega_a &= \frac{2\left[-4+\lambda^2+2\lambda\rho_a-4\rho_b(\rho_b-\rho_a)\right]\left[1+\lambda(\rho_a+\rho_b)+2(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right]}{\lambda^2\left[\lambda+2(\rho_a+\rho_b)\right]^2},\\
\omega_b &= \frac{2\left[-4+\lambda^2+2\lambda\rho_b-4\rho_a(\rho_a-\rho_b)\right]\left[1+\lambda(\rho_a+\rho_b)+2(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right]}{\lambda^2\left[\lambda+2(\rho_a+\rho_b)\right]^2},
\end{aligned} \tag{3.17}$$

which corresponds to a different general Bianchi type I metric, which will it become the z-rsBI metric if $\rho_a = \rho_b$.

D. Solution type III

In a case where A_μ, B_μ, and C_μ are all non-vanishing vector fields, i.e., $p_a \neq 0$, $p_b \neq 0$, and $p_c \neq 0$, the system (3.6) becomes

$$\begin{aligned}
-\lambda\xi = 2, \quad \rho_a\xi+2\zeta+\eta_b+\eta_c = 1, \quad \rho_b\xi+2\zeta-\eta_b = 1, \quad \rho_c\xi+2\zeta-\eta_c &= 1,\\
-3\zeta^2+\eta_b^2+\eta_c^2+\eta_b\eta_c+\frac{\xi^2}{2}+u+\frac{\omega_a}{2}+\frac{\omega_b}{2}+\frac{\omega_c}{2} &= 0,\\
-\zeta+3\zeta^2-u-\frac{\omega_a}{6}-\frac{\omega_b}{6}-\frac{\omega_c}{6} &= 0,\\
-\eta_b+3\zeta\eta_b-\frac{\omega_a}{3}+\frac{2\omega_b}{3}-\frac{\omega_c}{3} &= 0,\\
-\eta_c+3\zeta\eta_c-\frac{\omega_a}{3}-\frac{\omega_b}{3}+\frac{2\omega_c}{3} &= 0,\\
-\xi+3\zeta\xi+\lambda u-\rho_a\omega_a-\rho_b\omega_b-\rho_c\omega_c &= 0,
\end{aligned} \tag{3.18}$$

where the first four equations are given to ensure that all terms in each equation of the system (3.6) have the same time dependence. The system (3.18) has the following solution,

$$\begin{aligned}
\zeta &= \frac{3\lambda+2(\rho_a+\rho_b+\rho_c)}{6\lambda}, \quad \eta_b = \frac{2(\rho_a-2\rho_b+\rho_c)}{3\lambda}, \quad \eta_c = \frac{2(\rho_a+\rho_b-2\rho_c)}{3\lambda},\\
\xi &= -\frac{2}{\lambda}, \quad u = \frac{2+\lambda(\rho_a+\rho_b+\rho_c)+2(\rho_a^2+\rho_b^2+\rho_c^2)}{2\lambda^2},\\
\omega_a &= \frac{-4+\lambda^2+2\lambda\rho_a-4\left[\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)\right]}{2\lambda^2},\\
\omega_b &= \frac{-4+\lambda^2+2\lambda\rho_b-4\left[\rho_a^2+\rho_c^2-\rho_b(\rho_a+\rho_c)\right]}{2\lambda^2},\\
\omega_c &= \frac{-4+\lambda^2+2\lambda\rho_c-4\left[\rho_a^2+\rho_b^2-\rho_c(\rho_a+\rho_b)\right]}{2\lambda^2}.
\end{aligned} \tag{3.19}$$

Since (3.19) is the solution where all three vector fields A_μ, B_μ, and C_μ are non-vanishing, we name it the solution type III. In general, η_a, η_b, and η_c are different, so the solution type III corresponds to a general Bianchi type I metric with three different scale factors. If two of three coupling constants are equal to each other, say ρ_b and ρ_c, then $\eta_b = \eta_c$ and the metric will become x-rsBI one and so on. If all three coupling constant are equal to each other, i.e., $\rho_a = \rho_b = \rho_c$, then $\eta_a = \eta_b = \eta_c$ and the metric becomes the FLRW one.

For convenience, we summarize all the obtained solutions shown above in Table (I).

Solution type	**Specific formula**	**Non-vanishing vector field(s)**	**Corresponding metric**
0	(3.8)	None	FLRW
I_a	(3.11)	A_μ	x-rsBI
I_b	(3.12)	B_μ	y-rsBI
I_c	(3.13)	C_μ	z-rsBI
II_{bc}	(3.15)	B_μ and C_μ	general BI
II_{ac}	(3.16)	A_μ and C_μ	general BI
II_{ab}	(3.17)	A_μ and B_μ	general BI
III	(3.19)	A_μ, B_μ, and C_μ	general BI

TABLE I: List of the power-law solutions.

IV. STABILITY ANALYSIS

A. Dynamical system

For the stability analysis of the derived solutions, we would like to introduce the following dynamical variables [29],

$$\begin{aligned} X_b &\equiv \frac{\dot{\sigma}_b}{\dot{\alpha}}, \quad X_c \equiv \frac{\dot{\sigma}_c}{\dot{\alpha}}, \quad X_a \equiv -X_b - X_c = \frac{-\dot{\sigma}_b - \dot{\sigma}_c}{\dot{\alpha}}, \quad Y \equiv \frac{\dot{\phi}}{\dot{\alpha}}, \\ Z_a &\equiv \frac{p_a f_a^{-1}}{\dot{\alpha}} e^{-2\alpha - \sigma_b - \sigma_c}, \quad Z_b \equiv \frac{p_b f_b^{-1}}{\dot{\alpha}} e^{-2\alpha + \sigma_b}, \quad Z_c \equiv \frac{p_c f_c^{-1}}{\dot{\alpha}} e^{-2\alpha + \sigma_c}, \end{aligned} \tag{4.1}$$

where X_a is an auxiliary variable, Note that the variables X_a, X_b, and X_c, which measure the anisotropies along axes x, y and z, have been defined in (2.4). Thanks to the dynamical variables shown in (4.1) and the field equations (2.15), (2.16), (2.17), (2.18), and (2.19), we obtain the system of autonomous equations as follows (see Appendix A)

$$\frac{dX_b}{d\alpha} = X_b \left[-3 + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} \right] + \frac{(Z_a)^2}{3} - \frac{2(Z_b)^2}{3} + \frac{(Z_c)^2}{3}, \tag{4.2}$$

$$\frac{dX_c}{d\alpha} = X_c \left[-3 + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} \right] + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} - \frac{2(Z_c)^2}{3}, \tag{4.3}$$

$$\begin{aligned} \frac{dY}{d\alpha} &= (\lambda + Y) \left[-3 + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} \right] \\ &\quad + \left(\frac{\lambda}{6} + \rho_a \right) (Z_a)^2 + \left(\frac{\lambda}{6} + \rho_b \right) (Z_b)^2 + \left(\frac{\lambda}{6} + \rho_c \right) (Z_c)^2, \end{aligned} \tag{4.4}$$

$$\frac{dZ_a}{d\alpha} = Z_a \left[-2 - X_b - X_c - \rho_a Y + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} \right], \tag{4.5}$$

$$\frac{dZ_b}{d\alpha} = Z_b \left[-2 + X_b - \rho_b Y + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} \right], \tag{4.6}$$

$$\frac{dZ_c}{d\alpha} = Z_c \left[-2 + X_c - \rho_c Y + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} \right], \tag{4.7}$$

along with the Hamiltonian constraint coming from Eq. (2.15),

$$-\frac{V}{\dot{\alpha}^2} = -3 + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{2} + \frac{(Z_b)^2}{2} + \frac{(Z_c)^2}{2} < 0. \tag{4.8}$$

where the inequality was added to ensure the positivity of the potential. Next, we seek fixed points of the dynamical system by solving the corresponding set of equations,

$$\frac{dX_b}{d\alpha} = \frac{dX_c}{d\alpha} = \frac{dY}{d\alpha} = \frac{dZ_a}{d\alpha} = \frac{dZ_b}{d\alpha} = \frac{dZ_c}{d\alpha} = 0. \tag{4.9}$$

In the following, we only show the solutions of (4.9) without derivations. The details of the solving will be presented in Appendix A.

B. Fixed point type 0

For $Z_a = Z_b = Z_c = 0$, solving Eq. (4.9) gives us an isotropic fixed point given by

$$X_a = X_b = X_c = 0, \quad Y = -\lambda, \quad Z_a = Z_b = Z_c = 0. \tag{4.10}$$

As one can easily check, the fixed point (4.10) corresponds to the solutions type 0 shown in Eq. (3.8). Therefore, it is convenient to name it the fixed point type 0.

C. Fixed points type I

For $Z_a \neq 0$ and $Z_b = Z_c = 0$, solving (4.9) gives us an anisotropic fixed point given by

$$\begin{aligned} X_a &= \frac{4(4-\lambda^2-2\lambda\rho_a)}{8+\lambda^2+8\lambda\rho_a+12\rho_a^2}, \quad X_b = X_c = \frac{2(-4+\lambda^2+2\lambda\rho_a)}{8+\lambda^2+8\lambda\rho_a+12\rho_a^2}, \quad Y = \frac{-12(\lambda+2\rho_a)}{8+\lambda^2+8\lambda\rho_a+12\rho_a^2}, \\ (Z_a)^2 &= \frac{18(-4+\lambda^2+2\lambda\rho_a)(8-\lambda^2+4\lambda\rho_a+12\rho_a^2)}{(8+\lambda^2+8\lambda\rho_a+12\rho_a^2)^2}, \quad Z_b = Z_c = 0, \end{aligned} \tag{4.11}$$

whose existence condition is due to the following inequality,

$$(-4+\lambda^2+2\lambda\rho_a)(8-\lambda^2+4\lambda\rho_a+12\rho_a^2) > 0, \tag{4.12}$$

to ensure that $(Z_a)^2 > 0$. The fixed point (4.11) corresponds to the solution type I_a (3.11) as one can easily check. Therefore, it is convenient to name the fixed point (4.11) the fixed point type I_a.

Similarly, for $Z_b \neq 0$ and $Z_a = Z_c = 0$, we have the corresponding fixed point type I_b defined as

$$\begin{aligned} X_a = X_c &= \frac{2(-4+\lambda^2+2\lambda\rho_b)}{8+\lambda^2+8\lambda\rho_b+12\rho_b^2}, \quad X_b = \frac{4(4-\lambda^2-2\lambda\rho_b)}{8+\lambda^2+8\lambda\rho_b+12\rho_b^2}, \quad Y = \frac{-12(\lambda+2\rho_b)}{8+\lambda^2+8\lambda\rho_b+12\rho_b^2}, \\ (Z_a)^2 = (Z_c)^2 &= 0, \quad (Z_b)^2 = \frac{18(-4+\lambda^2+2\lambda\rho_b)(8-\lambda^2+4\lambda\rho_b+12\rho_b^2)}{(8+\lambda^2+8\lambda\rho_b+12\rho_b^2)^2}, \end{aligned} \tag{4.13}$$

which corresponds to the solution type I_b (3.12). The existence region of the fixed point type I_b is determined by the following inequality,

$$(-4+\lambda^2+2\lambda\rho_b)(8-\lambda^2+4\lambda\rho_b+12\rho_b^2) > 0. \tag{4.14}$$

Finally, we have the fixed point type I_c for $Z_c \neq 0$ and $Z_a = Z_b = 0$,

$$\begin{aligned} X_a = X_b &= \frac{2(-4+\lambda^2+2\lambda\rho_c)}{8+\lambda^2+8\lambda\rho_c+12\rho_c^2}, \quad X_c = \frac{4(4-\lambda^2-2\lambda\rho_c)}{8+\lambda^2+8\lambda\rho_c+12\rho_c^2}, \quad Y = \frac{-12(\lambda+2\rho_c)}{8+\lambda^2+8\lambda\rho_c+12\rho_c^2}, \\ (Z_a)^2 = (Z_b)^2 &= 0, \quad (Z_c)^2 = \frac{18(-4+\lambda^2+2\lambda\rho_c)(8-\lambda^2+4\lambda\rho_c+12\rho_c^2)}{(8+\lambda^2+8\lambda\rho_c+12\rho_c^2)^2}, \end{aligned} \tag{4.15}$$

which correspond to the solution type I_c (3.13) and have the existence region given by

$$(-4+\lambda^2+2\lambda\rho_c)(8-\lambda^2+4\lambda\rho_c+12\rho_c^2) > 0. \tag{4.16}$$

D. Fixed points type II

For $Z_a = 0$, $Z_b \neq 0$, and $Z_c \neq 0$, solving Eq. (4.9) gives us an anisotropic fixed point given by

$$\begin{aligned}
X_a &= \frac{-8 + 2\lambda^2 + 2\lambda(\rho_b + \rho_c) - 4(\rho_b^2 + \rho_c^2 - 2\rho_b\rho_c)}{2 + \lambda^2 + 4\left[\lambda(\rho_b + \rho_c) + \rho_b^2 + \rho_c^2 + \rho_b\rho_c\right]}, \\
X_b &= \frac{4 - \lambda^2 - 2\lambda(2\rho_b - \rho_c) - 4(\rho_b^2 - 2\rho_c^2 + \rho_b\rho_c)}{2 + \lambda^2 + 4\left[\lambda(\rho_b + \rho_c) + \rho_b^2 + \rho_c^2 + \rho_b\rho_c\right]}, \\
X_c &= \frac{4 - \lambda^2 - 2\lambda(2\rho_c - \rho_b) - 4(\rho_c^2 - 2\rho_b^2 + \rho_b\rho_c)}{2 + \lambda^2 + 4\left[\lambda(\rho_b + \rho_c) + \rho_b^2 + \rho_c^2 + \rho_b\rho_c\right]}, \\
Y &= \frac{-6\left[\lambda + 2(\rho_b + \rho_c)\right]}{2 + \lambda^2 + 4\left[\lambda(\rho_b + \rho_c) + \rho_b^2 + \rho_c^2 + \rho_b\rho_c\right]}, \quad Z_a = 0, \\
(Z_b)^2 &= \frac{18\left[-4 + \lambda^2 + 2\lambda\rho_b - 4\rho_c(\rho_c - \rho_b)\right]\left[1 + \lambda(\rho_b + \rho_c) + 2(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right]}{\left[2 + \lambda^2 + 4\lambda(\rho_b + \rho_c) + 4(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right]^2}, \\
(Z_c)^2 &= \frac{18\left[-4 + \lambda^2 + 2\lambda\rho_c - 4\rho_b(\rho_b - \rho_c)\right]\left[1 + \lambda(\rho_b + \rho_c) + 2(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right]}{\left[2 + \lambda^2 + 4\lambda(\rho_b + \rho_c) + 4(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right]^2},
\end{aligned} \tag{4.17}$$

of which the existence region is determined by two inequalities,

$$\begin{aligned}
&\left[-4 + \lambda^2 + 2\lambda\rho_b - 4\rho_c(\rho_c - \rho_b)\right]\left[1 + \lambda(\rho_b + \rho_c) + 2(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right] > 0, \\
&\left[-4 + \lambda^2 + 2\lambda\rho_c - 4\rho_b(\rho_b - \rho_c)\right]\left[1 + \lambda(\rho_b + \rho_c) + 2(\rho_b^2 + \rho_c^2 + \rho_b\rho_c)\right] > 0.
\end{aligned} \tag{4.18}$$

As a result, these constraints are due to the positivity of $(Z_b)^2$ and $(Z_c)^2 > 0$. Since the fixed point (4.17) is equivalent to the solution type II$_{bc}$ (3.15), we will it the fixed point type II$_{bc}$ for convenience.

Similarly, for $Z_b = 0$ and $Z_a, Z_c \neq 0$, we have the fixed point type II$_{ac}$

$$\begin{aligned}
X_a &= \frac{4 - \lambda^2 - 2\lambda(2\rho_a - \rho_c) - 4(\rho_a^2 - 2\rho_c^2 + \rho_a\rho_c)}{2 + \lambda^2 + 4\left[\lambda(\rho_a + \rho_c) + \rho_a^2 + \rho_c^2 + \rho_a\rho_c\right]}, \\
X_b &= \frac{-8 + 2\lambda^2 + 2\lambda(\rho_a + \rho_c) - 4(\rho_a^2 + \rho_c^2 - 2\rho_a\rho_c)}{2 + \lambda^2 + 4\left[\lambda(\rho_a + \rho_c) + \rho_a^2 + \rho_c^2 + \rho_a\rho_c\right]}, \\
X_c &= \frac{4 - \lambda^2 - 2\lambda(2\rho_c - \rho_a) - 4(\rho_c^2 - 2\rho_a^2 + \rho_a\rho_c)}{2 + \lambda^2 + 4\left[\lambda(\rho_a + \rho_c) + \rho_a^2 + \rho_c^2 + \rho_a\rho_c\right]}, \\
Y &= \frac{-6\left[\lambda + 2(\rho_a + \rho_c)\right]}{2 + \lambda^2 + 4\left[\lambda(\rho_a + \rho_c) + \rho_a^2 + \rho_c^2 + \rho_a\rho_c\right]}, \quad Z_b = 0, \\
(Z_a)^2 &= \frac{18\left[-4 + \lambda^2 + 2\lambda\rho_a - 4\rho_c(\rho_c - \rho_a)\right]\left[1 + \lambda(\rho_a + \rho_c) + 2(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right]}{\left[2 + \lambda^2 + 4\lambda(\rho_a + \rho_c) + 4(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right]^2}, \\
(Z_c)^2 &= \frac{18\left[-4 + \lambda^2 + 2\lambda\rho_c - 4\rho_a(\rho_a - \rho_c)\right]\left[1 + \lambda(\rho_a + \rho_c) + 2(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right]}{\left[2 + \lambda^2 + 4\lambda(\rho_a + \rho_c) + 4(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right]^2},
\end{aligned} \tag{4.19}$$

which corresponds to the solution type II$_{ac}$ (3.16). The existence region of the fixed point type II$_{ac}$ is

$$\begin{aligned}
&\left[-4 + \lambda^2 + 2\lambda\rho_a - 4\rho_c(\rho_c - \rho_a)\right]\left[1 + \lambda(\rho_a + \rho_c) + 2(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right] > 0, \\
&\left[-4 + \lambda^2 + 2\lambda\rho_c - 4\rho_a(\rho_a - \rho_c)\right]\left[1 + \lambda(\rho_a + \rho_c) + 2(\rho_a^2 + \rho_c^2 + \rho_a\rho_c)\right] > 0.
\end{aligned} \tag{4.20}$$

Similarly, for $Z_c = 0$ and $Z_a, Z_b \neq 0$, we have the fixed point type II$_{ab}$

$$\begin{aligned}
X_a &= \frac{4-\lambda^2-2\lambda(2\rho_a-\rho_b)-4(\rho_a^2-2\rho_b^2+\rho_a\rho_b)}{2+\lambda^2+4\left[\lambda(\rho_a+\rho_b)+\rho_a^2+\rho_b^2+\rho_a\rho_b\right]},\\
X_b &= \frac{4-\lambda^2-2\lambda(2\rho_b-\rho_a)-4(\rho_b^2-2\rho_a^2+\rho_a\rho_b)}{2+\lambda^2+4\left[\lambda(\rho_a+\rho_b)+\rho_a^2+\rho_b^2+\rho_a\rho_b\right]},\\
X_c &= \frac{-8+2\lambda^2+2\lambda(\rho_a+\rho_b)-4(\rho_a^2+\rho_b^2-2\rho_a\rho_b)}{2+\lambda^2+4\left[\lambda(\rho_a+\rho_b)+\rho_a^2+\rho_b^2+\rho_a\rho_b\right]},\\
Y &= \frac{-6\left[\lambda+2(\rho_a+\rho_b)\right]}{2+\lambda^2+4\left[\lambda(\rho_a+\rho_b)+\rho_a^2+\rho_b^2+\rho_a\rho_b\right]}, \quad Z_c = 0,\\
(Z_a)^2 &= \frac{18\left[-4+\lambda^2+2\lambda\rho_a-4\rho_b(\rho_b-\rho_a)\right]\left[1+\lambda(\rho_a+\rho_b)+2(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right]}{\left[2+\lambda^2+4\lambda(\rho_a+\rho_b)+4(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right]^2},\\
(Z_b)^2 &= \frac{18\left[-4+\lambda^2+2\lambda\rho_b-4\rho_a(\rho_a-\rho_b)\right]\left[1+\lambda(\rho_a+\rho_b)+2(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right]}{\left[2+\lambda^2+4\lambda(\rho_a+\rho_b)+4(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right]^2},
\end{aligned} \tag{4.21}$$

which corresponds to the solution type II$_{ab}$ (3.17). The existence region of the fixed point type II$_{ab}$ is

$$\begin{aligned}
&\left[-4+\lambda^2+2\lambda\rho_a-4\rho_b(\rho_b-\rho_a)\right]\left[1+\lambda(\rho_a+\rho_b)+2(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right] > 0,\\
&\left[-4+\lambda^2+2\lambda\rho_b-4\rho_a(\rho_a-\rho_b)\right]\left[1+\lambda(\rho_a+\rho_b)+2(\rho_a^2+\rho_b^2+\rho_a\rho_b)\right] > 0.
\end{aligned} \tag{4.22}$$

E. Fixed point type III

For Z_a, Z_b, and $Z_c \neq 0$, solving (4.9) gives us the anisotropic fixed point,

$$\begin{aligned}
X_a &= \frac{4(-2\rho_a+\rho_b+\rho_c)}{3\lambda+2(\rho_a+\rho_b+\rho_c)}, \quad X_b = \frac{4(\rho_a-2\rho_b+\rho_c)}{3\lambda+2(\rho_a+\rho_b+\rho_c)}, \quad X_c = \frac{4(\rho_a+\rho_b-2\rho_c)}{3\lambda+2(\rho_a+\rho_b+\rho_c)},\\
Y &= \frac{-12}{3\lambda+2(\rho_a+\rho_b+\rho_c)}, \quad (Z_a)^2 = \frac{18\left\{-4+\lambda^2+2\lambda\rho_a-4\left[\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)\right]\right\}}{\left[3\lambda+2(\rho_a+\rho_b+\rho_c)\right]^2},\\
(Z_b)^2 &= \frac{18\left\{-4+\lambda^2+2\lambda\rho_b-4\left[\rho_a^2+\rho_c^2-\rho_b(\rho_a+\rho_c)\right]\right\}}{\left[3\lambda+2(\rho_a+\rho_b+\rho_c)\right]^2},\\
(Z_c)^2 &= \frac{18\left\{-4+\lambda^2+2\lambda\rho_c-4\left[\rho_a^2+\rho_b^2-\rho_c(\rho_a+\rho_b)\right]\right\}}{\left[3\lambda+2(\rho_a+\rho_b+\rho_c)\right]^2},
\end{aligned} \tag{4.23}$$

of which the existence region is

$$\begin{aligned}
-4+\lambda^2+2\lambda\rho_a-4\left[\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)\right] &> 0,\\
-4+\lambda^2+2\lambda\rho_b-4\left[\rho_a^2+\rho_c^2-\rho_b(\rho_a+\rho_c)\right] &> 0,\\
-4+\lambda^2+2\lambda\rho_c-4\left[\rho_a^2+\rho_b^2-\rho_c(\rho_a+\rho_b)\right] &> 0,
\end{aligned} \tag{4.24}$$

to ensure that $(Z_a)^2, (Z_b)^2, (Z_c)^2 > 0$

F. Stability regions

We have found the fixed points of the system of autonomous equations (4.2)-(4.7) along with their existence regions. Now we would like to investigate whether the fixed points are stable. Particularly, we will find each fixed point's stability region (condition that makes the fixed point stable), which is described by a set of constraints of parameters

λ, ρ_a, ρ_b and ρ_c. We start our stability analysis by perturbing the system (4.2)-(4.7) to the first order as follows

$$\begin{pmatrix} d\delta X_b/d\alpha \\ d\delta X_c/d\alpha \\ d\delta Y/d\alpha \\ d\delta Z_a/d\alpha \\ d\delta Z_b/d\alpha \\ d\delta Z_c/d\alpha \end{pmatrix} = M \begin{pmatrix} \delta X_b \\ \delta X_c \\ \delta Y \\ \delta Z_a \\ \delta Z_b \\ \delta Z_c \end{pmatrix} \tag{4.25}$$

where $M \equiv [M_{ij}]_{6\times 6}$ is a matrix with entries given by

$$M_{11} = -3 + 3(X_b)^2 + (X_c)^2 + 2X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3}, \quad M_{12} = (X_b)^2 + 2X_bX_c, \quad M_{13} = X_bY,$$
$$M_{14} = \frac{2Z_a}{3}(X_b+1), \quad M_{15} = \frac{2Z_b}{3}(X_b-2), \quad M_{16} = \frac{2Z_c}{3}(X_b+1), \quad M_{21} = (X_c)^2 + 2X_bX_c,$$
$$M_{22} = -3 + (X_b)^2 + 3(X_c)^2 + 2X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3}, \quad M_{23} = X_cY, \quad M_{24} = \frac{2Z_a}{3}(X_c+1),$$
$$M_{25} = \frac{2Z_b}{3}(X_c+1), \quad M_{26} = \frac{2Z_c}{3}(X_c-2), \quad M_{31} = (\lambda+Y)(2X_b+X_c), \quad M_{32} = (\lambda+Y)(X_b+2X_c),$$
$$M_{33} = -3 + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{3Y^2}{2} + \lambda Y + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3}, \quad M_{34} = \left(\lambda + 2\rho_a + \frac{2Y}{3}\right) Z_a,$$
$$M_{35} = \left(\lambda + 2\rho_b + \frac{2Y}{3}\right) Z_b, \quad M_{36} = \left(\lambda + 2\rho_c + \frac{2Y}{3}\right) Z_c, \quad M_{41} = (-1 + 2X_b + X_c)Z_a, \quad M_{42} = (-1 + X_b + 2X_c)Z_a,$$
$$M_{43} = (Y - \rho_a)Z_a,$$
$$M_{44} = -2 - X_b - X_c - \rho_aY + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + (Z_a)^2 + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3},$$
$$M_{45} = \frac{2Z_aZ_b}{3}, \quad M_{46} = \frac{2Z_aZ_c}{3},$$
$$M_{51} = (1 + 2X_b + X_c)Z_b, \quad M_{52} = (X_b + 2X_c)Z_b, \quad M_{53} = (Y - \rho_b)Z_b, \quad M_{54} = \frac{2Z_aZ_b}{3},$$
$$M_{55} = -2 + X_b - \rho_bY + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + (Z_b)^2 + \frac{(Z_c)^2}{3}, \quad M_{56} = \frac{2Z_bZ_c}{3},$$
$$M_{61} = (2X_b + X_c)Z_c, \quad M_{62} = (1 + X_b + 2X_c)Z_c, \quad M_{63} = (Y - \rho_c)Z_c, \quad M_{64} = \frac{2Z_aZ_c}{3},$$
$$M_{65} = \frac{2Z_bZ_c}{3}, \quad M_{66} = -2 + X_c - \rho_cY + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + (Z_c)^2.$$

To find the stability regions of the fixed points, we insert each of them into the eigenvalue equation,

$$\det(M - sI_6) = 0, \tag{4.26}$$

where I_6 is the 6×6 identity matrix. The stability region then is the conditions on parameters that ensure that all the roots of Eq. (4.26) have negative real parts. For simplicity, we will assume all the parameters λ, ρ_a, ρ_b, and ρ_c are positive. The detailed derivation of the fixed point's stability regions using the Routh–Hurwitz criterion will be presented in Appendix B for convenience. In the following, we only describe the main results.

1. *Fixed point type 0*

The stability region of the fixed point type 0 is determined by the following inequalities,

$$\lambda^2 + 2\lambda\rho_a - 4 < 0, \quad \lambda^2 + 2\lambda\rho_b - 4 < 0, \quad \lambda^2 + 2\lambda\rho_c - 4 < 0. \tag{4.27}$$

For a fixed λ, the stability region is a cube bounded by three planes: $\rho_a = (4-\lambda^2)/(2\lambda)$, $\rho_b = (4-\lambda^2)/(2\lambda)$, and $\rho_c = (4-\lambda^2)/(2\lambda)$. Qualitatively, in order to make the fixed point type 0 stable, ρ_a, ρ_b, and ρ_c have to be sufficiently small. To be more transparent, the stability region of the fixed point type 0 for a specific value $\lambda = 1.5$ will be plotted in Fig.1.

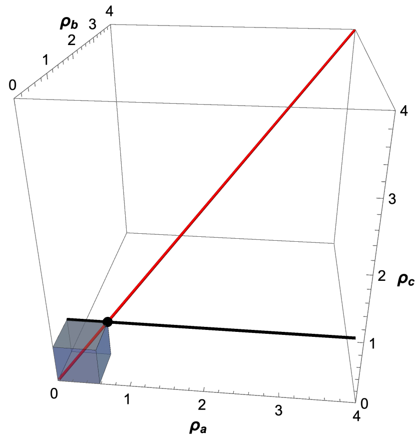


FIG. 1: Stability region of the fixed point type 0 for $\lambda = 1.5$ displayed as the dark blue cube. The red line corresponds to $\rho_a = \rho_b = \rho_c$. The black point is a special point, where $\rho_a = \rho_b = \rho_c = (4 - \lambda^2)/(2\lambda)$. The thick black line corresponds to $\rho_b = \rho_c = (4 - \lambda^2)/(2\lambda)$.

2. Fixed point type III

Contrast to the fixed point type 0, which clearly isotropic, the fixed point type III is generically anisotropic. The existence region of the fixed point type III is determined by three inequalities shown in Eq. (4.24) and depicted as a dark blue region for $\lambda = 1.5$ in Fig. 2. This region is bounded by three surfaces: $-4+\lambda^2+2\lambda\rho_a-4\left[\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)\right] = 0$, $-4+\lambda^2+2\lambda\rho_b-4\left[\rho_a^2+\rho_c^2-\rho_b(\rho_a+\rho_c)\right] = 0$, and $-4+\lambda^2+2\lambda\rho_c-4\left[\rho_a^2+\rho_b^2-\rho_c(\rho_a+\rho_b)\right] = 0$. Apparently, for the fixed point type III as well as the other fixed points, the stability region has to be included in the existence regions (we need the fixed point to exist before we investigate its stability). Interestingly, we have found that the stability region of the fixed point type III is identical to its existence region determined by Eq. (4.24). This means that the stability region is also depicted by the dark blue region in Fig. 2 for $\lambda = 1.5$.

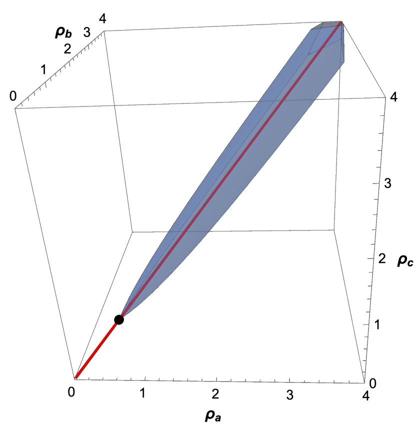


FIG. 2: Existence and stability region of the fixed point type III for $\lambda = 1.5$ displayed as the dark blue region. The red line corresponds to $\rho_a = \rho_b = \rho_c$, while the black point is shown for a special case, in which $\rho_a = \rho_b = \rho_c = (4 - \lambda^2)/(2\lambda)$.

It is constructive to have a qualitative description for that stability region. We see that it is a narrow region surrounding the line $\rho_a = \rho_b = \rho_c$, which means that the coupling constants ρ_a, ρ_b and ρ_c have to be sufficiently close to each other in order to make the fixed point type III stable. Additionally, the fact that the region terminates at the black point $\rho_a = \rho_b = \rho_c = (4 - \lambda^2)/(2\lambda)$ (note that it is identical to the black point in Fig. 1) implies that all the coupling constants ρ_a, ρ_b and ρ_c also have to be sufficiently large compared to $(4 - \lambda^2)/(2\lambda)$. In contrast, if the coupling constants are all smaller than $(4-\lambda^2)/(2\lambda)$, then the fixed point type III cannot be stable, regardless of how close they are.

3. Fixed points type II

Let us continue with the fixed point type II_{bc}. Its existence region, determined by Eq. (4.18), is depicted in Fig. 3a as the orange one, which is bounded by two surfaces: $-4+\lambda(\lambda+2\rho_b)-4\rho_c(\rho_c-\rho_b)=0$ and $-4+\lambda(\lambda+2\rho_c)-4\rho_b(\rho_b-\rho_c)=0$. However, the stability region of this fixed point must follow the corresponding inequalities,

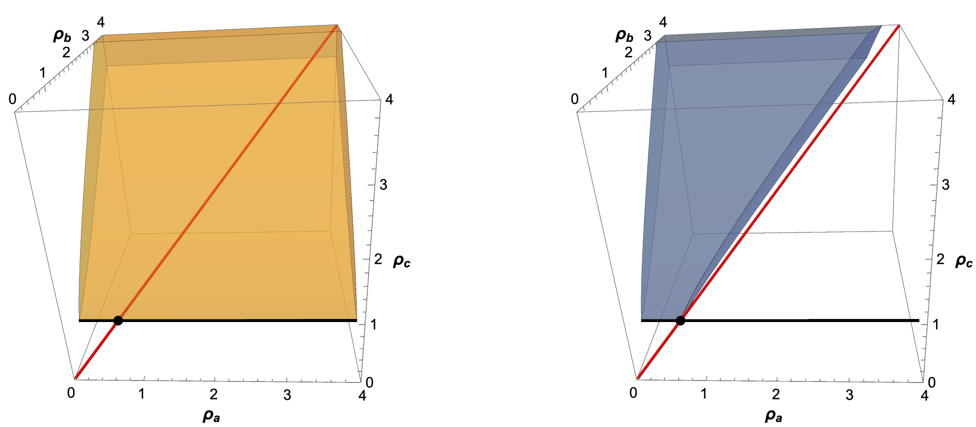
$$\begin{aligned} -4+\lambda^2+2\lambda\rho_b-4\rho_c(\rho_c-\rho_b)&>0, \\ -4+\lambda^2+2\lambda\rho_c-4\rho_b(\rho_b-\rho_c)&>0, \\ -4+\lambda^2+2\lambda\rho_a-4\left[\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)\right]&<0. \end{aligned} \tag{4.28}$$

It is apparent that this region differs from the existence region by the additional third inequality. To be more specific, this stability region is depicted in Fig. 3b as the blue region, which is of course a part of the orange region shown in Fig. 3a. Here, we notice that the surface $\lambda^2+2\lambda\rho_a-4\left[1+\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)\right]=0$ is also one of the surfaces that bound the stability region of the fixed point type III in Fig.2. Interestingly, we see that the stability region of the fixed point type II_{bc} is a thin layer surrounding the plane $\rho_b=\rho_c$, but the surface $\lambda^2+2\lambda\rho_a-4\left[1+\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)\right]=0$ prevents ρ_a from getting too close to ρ_b and ρ_c. Qualitatively speaking, therefore, ρ_b and ρ_c have to be sufficiently close to each other and significantly larger than ρ_a in order to make the fixed point II_{bc} stable. Moreover, the stability region terminates at the thick black line with $\rho_b=\rho_c=(4-\lambda^2)/(2\lambda)$ (note that it is identical with to the thick black line in Fig.1), which means that ρ_b and ρ_c also have to be sufficiently large. Similar arguments are applied for the fixed points type II_{ac} and II_{ab} of which the stability regions are depicted in Figs. 4a and 4b, respectively. It turns out that the stability region of the fixed point type II_{ac} obeys the corresponding inequalities,

$$\begin{aligned} -4+\lambda^2+2\lambda\rho_a-4\rho_c(\rho_c-\rho_a)&>0, \\ -4+\lambda^2+2\lambda\rho_c-4\rho_a(\rho_a-\rho_c)&>0, \\ -4+\lambda^2+2\lambda\rho_b-4\left[\rho_a^2+\rho_c^2-\rho_b(\rho_a+\rho_c)\right]&<0, \end{aligned} \tag{4.29}$$

while that of the fixed point type II_{ab} follows the corresponding inequalities,

$$\begin{aligned} -4+\lambda^2+2\lambda\rho_a-4\rho_b(\rho_b-\rho_a)&>0, \\ -4+\lambda^2+2\lambda\rho_b-4\rho_a(\rho_a-\rho_b)&>0, \\ -4+\lambda^2+2\lambda\rho_c-4\left[\rho_a^2+\rho_b^2-\rho_c(\rho_a+\rho_b)\right]&<0. \end{aligned} \tag{4.30}$$



(a) (b)

FIG. 3: (left) Existence region colored as orange and (right) stability region colored as dark blue of the fixed point type II_{bc} for $\lambda=1.5$. The red line corresponds to $\rho_a=\rho_b=\rho_c$. The black point is a special point, where $\rho_a=\rho_b=\rho_c=(4-\lambda^2)/(2\lambda)$. The thick black line corresponds to $\rho_b=\rho_c=(4-\lambda^2)/(2\lambda)$.

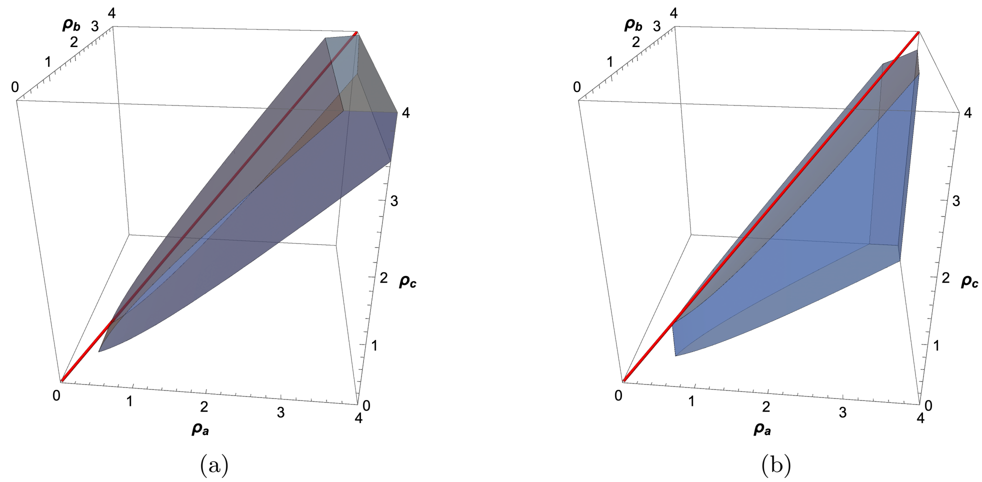


FIG. 4: Stability regions of the fixed points type II_{ac} (left) and II_{ab} (right) colored as dark blue for $\lambda = 1.5$.

4. Fixed points type I

In the next step, we would like to combine the stability regions of the fixed points found above into one figure to form a *three-bladed turbine-shape* region (see Fig. 5). The cube at the corner, the turbine shaft and turbine blades are the stability regions of the fixed points type 0, III, II_{bc}, II_{ac}, and II_{ab} respectively. One might expect that the stability regions of the fixed points type I_a, I_b, and I_c are the spaces left between the blades. In fact, we found that it is the case. Let us focus on the stability region of the fixed point type I_a, which is given by

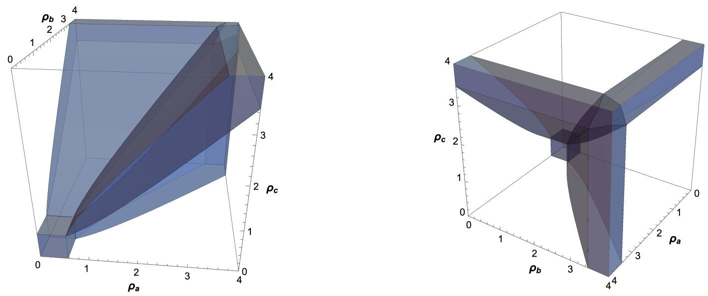
$$\begin{aligned} -4 + \lambda(\lambda + 2\rho_b) - 4\rho_a(\rho_a - \rho_b) &< 0, \\ -4 + \lambda(\lambda + 2\rho_c) - 4\rho_a(\rho_a - \rho_c) &< 0, \\ \lambda^2 + 2\lambda\rho_a - 4 &> 0. \end{aligned} \tag{4.31}$$

and is depicted in Fig. 6a for $\lambda = 1.5$. It is observed that this stability region is included in a region, in which $\rho_a > \rho_b$ and $\rho_a > \rho_c$. However, two surfaces: $-4+\lambda(\lambda+2\rho_b)-4\rho_a(\rho_a-\rho_b) = 0$ and $-4+\lambda(\lambda+2\rho_c)-4\rho_a(\rho_a-\rho_c) = 0$ prevent ρ_a from getting too close to ρ_b and ρ_c, respectively. Qualitatively speaking, therefore, ρ_a has to be significantly larger than ρ_b and ρ_c in order to make the fixed point type I_a stable. Moreover, the stability region terminates at the plane $\rho_a = (4 - \lambda^2)/(2\lambda)$, which means that ρ_a has to be sufficiently large, too. Similar arguments can be applied to the fixed points type I_b and I_c, of which the stability regions are depicted by Figs. 6b and 6c, respectively. It is noted that the stability region of the fixed point I_b is determined by the corresponding inequalities,

$$\begin{aligned} -4 + \lambda(\lambda + 2\rho_c) - 4\rho_b(\rho_b - \rho_c) &< 0, \\ -4 + \lambda(\lambda + 2\rho_a) - 4\rho_b(\rho_b - \rho_a) &< 0, \\ \lambda^2 + 2\lambda\rho_b - 4 &> 0, \end{aligned} \tag{4.32}$$

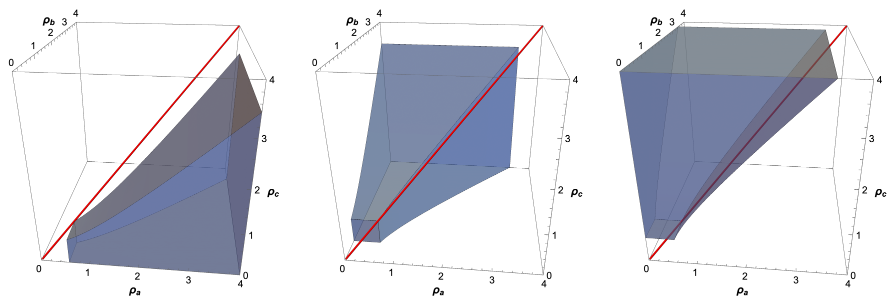
while that of the fixed point I_c is defined by the corresponding inequalities,

$$\begin{aligned} -4 + \lambda(\lambda + 2\rho_a) - 4\rho_c(\rho_c - \rho_a) &< 0, \\ -4 + \lambda(\lambda + 2\rho_b) - 4\rho_c(\rho_c - \rho_b) &< 0, \\ \lambda^2 + 2\lambda\rho_c - 4 &> 0. \end{aligned} \tag{4.33}$$



(a) (b)

FIG. 5: Combination of the stability regions (colored as dark blue) of the fixed points type 0, III, II_{bc}, II_{ac}, and II_{ab} for $\lambda = 1.5$. The two figures are identical but viewed from different angles.



(a) (b) (c)

FIG. 6: Stability regions colored as dark blue of the fixed points type I_a (left), I_b (middle), and I_c (right) for $\lambda = 1.5$.

For convenience, we summarize the stability regions of the fixed points in Table II.

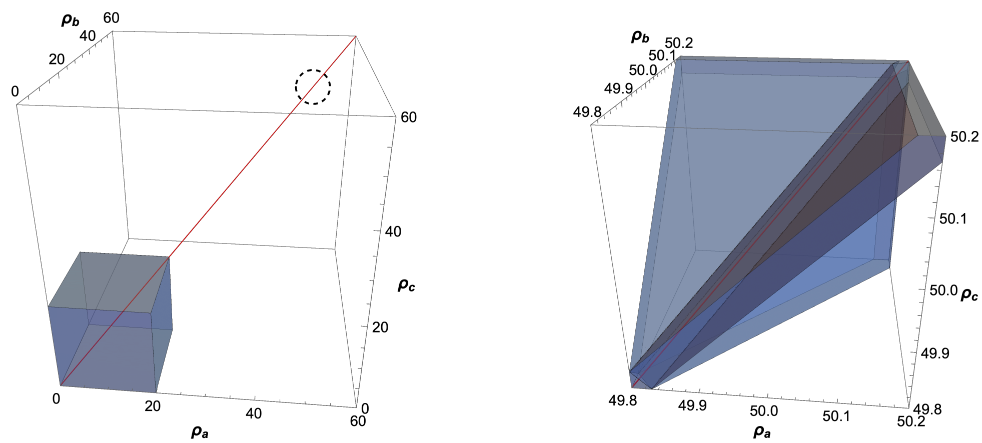
Fixed point type	**Stability regions**	
	Quantitative conditions	**Qualitative conditions (for a fixed λ)**
0	(4.27)	ρ_a, ρ_b, and ρ_c are sufficiently small
I_a	(4.31)	ρ_a is sufficiently large ρ_a is significantly larger than ρ_b and ρ_c
I_b	(4.32)	ρ_b is sufficiently large ρ_b is significantly larger than ρ_a and ρ_c
I_c	(4.33)	ρ_c is sufficiently large ρ_c is significantly larger than ρ_a and ρ_b
II_{bc}	(4.28)	ρ_b and ρ_c are sufficiently large and close ρ_b and ρ_c are significantly larger than ρ_a
II_{ac}	(4.29)	ρ_a and ρ_c are sufficiently large and close ρ_a and ρ_c are significantly larger than ρ_b
II_{ab}	(4.30)	ρ_a and ρ_b are sufficiently large and close ρ_a and ρ_b are significantly larger than ρ_c
III	(4.24)	ρ_a, ρ_b, and ρ_c are sufficiently large and close

TABLE II: Summary of the stability regions of the fixed points.

Furthermore, we can combine the results presented in Tables I and II to draw a general qualitative conclusion on the fate of the vector fields and the metric as follows. If the all coupling constants are sufficiently small, the vector fields eventually vanish regardless of their relative values to each other. But if at least one of the coupling constants are sufficiently large, the vector field with largest coupling constant, says $\rho_{\max}$, remains non-vanishing. Moreover, any vector field with coupling constant that is smaller but sufficiently close to $\rho_{\max}$ remains non-vanishing as well. However, any vector field with coupling constant that is significantly smaller than $\rho_{\max}$ eventually vanish. On the other hand, the fate of spacetime's metric depends on the number of non-vanishing vector fields. If there are three or two non-vanishing vector fields, the metric of spacetime in general approaches a general power-law BI metric. If only one vector field remains non-vanishing, the metric approaches a power-law rsBI metric. If all the vector fields vanish, then the metric approaches a power-law FLRW metric, which is consistent with the cosmic no-hair conjecture.

G. Numerical calculations

To provide support to our above arguments on the stability of fixed points, we would like to solve the system of autonomous equations (4.2)-(4.7) numerically. However, we should first note that the above stability analysis is quite general that it can be applied to arbitrary expansion rate and anisotropies. The choice $\lambda = 1.5$ above is merely for better visualization. To realize a realistic anisotropic inflation, we need ζ to be very large and the anisotropies to be very small [29]. We therefore focus on the value $\lambda = 0.1$ for numerical calculations, following the previous study in Ref. [29] while the values of the coupling constants will be appropriately chosen for each fixed point. The stability region of the fixed point type 0 is depicted in Fig. 7a. Unfortunately, the stability regions of the other fixed points are very difficult to distinguish when displayed in Fig. 7a. Therefore, we depict the stability regions of the fixed points type II_{bc}, II_{ac}, II_{ab}, and III in Fig. 7b, where the coupling constants are restricted between 49.8 and 50.2. It is noted that Figs. 7a and 7b are analogous to Fig. 5a. We will not show the stability regions of the fixed points I_a, I_b, and I_c for $\lambda = 0.1$ since they are analogous to Figs. 6a, 6b, and 6c, respectively.



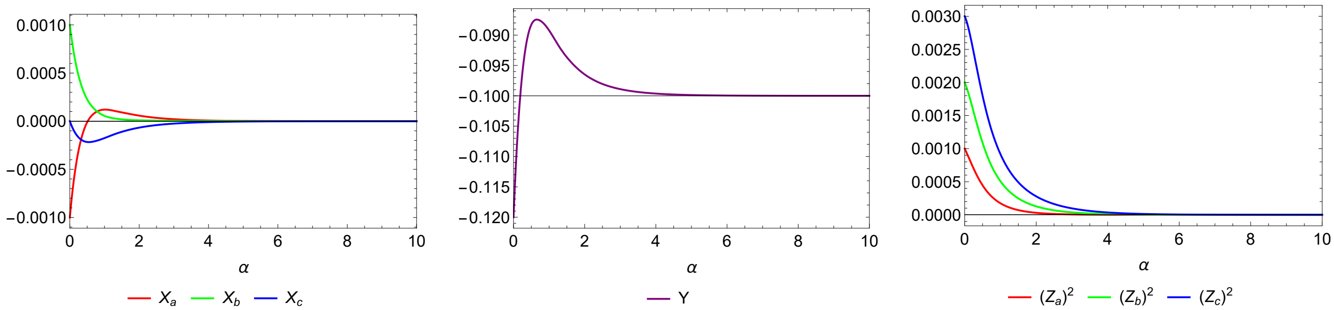
(a) (b)

FIG. 7: (left) Stability region of the fixed point type 0 and (right) a combination of stability regions of the fixed point types III, II_{bc}, II_{ac}, and II_{ab} (right) for $\lambda = 0.1$. The right figure is the enlarged version of the dashed circle region in the left figure.

We start the numerical calculations with the fixed point type 0 by choosing $\lambda = 0.1$, $\rho_a = 12$, $\rho_b = 14$, and $\rho_c = 15$, which satisfy the stability conditions shown in Eq. (4.27) (ρ_a, ρ_b, and ρ_c are sufficiently small). The fixed point is therefore defined to be

$$X_a = X_b = X_c = 0, \quad Y = -0.1, \quad Z_a = Z_b = Z_c = 0. \tag{4.34}$$

At the fixed point (4.34), the anisotropies vanish and $\zeta = 200 \gg 1$ according to the solution (3.8). In Fig. 8, we see that the dynamical variables X_a, X_b, X_c, Y, $(Z_a)^2$, $(Z_b)^2$, and $(Z_c)^2$ all tend to converge to the fixed point (4.34) as expected. This confirms the fixed point type 0 is apparently an attractor.



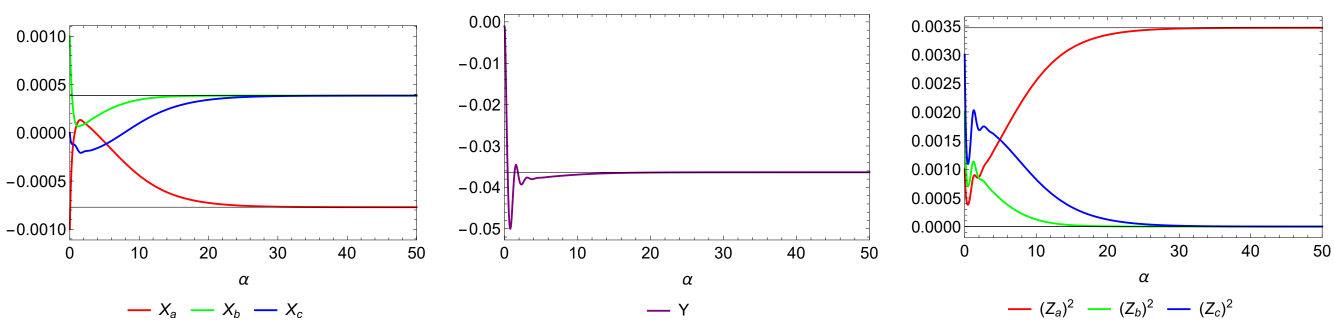
(a) (b) (c)

FIG. 8: Convergence of the dynamical variables to the fixed point type 0 for $\lambda = 0.1$, $\rho_a = 12$, $\rho_b = 14$, and $\rho_c = 15$.

Next, we move on to the fixed point type I_a by choosing $\lambda = 0.1$, $\rho_a = 55$, $\rho_b = 50$, and $\rho_c = 52$, which satisfy the corresponding stability conditions shown in Eq. (4.31) (ρ_a is sufficiently large and significantly larger than ρ_b and ρ_c). The fixed point is therefore given by

$$\begin{aligned} &X_a \approx -0.0007713, \quad X_b = X_c \approx 0.0003857, \\ &Y \approx -0.03634, \quad (Z_a)^2 \approx 0.003469, \quad (Z_b)^2 = (Z_c)^2 = 0. \end{aligned} \tag{4.35}$$

At the fixed point (4.35), the anisotropies are very small and $\zeta \approx 550 \gg 1$ according to the solution (3.11). In Fig. 9, we see that the dynamical variables X_a, X_b, X_c, Y, $(Z_a)^2$, $(Z_b)^2$, and $(Z_c)^2$ converge to the fixed point (4.35) as expected. Hence, we can conclude that this fixed point is an attractor. The similar results for the fixed points type I_b and I_c can be straightforward to achieve.



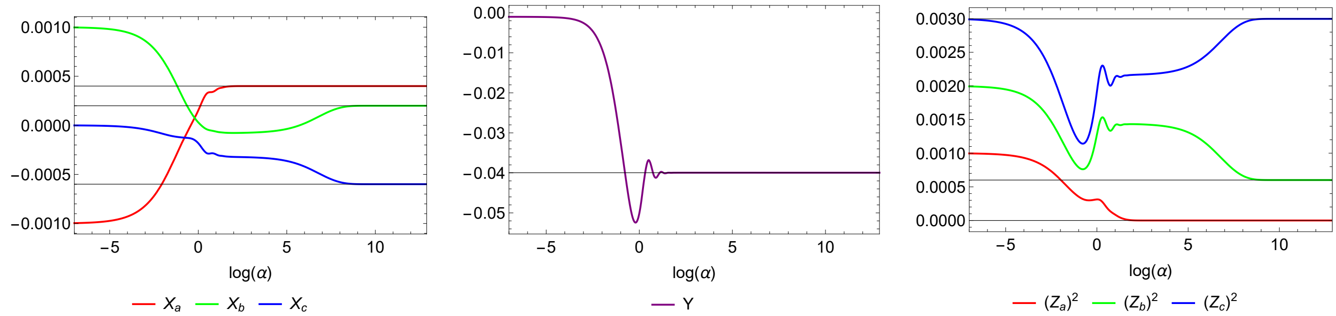
(a) (b) (c)

FIG. 9: Convergence of the dynamical variables to the fixed point type I_a for $\lambda = 0.1$, $\rho_a = 55$, $\rho_b = 50$, and $\rho_c = 52$.

We continue with the fixed point type II_{bc}. We choose $\lambda = 0.1$, $\rho_a = 40$, $\rho_b = 50$, and $\rho_c = 50.02$, which do satisfy the corresponding stability conditions shown in Eq. (4.28) (ρ_b and ρ_c are sufficiently large and close and significantly larger than ρ_a). The fixed point therefore takes the following value,

$$\begin{aligned} &X_a \approx 0.0004000, \quad X_b \approx 0.0001995, \quad X_c \approx -0.0005996, \\ &Y \approx -0.03996, \quad (Z_a)^2 = 0, \quad (Z_b)^2 \approx 0.000601, \quad (Z_c)^2 \approx 0.002997. \end{aligned} \tag{4.36}$$

At the fixed point (4.36), the anisotropies are very small and $\zeta \approx 500 \gg 1$ according to the solution (3.15). In Fig. 10, we see that the dynamical variables X_a, X_b, X_c, Y, $(Z_a)^2$, $(Z_b)^2$, and $(Z_c)^2$ tend to converge to the fixed point (4.36) as expected, confirming the attractor property of the fixed point type II_{bc}. Note that we choose $\log(\alpha)$ instead of α for the horizontal axis for a better visualization. The similar results for the fixed points type II_{ac} and II_{ab} can be done straightforwardly.



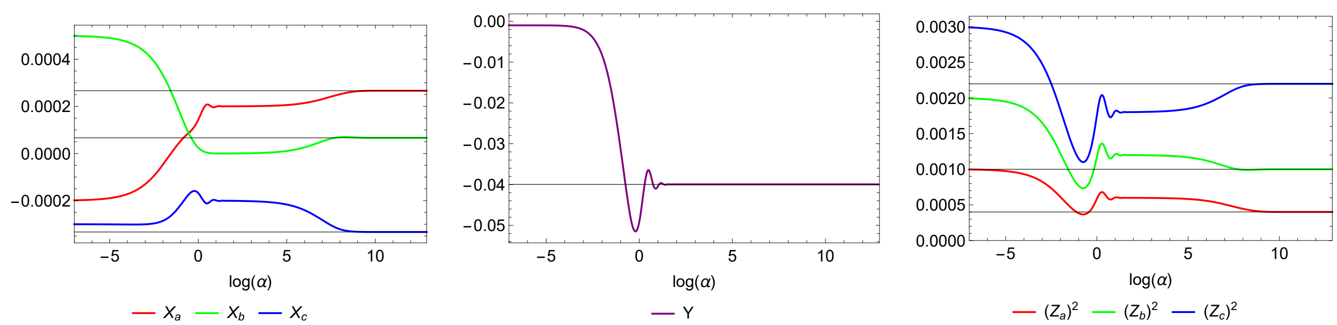
(a) (b) (c)

FIG. 10: Convergence of the dynamical variables to the fixed point type II_{bc} for $\lambda = 0.1$, $\rho_a = 40$, $\rho_b = 50$, and $\rho_c = 50.02$.

For the fixed point type III, we choose $\lambda = 0.1$, $\rho_a = 49.995$, $\rho_b = 50$, and $\rho_c = 50.01$, which clearly satisfy the stability conditions shown in Eq. (4.24) (ρ_a, ρ_b and ρ_c are sufficiently large and close). The fixed point therefore reads

$$\begin{aligned} X_a \approx 0.0002664, \quad X_b \approx 0.0000666, \quad X_c \approx -0.0003330, \quad Y \approx -0.03996, \\ (Z_a)^2 \approx 0.000401, \quad (Z_b)^2 \approx 0.001000, \quad (Z_c)^2 \approx 0.002198. \end{aligned} \tag{4.37}$$

At the fixed point (4.37), the anisotropies are very small and $\zeta \approx 500 \gg 1$ according to the solution (3.15). In the Fig. 11, we see that the dynamical variables X_a, X_b, X_c, Y, $(Z_a)^2$, $(Z_b)^2$, and $(Z_c)^2$ all tend to converge to the fixed point (4.37) as expected, displaying the attractor property of the fixed point type III.



(a) (b) (c)

FIG. 11: Convergence of the dynamical variables to the fixed point type III for $\lambda = 0.1$, $\rho_a = 49.995$, $\rho_b = 50$, and $\rho_c = 50.01$.

V. CONCLUSIONS

We investigated a model of Bianchi type I inflationary universe where the inflation field is non-minimally coupled to three vector fields that propagate along three axes. As a result, we found four types of power-law solutions (types 0, I, II, and III) that are classified based on the number of non-vanishing vector fields. All these solutions have been summarized in Table I. By investigating stability analysis, we showed that all solutions can be attractive and stable against perturbations under specific conditions for the stability regions. It has turned out that the stability properties of the obtained solutions strongly depend on the relative values of the coupling constants. This dependence has been described in Sect. IV both quantitatively and qualitatively. To have a systematical view of the stability, we have built Table II listing all possible cases. The attractor properties of the solutions have been confirmed by numerical calculations. Since our present study is to focus on the classification of all possible stable solutions of a generalized KSW model, in which we maximize the number of vector fields as well as the number of spatial anisotropies, we have not discussed about their CMB imprints. This issue will be our future studies. For now, we hope that our results would be useful for other studies on the early universe in general and the evolution of anisotropies and vector fields during inflation in particular.

ACKNOWLEDGMENTS

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Appendix A: Autonomous equations and fixed points

With the dynamical variables defined as Eq. (4.1), we can write the field equations (2.15), (2.16), (2.17), (2.18), and (2.19) as

$$\frac{V}{\dot{\alpha}^2} = 3 - (X_b)^2 - (X_c)^2 - X_b X_c - \frac{Y^2}{2} - \frac{(Z_a)^2}{2} - \frac{(Z_b)^2}{2} - \frac{(Z_c)^2}{2}, \tag{A1}$$

$$\frac{\ddot{\alpha}}{\dot{\alpha}^2} = -(X_b)^2 - (X_c)^2 - X_b X_c - \frac{Y^2}{2} - \frac{(Z_a)^2}{3} - \frac{(Z_b)^2}{3} - \frac{(Z_c)^2}{3}, \tag{A2}$$

$$\frac{\ddot{\sigma}_b}{\dot{\alpha}^2} = -3X_b + \frac{(Z_a)^2}{3} - \frac{2(Z_b)^2}{3} + \frac{(Z_c)^2}{3}, \tag{A3}$$

$$\frac{\ddot{\sigma}_c}{\dot{\alpha}^2} = -3X_c + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} - \frac{2(Z_c)^2}{3}, \tag{A4}$$

$$\frac{\ddot{\phi}}{\dot{\alpha}^2} = \lambda\left[-3 + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2}\right] - 3Y + \left(\frac{\lambda}{2} + \rho_a\right)(Z_a)^2 + \left(\frac{\lambda}{2} + \rho_b\right)(Z_b)^2 + \left(\frac{\lambda}{2} + \rho_c\right)(Z_c)^2. \tag{A5}$$

The corresponding autonomous equations are then defined as follows

$$\frac{dX_b}{d\alpha} = \frac{d}{dt}\left(\frac{\dot{\sigma}_b}{\dot{\alpha}}\right)\frac{1}{\dot{\alpha}} = \frac{\ddot{\sigma}_b}{\dot{\alpha}^2} - X_b\frac{\ddot{\alpha}}{\dot{\alpha}^2}, \tag{A6}$$

$$\frac{dX_c}{d\alpha} = \frac{d}{dt}\left(\frac{\dot{\sigma}_c}{\dot{\alpha}}\right)\frac{1}{\dot{\alpha}} = \frac{\ddot{\sigma}_c}{\dot{\alpha}^2} - X_c\frac{\ddot{\alpha}}{\dot{\alpha}^2}, \tag{A7}$$

$$\frac{dY}{d\alpha} = \frac{d}{dt}\left(\frac{\dot{\phi}}{\dot{\alpha}}\right)\frac{1}{\dot{\alpha}} = \frac{\ddot{\phi}}{\dot{\alpha}^2} - Y\frac{\ddot{\alpha}}{\dot{\alpha}^2}, \tag{A8}$$

$$\begin{aligned} \frac{dZ_a}{d\alpha} &= \frac{d}{dt}\left(\frac{p_a f_a^{-1}}{\dot{\alpha}} e^{-2\alpha-\sigma_b-\sigma_c}\right)\frac{1}{\dot{\alpha}} \\ &= -Z_a\left(2 + X_b + X_c + \rho_a Y + \frac{\ddot{\alpha}}{\dot{\alpha}^2}\right), \end{aligned} \tag{A9}$$

$$\begin{aligned} \frac{dZ_b}{d\alpha} &= \frac{d}{dt}\left(\frac{p_b f_b^{-1}}{\dot{\alpha}} e^{-2\alpha+\sigma_b}\right)\frac{1}{\dot{\alpha}} \\ &= -Z_b\left(2 - X_b + \rho_b Y + \frac{\ddot{\alpha}}{\dot{\alpha}^2}\right), \end{aligned} \tag{A10}$$

$$\begin{aligned} \frac{dZ_c}{d\alpha} &= \frac{d}{dt}\left(\frac{p_c f_c^{-1}}{\dot{\alpha}} e^{-2\alpha+\sigma_c}\right)\frac{1}{\dot{\alpha}} \\ &= -Z_c\left(2 - X_c + \rho_c Y + \frac{\ddot{\alpha}}{\dot{\alpha}^2}\right). \end{aligned} \tag{A11}$$

Now we solve the system (4.9) to find the fixed points. For $Z_a = Z_b = Z_c = 0$, the system (4.9) becomes

$$X_b\left[-3 + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2}\right] = 0, \tag{A12}$$

$$X_c\left[-3 + (X_b)^2 + (X_c)^2 + X_b X_c + \frac{Y^2}{2}\right] = 0, \tag{A13}$$

$$(Y+\lambda)\left[-3+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}\right]=0. \tag{A14}$$

With the condition (4.8), the system (A12), (A13), and (A14) can be solved to give the fixed point type 0 (4.10).

For $Z_a \neq 0$ and $Z_b = Z_c = 0$, the system (4.9) becomes

$$X_b\left[-3+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_a)^2}{3}\right]+\frac{(Z_a)^2}{3}=0, \tag{A15}$$

$$X_c\left[-3+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_a)^2}{3}\right]+\frac{(Z_a)^2}{3}=0, \tag{A16}$$

$$(\lambda+Y)\left[-3+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_a)^2}{3}\right]+\left(\frac{\lambda}{6}+\rho_a\right)(Z_a)^2=0, \tag{A17}$$

$$-2-X_b-X_c-\rho_aY+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_a)^2}{3}=0. \tag{A18}$$

From Eqs. (A15) and (A16), we obtain

$$X_c=X_b. \tag{A19}$$

Then Eqs. (A16) and (A17) become

$$X_b\left[-3+3(X_b)^2+\frac{Y^2}{2}+\frac{(Z_a)^2}{3}\right]+\frac{(Z_a)^2}{3}=0, \tag{A20}$$

$$(\lambda+Y)\left[-3+3(X_b)^2+\frac{Y^2}{2}+\frac{(Z_a)^2}{3}\right]+\left(\frac{\lambda}{6}+\rho_a\right)(Z_a)^2=0, \tag{A21}$$

respectively, from which we can write $(Z_a)^2$ and Y in terms of X_b as

$$Y=\left(\frac{\lambda}{2}+3\rho_a\right)X_b-\lambda, \tag{A22}$$

$$\frac{(Z_a)^2}{3}=(1+\lambda\rho_a)X_b-\left(2+\frac{\lambda\rho_a}{2}+3\rho_a^2\right)X_b^2. \tag{A23}$$

Inserting Eqs. (A19), (A22), and (A23) into Eq. (A18), we get a quadratic equation of X_b, which has solutions

$$X_b=\frac{2(-4+\lambda^2+2\lambda\rho_a)}{8+\lambda^2+8\lambda\rho_a+12\rho_a^2}, \tag{A24}$$

$$X_b=2. \tag{A25}$$

Putting Eq. (A24) back into Eqs. (A19), (A22), and (A23), we obtain the fixed point type I_a (4.11). We ignore the solution (A25) since it leads to $(Z_a)^2<0$. The derivations of the fixed points type I_b and I_c are similar.

For $Z_a=0$, $Z_b\neq 0$ and $Z_c\neq 0$, the system (4.9) becomes

$$X_b\left[-3+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_b)^2}{3}+\frac{(Z_c)^2}{3}\right]-\frac{2(Z_b)^2}{3}+\frac{(Z_c)^2}{3}=0, \tag{A26}$$

$$X_c\left[-3+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_b)^2}{3}+\frac{(Z_c)^2}{3}\right]+\frac{(Z_b)^2}{3}-\frac{2(Z_c)^2}{3}=0, \tag{A27}$$

$$(\lambda+Y)\left[-3+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_b)^2}{3}+\frac{(Z_c)^2}{3}\right]+\left(\frac{\lambda}{6}+\rho_b\right)(Z_b)^2+\left(\frac{\lambda}{6}+\rho_c\right)(Z_c)^2=0, \tag{A28}$$

$$-2+X_b-\rho_bY+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_b)^2}{3}+\frac{(Z_c)^2}{3}=0, \tag{A29}$$

$$-2+X_c-\rho_cY+(X_b)^2+(X_c)^2+X_bX_c+\frac{Y^2}{2}+\frac{(Z_b)^2}{3}+\frac{(Z_c)^2}{3}=0. \tag{A30}$$

From Eqs. (A29) and (A30), we obtain

$$X_c=X_b+(\rho_c-\rho_b)Y. \tag{A31}$$

From Eqs. (A26), (A27), (A29), and (A31), we can solve for $(Z_b)^2$ and $(Z_c)^2$ as

$$(Z_b)^2 = (1 + X_b - \rho_b Y)\left[-3X_b + (\rho_b - \rho_c)Y\right], \tag{A32}$$

$$(Z_c)^2 = (1 + X_b - \rho_b Y)\left[-3X_b + 2(\rho_b - \rho_c)Y\right]. \tag{A33}$$

Then inserting Eqs. (A29), (A32), and (A33) into Eq. (A28), we obtain a quadratic equation of Y that has two solutions given by

$$Y = \frac{2\left[\lambda + (\lambda + 3\rho_b + 3\rho_c)X_b\right]}{-2 + \lambda(\rho_b - \rho_c) + 2\rho_b^2 - 4\rho_c^2 + 2\rho_b\rho_c}, \tag{A34}$$

$$Y = \frac{1 + X_b}{\rho_b}. \tag{A35}$$

It turns out that the solution (A35) should be ignored since we find that it violates the constraint (4.8). Inserting Eqs. (A31), (A32), (A33), (A34) into Eq. (A29), we obtain a quadratic equation of X_b that has two solutions,

$$X_b = \frac{4 - \lambda^2 - 2\lambda(2\rho_b - \rho_c) - 4(\rho_b^2 - 2\rho_c^2 + \rho_b\rho_c)}{2 + \lambda^2 + 4\left[\lambda(\rho_b + \rho_c) + \rho_b^2 + \rho_c^2 + \rho_b\rho_c\right]}, \tag{A36}$$

$$X_b = \frac{2(-1 + \rho_b^2 - 2\rho_c^2 + \rho_b\rho_c)}{2 + (\rho_b - \rho_c)^2}. \tag{A37}$$

Putting the solution (A36) back into Eqs. (A31), (A32), (A33), and (A34), we obtain the fixed point type II_{bc} (4.17). On the other hand, the other solution (A37) is ignored since it violates the constraint (4.8).

For $Z_a \neq 0$, $Z_b \neq 0$, and $Z_c \neq 0$, the system (4.9) becomes

$$X_b\left[-3 + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3}\right] + \frac{(Z_a)^2}{3} - \frac{2(Z_b)^2}{3} + \frac{(Z_c)^2}{3} = 0, \tag{A38}$$

$$X_c\left[-3 + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3}\right] + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} - \frac{2(Z_c)^2}{3} = 0, \tag{A39}$$

$$(\lambda + Y)\left[-3 + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3}\right] \tag{A40}$$

$$+ \left(\frac{\lambda}{6} + \rho_a\right)(Z_a)^2 + \left(\frac{\lambda}{6} + \rho_b\right)(Z_b)^2 + \left(\frac{\lambda}{6} + \rho_c\right)(Z_c)^2 = 0, \tag{A41}$$

$$-2 - X_b - X_c - \rho_a Y + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} = 0, \tag{A42}$$

$$-2 + X_b - \rho_b Y + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} = 0, \tag{A43}$$

$$-2 + X_c - \rho_c Y + (X_b)^2 + (X_c)^2 + X_bX_c + \frac{Y^2}{2} + \frac{(Z_a)^2}{3} + \frac{(Z_b)^2}{3} + \frac{(Z_c)^2}{3} = 0. \tag{A44}$$

From Eqs. (A42), (A43), and (A44), we can write X_b and X_c in terms of Y as

$$X_b = \frac{-Y(\rho_a - 2\rho_b + \rho_c)}{3}, \tag{A45}$$

$$X_c = \frac{-Y(\rho_a + \rho_b - 2\rho_c)}{3}. \tag{A46}$$

Inserting Eqs. (A45) and (A46) into Eqs. (A38), (A39), and (A42), we obtain $(Z_a)^2$, $(Z_b)^2$, and $(Z_c)^2$ in terms of Y as

$$(Z_a)^2 = \frac{36 - 6\left[\rho_a - 2(\rho_b + \rho_c)\right]Y - \left[9 + 2\rho_a^2 + 8(\rho_b^2 + \rho_c^2) - 8\rho_a(\rho_b + \rho_c) - 2\rho_b\rho_c\right]Y^2}{18}, \tag{A47}$$

$$(Z_b)^2 = \frac{36 - 6\left[\rho_b - 2(\rho_a + \rho_c)\right]Y - \left[9 + 2\rho_b^2 + 8(\rho_a^2 + \rho_c^2) - 8\rho_b(\rho_a + \rho_c) - 2\rho_a\rho_c\right]Y^2}{18}, \tag{A48}$$

$$(Z_c)^2 = \frac{36 - 6\left[\rho_c - 2(\rho_a + \rho_b)\right]Y - \left[9 + 2\rho_c^2 + 8(\rho_a^2 + \rho_b^2) - 8\rho_c(\rho_a + \rho_b) - 2\rho_a\rho_b\right]Y^2}{18}. \tag{A49}$$

Inserting Eqs. (A45), (A46), (A47), (A48), and (A49) into Eq. (A41) will lead to a quadratic equation of Y, which can be solved to give two solutions,

$$Y = \frac{-12}{3\lambda + 2(\rho_a + \rho_b + \rho_c)}, \tag{A50}$$

$$Y = \frac{6(\rho_a + \rho_b + \rho_c)}{3 + 2(\rho_a^2 + \rho_b^2 + \rho_c^2 - \rho_a\rho_b - \rho_a\rho_c - \rho_b\rho_c)}. \tag{A51}$$

Putting the solution (A50) back into Eqs. (A45), (A46), (A47), (A48), and (A49), we obtain the fixed point type III (4.23). We ignore the other solution (A51) since it violates the condition (4.8).

Appendix B: Finding stability regions using the Routh–Hurwitz criterion

To find the stability regions of the fixed points, the standard procedure is to insert each of them into the eigenvalue equation $\det(M - sI) = 0$, which is a sixth-degree polynomial equation. The stability region then is the condition on parameters that ensures all the eigenvalue equation's roots have negative real parts. For the fixed point type 0, the task is simple since the corresponding matrix M has the diagonal form. For the other fixed points especially the fixed point type III, the eigenvalue equation becomes very complicated. We can overcome this difficulty by using the Routh–Hurwitz criterion, which tells us whether all the roots have negative real parts without solving the equation. For simplicity, we only consider the case λ, ρ_a, ρ_b, and $\rho_c > 0$ from now on. In particular, the Routh–Hurwitz criterion is stated as follows (see [66]). For a polynomial equation of degree n

$$a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + ... + a_1 s + a_0 = 0, \tag{B1}$$

where all coefficients are real and a_n is chosen to be positive, we define the $n \times n$ Hurwitz matrix as follows

$$\mathcal{H} \equiv \begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ 0 & a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}. \tag{B2}$$

Then all the roots of the equation have negative real parts if and only if all the leading principal minors of the matrix $\mathcal{H}$, defined as

$$\Delta_1 \equiv \begin{vmatrix} a_{n-1} \end{vmatrix}, \quad \Delta_2 \equiv \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}, \quad \Delta_3 \equiv \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}, \quad \ldots, \quad \Delta_n \equiv \begin{vmatrix} \mathcal{H} \end{vmatrix}, \tag{B3}$$

are positive. Remarkably, an equivalent way to state the Routh–Hurwitz criterion using Routh table can be found in Ref. [67].

We now can use the Routh–Hurwitz criterion to determine the stability regions of the fixed points. Let us start with the fixed point type 0. The corresponding eigenvalue equation is

$$\left(s - \frac{\lambda^2}{2} + 3\right)^3 \left(s - \frac{\lambda^2}{2} - \lambda\rho_a + 2\right)\left(s - \frac{\lambda^2}{2} - \lambda\rho_b + 2\right)\left(s - \frac{\lambda^2}{2} - \lambda\rho_c + 2\right) = 0. \tag{B4}$$

It is easy to see that all the roots of the equation have negative real parts if and only if the conditions shown in Eq. (4.27) are satisfied. Therefore, the stability region of the fixed point type 0 is described by the inequalities (4.27).

For the fixed point type I_a, the corresponding eigenvalue equation is

$$\begin{aligned} &\left(\frac{s}{3} + \frac{8 - \lambda^2 + 4\lambda\rho_a + 12\rho_a^2}{8 + \lambda^2 + 8\lambda\rho_a + 12\rho_a^2}\right)^2 \left[\frac{s}{6} - \frac{-4 + \lambda^2 + 2\lambda\rho_b - 4\rho_a(\rho_a - \rho_b)}{8 + \lambda^2 + 8\lambda\rho_a + 12\rho_a^2}\right]\left[\frac{s}{6} - \frac{-4 + \lambda^2 + 2\lambda\rho_c - 4\rho_a(\rho_a - \rho_c)}{8 + \lambda^2 + 8\lambda\rho_a + 12\rho_a^2}\right] \\ &\times \left[\frac{s^2}{3} + \frac{8 - \lambda^2 + 4\lambda\rho_a + 12\rho_a^2}{8 + \lambda^2 + 8\lambda\rho_a + 12\rho_a^2}s + \frac{6(-4 + \lambda^2 + 2\lambda\rho_a)(8 - \lambda^2 + 4\lambda\rho_a + 12\rho_a^2)(2 + \lambda\rho_a + 2\rho_a^2)}{(8 + \lambda^2 + 8\lambda\rho_a + 12\rho_a^2)^2}\right] = 0 \end{aligned} \tag{B5}$$

All the roots of the equation have negative real parts if and only if

$$-4+\lambda^2+2\lambda\rho_b-4\rho_a(\rho_a-\rho_b)<0, \tag{B6}$$

$$-4+\lambda^2+2\lambda\rho_c-4\rho_a(\rho_a-\rho_c)<0, \tag{B7}$$

$$-4+\lambda^2+2\lambda\rho_a>0, \tag{B8}$$

$$8-\lambda^2+4\lambda\rho_a+12\rho_a^2>0. \tag{B9}$$

according to the Routh–Hurwitz criterion. However, we can remove the fourth constraint since

$$8-\lambda^2+4\lambda\rho_a+12\rho_a^2=[4+2(2\rho_a+\rho_b)(\lambda+2\rho_a)]+\left[4-\lambda^2-2\lambda\rho_b+4\rho_a(\rho_a-\rho_b)\right]>0, \tag{B10}$$

given the first constraint. The stability region of the fixed point type I_a therefore is given by the inequalities (4.31). Similarly, the stability region of the fixed points type I_b and I_c are shown in Eqs. (4.32) and (4.33), respectively.

We continue with the fixed point type II_{bc}. The corresponding eigenvalue equation is

$$\left\{\frac{s}{3}-\frac{-4+\lambda^2+2\lambda\rho_a-4[\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)]}{2+\lambda^2+4\lambda(\rho_b+\rho_c)+4(\rho_b^2+\rho_c^2+\rho_b\rho_c)}\right\}\left[\frac{s}{6}+\frac{1+\lambda(\rho_b+\rho_c)+2(\rho_b^2+\rho_c^2+\rho_b\rho_c)}{2+\lambda^2+4\lambda(\rho_b+\rho_c)+4(\rho_b^2+\rho_c^2+\rho_b\rho_c)}\right]$$
$$\times\,[\text{a polynomial of degree 4}]=0 \tag{B11}$$

All the roots of the equation have negative real parts if and only if

$$-4+\lambda^2+2\lambda\rho_a-4[\rho_b^2+\rho_c^2-\rho_a(\rho_b+\rho_c)]<0, \tag{B12}$$

and

$$\text{all roots of the polynomial of degree 4 have negative real parts.} \tag{B13}$$

Applying the Routh–Hurwitz criterion to Eq. (B13) is straightforward but the condition we found turns out to be very cumbersome. In principle, we can simplify the condition (for example, some constraints in the condition might be redundant and therefore can be removed) but it is not an easy task. We therefore try another approach. In particular, we plot the unsimplified stability region according to the Routh–Hurwitz criterion in parameter space and try to guess its simplified version by comparing it to the existence region. The condition for the requirement (B13) is depicted in Fig. 12. We see that it is very similar to Fig. 3a, which is the visualization of the fixed point type II_{bc}'s existence region (4.18). After examining other values of λ, we can conclude that the condition for the requirement (B13) is exactly described by the inequalities shown in Eq. (4.18), while the stability region of the fixed point type II_{bc} is displayed by Eq. (4.28).

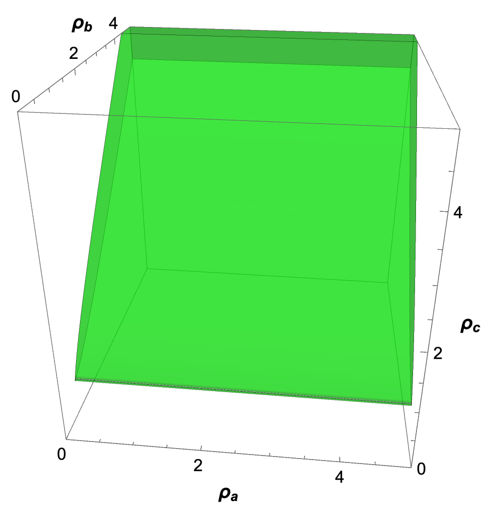

FIG. 12: Condition region (colored as green) for the requirement (B13) for $\lambda = 1.5$ according to the Routh–Hurwitz criterion. Notte that it is very similar to Fig. 3a.

For the fixed point type III, the eigenvalue equation is a very cumbersome polynomial equation of degree 6 that is not worth writing down here. However, we can plot its stability region in Fig. 13. We can clearly see that it is very similar to Fig. 2. After examining other values of λ, we can conclude that the stability region of the fixed point type III is determined by Eq. (4.24).

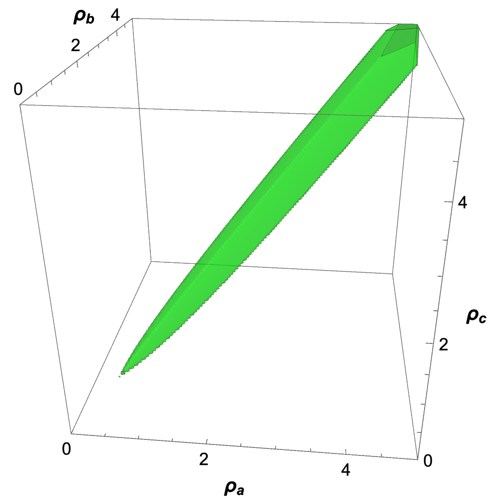

FIG. 13: Stability region (colored as green) of the fixed point type III for $\lambda = 1.5$ according to the Routh–Hurwitz criterion.

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