COMPUTABLE SHEAF INVARIANTS FOR LEGENDRIAN RAINBOW CLOSURES

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Abstract. For any Legendrian link in $\mathbb{R}^3$ given by the rainbow closure of a positive braid word, we develop an explicit and computable description of a Legendrian isotopy invariant associated with it, namely the cohomological category of compactly supported, microlocal rank-one sheaves with singular support on the Legendrian link. In particular, we parametrize the objects of the category by points of a braid variety, and for any pair of objects, we provide a linear map that algebraically characterizes their possible non-trivial graded morphism spaces. In addition, we provide combinatorial rules governing the compositions of graded morphisms in the category under consideration. Finally, we present several applications of our results, highlighting the structural features captured by the categorical invariant of interest.

Contents

1. Introduction

In this article, we investigate a broad family of Legendrian links in the standard contact three-dimensional space through the lens of microlocal sheaf theory. More precisely, for any Legendrian link obtained as the rainbow closure of a positive braid word, we analyze in depth a categorical invariant associated with it, namely a category of sheaves whose singular support lies on the Legendrian link under study. The main contribution of this work is the development of an explicit sheaf-theoretic framework that, for any Legendrian link in the family of interest, provides an algebraic, combinatorial, and computable characterization of the objects, morphisms, and compositions of its associated category. Furthermore, to illustrate our methods, we present several examples demonstrating the range and effectiveness of our approach.

1.1. **Scientific Context.** In recent years, microlocal sheaf-theoretic methods [KS90] have emerged as a powerful framework in contact and symplectic topology [GKS12, Gui23, NZ09, Nad09, GPS24, JT24], providing new categorical invariants [STZ17] and novel approaches to long-standing problems in the theory of Legendrian links [CG22, CZ22], thereby revealing rich connections with cluster theory, algebraic geometry, and representation theory [STWZ19, CW24, TZ18, CGG+25, CGGS24, CGGS21].

For a Legendrian link, the study of its associated microlocal-sheaf categorical invariants has evolved along two main directions. The first explores the relationship between these invariants and those arising from Floer-theoretic constructions [STZ17, NRS+20, CNS19], while the second focuses on the algebraic and geometric structures of their moduli spaces of objects and their applications to the problem of existence and classification of exact Lagrangian fillings of the underlying Legendrian link [CG22, CZ22, CW24, CL22, CL24, SW21, Hug23, Hug24]. Despite the substantial progress achieved along these two research directions, several aspects of these categorical invariants remain largely unexplored, particularly those concerning their graded morphism spaces and their compositions. In particular, from the general framework developed by Nadler and Zaslow [NZ09] and by Nadler [Nad09], one can deduce that the structure of the graded morphism spaces in these invariants encodes significant geometric and topological information about the underlying Legendrian link [GPS24]. For example, in [STZ17], Shende, Treumann, and Zaslow studied a concrete microlocal-sheaf categorical invariant for the Chekanov pair, a pair of Legendrian knots with identical classical invariants, and used the structure of the associated graded morphism spaces to show that the two Chekanov knots are not Legendrian isotopic. Apart from this result, the work of Chantraine, Ng, and Sivek on Legendrian $(2, m)$ torus links [CNS19] constitutes essentially the only study in the literature where the structure of the graded morphism spaces and their compositions has been treated in detail for a microlocal-sheaf categorical invariant.

Bearing this in mind, the main goal in this paper is to investigate in detail a microlocal-sheaf categorical invariant for a broad family of Legendrian links and to shed light on the rich algebraic, combinatorial, geometric, and topological structures encoded by the graded morphism spaces and their compositions within the categorical invariant of interest.

1.2. **Main Results.** Before stating our main results, we briefly introduce the notation and conventions necessary for their formulation. Let (x, y, z) be coordinates on $\mathbb{R}^3$. The standard contact structure ξ_{std} on $\mathbb{R}^3$ is given by $\xi_{\text{std}} := \ker(dz - ydx)$ [EN22]. Let $n \geq 2$ be an integer. We denote by Br_n the braid group on n strands with Artin generators $\sigma_1, \ldots, \sigma_{n-1}$, and by S_n its associated Coxeter group—the symmetric group on n elements—with generators $s_1, \ldots, s_{n-1}$, the adjacent transpositions [CGGS24]. Accordingly, we denote by $\mathrm{Br}_n^+ \subset \mathrm{Br}_n$ the monoid generated by the positive powers of the Artin generators, and call its elements positive braid words. Finally, let $\mathbb{K}$ be a ground field. For any integer $m \geq 1$, we denote by $\mathbb{K}^m$ the m-dimensional Cartesian vector space over $\mathbb{K}$ with no basis specified, and by $\mathbb{K}^m_{\text{std}}$ the same vector space equipped with its standard ordered basis.

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. Following [Kálmán05, Kálmán06, STZ17], we associate to it the Legendrian link $\Lambda(\beta)$ in $(\mathbb{R}^3, \xi_{\text{std}})$—the rainbow closure of β—whose front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ is depicted in Figure 1.

Subsequently, we denote by $\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0$ the dg–category of microlocal-rank-one, compactly supported, constructible sheaves of $\mathbb{K}$–modules on $\mathbb{R}^2$ whose singular support lies on $\Lambda(\beta)$ [STZ17]. By the work of Guillermou, Kashiwara, and Schapira [GKS12], this category is an invariant of the Legendrian isotopy class of $\Lambda(\beta)$, and in [STZ17], Shende, Treumann, and Zaslow provided an explicit local combinatorial description of its objects. In particular, building on these foundations and the seminal work of Chantraine, Ng, and Sivek [CNS19], the main results in this paper lead to a concrete and computable characterization of a categorical invariant of $\Lambda(\beta)$ closely associated with $\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0$, namely its cohomological category $H^\bullet\left(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0\right)$.

Definition 1.1 (Cohomological Category for Rainbow Closures)**.** *Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. According to* [STZ17, CNS19]*, the category $H^\bullet\left(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0\right)$ admits the following presentation:*

- ***Objects**: The objects of the category $H^\bullet\left(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0\right)$ are those of the category $\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0$.*
- ***Graded Morphisms**: Let $\mathscr{F}$, $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. The graded morphism space associated with the pair $(\mathscr{F},\mathscr{G})$ is given by the Yoneda graded vector space:*

$$\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{G}) = \bigoplus_{p\geq 0} \mathrm{Ext}^{\,p}(\mathscr{F},\mathscr{G})\,.$$

- ***Graded Composition**: Let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$, and let $p,\, q \geq 0$ be integers. The graded composition $\circ : \mathrm{Ext}^{\,p}(\mathscr{G},\mathscr{H}) \times \mathrm{Ext}^{\,q}(\mathscr{F},\mathscr{G}) \to \mathrm{Ext}^{\,p+q}(\mathscr{F},\mathscr{H})$ is given by the Yoneda composition (see, for instance,* [HS71]*).*

In recent years, the category $H^\bullet\left(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0\right)$ has played a pivotal role in addressing several long-standing problems in symplectic topology and algebraic geometry. For instance, the study of its moduli space of objects has yielded fundamental results on the existence and classification of the exact Lagrangian fillings of $\Lambda(\beta)$ [STWZ19, CG22, CL22] and led to the definition of the braid varieties [CGGS24, CGGS21, CGG+25], thereby revealing deep connections between the theory of Legendrian links, cluster theory, algebraic geometry, and representation theory [CW24, Hug23, Hug24].

Definition 1.2 (A-type Braid Variety)**.** *Let $n \geq 2$ be an integer, and let $\beta = \sigma_{i_1}\dots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. The braid variety $X(\beta,\mathbb{K}) \subset \mathbb{K}^\ell_{\mathrm{std}}$ associated with β is the affine variety defined by*

$$X(\beta,\mathbb{K}) := \left\{ \vec{x} \in \mathbb{K}^\ell_{\mathrm{std}} \;\middle|\; P_\beta(\vec{x}) \text{ admits an } LU \text{ decomposition} \right\},$$

where, for any $\vec{x} = (x_1,\dots,x_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$, we denote by $P_\beta(\vec{x}) \in \mathrm{GL}(n,\mathbb{K})$ the ordered matrix product of the n-dimensional braid matrices associated with the generators of β and the entries of $\vec{x}$:

$$P_\beta(\vec{x}) := B^{(n)}_{i_1}(x_1)\,\cdots\, B^{(n)}_{i_\ell}(x_\ell)\,.$$

Specifically, given an Artin generator $\sigma_k \in \mathrm{Br}_n^+$, with $k \in [1, n-1]$, and a parameter $x \in \mathbb{K}$, the n-dimensional braid matrix $B_k^{(n)}(x) \in \mathrm{GL}(n, \mathbb{K})$ associated with σ_k and x is defined by

$$\left[B_k^{(n)}(x) \right]_{i,j} := \begin{cases} 1\,, & \text{if } \; i = j \; \text{ and } \; i \neq k,\, k+1\,, \\ 1\,, & \text{if } \; (i,j) = (k, k+1) \; \text{ or } \; (k+1, k)\,, \\ x\,, & \text{if } \; i = j = k\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \qquad i, j \in [1, n]\,.$$

In [STZ17], Shende, Treumann, and Zaslow showed that the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ can be geometrically described by linear configurations of complete flags in $\mathbb{K}^n$ that satisfy certain transversality conditions determined by the positive braid word β. By the work of Casals, Gorsky, Gorsky, and Simental [CGGS24, CGGS21], it is known that such configurations of flags are parametrized by points in the braid variety $X(\beta, \mathbb{K})$. In particular, combining these results yields the following algebraic characterization of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Theorem 1.3 (Algebraic Characterization of the Objects, [STZ17, CGGS24])**.** *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Then $\mathscr{F}$ is algebraically parametrized by a basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$ and a point $\vec{x}$ in the braid variety $X(\beta, \mathbb{K})$.*

Prior to this work, the only known results regarding the graded morphism spaces and their compositions in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ were those due to Chantraine, Ng, and Sivek [CNS19]. Specifically, they provided an explicit characterization of the category for Legendrian $(2, m)$ torus links—positive braid words on two strands—and proved the hereditary-type property in full generality.

Theorem 1.4 (Hereditary-Type Property, [CNS19])**.** *Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}, \mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Then, for any $p \geq 2$,*

$$\mathrm{Ext}^p(\mathscr{F}, \mathscr{G}) = 0.$$

Building on the above foundations, we now present the first main result of this paper: a theorem providing a precise algebraic description of the lower-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. To this end, the following definition plays a central role.

Definition 1.5 (δ–map)**.** *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}, \mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$ be bases for $\mathbb{K}^n$, and let $\vec{x} = (x_1, \ldots, x_\ell)$, $\vec{y} = (y_1, \ldots, y_\ell) \in X(\beta, \mathbb{K})$ be points such that the pairs $\left(\hat{\mathbf{f}}^{(n)}, \vec{x}\right)$ and $\left(\hat{\mathbf{g}}^{(n)}, \vec{y}\right)$ algebraically characterize $\mathscr{F}$ and $\mathscr{G}$ according to Theorem (1.3),* *respectively. Then, we assign to the pair* $(\mathscr{F},\mathscr{G})$ *the linear map* $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{K}^n_{\rm std} \to \mathbb{K}^\ell_{\rm std}$ *defined by*

$$\delta_{\mathscr{F},\mathscr{G}}(\vec{u}\,) := \big(\delta_1(\vec{u}\,), \ldots, \delta_\ell(\vec{u}\,)\big) \in \mathbb{K}^\ell_{\rm std}\,, \qquad \vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\rm std}\,,$$

where, for each $j \in [1,\ell]$,

$$\delta_j(\vec{u}\,) := \Big[\, \big(B^{(n)}_{i_j}(y_j)\big)^{-1} D\big(\pi_{\beta_{j-1}}(\vec{u}\,)\big)\, B^{(n)}_{i_j}(x_j) \,\Big]_{i_j+1,\, i_j}\,.$$

The components appearing in this formula are defined as follows:

- $\pi_{\beta_{j-1}} \in \mathrm{S}_n$ *is the permutation associated with* $\beta_{j-1} = \sigma_{i_1}\cdots\sigma_{i_{j-1}} \in \mathrm{Br}^+_n$, *the truncation of* β *at the* $(j-1)$*-th crossing; in particular,* π_{β_0} *is the trivial permutation.*
- $\pi_{\beta_{j-1}}(\vec{u}\,) := (u_{\pi_{\beta_{j-1}}(1)}, \ldots, u_{\pi_{\beta_{j-1}}(n)}) \in \mathbb{K}^n_{\rm std}$ *is the vector obtained by permuting the entries of* $\vec{u}$ *via* $\pi_{\beta_{j-1}}$*; accordingly,* $\pi_{\beta_0}(\vec{u}\,) = \vec{u}$.
- $D(\pi_{\beta_{j-1}}(\vec{u}\,)) := \mathrm{diag}\big\{u_{\pi_{\beta_{j-1}}(1)}, \ldots, u_{\pi_{\beta_{j-1}}(n)}\big\} \in \mathrm{M}(n,\mathbb{K})$ *is the* $n\times n$ *diagonal matrix whose diagonal entries are given by* $\pi_{\beta_{j-1}}(\vec{u}\,)$.
- $B^{(n)}_{i_j}(x_j)$, $B^{(n)}_{i_j}(y_j) \in \mathrm{GL}(n,\mathbb{K})$ *are* n*-dimensional braid matrices associated with* σ_{i_j}*—the* j*-th crossing of* β*—and* x_j, $y_j \in \mathbb{K}$*—the* j*-th entries of* $\vec{x}$ *and* $\vec{y}$*—respectively.*

Theorem A (Algebraic Description of the Lower-Degree Morphism Spaces)**.** *Let* $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}^+_n$ *be a positive braid word, and let* $\mathscr{F}$, $\mathscr{G}$ *be objects of the category* $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. *Let* $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$ *be bases for* $\mathbb{K}^n$, *and let* $\vec{x}$, $\vec{y} \in X(\beta,\mathbb{K})$ *be points such that the pairs* $\big(\hat{\mathbf{f}}^{(n)},\, \vec{x}\,\big)$ *and* $\big(\hat{\mathbf{g}}^{(n)},\, \vec{y}\,\big)$ *algebraically characterize* $\mathscr{F}$ *and* $\mathscr{G}$ *according to Theorem* (1.3)*, respectively.*

Following Definition (1.5)*, let* $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{K}^n_{\rm std} \to \mathbb{K}^\ell_{\rm std}$ *be the linear map associated with the pair* $(\mathscr{F},\mathscr{G})$. *Then there are isomorphisms of vector spaces*

$$\begin{aligned} \mathrm{Ext}^0(\mathscr{F},\mathscr{G}) \;&\cong\; \ker\delta_{\mathscr{F},\mathscr{G}}\,, \\ \mathrm{Ext}^1(\mathscr{F},\mathscr{G}) \;&\cong\; \mathrm{coker}\,\delta_{\mathscr{F},\mathscr{G}}\,. \end{aligned}$$

Having established this, we proceed to present the second main result of this paper: a theorem providing a precise algebraic and combinatorial description of the composition between the lower-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. In particular, the following definition is fundamental to its formulation.

Definition 1.6 (Braided Compositions)**.** *Let* $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}^+_n$ *be a positive braid word. We introduce three bilinear operations associated with* β*: the* ***Hadamard*** $\odot : \mathbb{K}^n_{\rm std}\times\mathbb{K}^n_{\rm std} \to \mathbb{K}^n_{\rm std}$, *the* ***left braided*** $\circ_{\beta_{\rm L}} : \mathbb{K}^n_{\rm std}\times\mathbb{K}^\ell_{\rm std} \to \mathbb{K}^\ell_{\rm std}$, *and the* ***right braided*** $\circ_{\beta_{\rm R}} : \mathbb{K}^\ell_{\rm std}\times\mathbb{K}^n_{\rm std} \to \mathbb{K}^\ell_{\rm std}$ *compositions.*

Specifically, for any $\vec{u} = (u_1, \ldots, u_n)$, $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{K}^n_{\mathrm{std}}$, *and any* $\vec{p} = (p_1, \ldots, p_\ell)$, $\vec{q} = (q_1, \ldots, q_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$, *we define:*

$$\begin{aligned} \vec{v} \odot \vec{u} &:= (v_1 u_1, \ldots, v_n u_n) \in \mathbb{K}^n_{\mathrm{std}} \,, \\ \vec{v} \circ_{\beta_{\mathrm{L}}} \vec{p} &:= (v_{\pi_{\beta_1}(i_1+1)} p_1, \ldots, v_{\pi_{\beta_\ell}(i_\ell+1)} p_\ell) \in \mathbb{K}^\ell_{\mathrm{std}} \,, \\ \vec{q} \circ_{\beta_{\mathrm{R}}} \vec{u} &:= (q_1 u_{\pi_{\beta_1}(i_1)}, \ldots, q_\ell u_{\pi_{\beta_\ell}(i_\ell)}) \in \mathbb{K}^\ell_{\mathrm{std}} \,, \end{aligned}$$

where, for each $j \in [1, \ell]$*:*

- $\pi_{\beta_j} \in \mathrm{S}_n$ *is the permutation associated with* $\beta_j = \sigma_{i_1} \ldots \sigma_{i_j} \in \mathrm{Br}^+_n$, *the truncation of* β *at the* j*-th crossing.*
- $u_{\pi_{\beta_j}(i_j)} \in \mathbb{K}$ *and* $v_{\pi_{\beta_j}(i_j+1)} \in \mathbb{K}$ *are the* i_j*-th and* (i_j+1)*-th entries of* $\pi_{\beta_j}(\vec{u}\,) \in \mathbb{K}^n_{\mathrm{std}}$ *and* $\pi_{\beta_j}(\vec{v}\,) \in \mathbb{K}^n_{\mathrm{std}}$*—the vectors obtained by permuting the entries of* $\vec{u}$ *and* $\vec{v}$ *via* π_{β_j}*—respectively, where* $i_j \in [1, n-1]$ *denotes the index of* σ_{i_j}*—the* j*-th crossing of* β*.*

Theorem B (Algebraic Description of the Graded Composition)**.** *Let* $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell} \in \mathrm{Br}^+_n$ *be a positive braid word, and let* $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ *be objects of the category* $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$*. Let* $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$, $\hat{\mathbf{h}}^{(n)}$ *be bases for* $\mathbb{K}^n$*, and let* $\vec{x}$, $\vec{y}$, $\vec{z} \in X(\beta, \mathbb{K})$ *be points such that the pairs* $(\hat{\mathbf{f}}^{(n)}, \vec{x}\,)$, $(\hat{\mathbf{g}}^{(n)}, \vec{y}\,)$, $(\hat{\mathbf{h}}^{(n)}, \vec{z}\,)$ *algebraically characterize* $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ *according to Theorem (1.3), respectively.*

Following Definition (1.5), let $\delta_{\mathscr{F},\mathscr{G}}$, $\delta_{\mathscr{G},\mathscr{H}}$, $\delta_{\mathscr{F},\mathscr{H}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ *be the linear maps associated with the pairs* $(\mathscr{F},\mathscr{G})$, $(\mathscr{G},\mathscr{H})$, $(\mathscr{F},\mathscr{H})$*. In light of Theorem (A), let* $\mu \in \mathrm{Ext}^0(\mathscr{F},\mathscr{G})$, $\nu \in \mathrm{Ext}^0(\mathscr{G},\mathscr{H})$, $\Theta \in \mathrm{Ext}^1(\mathscr{F},\mathscr{G})$, $\Phi \in \mathrm{Ext}^1(\mathscr{G},\mathscr{H})$*, and suppose that, for some representatives* $\vec{p}, \vec{q} \in \mathbb{K}^\ell_{\mathrm{std}}$*, the vectors* $\vec{u} \in \ker \delta_{\mathscr{F},\mathscr{G}} \subseteq \mathbb{K}^n_{\mathrm{std}}$, $\vec{v} \in \ker \delta_{\mathscr{G},\mathscr{H}} \subseteq \mathbb{K}^n_{\mathrm{std}}$*, and the classes* $[\vec{p}\,] \in \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}}$, $[\vec{q}\,] \in \operatorname{coker} \delta_{\mathscr{G},\mathscr{H}}$ *determine* μ, ν, Θ, Φ*, respectively, under the isomorphisms*

$$\mathrm{Ext}^0(\mathscr{F},\mathscr{G}) \cong \ker \delta_{\mathscr{F},\mathscr{G}} \,, \qquad \mathrm{Ext}^0(\mathscr{G},\mathscr{H}) \cong \ker \delta_{\mathscr{G},\mathscr{H}} \,,$$
$$\mathrm{Ext}^1(\mathscr{F},\mathscr{G}) \cong \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}} \,, \qquad \mathrm{Ext}^1(\mathscr{G},\mathscr{H}) \cong \operatorname{coker} \delta_{\mathscr{G},\mathscr{H}} \,.$$

Let $i, j \geq 0$ *be integers such that* $i + j \in \{0, 1\}$*. Then, building on Definition (1.6), the graded composition* $\circ : \mathrm{Ext}^i(\mathscr{G},\mathscr{H}) \times \mathrm{Ext}^j(\mathscr{F},\mathscr{G}) \to \mathrm{Ext}^{i+j}(\mathscr{F},\mathscr{H})$ *admits the following description:*

- $(0,0)$***-degree composition****: Under the isomorphism* $\mathrm{Ext}^0(\mathscr{F},\mathscr{H}) \cong \ker \delta_{\mathscr{F},\mathscr{H}}$*, the graded composition* $\nu \circ \mu \in \mathrm{Ext}^0(\mathscr{F},\mathscr{H})$ *is determined by* $\vec{v} \odot \vec{u} \in \ker \delta_{\mathscr{F},\mathscr{H}}$*.*
- $(0,1)$***-degree composition****: Under the isomorphism* $\mathrm{Ext}^1(\mathscr{F},\mathscr{H}) \cong \operatorname{coker} \delta_{\mathscr{F},\mathscr{H}}$*, the graded composition* $\nu \circ \Theta \in \mathrm{Ext}^1(\mathscr{F},\mathscr{H})$ *is determined by the class* $[\,\vec{v} \circ_{\beta_{\mathrm{L}}} \vec{p}\,] \in \operatorname{coker} \delta_{\mathscr{F},\mathscr{H}}$*.*
- $(1,0)$***-degree composition****: Under the isomorphism* $\mathrm{Ext}^1(\mathscr{F},\mathscr{H}) \cong \operatorname{coker} \delta_{\mathscr{F},\mathscr{H}}$*, the graded composition* $\Phi \circ \mu \in \mathrm{Ext}^1(\mathscr{F},\mathscr{H})$ *is determined by the class* $[\,\vec{q} \circ_{\beta_{\mathrm{R}}} \vec{u}\,] \in \operatorname{coker} \delta_{\mathscr{F},\mathscr{H}}$*.*

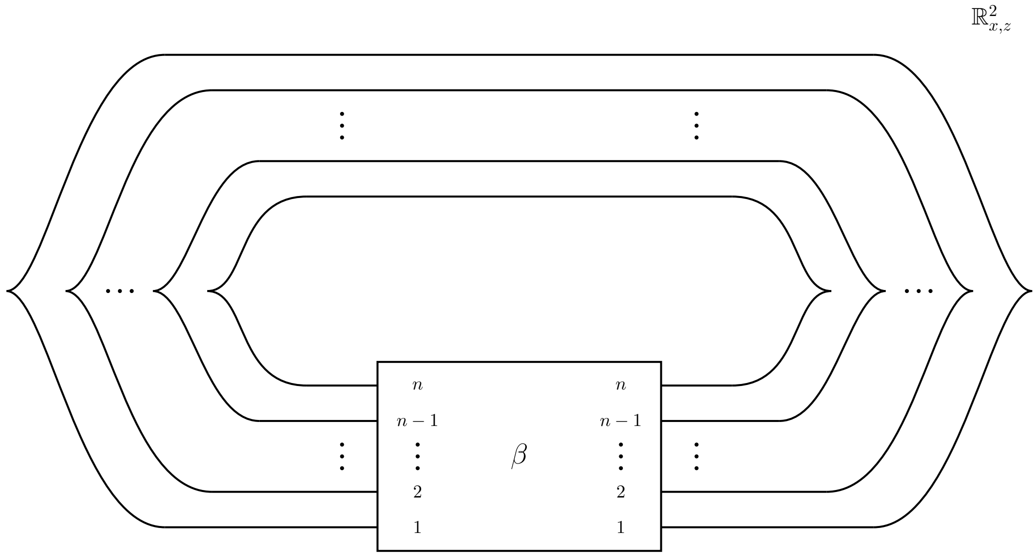

FIGURE 1. Front projection $\Pi_{x,z}(\Lambda(\beta))$ of the Legendrian link $\Lambda(\beta)$.

The above results are of fundamental relevance, as together with Theorems (1.3) and (1.4), they provide an explicit and computable characterization of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Furthermore, together with the work of Chantraine, Ng, and Sivek [CNS19] on Legendrian $(2, m)$ torus links, our results constitute one of the first comprehensive studies of a microlocal-sheaf categorical invariant for a broad family of Legendrian links.

Having presented the main results of this paper, we proceed to describe the structure of the paper.

1.3. **Structure of the Manuscript.** In this subsection, we briefly outline the structure of the paper. In particular, the remainder of Section 1 introduces the notation and conventions used throughout the manuscript.

- Section 2 presents the basic geometric constructions relevant to our study and gives a concise definition of the family of Legendrian links of interest.
- Section 3 briefly reviews some aspects of the microlocal theory of Legendrian links developed by Shende, Treumann, and Zaslow [STZ17].
- Section 4 provides a detailed analysis of the objects of the category under study and derives Theorem (1.3) from first principles.
- Section 5 discusses the graded morphism spaces and their compositions in the category of interest, and provides the proofs of Theorems (A) and (B).
- Section 6 presents applications of our main results and illustrates the scope and effectiveness of our methods through a detailed study of selected examples.

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1.4. **Notational Conventions.** To ensure clarity and consistency, this subsection sets forth the notation and conventions used throughout the manuscript.

Let $a, b \in \mathbb{N}$ with $a < b$. We define $[a,b] := \{c \in \mathbb{N} \mid a \leq c \leq b\}$. Let $n \geq 2$ be an integer. We denote by Br_n the braid group on n strands; more precisely, the group on $n-1$ generators whose presentation is given by:

$$\mathrm{Br}_n := \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,, & \text{for all } i \in [1, n-2] \\ \sigma_i\sigma_j = \sigma_j\sigma_i\,, & \text{for all } i, j \in [1, n-1] \text{ such that } |i-j| \geq 2 \end{array} \right\rangle,$$

where $\sigma_i \in \mathrm{Br}_n$ corresponds to the standard i-th Artin generator for all $i \in [1, n-1]$. Accordingly, we denote by e_n the identity element of Br_n, and by $\mathrm{Br}_n^+ \subset \mathrm{Br}_n$ the monoid generated by the positive powers of the Artin generators. Furthermore, a product expression of the form $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ is referred to as a <u>positive braid word</u> of length $\ell \in \mathbb{N}$.

When drawing a braid diagram on n strands, our convention is that strands are enumerated from bottom to top by $1, \ldots, n$. In addition, we multiply braids from left to right, and depict the braid diagram of a positive braid word by reading its generators from left to right, drawing the corresponding crossings in the same order. For example, the braid diagram on three strands corresponding to the positive braid word $\sigma_1\sigma_2 \in \mathrm{Br}_3^+$ is depicted in Figure (2).

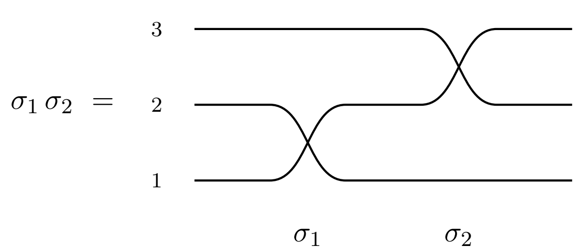


FIGURE 2. Braid diagram on three strands of the positive braid word $\sigma_1\sigma_2 \in \mathrm{Br}_3^+$.

Next, we denote by S_n the Coxeter group associated with Br_n—the symmetric group on n elements. More precisely,

$$\mathrm{S}_n := \left\langle \; s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{cl} s_i^2 = 1\,, & \text{for all } i \in [1, n-1] \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\,, & \text{for all } i \in [1, n-2] \\ s_i s_j = s_j s_i\,, & \text{for all } i, j \in [1, n-1] \text{ such that } |i-j| \geq 2 \end{array} \right\rangle ,$$

where $s_i \in \mathrm{S}_n$ denotes the i-th adjacent transposition for all $i \in [1, n-1]$. In particular, by a slight abuse of notation, we also use e_n to denote the identity element in S_n. Throughout this paper, we compose permutations from left to right, so that the map $\mathrm{Br}_n \to \mathrm{S}_n$ given by $\sigma_i \mapsto s_i$ defines a group homomorphism. In other words, for any $\pi_1, \pi_2 \in \mathrm{S}_n$ and any $k \in [1, n]$, we set $(\pi_1\pi_2)(k) := \pi_2(\pi_1(k))$. For instance, the permutation $s_1 s_2 \in \mathrm{S}_3$ is given by $s_1s_2(1) = 3$, $s_1s_2(2) = 1$, $s_1s_2(3) = 2$. Furthermore, for any positive braid word $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$, we denote by $\pi_\beta := s_{i_1} \cdots s_{i_\ell} \in \mathrm{S}_n$ the permutation associated with β.

Let $m, n \in \mathbb{N}$, and let R be a commutative ring. We denote by $\mathrm{M}(n, m, R)$ the set of $n \times m$ matrices over R, by $\mathrm{M}(n, R)$ the ring of $n \times n$ matrices over R, and by $\mathrm{GL}(n, R)$ the group of invertible $n \times n$ matrices over R. Moreover, given a matrix $A \in \mathrm{M}(n, m, R)$, we denote by $A_{i,j} \in R$ the (i, j)-entry of A, for all $i \in [1, n]$ and $j \in [1, m]$. In certain contexts, we also write

$$A = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix},$$

where c_i and r_j denote the i-th column and j-th row vectors of A, respectively, for all $i \in [1, m]$ and $j \in [1, n]$.

Throughout the manuscript, we identify $\mathbb{K}$ with an arbitrary ground field. Accordingly, for any integer $m \geq 1$, we denote by $\mathbb{K}^m$ the m-dimensional Cartesian vector space over $\mathbb{K}$ with no basis specified, and by $\mathbb{K}^m_{\mathrm{std}}$ the same vector space equipped with its standard ordered basis. Now, let X and Y be finite-dimensional vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} X = p$ and $\dim_{\mathbb{K}} Y = q$, for some $p, q \in \mathbb{N}$. Let $T : X \to Y$ be a linear map, and let $\hat{\mathbf{x}} := \{\hat{x}_i\}_{i=1}^p$ and $\hat{\mathbf{y}} := \{\hat{y}_i\}_{i=1}^q$ be bases for X and Y, respectively. Then, we denote by ${}_{\hat{\mathbf{y}}}\big[\, T \,\big]_{\hat{\mathbf{x}}} \in \mathrm{M}(q, p, \mathbb{K})$ the matrix representing T with respect to the bases $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

Finally, let $\mathcal{M}$ be a smooth manifold, and let $\mathscr{F}$ be a sheaf on $\mathcal{M}$ with values in an abelian category $\mathcal{C}$ [Har77, KS90, HS71]. For any point $x \in \mathcal{M}$, we denote by $\mathscr{F}_x$ the stalk of $\mathscr{F}$ at x. Furthermore, we denote by $SS(\mathscr{F}) \subseteq T^*\mathcal{M}$ the singular support of $\mathscr{F}$. For the definition of singular support of a sheaf and a detailed treatment of its geometric and topological properties, see [KS90].

With our notation and conventions in place, we proceed to introduce the family of Legendrian links in $\mathbb{R}^3$ that constitutes the essential geometric foundation for the main categorical invariant under study in this article.

2. The Rainbow Closure of Positive Braids

This section is devoted to defining the rainbow closure of positive braid words—that is, a geometric construction that yields a distinguished family of Legendrian links in $\mathbb{R}^3$.

Let (x, y, z) be coordinates on $\mathbb{R}^3$. In this setting, the *standard contact structure* ξ_{std} on $\mathbb{R}^3$ is defined by $\xi_{\text{std}} := \ker(dz - ydx)$. Given this contact distribution, a *Legendrian knot* in $(\mathbb{R}^3, \xi_{\text{std}})$ is a knot in $\mathbb{R}^3$ with a smooth parametrization $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ such that, for all $t \in \mathbb{S}^1$, the tangent vector $\gamma'(t)$ lies in ξ_{std} at the point $\gamma(t)$. Extending this notion, a *Legendrian link* in $(\mathbb{R}^3, \xi_{\text{std}})$ is a finite collection of disjoint Legendrian knots [EN22].

Next, we introduce a powerful representation for visualizing Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$, namely the front projection. To this end, let $\Lambda \subset (\mathbb{R}^3, \xi_{\text{std}})$ be a Legendrian knot, and let $\Pi_{x,z} : \mathbb{R}^3 \to \mathbb{R}^2$ be the smooth map given by $\Pi_{x,z}(x, y, z) := (x, z)$, for all $(x, y, z) \in \mathbb{R}^3$. Accordingly, the front projection of Λ, denoted $\Pi_{x,z}(\Lambda) \subset \mathbb{R}^2$, is defined as the image of Λ under the map $\Pi_{x,z}$. Notably, the front projection fully characterizes Λ. More precisely, since the 1-form $dz - ydx$ vanishes along Λ, the y-coordinate can be recovered from the front projection $\Pi_{x,z}(\Lambda)$ via the relation $y = dz/dx$. The front projection possesses several powerful properties; in particular, any immersed curve in $\mathbb{R}^2$ with no vertical tangencies lifts to a unique Legendrian link in $(\mathbb{R}^3, \xi_{\text{std}})$ [EN22]. Consequently, certain distinguished families of Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$ can be constructed via the process of *cusping off* braid diagrams of positive braid words on n strands. Following [Ḱ05, Ḱ06, STZ17], we introduce the following definition.

Definition 2.7. *Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. We denote by $\Lambda(\beta)$ the Legendrian link in $(\mathbb{R}^3, \xi_{\text{std}})$ whose front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ is illustrated in Figure (1). We refer to $\Lambda(\beta)$ as the rainbow closure of β.*

Remark 2.8. *Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word. The Legendrian link $\Lambda(e_n) \subset (\mathbb{R}^3, \xi_{\text{std}})$ corresponds to the Legendrian unlink on n strands. In particular, the front projection $\Pi_{x,z}(\Lambda(e_n)) \subset \mathbb{R}^2$ of $\Lambda(e_n)$ is depicted in Figure (3).*

The family of Legendrian links introduced in Definition (2.7) constitutes the essential geometric foundation for the main categorical invariant under study in this manuscript. This family of Legendrian links has also been studied from a variety of algebraic and geometric perspectives in the literature; see, for instance, [Ḱ05, Ḱ06, STZ17, STWZ19, CG22, Hug23, CG24, CN22, GSW24, SW21].

We conclude this section by recalling a construction from [STZ17] that associates to any Legendrian link $\Lambda \subset (\mathbb{R}^3, \xi_{\text{std}})$ a closed conic Lagrangian $L(\Lambda) \subset (T^*\mathbb{R}^2, \omega_{\text{std}})$, where ω_{std} denotes the standard symplectic structure on $T^*\mathbb{R}^2$.

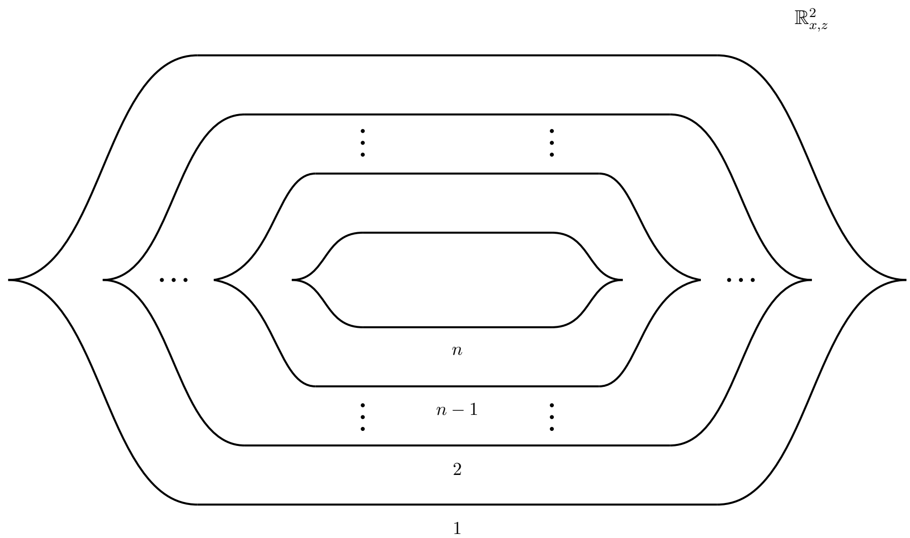


FIGURE 3. Front projection $\Pi_{x,z}(\Lambda(e_n)) \subset \mathbb{R}^2$ of the Legendrian unlink $\Lambda(e_n) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ on n strands.

Construction 2.9. *Let (x, z) be coordinates on $\mathbb{R}^2$, and let (x, z, p_x, p_z) be induced coordinates on $T^*\mathbb{R}^2$. In this setting, $T^*\mathbb{R}^2$ carries the standard symplectic structure $\omega_{\mathrm{std}} := d\,\theta_{\mathrm{std}}$, where the standard Liouville 1-form θ_{std} on $T^*\mathbb{R}^2$ is given by $\theta_{\mathrm{std}} := -(p_x\, dx + p_z\, dz)$.*

Let (x, y, z) be coordinates on $\mathbb{R}^3$, and recall that the standard contact structure ξ_{std} on $\mathbb{R}^3$ is defined by $\xi_{\mathrm{std}} = \ker(dz - y\, dx)$. Now, let $\kappa : \mathbb{R}^3 \to T^*\mathbb{R}^2$ be the smooth map given by $\kappa(x, y, z) := (x, z, y, -1)$, for all $(x, y, z) \in \mathbb{R}^3$. By construction, $\xi_{\mathrm{std}} = \ker\left(\kappa^*\theta_{\mathrm{std}}\right)$, where $\kappa^*\theta_{\mathrm{std}}$ denotes the pull-back of θ_{std} via κ. It follows that, since κ is injective, has an injective differential, and $\mathbb{R}^3$ is diffeomorphic to $\kappa(\mathbb{R}^3)$, κ is an embedding [STZ17]. *As a result, κ realizes $(\mathbb{R}^3, \xi_{\mathrm{std}})$ as an embedded contact submanifold of $(T^*\mathbb{R}^2, \omega_{\mathrm{std}})$.*

Let $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ be a Legendrian link. We denote by $\tilde{\Lambda} \subset T^*\mathbb{R}^2$ the image of Λ under κ, namely $\tilde{\Lambda} := \kappa(\Lambda)$, and introduce $L(\Lambda) \subset (T^*\mathbb{R}^2, \omega_{\mathrm{std}})$ to denote the closed conic Lagrangian defined by $L(\Lambda) := \mathbb{R}_{>0}\,\tilde{\Lambda} \cup 0_{\mathbb{R}^2}$, where $\mathbb{R}_{>0}\,\tilde{\Lambda} \subset T^*\mathbb{R}^2$ is the positive cone over $\tilde{\Lambda}$ and $0_{\mathbb{R}^2} \subset T^*\mathbb{R}^2$ is the zero section.

Notation 2.10. *Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word, and let $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ be its associated Legendrian link. Building on Construction (2.9), we introduce $L(\Lambda(\beta)) \subset (T^*\mathbb{R}^2, \omega_{\mathrm{std}})$ to denote the closed conic Lagrangian associated with $\Lambda(\beta)$.*

Having introduced the distinguished family of Legendrian links and the main contact and symplectic constructions relevant to our study, we now turn to the definition of the associated microlocal-sheaf categorical invariant of interest in this article.

3. Microlocal Theory of Legendrian Links

In this section, we briefly review the microlocal theory of Legendrian links developed by Shende, Treumann, and Zaslow [STZ17]. More precisely, given a ground field $\mathbb{K}$, an integer $r \geq 1$, and a Legendrian link $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ whose front projection $\Pi_{x,z}(\Lambda) \subset \mathbb{R}^2$ is generic and carries a binary Maslov potential, we focus on the combinatorial description of the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$—that is, the dg-derived category of microlocal rank r, compactly supported, constructible sheaves of $\mathbb{K}$-modules on $\mathbb{R}^2$ whose singular support is contained in the closed conic Lagrangian $L(\Lambda) \subset (T^*\mathbb{R}^2, \omega_{\mathrm{std}})$ associated with Λ (see Construction (2.9)). In addition, we define the main object of interest in this manuscript: the cohomological category $H^\bullet\big(\mathcal{S}h_r(\Lambda, \mathbb{K})_0\big)$. To this end, we begin by describing certain stratifications of $\mathbb{R}^2$, which are essential for the combinatorial characterization of the objects of the categories under consideration.

3.1. Regular Stratifications of $\mathbb{R}^2$ Induced by Legendrian Links.

As previously mentioned, we aim to describe a specific dg-derived category of constructible sheaves of $\mathbb{K}$-modules on $\mathbb{R}^2$. Accordingly, the main goal of this subsection is to introduce the stratifications of $\mathbb{R}^2$ relevant to our study.

Let $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ be a Legendrian link. We say that the front projection $\Pi_{x,z}(\Lambda) \subset \mathbb{R}^2$ of Λ is *generic* if $\Pi_{x,z}(\Lambda)$ has only cusps and crossings as singularities. In particular, when Λ has a generic front projection, it induces a regular stratification $\mathcal{S}_\Lambda$ of $\mathbb{R}^2$ defined as follows [STZ17]:

- (i) The 0-dimensional strata correspond to the cusps and crossings in $\Pi_{x,z}(\Lambda)$.
- (ii) The 1-dimensional strata are given by the arcs in $\Pi_{x,z}(\Lambda)$.
- (iii) The 2-dimensional strata consist of the disjoint open subsets of $\mathbb{R}^2$ whose union is equal to the complement of $\Pi_{x,z}(\Lambda)$ in $\mathbb{R}^2$. Consequently, there is only one unbounded 2-dimensional stratum in the stratification $\mathcal{S}_\Lambda$.

Next, let $a \in \mathcal{S}_\Lambda$ be a stratum. We define the star of a, denoted $s(a) \subset \mathcal{S}_\Lambda$, as the union of all the strata of $\mathcal{S}_\Lambda$ whose closure contains a. By construction, $\mathcal{S}_\Lambda$ forms a regular cell complex; that is, each stratum $a \in \mathcal{S}_\Lambda$ and its star $s(a)$ are contractible in $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$, the one-point compactification of $\mathbb{R}^2$.

It is important to emphasize that, for any positive braid word $\beta \in \mathrm{Br}_n^+$, the front projection $\Pi_{x,z}(\Lambda(\beta))$ of the associated Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ is generic. Bearing this in mind, we henceforth restrict our discussion to Legendrian links with generic front projections.

Having introduced the stratifications of $\mathbb{R}^2$ relevant to our discussion, we now turn to the definition of a binary Maslov potential for any Legendrian link in $(\mathbb{R}^3, \xi_{\mathrm{std}})$ arising as the rainbow closure of a positive braid word. In particular, this binary Maslov potential will constitute an essential ingredient in the combinatorial description of the objects of the categories of interest.

3.2. **A Binary Maslov Potential for the Rainbow Closure of Positive Braids.** Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word, and let $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ be its associated Legendrian link. In this subsection, our goal is to equip $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ with a binary Maslov potential; specifically, a locally constant function $\mu_{\mathrm{Maslov}} : \Pi_{x,z}(\Lambda(\beta)) \to \mathbb{Z}$ satisfying certain combinatorial properties.

To begin, observe that, given a generic positive braid word $\beta \in \mathrm{Br}_n^+$, the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ is illustrated in Figure (1). In this article, we call strands the smooth curves connecting a left cusp to a right cusp in $\Pi_{x,z}(\Lambda(\beta))$. It follows that, for each pair of left and right cusps c_l and c_r in $\Pi_{x,z}(\Lambda(\beta))$, exactly two strands meet at this pair of cusps. Consequently, we denote by $L^-_{c_l,c_r}$ and $L^+_{c_l,c_r}$ the bottom and top strands meeting at c_l and c_r, respectively. Having established these notions, we define a binary Maslov potential $\mu_{\mathrm{Maslov}} : \Pi_{x,z}(\Lambda(\beta)) \to \mathbb{Z}$ via the assignment $\mu_{\mathrm{Maslov}}(L^-_{c_l,c_r}) = 0$ and $\mu_{\mathrm{Maslov}}(L^+_{c_l,c_r}) = 1$, for every pair of left and right cusps c_l and c_r in $\Pi_{x,z}(\Lambda(\beta))$.

In particular, by closely inspecting the front projection $\Pi_{x,z}(\Lambda(\beta))$ of the Legendrian link $\Lambda(\beta)$, illustrated in Figure (1), we observe that $\Pi_{x,z}(\Lambda(\beta))$ has n left cusps, n right cusps, and $2n$ strands. Then, according to our previous discussion, we have that:

- The n strands forming the braid diagram of β in the front projection $\Pi_{x,z}(\Lambda(\beta))$ have Maslov potential 0. These strands are referred to as the *bottom strands* in $\Pi_{x,z}(\Lambda(\beta))$.
- The n strands comprising the *rainbow-like shape* that closes the braid diagram of β to form the front projection $\Pi_{x,z}(\Lambda(\beta))$ have Maslov potential 1. These strands are referred to as the *top strands* in $\Pi_{x,z}(\Lambda(\beta))$.

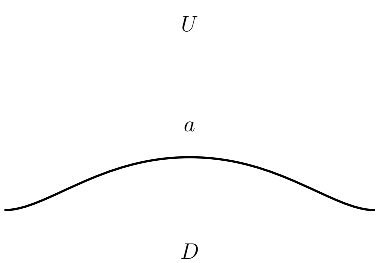
Having introduced a binary Maslov potential for the family of Legendrian links under study, we now review the combinatorial description of the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$, which constitutes a robust Legendrian isotopy invariant of a given Legendrian link $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ [STZ17].

3.3. **Combinatorial Description of the Category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$.** Let $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ be a Legendrian link. Given a field $\mathbb{K}$ and an integer $r \geq 1$, we denote by $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$ the dg-derived category of microlocal rank r, compactly supported, constructible sheaves of $\mathbb{K}$-modules on $\mathbb{R}^2$ whose singular support is contained in the closed conic Lagrangian $L(\Lambda) \subset (T^*\mathbb{R}^2, \omega_{\mathrm{std}})$ associated with Λ (see Construction (2.9)). From the perspective of contact topology, the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$ is of fundamental relevance: more precisely, the work of Guillermou, Kashiwara, and Schapira [GKS12] establishes that it is a Legendrian isotopy invariant of Λ. Moreover, Shende, Treumann, and Zaslow [STZ17] proved that when the front projection $\Pi_{x,z}(\Lambda) \subset \mathbb{R}^2$ is generic and equipped with a binary Maslov potential, the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$ admits an explicit local combinatorial description. In this subsection, our goal is to briefly review this combinatorial characterization of such a microlocal-sheaf categorical invariant of

Λ. Accordingly, we henceforth assume that Λ is a Legendrian link whose front projection $\Pi_{x,z}(\Lambda)$ is generic and equipped with a binary Maslov potential.

To begin, let $\mathcal{S}_\Lambda$ denote the regular stratification of $\mathbb{R}^2$ induced by Λ. As shown in [STZ17], when Λ carries a binary Maslov potential, the objects of the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$ are quasi-isomorphic to their zeroth cohomology sheaf, and therefore can be represented by honest sheaves of $\mathbb{K}$-modules. This simplification is particularly convenient, as it enables us to work with sheaves of $\mathbb{K}$-modules rather than complexes of such sheaves. Next, let $\mathscr{F}$ be an object of the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$. Then, according to [STZ17], $\mathscr{F}$ is fully characterized by the microlocal support conditions—namely, the set of conditions relative to the stratification $\mathcal{S}_\Lambda$ that determine $\mathscr{F}$ according to the following local models:

a) *Arcs*: Let a be an arc in the stratification $\mathcal{S}_\Lambda$. Locally near a, $\mathcal{S}_\Lambda$ consists of the arc a, an upper two-dimensional stratum U, and a lower two-dimensional stratum D, as illustrated in Sub-figure (4a). With respect to this local configuration, the behavior of $\mathscr{F}$ is described by the diagram in Sub-figure (4b).



$$\begin{tikzcd} \mathscr{F}(U) \\ \mathscr{F}(s(a)) \arrow[u] \arrow[d, "\cong"] \\ \mathscr{F}(D) \end{tikzcd}$$

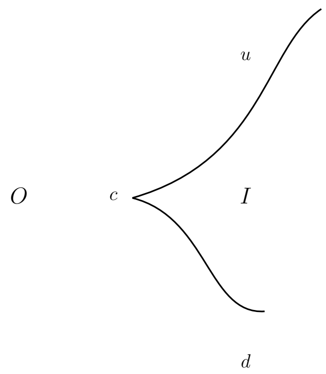
(A) Strata configuration near an arc. (B) Micro-support condition near an arc.

FIGURE 4. Strata configuration (left) and micro-support condition (right) near an arc.

b) *Cusps*: Let c be a cusp in the stratification $\mathcal{S}_\Lambda$. Locally near c, $\mathcal{S}_\Lambda$ consists of the cusp c, an upper arc u, a lower arc d, an inside region I, and an outside region O, as shown in Sub-figure (5a). With respect to this local configuration, the behavior of $\mathscr{F}$ is characterized by the commutative diagram in Sub-figure (5b).

c) *Crossings*: Let x be a crossing in the stratification $\mathcal{S}_\Lambda$. Locally near x, $\mathcal{S}_\Lambda$ consists of the crossing x, four two-dimensional strata N, S, E, and W, along with four arcs nw, ne, sw, and se, as depicted in Sub-figure (6a). With respect to this local configuration, the behavior of $\mathscr{F}$ is described by the commutative diagram in Sub-figure (6b). Moreover, the sequence

$$0 \longrightarrow \mathscr{F}(s(x)) \longrightarrow \mathscr{F}(s(nw)) \oplus \mathscr{F}(s(ne)) \longrightarrow \mathscr{F}(N) \longrightarrow 0,$$



(A) Strata configuration near a cusp.

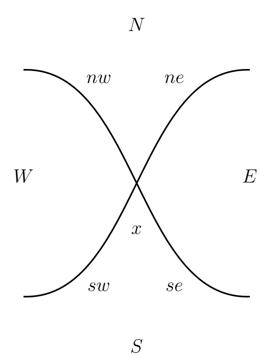
$$\begin{tikzcd}
 & & \mathscr{F}(s(u)) \arrow[d, "\cong"] \arrow[dll, bend right] \\
\mathscr{F}(O) & \mathscr{F}(s(c)) \arrow[l, "\cong"'] \arrow[r] \arrow[ur] \arrow[dr, "\cong"] & \mathscr{F}(I) \\
 & & \mathscr{F}(s(d)) \arrow[u] \arrow[ull, bend left, "\cong"]
\end{tikzcd}$$

(B) Micro-support condition near a cusp.

FIGURE 5. Strata configuration (left) and micro-support condition (right) near a cusp.

is required to be short exact.

Remark 3.11. *In the diagrams of Sub-figures* (4b)*,* (5b)*, and* (6b)*, all maps labeled with the symbol "$\cong$" are required to be isomorphisms. In this article, we further require these maps to be the identity maps, a convenient convention often adopted in the literature when working with concrete examples (see, for instance,* [STZ17, CNS19, CW24]*).*



(A) Strata configuration near a crossing.

$$\begin{tikzcd}
 & & \mathscr{F}(N) & & \\
 & \mathscr{F}(s(nw)) \arrow[ur] \arrow[dl, "\cong"'] & & \mathscr{F}(s(ne)) \arrow[ul] \arrow[dr, "\cong"] & \\
\mathscr{F}(W) & & \mathscr{F}(s(x)) \arrow[uu] \arrow[ul] \arrow[ur] \arrow[ll] \arrow[rr] \arrow[dl, "\cong"] \arrow[dr, "\cong"'] \arrow[dd, "\cong"] & & \mathscr{F}(E) \\
 & \mathscr{F}(s(sw)) \arrow[ul] \arrow[dr, "\cong"'] & & \mathscr{F}(s(se)) \arrow[ur] \arrow[dl, "\cong"] & \\
 & & \mathscr{F}(S) & &
\end{tikzcd}$$

(B) Micro-support condition near a crossing.

FIGURE 6. Strata configuration (left) and micro-support condition (right) near a crossing.

d) *Microlocal Rank for Binary Maslov Potentials*: Let $r \geq 1$ be an integer, and let a be an arc in the stratification $\mathcal{S}_\Lambda$. In particular, observe that locally near a, $\mathcal{S}_\Lambda$ consists of the arc a, an upper two-dimensional stratum U, and a lower two-dimensional stratum D, as illustrated in Sub-figure (4a). Given this local configuration, let $p \in U$ and $q \in D$ be arbitrary points, and denote by $\mathscr{F}_p$ and $\mathscr{F}_q$ the stalks of $\mathscr{F}$ at p and q, respectively. Then, depending on the binary Maslov potential, the microlocal-rank-r condition impose the following constraints on the dimensions of these stalks:

- If the arc a belongs to a strand with Maslov potential 0, we require $\dim_{\mathbb{K}} \mathscr{F}_p - \dim_{\mathbb{K}} \mathscr{F}_q = r$.
- If the arc a belongs to a strand with Maslov potential 1, we require $\dim_{\mathbb{K}} \mathscr{F}_q - \dim_{\mathbb{K}} \mathscr{F}_p = r$.

In addition, let U_0 denote the unique unbounded two-dimensional stratum of $\mathcal{S}_\Lambda$. Then, the compact support condition establishes that for any $x \in U_0$, the stalk of $\mathscr{F}$ at x must be trivial; that is, $\mathscr{F}_x = 0$.

With the local combinatorial description of the objects of the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$ in place, we now introduce one of the core concepts of this manuscript: the cohomological category $H^\bullet(\mathcal{S}h_r(\Lambda, \mathbb{K})_0)$. To this end, let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$. Following [KS90], we define $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G})$ to be the i-th sheaf cohomology of the right-derived internal Hom sheaf $R\mathscr{H}om(\mathscr{F}, \mathscr{G})$; that is,

$$\operatorname{Ext}^i(\mathscr{F}, \mathscr{G}) := H^i\big(R\Gamma(\mathbb{R}^2; R\mathscr{H}om(\mathscr{F}, \mathscr{G}))\big)\,, \tag{3.1}$$

for all $i \in \mathbb{Z}$. In particular, as shown in [STZ17], when the front projection $\Pi_{x,z}(\Lambda)$ is generic and equipped with a binary Maslov potential, $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i < 0$. In this setting, the category $H^\bullet(\mathcal{S}h_r(\Lambda, \mathbb{K})_0)$ admits the following presentation:

Definition 3.12. *Let $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ be a Legendrian link whose front projection $\Pi_{x,z}(\Lambda) \subset \mathbb{R}^2$ is generic and equipped with a binary Maslov potential. Then, building on [STZ17], the cohomological category $H^\bullet(\mathcal{S}h_r(\Lambda, \mathbb{K})_0)$ is characterized by the following data:*

- *Objects: The objects of the category $H^\bullet(\mathcal{S}h_r(\Lambda, \mathbb{K})_0)$ are those of the category $\mathcal{S}h_r(\Lambda, \mathbb{K})_0$.*
- *Morphisms: The morphism spaces are positively graded $\mathbb{K}$-modules. More precisely, let $\mathscr{F}$, $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_r(\Lambda, \mathbb{K})_0)$. Then,*

$$Hom_{H^\bullet(\mathcal{S}h_r(\Lambda,\mathbb{K})_0)}(\mathscr{F}, \mathscr{G}) := H^\bullet\big(\, Hom_{\mathcal{S}h_r(\Lambda,\mathbb{K})_0}(\mathscr{F}, \mathscr{G})\,\big) = \operatorname{Ext}^\bullet(\mathscr{F}, \mathscr{G}) = \bigoplus_{i \geq 0} \operatorname{Ext}^i(\mathscr{F}, \mathscr{G})\,.$$

- *Composition: The morphism spaces are equipped with a graded composition. Specifically, let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ be objects of the category $H^\bullet(\mathcal{S}h_r(\Lambda, \mathbb{K})_0)$, and let i, $j \geq 0$ be integers. Then, the graded composition*

$$\circ : Hom^i_{H^\bullet(\mathcal{S}h_r(\Lambda,\mathbb{K})_0)}(\mathscr{G}, \mathscr{H}) \times Hom^j_{H^\bullet(\mathcal{S}h_r(\Lambda,\mathbb{K})_0)}(\mathscr{F}, \mathscr{G}) \longrightarrow Hom^{i+j}_{H^\bullet(\mathcal{S}h_r(\Lambda,\mathbb{K})_0)}(\mathscr{F}, \mathscr{H})\,,$$

is defined by the Yoneda composition on Ext groups (see, for instance, [HS71]).

With the above definition at hand, we conclude our brief review of the microlocal theory of Legendrian links and turn our attention to the main goal of this manuscript. More precisely, given a positive braid word $\beta \in \mathrm{Br}_n^+$, the remainder of the paper is devoted to providing an explicit and computable characterization of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. In the next section, we present both a geometric and an algebraic description of the objects of this category.

4. The Objects of the Category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. Then, given a fixed ground field $\mathbb{K}$, the main purpose of this section is to provide both a geometric and an algebraic characterization of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. To this end, we first establish some technical definitions and lemmas concerning linear maps and complete flags, which will be instrumental in the discussion ahead.

4.1. **Some Technical Results on Linear Maps and Complete Flags.** Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. In this subsection, we collect several technical lemmas concerning linear maps subject to specific constraints. Furthermore, we introduce some relevant definitions and properties of complete flags in finite-dimensional vector spaces. The importance of the concepts we present below lies in the fact that they provide the basic framework for the geometric and algebraic description of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.

We begin by introducing the following lemma, which will be fundamental in studying the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ near the cusps in the front projection of the Legendrian link $\Lambda(\beta)$.

Lemma 4.13 (Cusp Condition)**.** *Let S and N be finite-dimensional vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} S = p$ and $\dim_{\mathbb{K}} N = q$, where p and q are two positive integers subject to the constraints $1 \leq p \leq q-1$ and $q \geq 2$. Let $\phi : S \to N$ and $\psi : N \to S$ be linear maps such that $\psi \circ \phi = \mathrm{id}_S$. Then, the following statements hold:*

(i) *ψ is surjective.*

(ii) *ϕ is injective.*

(iii) *$N = \ker\psi \oplus \mathrm{im}\,\phi$.*

Proof. First, we prove part (i). To this end, let $s \in S$. Then, we obtain that $s = \mathrm{id}_S(s) = \psi(\phi(s))$, which shows that $s \in \mathrm{im}\,\psi$. As a result, we conclude that $S = \mathrm{im}\,\psi$. Next, we verify part (ii). For this purpose, let $x \in \ker\phi \subset S$. Hence, we have that $x = \mathrm{id}_S(x) = \psi(\phi(x)) = \psi(0) = 0$. It follows that $\ker\phi = 0$. Finally, we prove part (iii). To this end, let $n \in \ker\psi \cap \mathrm{im}\,\phi \subset N$. By definition, we know that there exists $s \in S$ such that $n = \phi(s)$. In particular, we observe that $s = \mathrm{id}_S(s) = \psi(\phi(s)) = \psi(n) = 0$, which implies that $\ker\psi \cap \mathrm{im}\,\phi = 0$. Furthermore, by parts (i) and (ii) above, we know that ψ is surjective and ϕ is injective. Therefore, by the

$$\begin{tikzcd} & N & \\ W \arrow[ur, "\beta_1"] & & E \arrow[ul, "\beta_2"'] \\ & S \arrow[ul, "\alpha_1"] \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

FIGURE 7. Commutative diagram for a collection of linear maps α_1, β_1, α_2, and β_2 satisfying the crossing condition.

rank-nullity theorem, we get that $\dim_{\mathbb{K}} \ker \psi = q - p$ and $\dim_{\mathbb{K}} \operatorname{im} \phi = p$. Bearing this in mind, we deduce that $N = \ker \psi \oplus \operatorname{im} \phi$. □

We now present a lemma that will play a key role in the description of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ near the crossings in the front projection of the Legendrian link $\Lambda(\beta)$.

Lemma 4.14 (Crossing Condition)**.** *Let S, W, E, and N be vector spaces over $\mathbb{K}$. In addition, suppose that $\alpha_1 : S \to W$, $\beta_1 : W \to N$, $\alpha_2 : S \to E$, and $\beta_2 : E \to N$ are a collection of linear maps subject to the following constraints:*

- *The diagram in Figure (7) commutes.*
- *The sequence in Equation (4.1) is short exact.*

$$0 \longrightarrow S \xrightarrow{(\alpha_1, \alpha_2)} W \oplus E \xrightarrow{\beta_1 \oplus \big(- \beta_2 \big)} N \longrightarrow 0\ . \tag{4.1}$$

Then, the following statements hold:

(i) $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$.

(ii) $\operatorname{im}\big(\beta_1 \circ \alpha_1\big) = \operatorname{im} \beta_1 \cap \operatorname{im} \beta_2 = \operatorname{im}\big(\beta_2 \circ \alpha_2\big)$.

(iii) $N = \operatorname{im} \beta_1 + \operatorname{im} \beta_2$.

(iv) If the maps α_1 and β_1 are injective, then the maps α_2 and β_2 are also injective.

Proof. First, we prove part (i). To this end, observe that, by assumption, the diagram in Figure (7) commutes. In consequence, we obtain that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$.

Next, we verify part (ii). For this purpose, let $n \in \operatorname{im}\beta_1 \cap \operatorname{im}\beta_2 \subset N$. Then, there exist $w \in W$ and $e \in E$ such that $n = \beta_1(w)$ and $n = \beta_2(e)$. In particular, observe that $\beta_1(w) - \beta_2(e) = n - n = 0$, which implies that $(w, e) \in \ker\big(\beta_1 \oplus (-\beta_2)\big)$. Consequently, since the sequence in Equation (4.1) is short exact, there exists $s \in S$ such that $(w, e) = (\alpha_1(s), \alpha_2(s))$, which shows that $n = \beta_1(\alpha_1(s))$ and $n = \beta_2(\alpha_2(s))$. Therefore, by part (i) above, we conclude that $\operatorname{im}\big(\beta_1 \circ \alpha_1\big) = \operatorname{im}\beta_1 \cap \operatorname{im}\beta_2 = \operatorname{im}\big(\beta_2 \circ \alpha_2\big)$.

Now, we prove part (iii). To this end, let $n \in N$. Then, since the sequence in Equation (4.1) is short exact, there exist $w \in W$ and $e \in E$ such that $n = \beta_1(w) - \beta_2(e)$. It follows that $N = \operatorname{im}\beta_1 + \operatorname{im}\beta_2$.

Finally, we verify part (iv). For this purpose, let $x \in \ker\alpha_2 \subset S$. Then, by part (i) above, we have that $\beta_1(\alpha_1(x)) = \beta_2(\alpha_2(x)) = \beta_2(0) = 0$. In particular, since the linear maps α_1 and β_1 are assumed to be injective, we conclude that $x = 0$, and therefore $\ker\alpha_2 = 0$. In addition, let $y \in \ker\beta_2 \subset E$. Then, we have that $\beta_1(0) - \beta_2(y) = 0 - 0 = 0$, which implies that $(0, y) \in \ker\big(\beta_1 \oplus (-\beta_2)\big)$. Thus, since the sequence in Equation (4.1) is short exact, there exists $s \in S$ such that $(\alpha_1(s), \alpha_2(s)) = (0, y)$. Consequently, since the linear map α_1 is assumed to be injective, we obtain that $s = 0$. It follows that $y = \alpha_2(s) = \alpha_2(0) = 0$. Bearing this in mind, we deduce that $\ker\beta_2 = 0$. This completes the proof. □

Next, we introduce some key definitions and lemmas concerning complete flags in finite-dimensional vector spaces.

Definition 4.15. *Let V be a finite-dimensional vector space over $\mathbb{K}$ with $\dim_{\mathbb{K}} V = n$, for some integer $n \geq 1$. Then, a complete flag $\mathcal{F}^\bullet = \big\{F^{(0)} \subset \cdots \subset F^{(n)}\big\}$ in V is defined to be a nested sequence of vector subspaces $F^{(p)}$ of V such that $\dim_{\mathbb{K}} F^{(p)} = p$ for all $p \in [0, n]$. With this definition in place, we adopt the following terminology:*

- *Two complete flags $\mathcal{F}^\bullet = \big\{F^{(0)} \subset \cdots \subset F^{(n)}\big\}$ and $\mathcal{G}^\bullet = \big\{G^{(0)} \subset \cdots \subset G^{(n)}\big\}$ in V are said to be completely opposite if and only if*

$$V = F^{(p)} \oplus G^{(n-p)}\,,$$

  for each $p \in [0, n]$.

- *Let $k \in [1, n-1]$ be an integer. Two complete flags $\mathcal{F}^\bullet = \big\{F^{(0)} \subset \cdots \subset F^{(n)}\big\}$ and $\mathcal{G}^\bullet = \big\{G^{(0)} \subset \cdots \subset G^{(n)}\big\}$ in V are said to be in s_k-relative position if and only if $F^{(k)} \neq G^{(k)}$ and $F^{(p)} = G^{(p)}$ for each $p \neq k$, where $p \in [0, n]$.*

- *Let* $\hat{\mathbf{x}} := \{\hat{x}_i\}_{i=1}^n$ *be a basis for* V. *We define the standard flag in* V *relative to the basis* $\hat{\mathbf{x}}$ *to be the complete flag* $\mathcal{F}^\bullet_{\mathrm{std}}[\hat{\mathbf{x}}] := \{F^{(0)}_{\mathrm{std}} \subset \cdots \subset F^{(n)}_{\mathrm{std}}\}$ *given by*

$$F^{(p)}_{\mathrm{std}}[\hat{\mathbf{x}}] := \begin{cases} 0\,, & \text{if } p = 0 \\ \langle \hat{x}_1, \ldots, \hat{x}_p \rangle\,, & \text{if } p \in [1,n] \end{cases} .$$

Similarly, we define the anti-standard flag in V *relative to the basis* $\hat{\mathbf{x}}$ *to be the complete flag* $\mathcal{F}^\bullet_{\mathrm{astd}}[\hat{\mathbf{x}}] := \{F^{(0)}_{\mathrm{astd}} \subset \cdots \subset F^{(n)}_{\mathrm{astd}}\}$ *given by*

$$F^{(p)}_{\mathrm{astd}}[\hat{\mathbf{x}}] := \begin{cases} 0\,, & \text{if } p = 0\,, \\ \langle \hat{x}_n, \ldots, \hat{x}_{n+1-p} \rangle\,, & \text{if } p \in [1,n] \end{cases} .$$

- *Let* $\hat{\mathbf{x}} := \{\hat{x}_i\}_{i=1}^n$ *be a basis for* V, *and let* $A \in \mathrm{GL}(n,\mathbb{K})$ *be an* $n \times n$ *invertible matrix over* $\mathbb{K}$. *We introduce* $\{\vec{a}_j\}_{j=1}^n$ *to denote the collection of vectors in* V *given by* $\vec{a}_j := \sum_{i=1}^n A_{i,j}\,\hat{x}_i$ *for each* $j \in [1,n]$. *In particular, let* $\mathcal{F}^\bullet = \{F^{(0)} \subset \cdots \subset F^{(n)}\}$ *be a complete flag in* V *such that*

$$F^{(p)} = \begin{cases} 0\,, & \text{if } p = 0\,, \\ \langle \vec{a}_1, \ldots, \vec{a}_p \rangle\,, & \text{if } p \in [1,n] \end{cases} .$$

Then, we say that, relative to the basis $\hat{\mathbf{x}}$ *for* V, *the complete flag* $\mathcal{F}^\bullet$ *is represented by the matrix* A.

Notation 4.16. *Let* n, $m \geq 1$ *be integers. Throughout this manuscript, we implement the following standard matrices:*

- $\mathbf{1}_n \in \mathrm{GL}(n,\mathbb{K})$*: the identity matrix of size* $n \times n$.
- $\mathbf{w}_n \in \mathrm{GL}(n,\mathbb{K})$*: the anti-diagonal identity matrix of size* $n \times n$*; that is, the matrix with ones on the anti-diagonal and zeros elsewhere.*
- $\mathbf{0}_{n\times m} \in M(n,m,\mathbb{K})$*: the zero matrix of size* $n \times m$.

Explicitly,

$$\mathbf{1}_n := \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{n\times n}, \qquad \mathbf{w}_n := \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}_{n\times n}, \qquad \mathbf{0}_{n\times m} := \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{n\times m} .$$

In addition, for any pair of integers $p, q \geq 1$ *with* $q > p$, *we denote by* $\iota^{(q,p)} \in M(q,p,\mathbb{K})$ *the standard inclusion matrix, and by* $\pi^{(p,q)} \in M(p,q,\mathbb{K})$ *the standard projection matrix, defined by*

$$\iota^{(q,p)} := \left[\begin{array}{c} \mathbf{1}_p \\ \hline \mathbf{0}_{(q-p)\times p} \end{array}\right], \qquad \pi^{(p,q)} := \left[\begin{array}{c|c} \mathbf{1}_p & \mathbf{0}_{p\times(q-p)} \end{array}\right] .$$

Accordingly, for all $k \geq 2$, the following identities hold:

$$\pi^{(1,2)} \cdots \pi^{(k-1,k)} = \pi^{(1,k)}\,, \qquad \pi^{(p,q)} \cdot \iota^{(q,p)} = \mathbf{1}_p\,, \qquad \iota^{(k,k-1)} \cdots \iota^{(2,1)} = \iota^{(k,1)}\,.$$

Remark 4.17. *Let V be a finite-dimensional vector space over $\mathbb{K}$ with $\dim_{\mathbb{K}} V = n$, for some integer $n \geq 1$. Furthermore, suppose that $\hat{\mathbf{x}} := \{\hat{x}_i\}_{i=1}^n$ is a basis for V. Then, according to Definition (4.15), we have that:*

- *Relative to the basis $\hat{\mathbf{x}}$ for V, the standard flag $\mathcal{F}^\bullet_{\mathrm{std}}[\hat{\mathbf{x}}]$ in V is represented by the matrix $\mathbf{1}_n \in \mathrm{GL}(n, \mathbb{K})$.*
- *Relative to the basis $\hat{\mathbf{x}}$ for V, the anti-standard flag $\mathcal{F}^\bullet_{\mathrm{astd}}[\hat{\mathbf{x}}]$ in V is represented by the matrix $\mathbf{w}_n \in \mathrm{GL}(n, \mathbb{K})$.*

With the basic notions of complete flags and standard matrices in place, we now formalize how complete flags arise naturally from collections of vector spaces and linear maps. Moreover, we introduce the notion of bases adapted to such collections, which provide concrete matrix realizations of the resulting flags.

Definition 4.18. *Let $n \geq 2$ be an integer, and let $\{V^{(i)}\}_{i=1}^n$ be a collection of vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} V^{(i)} = i$ for all $i \in [1, n]$. Let $\{\phi^{(i)} : V^{(i)} \to V^{(i+1)}\}_{i=1}^{n-1}$ and $\{\psi^{(i)} : V^{(i+1)} \to V^{(i)}\}_{i=1}^{n-1}$ be collections of injective and surjective linear maps, respectively. Then, we introduce the following definitions:*

(1) ***Type $\mathcal{I}$ Flag**: Consider the collection $\{\phi^{(i)}\}_{i=1}^{n-1}$. We associate to this data the filtration in $V^{(n)}$ given by*

$${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)}) := \{F^{(0)} \subset \cdots \subset F^{(n)}\}\,,$$

where

$$F^{(p)} := \begin{cases} 0, & \text{if } p = 0\,, \\ \mathrm{im}\big(\phi^{(n-1)} \circ \cdots \circ \phi^{(p)}\big), & \text{if } p \in [1, n-1]\,, \\ V^{(n)}, & \text{if } p = n\,. \end{cases} \tag{4.2}$$

We refer to ${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)})$ as the <u>type $\mathcal{I}$ flag</u> in $V^{(n)}$ associated with $\{\phi^{(i)}\}_{i=1}^{n-1}$.

(2) ***Type $\mathcal{K}$ Flag**: Consider the collection $\{\psi^{(i)}\}_{i=1}^{n-1}$. We associate to this data the filtration in $V^{(n)}$ given by*

$${}_{\mathcal{K}}\mathcal{F}^\bullet(\psi^{(1)}, \ldots, \psi^{(n-1)}) := \{F^{(0)} \subset \cdots \subset F^{(n)}\}\,,$$

where

$$F^{(p)} := \begin{cases} 0, & \text{if } p = 0\,, \\ \ker\big(\psi^{(n-p)} \circ \cdots \circ \psi^{(n-1)}\big), & \text{if } p \in [1, n-1]\,, \\ V^{(n)}, & \text{if } p = n\,. \end{cases} \tag{4.3}$$

We refer to ${}_{\mathcal{K}}\mathcal{F}^\bullet(\psi^{(1)}, \ldots, \psi^{(n-1)})$ as the <u>type $\mathcal{K}$ flag</u> in $V^{(n)}$ associated with $\{\psi^{(i)}\}_{i=1}^{n-1}$.

(3) ***Adapted System of Bases I****: Consider the collection* $\{\phi^{(i)}\}_{i=1}^{n-1}$*. For each* $i \in [1, n]$*, let* $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_k^{(i)}\}_{k=1}^{i}$ *be a basis for* $V^{(i)}$*, and denote by* $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ *the collection of such bases. We say that* $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ *is a* <u>*system of bases adapted*</u> *to* $\{\phi^{(i)}\}_{i=1}^{n-1}$ *if, for each* $i \in [1, n-1]$*, the matrix* ${}_{\hat{\mathbf{f}}^{(i+1)}}[\,\phi^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}} \in M(i+1, i, \mathbb{K})$ *representing* $\phi^{(i)}$ *is the standard inclusion matrix; namely,*

$$ {}_{\hat{\mathbf{f}}^{(i+1)}}[\,\phi^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)} \,. $$

(4) ***Adapted System of Bases II****: Consider the collection* $\{\psi^{(i)}\}_{i=1}^{n-1}$*. For each* $i \in [1, n]$*, let* $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_k^{(i)}\}_{k=1}^{i}$ *be a basis for* $V^{(i)}$*, and denote by* $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ *the collection of such bases. We say that* $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ *is a* <u>*system of bases adapted*</u> *to* $\{\psi^{(i)}\}_{i=1}^{n-1}$ *if, for each* $i \in [1, n-1]$*, the matrix* ${}_{\hat{\mathbf{f}}^{(i)}}[\,\psi^{(i)}\,]_{\hat{\mathbf{f}}^{(i+1)}} \in M(i, i+1, \mathbb{K})$ *representing* $\psi^{(i)}$ *is the standard projection matrix; specifically,*

$$ {}_{\hat{\mathbf{f}}^{(i)}}[\,\psi^{(i)}\,]_{\hat{\mathbf{f}}^{(i+1)}} = \pi^{(i,i+1)} \,. $$

Lemma 4.19. *Let* $n \geq 2$ *be an integer, and let* $\{V^{(i)}\}_{i=1}^{n}$ *be a collection of vector spaces over* $\mathbb{K}$ *such that* $\dim_{\mathbb{K}} V^{(i)} = i$ *for all* $i \in [1, n]$*. In addition, let* $\{\phi^{(i)} : V^{(i)} \to V^{(i+1)}\}_{i=1}^{n-1}$ *be a collection of injective linear maps. Then the type* $\mathcal{I}$ *flag* ${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)})$ *in* $V^{(n)}$ *associated with* $\{\phi^{(i)}\}_{i=1}^{n-1}$ *is a complete flag.*

Proof. By Definition (4.18)–(1), the filtration ${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)}) := \{F^{(0)} \subset \cdots \subset F^{(n)}\}$ in $V^{(n)}$ is given by

$$ F^{(p)} := \begin{cases} 0, & \text{if } \ p = 0 \,, \\ \operatorname{im}\big(\phi^{(n-1)} \circ \cdots \circ \phi^{(p)}\big), & \text{if } \ p \in [1, n-1] \,, \\ V^{(n)}, & \text{if } \ p = n \,. \end{cases} $$

In particular, observe that since each $\phi^{(i)}$ is injective, the composition $\phi^{(n-1)} \circ \cdots \circ \phi^{(p)} : V^{(p)} \to V^{(n)}$ is also injective for every $p \in [1, n-1]$. Consequently, by the rank-nullity theorem, we deduce that

$$ \dim_{\mathbb{K}} \operatorname{im}\big(\phi^{(n-1)} \circ \cdots \circ \phi^{(p)}\big) = p \,, $$

for all $p \in [1, n-1]$, which shows that ${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)})$ is a complete flag. $\square$

Lemma 4.20. *Let* $n \geq 2$ *be an integer, and let* $\{V^{(i)}\}_{i=1}^{n}$ *be a collection of vector spaces over* $\mathbb{K}$ *such that* $\dim_{\mathbb{K}} V^{(i)} = i$ *for all* $i \in [1, n]$*. In addition, let* $\{\psi^{(i)} : V^{(i+1)} \to V^{(i)}\}_{i=1}^{n-1}$ *be a collection of surjective linear maps. Then the type* $\mathcal{K}$ *flag* ${}_{\mathcal{K}}\mathcal{F}^\bullet(\psi^{(1)}, \ldots, \psi^{(n-1)})$ *in* $V^{(n)}$ *associated with* $\{\psi^{(i)}\}_{i=1}^{n-1}$ *is a complete flag.*

Proof. By Definition (4.18)–(2), the filtration ${}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)}) := \{F^{(0)} \subset \cdots \subset F^{(n)}\}$ in $V^{(n)}$ is given by

$$F^{(p)} := \begin{cases} 0, & \text{if } p = 0\,, \\ \ker\big(\psi^{(n-p)} \circ \cdots \circ \psi^{(n-1)}\big), & \text{if } p \in [1, n-1]\,, \\ V^{(n)}, & \text{if } p = n\,. \end{cases} \tag{4.4}$$

In addition, note that since each $\psi^{(i)}$ is surjective, the composition $\psi^{(n-p)} \circ \cdots \circ \psi^{(n-1)} : V^{(n)} \to V^{(n-p)}$ is also surjective for every $p \in [1, n-1]$. Consequently, by the rank-nullity theorem, we deduce that

$$\dim_{\mathbb{K}} \ker\big(\psi^{(n-p)} \circ \cdots \circ \psi^{(n-1)}\big) = p\,,$$

for all $p \in [1, n-1]$, which confirms that ${}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)})$ is a complete flag. □

Lemma 4.21. *Let $n \geq 2$ be an integer, and let $\{V^{(i)}\}_{i=1}^{n}$ be a collection of vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} V^{(i)} = i$ for all $i \in [1, n]$. Let $\{\psi^{(i)} : V^{(i+1)} \to V^{(i)}\}_{i=1}^{n-1}$ be a collection of surjective linear maps, and let $\{\phi^{(i)} : V^{(i)} \to V^{(i+1)}\}_{i=1}^{n-1}$ be a collection of injective linear maps satisfying:*

$$\psi^{(i)} \circ \phi^{(i)} = \mathrm{id}_{V^{(i)}}\,,$$

for each $i \in [1, n-1]$. In particular, denote by

$$\mathcal{F}_0^{\bullet} := {}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)})\,,$$

$$\mathcal{F}_1^{\bullet} := {}_{\mathcal{I}}\mathcal{F}^{\bullet}(\phi^{(1)},\ldots,\phi^{(n-1)})\,,$$

the type $\mathcal{K}$ and type $\mathcal{I}$ flags in $V^{(n)}$ associated with $\{\psi^{(i)}\}_{i=1}^{n-1}$ and $\{\phi^{(i)}\}_{i=1}^{n-1}$, respectively. Then, $\mathcal{F}_0^{\bullet}$ and $\mathcal{F}_1^{\bullet}$ are completely opposite flags.

Proof. By assumption, $\psi^{(i)} \circ \phi^{(i)} = \mathrm{id}_{V^{(i)}}$ for all $i \in [1, n-1]$. Thus, for each $i \in [1, n-1]$, the compositions $\psi^{(i)} \circ \cdots \circ \psi^{(n-1)} : V^{(n)} \to V^{(i)}$ and $\phi^{(n-1)} \circ \cdots \circ \phi^{(i)} : V^{(i)} \to V^{(n)}$ satisfy

$$\Big(\psi^{(i)} \circ \cdots \circ \psi^{(n-1)}\Big) \circ \Big(\phi^{(n-1)} \circ \cdots \circ \phi^{(i)}\Big) = \mathrm{id}_{V^{(i)}}\,.$$

Consequently, a direct application of Lemma (4.13) yields, for each $i \in [1, n-1]$, the direct sum decomposition

$$V^{(n)} = \ker\Big(\psi^{(i)} \circ \cdots \circ \psi^{(n-1)}\Big) \oplus \mathrm{im}\Big(\phi^{(n-1)} \circ \cdots \circ \phi^{(i)}\Big)\,. \tag{4.5}$$

Now, consider

$$\mathcal{F}_0^{\bullet} = {}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)}) = \{F_0^{(0)} \subset \cdots \subset F_0^{(n)}\}\,,$$

$$\mathcal{F}_1^{\bullet} = {}_{\mathcal{I}}\mathcal{F}^{\bullet}(\phi^{(1)},\ldots,\phi^{(n-1)}) = \{F_1^{(0)} \subset \cdots \subset F_1^{(n)}\}\,.$$

the type $\mathcal{K}$ and type $\mathcal{I}$ flags in $V^{(n)}$ associated with $\{\psi^{(i)}\}_{i=1}^{n-1}$ and $\{\phi^{(i)}\}_{i=1}^{n-1}$, respectively. By Definition (4.18)–(1)–(2), we have that

$$F_0^{(p)} := \begin{cases} 0, & \text{if } p = 0\,, \\ \ker\big(\psi^{(n-p)} \circ \cdots \circ \psi^{(n-1)}\big), & \text{if } p \in [1, n-1]\,, \\ V^{(n)}, & \text{if } p = n\,. \end{cases}$$

$$F_1^{(p)} := \begin{cases} 0, & \text{if } \; p = 0\,, \\ \operatorname{im}\big(\phi^{(n-1)} \circ \cdots \circ \phi^{(p)}\big), & \text{if } \; p \in [1, n-1]\,, \\ V^{(n)}, & \text{if } \; p = n\,. \end{cases}$$

Therefore, by the direct sum decompositions (4.5), we deduce that

$$V^{(n)} = F_0^{(p)} \oplus F_1^{(n-p)}\,,$$

for all $p \in [1, n-1]$, which shows that $\mathcal{F}_0^\bullet$ and $\mathcal{F}_1^\bullet$ are completely opposite flags, as desired. □

Lemma 4.22. *Let $n \geq 2$ be an integer, and let $\{V^{(i)}\}_{i=1}^{n}$ be a collection of vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} V^{(i)} = i$ for all $i \in [1, n]$. Let $\{\phi^{(i)} : V^{(i)} \to V^{(i+1)}\}_{i=1}^{n-1}$ be a collection of injective linear maps, and for each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_k^{(i)}\}_{k=1}^{i}$ be a basis for $V^{(i)}$ such that $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to $\{\phi^{(i)}\}_{i=1}^{n-1}$ (see Definition (4.18)–(3)). Then, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $V^{(n)}$, the type $\mathcal{I}$ flag ${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)})$ in $V^{(n)}$ associated with $\{\phi^{(i)}\}_{i=1}^{n-1}$ coincides with the standard flag, ${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)})\big[\,\hat{\mathbf{f}}^{(n)}\,\big] = \mathcal{F}_{\text{std}}^\bullet\big[\,\hat{\mathbf{f}}^{(n)}\,\big]$, and is therefore represented by the matrix $\mathbf{1}_n$.*

Proof. By Definition (4.18)–(1), the filtration ${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)}) := \{F^{(0)} \subset \cdots \subset F^{(n)}\}$ in $V^{(n)}$ is given by

$$F^{(p)} := \begin{cases} 0, & \text{if } \; p = 0\,, \\ \operatorname{im}\big(\phi^{(n-1)} \circ \cdots \circ \phi^{(p)}\big), & \text{if } \; p \in [1, n-1]\,, \\ V^{(n)}, & \text{if } \; p = n\,. \end{cases}$$

Now, since $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to $\{\phi^{(i)}\}_{i=1}^{n-1}$, Definition (4.18)–(3) asserts that for each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{f}}^{(i+1)}}\big[\,\phi^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i+1, i, \mathbb{K})$ representing $\phi^{(i)}$ is the standard inclusion matrix; that is, ${}_{\hat{\mathbf{f}}^{(i+1)}}\big[\,\phi^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)}$. As a result, we deduce that

$$F^{(p)} = \left\langle \hat{f}_1^{(n)}, \ldots, \hat{f}_p^{(n)} \right\rangle\,,$$

for all $p \in [1, n-1]$, which confirms that ${}_{\mathcal{I}}\mathcal{F}^\bullet(\phi^{(1)}, \ldots, \phi^{(n-1)})\big[\,\hat{\mathbf{f}}^{(n)}\,\big] = \mathcal{F}_{\text{std}}^\bullet\big[\,\hat{\mathbf{f}}^{(n)}\,\big]$, as claimed. □

Remark 4.23. *Let $n \geq 2$ be an integer, and let $\{V^{(i)}\}_{i=1}^{n}$ be a collection of vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} V^{(i)} = i$ for all $i \in [1,n]$. Let $\{\phi^{(i)} : V^{(i)} \to V^{(i+1)}\}_{i=1}^{n-1}$ be a collection of injective linear maps, and let $\hat{\mathbf{f}}^{(n)} := \{\hat{f}_k^{(n)}\}_{k=1}^{n}$ be a basis for $V^{(n)}$ such that the type $\mathcal{I}$ flag ${}_{\mathcal{I}}\mathcal{F}^{\bullet}(\phi^{(1)},\ldots,\phi^{(n-1)})$ in $V^{(n)}$ associated with $\{\phi^{(i)}\}_{i=1}^{n-1}$ coincides with the standard flag, ${}_{\mathcal{I}}\mathcal{F}^{\bullet}(\phi^{(1)},\ldots,\phi^{(n-1)})\big[\,\hat{\mathbf{f}}^{(n)}\,\big] = \mathcal{F}^{\bullet}_{\mathrm{std}}\big[\,\hat{\mathbf{f}}^{(n)}\,\big]$. Then, an inductive argument establishes that, for each $i \in [1,n-1]$, there is a unique basis $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_k^{(i)}\}_{k=1}^{i}$ for $V^{(i)}$ such that the collection of bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases for $\{V^{(i)}\}_{i=1}^{n}$ adapted to $\{\phi^{(i)}\}_{i=1}^{n-1}$ (see Definition (4.18)–(3)). In particular, this observation will play a key role in our discussion ahead.*

Lemma 4.24. *Let $n \geq 2$ be an integer, and let $\{V^{(i)}\}_{i=1}^{n}$ be a collection of vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} V^{(i)} = i$ for all $i \in [1,n]$. Let $\{\psi^{(i)} : V^{(i+1)} \to V^{(i)}\}_{i=1}^{n-1}$ be a collection of surjective linear maps, and for each $i \in [1,n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_k^{(i)}\}_{k=1}^{i}$ be a basis for $V^{(i)}$ such that $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to $\{\psi^{(i)}\}_{i=1}^{n-1}$ (see Definition (4.18)–(4)). Then, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $V^{(n)}$, the type $\mathcal{K}$ flag ${}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)})$ in $V^{(n)}$ associated with $\{\psi^{(i)}\}_{i=1}^{n-1}$ coincides with the anti-standard flag, ${}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)})\big[\,\hat{\mathbf{f}}^{(n)}\,\big] = \mathcal{F}^{\bullet}_{\mathrm{astd}}\big[\,\hat{\mathbf{f}}^{(n)}\,\big]$, and is therefore represented by the matrix $\mathbf{w}_n$.*

Proof. By Definition (4.18)–(2), the filtration ${}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)}) := \{F^{(0)} \subset \cdots \subset F^{(n)}\}$ in $V^{(n)}$ is given by

$$F^{(p)} := \begin{cases} 0, & \text{if } \ p = 0\,, \\ \ker\big(\psi^{(n-p)} \circ \cdots \circ \psi^{(n-1)}\big), & \text{if } \ p \in [1,n-1]\,, \\ V^{(n)}, & \text{if } \ p = n\,. \end{cases}$$

Now, since $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to $\{\psi^{(i)}\}_{i=1}^{n-1}$, Definition (4.18)–(4) asserts that for each $i \in [1,n-1]$, the matrix ${}_{\hat{\mathbf{f}}^{(i)}}\big[\,\psi^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i+1)}} \in M(i,i+1,\mathbb{K})$ representing $\psi^{(i)}$ is the standard projection matrix; that is, ${}_{\hat{\mathbf{f}}^{(i)}}\big[\,\psi^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i+1)}} = \pi^{(i,i+1)}$. Consequently, we obtain that

$$F^{(p)} = \left\langle \hat{f}_n^{(n)},\ldots,\hat{f}_{n-p+1}^{(n)} \right\rangle\,,$$

for all $p \in [1,n-1]$, which establishes that ${}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)})\big[\,\hat{\mathbf{f}}^{(n)}\,\big] = \mathcal{F}^{\bullet}_{\mathrm{std}}\big[\,\hat{\mathbf{f}}^{(n)}\,\big]$, as desired. ☐

Remark 4.25. *Let $n \geq 2$ be an integer, and let $\{V^{(i)}\}_{i=1}^{n}$ be a collection of vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} V^{(i)} = i$ for all $i \in [1,n]$. Let $\{\psi^{(i)} : V^{(i+1)} \to V^{(i)}\}_{i=1}^{n-1}$ be a collection of surjective linear maps, and let $\hat{\mathbf{f}}^{(n)} := \{\hat{f}_k^{(n)}\}_{k=1}^{n}$ be a basis for $V^{(n)}$ such that the type $\mathcal{K}$ flag ${}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)})$ in $V^{(n)}$ associated with $\{\psi^{(i)}\}_{i=1}^{n-1}$ coincides with the anti-standard flag, ${}_{\mathcal{K}}\mathcal{F}^{\bullet}(\psi^{(1)},\ldots,\psi^{(n-1)})\big[\,\hat{\mathbf{f}}^{(n)}\,\big] = \mathcal{F}^{\bullet}_{\mathrm{astd}}\big[\,\hat{\mathbf{f}}^{(n)}\,\big]$. Then, an inductive argument shows that, for each $i \in [1,n-1]$, there is a unique basis $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_k^{(i)}\}_{k=1}^{i}$ for $V^{(i)}$ such that the collection of bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases for $\{V^{(i)}\}_{i=1}^{n}$ adapted to $\{\psi^{(i)}\}_{i=1}^{n-1}$ (see Definition (4.18)–(4)). In particular, this observation will play a key role in our discussion ahead.*

Before proceeding further, we state a lemma that describes how the matrix representation of a complete flag in a vector space transforms under a change of basis.

Lemma 4.26. *Let V be a finite-dimensional vector space over $\mathbb{K}$ with $\dim_{\mathbb{K}} V = n$ for some integer $n \geq 1$, and let $\hat{\mathbf{x}} := \big\{\hat{x}_i\big\}_{i=1}^{n}$ and $\hat{\mathbf{y}} := \big\{\hat{y}_j\big\}_{j=1}^{n}$ be bases for V related via the change-of-basis matrix $M \in \mathrm{GL}(n, \mathbb{K})$; that is, $\hat{y}_j = \sum_{i=1}^{n} M_{i,j}\, \hat{x}_i$ for each $j \in [1, n]$. Furthermore, let $\mathcal{F}^{\bullet} := \big\{F^0 \subset \cdots \subset F^n\big\}$ be a complete flag in V, and suppose that relative to the basis $\hat{\mathbf{y}}$ for V, the flag is represented by the matrix $A \in \mathrm{GL}(n, \mathbb{K})$. Then, relative to the basis $\hat{\mathbf{x}}$ for V, the complete flag $\mathcal{F}^{\bullet}$ is represented by the matrix product $M \cdot A \in \mathrm{GL}(n, \mathbb{K})$.*

Proof. To begin with, let $\big\{\vec{v}_k\big\}_{k=1}^{n}$ be the collection of vectors in V given by $\vec{v}_k = \sum_{j=1}^{n} A_{j,k}\, \hat{y}_j$ for each $k \in [1, n]$. Thus, since relative to the basis $\hat{\mathbf{y}}$ for V, the flag $\mathcal{F}^{\bullet}$ is represented by the matrix A, we have that

$$F^{(p)} = \begin{cases} 0\,, & \text{if } p = 0\,, \\ \big\langle \vec{v}_1, \ldots, \vec{v}_p \big\rangle\,, & \text{if } p = 1, \ldots, n\,. \end{cases}$$

In addition, recall that $\hat{y}_j = \sum_{i=1}^{n} M_{i,j}\, \hat{x}_i$ for each $j \in [1, n]$. As a result, we obtain that

$$\begin{aligned} \vec{v}_k &= \sum_{j=1}^{n} \sum_{i=1}^{n} A_{j,k} M_{i,j}\, \hat{x}_i\,, \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} M_{i,j} A_{j,k} \right) \hat{x}_i\,, \\ &= \sum_{i=1}^{n} \big(M \cdot A\big)_{i,k}\, \hat{x}_i\,, \end{aligned}$$

for each $k \in [1, n]$. Bearing this in mind, we conclude that, relative to the basis $\hat{\mathbf{x}}$ for V, the flag $\mathcal{F}^{\bullet}$ is represented by the matrix product $M \cdot A$. This completes the proof. $\square$

We now introduce a collection of matrices that parametrize complete flags in relative position, the so-called braid matrices. These matrices are of central importance, as they will play a key role in the algebraic description of the objects of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Definition 4.27 (Braid Matrices)**.** *Let $n \geq 2$ be an integer. Given an Artin generator $\sigma_k \in \mathrm{Br}_n^{+}$, with $k \in [1, n-1]$, and a parameter $z \in \mathbb{K}$, the n-dimensional braid matrix $B_k^{(n)}(z) \in \mathrm{GL}\big(n, \mathbb{K}\big)$ associated with σ_k and z is defined by*

$$\Big[B_k^{(n)}(z) \Big]_{i,j} := \begin{cases} 1\,, & \textit{if } \ i = j \ \textit{ and } \ i \neq k,\ k+1\,, \\ 1\,, & \textit{if } \ (i,j) = (k, k+1) \ \textit{ or } \ (k+1, k)\,, \\ z\,, & \textit{if } \ i = j = k\,, \\ 0\,, & \textit{otherwise}\,, \end{cases} \qquad i, j \in [1, n]\,.$$

More precisely,

$$B_k^{(n)}(z) := \begin{array}{cl}
\begin{array}{ccc} \quad\quad k \quad\quad & \quad 2 \quad & n-(k+1) \end{array} & \\
\left[\begin{array}{ccc|cc|ccc}
1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \hline
0 & \cdots & 0 & z & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \hline
0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1
\end{array}\right] & \begin{array}{l} \\ k \\ \\ \\ 2 \\ \\ \\ n-(k+1) \\ \\ \end{array}
\end{array}$$

In this manuscript, we refer to $B_k^{(n)}(z)$ *as the* k*-th braid matrix of dimension* n *associated with* σ_k *and* z*.*

The braid matrices have been extensively studied and employed in the literature (see, for instance, [KÓ6, CGGS24, CGGS21, CGGS24, GSW24, CN22, CNS19]). In particular, in Appendix A, we list some of their properties, focusing on those most relevant for this manuscript.

Lemma 4.28. *Let* V *be a finite-dimensional vector space over* $\mathbb{K}$ *with* $\dim_{\mathbb{K}} V = n$ *for some integer* $n \geq 1$*. Let* $\mathcal{F}^\bullet = \{F^{(0)} \subset \cdots \subset F^{(n)}\}$ *and* $\mathcal{G}^\bullet = \{G^{(0)} \subset \cdots \subset G^{(n)}\}$ *be a pair of complete flags in* V *such that* $\mathcal{G}^\bullet$ *is in* s_k*-relative position with respect to* $\mathcal{F}^\bullet$ *for some* $k \in [1, n-1]$*. Furthermore, let* $\hat{\mathbf{x}} := \{\hat{x}_i\}_{i=1}^n$ *be a basis for* V*, and suppose that relative to the basis* $\hat{\mathbf{x}}$*, the flag* $\mathcal{F}^\bullet$ *is represented by the matrix* $A \in \mathrm{GL}(n, \mathbb{K})$*. Then there exists* $z \in \mathbb{K}$ *such that, relative to the basis* $\hat{\mathbf{x}}$*, the flag* $\mathcal{G}^\bullet$ *is represented by the matrix product* $A \cdot B_k^{(n)}(z)$*, where* $B_k^{(n)}(z) \in \mathrm{GL}(n, \mathbb{K})$ *denotes a* k*-th braid matrix of dimension* n*.*

Proof. Let $\{\vec{v}_j\}_{j=1}^n$ be the collection of vectors in V given by $\vec{v}_j = \sum_{i=1}^n A_{i,j}\, \hat{x}_i$ for each $j \in [1, n]$. By hypothesis, we know that, relative to the basis $\hat{\mathbf{x}}$, the flag $\mathcal{F}^\bullet$ is represented by the matrix A. In other words, we have that

$$F^{(p)} = \begin{cases} 0\,, & \text{if } p = 0\,, \\ \langle \vec{v}_1, \ldots, \vec{v}_p \rangle\,, & \text{if } p = 1, \ldots, n\,. \end{cases}$$

Moreover, recall that the flag $\mathcal{G}^\bullet$ is in s_k-relative position with respect to $\mathcal{F}^\bullet$. To be more precise, we know that $F^{(k)} \neq G^{(k)}$ and $F^{(p)} = G^{(p)}$ for all $p \neq k$, where $p \in [0, n]$. Here, observe that for each $p \in [1, n]$, the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis for $F^{(p)}$. Thus, since $F^{(k-1)} \subset G^{(k)}$, we can extend the basis $\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$ for $F^{(k-1)}$ to a basis $\{\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{g}\}$ for $G^{(k)}$, for some non-zero vector $\vec{g} \in G^{(k)}$. Moreover, since $\{\vec{v}_1, \ldots, \vec{v}_{k+1}\}$ spans $F^{(k+1)}$ and $G^{(k)} \subset F^{(k+1)}$, we can write $\vec{g} = \alpha_1\, \vec{v}_1 + \cdots + \alpha_{k+1}\, \vec{v}_{k+1}$ for some $\alpha_j \in \mathbb{K}$, where $j \in [1, k+1]$.

In particular, since $G^{(k)} \neq F^{(k)}$, we know that $\vec{g}$ does not belong to $F^{(k)}$, and therefore $\alpha_{k+1} \neq 0$. Thus, we obtain that $\vec{g} = \alpha_1 \vec{v}_1 + \cdots + \alpha_{k-1} \vec{v}_{k-1} + \alpha_{k+1}\big(\vec{v}_{k+1} + z\,\vec{v}_k\big)$, where $z := \alpha_{k+1}^{-1}\alpha_k \in \mathbb{K}$. As a result, we can see that the set $\big\{\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1} + z\,\vec{v}_k\big\}$ spans $G^{(k)}$. Bearing this in mind, we conclude that, there is $z \in \mathbb{K}$ such that, relative to the basis $\hat{\mathbf{x}}$ for V, the flag $\mathcal{G}^\bullet$ is represented by the matrix whose column vectors are given by $\big[\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1} + z\,\vec{v}_k, \vec{v}_k, \vec{v}_{k+2}, \ldots, \vec{v}_n\big]$. Finally, in light of Claim (7.111), we readily verify that

$$A \cdot B_k^{(n)}(z) = \big[\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1} + z\,\vec{v}_k, \vec{v}_k, \vec{v}_{k+2}, \ldots, \vec{v}_n\big]\,,$$

where $B_k^{(n)}(z) \in \mathrm{GL}(n, \mathbb{K})$ denotes a k-th braid matrix of dimension n. Therefore, we deduce that, relative to the basis $\hat{\mathbf{x}}$, the flag $\mathcal{G}^\bullet$ is represented by the matrix product $A \cdot B_k^{(n)}(z)$, for some $z \in \mathbb{K}$. $\square$

Finally, we introduce the notion of braid-transformed bases, which captures how a positive braid word $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$, together with a tuple $\vec{z} \in \mathbb{K}^\ell_{\mathrm{std}}$, acts on a collection of bases for a family of vector spaces and produces a new set of bases for the corresponding family.

Definition 4.29 (Braid-Transformed Bases). *Let $n \geq 2$ be an integer, and let $\big\{V^{(i)}\big\}_{i=1}^n$ be a collection of vector spaces over $\mathbb{K}$ such that $\dim_{\mathbb{K}} V^{(i)} = i$ for all $i \in [1, n]$. For each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \big\{\hat{f}_k^{(i)}\big\}_{k=1}^i$ be basis for $V^{(i)}$, and denote by $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^n$ the collection of such bases.*

Let $\sigma_p \in \mathrm{Br}_n^+$ be the p-th Artin generator for some $p \in [1, n-1]$, and let $z \in \mathbb{K}$ be a fixed parameter. For each $i \in [1, n]$, we define a new basis for $V^{(i)}$, $\hat{\mathbf{f}}^{(i)}[\sigma_p, z] := \big\{\hat{f}_k^{(i)}[\sigma_p, z]\big\}_{k=1}^i$, as follows: for each $k \in [1, i]$, we set

$$\hat{f}_k^{(i)}[\sigma_p, z] := \begin{cases} \hat{f}_k^{(i)}\,, & \text{if } \ i \in [1, p]\,, \\ \sum_{j=1}^i \big(B_p^{(i)}(z)\big)_{j,k}\, \hat{f}_j^{(i)}\,, & \text{if } \ i \in [p+1, n]\,, \end{cases}$$

where $B_p^{(i)}(z)$ denotes the p-th braid matrix of dimension i associated with σ_p and z. In other words, for each $i \in [1, n]$, the new basis for $V^{(i)}$ is obtained from the old basis via the change-of-basis matrix $B_p^{(i)}(z)$ if $i \geq p+1$; otherwise, the new and old bases agree.

Accordingly, let $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\vec{z} = (z_1, \ldots, z_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$ be a fixed tuple. Then, we define the braid-transformed bases $\big\{\,\hat{\mathbf{f}}^{(i)}[\,\beta, \vec{z}\,]\,\big\}_{i=1}^n$, a new collection of bases for the family of vector spaces $\big\{V^{(i)}\big\}_{i=1}^n$, by recursively applying, in order, the single-step transformations corresponding to each Artin generator in β and the associated entry of $\vec{z}$ to the old bases $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^n$, namely:

$$\hat{\mathbf{f}}^{(i)}[\,\beta, \vec{z}\,] := \Big(\cdots\Big(\hat{\mathbf{f}}^{(i)}[\sigma_{i_1}, z_1]\Big)\cdots\Big)[\sigma_{i_\ell}, z_\ell]\,, \quad i \in [1, n].$$

Having established the necessary technical background, we now turn to the explicit characterization of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ in the case $\beta = e_n$.

4.2. **The Case of the Trivial Braid.** Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word. In this subsection, we study the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n), \mathbb{K})_0)$. In particular, this concrete example, corresponding to the Legendrian unlink $\Lambda(e_n) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ on n strands, will serve as a prototype for the general theory developed later in this section.

Let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n), \mathbb{K})_0)$. Next, we construct an open cover of $\mathbb{R}^2$ adapted to the front projection $\Pi_{x,z}(\Lambda(e_n)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(e_n) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$. The purpose of this construction is to decompose the plane into manageable regions where the local behavior of $\mathscr{F}$ can be analyzed independently and later integrated into a concise global description.

Construction 4.30. *Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word. Given the front projection $\Pi_{x,z}(\Lambda(e_n)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(e_n) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, we denote by $\mathcal{U}_{\Lambda(e_n)} := \{U_0, U_{\mathrm{B}}, U_{\mathrm{T}}\}$ the finite open cover of $\mathbb{R}^2$ illustrated in Figure (8). Specifically, we assume that:*

- *U_0 is an unbounded open subset of $\mathbb{R}^2$ such that $U_0 \cap \Pi_{x,z}(\Lambda(e_n)) = \emptyset$. In Figure (8), the region U_0 is depicted in purple.*
- *U_{B} is a bounded open subset of $\mathbb{R}^2$ such that the intersection $U_{\mathrm{B}} \cap \Pi_{x,z}(\Lambda(e_n))$ consists of the n strands at the bottom of the front projection $\Pi_{x,z}(\Lambda(e_n))$. In Figure (8), the region U_{B} is shown in pink.*
- *U_{T} is a bounded open subset of $\mathbb{R}^2$ such that the intersection $U_{\mathrm{T}} \cap \Pi_{x,z}(\Lambda(e_n))$ consists the n strands at the top of the front projection $\Pi_{x,z}(\Lambda(e_n))$. In Figure (8), the region U_{T} is illustrated in green.*

Next, by applying the sheaf axioms to the open cover $\mathcal{U}_{\Lambda(e_n)}$ of $\mathbb{R}^2$ introduced in Construction (4.30), we provide a global characterization of $\mathscr{F}$ in terms of its local data and the associated gluing conditions.

Proposition 4.31. *Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word, $\mathscr{F}$ an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n), \mathbb{K})_0)$, and $\mathcal{U}_{\Lambda(e_n)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ introduced in Construction (4.30). Then, the following statements hold:*

- *On U_0, $\mathscr{F}$ is identically zero.*
- *On U_{T}, $\mathscr{F}$ is specified by a collection of $n-1$ surjective linear maps $\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}$, as illustrated in Figure (9).*
- *On U_{B}, $\mathscr{F}$ is specified by a collection of $n-1$ injective linear maps $\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$, as illustrated in Figure (10).*

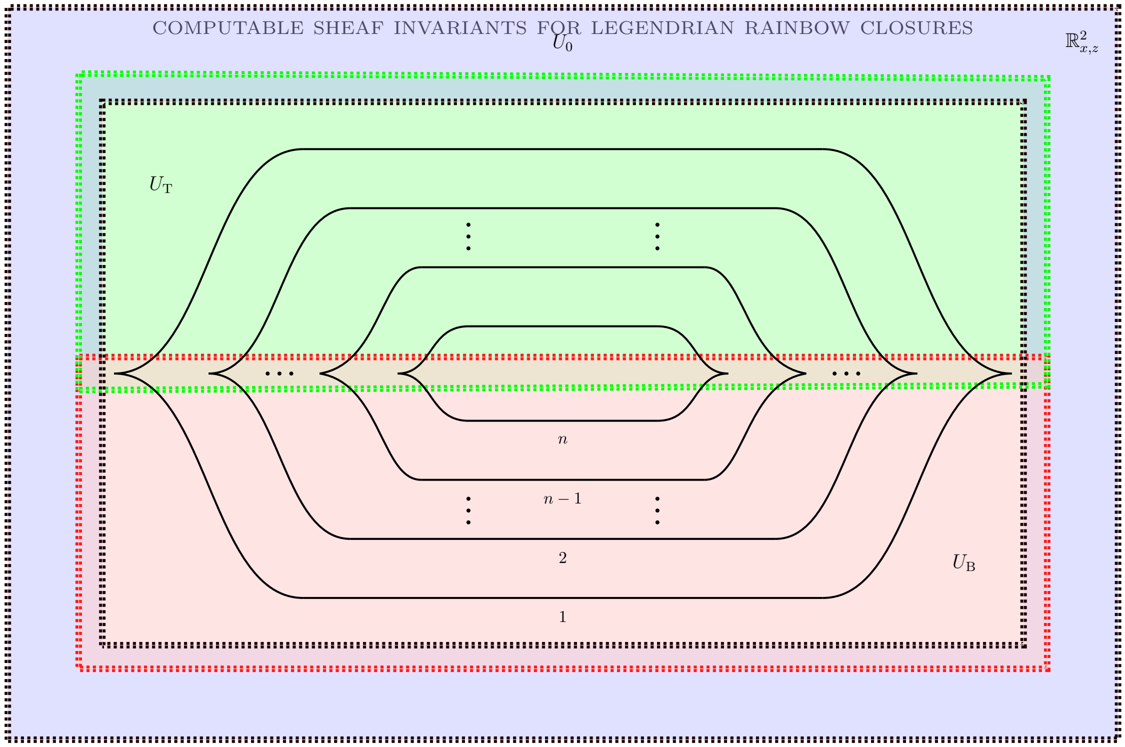


FIGURE 8. The front projection $\Pi_{x,z}(\Lambda(e_n)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(e_n) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, along with the finite open cover $\mathcal{U}_{\Lambda(e_n)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{T}}\}$ of $\mathbb{R}^2$. The regions U_0, U_{B}, and U_{T} are depicted in purple, pink, and green, respectively.

- ***Compatibility conditions***: *For each $i \in [1, n-1]$,*

$$\psi^{(i)}_{\mathscr{F}} \circ \phi^{(i)}_{\mathscr{F}} = \mathrm{id}_{\mathbb{K}^i} .$$

Proof. To begin, consider the front projection $\Pi_{x,z}(\Lambda(e_n)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(e_n) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, which is depicted in Figure (3). In particular, recall that in $\Pi_{x,z}(\Lambda(e_n))$, the strands at the bottom have Maslov potential 0 and the strands at the top have Maslov potential 1. Thus, by the microlocal rank conditions and the microlocal support conditions near the arcs, we obtain that:

- For all $q \in U_0$, the stalk of $\mathscr{F}$ at q is trivial; that is, $\mathscr{F}_q = 0$. In other words, $\mathscr{F}$ is identically zero on U_0.
- On U_{T}, $\mathscr{F}$ is defined by a collections of $n-1$ linear maps $\left\{\psi^{(i)}_{\mathscr{F}} : \mathbb{K}^{i+1} \to \mathbb{K}^i\right\}_{i=1}^{n-1}$, as shown in Figure (9).
- On U_{B}, $\mathscr{F}$ is defined by a collections of $n-1$ linear maps $\left\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1}$, as shown in Figure (10).

Now, note that the intersection $U_{\mathrm{B}} \cap U_{\mathrm{T}} \cap \Pi_{x,z}(\Lambda(e_n))$ consists of the n left cusps and the n right cusps in $\Pi_{x,z}(\Lambda(e_n))$. Thus, on $U_{\mathrm{B}} \cap U_{\mathrm{T}}$, $\mathscr{F}$ is described by the diagram in Figure (11). Bearing this in mind, the microlocal support conditions near the cusps establish that:

$$\psi^{(i)}_{\mathscr{F}} \circ \phi^{(i)}_{\mathscr{F}} = \mathrm{id}_{\mathbb{K}^i} , \tag{4.6}$$

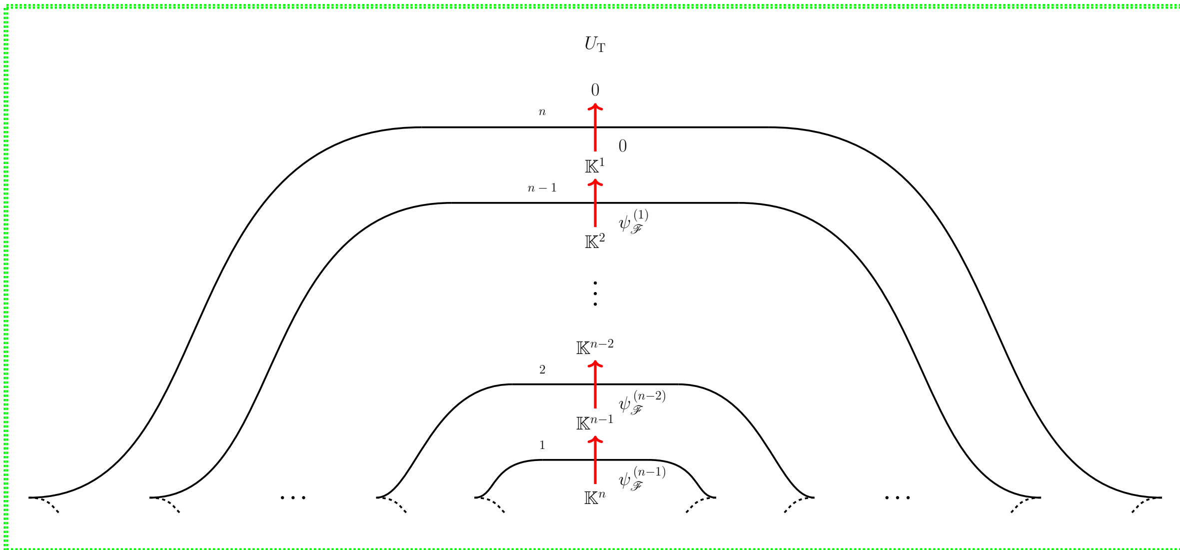


FIGURE 9. An object $\mathscr{F}$ of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(e_n), \mathbb{K})_0)$ on the region U_{T}.

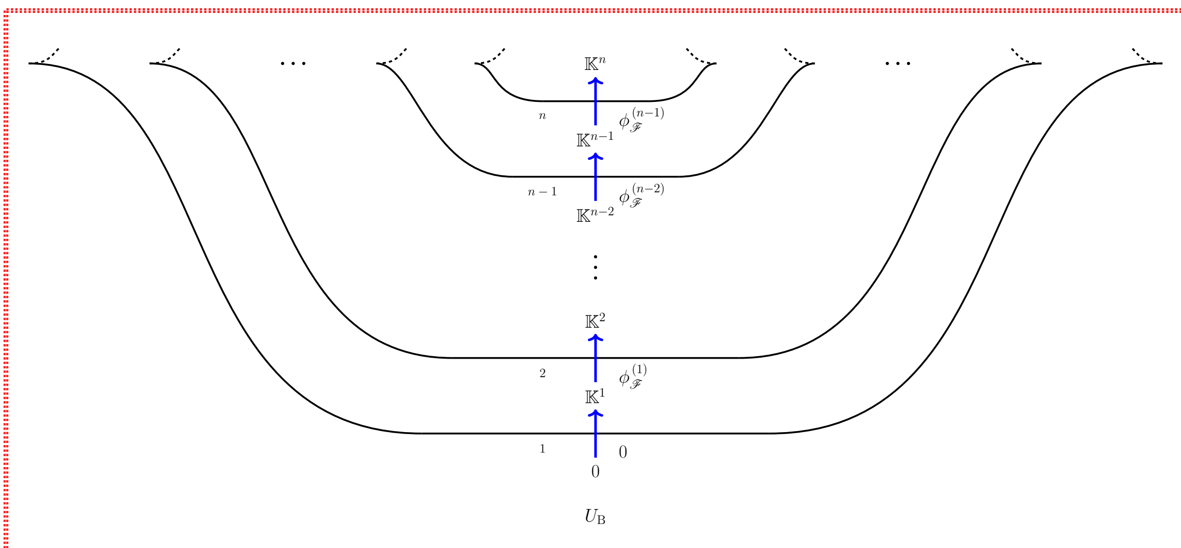


FIGURE 10. An object $\mathscr{F}$ of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(e_n), \mathbb{K})_0)$ on the region U_{B}.

for each $i \in [1, n-1]$. Consequently, a direct application of Lemma (4.13) yields that the maps $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are surjective, whereas the maps $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are injective. This concludes the proof.

□

Finally, we formalize how the description of $\mathscr{F}$ in terms of linear maps, as established in Proposition (4.31), naturally gives rise to a geometric characterization via complete flags in $\mathbb{K}^n$.

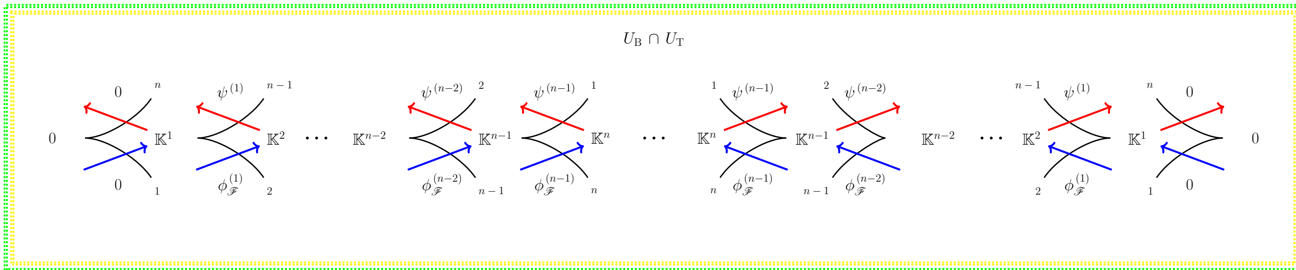


FIGURE 11. An object $\mathscr{F}$ of the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$ on the intersection $U_{\mathrm{B}} \cap U_{\mathrm{T}}$.

Proposition 4.32. *Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word, $\mathscr{F}$ an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$, and $\mathcal{U}_{\Lambda(e_n)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ introduced in Construction (4.30). Then, the following statements hold:*

- *On U_0, $\mathscr{F}$ is identically zero.*
- *On U_{T}, $\mathscr{F}$ is characterized by a complete flag $\mathcal{F}_0^\bullet$ in $\mathbb{K}^n$.*
- *On U_{B}, $\mathscr{F}$ is characterized by a complete flag $\mathcal{F}_1^\bullet$ in $\mathbb{K}^n$.*
- ***Compatibility condition**: $\mathcal{F}_0^\bullet$ and $\mathcal{F}_1^\bullet$ are completely opposite.*

Proof. By Proposition (4.31), we know that:

- On U_0, $\mathscr{F}$ is identically zero.
- On U_{T}, $\mathscr{F}$ is specified by a collection of $n-1$ surjective linear maps $\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}$.
- On U_{B}, $\mathscr{F}$ is specified by a collection of $n-1$ injective linear maps $\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$.
- **Compatibility conditions**: For each $i \in [1, n-1]$,

$$\psi_{\mathscr{F}}^{(i)} \circ \phi_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i} .$$

To verify the result, it suffices to show that the above data determines a pair of completely opposite flags $\mathcal{F}_0^\bullet$ and $\mathcal{F}_1^\bullet$ in $\mathbb{K}^n$, thereby providing a geometric description of $\mathscr{F}$. To this end, relying on the notions of type $\mathcal{I}$ and type $\mathcal{K}$ flags introduced in Definition (4.18)–(1)–(2), we proceed as follows:

(1) Since the maps $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are surjective, we assign to $\mathscr{F}$ on U_{T} the complete flag $\mathcal{F}_0^\bullet$ in $\mathbb{K}^n$ given by

$$\mathcal{F}_0^\bullet := {}_{\mathcal{K}}\mathcal{F}^\bullet\big(\psi_{\mathscr{F}}^{(1)}, \ldots, \psi_{\mathscr{F}}^{(n-1)}\big) .$$

(2) Since the maps $\{\phi^{(i)}_{\mathscr{F}}\}_{i=1}^{n-1}$ are injective, we assign to $\mathscr{F}$ on U_{B} the complete flag $\mathcal{F}^\bullet_1$ in $\mathbb{K}^n$ given by

$$\mathcal{F}^\bullet_1 := {}_{\mathcal{I}}\mathcal{F}^\bullet\big(\,\phi^{(1)}_{\mathscr{F}}, \dots, \phi^{(n-1)}_{\mathscr{F}}\,\big)\,.$$

Finally, observe that the compatibility conditions on the maps $\{\psi^{(i)}_{\mathscr{F}}\}_{i=1}^{n-1}$ and $\{\phi^{(i)}_{\mathscr{F}}\}_{i=1}^{n-1}$ allow a direct application of Lemma (4.21), which shows that $\mathcal{F}^\bullet_0$ and $\mathcal{F}^\bullet_1$ are completely opposite, as desired. □

With the above results at hand, we consider the analysis of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$ complete. Next, we extend our discussion to general positive braid words $\beta \in \mathrm{Br}^+_n$, aiming to establish a geometric characterization of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.

4.3. **A Geometric Characterization of the Objects for General Positive Braids.** Let $\beta := \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}^+_n$ be a positive braid word. In this subsection, we focus our attention on the study of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. In particular, our goal is to describe these objects geometrically, thereby extending the approach previously developed for the Legendrian unlink $\Lambda(e_n) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ on n strands. Following [STZ17], we provide a characterization of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ using configurations of complete flags in $\mathbb{K}^n$ subject to certain compatibility conditions determined by the singularities in the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$.

Let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Next, we construct an open cover of $\mathbb{R}^2$ adapted to the front projection $\Pi_{x,z}(\Lambda(\beta))$. The purpose of this construction is to decompose the plane into simple regions where the local behavior of $\mathscr{F}$ can be examined independently and subsequently assembled into a concise global description.

Construction 4.33. *Let $\beta := \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}^+_n$ be a positive braid word. Given the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, we introduce $\mathcal{U}_{\Lambda(\beta)} := \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ to denote the finite open cover of $\mathbb{R}^2$ illustrated in Figure (12). Specifically, we assume that:*

- *U_0 is an unbounded open subset of $\mathbb{R}^2$ such that $U_0 \cap \Pi_{x,z}(\Lambda(\beta)) = \emptyset$. In Figure (12), the region U_0 is depicted in purple.*
- *U_{B} is a bounded open subset of $\mathbb{R}^2$ such that the intersection $U_{\mathrm{B}} \cap \Pi_{x,z}(\Lambda(\beta))$ comprises the braid diagram of β. In Figure (12), the region U_{B} is illustrated in pink.*
- *U_{L} is a bounded open subset of $\mathbb{R}^2$ such that the intersection $U_{\mathrm{L}} \cap \Pi_{x,z}(\Lambda(\beta))$ consists of the n arcs on the left side of the braid diagram of β. In Figure (12), the region U_{L} is shown in yellow.*
- *U_{R} is a bounded open subset of $\mathbb{R}^2$ such that the intersection $U_{\mathrm{R}} \cap \Pi_{x,z}(\Lambda(\beta))$ consists of the n arcs on the right side of the braid diagram of β. In Figure (12), the region U_{R} is also depicted in yellow.*

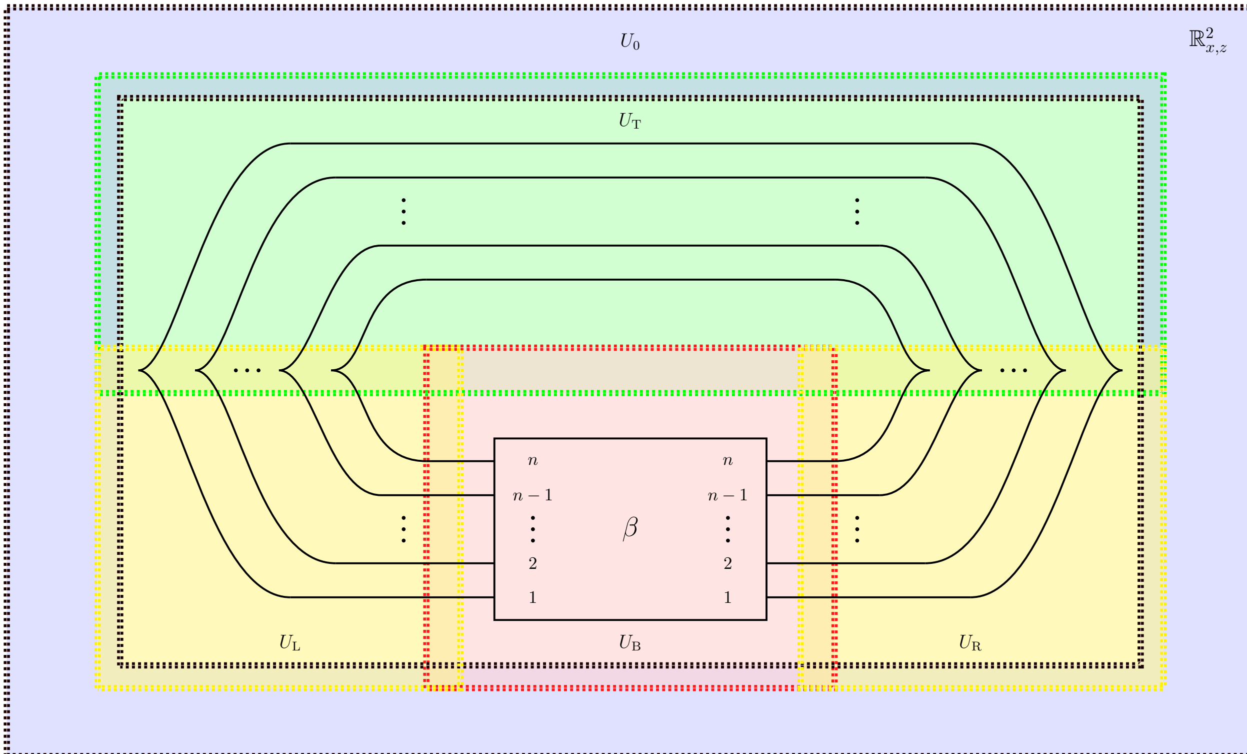


FIGURE 12. The front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, along with the finite open cover $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ of $\mathbb{R}^2$. The regions U_{L} and U_{R} are shown in yellow, while U_0, U_{B}, and U_{T} are depicted in purple, pink, and green, respectively.

- *U_{T} is a bounded open subset of $\mathbb{R}^2$ such that the intersection $U_{\mathrm{T}} \cap \Pi_{x,z}(\Lambda(\beta))$ consists of the rainbow on n strands that closes the braid diagram of β in $\mathbb{R}^2$ to form the front projection $\Pi_{x,z}(\Lambda(\beta))$. In Figure (12), the region U_{T} is illustrated in green.*

Next, by applying the sheaf axioms to the open cover $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ of $\mathbb{R}^2$ introduced in Construction (4.33), we provide a global description of $\mathscr{F}$ in terms of its local data and the associated gluing conditions. Among the subsets of the cover $\mathcal{U}_{\Lambda(\beta)}$, the region U_{B} is particularly important, as it contains all the crossings in the front projection $\Pi_{x,z}(\Lambda(\beta))$, and consequently, the behavior of $\mathscr{F}$ on U_{B} is more intricate and warrants a detailed analysis. We therefore begin by analyzing $\mathscr{F}$ on the remaining subsets, postponing the study of its behavior on U_{B} until afterward. With this strategy in place, we now present the following result.

Lemma 4.34. *Let $\beta := \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathscr{F}$ an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ introduced in Construction (4.33). Then, the following statements hold:*

- *On U_0, $\mathscr{F}$ is identically zero.*

- *On* U_{T}*,* $\mathscr{F}$ *is specified by a collection of* $n-1$ *surjective linear maps* $\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}$*, as illustrated in Figure* (15)*.*

- *On* U_{L}*,* $\mathscr{F}$ *is specified by a collection of* $n-1$ *injective linear maps* $\{\alpha_{\mathscr{F}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$*, as illustrated in Figure* (13)*.*

- *On* U_{R}*,* $\mathscr{F}$ *is specified by a collection of* $n-1$ *injective linear maps* $\{\beta_{\mathscr{F}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$*, as illustrated in Figure* (14)*.*

- ***Compatibility conditions****: For each* $i \in [1, n-1]$*,*

$$\psi_{\mathscr{F}}^{(i)} \circ \alpha_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}\,, \quad \textit{and} \quad \psi_{\mathscr{F}}^{(i)} \circ \beta_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}\,.$$

Proof. To begin, consider the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, which is depicted in Figure (1). In particular, recall that in $\Pi_{x,z}(\Lambda(\beta))$, the strands at bottom have Maslov potential 0 and the strands at the top have Maslov potential 1. Thus, by the microlocal rank conditions and the microlocal support conditions near the arcs, we obtain that:

- For all $q \in U_0$, the stalk of $\mathscr{F}$ at q is trivial; that is, $\mathscr{F}_q = 0$. In other words, $\mathscr{F}$ is identically zero on U_0.
- On U_{T}, $\mathscr{F}$ is defined by a collection of $n-1$ linear maps $\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}$, as shown in Figure (15).
- On U_{L}, $\mathscr{F}$ is defined by a collection of $n-1$ linear maps $\{\alpha_{\mathscr{F}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$, as shown in Figure (13).
- On U_{R}, $\mathscr{F}$ is defined by a collection of $n-1$ linear maps $\{\beta_{\mathscr{F}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$, as shown in Figure (14).

Now, note that the intersection $U_{\mathrm{L}} \cap U_{\mathrm{T}} \cap \Pi_{x,z}(\Lambda(\beta))$ consists of the n left cusps in $\Pi_{x,z}(\Lambda(\beta))$. Thus, on $U_{\mathrm{L}} \cap U_{\mathrm{T}}$, $\mathscr{F}$ is described by the diagram in Figure (16). Similarly, the intersection $U_{\mathrm{R}} \cap U_{\mathrm{T}} \cap \Pi_{x,z}(\Lambda(\beta))$ consists of the n right cusps in $\Pi_{x,z}(\Lambda(\beta))$, and hence, on $U_{\mathrm{R}} \cap U_{\mathrm{T}}$, $\mathscr{F}$ is described by the diagram in Figure (17). Bearing this in mind, the microlocal support conditions near the cusps establish that:

$$\psi_{\mathscr{F}}^{(i)} \circ \alpha_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}\,, \quad \text{and} \quad \psi_{\mathscr{F}}^{(i)} \circ \beta_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}\,, \tag{4.7}$$

for each $i \in [1, n-1]$. Consequently, a straightforward application of Lemma (4.13) yields that the maps $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are surjective, whereas the maps $\{\alpha_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\beta_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are injective. This completes the proof. $\square$

Next, we address how the description of $\mathscr{F}$ on the regions U_{T}, U_{L}, and U_{R} in terms of linear maps, as specified in Lemma (4.34), naturally leads to a geometric characterization via complete flags in $\mathbb{K}^n$.

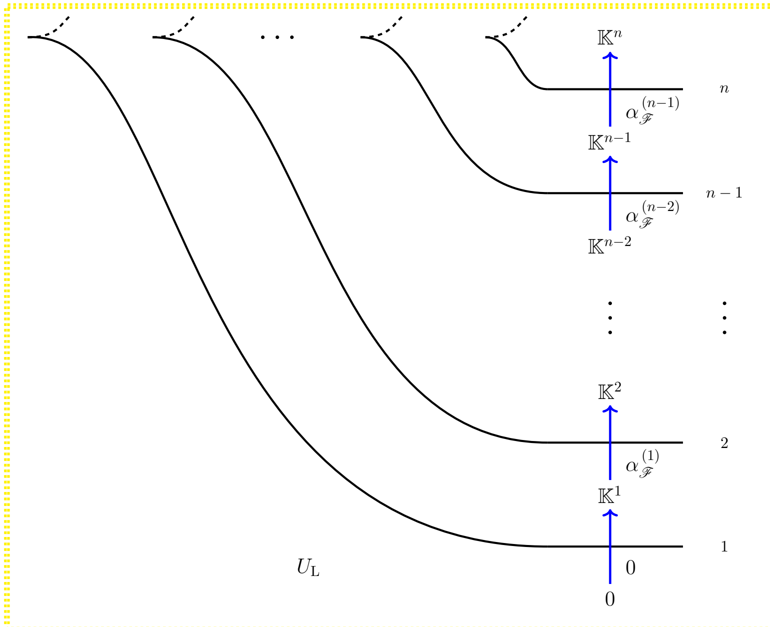


FIGURE 13. An object $\mathscr{F}$ of the cohomological category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ on the region U_{L}.

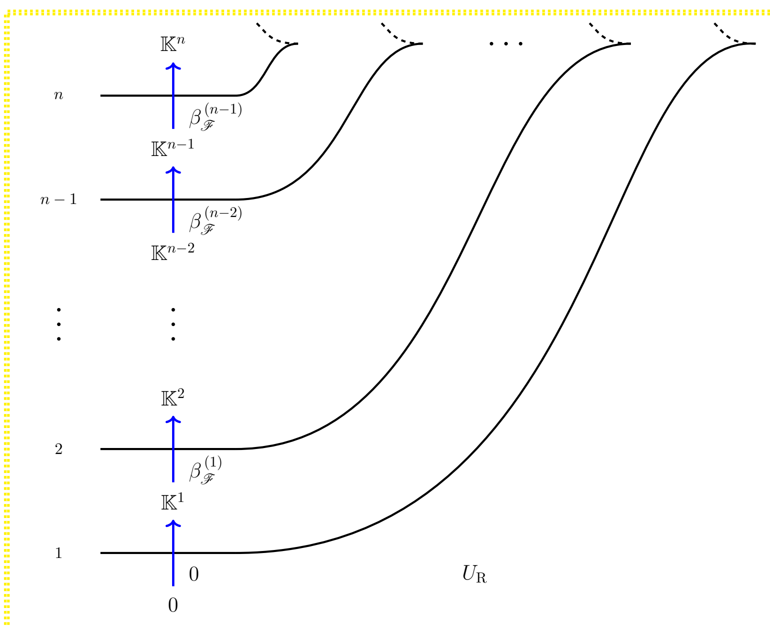


FIGURE 14. An object $\mathscr{F}$ of the cohomological category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ on the region U_{R}.

Lemma 4.35. *Let $\beta := \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathscr{F}$ an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ introduced in Construction (4.33). Then, the following statements hold:*

- *On U_0, $\mathscr{F}$ is identically zero.*

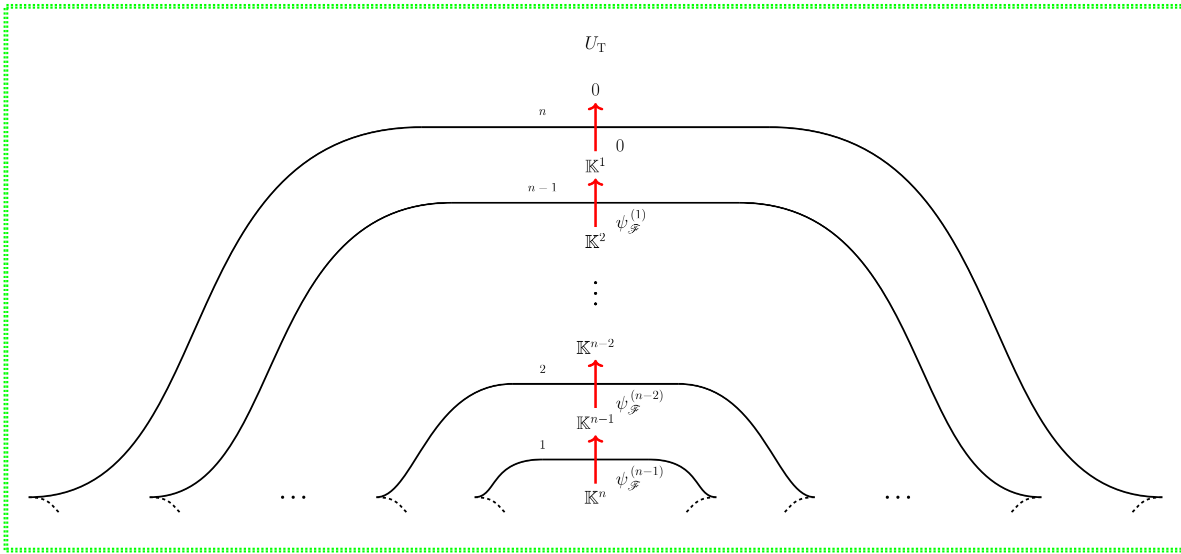


FIGURE 15. An object $\mathscr{F}$ of the cohomological category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ on the region U_{T}.

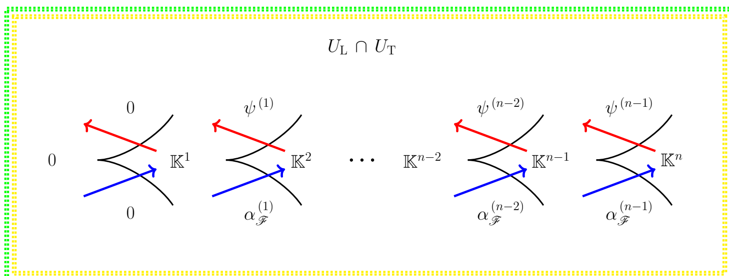


FIGURE 16. An object $\mathscr{F}$ of the cohomological category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ on the intersection $U_{\mathrm{L}} \cap U_{\mathrm{T}}$.

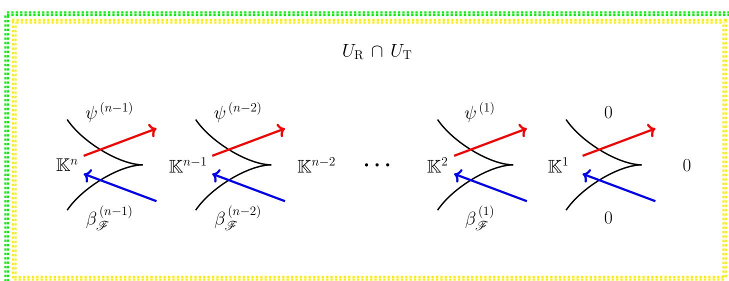


FIGURE 17. An object $\mathscr{F}$ of the cohomological category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ on the intersection $U_{\mathrm{R}} \cap U_{\mathrm{T}}$.

- *On U_{T}, $\mathscr{F}$ is characterized by a complete flag $\mathcal{F}^\bullet_0$ in $\mathbb{K}^n$.*
- *On U_{L}, $\mathscr{F}$ is characterized by a complete flag $\mathcal{F}^\bullet_{\text{left}}$ in $\mathbb{K}^n$.*
- *On U_{R}, $\mathscr{F}$ is characterized by a complete flag $\mathcal{F}^\bullet_{\text{right}}$ in $\mathbb{K}^n$.*
- ***Compatibility conditions**: $\mathcal{F}^\bullet_0$ is completely opposite to both $\mathcal{F}^\bullet_{\text{left}}$ and $\mathcal{F}^\bullet_{\text{right}}$.*

Proof. By Lemma (4.34), we know that:

- On U_0, $\mathscr{F}$ is identically zero.
- On U_{T}, $\mathscr{F}$ is defined by a collection of $n-1$ surjective linear maps $\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}$.
- On U_{L}, $\mathscr{F}$ is defined by a collection of $n-1$ injective linear maps $\{\alpha_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$.
- On U_{R}, $\mathscr{F}$ is defined by a collection of $n-1$ injective linear maps $\{\beta_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$.
- **Compatibility conditions**: For each $i \in [1, n-1]$,

$$\psi_{\mathscr{F}}^{(i)} \circ \alpha_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}, \quad \text{and} \quad \psi_{\mathscr{F}}^{(i)} \circ \beta_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}.$$

To prove the claim, it suffices to show that the above data determines three complete flags $\mathcal{F}_0^\bullet$, $\mathcal{F}_{\text{left}}^\bullet$, and $\mathcal{F}_{\text{right}}^\bullet$ in $\mathbb{K}^n$, with $\mathcal{F}_0^\bullet$ completely opposite to both $\mathcal{F}_{\text{left}}^\bullet$ and $\mathcal{F}_{\text{right}}^\bullet$, thereby providing a geometric description of $\mathscr{F}$. To this end, relying on the notions of type $\mathcal{I}$ and type $\mathcal{K}$ flags introduced in Definition (4.18)–(1)–(2), we proceed as follows:

(1) Since the maps $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are surjective, we assign to $\mathscr{F}$ on U_{T} the complete flag $\mathcal{F}_0^\bullet$ in $\mathbb{K}^n$ given by

$$\mathcal{F}_0^\bullet := {}_{\mathcal{K}}\mathcal{F}^\bullet\left(\psi_{\mathscr{F}}^{(1)}, \ldots, \psi_{\mathscr{F}}^{(n-1)}\right).$$

(2) Since the maps $\{\alpha_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are injective, we assign to $\mathscr{F}$ on U_{L} the complete flag $\mathcal{F}_{\text{left}}^\bullet$ in $\mathbb{K}^n$ given by

$$\mathcal{F}_{\text{left}}^\bullet := {}_{\mathcal{I}}\mathcal{F}^\bullet\left(\alpha_{\mathscr{F}}^{(1)}, \ldots, \alpha_{\mathscr{F}}^{(n-1)}\right).$$

(3) Since the maps $\{\beta_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are injective, we assign to $\mathscr{F}$ on U_{R} the complete flag $\mathcal{F}_{\text{right}}^\bullet$ in $\mathbb{K}^n$ given by

$$\mathcal{F}_{\text{right}}^\bullet := {}_{\mathcal{I}}\mathcal{F}^\bullet\left(\beta_{\mathscr{F}}^{(1)}, \ldots, \beta_{\mathscr{F}}^{(n-1)}\right).$$

Finally, the compatibility conditions on the maps $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$, $\{\alpha_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$, and $\{\beta_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ allow a direct application of Lemma (4.21), which shows that $\mathcal{F}_0^\bullet$ is completely opposite to both $\mathcal{F}_{\text{left}}^\bullet$ and $\mathcal{F}_{\text{right}}^\bullet$, as desired. □

We now turn to the analysis of $\mathscr{F}$ on the region U_{B}, the last remaining element of the open cover $\mathcal{U}_{\Lambda(\beta)}$ of $\mathbb{R}^2$ introduced in Construction (4.33). As previously mentioned, U_{B} contains all the crossings in the front projection $\Pi_{x,z}(\Lambda(\beta))$, and consequently, the behavior of $\mathscr{F}$ on this region is inherently more intricate. To facilitate the exposition, we construct a finite open cover of U_{B} adapted to the front projection $\Pi_{x,z}(\Lambda(\beta))$, allowing us to study $\mathscr{F}$ on this region in terms of tractable local data and the corresponding gluing conditions.

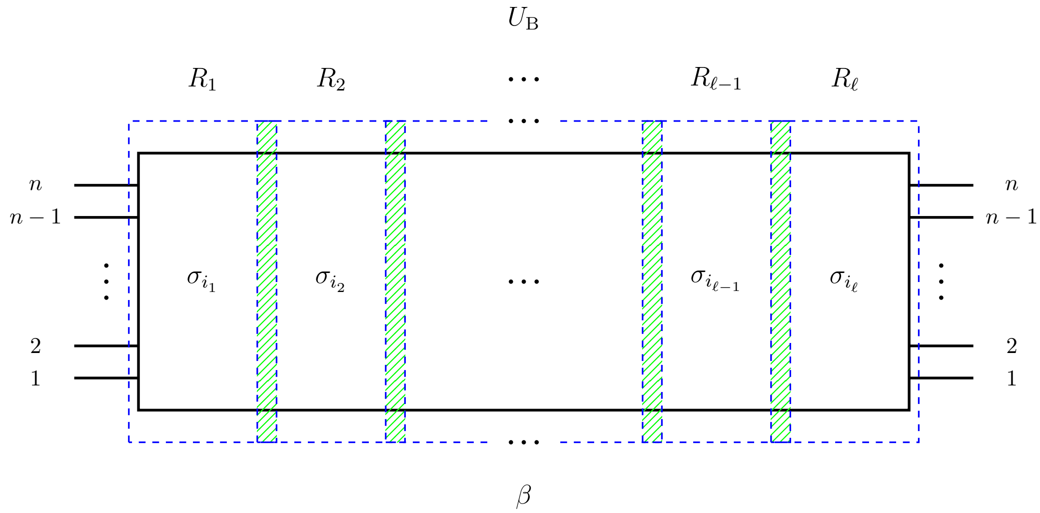


FIGURE 18. Region U_{B} in $\mathbb{R}^2$ decomposed into a collection $\mathcal{R}_{\Lambda(\beta)} := \{R_i\}_{i=1}^{\ell}$ of ℓ open vertical straps R_j in $\mathbb{R}^2$.

Construction 4.36. *Let $\beta := \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ be the open cover of $\mathbb{R}^2$ introduced in Construction (4.33). Then, we decompose the region $U_{\mathrm{B}} \subset \mathbb{R}^2$ into a collection $\mathcal{R}_{\Lambda(\beta)} := \{R_j\}_{j=1}^{\ell}$ of ℓ open vertical straps $R_j \subset \mathbb{R}^2$ as illustrated in Figure (18). In particular, given $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ the front projection of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, we assume that:*

- *For each $j \in [1, \ell]$, the intersection $R_j \cap \Pi_{x,z}(\Lambda(\beta))$ consists of the braid on n strands associated with σ_{i_j}—the j-th crossing of β—as shown in Figure (19).*
- *For each $j \in [1, \ell-1]$, the intersection $R_j \cap R_{j+1} \cap \Pi_{x,z}(\Lambda(\beta))$ consists of the trivial braid on n strands as depicted in Figure (20).*

Next, by applying the sheaf axioms to the open cover $\mathcal{R}_{\Lambda(\beta)} = \{R_j\}_{j=1}^{\ell}$ of U_{B} introduced in Construction (4.36), we provide an explicit characterization of $\mathscr{F}$ on the region U_{B} as follows: First, we give a geometric description of $\mathscr{F}$ on each vertical strap R_j independently, for every $j \in [1, \ell]$. Second, we proceed by gluing the local data that characterizes $\mathscr{F}$ across adjacent straps R_j and R_{j+1}, for every $j \in [1, \ell-1]$. Accordingly, we present the following results.

Lemma 4.37. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Fix $j \in [1, \ell]$, and let $R_j \subset \mathbb{R}^2$ be the open vertical strap whose intersection with the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ consists of the braid diagram on n strands associated with σ_{i_j}, i.e., the j-th crossing of β (see Figure (19)). Denote by $k := i_j \in [1, n-1]$ the index of σ_{i_j}, and suppose that $k = 1$. Then, on the region R_j, the sheaf $\mathscr{F}$ is specified by a collection of n linear maps*

$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}_{\mathscr{F}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2\right\},$$

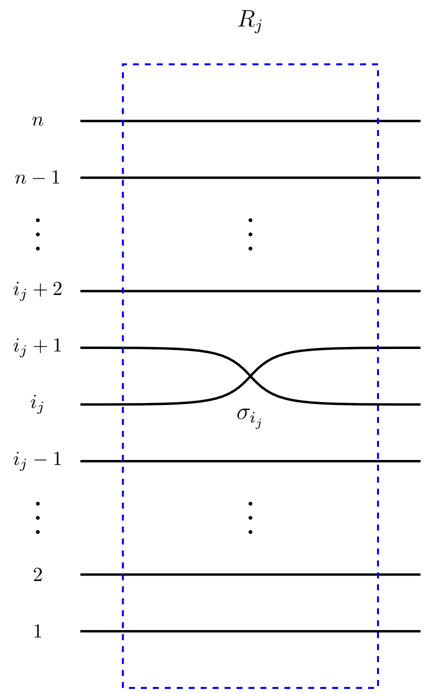


FIGURE 19. Intersection of an open vertical strap R_j in $\mathbb{R}^2$ with the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$.

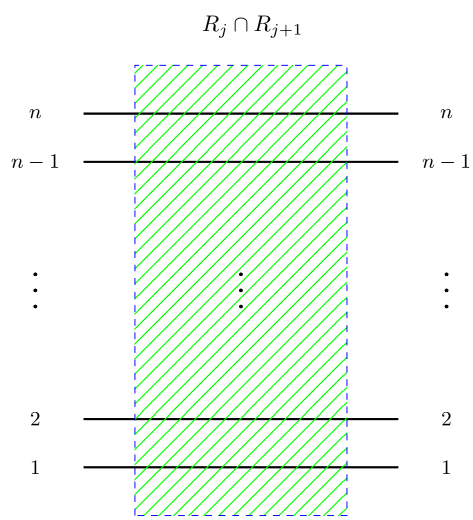


FIGURE 20. Intersection of the overlap $R_j \cap R_{j+1}$ of two adjacent straps R_j and R_{j+1} in $\mathbb{R}^2$ with the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$.

as illustrated in Figure (21).

***Compatibility conditions**: The maps $\phi^{(1)}_{\mathscr{F}}$ and $\widetilde{\phi}^{(1)}_{\mathscr{F}}$ are injective and have complementary image in $\mathbb{K}^2$; that is, $\mathbb{K}^2 = \operatorname{im}\phi^{(1)}_{\mathscr{F}} \oplus \operatorname{im}\widetilde{\phi}^{(1)}_{\mathscr{F}}$.*

In particular, if we assume that the maps $\{\phi_{\mathscr{F}}^{(i)}\}_{i=2}^{n-1}$ *are injective, we further obtain that, on* R_j*,* $\mathscr{F}$ *is characterized by two complete flags* $\mathcal{F}_j^{\bullet}$ *and* $\mathcal{F}_{j+1}^{\bullet}$ *in* $\mathbb{K}^n$*, which are in* s_1*–relative position.*

Proof. To begin, consider the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, which is depicted in Figure (1). In particular, recall that in $\Pi_{x,z}(\Lambda(\beta))$, the strands at the bottom have Maslov potential 0. Thus, since $k = 1$, the microlocal rank conditions and the microlocal support conditions near the arcs imply that, on R_j, $\mathscr{F}$ is defined by a collection of n linear maps

$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}_{\mathscr{F}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2\right\},$$

as shown in Figure (21).

Now, let $U_j \subset \mathbb{R}^2$ be the small open ball—centered at the crossing σ_{i_j}—highlighted in yellow in Figure (21), where $k = i_j = 1$. Then, by the microlocal support conditions near the crossings, we obtain that the maps $\phi_{\mathscr{F}}^{(1)}$ and $\widetilde{\phi}_{\mathscr{F}}^{(1)}$ defining $\mathscr{F}$ on U_j satisfy the following compatibility conditions:

(i) The diagram in Figure (22) commutes.

(ii) The sequence in Equation (4.8) is short exact.

$$0 \longrightarrow 0 \xrightarrow{\quad (0,0) \quad} \mathbb{K}^1 \oplus \mathbb{K}^1 \xrightarrow{\quad \phi_{\mathscr{F}}^{(1)} \oplus \left(-\widetilde{\phi}_{\mathscr{F}}^{(1)}\right) \quad} \mathbb{K}^2 \longrightarrow 0\,. \tag{4.8}$$

Consequently, since $0 \leq \dim_{\mathbb{K}} \operatorname{im} \phi_{\mathscr{F}}^{(1)} \leq 1$ and $0 \leq \dim_{\mathbb{K}} \operatorname{im} \widetilde{\phi}_{\mathscr{F}}^{(1)} \leq 1$, the short exact sequence in Equation (4.8) guarantees that the maps $\phi_{\mathscr{F}}^{(1)}$ and $\widetilde{\phi}_{\mathscr{F}}^{(1)}$ are injective and their images are complementary subspaces in $\mathbb{K}^2$; that is, $\mathbb{K}^2 = \operatorname{im} \phi_{\mathscr{F}}^{(1)} \oplus \operatorname{im} \widetilde{\phi}_{\mathscr{F}}^{(1)}$.

Finally, assuming the maps $\{\phi_{\mathscr{F}}^{(i)}\}_{i=2}^{n-1}$ are injective, we show that the linear maps defining $\mathscr{F}$ on R_j determine two complete flags $\mathcal{F}_j^{\bullet}$ and $\mathcal{F}_{j+1}^{\bullet}$ in $\mathbb{K}^n$, which are in s_1–relative position, thereby providing a geometric characterization of $\mathscr{F}$ on R_j. To this end, relying on the notion of type $\mathcal{I}$ flag introduced in Definition (4.18)–(1), we proceed as follows:

(1) Since the maps $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ are injective, we assign to $\mathscr{F}$ on R_j the complete flag $\mathcal{F}_j^{\bullet}$ in $\mathbb{K}^n$ given by

$$\mathcal{F}_j^{\bullet} := {}_{\mathcal{I}}\mathcal{F}^{\bullet}\left(\phi_{\mathscr{F}}^{(1)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\right).$$

(2) Since the maps $\left\{\widetilde{\phi}_{\mathscr{F}}^{(1)}, \phi_{\mathscr{F}}^{(2)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\right\}$ are injective, we assign to $\mathscr{F}$ on R_j the complete flag $\mathcal{F}_{j+1}^{\bullet}$ in $\mathbb{K}^n$ given by

$$\mathcal{F}_{j+1}^{\bullet} := {}_{\mathcal{I}}\mathcal{F}^{\bullet}\left(\widetilde{\phi}_{\mathscr{F}}^{(1)}, \phi_{\mathscr{F}}^{(2)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\right).$$

In particular, by Definition (4.18)–(1), we know that, as filtrations of vector subspaces in $\mathbb{K}^n$, the flags

$$\mathcal{F}_j^\bullet = \left\{F_j^{(0)} \subset \cdots \subset F_j^{(n)}\right\},$$

$$\mathcal{F}_{j+1}^\bullet = \left\{F_{j+1}^{(0)} \subset \cdots \subset F_{j+1}^{(n)}\right\},$$

are given by

$$F_j^{(p)} := \begin{cases} 0\,, & \text{if } \ p = 0\,, \\ \operatorname{im}\Big(\phi_{\mathscr{F}}^{(n-1)} \circ \phi_{\mathscr{F}}^{(n-2)} \circ \cdots \circ \phi_{\mathscr{F}}^{(2)} \circ \phi_{\mathscr{F}}^{(1)}\Big)\,, & \text{if } \ p = 1\,, \\ \operatorname{im}\Big(\phi_{\mathscr{F}}^{(n-1)} \circ \phi_{\mathscr{F}}^{(n-2)} \circ \cdots \circ \phi_{\mathscr{F}}^{(p+1)} \circ \phi_{\mathscr{F}}^{(p)}\Big)\,, & \text{if } \ p \in [2, n-1]\,, \\ \mathbb{K}^n\,, & \text{if } \ p = n\,. \end{cases}$$

$$F_{j+1}^{(p)} := \begin{cases} 0\,, & \text{if } \ p = 0\,, \\ \operatorname{im}\Big(\phi_{\mathscr{F}}^{(n-1)} \circ \phi_{\mathscr{F}}^{(n-2)} \circ \cdots \circ \phi_{\mathscr{F}}^{(2)} \circ \widetilde{\phi}_{\mathscr{F}}^{(1)}\Big)\,, & \text{if } \ p = 1\,, \\ \operatorname{im}\Big(\phi_{\mathscr{F}}^{(n-1)} \circ \phi_{\mathscr{F}}^{(n-2)} \circ \cdots \circ \phi_{\mathscr{F}}^{(p+1)} \circ \phi_{\mathscr{F}}^{(p)}\Big)\,, & \text{if } \ p \in [2, n-1]\,, \\ \mathbb{K}^n\,, & \text{if } \ p = n\,. \end{cases}$$

Thus, since $\operatorname{im}\phi_{\mathscr{F}}^{(1)} \neq \operatorname{im}\widetilde{\phi}_{\mathscr{F}}^{(1)}$ and the composition $\phi_{\mathscr{F}}^{(n-1)} \circ \cdots \circ \phi_{\mathscr{F}}^{(2)} : \mathbb{K}^2 \longrightarrow \mathbb{K}^n$ is injective, we deduce that $F_j^{(1)} \neq F_{j+1}^{(1)}$ and $F_j^{(p)} = F_{j+1}^{(p)}$ for all $p \in [0, n]$ with $p \neq 1$, confirming that $\mathcal{F}_j^\bullet$ and $\mathcal{F}_{j+1}^\bullet$ are in s_1–relative position. This concludes the proof. $\square$

Lemma 4.38. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Fix $j \in [1, \ell]$, and let $R_j \subset \mathbb{R}^2$ be the open vertical strap whose intersection with the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ consists of the braid diagram on n strands associated with σ_{i_j}, i.e., the j-th crossing of β (see Figure (19)). Denote by $k := i_j \in [1, n-1]$ the index of σ_{i_j}, and suppose that $k \geq 2$. Then, on the region R_j, the sheaf $\mathscr{F}$ is specified by a collection of $n+1$ linear maps*

$$\Big\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\Big\}_{i=1}^{n-1} \cup \Big\{\widetilde{\phi}_{\mathscr{F}}^{(k-1)} : \mathbb{K}^{k-1} \to \mathbb{K}^k\,,\ \widetilde{\phi}_{\mathscr{F}}^{(k)} : \mathbb{K}^k \to \mathbb{K}^{k+1}\Big\}\,,$$

as illustrated in Figure (23).

Compatibility conditions*: The maps $\phi_{\mathscr{F}}^{(k)}$, $\phi_{\mathscr{F}}^{(k-1)}$, $\widetilde{\phi}_{\mathscr{F}}^{(k)}$, and $\widetilde{\phi}_{\mathscr{F}}^{(k-1)}$ satisfy the following properties:*

- $\phi_{\mathscr{F}}^{(k)} \circ \phi_{\mathscr{F}}^{(k-1)} = \widetilde{\phi}_{\mathscr{F}}^{(k)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k-1)}$.
- $\operatorname{im}\Big(\phi_{\mathscr{F}}^{(k)} \circ \phi_{\mathscr{F}}^{(k-1)}\Big) = \operatorname{im}\phi_{\mathscr{F}}^{(k)} \cap \operatorname{im}\widetilde{\phi}_{\mathscr{F}}^{(k)} = \operatorname{im}\Big(\widetilde{\phi}_{\mathscr{F}}^{(k)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k-1)}\Big)$.
- $\mathbb{K}^{k+1} = \operatorname{im}\phi_{\mathscr{F}}^{(k)} + \operatorname{im}\widetilde{\phi}_{\mathscr{F}}^{(k)}$.

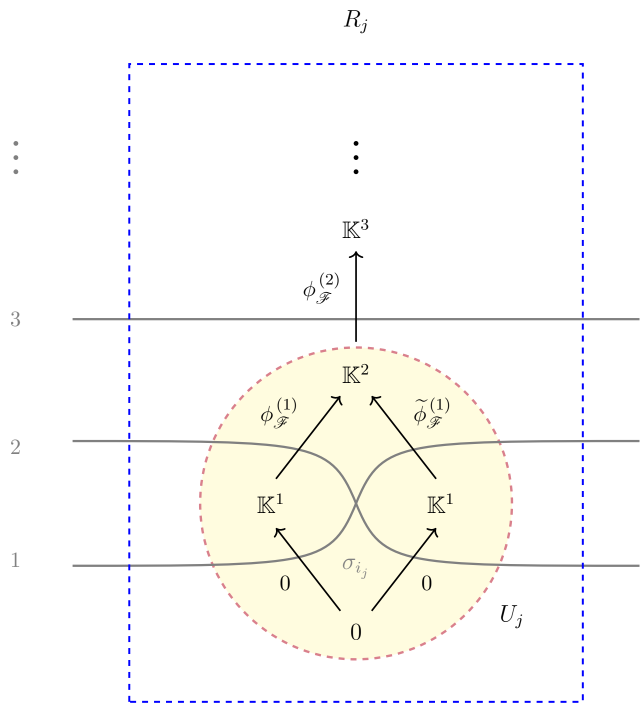


FIGURE 21. An object $\mathscr{F}$ on the vertical strap R_j containing a crossing σ_{i_j} in the case $k = i_j = 1$.

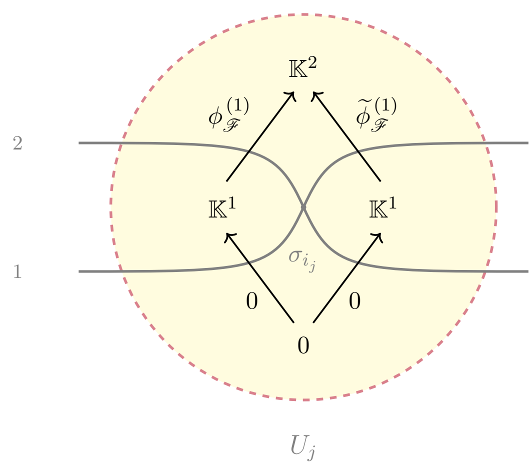


FIGURE 22. Commutative diagram for the linear maps defining a sheaf $\mathscr{F}$ on the region U_j in the case $k = i_j = 1$.

In particular, if we assume that the maps $\{\phi^{(i)}_{\mathscr{F}}\}_{i=1}^{n-1}$ *are injective, we further obtain that:*

- *The maps* $\widetilde{\phi}^{(k-1)}_{\mathscr{F}}$ *and* $\widetilde{\phi}^{(k)}_{\mathscr{F}}$ *are injective.*
- *On* R_j*,* $\mathscr{F}$ *is characterized by two complete flags* $\mathcal{F}^\bullet_j$ *and* $\mathcal{F}^\bullet_{j+1}$ *in* $\mathbb{K}^n$*, which are in* s_k*–relative position.*

Proof. To begin, consider the front projection $\Pi_{x,z}(\Lambda(\beta)) \subset \mathbb{R}^2$ of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, which is depicted in Figure (1). In particular, recall that in $\Pi_{x,z}(\Lambda(\beta))$, the strands at the bottom have Maslov potential

0. Thus, since $k \geq 2$, the microlocal rank conditions and the microlocal support conditions near the arcs establish that, on R_j, $\mathscr{F}$ is defined by a collection of $n+1$ linear maps

$$\big\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\big\}_{i=1}^{n-1} \cup \big\{\widetilde{\phi}^{(k-1)}_{\mathscr{F}} : \mathbb{K}^{k-1} \to \mathbb{K}^k\,,\ \widetilde{\phi}^{(k)}_{\mathscr{F}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\big\}\,,$$

as shown in Figure (23).

Now, let $U_j \subset \mathbb{R}^2$ be the small open ball—centered at the crossing σ_{i_j}—highlighted in yellow in Figure (23). Then, by the microlocal support conditions near the crossings, we obtain that the maps $\phi^{(k)}_{\mathscr{F}}$, $\phi^{(k-1)}_{\mathscr{F}}$, $\widetilde{\phi}^{(k)}_{\mathscr{F}}$, and $\widetilde{\phi}^{(k-1)}_{\mathscr{F}}$ defining $\mathscr{F}$ on U_j satisfy the following compatibility conditions:

(i) The diagram in Figure (24) commutes.

(ii) The sequence in Equation (4.9) is short exact.

$$0 \longrightarrow \mathbb{K}^{k-1} \xrightarrow{\ \big(\phi^{(k-1)}_{\mathscr{F}},\, \widetilde{\phi}^{(k-1)}_{\mathscr{F}}\big)\ } \mathbb{K}^k \oplus \mathbb{K}^k \xrightarrow{\ \phi^{(k)}_{\mathscr{F}} \oplus \big(-\widetilde{\phi}^{(k)}_{\mathscr{F}}\big)\ } \mathbb{K}^{k+1} \longrightarrow 0\,. \tag{4.9}$$

Consequently, a straightforward application of Lemma (4.14) ensures that:

- *(i)* $\phi^{(k)}_{\mathscr{F}} \circ \phi^{(k-1)}_{\mathscr{F}} = \widetilde{\phi}^{(k)}_{\mathscr{F}} \circ \widetilde{\phi}^{(k-1)}_{\mathscr{F}}$.
- *(ii)* $\operatorname{im}\big(\phi^{(k)}_{\mathscr{F}} \circ \phi^{(k-1)}_{\mathscr{F}}\big) = \operatorname{im}\phi^{(k)}_{\mathscr{F}} \cap \operatorname{im}\widetilde{\phi}^{(k)}_{\mathscr{F}} = \operatorname{im}\big(\widetilde{\phi}^{(k)}_{\mathscr{F}} \circ \widetilde{\phi}^{(k-1)}_{\mathscr{F}}\big)$.
- *(iii)* $\mathbb{K}^{k+1} = \operatorname{im}\phi^{(k)}_{\mathscr{F}} + \operatorname{im}\widetilde{\phi}^{(k)}_{\mathscr{F}}$.

Finally, assume that the maps $\big\{\phi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}$ are injective. Then, the compatibility conditions for the maps $\phi^{(k)}_{\mathscr{F}}$, $\phi^{(k-1)}_{\mathscr{F}}$, $\widetilde{\phi}^{(k)}_{\mathscr{F}}$, and $\widetilde{\phi}^{(k-1)}_{\mathscr{F}}$ further imply, via Lemma (4.14), that $\widetilde{\phi}^{(k-1)}_{\mathscr{F}}$ and $\widetilde{\phi}^{(k)}_{\mathscr{F}}$ are also injective.

Bearing this in mind, we now show that, under the above injectivity assumptions, the linear maps defining $\mathscr{F}$ on R_j determine two complete flags $\mathcal{F}^\bullet_j$ and $\mathcal{F}^\bullet_{j+1}$ in $\mathbb{K}^n$, which are in s_k–relative position, thereby providing a geometric characterization of $\mathscr{F}$ on R_j. To this end, relying on the notion of type $\mathcal{I}$ flag introduced in Definition (4.18)–(1), we proceed as follows:

(1) Since the maps $\big\{\phi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}$ are injective, we assign to $\mathscr{F}$ on R_j the complete flag $\mathcal{F}^\bullet_j$ in $\mathbb{K}^n$ given by

$$\mathcal{F}^\bullet_j := {}_{\mathcal{I}}\mathcal{F}^\bullet\big(\phi^{(1)}_{\mathscr{F}}, \ldots, \phi^{(n-1)}_{\mathscr{F}}\big)\,.$$

(2) Since the maps $\big\{\phi^{(1)}_{\mathscr{F}}, \ldots, \phi^{(k-2)}_{\mathscr{F}}, \widetilde{\phi}^{(k-1)}_{\mathscr{F}}, \widetilde{\phi}^{(k)}_{\mathscr{F}}, \phi^{(k+1)}_{\mathscr{F}}, \ldots, \phi^{(n-1)}_{\mathscr{F}}\big\}$ are injective, we assign to $\mathscr{F}$ on R_j the complete flag $\mathcal{F}^\bullet_{j+1}$ in $\mathbb{K}^n$ given by

$$\mathcal{F}^\bullet_{j+1} := {}_{\mathcal{I}}\mathcal{F}^\bullet\big(\phi^{(1)}_{\mathscr{F}}, \ldots, \phi^{(k-2)}_{\mathscr{F}}, \widetilde{\phi}^{(k-1)}_{\mathscr{F}}, \widetilde{\phi}^{(k)}_{\mathscr{F}}, \phi^{(k+1)}_{\mathscr{F}}, \ldots, \phi^{(n-1)}_{\mathscr{F}}\big)\,.$$

In particular, by Definition (4.18)–(1), we know that, as filtrations of vector subspaces in $\mathbb{K}^n$, the flags

$$\mathcal{F}_j^{\bullet} = \left\{ F_j^{(0)} \subset \cdots \subset F_j^{(n)} \right\},$$

$$\mathcal{F}_{j+1}^{\bullet} = \left\{ F_{j+1}^{(0)} \subset \cdots \subset F_{j+1}^{(n)} \right\},$$

are given by

$$F_j^{(p)} := \begin{cases} \{0\}\,, & \text{if } p = 0\,, \\ \operatorname{im}\left(\phi_{\mathscr{F}}^{(n-1)} \circ \cdots \circ \phi_{\mathscr{F}}^{(k+1)} \circ \phi_{\mathscr{F}}^{(k)} \circ \phi_{\mathscr{F}}^{(k-1)} \circ \phi_{\mathscr{F}}^{(k-2)} \circ \cdots \circ \phi_{\mathscr{F}}^{(p)} \right), & \text{if } p \in [1, n-1]\,, \\ \mathbb{K}^n\,, & \text{if } p = n\,. \end{cases}$$

$$F_{j+1}^{(p)} := \begin{cases} \{0\}\,, & \text{if } p = 0\,, \\ \operatorname{im}\left(\phi_{\mathscr{F}}^{(n-1)} \circ \cdots \circ \phi_{\mathscr{F}}^{(k+1)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k-1)} \circ \phi_{\mathscr{F}}^{(k-2)} \circ \cdots \circ \phi_{\mathscr{F}}^{(p)} \right), & \text{if } p \in [1, n-1]\,, \\ \mathbb{K}^n\,, & \text{if } p = n\,. \end{cases}$$

Thus, since $\phi_{\mathscr{F}}^{(k)} \circ \phi_{\mathscr{F}}^{(k-1)} = \widetilde{\phi}_{\mathscr{F}}^{(k)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k-1)}$, we have that $F_j^{(p)} = F_{j+1}^{(p)}$ for all $p \in [0, n]$ with $p \neq k$. Moreover, since $\operatorname{im} \phi_{\mathscr{F}}^{(k)} \neq \operatorname{im} \widetilde{\phi}_{\mathscr{F}}^{(k)}$ and the composition map $\phi_{\mathscr{F}}^{(n-1)} \circ \cdots \circ \phi_{\mathscr{F}}^{(k+1)} : \mathbb{K}^{k+1} \longrightarrow \mathbb{K}^n$ is injective, we deduce that $F_j^{(k)} \neq F_{j+1}^{(k)}$, confirming that $\mathcal{F}_{j+1}^{\bullet}$ and $\mathcal{F}_j^{\bullet}$ are in s_k-relative position. This concludes the proof. □

Building on the previous results, we now present a concise geometric description of $\mathscr{F}$ on the region U_{B}.

Lemma 4.39. *Let $\beta := \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathscr{F}$ an object of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and $\mathcal{U}_{\Lambda(\beta)} = \left\{ U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}} \right\}$ the open cover of $\mathbb{R}^2$ introduced in Construction (4.33). Then, on the region U_{B}, $\mathscr{F}$ is characterized by a sequence of complete flags $\left\{ \mathcal{F}_j^{\bullet} \right\}_{j=1}^{\ell+1}$ in $\mathbb{K}^n$, such that for each $j \in [1, \ell]$, $\mathcal{F}_{j+1}^{\bullet}$ is in s_{i_j}-relative position with respect to $\mathcal{F}_j^{\bullet}$.*

Proof. Let $\mathcal{R}_{\Lambda(\beta)} = \left\{ R_j \right\}_{j=1}^{\ell}$ be the collection of ℓ open vertical straps in $\mathbb{R}^2$ that partition the region $U_{\mathrm{B}} \subset \mathbb{R}^2$, as specified in Construction (4.36). Specifically, for each $j \in [1, \ell]$, $R_j \subset \mathbb{R}^2$ denotes the open vertical strap whose intersection with the front projection $\Pi_{x,z}(\Lambda(\beta))$ consists of the braid diagram on n strands associated with σ_{i_j}, i.e., the j-th crossing of β, as illustrated in Figure (19).

Next, we develop a recursive argument to analyze the behavior of $\mathscr{F}$ on each successive vertical strap R_j, thereby obtaining a geometric characterization of $\mathscr{F}$ on U_{B}. With this strategy in place, we begin by studying $\mathscr{F}$ on the first region R_1. Let $k := i_1 \in [1, n-1]$ denote the index of σ_{i_1}—the first crossing of β and the only crossing contained in R_1. Accordingly, we distinguish two cases:

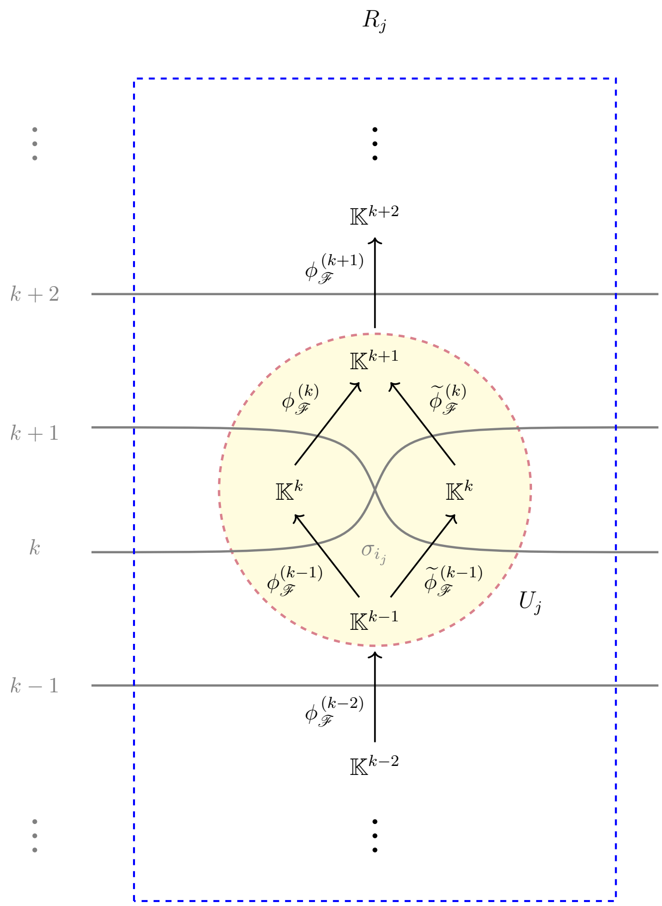


FIGURE 23. An object $\mathscr{F}$ on the vertical strap R_j containing the crossing σ_{i_j} in the case $k = i_j \geq 2$.

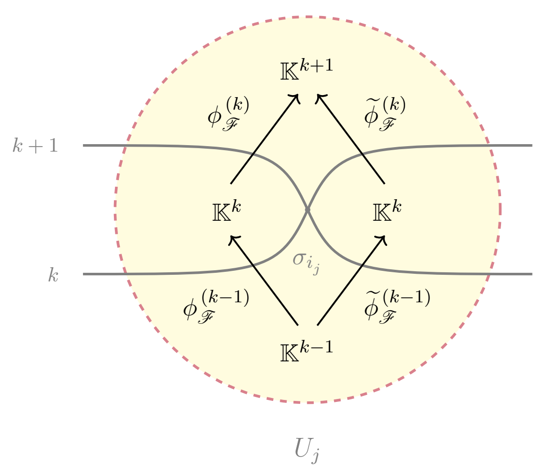


FIGURE 24. Commutative diagram for the linear maps that define a sheaf $\mathscr{F}$ on the region U_j in the case $k = i_j \geq 2$.

- *Case 1*: Suppose that $k = 1$. Then, on R_1, $\mathscr{F}$ is defined by a collection of n linear maps

$$\left\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}^{(1)}_{\mathscr{F}} : \mathbb{K}^1 \to \mathbb{K}^2\right\},$$

as shown in Figure (21), where $j = 1$. In particular, these maps satisfy the compatibility conditions stated in Lemma (4.37).

- *Case 2*: Suppose that $k \geq 2$. Then, on R_j, $\mathscr{F}$ is defined by a collection of $n+1$ linear maps

$$\left\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}^{(k-1)}_{\mathscr{F}} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{F}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\right\},$$

as shown in Figure (23), where $j = 1$. Specifically, these maps satisfy the compatibility conditions stated in Lemma (4.38).

Now, let U_{L} be the region of the open cover $\mathcal{U}_{\Lambda(\beta)}$ of $\mathbb{R}^2$ intersecting U_{B} on the left, as illustrated in Figure (12). By construction, R_1 is the only vertical strap in the partition $\mathcal{R}_{\Lambda(\beta)}$ of U_{B} whose intersection with U_{L} is nontrivial. Consequently, in either of the above cases, constructibility and the sheaf axioms imply that, on U_{L}, $\mathscr{F}$ is determined by the collection of linear maps $\left\{\phi^{(i)}_{\mathscr{F}}\right\}_{i=1}^{n-1}$, and hence, by Lemma (4.34), these maps are injective. Bearing this in mind, depending on the value of k, we can apply Lemma (4.37) or Lemma (4.38) to obtain that:

- If $k = 1$, the map $\widetilde{\phi}^{(1)}_{\mathscr{F}}$ is injective.
- If $k \geq 2$, the maps $\widetilde{\phi}^{(k-1)}_{\mathscr{F}}$ and $\widetilde{\phi}^{(k)}_{\mathscr{F}}$ are injective.
- On R_1, $\mathscr{F}$ is characterized by two complete flags $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ in $\mathbb{K}^n$, which are in s_k-relative position (recalling that $k = i_1$ as fixed earlier).

In particular, note that regardless of the value of i_1, all the linear maps defining $\mathscr{F}$ on R_1 are injective.

Next, consider the region R_2 in the partition $\mathcal{R}_{\Lambda(\beta)}$ of U_{B}; that is, the open vertical strap adjacent to R_1 on the right. By construction, the intersection $R_1 \cap R_2 \cap \Pi_{x,z}(\Lambda(\beta))$ consists of the trivial braid on n strands, as illustrated in Figure (20). Thus, since all the linear maps defining $\mathscr{F}$ on R_1 are injective, the sheaf axioms ensure that the linear maps defining $\mathscr{F}$ on the intersection $R_1 \cap R_2$ are also injective. Consequently, depending on the value of the index i_2 of the second crossing σ_{i_2} of β, we can apply Lemma (4.37) or Lemma (4.38) to conclude that:

- The linear maps determining $\mathscr{F}$ on R_2 are injective.
- On R_2, $\mathscr{F}$ is characterized by two complete flags $\mathcal{F}_2^\bullet$ and $\mathcal{F}_3^\bullet$ in $\mathbb{K}^n$, which are in s_{i_2}–relative position.

In particular, by applying the above argument recursively to each vertical strap R_j in the partition $\mathcal{R}_{\Lambda(\beta)}$ of U_{B}, we deduce that on U_{B}, the object $\mathscr{F}$ is determined by a sequence of complete flags $\{\mathcal{F}_j^\bullet\}_{j=1}^{\ell+1}$ in $\mathbb{K}^n$, such that for each $j \in [1, \ell]$, $\mathcal{F}_{j+1}^\bullet$ is in s_{i_j}-relative position with respect to $\mathcal{F}_j^\bullet$. □

With the preceding results in place, we now present a global geometric characterization of $\mathscr{F}$, a result which also follows from [STZ17] and is included here for completeness.

Theorem 4.40. *Let $\beta := \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Then, as illustrated in Figure (25), $\mathscr{F}$ is determined by a sequence of complete flags $\{\mathcal{F}_j^\bullet\}_{j=0}^{\ell+1}$ in $\mathbb{K}^n$ satisfying the following compatibility conditions:*

- *For each $j \in [1, \ell]$, $\mathcal{F}_{j+1}^\bullet$ is in s_{i_j}-relative position with respect to $\mathcal{F}_j^\bullet$.*
- *$\mathcal{F}_0^\bullet$ is completely opposite to both $\mathcal{F}_1^\bullet$ and $\mathcal{F}_{\ell+1}^\bullet$.*

Proof. Let $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ denote the finite open cover of $\mathbb{R}^2$ introduced in Construction (4.33). By Lemma (4.35), we know that:

- On U_0, $\mathscr{F}$ is identically zero.
- On U_{T}, $\mathscr{F}$ is characterized by a complete flag $\mathcal{F}_0^\bullet$ in $\mathbb{K}^n$.
- On U_{L}, $\mathscr{F}$ is characterized by a complete flag $\mathcal{F}_{\mathrm{left}}^\bullet$ in $\mathbb{K}^n$.
- On U_{R}, $\mathscr{F}$ is characterized by a complete flag $\mathcal{F}_{\mathrm{right}}^\bullet$ in $\mathbb{K}^n$.
- **Compatibility conditions**: $\mathcal{F}_0^\bullet$ is completely opposite to both $\mathcal{F}_{\mathrm{left}}^\bullet$ and $\mathcal{F}_{\mathrm{right}}^\bullet$.

Furthermore, by Lemma (4.39), we have that on U_{B}, $\mathscr{F}$ is characterized by a sequence of complete flags $\{\mathcal{F}_j^\bullet\}_{j=1}^{\ell+1}$ in $\mathbb{K}^n$ such that, for each $j \in [1, \ell]$, $\mathcal{F}_{j+1}^\bullet$ is in s_{i_j}-relative position with respect to $\mathcal{F}_j^\bullet$. In particular, observe that on the left and right edges of the region U_{B}, $\mathscr{F}$ is determined by the complete flags $\mathcal{F}_1^\bullet$ and $\mathcal{F}_{\ell+1}^\bullet$, respectively. Hence, by applying the sheaf axioms to the intersections $U_{\mathrm{L}} \cap U_{\mathrm{B}}$ and $U_{\mathrm{R}} \cap U_{\mathrm{B}}$, we obtain that $\mathcal{F}_{\mathrm{left}}^\bullet = \mathcal{F}_1^\bullet$ and $\mathcal{F}_{\mathrm{right}}^\bullet = \mathcal{F}_{\ell+1}^\bullet$. Bearing this in mind, we conclude that $\mathscr{F}$ is determined by a sequence of complete flags $\{\mathcal{F}_j^\bullet\}_{j=0}^{\ell+1}$ in $\mathbb{K}^n$ satisfying the compatibility conditions stated in the theorem. □

Having established a geometric description of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, we now turn to an algebraic characterization, which is particularly well-suited for concrete computations and will play a key role in the analysis of the graded morphism spaces and their compositions in the category under study.

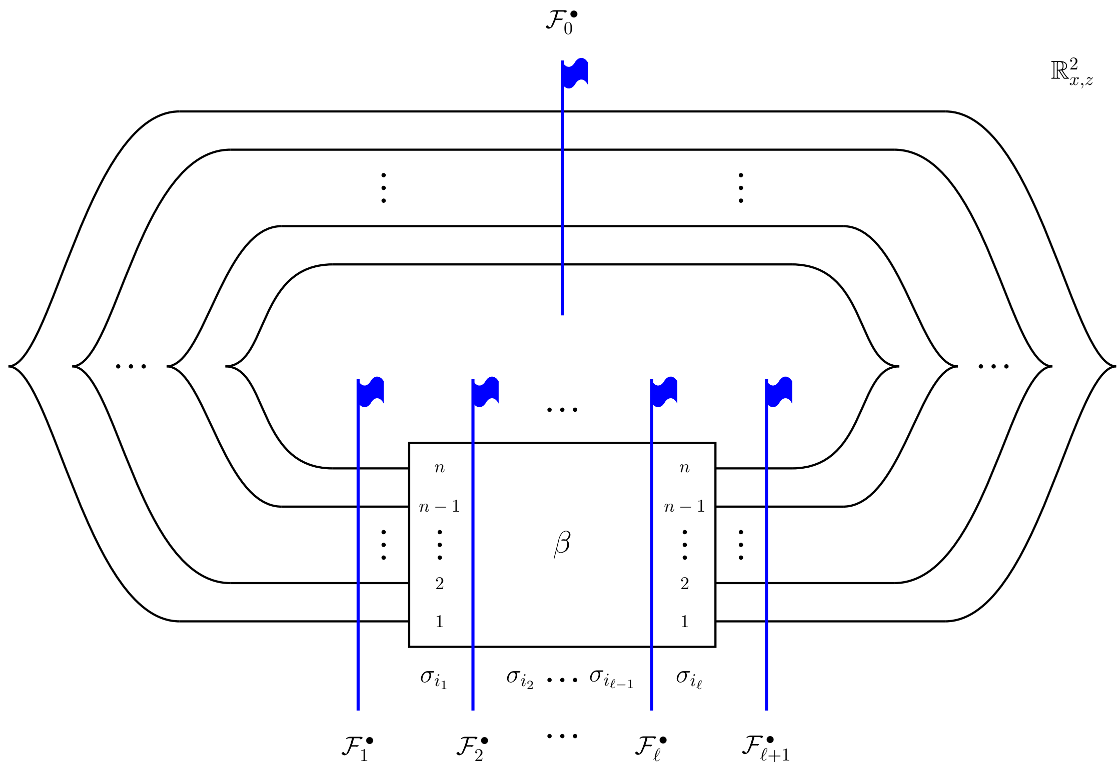


FIGURE 25. An object $\mathscr{F}$ of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.

4.4. **An Algebraic Characterization of the Objects for General Positive Braids.** Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. Building on the previous subsection, we now develop a correspondence between the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ and the points of the so-called <u>braid variety</u> $X(\beta,\mathbb{K})$, thereby providing an algebraic description of these objects. With this goal in mind, we begin by introducing the following definitions.

Definition 4.41. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. For each $j \in [1,\ell]$, we define $\beta_j := \sigma_{i_1}\cdots\sigma_{i_j} \in \mathrm{Br}_n^+$ to be the truncation of β at its j-th crossing. In particular, we have that $\beta_\ell = \beta$.*

Definition 4.42. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\vec{z} = (z_1,\dots,z_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$ be a fixed tuple. Then, the <u>path matrix</u> $P_\beta(z_1,\dots,z_\ell) \in \mathrm{GL}(n,\mathbb{K})$ associated with β and $\vec{z}$ is defined by*

$$P_\beta(z_1,\dots,z_\ell) := B^{(n)}_{i_1}(z_1)\cdots B^{(n)}_{i_\ell}(z_\ell),$$

where, for each $j \in [1,\ell]$, the matrix $B^{(n)}_{i_j}(z_j) \in \mathrm{GL}(n,\mathbb{K})$ denotes the i_j-th braid matrix of dimension n associated with the j-th crossing σ_{i_j} of β and the j-th entry z_j of $\vec{z}$.

Remark 4.43. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. For each $k \in [1,\ell-1]$, the truncations satisfy $\beta_{k+1} = \beta_k\cdot\sigma_{i_{k+1}}$. Consequently, the associated path matrices satisfy $P_{\beta_{k+1}}(z_1,\dots,z_{k+1}) = P_{\beta_k}(z_1,\dots,z_k)\cdot B^{(n)}_{i_{k+1}}(z_{k+1})$.*

Definition 4.44. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. The* <u>*braid variety*</u> *$X(\beta, \mathbb{K})$ associated with β corresponds to the affine variety defined by*

$$X(\beta, \mathbb{K}) := \Big\{ (z_1, \ldots, z_\ell) \in \mathbb{K}^\ell \,\Big|\, P_\beta(z_1, \ldots, z_\ell) \text{ admits an LU decomposition} \Big\}. \tag{4.10}$$

Braid varieties have been the focus of considerable research (see, for instance, [KÓ6, CN22, GSW24]). In particular, Casals, Gorsky, Gorsky, and Simental, along with their collaborators, have explored their cluster-algebraic structures and their connections with classical varieties in algebraic geometry, including positroid varieties [CGGS24, CGG+25, CGGS21].

Next, we proceed to reformulate Theorem (4.40) in terms of the braid matrices. To facilitate this, we present the following lemma.

Lemma 4.45. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\{\mathcal{F}_j^\bullet\}_{j=1}^{\ell+1}$ be a sequence of complete flags in $\mathbb{K}^n$ such that, for each $j \in [1, \ell]$, $\mathcal{F}_{j+1}^\bullet$ is in s_{i_j}-relative position with respect to $\mathcal{F}_j^\bullet$.*

Let $\hat{\mathbf{f}}^{(n)} := \{\hat{f}_i^{(n)}\}_{i=1}^n$ be a basis for $\mathbb{K}^n$, and suppose that, relative to this basis, $\mathcal{F}_1^\bullet$ is the standard flag, that is, $\mathcal{F}_1^\bullet[\hat{\mathbf{f}}^{(n)}] = \mathcal{F}_{\mathrm{std}}^\bullet[\hat{\mathbf{f}}^{(n)}]$. Then there exists a tuple $\vec{z} = (z_1, \ldots, z_\ell) \in \mathbb{K}_{\mathrm{std}}^\ell$ such that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flag $\mathcal{F}_{j+1}^\bullet$ is represented by the path matrix $P_{\beta_j}(\vec{z}_j) = B_{i_1}^{(n)}(z_1) \cdots B_{i_j}^{(n)}(z_j) \in \mathrm{GL}(n, \mathbb{K})$ associated with the truncated braid word $\beta_j = \sigma_{i_1} \cdots \sigma_{i_j} \in \mathrm{Br}_n^+$ and the truncated tuple $\vec{z}_j = (z_1, \ldots, z_j) \in \mathbb{K}_{\mathrm{std}}^j$, for each $j \in [1, \ell]$.

Proof. To prove the claim, we proceed recursively. To begin, observe that $\hat{\mathbf{f}}^{(n)}$ is a basis for $\mathbb{K}^n$ such that $\mathcal{F}_1^\bullet[\hat{\mathbf{f}}^{(n)}] = \mathcal{F}_{\mathrm{std}}^\bullet[\hat{\mathbf{f}}^{(n)}]$. Hence, relative to this basis, the flag $\mathcal{F}_1^\bullet$ is represented by the matrix $\mathbf{1}_n$.

Now, consider the flag $\mathcal{F}_2^\bullet$. By assumption, $\mathcal{F}_2^\bullet$ is in s_{i_1}–relative position with respect to $\mathcal{F}_1^\bullet$. Then, by Lemma (4.28), there exists $z_1 \in \mathbb{K}$ such that, relative to the basis $\hat{\mathbf{f}}^{(n)}$, the flag $\mathcal{F}_2^\bullet$ is represented by the matrix product $\mathbf{1}_n \cdot B_{i_1}^{(n)}(z_1)$. In particular, since $\beta_1 = \sigma_{i_1}$, we have that $P_{\beta_1}(z_1) = B_{i_1}^{(n)}(z_1)$, implying that the flag $\mathcal{F}_2^\bullet$ is represented by the path matrix $P_{\beta_1}(z_1)$.

Next, consider the flag $\mathcal{F}_3^\bullet$. By hypothesis, $\mathcal{F}_3^\bullet$ is in s_{i_2}–relative position with respect to $\mathcal{F}_2^\bullet$. Then, by Lemma (4.28), there exists $z_2 \in \mathbb{K}$ such that, relative to the basis $\hat{\mathbf{f}}^{(n)}$, the flag $\mathcal{F}_3^\bullet$ is represented by the matrix product $P_{\beta_1}(z_1) \cdot B_{i_2}^{(n)}(z_2)$. Furthermore, by Remark (4.43), we know that $P_{\beta_2}(z_1, z_2) = P_{\beta_1}(z_1) \cdot B_{i_2}^{(n)}(z_2)$, confirming that the flag $\mathcal{F}_3^\bullet$ is represented by the path matrix $P_{\beta_2}(z_1, z_2)$.

Continuing this process recursively, we conclude that there exists a tuple $\vec{z} = (z_1, \ldots, z_\ell) \in \mathbb{K}_{\mathrm{std}}^\ell$ such that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flag $\mathcal{F}_{j+1}^\bullet$ is represented by the path matrix $P_{\beta_j}(\vec{z}_j) = B_{i_1}^{(n)}(z_1) \cdots B_{i_j}^{(n)}(z_j) \in$

$\mathrm{GL}(n,\mathbb{K})$ associated with the truncated braid word $\beta_j = \sigma_{i_1}\cdots\sigma_{i_j} \in \mathrm{Br}_n^+$ and the truncated tuple $\vec{z}_j = (z_1,\ldots,z_j) \in \mathbb{K}^j_{\mathrm{std}}$, for each $j \in [1,\ell]$. This completes the proof. □

Building on the above preliminary result, we now present the following theorem, which algebraically characterizes an object $\mathscr{F}$ of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ in terms of a basis for $\mathbb{K}^n$ and a point in the braid variety $X(\beta,\mathbb{K})$.

Theorem 4.46. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. In accordance with Theorem (4.40), we denote by $\{\mathcal{F}_j^\bullet\}_{j=0}^{\ell+1}$ the sequence of complete flags in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$.*

Let $\hat{\mathbf{f}}^{(n)} := \{\hat{f}_i^{(n)}\}_{i=1}^n$ be a basis for $\mathbb{K}^n$, and suppose that, relative to this basis, $\mathcal{F}_0^\bullet$ and $\mathcal{F}_1^\bullet$ are the anti-standard and standard flags, respectively, that is, $\mathcal{F}_0^\bullet[\hat{\mathbf{f}}^{(n)}] = \mathcal{F}^\bullet_{\mathrm{astd}}[\hat{\mathbf{f}}^{(n)}]$ and $\mathcal{F}_1^\bullet[\hat{\mathbf{f}}^{(n)}] = \mathcal{F}^\bullet_{\mathrm{std}}[\hat{\mathbf{f}}^{(n)}]$.

Then $\mathscr{F}$ is algebraically parametrized by the basis $\hat{\mathbf{f}}^{(n)}$ and a point $\vec{z} = (z_1,\ldots,z_\ell)$ in the braid variety $X(\beta,\mathbb{K}) \subset \mathbb{K}^\ell_{\mathrm{std}}$. Furthermore, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flag $\mathcal{F}^\bullet_{j+1}$ is represented by the path matrix $P_{\beta_j}(\vec{z}_j) := B^{(n)}_{i_1}(z_1)\cdots B^{(n)}_{i_j}(z_j) \in \mathrm{GL}(n,\mathbb{K})$ associated with the truncated braid word $\beta_j = \sigma_{i_1}\cdots\sigma_{i_j} \in \mathrm{Br}_n^+$ and the truncated tuple $\vec{z}_j = (z_1,\ldots,z_j) \in \mathbb{K}^j_{\mathrm{std}}$, for each $j \in [1,\ell]$.

Proof. By Theorem (4.40), $\{\mathcal{F}_j^\bullet\}_{j=0}^{\ell+1}$ is a sequence of complete flags in $\mathbb{K}^n$ such that: $\mathcal{F}_0^\bullet$ is completely opposite to both $\mathcal{F}_1^\bullet$ and $\mathcal{F}^\bullet_{\ell+1}$, and for each $j \in [1,\ell]$, $\mathcal{F}^\bullet_{j+1}$ is in s_{i_j}-relative position with respect to $\mathcal{F}_j^\bullet$.

Observe that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, $\mathcal{F}_0^\bullet$ and $\mathcal{F}_1^\bullet$ are the anti-standard and standard flags, respectively. Consequently, Lemma (4.45) asserts that there exists a tuple $\vec{z} = (z_1,\ldots,z_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$ such that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flag $\mathcal{F}^\bullet_{j+1}$ is represented by the path matrix $P^{(n)}_{\beta_j}(\vec{z}_j) = B^{(n)}_{i_1}(z_1)\cdots B^{(n)}_{i_j}(z_j) \in \mathrm{GL}(n,\mathbb{K})$ associated with the truncated braid word $\beta_j = \sigma_{i_1}\cdots\sigma_{i_j} \in \mathrm{Br}_n^+$ and the truncated tuple $\vec{z}_j = (z_1,\ldots,z_j) \in \mathbb{K}^j$, for each $j \in [1,\ell]$. Thus, since $\beta_\ell = \beta$ and $\vec{z}_\ell = \vec{z}$, we have that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flag $\mathcal{F}^\bullet_{\ell+1}$ is represented by the path matrix $P_\beta(\vec{z})$.

Finally, recall that $\mathcal{F}_0^\bullet$ is completely opposite to $\mathcal{F}^\bullet_{\ell+1}$. Then, since relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, $\mathcal{F}_0^\bullet$ is the anti-standard flag, the Bruhat decomposition of $\mathrm{GL}(n,\mathbb{K})$ implies that $P_\beta(\vec{z}) = \mathbf{w}_n\tilde{U}\mathbf{w}_n U$ for some upper triangular matrices $\tilde{U}, U \in \mathrm{GL}(n,\mathbb{K})$, where $\mathbf{w}_n$ corresponds to the permutation matrix associated with the longest element in S_n [CGGS24, CGG+25, SW21]. In particular, note that the matrix $L = \mathbf{w}_n\tilde{U}\mathbf{w}_n \in \mathrm{GL}(n,\mathbb{K})$ is lower triangular, and hence $P_\beta(\vec{z}) = LU$, which shows $P_\beta(\vec{z})$ admits an LU factorization. Bearing this in mind, it follows from Definition (4.44) that $\vec{z}$ is a point in the braid variety $X(\beta,\mathbb{K})$, which completes the proof. □

Lastly, let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$, and consider the open cover $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ of $\mathbb{R}^2$ (cf. Construction (4.33)), together with the partition $\mathcal{R}_{\Lambda(\beta)} = \{R_j\}_{j=1}^{\ell}$ of U_{B} into ℓ open vertical straps (cf. Construction (4.36)). We conclude this subsection by describing how the global correspondence established in Theorem (4.46) manifests locally on each vertical strap R_j. More precisely, following the approach developed by Chantraine, Ng, and Sivek in [CNS19], we construct algebraic local models for $\mathscr{F}$ on the regions R_j, employing braid matrices. In particular, these local models will be instrumental in the explicit computation of the graded morphism spaces and their compositions in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$, which constitutes the main subject of study in the subsequent section. Accordingly, we present the following results.

Lemma 4.47 (Algebraic Local Model I)**.** ***Setup***: *Let $\beta = \sigma_{i_1}\dots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Fix $j \in [1,\ell]$, and let R_j be the open vertical strap in $\mathbb{R}^2$ containing σ_{i_j}—the j-th crossing of β (see Figure (19)).*

$\star$ *Assumption 1: Let $k := i_j \in [1, n-1]$ denote the index of σ_{i_j}, and suppose that $k = 1$.*

In this setting, we have that on R_j, the sheaf $\mathscr{F}$ is specified by a collection of n injective linear maps

$$\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(1)}_{\mathscr{F}} : \mathbb{K}^1 \to \mathbb{K}^2\},$$

as illustrated in Figure (21). According to Lemma (4.37), the compatibility conditions for these maps ensure that the induced complete flags in $\mathbb{K}^n$,

$$\mathcal{F}^\bullet_j := {}_{\mathcal{I}}\mathcal{F}^\bullet\big(\phi^{(1)}_{\mathscr{F}}, \phi^{(2)}_{\mathscr{F}}, \dots, \phi^{(n-1)}_{\mathscr{F}}\big), \quad \text{and} \quad \mathcal{F}^\bullet_{j+1} := {}_{\mathcal{I}}\mathcal{F}^\bullet\big(\widetilde{\phi}^{(1)}_{\mathscr{F}}, \phi^{(2)}_{\mathscr{F}}, \dots, \phi^{(n-1)}_{\mathscr{F}}\big),$$

are in s_1-relative position.

$\star$ *Assumption 2: For each $i \in [1,n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}^{(i)}_j\}_{j=1}^{i}$ be a basis for $\mathbb{K}^i$, and suppose that $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to $\{\phi^{(i)}_{\mathscr{F}}\}_{i=1}^{n-1}$ (cf. Definition (4.18)-(3)).*

Within this framework and relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, Lemmas (4.22) and (4.28) algebraically capture the geometric compatibility between the flags $\mathcal{F}^\bullet_j$ and $\mathcal{F}^\bullet_{j+1}$ as follows:

- *$\mathcal{F}^\bullet_j$ is the standard flag, that is, $\mathcal{F}^\bullet_j[\hat{\mathbf{f}}^{(n)}] = \mathcal{F}^\bullet_{\mathrm{std}}[\hat{\mathbf{f}}^{(n)}]$, and is therefore represented by the matrix $\mathbf{1}_n$.*
- *$\mathcal{F}^\bullet_{j+1}$ is represented by the matrix product $\mathbf{1}_n \cdot B^{(n)}_1(z)$, where $B^{(n)}_1(z)$ denotes the first braid matrix of dimension n, for some $z \in \mathbb{K}$ parameterizing the s_1-relative position between $\mathcal{F}^\bullet_j$ and $\mathcal{F}^\bullet_{j+1}$.*

Main Conclusion: *Under the given **setup**, the following statements hold:*

- *For each* $i \in [1, n-1]$, *the matrix* ${}_{\hat{\mathbf{f}}^{(i+1)}}\big[\, \phi^{(i)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i+1, i, \mathbb{K})$ *representing* $\phi^{(i)}_{\mathscr{F}}$ *is given by*

$$ {}_{\hat{\mathbf{f}}^{(i+1)}}\big[\, \phi^{(i)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)} \,. $$

- *The matrix* ${}_{\hat{\mathbf{f}}^{(2)}}\big[\, \widetilde{\phi}^{(1)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{f}}^{(1)}} \in \mathrm{M}(2, 1, \mathbb{K})$ *representing* $\widetilde{\phi}^{(1)}_{\mathscr{F}}$ *can be written as*

$$ {}_{\hat{\mathbf{f}}^{(2)}}\big[\, \widetilde{\phi}^{(1)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{f}}^{(1)}} = B^{(2)}_1(z) \cdot \iota^{(2,1)} \,, $$

  where $B^{(2)}_1(z)$ *denotes the first braid matrix of dimension* 2*.*

Definition (system of bases adapted to $\mathscr{F}$ ***on*** R_j***):*** *For each* $i \in [1, n]$, *let* $\hat{\mathbf{h}}^{(i)} := \big\{ \hat{h}^{(i)}_j \big\}_{j=1}^{i}$ *be a basis for* $\mathbb{K}^i$, *and denote by* $\big\{ \hat{\mathbf{h}}^{(i)} \big\}_{i=1}^{n}$ *the collection of these bases.*

We say that a pair $\big(\big\{ \hat{\mathbf{h}}^{(i)} \big\}_{i=1}^{n}, \lambda \big)$, *with* $\lambda \in \mathbb{K}$, *is a* <u>*system of bases adapted to* $\mathscr{F}$ *on* R_j</u> *if, with respect to these bases, the linear maps defining* $\mathscr{F}$ *on* R_j *have the following matrix representations:*

$$ {}_{\hat{\mathbf{h}}^{(i+1)}}\big[\, \phi^{(i)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{h}}^{(i)}} = \iota^{(i+1,i)} \,, \qquad \textit{for all } i \in [1, n-1] \,, $$

$$ {}_{\hat{\mathbf{h}}^{(2)}}\big[\, \widetilde{\phi}^{(1)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{h}}^{(1)}} = B^{(2)}_1(\lambda) \cdot \iota^{(2,1)} \,. $$

In particular, under the given ***setup****, the* ***main conclusion*** *implies that* $\big(\big\{ \hat{\mathbf{f}}^{(i)} \big\}_{i=1}^{n}, z \big)$ *is a system of bases adapted to* $\mathscr{F}$ *on* R_j*.*

Proof. By assumption, $\big\{ \hat{\mathbf{f}}^{(i)} \big\}_{i=1}^{n}$ is a system of bases adapted to $\big\{ \phi^{(i)}_{\mathscr{F}} \big\}_{i=1}^{n-1}$. Then, Definition (4.18)–(3) asserts that, for each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{f}}^{(i+1)}}\big[\, \phi^{(i)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i+1, i, \mathbb{K})$ representing $\phi^{(i)}_{\mathscr{F}}$ is the standard inclusion matrix:

$$ {}_{\hat{\mathbf{f}}^{(i+1)}}\big[\, \phi^{(i)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)} \,. $$

Here, recall that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flags $\mathcal{F}^{\bullet}_j$ and $\mathcal{F}^{\bullet}_{j+1}$ that geometrically characterize $\mathscr{F}$ on R_j are represented by the matrices $\mathbf{1}_n$ and $B^{(n)}_1(z)$, respectively, where $B^{(n)}_1(z)$ denotes the first braid matrix of dimension n, for some $z \in \mathbb{K}$ parameterizing the s_1-relative position between $\mathcal{F}^{\bullet}_j$ and $\mathcal{F}^{\bullet}_{j+1}$. In particular, this suggests that the matrix ${}_{\hat{\mathbf{f}}^{(2)}}\big[\, \widetilde{\phi}^{(1)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{f}}^{(1)}} \in \mathrm{M}(2, 1, \mathbb{K})$ representing $\widetilde{\phi}^{(1)}_{\mathscr{F}}$ can be expressed as

$$ {}_{\hat{\mathbf{f}}^{(2)}}\big[\, \widetilde{\phi}^{(1)}_{\mathscr{F}} \,\big]_{\hat{\mathbf{f}}^{(1)}} = B^{(2)}_1(z) \cdot \iota^{(2,1)} \,, $$

where $B^{(2)}_1(z)$ denotes the first braid matrix of dimension 2.

Next, we verify that the linear map $\widetilde{\phi}^{(1)}_{\mathscr{F}}$, henceforth thought of as defined by the above matrix expression in the given bases, is compatible with the microlocal support conditions for $\mathscr{F}$ on R_j and induces the flag $\mathcal{F}^{\bullet}_{j+1}$ in $\mathbb{K}^n$.

To begin, let $\vec{x} := \alpha_1 \hat{f}^{(2)}_1 + \alpha_2 \hat{f}^{(2)}_2 \in \mathbb{K}^2$, for some $\alpha_1, \alpha_2 \in \mathbb{K}$. Then, we have that

$$\begin{aligned} \vec{x} &= \alpha_1 \hat{f}^{(2)}_1 - z\,\alpha_2 \hat{f}^{(2)}_1 + z\,\alpha_2 \hat{f}^{(2)}_1 + \alpha_2 \hat{f}^{(2)}_2 \,, \\ &= \phi^{(1)}_{\mathscr{F}}\big(\alpha_1 \hat{f}^{(1)}_1 - z\,\alpha_2 \hat{f}^{(1)}_1\big) - \widetilde{\phi}^{(1)}_{\mathscr{F}}\big(- \alpha_2 \hat{f}^{(1)}_1\big)\,, \end{aligned}$$

which implies that the linear map $\phi^{(1)}_{\mathscr{F}} \oplus \big(- \widetilde{\phi}^{(1)}_{\mathscr{F}}\big) : \mathbb{K} \oplus \mathbb{K} \longrightarrow \mathbb{K}^2$ is surjective. Consequently, since the linear maps $\phi^{(1)}_{\mathscr{F}}$ and $\widetilde{\phi}^{(1)}_{\mathscr{F}}$ are injective, we conclude that:

(i) The diagram in Figure (22) commutes.

(ii) The sequence in Equation (4.8) is short exact.

Thus, $\widetilde{\phi}^{(1)}_{\mathscr{F}}$ is compatible with the microlocal support conditions for $\mathscr{F}$ on R_j.

Finally, let $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\big\}^n_{i=1}$ be the braid-transformed bases obtained from $\big\{\hat{\mathbf{f}}^{(i)}\big\}^n_{i=1}$ via σ_1 and the parameter z (cf, Definition (4.29)). In particular, observe that:

- By construction, $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\big\}^n_{i=1}$ is a system of bases adapted to $\big\{\widetilde{\phi}^{(1)}_{\mathscr{F}}, \phi^{(2)}_{\mathscr{F}}, \ldots, \phi^{(n-1)}_{\mathscr{F}}\big\}$ in the sense of Definition (4.18)–(3).
- $\mathcal{F}^{\bullet}_{j+1} = {}_{\mathcal{I}}\mathcal{F}^{\bullet}\big(\widetilde{\phi}^{(1)}_{\mathscr{F}}, \phi^{(2)}_{\mathscr{F}}, \ldots, \phi^{(n-1)}_{\mathscr{F}}\big)$ is the type $\mathcal{I}$ flag in $\mathbb{K}^n$ associated with the indicated collection of maps (cf. Definition (4.18)–(1)).

Hence, by Lemma (4.22), we obtain that, relative to the basis $\hat{\mathbf{f}}^{(n)}[\sigma_1, z]$ for $\mathbb{K}^n$, $\mathcal{F}^{\bullet}_{j+1}$ is the standard flag, that is, $\mathcal{F}^{\bullet}_{j+1}\big[\hat{\mathbf{f}}^{(n)}[\sigma_1, z]\big] = \mathcal{F}^{\bullet}_{\text{std}}\big[\hat{\mathbf{f}}^{(n)}[\sigma_1, z]\big]$, and is therefore represented by the matrix $\mathbf{1}_n$.

Furthermore, by Definition (4.29), the bases $\hat{\mathbf{f}}^{(n)}$ and $\hat{\mathbf{f}}^{(n)}[\sigma_1, z]$ for $\mathbb{K}^n$ are related via the change-of-basis matrix $B^{(n)}_1(z)$; specifically, for each $j \in [1, n]$,

$$\hat{f}^{(n)}_j[\sigma_1, z] = \sum_{q=1}^{n} \big(B^{(n)}_1(z)\big)_{q,j}\, \hat{f}^{(n)}_q \,.$$

Therefore, by Lemma (4.26), we conclude that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flag $\mathcal{F}^{\bullet}_{j+1}$ is represented by the matrix product $B^{(n)}_1(z) \cdot \mathbf{1}_n$, confirming that the linear map $\widetilde{\phi}^{(1)}_{\mathscr{F}}$ induces the flag $\mathcal{F}^{\bullet}_{j+1}$, as required. □

Lemma 4.48 (Algebraic Local Model II)**.** ***Setup****:* *Let $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell} \in \mathrm{Br}^+_n$ be a positive braid word, and let $\mathscr{F}$ be an object of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Fix $j \in [1, \ell]$, and let R_j be the open vertical strap in $\mathbb{R}^2$ containing σ_{i_j}—the j-th crossing of β (see Figure (19)).*

⋆ *Assumption 1: Let $k := i_j \in [1, n-1]$ denote the index of σ_{i_j}, and suppose that $k \geq 2$.*

In this setting, we have that on R_j, the sheaf $\mathscr{F}$ is specified by a collection of $n+1$ injective linear maps

$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}_{\mathscr{F}}^{(k-1)} : \mathbb{K}^{k-1} \to \mathbb{K}^k ,\ \widetilde{\phi}_{\mathscr{F}}^{(k)} : \mathbb{K}^k \to \mathbb{K}^{k+1}\right\},$$

as illustrated in Figure (23). According to Lemma (4.38), the compatibility conditions for these maps ensure that the induced complete flags in $\mathbb{K}^n$,

$$\mathcal{F}_j^\bullet := {}_{\mathcal{I}}\mathcal{F}^\bullet\left(\phi_{\mathscr{F}}^{(1)}, \ldots, \phi_{\mathscr{F}}^{(k-2)}, \phi_{\mathscr{F}}^{(k-1)}, \phi_{\mathscr{F}}^{(k)}, \phi_{\mathscr{F}}^{(k+1)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\right),$$

$$\mathcal{F}_{j+1}^\bullet := {}_{\mathcal{I}}\mathcal{F}^\bullet\left(\phi_{\mathscr{F}}^{(1)}, \ldots, \phi_{\mathscr{F}}^{(k-2)}, \widetilde{\phi}_{\mathscr{F}}^{(k-1)}, \widetilde{\phi}_{\mathscr{F}}^{(k)}, \phi_{\mathscr{F}}^{(k+1)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\right),$$

are in s_k-relative position.

⋆ *Assumption 2: For each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \left\{\hat{f}_j^{(i)}\right\}_{j=1}^{i}$ be a basis for $\mathbb{K}^i$, and suppose that $\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^{n}$ is a system of bases adapted to $\left\{\phi_{\mathscr{F}}^{(i)}\right\}_{i=1}^{n-1}$ (cf. Definition (4.18)–(3)).*

Within this framework and relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, Lemmas (4.22) and (4.28) algebraically capture the geometric compatibility between the flags $\mathcal{F}_j^\bullet$ and $\mathcal{F}_{j+1}^\bullet$ as follows:

i) $\mathcal{F}_j^\bullet$ is the standard flag, that is, $\mathcal{F}_j^\bullet\left[\hat{\mathbf{f}}^{(n)}\right] = \mathcal{F}_{\mathrm{std}}^\bullet\left[\hat{\mathbf{f}}^{(n)}\right]$, and is therefore represented by the matrix $\mathbf{1}_n$.

ii) $\mathcal{F}_{j+1}^\bullet$ is represented by the matrix product $\mathbf{1}_n \cdot B_k^{(n)}(z)$, where $B_k^{(n)}(z)$ denotes the k-th braid matrix of dimension n, for some $z \in \mathbb{K}$ parameterizing the s_k-relative position between $\mathcal{F}_j^\bullet$ and $\mathcal{F}_{j+1}^\bullet$.

***Main Result**: Under the given **setup**, the following statements hold:*

- *For each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{f}}^{(i+1)}}\left[\phi_{\mathscr{F}}^{(i)}\right]_{\hat{\mathbf{f}}^{(i)}} \in M(i+1, i, \mathbb{K})$ representing $\phi_{\mathscr{F}}^{(i)}$ is given by*

$${}_{\hat{\mathbf{f}}^{(i+1)}}\left[\phi_{\mathscr{F}}^{(i)}\right]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)}.$$

- *The matrices ${}_{\hat{\mathbf{f}}^{(k)}}\left[\widetilde{\phi}_{\mathscr{F}}^{(k-1)}\right]_{\hat{\mathbf{f}}^{(k-1)}} \in \mathrm{M}(k, k-1, \mathbb{K})$ and ${}_{\hat{\mathbf{f}}^{(k+1)}}\left[\widetilde{\phi}_{\mathscr{F}}^{(k)}\right]_{\hat{\mathbf{f}}^{(k)}} \in \mathrm{M}(k+1, k, \mathbb{K})$ representing $\widetilde{\phi}_{\mathscr{F}}^{(k-1)}$ and $\widetilde{\phi}_{\mathscr{F}}^{(k)}$, respectively, can be written as*

$${}_{\hat{\mathbf{f}}^{(k)}}\left[\widetilde{\phi}_{\mathscr{F}}^{(k-1)}\right]_{\hat{\mathbf{f}}^{(k-1)}} = \iota^{(k,k-1)}, \quad \text{and} \quad {}_{\hat{\mathbf{f}}^{(k+1)}}\left[\widetilde{\phi}_{\mathscr{F}}^{(k)}\right]_{\hat{\mathbf{f}}^{(k)}} = B_k^{(k+1)}(z) \cdot \iota^{(k+1,k)},$$

where $B_k^{(k+1)}(z)$ denotes the k-th braid matrix of dimension $k+1$.

***Definition (system of bases adapted to $\mathscr{F}$ on R_j)**: For each $i \in [1, n]$, let $\hat{\mathbf{h}}^{(i)} := \left\{\hat{h}_j^{(i)}\right\}_{j=1}^{i}$ be a basis for $\mathbb{K}^i$, and denote by $\left\{\hat{\mathbf{h}}^{(i)}\right\}_{i=1}^{n}$ the collection of these bases.*

We say that a pair $\big(\{\hat{\mathbf{h}}^{(i)}\}_{i=1}^{n}, \lambda\big)$, with $\lambda \in \mathbb{K}$, is a system of bases adapted to $\mathscr{F}$ on R_j if, with respect to these bases, the linear maps defining $\mathscr{F}$ on R_j have the following matrix representations:

$$_{\hat{\mathbf{h}}^{(i+1)}}\big[\,\phi_{\mathscr{F}}^{(i)}\,\big]_{\hat{\mathbf{h}}^{(i)}} = \iota^{(i+1,i)}\,, \qquad \textit{for all } i \in [1, n-1]\,,$$

$$_{\hat{\mathbf{h}}^{(k)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k-1)}\,\big]_{\hat{\mathbf{h}}^{(k-1)}} = \iota^{(k,k-1)}\,, \qquad _{\hat{\mathbf{h}}^{(k+1)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k)}\,\big]_{\hat{\mathbf{h}}^{(k)}} = B_k^{(k+1)}(\lambda)\cdot\iota^{(k+1,k)}\,.$$

In particular, under the given ***setup****, the* ***main conclusion*** *implies that $\big(\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}, z\big)$ is a system of bases adapted to $\mathscr{F}$ on R_j.*

Proof. By assumption, $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$. Then, Definition (4.18)–(3) asserts that, for each $i \in [1, n-1]$, the matrix $_{\hat{\mathbf{f}}^{(i+1)}}\big[\,\phi_{\mathscr{F}}^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i+1, i, \mathbb{K})$ representing $\phi_{\mathscr{F}}^{(i)}$ is the standard inclusion matrix:

$$_{\hat{\mathbf{f}}^{(i+1)}}\big[\,\phi_{\mathscr{F}}^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)}\,.$$

Here, recall that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flags $\mathcal{F}_j^{\bullet}$ and $\mathcal{F}_{j+1}^{\bullet}$ that geometrically characterize $\mathscr{F}$ on R_j are represented by the matrices $\mathbf{1}_n$ and $B_k^{(n)}(z)$, respectively, where $B_k^{(n)}(z)$ denotes the k-th braid matrix of dimension n, for some $z \in \mathbb{K}$ parameterizing the s_k-relative position between $\mathcal{F}_j^{\bullet}$ and $\mathcal{F}_{j+1}^{\bullet}$. In particular, this suggests that the matrices $_{\hat{\mathbf{f}}^{(k)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}} \in \mathrm{M}(k, k-1, \mathbb{K})$ and $_{\hat{\mathbf{f}}^{(k+1)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}} \in \mathrm{M}(k+1, k, \mathbb{K})$ representing $\widetilde{\phi}_{\mathscr{F}}^{(k-1)}$ and $\widetilde{\phi}_{\mathscr{F}}^{(k)}$, respectively, can be expressed as

$$_{\hat{\mathbf{f}}^{(k)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}} = \iota^{(k,k-1)}, \quad \text{and} \quad _{\hat{\mathbf{f}}^{(k+1)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}} = B_k^{(k+1)}(z)\cdot\iota^{(k+1,k)}\,,$$

where $B_k^{(k+1)}(z)$ denotes the k-th braid matrix of dimension $k+1$.

Next, we verify that the linear maps $\widetilde{\phi}_{\mathscr{F}}^{(k-1)}$ and $\widetilde{\phi}_{\mathscr{F}}^{(k)}$, henceforth thought of as defined by the above matrix expressions in the given bases, are compatible with the microlocal support conditions for $\mathscr{F}$ on R_j and induce the flag $\mathcal{F}_{j+1}^{\bullet}$ in $\mathbb{K}^n$.

To begin, a direct calculation shows that

$$\begin{aligned} _{\hat{\mathbf{f}}^{(k+1)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}}\cdot{}_{\hat{\mathbf{f}}^{(k)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}} &= B_k^{(k+1)}(z)\cdot\iota^{(k+1,k)}\cdot\iota^{(k,k-1)}\,,\\ &= B_k^{(k+1)}(z)\cdot\iota^{(k+1,k-1)}\,,\\ &= \iota^{(k+1,k-1)}\,, \end{aligned}$$

and similarly,

$$
\begin{aligned}
{}_{\hat{\mathbf{f}}^{(k+1)}}\big[\,\phi_{\mathscr{F}}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}} \cdot {}_{\hat{\mathbf{f}}^{(k)}}\big[\,\phi_{\mathscr{F}}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}} &= \iota^{(k+1,k)} \cdot \iota^{(k,k-1)}\,, \\
&= \iota^{(k+1,k-1)}\,.
\end{aligned}
$$

It follows that

$$
\phi_{\mathscr{F}}^{(k)} \circ \phi_{\mathscr{F}}^{(k-1)}(y) = \widetilde{\phi}_{\mathscr{F}}^{(k)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k-1)}(y)\,,
$$

for all $y := \beta_1 \hat{f}_1^{(k-1)} + \cdots + \beta_{k-1}\hat{f}_{k-1}^{(k-1)} \in \mathbb{K}^{k-1}$, with $\beta_1, \ldots, \beta_{k-1} \in \mathbb{K}$.

Now, let $\vec{x} := \alpha_1 \hat{f}_1^{(k+1)} + \cdots + \alpha_{k+1}\hat{f}_{k+1}^{(k+1)} \in \mathbb{K}^{k+1}$, for some $\alpha_1, \ldots, \alpha_{k+1} \in \mathbb{K}$. Then, we have that

$$
\begin{aligned}
\vec{x} &= \alpha_1 \hat{f}_1^{(k+1)} + \cdots + \alpha_k \hat{f}_k^{(k+1)} - z\,\alpha_{k+1}\hat{f}_k^{(k+1)} + z\,\alpha_{k+1}\hat{f}_k^{(k+1)} + \alpha_{k+1}\hat{f}_{k+1}^{(k+1)}\,, \\
&= \phi_{\mathscr{F}}^{(k)}\big(\alpha_1 \hat{f}_1^{(k)} + \cdots + \alpha_k \hat{f}_k^{(k)} - z\,\alpha_{k+1}\hat{f}_k^{(k)}\big) - \widetilde{\phi}_{\mathscr{F}}^{(k)}\big(-\alpha_{k+1}\hat{f}_k^{(k)}\big)\,,
\end{aligned}
$$

which implies that the linear map $\phi_{\mathscr{F}}^{(k)} \oplus (-\widetilde{\phi}_{\mathscr{F}}^{(k)}) : \mathbb{K}^k \oplus \mathbb{K}^k \longrightarrow \mathbb{K}^{k+1}$ is surjective.

In particular, since the linear maps $\phi_{\mathscr{F}}^{(k-1)}$, $\phi_{\mathscr{F}}^{(k)}$, $\widetilde{\phi}_{\mathscr{F}}^{(k-1)}$, and $\widetilde{\phi}_{\mathscr{F}}^{(k)}$ are injective, the above results guarantee that:

(i) The diagram in Figure (24) commutes.

(ii) The sequence in Equation (4.9) is short exact.

Hence, $\widetilde{\phi}_{\mathscr{F}}^{(k-1)}$ and $\widetilde{\phi}_{\mathscr{F}}^{(k)}$ are compatible with the microlocal support conditions for $\mathscr{F}$ on R_j.

Finally, let $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\big\}_{i=1}^n$ be the braid-transformed bases obtained from $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^n$ via σ_k and the parameter z (cf. Definition (4.29)). In particular, observe that:

- By construction, $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\big\}_{i=1}^n$ is a system of bases adapted to the collection of linear maps

$$
\big\{\phi_{\mathscr{F}}^{(1)}, \ldots, \phi_{\mathscr{F}}^{(k-2)}, \widetilde{\phi}_{\mathscr{F}}^{(k-1)}, \widetilde{\phi}_{\mathscr{F}}^{(k)}, \phi_{\mathscr{F}}^{(k+1)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\big\}
$$

  in the sense of Definition (4.18)–(3).

- $\mathcal{F}_{j+1}^{\bullet} = {}_{\mathcal{I}}\mathcal{F}^{\bullet}\big(\,\phi_{\mathscr{F}}^{(1)}, \ldots, \phi_{\mathscr{F}}^{(k-2)}, \widetilde{\phi}_{\mathscr{F}}^{(k-1)}, \widetilde{\phi}_{\mathscr{F}}^{(k)}, \phi_{\mathscr{F}}^{(k+1)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\,\big)$ is the type $\mathcal{I}$ flag in $\mathbb{K}^n$ associated with the indicated collection of maps (cf. Definition (4.18)–(1)).

Hence, by Lemma (4.22), we obtain that, relative to the basis $\hat{\mathbf{f}}^{(n)}[\sigma_k, z]$ for $\mathbb{K}^n$, the flag $\mathcal{F}_{j+1}^{\bullet}$ is the standard flag, that is, $\mathcal{F}_{j+1}^{\bullet}\big[\,\hat{\mathbf{f}}^{(n)}[\sigma_k, z]\,\big] = \mathcal{F}_{\mathrm{std}}^{\bullet}\big[\,\hat{\mathbf{f}}^{(n)}[\sigma_k, z]\,\big]$, and is therefore represented by the matrix $\mathbf{1}_n$.

Furthermore, by Definition (4.29), the bases $\hat{\mathbf{f}}^{(n)}$ and $\hat{\mathbf{f}}^{(n)}[\sigma_k, z]$ for $\mathbb{K}^n$ are related via the change-of-basis matrix $B_k^{(n)}(z)$; specifically, for each $j \in [1, n]$,

$$\hat{f}_j^{(n)}[\sigma_k, z] = \sum_{q=1}^{n} \left(B_k^{(n)}(z)\right)_{q,j} \hat{f}_q^{(n)}\,.$$

Therefore, by Lemma (4.26), we conclude that, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flag $\mathcal{F}_{j+1}^{\bullet}$ is represented by the matrix product $B_k^{(n)}(z) \cdot \mathbf{1}_n$, confirming that the linear maps $\widetilde{\phi}_{\mathscr{F}}^{(k-1)}$ and $\widetilde{\phi}_{\mathscr{F}}^{(k)}$ induce the flag $\mathcal{F}_{j+1}^{\bullet}$, as required. □

Remark 4.49. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathscr{F}$ an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and $\mathcal{R}_{\Lambda(\beta)} = \left\{R_j\right\}_{j=1}^{\ell}$ the collection of ℓ open vertical straps in $\mathbb{R}^2$ introduced in Construction (4.36). For each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \left\{\hat{f}_j^{(i)}\right\}_{j=1}^{i}$ be a basis for $\mathbb{K}^i$, and denote by $\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^{n}$ the collection of these bases.*

Intuitively, Lemmas (4.47) and (4.48) assert the following. If $\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^{n}$ is a system of bases adapted to the $n-1$ injective linear maps defining $\mathscr{F}$ to the left of a crossing σ_{i_j} within a vertical strap R_j, then the pair $\left(\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^{n}, z\right)$ defines a system of bases adapted to $\mathscr{F}$ on R_j, where $z \in \mathbb{K}$ algebraically encodes, relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the s_{i_j}-relative position between the complete flags in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$ on R_j. In other words, the behavior of $\mathscr{F}$ on R_j is completely determined by the single parameter z together with the adapted bases $\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^{n}$ on the left of the crossing σ_{i_j}.

Building on the local algebraic descriptions established in Lemmas (4.47) and (4.48), we now state the following result.

Theorem 4.50. ***Setup**: Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ be an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\mathcal{U}_{\Lambda(\beta)} = \left\{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\right\}$ be the open cover of $\mathbb{R}^2$ from Construction (4.33), and let $\mathcal{R}_{\Lambda(\beta)} = \left\{R_j\right\}_{j=1}^{\ell}$ be the partition of U_{B} into ℓ open vertical straps from Construction (4.36).*

⋆ *Local descriptions on U_{T} and U_{L}: According to Lemma (4.34), $\mathscr{F}$ admits the following local descriptions:*

- *On U_{T}, $\mathscr{F}$ is specified by a collection of $n-1$ surjective linear maps $\left\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\right\}_{i=1}^{n-1}$ (see Figure (15)).*
- *On U_{L}, $\mathscr{F}$ is specified by a collection of $n-1$ injective linear maps $\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1}$ (see Figure (13)).*
- ***Compatibility conditions**: For each $i \in [1, n-1]$,*

$$\psi_{\mathscr{F}}^{(i)} \circ \phi_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}\,.$$

⋆ *Global flag data: By Theorem (4.40), $\mathscr{F}$ is geometrically characterized by a sequence of complete flags $\left\{\mathcal{F}_j^{\bullet}\right\}_{j=0}^{\ell+1}$ in $\mathbb{K}^n$ such that: $\mathcal{F}_0^{\bullet}$ is completely opposite to both $\mathcal{F}_1^{\bullet}$ and $\mathcal{F}_{\ell+1}^{\bullet}$, and for each $j \in [1, \ell]$, $\mathcal{F}_j^{\bullet}$ is in s_{i_j}-relative* *position with respect to $\mathcal{F}^\bullet_{j+1}$. In particular, by Lemma (4.35), we know that:*

$$\mathcal{F}^\bullet_0 := {}_{\mathcal{K}}\mathcal{F}^\bullet\big(\psi^{(1)}_{\mathscr{F}}, \dots, \psi^{(n-1)}_{\mathscr{F}}\big), \quad \textit{and} \quad \mathcal{F}^\bullet_1 := {}_{\mathcal{I}}\mathcal{F}^\bullet\big(\phi^{(1)}_{\mathscr{F}}, \dots, \phi^{(n-1)}_{\mathscr{F}}\big),$$

are the type $\mathcal{K}$ and type $\mathcal{I}$ flags in $\mathbb{K}^n$ associated with $\big\{\psi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}$ and $\big\{\phi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}$, respectively (cf. Definition (4.18)–(1)–(2)).

⋆ *<u>Main assumption</u> (Adapted bases): Let $\hat{\mathbf{f}}^{(n)}$ be a basis for $\mathbb{K}^n$, and let $\vec{z} = (z_1, \dots, z_\ell) \in X(\beta, \mathbb{K})$ be a point such that the pair $\big(\hat{\mathbf{f}}^{(n)}, \vec{z}\big)$ algebraically characterizes $\mathscr{F}$ according to Theorem (4.46).*

In this setting, we have that relative to the basis $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, the flags $\mathcal{F}^\bullet_0$ and $\mathcal{F}^\bullet_1$ are the anti-standard and standard flags, respectively, and for each $j \in [1, \ell]$, the flag $\mathcal{F}^\bullet_{j+1}$ is represented by the path matrix $P_{\beta_j}(\vec{z}_j) = B^{(n)}_{i_1}(z_1) \cdots B^{(n)}_{i_j}(z_j) \in \mathrm{GL}(n, \mathbb{K})$ associated with the truncated braid word $\beta_j = \sigma_{i_1} \cdots \sigma_{i_j} \in \mathrm{Br}^+_n$ and the truncated tuple $\vec{z}_j = (z_1, \dots, z_j) \in \mathbb{K}^j_{\mathrm{std}}$.

*Furthermore, under this assumption, the **compatibility conditions** and an inductive argument assert that, for each $i \in [1, n-1]$, there is a unique basis $\hat{\mathbf{f}}^{(i)} := \big\{\hat{f}^{(i)}_j\big\}_{j=1}^{i}$ for $\mathbb{K}^i$ such that the collection $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}$ is a system of bases adapted to both $\big\{\psi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}$ and $\big\{\phi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}$ (cf. Definition (4.18)–(3)–(4)).*

***Main conclusion**: For each $j \in [1, \ell]$, let $\big\{\hat{\mathbf{f}}^{(i)}[\beta_j, \vec{z}_j]\big\}_{i=1}^{n}$ denote the braid-transformed bases obtained from $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}$ via the truncated braid word β_j and the truncated tuple $\vec{z}_j$ (cf. Definition (4.29)). Then:*

- *$\big(\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}, z_1\big)$ is a system of bases adapted to $\mathscr{F}$ on R_1.*
- *For each $j \in [1, \ell-1]$, $\big(\big\{\hat{\mathbf{f}}^{(i)}[\beta_j, \vec{z}_j]\big\}_{i=1}^{n}, z_{j+1}\big)$ is a system of bases adapted to $\mathscr{F}$ on R_{j+1}.*

Proof. To begin, let R_1 be the open vertical strap containing σ_{i_1}, the first crossing of β, and let z_1 be the first entry of $\vec{z}$, which parametrizes the s_{i_1}-relative position between the complete flags $\mathcal{F}^\bullet_1$ and $\mathcal{F}^\bullet_2$ in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$ on R_1. By assumption, $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}$ is a system of bases adapted to the $n-1$ injective linear maps defining $\mathscr{F}$ on the intersection $U_{\mathrm{L}} \cap R_1$, namely $\big\{\phi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}$. Then, depending on whether $i_1 = 1$ or $i_1 \geq 2$, we can apply Lemma (4.47) or Lemma (4.48) to deduce that $\big(\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}, z_1\big)$ is a system of bases adapted to $\mathscr{F}$ on R_1.

Now, consider R_2, the open vertical strap containing σ_{i_2}, the second crossing of β, and let z_2 be the second entry of $\vec{z}$, which parametrizes the s_{i_2}-relative position between the complete flags $\mathcal{F}^\bullet_2$ and $\mathcal{F}^\bullet_3$ in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$ on R_2. In addition, let $\big\{\hat{\mathbf{f}}^{(i)}[\beta_1, \vec{z}_1]\big\}_{i=1}^{n}$ be the braid-transformed bases obtained from $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}$ via the truncated braid word $B_1 = \sigma_{i_1}$ and the truncated tuple $\vec{z}_1 = z_1$ (cf. Definition (4.29)). By construction, $\big\{\hat{\mathbf{f}}^{(i)}[\beta_1, \vec{z}_1]\big\}_{i=1}^{n}$ is a system of bases adapted to the $n-1$ injective linear maps defining $\mathscr{F}$ on the intersection $R_1 \cap R_2$. Then, depending on whether $i_2 = 1$ or $i_2 \geq 2$, we can apply Lemma (4.47) or Lemma (4.48) to obtain that $\big(\{\hat{\mathbf{f}}^{(i)}[\beta_1, \vec{z}_1]\}_{i=1}^{n}, z_2\big)$ is a system of bases adapted to $\mathscr{F}$ on R_2.

Finally, for each $j \in [1, \ell]$, let $\{\hat{\mathbf{f}}^{(i)}[\beta_j, \vec{z}_j]\}_{i=1}^{n}$ be the braid-transformed bases obtained from $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ via the truncated braid word $B_j = \sigma_{i_1} \cdots \sigma_{i_j} \in \mathrm{Br}_n^+$ and the truncated tuple $\vec{z}_j = (z_1, \ldots, z_j) \in \mathbb{K}^j_{\mathrm{std}}$. Then, iterating the above argument over all the straps R_j shows that, for each $j \in [1, \ell - 1]$, the pair $\big(\{\hat{\mathbf{f}}^{(i)}[\beta_j, \vec{z}_j]\}_{i=1}^{n}, z_{j+1}\big)$ is a system of bases adapted to $\mathscr{F}$ on R_{j+1}, as desired. □

Remark 4.51. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathscr{F}$ an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and $\mathcal{R}_{\Lambda(\beta)} = \{R_j\}_{j=1}^{\ell}$ the collection of ℓ open vertical straps in $\mathbb{R}^2$ introduced in Construction (4.36). Let $\hat{\mathbf{f}}^{(n)}$ be a basis for $\mathbb{K}^n$, and let $\vec{z} = (z_1, \ldots, z_\ell) \in X(\beta, \mathbb{K})$ be a point such that the pair $(\hat{\mathbf{f}}^{(n)}, \vec{z})$ algebraically characterizes $\mathscr{F}$ according to Theorem (4.46). Then, Theorem (4.50) illustrates how the algebraic data $(\hat{\mathbf{f}}^{(n)}, \vec{z})$ locally determines the behavior of $\mathscr{F}$ on each vertical strap R_j via the notion of braid-transformed bases, which will play a key role in the computations and discussions ahead.*

Having completed the algebraic characterization of the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, we now turn to the study of its graded morphism spaces and their compositions, which will further elucidate the rich algebraic, geometric, and combinatorial structures of the categorical invariant under consideration.

5. The Morphism Spaces and Their Compositions in the Category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. In this section, we present an explicit description of the graded morphism spaces and their compositions in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. To lay the groundwork, we begin by reviewing some key results due to Chantraine, Ng, and Sivek [CNS19].

5.1. Higher-Degree Morphism Spaces.

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. In this subsection, we analyze the structure of the higher-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. As a first step, we recall the following fundamental result of Chantraine, Ng, and Sivek [CNS19].

Proposition 5.52 ([Propositions 6.5 & 6.6, [CNS19]). *Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. Let $r, s \geq 1$ be integers, and let $\mathscr{F}$ and $\mathscr{G}$ be objects of the categories $H^\bullet(\mathcal{S}h_r(\Lambda(\beta), \mathbb{K})_0)$ and $H^\bullet(\mathcal{S}h_s(\Lambda(\beta), \mathbb{K})_0)$, respectively. Then, the following statements hold:*

a) For all $p \geq 2$,

$$H^p\Big(R\Gamma\big(\mathbb{R}^2; \mathscr{H}om(\mathscr{F}, \mathscr{G})\big)\Big) = 0\,.$$

b) For all $q \geq 1$,

$$H^q\big(R\mathscr{H}om(\mathscr{F},\mathscr{G})\big) = 0\,.$$

In other words, $R\mathscr{H}om(\mathscr{F},\mathscr{G}) \simeq \mathscr{H}om(\mathscr{F},\mathscr{G})$.

Next, let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. The local–to–global Ext spectral sequence establishes that

$$E_2^{p,q} = H^p\big(R\Gamma(\mathbb{R}^2; H^q(R\mathscr{H}om(\mathscr{F},\mathscr{G})))\big) \Longrightarrow \operatorname{Ext}^{p+q}(\mathscr{F},\mathscr{G})\,,$$

for all $p,q \geq 0$. By Proposition (5.52)-(b), we have that $H^q\big(R\mathscr{H}om(\mathscr{F},\mathscr{G})\big) = 0$ for all $q \geq 1$, and therefore $E_2^{p,q} = 0$ whenever $q \geq 1$. It follows that the E_2-page is supported entirely along the axis $q = 0$, and as a result, the spectral sequence collapses at this page. Consequently, we obtain that

$$\operatorname{Ext}^p(\mathscr{F},\mathscr{G}) = E_2^{p,0} = H^p\Big(R\Gamma\big(\mathbb{R}^2; H^0(R\mathscr{H}om(\mathscr{F},\mathscr{G}))\big)\Big) = H^p\Big(R\Gamma\big(\mathbb{R}^2;\mathscr{H}om(\mathscr{F},\mathscr{G})\big)\Big)\,,$$

for all $p \geq 0$ [CNS19]. Proposition (5.52)-(a) then implies that

$$\operatorname{Ext}^p(\mathscr{F},\mathscr{G}) = 0\,,$$

for all $p \geq 2$. In particular, this shows that the higher-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ are trivial.

Having established the vanishing of the higher-degree morphism spaces in category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$, we now focus on analyzing the structure of its lower-degree morphism spaces.

5.2. **Lower-Degree Morphism Spaces.** Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. In this subsection, our goal is to provide an explicit and computable description of the lower-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. To this end, we begin by introducing some notation.

Notation 5.53. *Let $n \geq 1$ be an integer. For any tuple $\vec{u} = (u_1,\ldots,u_n) \in \mathbb{K}^n_{\mathrm{std}}$, we denote by $D(\vec{u}) \in \mathrm{M}(n,\mathbb{K})$ the $n \times n$ diagonal matrix given by*

$$D(\vec{u}) := \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix}.$$

Notation 5.54. *Let $n \geq 2$ be an integer, and let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. For each $j \in [0,\ell]$, we denote by $\beta_j := \sigma_{i_1}\ldots\sigma_{i_j} \in \mathrm{Br}_n^+$ the truncation of β at the j-th crossing, with the convention $\beta_0 := e_n$, where $e_n \in \mathrm{Br}^+_{\mathrm{std}}$ is the trivial braid on n strands. Bearing this in mind, we adopt the following notation:*

- *For each $j \in [0,\ell]$, we introduce $\pi_{\beta_j} := s_{i_1} \cdots s_{i_j} \in \mathrm{S}_n$ to denote the permutation associated with β_j, with the convention $\pi_{\beta_0} := e_n$, where by slight abuse of notation $e_n \in \mathrm{S}_n$ denotes within this context the trivial permutation on n elements.*

- *For any tuple $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}$, we set $\pi_{\beta_j}(\vec{u}) := (u_{\pi_{\beta_j}(1)}, \ldots, u_{\pi_{\beta_j}(n)}) \in \mathbb{K}^n_{\mathrm{std}}$ for each $j \in [0,\ell]$, with the convention $\pi_{\beta_0}(\vec{u}) := \vec{u}$.*

With the preceding notation in place, we now introduce a linear map that will be central to the explicit description of the lower-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Definition 5.55. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$, $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$ be bases for $\mathbb{K}^n$, and let $\vec{z} = (z_1, \ldots, z_\ell)$, $\vec{z}\,' = (z'_1, \ldots, z'_\ell) \in X(\beta, \mathbb{K})$ be points such that the pairs $\big(\hat{\mathbf{f}}^{(n)}, \vec{z}\big)$ and $\big(\hat{\mathbf{g}}^{(n)}, \vec{z}\,'\big)$ algebraically characterize $\mathscr{F}$ and $\mathscr{G}$ according to Theorem (4.46), respectively. Then, we associate to the pair $(\mathscr{F}, \mathscr{G})$ the linear $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ defined by*

$$\delta_{\mathscr{F},\mathscr{G}}(\vec{u}) := \big(\delta_1(\vec{u}), \ldots, \delta_\ell(\vec{u})\big)\,, \qquad \vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}\,,$$

where, for each $j \in [1,\ell]$,

$$\delta_j(\vec{u}) := \Big[\,(B^{(n)}_{i_j}(z'_j))^{-1} D\big(\pi_{\beta_{j-1}}(\vec{u})\big)\, B^{(n)}_{i_j}(z_j)\,\Big]_{i_j+1, i_j}\,.$$

Building on the previous definition, we now state one of the main results of this manuscript: a theorem providing an explicit and computable characterization of the lower-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Theorem 5.56. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$, $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$ be bases for $\mathbb{K}^n$, and let $\vec{z}$, $\vec{z}\,' \in X(\beta, \mathbb{K})$ be points such that the pairs $\big(\hat{\mathbf{f}}^{(n)}, \vec{z}\big)$ and $\big(\hat{\mathbf{g}}^{(n)}, \vec{z}\,'\big)$ algebraically characterize $\mathscr{F}$ and $\mathscr{G}$ according to Theorem (1.3), respectively.*

Following Definition (5.55), let $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ be the linear map associated with the pair $(\mathscr{F}, \mathscr{G})$. Then there are isomorphisms of vector spaces

$$\mathrm{Ext}^0(\mathscr{F}, \mathscr{G}) \cong \ker \delta_{\mathscr{F},\mathscr{G}}\,, \tag{5.1}$$

$$\mathrm{Ext}^1(\mathscr{F}, \mathscr{G}) \cong \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}}\,. \tag{5.2}$$

In particular, the following subsections are devoted to proving the preceding statement. Bearing this in mind, we now turn to the study of the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

5.3. **Zero-Degree Morphism Spaces in the Category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.** Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. The main goal of this subsection is to establish the first part of Theorem (5.56), namely, the isomorphism of vector spaces in Equation (5.1), thereby providing an explicit algebraic characterization of the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. To this end, we begin by introducing some preliminaries.

5.3.1. *Technical Background.* Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. We now collect some preliminaries that will play a fundamental role in the explicit computation of the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. In particular, we open the discussion with the following definition.

Definition 5.57. *Let A, B, X, Y be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{F}} : A \to B$ and $\gamma_{\mathscr{G}} : X \to Y$ be linear maps. A morphism $\lambda_{\xi,\delta} := (\lambda_\xi, \lambda_\delta)$ between $\gamma_{\mathscr{F}}$ and $\gamma_{\mathscr{G}}$ consists of a pair of linear maps $\lambda_\xi : A \to X$ and $\lambda_\delta : B \to Y$ such that the diagram in Figure (26) commutes. Equivalently,*

$$\lambda_\delta \circ \gamma_{\mathscr{F}} = \gamma_{\mathscr{G}} \circ \lambda_\xi \,.$$

$$\begin{tikzcd} B \arrow[r, "\lambda_\delta"] & Y \\ A \arrow[u, "\gamma_{\mathscr{F}}"] \arrow[r, "\lambda_\xi"'] & X \arrow[u, "\gamma_{\mathscr{G}}"'] \end{tikzcd}$$

FIGURE 26. A morphism $\lambda_{\xi,\delta}$ between $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.

Lemma 5.58. *Let $\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}$ and $\{\psi_{\mathscr{G}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}$ be two collections of surjective linear maps, and let $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^{n}$ be a collection of linear maps such that for each $i \in [1, n-1]$, the pair $(T_\lambda^{(i)}, T_\lambda^{(i+1)})$ defines a morphism between $\psi_{\mathscr{F}}^{(i)}$ and $\psi_{\mathscr{G}}^{(i)}$, i.e.,*

$$T_\lambda^{(i)} \circ \psi_{\mathscr{F}}^{(i)} = \psi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i+1)} \,. \tag{5.3}$$

For each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_j^{(i)}\}_{j=1}^{i}$ and $\hat{\mathbf{g}}^{(i)} := \{\hat{g}_j^{(i)}\}_{j=1}^{i}$ be bases for $\mathbb{K}^i$ such that the collections $\{\hat{\mathbf{f}}^{(j)}\}_{j=1}^{n}$ and $\{\hat{\mathbf{g}}^{(j)}\}_{j=1}^{n}$ are systems of bases adapted to $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$, respectively (cf. Definition (4.18)–(4)). Then:

- *The matrix ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} \in M(n,\mathbb{K})$ representing $T_\lambda^{(n)}$ is lower triangular.*
- *For each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i,\mathbb{K})$ representing $T_\lambda^{(i)}$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$.*

Proof. By assumption, $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$ are systems of bases adapted to $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$, respectively. Then, Definition (4.18)–(4) asserts that for each $i \in [1, n-1]$, the matrices ${}_{\hat{\mathbf{f}}^{(i)}}[\,\psi_{\mathscr{F}}^{(i)}\,]_{\hat{\mathbf{f}}^{(i+1)}} \in M(i, i+1, \mathbb{K})$ and ${}_{\hat{\mathbf{g}}^{(i)}}[\,\psi_{\mathscr{G}}^{(i)}\,]_{\hat{\mathbf{g}}^{(i+1)}} \in M(i, i+1, \mathbb{K})$ representing $\psi_{\mathscr{F}}^{(i)}$ and $\psi_{\mathscr{G}}^{(i)}$ are the standard projection matrices:

$$ {}_{\hat{\mathbf{f}}^{(i)}}[\,\psi_{\mathscr{F}}^{(i)}\,]_{\hat{\mathbf{f}}^{(i+1)}} = \pi^{(i,i+1)}, \quad \text{and} \quad {}_{\hat{\mathbf{g}}^{(i)}}[\,\psi_{\mathscr{G}}^{(i)}\,]_{\hat{\mathbf{g}}^{(i+1)}} = \pi^{(i,i+1)}. $$

Now, for each $i \in [1, n]$, denote by ${}_{\hat{\mathbf{g}}^{(i)}}[\,T_\lambda^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}} \in M(i, \mathbb{K})$ the matrix representing $T_\lambda^{(i)}$. Then, for each $i \in [1, n-1]$, the morphism condition (5.3) translates into the matrix equation

$$ {}_{\hat{\mathbf{g}}^{(i)}}[\,T_\lambda^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}} \cdot {}_{\hat{\mathbf{f}}^{(i)}}[\,\psi_{\mathscr{F}}^{(i)}\,]_{\hat{\mathbf{f}}^{(i+1)}} = {}_{\hat{\mathbf{g}}^{(i)}}[\,\psi_{\mathscr{G}}^{(i)}\,]_{\hat{\mathbf{g}}^{(i+1)}} \cdot {}_{\hat{\mathbf{g}}^{(i+1)}}[\,T_\lambda^{(i+1)}\,]_{\hat{\mathbf{f}}^{(i+1)}}. $$

Consequently, a direct substitution of the standard projection matrices into the above relation shows that

$$ \left[\ {}_{\hat{\mathbf{g}}^{(i)}}[\,T_\lambda^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}} \ \middle|\ \mathbf{0}_{i\times 1}\ \right]_{i\times(i+1)} = \begin{array}{c}\text{the upper-left}\\ (i, i+1) \text{ submatrix of}\end{array} \quad {}_{\hat{\mathbf{g}}^{(i+1)}}[\,T_\lambda^{(i+1)}\,]_{\hat{\mathbf{f}}^{(i+1)}}, $$

for each $i \in [1, n-1]$. Hence, by applying the above identity recursively, we conclude that the matrix ${}_{\hat{\mathbf{g}}^{(n)}}[\,T_\lambda^{(n)}\,]_{\hat{\mathbf{f}}^{(n)}}$ representing $T_\lambda^{(n)}$ is lower triangular, and that for each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{g}}^{(i)}}[\,T_\lambda^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}}$ representing $T_\lambda^{(i)}$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}[\,T_\lambda^{(n)}\,]_{\hat{\mathbf{f}}^{(n)}}$, as claimed. ☐

Lemma 5.59. *Let $\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$ be two collections of injective linear maps, and let $\{B_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^{n}$ be a collection of linear maps such that for each $i \in [1, n-1]$, the pair $(B_\lambda^{(i+1)}, B_\lambda^{(i)})$ defines a morphism between $\phi_{\mathscr{F}}^{(i)}$ and $\phi_{\mathscr{G}}^{(i)}$, i.e.,*

$$ B_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} = \phi_{\mathscr{G}}^{(i)} \circ B_\lambda^{(i)}. \tag{5.4} $$

For each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_j^{(i)}\}_{j=1}^{i}$ and $\hat{\mathbf{g}}^{(i)} := \{\hat{g}_j^{(i)}\}_{j=1}^{i}$ be bases for $\mathbb{K}^i$ such that the collections $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$ are systems of bases adapted to $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$, respectively (cf. Definition (4.18)–(3)). Then:

- *The matrix ${}_{\hat{\mathbf{g}}^{(n)}}[\,B_\lambda^{(n)}\,]_{\hat{\mathbf{f}}^{(n)}} \in M(n, \mathbb{K})$ representing $B_\lambda^{(n)}$ is upper triangular.*
- *For each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{g}}^{(i)}}[\,B_\lambda^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}} \in M(i, \mathbb{K})$ representing $B_\lambda^{(i)}$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}[\,B_\lambda^{(n)}\,]_{\hat{\mathbf{f}}^{(n)}}$.*

Proof. By assumption, $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$ are systems of bases adapted to $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$, respectively. Then, Definition (4.18)–(3) asserts that, for each $i \in [1, n-1]$, the matrices ${}_{\hat{\mathbf{f}}^{(i+1)}}[\,\phi_{\mathscr{F}}^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}} \in M(i+1, i, \mathbb{K})$ and ${}_{\hat{\mathbf{g}}^{(i+1)}}[\,\phi_{\mathscr{G}}^{(i)}\,]_{\hat{\mathbf{g}}^{(i)}} \in M(i+1, i, \mathbb{K})$ representing $\phi_{\mathscr{F}}^{(i)}$ and $\phi_{\mathscr{G}}^{(i)}$ are the standard inclusion matrices:

$$ {}_{\hat{\mathbf{f}}^{(i+1)}}[\,\phi_{\mathscr{F}}^{(i)}\,]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)}, \quad \text{and} \quad {}_{\hat{\mathbf{g}}^{(i+1)}}[\,\phi_{\mathscr{G}}^{(i)}\,]_{\hat{\mathbf{g}}^{(i)}} = \iota^{(i+1,i)}. $$

Now, for each $i \in [1,n]$, denote by ${}_{\hat{\mathbf{g}}^{(i)}}\big[\, B_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i,\mathbb{K})$ the matrix representing $B_\lambda^{(i)}$. Then, for each $i \in [1,n-1]$, the morphism condition (5.4) translates into the matrix equation

$$ {}_{\hat{\mathbf{g}}^{(i+1)}}\big[\, B_\lambda^{(i+1)} \,\big]_{\hat{\mathbf{f}}^{(i+1)}} \cdot {}_{\hat{\mathbf{f}}^{(i+1)}}\big[\, \phi_{\mathscr{F}}^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}} = {}_{\hat{\mathbf{g}}^{(i+1)}}\big[\, \phi_{\mathscr{G}}^{(i)} \,\big]_{\hat{\mathbf{g}}^{(i)}} \cdot {}_{\hat{\mathbf{g}}^{(i)}}\big[\, B_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}} . $$

Consequently, a direct substitution of the standard inclusion matrices into the above relation shows that

$$ \begin{array}{c} \text{the upper-left} \\ (i+1,i) \text{ submatrix of} \end{array} \quad {}_{\hat{\mathbf{g}}^{(i+1)}}\big[\, B_\lambda^{(i+1)} \,\big]_{\hat{\mathbf{f}}^{(i+1)}} = \left[\begin{array}{c} {}_{\hat{\mathbf{g}}^{(i)}}\big[\, B_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}} \\ \hline \mathbf{0}_{1\times i} \end{array} \right]_{(i+1)\times i} , $$

for each $i \in [1,n-1]$. Hence, by applying the above identity recursively, we conclude that the matrix ${}_{\hat{\mathbf{g}}^{(n)}}\big[\, B_\lambda^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}}$ representing $B_\lambda^{(n)}$ is upper triangular, and that for each $i \in [1,n-1]$, the matrix ${}_{\hat{\mathbf{g}}^{(i)}}\big[\, B_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}}$ representing $B_\lambda^{(i)}$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}\big[\, B_\lambda^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}}$, as claimed. $\square$

Having established the necessary groundwork, we conclude this part of the manuscript with the following observation, which sheds light on the local structure of the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.

Observation 5.60. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathcal{S}_{\Lambda(\beta)}$ the stratification of $\mathbb{R}^2$ induced by $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, and $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. By the microlocal support conditions, the global structure of $\mathscr{F}$ and $\mathscr{G}$ is determined by their local behavior along the arcs in $\mathcal{S}_{\Lambda(\beta)}$, together with compatibility conditions imposed at cusps and crossings. Now, recall that elements of $\mathrm{Ext}^0(\mathscr{F},\mathscr{G})$ are global sheaf homomorphisms between $\mathscr{F}$ and $\mathscr{G}$. Hence, the constructibility of $\mathscr{F}$ and $\mathscr{G}$ with respect to $\mathcal{S}_{\Lambda(\beta)}$, together with the sheaf axioms, implies that the global structure of the elements of $\mathrm{Ext}^0(\mathscr{F},\mathscr{G})$ is likewise determined by their local behavior along the arcs in $\mathcal{S}_{\Lambda(\beta)}$, together with compatibility conditions required at cusps and crossings.*

Next, we explicitly describe the structure of the elements of $\mathrm{Ext}^0(\mathscr{F},\mathscr{G})$ near an arc a in $\mathcal{S}_{\Lambda(\beta)}$. To this end, observe that in a neighborhood of a, the stratification $\mathcal{S}_{\Lambda(\beta)}$ consists of an upper 2-dimensional stratum U and a lower 2-dimensional stratum D, as illustrated in Sub-figure (4a). Given this local configuration, let $p \in U$ and $q \in D$ be arbitrary points, and denote by $A = \mathscr{F}_p$, $B = \mathscr{F}_q$, $X = \mathscr{G}_p$, and $Y = \mathscr{G}_q$ the stalks of $\mathscr{F}$ and $\mathscr{G}$ at these points, respectively. It then follows from the microlocal support conditions near the arcs that, near a, $\mathscr{F}$ and $\mathscr{G}$ are determined by a pair of linear maps $\gamma_{\mathscr{F}} : A \to B$ and $\gamma_{\mathscr{G}} : X \to Y$, respectively. According to our previous discussion, we have that a sheaf homomorphism $\lambda \in \mathrm{Ext}^0(\mathscr{F},\mathscr{G})$ between $\mathscr{F}$ and $\mathscr{G}$ is specified near a by two linear maps $\lambda_\xi : A \to X$ and $\lambda_\delta : B \to Y$ such that

$$ \lambda_\delta \circ \gamma_{\mathscr{F}} = \gamma_{\mathscr{G}} \circ \lambda_\xi \, . $$

In other words, near a, λ is determined by a morphism $\lambda_{\xi,\delta} := (\lambda_\xi, \lambda_\delta)$ between $\gamma_{\mathscr{F}}$ and $\gamma_{\mathscr{G}}$ (cf. Definition (5.57)).

Building on the preceding local description of the elements of the zero-degree morphism spaces in the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$, we proceed to analyze the global structure of these elements, yielding an explicit characterization of the zero-degree morphism spaces under consideration. In particular, we begin our analysis by considering the case $\beta = e_n$.

5.3.2. *The Case of the Trivial Braid.* Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word. Next, we study the zero-degree morphism spaces in the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$.

To begin, consider the open cover $\mathcal{U}_{\Lambda(e_n)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{T}}\}$ of $\mathbb{R}^2$ introduced in Construction (4.30). Let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$, and let $\lambda \in \mathrm{Ext}^0(\mathscr{F},\mathscr{G})$ be a sheaf homomorphism between them. By Proposition (4.32), $\mathscr{F}$ and $\mathscr{G}$ are identically zero on U_0, and as a result, λ is also identically zero on this region. It follows that λ is entirely determined by its behavior on the regions U_{B} and U_{T}. Having established this, we present the following proposition.

Proposition 5.61. ***Setup****: Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word, $\mathcal{U}_{\Lambda(e_n)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ introduced in Construction* (4.30)*, and $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$. Accordingly, Proposition* (4.31) *asserts that:*

- *On U_{T}, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of surjective linear maps*
$$\left\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^{i}\right\}_{i=1}^{n-1}, \quad \textit{and} \quad \left\{\psi_{\mathscr{G}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^{i}\right\}_{i=1}^{n-1},$$
respectively (see Figure (9) *for a schematic representation).*

- *On U_{B}, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of injective linear maps*
$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1}, \quad \textit{and} \quad \left\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1},$$
respectively (see Figure (10) *for a schematic representation).*

- ***Compatibility conditions****: For each $i \in [1, n-1]$,*
$$\psi_{\mathscr{F}}^{(i)} \circ \phi_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}, \quad \textit{and} \quad \psi_{\mathscr{G}}^{(i)} \circ \phi_{\mathscr{G}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}.$$

$\star$ *Assumption: For each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_j^{(i)}\}_{j=1}^{i}$ and $\hat{\mathbf{g}}^{(i)} := \{\hat{g}_j^{(i)}\}_{j=1}^{i}$ be bases for $\mathbb{K}^i$. In particular, building on Definition* (4.18)*–*(3)*–*(4)*, we assume that:*

- *$\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to both $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$.*
- *$\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to both $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$.*

***Main Conclusion**: Let $\lambda \in \operatorname{Ext}^0(\mathscr{F}, \mathscr{G})$ be a sheaf homomorphism between $\mathscr{F}$ and $\mathscr{G}$. Then, under the given **setup**, the following statements hold:*

- *λ is characterized by a collection of n linear maps $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^n$ such that, for each $i \in [1, n-1]$:*

$$T_\lambda^{(i)} \circ \psi_{\mathscr{F}}^{(i)} = \psi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i+1)}, \quad \text{and} \quad T_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} = \phi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i)}.$$

- *The matrix ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} \in M(n, \mathbb{K})$ representing $T_\lambda^{(n)}$ is diagonal.*
- *For each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i, \mathbb{K})$ representing $T_\lambda^{(i)}$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$.*

Consequently, we deduce that

$$\operatorname{Ext}^0(\mathscr{F}, \mathscr{G}) \cong \mathbb{K}^n_{\text{std}}.$$

Proof. Building on our previous discussion, we have that:

- On U_{T}, λ is specified by a collection of linear maps $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^n$ such that for each $i \in [1, n-1]$, the pair $\big(T_\lambda^{(i+1)}, T_\lambda^{(i)}\big)$ defines a morphism between $\psi_{\mathscr{F}}^{(i)}$ and $\psi_{\mathscr{G}}^{(i)}$, i.e.,

$$T_\lambda^{(i)} \circ \psi_{\mathscr{F}}^{(i)} = \psi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i+1)}. \tag{5.5}$$

- On U_{B}, λ is specified by a collection of linear maps $\{B_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^n$ such that for each $i \in [1, n-1]$, the pair $\big(B_\lambda^{(i)}, B_\lambda^{(i+1)}\big)$ defines a morphism between $\phi_{\mathscr{F}}^{(i)}$ and $\phi_{\mathscr{G}}^{(i)}$, i.e.,

$$B_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} = \phi_{\mathscr{G}}^{(i)} \circ B_\lambda^{(i)}. \tag{5.6}$$

In particular, note that a direct application of the sheaf axioms to the intersection $U_{\mathrm{B}} \cap U_{\mathrm{T}} \subset \mathbb{R}^2$ implies that $T_\lambda^{(n)} = B_\lambda^{(n)}$. Hence, combining the morphism conditions (5.5) and (5.6) with the **compatibility conditions** for the maps characterizing $\mathscr{F}$ and $\mathscr{G}$ on U_{B} and U_{T}, we further deduce that $T_\lambda^{(i)} = B_\lambda^{(i)}$ for every $i \in [1, n-1]$. It follows that λ is determined by a collection of linear maps $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^n$ satisfying the stated conditions.

Finally, recall that $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ are systems of bases adapted to the pairs $\big(\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}, \{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}\big)$ and $\big(\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}, \{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}\big)$, respectively. Thus, relying on our previous results, Lemmas (5.58) and (5.59) guarantee that:

- The matrix ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} \in M(n, \mathbb{K})$ representing $T_\lambda^{(n)}$ is both lower and upper triangular, and hence diagonal.

- For each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{g}}^{(i)}}\big[\, T_{\lambda}^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i, \mathbb{K})$ representing $T_{\lambda}^{(i)}$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}\big[\, T_{\lambda}^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}}$.

Bearing this in mind, we conclude that

$$\mathrm{Ext}^0(\mathscr{F}, \mathscr{G}) \cong \mathbb{K}^n_{\mathrm{std}} \,.$$

This completes the proof. □

Having established the preceding result, we conclude our study of the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n), \mathbb{K})_0)$, and turn to their analysis in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ for an arbitrary positive braid word $\beta \in \mathrm{Br}_n^+$.

5.3.3. *General Positive Braids.* Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. Next, we analyze the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

To being, we establish an observation that will facilitate the study the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Observation 5.62. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathcal{S}_{\Lambda(\beta)}$ the stratification of $\mathbb{R}^2$ induced by $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, and $\mathscr{F}$ an object of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\mathcal{U}_{\Lambda(\beta)} = \big\{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\big\}$ be the open cover of $\mathbb{R}^2$ from Construction (4.33), and let $\mathcal{R}_{\Lambda(\beta)} = \big\{R_j\big\}_{j=1}^{\ell}$ be the partition of U_{B} into ℓ open vertical straps from Construction (4.36).*

In particular, observe that the only singularities of $\Pi_{x,z}(\Lambda)$ contained in U_{L} are the n left cusps, and R_1 is its only adjacent vertical strap. Hence, since U_{L} contains no crossings of $\Pi_{x,z}(\Lambda)$, the constructibility of $\mathscr{F}$ with respect to $\mathcal{S}_{\Lambda(\beta)}$ and the sheaf axioms guarantee that the maps defining $\mathscr{F}$ on U_{L} coincide with those specifying it on the intersection $U_{\mathrm{L}} \cap R_1$. It follows that, for the purpose of studying $\mathscr{F}$, instead of considering the region U_{L}, it suffices to restrict to R_1, with the understanding that the data characterizing $\mathscr{F}$ on R_1 inherits the compatibility conditions associated with the left cusps in U_{L}.

Finally, note that a completely analogous argument applies to the region U_{R} and the vertical strap R_ℓ, with the data characterizing $\mathscr{F}$ on R_ℓ inheriting the compatibility conditions associated with the right cusps in U_{R}. Consequently, we deduce that the global structure of $\mathscr{F}$ can be fully recovered from its local description on the region U_{T} and the collection of vertical straps $\mathcal{R}_{\Lambda(\beta)}$, together with the appropriate compatibility conditions.

Now, let $\mathcal{U}_{\Lambda(\beta)} = \big\{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\big\}$ be the open cover of $\mathbb{R}^2$ from Construction (4.33), $\mathcal{R}_{\Lambda(\beta)} = \big\{R_j\big\}_{j=1}^{\ell}$ be the partition of U_{B} into ℓ open vertical straps from Construction (4.36), $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and $\lambda \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{G})$ a sheaf homomorphism between them. Building on Observation (5.62), we have that the global structure of λ can be fully recovered from its local description on the region U_{T} and the collection of vertical straps $\mathcal{R}_{\Lambda(\beta)}$, together with the appropriate compatibility conditions. Accordingly, we present the following result.

Lemma 5.63. ***Setup***: *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ from Construction (4.33), and $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. According to Lemma (4.34), $\mathscr{F}$ and $\mathscr{G}$ have the following local descriptions:*

- *On U_{T}, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of surjective linear maps*
$$\left\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\right\}_{i=1}^{n-1}, \quad \text{and} \quad \left\{\psi_{\mathscr{G}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\right\}_{i=1}^{n-1},$$
respectively. For a schematic illustration of a generic representative of one of these sheaves on U_{T}, see Figure (15).

- *On U_{L}, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of injective linear maps*
$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1}, \quad \text{and} \quad \left\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1},$$
respectively. For a schematic illustration of a generic representative of one of these sheaves on U_{L}, see Figure (13).

- ***Compatibility conditions***: *For each $i \in [1, n-1]$,*
$$\psi_{\mathscr{F}}^{(i)} \circ \phi_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}, \quad \text{and} \quad \psi_{\mathscr{G}}^{(i)} \circ \phi_{\mathscr{G}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}.$$

⋆ *Assumption: For each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_j^{(i)}\}_{j=1}^i$ and $\hat{\mathbf{g}}^{(i)} := \{\hat{g}_j^{(i)}\}_{j=1}^i$ be bases for $\mathbb{K}^i$. Based on Definition (4.18)–(3)–(4), we assume that:*

- *$\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ is a system of bases adapted to both $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$.*
- *$\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ is a system of bases adapted to both $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$.*

Main Conclusion: *Let $\lambda \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{G})$. Under the given **setup**, the following statements hold:*

⋆ *Local characterization of λ on U_{T} and U_{L}: On U_{T} and U_{L}, λ is characterized by a collection of n linear maps $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^n$ such that for each $i \in [1, n-1]$:*
$$T_\lambda^{(i)} \circ \psi_{\mathscr{F}}^{(i)} = \psi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i+1)}, \quad \text{and} \quad T_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} = \phi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i)}.$$

⋆ *Block-diagonal properties: With respect to the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$, we have that:*

- ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} \in M(n,\mathbb{K})$ *is a diagonal matrix.*
- *For each* $i \in [1, n-1]$*, the matrix* ${}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i,\mathbb{K})$ *coincides with the principal* $i \times i$ *submatrix of* ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$*.*

Proof. The argument parallels that of Proposition (5.61). More precisely, given the local descriptions of $\mathscr{F}$ and $\mathscr{G}$ on U_{T} and U_{L} from Lemma (4.34), the local characterization of λ on U_{T} and U_{L} follows from Observation (5.60). Furthermore, this characterization, combined with the standing assumption and Lemmas (5.58) and (5.59), yields the block-diagonal properties. □

Fix $j \in [1, \ell]$, and let R_j be the vertical strap in $\mathbb{R}^2$ containing σ_{i_j}—the j-th crossing of β (see Figure (19)). Building on the algebraic local models established in Lemmas (4.47) and (4.48), we now give an explicit local characterization of the elements of the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ on R_j.

Lemma 5.64. ***Setup****: Let* $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ *be a positive braid word, and let* $\mathscr{F}$ *and* $\mathscr{G}$ *be objects of the category* $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$*. Fix* $j \in [1,\ell]$*, and let* R_j *be the vertical strap in* $\mathbb{R}^2$ *containing* σ_{i_j}*—the* j*-th crossing of* β *(see Figure (19)).*

⋆ *Assumption 1: Let* $k := i_j \in [1, n-1]$ *denote the index of* σ_{i_j}*, and suppose that* $k = 1$*.*

Under this assumption, on R_j*,* $\mathscr{F}$ *and* $\mathscr{G}$ *are specified by two collections of* n *injective linear maps*

$$\begin{aligned} &\big\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\big\}_{i=1}^{n-1} \cup \big\{\widetilde{\phi}_{\mathscr{F}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2\big\}, \\ &\big\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\big\}_{i=1}^{n-1} \cup \big\{\widetilde{\phi}_{\mathscr{G}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2\big\}, \end{aligned} \tag{5.7}$$

respectively. For a schematic illustration of a generic representative of one of these sheaves, see Figure (21).

⋆ *Assumption 2: For each* $i \in [1, n]$*, let* $\hat{\mathbf{f}}^{(i)} := \big\{\hat{f}_j^{(i)}\big\}_{j=1}^{i}$ *and* $\hat{\mathbf{g}}^{(i)} := \big\{\hat{g}_j^{(i)}\big\}_{j=1}^{i}$ *be bases for* $\mathbb{K}^i$*. Based on Definition (4.18)–(3), we assume that:*

- $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}$ *is a system of bases adapted to* $\big\{\phi_{\mathscr{F}}^{(i)}\big\}_{i=1}^{n-1}$*.*
- $\big\{\hat{\mathbf{g}}^{(i)}\big\}_{i=1}^{n}$ *is a system of bases adapted to* $\big\{\phi_{\mathscr{G}}^{(i)}\big\}_{i=1}^{n-1}$*.*

Under this assumption, Lemma (4.47) ensures that $\big(\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}, z\big)$ *and* $\big(\big\{\hat{\mathbf{g}}^{(i)}\big\}_{i=1}^{n}, z'\big)$ *are system of bases adapted to* $\mathscr{F}$ *and* $\mathscr{G}$ *on* R_j*, where* $z,\, z' \in \mathbb{K}$ *algebraically parameterize, relative to the bases* $\hat{\mathbf{f}}^{(n)}$ *and* $\hat{\mathbf{g}}^{(n)}$ *for* $\mathbb{K}^n$*, the* s_1*-relative position between the pairs of complete flags in* $\mathbb{K}^n$ *that geometrically characterize* $\mathscr{F}$ *and* $\mathscr{G}$ *on* R_j*, respectively. In particular, following Definition (4.29), we denote by* $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\big\}_{i=1}^{n}$ *and* $\big\{\hat{\mathbf{g}}^{(i)}[\sigma_1, z']\big\}_{i=1}^{n}$ *the corresponding braid-transformed bases.*

Main Conclusion: *Let $\lambda \in \mathrm{Ext}^0(\mathscr{F},\mathscr{G})$. Under the given* ***setup****, the following statements hold:*

⋆ *Local characterization of λ on R_j: On R_j, λ is specified by a collection of $n+1$ linear maps*

$$\big\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\big\}_{i=1}^n \cup \big\{\widetilde{T}_\lambda^{(1)} : \mathbb{K}^1 \to \mathbb{K}^1\big\}\,, \tag{5.8}$$

such that

$$T_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} = \phi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i)}\,, \quad \text{and} \quad T_\lambda^{(2)} \circ \widetilde{\phi}_{\mathscr{F}}^{(1)} = \widetilde{\phi}_{\mathscr{G}}^{(1)} \circ \widetilde{T}_\lambda^{(1)}\,, \tag{5.9}$$

for each $i \in [1, n-1]$.

⋆ *Block-Triangular Properties I: With respect to the bases $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^n$ and $\big\{\hat{\mathbf{g}}^{(i)}\big\}_{i=1}^n$, we have that:*

- ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} \in M(n,\mathbb{K})$ *is an upper-triangular matrix.*
- *For each $i \in [1, n-1]$, the matrix ${}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i,\mathbb{K})$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$,*

⋆ *Block-Triangular Properties II: With respect to the bases $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\big\}_{i=1}^n$ and $\big\{\hat{\mathbf{g}}^{(i)}[\sigma_1, z']\big\}_{i=1}^n$, we have that:*

- ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]} \in M(n,\mathbb{K})$ *is upper-triangular.*
- ${}_{\hat{\mathbf{g}}^{(1)}[\sigma_1,z']}\big[\widetilde{T}_\lambda^{(1)}\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} \in M(1,\mathbb{K})$ *coincides with the principal 1×1 submatrix of ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]}$.*
- *For each $i \in [2, n-1]$, ${}_{\hat{\mathbf{g}}^{(i)}[\sigma_1,z']}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_1,z]} \in M(i,\mathbb{K})$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]}$.*

⋆ *Refinement: Suppose that ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$ is diagonal, that is,*

$${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} = D(\vec{u})\,,$$

for some tuple $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}$. Then:

- ***Compatibility condition for*** λ*: The $(1+1, 1)$-entry of the matrix product*

  $$\big(B_1^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_1^{(n)}(z)$$

  must vanish.

- *Under the above condition,*

  $${}_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]} = D(\pi_1(\vec{u}))\,,$$

  where $\pi_1(\vec{u})$ denotes the permutation of $\vec{u}$ associated with σ_1.

Proof. According to Assumption 1, on R_j, $\mathscr{F}$ and $\mathscr{G}$ are defined by the collections of n injective linear maps listed in (5.7). Hence, the Local characterization of λ on R_j, specified by the $n+1$ linear maps listed in (5.8) and subject to the morphism conditions (5.9), follows from Observation (5.60), which establishes the local description of the elements of $\mathrm{Ext}^0(\mathscr{F},\mathscr{G})$ near an arc of the stratification $\mathcal{S}_{\Lambda(\beta)}$ of $\mathbb{R}^2$ induced by $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$.

By Assumption 2, $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ are systems of bases adapted to $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$, respectively. Therefore, by virtue of the morphism conditions (5.9), the Block-Triangular Properties I follow directly from Lemma (5.59).

Moreover, under Assumption 2, Lemma (4.47) asserts that $\big(\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n, z\big)$ and $\big(\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n, z'\big)$ are system of bases adapted to $\mathscr{F}$ and $\mathscr{G}$ on R_j, where $z, z' \in \mathbb{K}$ algebraically parameterize, relative to the bases $\hat{\mathbf{f}}^{(n)}$ and $\hat{\mathbf{g}}^{(n)}$ for $\mathbb{K}^n$, the s_1-relative position between the pairs of complete flags in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$ and $\mathscr{G}$ on R_j, respectively. Accordingly, we have that:

- With respect to the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$, the linear maps determining $\mathscr{F}$ on R_j have the matrix representations:

$$_{\hat{\mathbf{f}}^{(i+1)}}\big[\,\phi_{\mathscr{F}}^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)}\,, \quad \text{for all } i \in [1, n-1]\,, \quad \text{and} \quad _{\hat{\mathbf{f}}^{(2)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(1)}\,\big]_{\hat{\mathbf{f}}^{(1)}} = B_1^{(2)}(z)\cdot\iota^{(2,1)}\,.$$

- With respect to the bases $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$, the linear maps determining $\mathscr{G}$ on R_j have the matrix representations:

$$_{\hat{\mathbf{g}}^{(i+1)}}\big[\,\phi_{\mathscr{G}}^{(i)}\,\big]_{\hat{\mathbf{g}}^{(i)}} = \iota^{(i+1,i)}\,, \quad \text{for all } i \in [1, n-1]\,, \quad \text{and} \quad _{\hat{\mathbf{g}}^{(2)}}\big[\,\widetilde{\phi}_{\mathscr{G}}^{(1)}\,\big]_{\hat{\mathbf{g}}^{(1)}} = B_1^{(2)}(z')\cdot\iota^{(2,1)}\,.$$

Next, consider the braid-transformed bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_1, z']\}_{i=1}^n$ (cf. Definition (4.29)). In particular, a direct calculation shows that:

- With respect to the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\}_{i=1}^n$,

$$_{\hat{\mathbf{f}}^{(i+1)}[\sigma_1,z]}\big[\,\phi_{\mathscr{F}}^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_1,z]} = \iota^{(i+1,i)}\,, \quad \text{for all } 2 \in [1, n-1]\,, \quad \text{and} \quad _{\hat{\mathbf{f}}^{(2)}[\sigma_1,z]}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(1)}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = \iota^{(2,1)}\,.$$

- With respect to the bases $\{\hat{\mathbf{g}}^{(i)}[\sigma_1, z']\}_{i=1}^n$,

$$_{\hat{\mathbf{g}}^{(i+1)}[\sigma_1,z']}\big[\,\phi_{\mathscr{G}}^{(i)}\,\big]_{\hat{\mathbf{g}}^{(i)}[\sigma_1,z']} = \iota^{(i+1,i)}\,, \quad \text{for all } 2 \in [1, n-1]\,, \quad \text{and} \quad _{\hat{\mathbf{g}}^{(2)}[\sigma_1,z']}\big[\,\widetilde{\phi}_{\mathscr{G}}^{(1)}\,\big]_{\hat{\mathbf{g}}^{(1)}[\sigma_1,z']} = \iota^{(2,1)}\,.$$

It follows from Definition (4.18)–(3) that $\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_1, z']\}_{i=1}^n$ are systems of bases adapted to $\{\widetilde{\phi}_{\mathscr{F}}^{(1)}, \phi_{\mathscr{F}}^{(2)}, \dots, \phi_{\mathscr{F}}^{(n-1)}\}$ and $\{\widetilde{\phi}_{\mathscr{G}}^{(1)}, \phi_{\mathscr{G}}^{(2)}, \dots, \phi_{\mathscr{G}}^{(n-1)}\}$, respectively. Hence, by virtue of the morphism conditions (5.9), the Block-Triangular Properties II follow directly from Lemma (5.59).

Finally, assume that $_{\hat{\mathbf{g}}^{(n)}}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}} = D(\vec{u})$ for some tuple $\vec{u} = (u_1, \dots, u_n) \in \mathbb{K}^n_{\mathrm{std}}$. In particular, observe that the bases $\hat{\mathbf{f}}^{(n)}[\sigma_1, z]$ and $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, as well as the bases $\hat{\mathbf{g}}^{(n)}[\sigma_1, z']$ and $\hat{\mathbf{g}}^{(n)}$, are related by the change-of-basis matrices $B_1^{(n)}(z)$ and $B_1^{(n)}(z')$, respectively. Therefore, by the *change-of-basis formula*, we deduce that

$$ {}_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]} = \big(B_1^{(n)}(z')\big)^{-1}\cdot D(\vec{u})\cdot B_1^{(n)}(z)\,. $$

Consequently, since ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]}$ must be upper-triangular, a direct application of Lemma (7.115) yields that:

- The $(1+1,1)$-entry of the matrix $\big(B_1^{(n)}(z')\big)^{-1}\cdot D(\vec{u})\cdot B_1^{(n)}(z)$ must vanish.
- Under the above condition, ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]} = D(\pi_1(\vec{u}))$,

where $\pi_1(\vec{u})$ denotes the permutation of $\vec{u}$ associated with σ_1. This completes the proof. $\square$

Lemma 5.65. ***Setup***: *Let $\beta=\sigma_{i_1}\cdots\sigma_{i_\ell}\in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Fix $j\in[1,\ell]$, and let R_j be the vertical strap in $\mathbb{R}^2$ containing σ_{i_j}—the j-th crossing of β (see Figure (19)).*

⋆ *<u>Assumption 1</u>: Let $k:=i_j\in[1,n-1]$ denote the index of σ_{i_j}, and suppose that $k\geq 2$.*

Under this assumption, on R_j, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of $n+1$ injective linear maps

$$ \begin{gathered} \big\{\phi_{\mathscr{F}}^{(i)}:\mathbb{K}^i\to\mathbb{K}^{i+1}\big\}_{i=1}^{n-1}\,\cup\,\big\{\widetilde{\phi}_{\mathscr{F}}^{(k-1)}:\mathbb{K}^{k-1}\to\mathbb{K}^k\,,\ \widetilde{\phi}_{\mathscr{F}}^{(k)}:\mathbb{K}^k\to\mathbb{K}^{k+1}\big\}\,,\\ \big\{\phi_{\mathscr{G}}^{(i)}:\mathbb{K}^i\to\mathbb{K}^{i+1}\big\}_{i=1}^{n-1}\,\cup\,\big\{\widetilde{\phi}_{\mathscr{G}}^{(k-1)}:\mathbb{K}^{k-1}\to\mathbb{K}^k\,,\ \widetilde{\phi}_{\mathscr{G}}^{(k)}:\mathbb{K}^k\to\mathbb{K}^{k+1}\big\}\,, \end{gathered} \tag{5.10} $$

respectively. For a schematic illustration of a generic representative of one of these sheaves, see Figure (23).

⋆ *<u>Assumption 2</u>: For each $i\in[1,n]$, let $\hat{\mathbf{f}}^{(i)}:=\big\{\hat{f}_j^{(i)}\big\}_{j=1}^i$ and $\hat{\mathbf{g}}^{(i)}:=\big\{\hat{g}_j^{(i)}\big\}_{j=1}^i$ be bases for $\mathbb{K}^i$. Based on Definition (4.18)–(3), we assume that:*

- *$\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^n$ is a system of bases adapted to $\big\{\phi_{\mathscr{F}}^{(i)}\big\}_{i=1}^{n-1}$.*
- *$\big\{\hat{\mathbf{g}}^{(i)}\big\}_{i=1}^n$ is a system of bases adapted to $\big\{\phi_{\mathscr{G}}^{(i)}\big\}_{i=1}^{n-1}$.*

Under this assumption, Lemma (4.48) ensures that $\big(\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^n,z\big)$ and $\big(\big\{\hat{\mathbf{g}}^{(i)}\big\}_{i=1}^n,z'\big)$ are system of bases adapted to $\mathscr{F}$ and $\mathscr{G}$ on R_j, where $z,z'\in\mathbb{K}$ algebraically parameterize, relative to the bases $\hat{\mathbf{f}}^{(n)}$ and $\hat{\mathbf{g}}^{(n)}$ for $\mathbb{K}^n$, the s_k-relative position between the pairs of complete flags in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$ and $\mathscr{G}$ on R_j, respectively. In particular, following Definition (4.29), we denote by $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_k,z]\big\}_{i=1}^n$ and $\big\{\hat{\mathbf{g}}^{(i)}[\sigma_k,z']\big\}_{i=1}^n$ the corresponding braid-transformed bases.

Main Conclusion: *Let $\lambda\in\mathrm{Ext}^0(\mathscr{F},\mathscr{G})$. Under the given **setup**, the following statements hold:*

⋆ *Local characterization of λ on R_j: On R_j, λ is specified by a collection of $n+1$ linear maps*

$$\left\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\right\}_{i=1}^n \cup \left\{\widetilde{T}_\lambda^{(k)} : \mathbb{K}^k \to \mathbb{K}^k\right\}, \tag{5.11}$$

such that

$$\begin{aligned} T_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} &= \phi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i)}, \\ \widetilde{T}_\lambda^{(k)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k-1)} &= \widetilde{\phi}_{\mathscr{G}}^{(k-1)} \circ T_\lambda^{(k-1)}, \\ T_\lambda^{(k+1)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k)} &= \widetilde{\phi}_{\mathscr{G}}^{(k)} \circ \widetilde{T}_\lambda^{(k)}, \end{aligned} \tag{5.12}$$

for each $i \in [1, n-1]$.

⋆ *Block-Triangular Properties I: With respect to the bases $\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^n$ and $\left\{\hat{\mathbf{g}}^{(i)}\right\}_{i=1}^n$, we have that:*

- ${}_{\hat{\mathbf{g}}^{(n)}}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}} \in M(n, \mathbb{K})$ *is upper-triangular.*
- *For each $i \in [1, n-1]$, ${}_{\hat{\mathbf{g}}^{(i)}}\left[T_\lambda^{(i)}\right]_{\hat{\mathbf{f}}^{(i)}} \in M(i, \mathbb{K})$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}}$.*

⋆ *Block-Triangular Properties II: With respect to the bases $\left\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\right\}_{i=1}^n$ and $\left\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\right\}_{i=1}^n$, we have that:*

- ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_k, z']}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}[\sigma_k, z]} \in M(n, \mathbb{K})$ *is upper-triangular.*
- ${}_{\hat{\mathbf{g}}^{(k)}[\sigma_k, z']}\left[\widetilde{T}_\lambda^{(k)}\right]_{\hat{\mathbf{f}}^{(k)}[\sigma_k, z]} \in M(k, \mathbb{K})$ *coincides with the principal $k \times k$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_k, z']}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}[\sigma_k, z]}$.*
- *For each $i \in [1, n-1]$ with $i \neq k$, ${}_{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']}\left[T_\lambda^{(i)}\right]_{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]} \in M(i, \mathbb{K})$ coincides with the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_k, z']}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}[\sigma_k, z]}$.*

⋆ *Refinement: Suppose that ${}_{\hat{\mathbf{g}}^{(n)}}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}}$ is diagonal, that is,*

$${}_{\hat{\mathbf{g}}^{(n)}}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}} = D(\vec{u}),$$

for some tuple $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\text{std}}$. Then:

- ***Compatibility condition for** λ: The $(k+1, k)$-entry of the matrix product*

  $$\left(B_k^{(n)}(z')\right)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)$$

  must vanish.

- *Under the above condition,*

  $${}_{\hat{\mathbf{g}}^{(n)}[\sigma_k, z']}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}[\sigma_k, z]} = D(\pi_k(\vec{u})),$$

  where $\pi_k(\vec{u})$ denotes the permutation of $\vec{u}$ associated with σ_k.

Proof. According to Assumption 1, on R_j, $\mathscr{F}$ and $\mathscr{G}$ are defined by the collections of $n+1$ injective linear maps listed in (5.10). Hence, the Local characterization of λ on R_j, specified by the $n+1$ linear maps listed in (5.11) and subject to the morphism conditions (5.12), follows from Observation (5.60), which establishes the local description of the elements of $\mathrm{Ext}^0(\mathscr{F},\mathscr{G})$ near an arc of the stratification $\mathcal{S}_{\Lambda(\beta)}$ of $\mathbb{R}^2$ induced by $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$.

By Assumption 2, $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$ are systems of bases adapted to $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$, respectively. Therefore, by virtue of the morphism conditions (5.12), the Block-Triangular Properties I follow directly from Lemma (5.59).

Moreover, under Assumption 2, Lemma (4.48) asserts that $\big(\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}, z\big)$ and $\big(\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}, z'\big)$ are system of bases adapted to $\mathscr{F}$ and $\mathscr{G}$ on R_j, where $z, z' \in \mathbb{K}$ algebraically parameterize, relative to the bases $\hat{\mathbf{f}}^{(n)}$ and $\hat{\mathbf{g}}^{(n)}$ for $\mathbb{K}^n$, the s_k-relative position between the pairs of complete flags in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$ and $\mathscr{G}$ on R_j, respectively. Accordingly, we have that:

- With respect to the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$, the linear maps determining $\mathscr{F}$ on R_j have the matrix representations:

$$
\begin{gathered}
{}_{\hat{\mathbf{f}}^{(i+1)}}\big[\,\phi_{\mathscr{F}}^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} = \iota^{(i+1,i)}\,, \qquad \text{for all } i \in [1, n-1]\,,\\
{}_{\hat{\mathbf{f}}^{(k)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}} = i^{(k,k-1)}\,, \quad {}_{\hat{\mathbf{f}}^{(k+1)}}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}} = B_k^{(k+1)}(z)\cdot \iota^{(k+1,k)}\,.
\end{gathered}
$$

- With respect to the bases $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$, the linear maps determining $\mathscr{G}$ on R_j have the matrix representations:

$$
\begin{gathered}
{}_{\hat{\mathbf{g}}^{(i+1)}}\big[\,\phi_{\mathscr{G}}^{(i)}\,\big]_{\hat{\mathbf{g}}^{(i)}} = \iota^{(i+1,i)}\,, \qquad \text{for all } i \in [1, n-1]\,,\\
{}_{\hat{\mathbf{g}}^{(k)}}\big[\,\widetilde{\phi}_{\mathscr{G}}^{(k-1)}\,\big]_{\hat{\mathbf{g}}^{(k-1)}} = i^{(k,k-1)}\,, \quad {}_{\hat{\mathbf{g}}^{(k+1)}}\big[\,\widetilde{\phi}_{\mathscr{G}}^{(k)}\,\big]_{\hat{\mathbf{g}}^{(k)}} = B_k^{(k+1)}(z')\cdot \iota^{(k+1,k)}\,.
\end{gathered}
$$

Next, consider the braid-transformed bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^{n}$ (cf. Definition (4.29)). In particular, a direct calculation shows that:

- With respect to the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^{n}$,

$$
\begin{aligned}
{}_{\hat{\mathbf{f}}^{(i+1)}[\sigma_k,z]}\big[\,\phi_{\mathscr{F}}^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_k,z]} &= \iota^{(i+1,i)}\,, &&\text{for all } i \in [1, n-1] \text{ with } i \neq k-1, k,\\
{}_{\hat{\mathbf{f}}^{(k)}[\sigma_k,z]}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}[\sigma_k,z]} &= \iota^{(k,k-1)}\,,\\
{}_{\hat{\mathbf{f}}^{(k+1)}[\sigma_k,z]}\big[\,\widetilde{\phi}_{\mathscr{F}}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k,z]} &= \iota^{(k+1,k)}\,.
\end{aligned}
$$

- With respect to the bases $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^{n}$,

$$
\begin{aligned}
{}_{\hat{\mathbf{g}}^{(i+1)}[\sigma_k,z']}\big[\,\phi_{\mathscr{G}}^{(i)}\,\big]_{\hat{\mathbf{g}}^{(i)}[\sigma_k,z']} &= \iota^{(i+1,i)}\,, \qquad\qquad \text{for all } i \in [1, n-1] \text{ with } i \neq k-1, k,\\
{}_{\hat{\mathbf{g}}^{(k)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{G}}^{(k-1)}\,\big]_{\hat{\mathbf{g}}^{(k-1)}[\sigma_k,z']} &= \iota^{(k,k-1)}\,,\\
{}_{\hat{\mathbf{g}}^{(k+1)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{G}}^{(k)}\,\big]_{\hat{\mathbf{g}}^{(k)}[\sigma_k,z']} &= \iota^{(k+1,k)}\,.
\end{aligned}
$$

It follows from Definition (4.18)–(3) that $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^{n}$ are systems of bases adapted to the collections

$$
\big\{\phi_{\mathscr{F}}^{(1)}, \ldots, \phi_{\mathscr{F}}^{(k-2)}, \widetilde{\phi}_{\mathscr{F}}^{(k-1)}, \widetilde{\phi}_{\mathscr{F}}^{(k)}, \phi_{\mathscr{F}}^{(k+1)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\big\}\,,
$$

$$
\big\{\phi_{\mathscr{G}}^{(1)}, \ldots, \phi_{\mathscr{G}}^{(k-2)}, \widetilde{\phi}_{\mathscr{G}}^{(k-1)}, \widetilde{\phi}_{\mathscr{G}}^{(k)}, \phi_{\mathscr{G}}^{(k+1)}, \ldots, \phi_{\mathscr{G}}^{(n-1)}\big\}\,,
$$

respectively. Hence, by virtue of the morphism conditions (5.12), the Block-Triangular Properties II follow directly from Lemma (5.59).

Finally, assume that ${}_{\hat{\mathbf{g}}^{(n)}}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}} = D(\vec{u})$ for some tuple $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}$. In particular, observe that the bases $\hat{\mathbf{f}}^{(n)}[\sigma_k, z]$ and $\hat{\mathbf{f}}^{(n)}$ for $\mathbb{K}^n$, as well as the bases $\hat{\mathbf{g}}^{(n)}[\sigma_k, z']$ and $\hat{\mathbf{g}}^{(n)}$, are related by the change-of-basis matrices $B_k^{(n)}(z)$ and $B_k^{(n)}(z')$, respectively. Therefore, by the *change-of-basis formula*, we deduce that

$$
{}_{\hat{\mathbf{g}}^{(n)}[\sigma_k,z']}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_k,z]} = \big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)\,.
$$

Consequently, since ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_k,z']}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_k,z]}$ must be upper-triangular, a direct application of Lemma (7.115) yields that:

- The $(k+1,k)$-entry of the matrix product $\big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)$ must vanish.
- Under the above condition, ${}_{\hat{\mathbf{g}}^{(n)}[\sigma_k,z']}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_k,z]} = D(\pi_k(\vec{u}))$,

where $\pi_k(\vec{u})$ denotes the permutation of $\vec{u}$ associated with σ_k. This completes the proof. □

Having established the preceding lemmas, we are now in a position to state and prove the first part of one of our main theorems.

Theorem 5.66. ***Setup**: Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ from Construction (4.33), and $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.*

⋆ *Local descriptions on U_{T} and U_{L}: According to Lemma (4.34), $\mathscr{F}$ and $\mathscr{G}$ have the following local descriptions:*

- *On U_{T}, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of $n-1$ surjective linear maps*

$$
\big\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\big\}_{i=1}^{n-1}\,, \quad \textit{and} \quad \big\{\psi_{\mathscr{G}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\big\}_{i=1}^{n-1}\,,
$$

respectively. For a schematic illustration of a generic representative of one of these sheaves on U_{T}*, see Figure* (15)*.*

- *On* U_{L}*,* $\mathscr{F}$ *and* $\mathscr{G}$ *are specified by two collections of* $n-1$ *injective linear maps*

$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1}, \quad \textit{and} \quad \left\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1},$$

respectively. For a schematic illustration of a generic representative of one of these sheaves on U_{L}*, see Figure* (13)*.*

- ***Compatibility conditions****: For each* $i \in [1, n-1]$*,*

$$\psi_{\mathscr{F}}^{(i)} \circ \phi_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}, \quad \textit{and} \quad \psi_{\mathscr{G}}^{(i)} \circ \phi_{\mathscr{G}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}.$$

$\star$ *Global flag data: By Theorem* (4.40)*,* $\mathscr{F}$ *and* $\mathscr{G}$ *are geometrically characterized by two sequence of complete flags* $\left\{\mathcal{F}_j^{\bullet}\right\}_{j=0}^{\ell+1}$ *and* $\left\{\mathcal{G}_j^{\bullet}\right\}_{j=0}^{\ell+1}$ *in* $\mathbb{K}^n$*, respectively, such that:*

- $\mathcal{F}_0^{\bullet}$ *is completely opposite to both* $\mathcal{F}_1^{\bullet}$ *and* $\mathcal{F}_{\ell+1}^{\bullet}$*, and for each* $j \in [1, \ell]$*,* $\mathcal{F}_j^{\bullet}$ *is in* s_{i_j}*-relative position with respect to* $\mathcal{F}_{j+1}^{\bullet}$*.*
- $\mathcal{G}_0^{\bullet}$ *is completely opposite to both* $\mathcal{G}_1^{\bullet}$ *and* $\mathcal{G}_{\ell+1}^{\bullet}$*, and for each* $j \in [1, \ell]$*,* $\mathcal{G}_j^{\bullet}$ *is in* s_{i_j}*-relative position with respect to* $\mathcal{G}_{j+1}^{\bullet}$*.*

In particular, by Lemma (4.35)*, we know that:*

$$\mathcal{F}_0^{\bullet} := {}_{\mathcal{K}}\mathcal{F}^{\bullet}\big(\psi_{\mathscr{F}}^{(1)}, \ldots, \psi_{\mathscr{F}}^{(n-1)}\big), \quad \textit{and} \quad \mathcal{F}_1^{\bullet} := {}_{\mathcal{I}}\mathcal{F}^{\bullet}\big(\phi_{\mathscr{F}}^{(1)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\big),$$

$$\mathcal{G}_0^{\bullet} := {}_{\mathcal{K}}\mathcal{F}^{\bullet}\big(\psi_{\mathscr{G}}^{(1)}, \ldots, \psi_{\mathscr{G}}^{(n-1)}\big), \quad \textit{and} \quad \mathcal{G}_1^{\bullet} := {}_{\mathcal{I}}\mathcal{F}^{\bullet}\big(\phi_{\mathscr{G}}^{(1)}, \ldots, \phi_{\mathscr{G}}^{(n-1)}\big),$$

are the type $\mathcal{K}$ *and type* $\mathcal{I}$ *flags in* $\mathbb{K}^n$ *associated with* $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$*,* $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$*,* $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$*, and* $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$*, respectively (cf. Definition* (4.18)*–*(1)*–*(2)*).*

$\star$ *Main assumption (Adapted bases): Let* $\hat{\mathbf{f}}^{(n)}$*,* $\hat{\mathbf{g}}^{(n)}$ *be bases for* $\mathbb{K}^n$*, and let* $\vec{z} = (z_1, \ldots, z_\ell)$*,* $\vec{z}\,' = (z_1', \ldots, z_\ell') \in X(\beta, \mathbb{K})$ *be points such that the pairs* $(\hat{\mathbf{f}}^{(n)}, \vec{z})$ *and* $(\hat{\mathbf{g}}^{(n)}, \vec{z}\,')$ *algebraically characterizes* $\mathscr{F}$ *and* $\mathscr{G}$ *according to Theorem* (4.46)*, respectively. In this setting, we have that:*

- *Relative to the basis* $\hat{\mathbf{f}}^{(n)}$ *for* $\mathbb{K}^n$*:* $\mathcal{F}_0^{\bullet}$ *and* $\mathcal{F}_1^{\bullet}$ *are the anti-standard and standard flags, respectively. For each* $j \in [1, \ell]$*, the flag* $\mathcal{F}_{j+1}^{\bullet}$ *is represented by the path matrix* $P_{\beta_j}(\vec{z}_j) = B_{i_1}^{(n)}(z_1) \cdots B_{i_j}^{(n)}(z_j) \in \mathrm{GL}(n, \mathbb{K})$ *associated with the truncated braid word* $\beta_j = \sigma_{i_1} \cdots \sigma_{i_j} \in \mathrm{Br}_n^+$ *and the truncated tuple* $\vec{z}_j = (z_1, \ldots, z_j) \in \mathbb{K}^j_{\mathrm{std}}$*.*

- *Relative to the basis $\hat{\mathbf{g}}^{(n)}$ for $\mathbb{K}^n$: $\mathcal{G}_0^\bullet$ and $\mathcal{G}_1^\bullet$ are the anti-standard and standard flags, respectively. For each $j \in [1, \ell]$, the flag $\mathcal{G}_{j+1}^\bullet$ is represented by the path matrix $P_{\beta_j}(\vec{z}'_j) = B_{i_1}^{(n)}(z'_1) \cdots B_{i_j}^{(n)}(z'_j) \in \mathrm{GL}(n, \mathbb{K})$ associated with the truncated braid word β_j and the truncated tuple $\vec{z}'_j = (z'_1, \dots, z'_j) \in \mathbb{K}^j_{\mathrm{std}}$.*

Furthermore, under this assumption, the ***compatibility conditions*** *and an inductive argument ensure that, for each $i \in [1, n-1]$, there are unique bases $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_j^{(i)}\}_{j=1}^i$ and $\hat{\mathbf{g}}^{(i)} := \{\hat{g}_j^{(i)}\}_{j=1}^i$ for $\mathbb{K}^i$ such that (cf. Definition (4.18)–(3)–(4)):*

- *The collection $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ is a system of bases adapted to both $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$.*
- *The collection $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ is a system of bases adapted to both $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$.*

Building on this, for each $j \in [1, \ell]$, we denote by $\{\hat{\mathbf{f}}^{(i)}[\beta_j, \vec{z}_j]\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}[\beta_j, \vec{z}'_j]\}_{i=1}^n$ the braid-transformed bases obtained from $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ via the truncated braid word β_j and the truncated tuples $\vec{z}_j$ and $\vec{z}'_j$, respectively (cf. Definition (4.29)). Accordingly, by Theorem (4.50), we have that:

- *$\left(\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n, z_1\right)$ is a system of bases adapted to $\mathscr{F}$ on R_1.*
- *For each $j \in [1, \ell-1]$, $\left(\{\hat{\mathbf{f}}^{(i)}[\beta_j, \vec{z}_j]\}_{i=1}^n, z_{j+1}\right)$ is a system of bases adapted to $\mathscr{F}$ on R_{j+1}.*
- *$\left(\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n, z'_1\right)$ is a system of bases adapted to $\mathscr{G}$ on R_1.*
- *For each $j \in [1, \ell-1]$, $\left(\{\hat{\mathbf{g}}^{(i)}[\beta_j, \vec{z}'_j]\}_{i=1}^n, z'_{j+1}\right)$ is a system of bases adapted to $\mathscr{G}$ on R_{j+1}.*

$\star$ *Main Conclusion: Following Definition (5.55), let $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ be the linear map associated with the pair $(\mathscr{F}, \mathscr{G})$. Then, under the given* ***setup****, there is an isomorphism of vector spaces*

$$\mathrm{Ext}^0(\mathscr{F}, \mathscr{G}) \cong \ker \delta_{\mathscr{F},\mathscr{G}} \,.$$

Proof. To begin, let $\mathcal{S}_{\Lambda(\beta)}$ be the stratification of $\mathbb{R}^2$ induced by $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_\mathrm{B}, U_\mathrm{L}, U_\mathrm{R}, U_\mathrm{T}\}$ the open cover of $\mathbb{R}^2$ from Construction (4.33), and $\mathcal{R}_{\Lambda(\beta)} = \{R_j\}_{j=1}^\ell$ the partition of U_B into ℓ open vertical straps from Construction (4.36).

Let $\lambda \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{G})$. By Lemma (5.63), on U_T and U_L, λ is characterized by a collection of n linear maps $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^n$ such that for each $i \in [1, n-1]$:

$$T_\lambda^{(i)} \circ \psi_{\mathscr{F}}^{(i)} = \psi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i+1)} \,, \quad \text{and} \quad T_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} = \phi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i)} \,.$$

By assumption, $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ are systems of bases adapted to the pairs $\left(\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}, \{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}\right)$ and $\left(\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}, \{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}\right)$, respectively. Hence, by Lemma (5.63), we obtain that, ${}_{\hat{\mathbf{g}}^{(n)}}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}} \in M(n, \mathbb{K})$ is diagonal, that is, ${}_{\hat{\mathbf{g}}^{(n)}}\left[T_\lambda^{(n)}\right]_{\hat{\mathbf{f}}^{(n)}} = D(\vec{u})$ for some tuple $\vec{u} = (u_1, \dots, u_n) \in \mathbb{K}^n_{\mathrm{std}}$.

Next, consider R_1, the vertical strap in $\mathbb{R}^2$ containing σ_{i_1}—the first crossing of β. In particular, we denote by $k = i_1 \in [1, n-1]$ the index of σ_{i_1}. By construction, R_1 is the vertical strap immediately to the right of U_{L}. Thus, by the constructibility of $\mathscr{F}$ and $\mathscr{G}$ with respect to $\mathcal{S}_{\Lambda(\beta)}$, we obtain one of the following two cases:

- *Case 1*: Suppose that $k = 1$. Then, on R_1, $\mathscr{F}$ and $\mathscr{G}$ are described by two collections of n injective linear maps:
$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}_{\mathscr{F}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2\right\},$$
$$\left\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}_{\mathscr{G}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2\right\},$$
respectively. In this local configuration, Lemma (5.64) asserts that λ is described by a collection of $n+1$ linear maps
$$\left\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\right\}_{i=1}^{n} \cup \left\{\widetilde{T}_\lambda^{(1)} : \mathbb{K}^1 \to \mathbb{K}^1\right\}.$$

- *Case 2*: Suppose that $k \geq 2$. Then, on R_1, $\mathscr{F}$ and $\mathscr{G}$ are characterized by two collections of $n+1$ injective linear maps:
$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}_{\mathscr{F}}^{(k-1)} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}_{\mathscr{F}}^{(k)} : \mathbb{K}^k \to \mathbb{K}^{k+1}\right\},$$
$$\left\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}_{\mathscr{G}}^{(k-1)} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}_{\mathscr{G}}^{(k)} : \mathbb{K}^k \to \mathbb{K}^{k+1}\right\},$$
respectively. In this local configuration, Lemma (5.65) asserts that λ is described by a collection of $n+1$ linear maps
$$\left\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\right\}_{i=1}^{n} \cup \left\{\widetilde{T}_\lambda^{(k)} : \mathbb{K}^k \to \mathbb{K}^k\right\}.$$

In any of the above cases, $\left\{\phi_{\mathscr{F}}^{(i)}\right\}_{i=1}^{n-1}$, $\left\{\phi_{\mathscr{G}}^{(i)}\right\}_{i=1}^{n-1}$, and $\left\{T_\lambda^{(i)}\right\}_{i=1}^{n}$ are precisely the linear maps characterizing $\mathscr{F}$, $\mathscr{G}$, and λ on U_{L}.

Now, let $\vec{z} = (z_1, \dots, z_\ell)$ and $\vec{z}\,' = (z_1', \dots, z_\ell')$ be the points in the braid variety $X(\beta, \mathbb{K})$ such that the pairs $\left(\hat{\mathbf{f}}^{(n)}, \vec{z}\right)$ and $\left(\hat{\mathbf{g}}^{(n)}, \vec{z}\,'\right)$ algebraically characterize $\mathscr{F}$ and $\mathscr{G}$ according to Theorem (4.46), respectively. By Theorem (4.50), we know that:

- $\left(\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^{n}, z_1\right)$ is a system of bases adapted to $\mathscr{F}$ on R_1.
- $\left(\left\{\hat{\mathbf{g}}^{(i)}\right\}_{i=1}^{n}, z_1'\right)$ is a system of bases adapted to $\mathscr{G}$ on R_1.

Then, a direct application of either Lemma (5.64) or Lemma (5.65) depending on the value of k, considering that $k = i_1$, implies that:

- **Compatibility condition for λ at σ_{i_1}**: The (i_1+1, i_1)-entry of $\left(B_{i_1}^{(n)}(z_1')\right)^{-1} D(\vec{u})\, B_{i_1}^{(n)}(z_1)$ must vanish.

- Under the above condition,

$$\hat{\mathbf{g}}^{(n)}[\sigma_{i_1}, z_1'] \big[T_\lambda^{(n)} \big]_{\hat{\mathbf{f}}^{(n)}[\sigma_{i_1}, z_1]} = D(\pi_{i_1}(\vec{u}))\,,$$

where $\pi_{i_1}(\vec{u})$ denotes the permutation of $\vec{u}$ associated with σ_{i_1}.

Applying the same reasoning to R_2, the vertical strap in $\mathbb{R}^2$ containing σ_{i_2}—the second crossing of β—we obtain that:

- **Compatibility condition for λ at σ_{i_2}:** The (i_2+1, i_2)-entry of $\big(B_{i_2}^{(n)}(z_2')\big)^{-1} D(\pi_{i_1}(\vec{u}))\, B_{i_2}^{(n)}(z_2)$ must vanish.

- Under the above condition,

$$\begin{aligned}
\hat{\mathbf{g}}^{(n)}[\beta_2, \vec{z}\,'_2] \big[T_\lambda^{(n)} \big]_{\hat{\mathbf{f}}^{(n)}[\beta_2, \vec{z}_2]} &= D\big(\pi_{i_2}(\pi_{i_1}(\vec{u}))\big)\,, \\
&= D(\pi_{i_1 \cdot i_2}(\vec{u}))\,, \\
&= D(\pi_{\beta_2}(\vec{u}))\,,
\end{aligned}$$

where $\pi_{\beta_2}(\vec{u})$ denotes the permutation of $\vec{u}$ associated with $\beta_2 = \sigma_{i_1}\sigma_{i_2} \in \mathrm{Br}_n^+$, the truncation of β at its second crossing, and $\vec{z}_2 = (z_1, z_2)$, $\vec{z}\,'_2 = (z_1', z_2') \in \mathbb{K}^2_{\mathrm{std}}$ denote the truncation of the tuples $\vec{z}$ and $\vec{z}\,'$ at their second entry, respectively.

Proceeding iteratively through all the vertical straps R_j, we obtain ℓ compatibility conditions for λ, one for each crossing σ_{i_j} of β. More precisely, for each $j \in [1, \ell]$, we deduce that

$$\Big[\big(B_{i_j}^{(n)}(z_j')\big)^{-1} D(\pi_{\beta_{j-1}}(\vec{u}))\, B_{i_j}^{(n)}(z_j) \Big]_{i_j+1, i_j} = 0\,.$$

In particular, building on Definition (5.55), we observe that the above compatibility conditions are precisely those characterizing the elements of the kernel of the linear map $\delta_{\mathscr{F},\mathscr{G}}$, which allows us to conclude that

$$\mathrm{Ext}^0(\mathscr{F}, \mathscr{G}) \cong \ker \delta_{\mathscr{F},\mathscr{G}}\,.$$

□

Having established the previous result, we conclude the analysis of the zero-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ and proceed to study the structure of its one-degree morphism spaces.

5.4. **One-Degree Morphism Spaces in the Category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.** Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. The main goal of this subsection is to establish the second part of Theorem (5.56), namely, the isomorphism of vector spaces in Equation (5.2), thereby providing an explicit algebraic characterization of the one-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. To this end, we begin by introducing some preliminaries.

5.4.1. *Technical Background.* Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. We now collect some preliminaries that will play a fundamental role in the explicit computation of the one-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. In particular, we begin by establishing some notation.

Notation 5.67. *Let $\mathscr{C}$ be a category, and suppose that A is an object of the category $\mathscr{C}$. Then, when drawing diagrams, we introduce $A =\!=\!= A$ to denote the identity morphism between A and A.*

Definition 5.68. *Let $\mathcal{M}$ be a smooth manifold, and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of $\mathbb{K}$-modules on $\mathcal{M}$. A sheaf $\mathscr{H}$ of $\mathbb{K}$-modules on $\mathcal{M}$ is called an extension of $\mathscr{F}$ by $\mathscr{G}$ if there exists a short exact sequence of the form*

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow \mathscr{F} \longrightarrow 0\,. \tag{5.13}$$

In particular, two extensions $\mathscr{H}$ and $\mathscr{H}'$ of $\mathscr{F}$ by $\mathscr{G}$ are said to be equivalent if there exists a sheaf isomorphism $\lambda : \mathscr{H} \to \mathscr{H}'$ such that the diagram in Figure (27) commutes in each square.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d, equal] & \mathscr{H} \arrow[r] \arrow[d, "\lambda"] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{H}' \arrow[r] & \mathscr{F} \arrow[r] & 0
\end{tikzcd}$$

FIGURE 27. Two equivalent extensions $\mathscr{H}$ and $\mathscr{H}'$ of $\mathscr{F}$ by $\mathscr{G}$.

Now, recall that the category of sheaves of $\mathbb{K}$-modules on a smooth manifold is an abelian category with enough injectives [Har77, KS90]. The following result is standard in classical sheaf theory and also follows from the general theory of abelian categories (see, for instance, [HS71]).

Lemma 5.69. *Let $\mathcal{M}$ be a smooth manifold, and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of $\mathbb{K}$-modules on $\mathcal{M}$. Then there exists a canonical isomorphism*

$$\mathrm{Ext}^1(\mathscr{F},\mathscr{G}) \cong \Big\{\textit{equivalence classes of extensions of } \mathscr{F} \textit{ by } \mathscr{G}\Big\}\,. \tag{5.14}$$

Let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Building on Lemma (5.69), we proceed to characterize $\mathrm{Ext}^1(\mathscr{F},\mathscr{G})$ via the equivalence classes of extensions of $\mathscr{F}$ by $\mathscr{G}$. To this end, we introduce several useful definitions and preliminary results.

Definition 5.70. *Let A, B, C, X, Y, Z be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\Gamma_{\mathscr{H}} : B \to Y$, and $\gamma_{\mathscr{F}} : C \to Z$ be linear maps. We say that $\Gamma_{\mathscr{H}}$ is an <u>extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$</u> if there exists a diagram as in Figure (28) where the horizontal rows are short exact sequences and each square commutes.*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\gamma_{\mathscr{G}}"'] & B \arrow[r] \arrow[u, "\Gamma_{\mathscr{H}}"'] & C \arrow[r] \arrow[u, "\gamma_{\mathscr{F}}"'] & 0
\end{tikzcd}$$

FIGURE 28. An extension $\Gamma_{\mathscr{H}}$ of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.

Definition 5.71. *Let A, B, B', C, X, Y, Y', Z be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\Gamma_{\mathscr{H}} : B \to Y$, $\Gamma_{\mathscr{H}'} : B' \to Y'$, and $\gamma_{\mathscr{F}} : C \to Z$ be linear maps, with $\Gamma_{\mathscr{H}}$ and $\Gamma_{\mathscr{H}'}$ realizing extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$. We say that $\Gamma_{\mathscr{H}}$ and $\Gamma_{\mathscr{H}'}$ are <u>equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$</u> if there exist linear isomorphisms $\xi : B \to B'$ and $\delta : Y \to Y'$ making the diagram in Figure (29) commute in each square.*

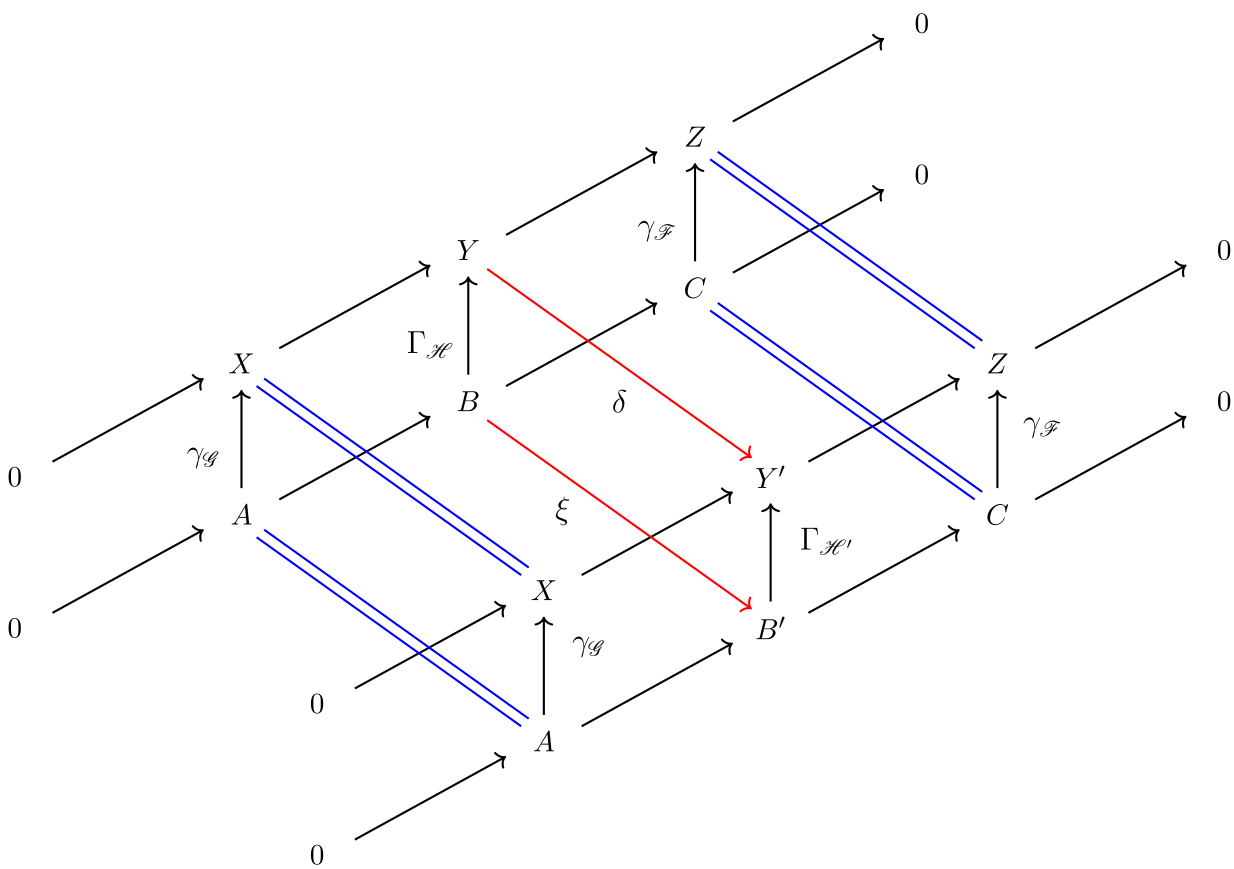


FIGURE 29. Two equivalent extensions $\Gamma_{\mathscr{H}}$ and $\Gamma_{\mathscr{H}'}$ of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.

Throughout this subsection, we will implement the following notation.

Notation 5.72. *Let A, B, X, Y be vector spaces over $\mathbb{K}$. Then we know that there exists a canonical isomorphism*

$$\mathrm{Hom}_{\mathbb{K}}(A \oplus B, X \oplus Y) \cong \mathrm{Hom}_{\mathbb{K}}(A, X) \oplus \mathrm{Hom}_{\mathbb{K}}(B, X) \oplus \mathrm{Hom}_{\mathbb{K}}(A, Y) \oplus \mathrm{Hom}_{\mathbb{K}}(B, Y)\,.$$

More precisely, we have that, for any linear map $\xi : A \oplus B \to X \oplus Y$, there exist linear maps $\lambda_\xi^{(1)} : A \to X$, $\lambda_\xi^{(2)} : B \to X$, $\lambda_\xi^{(3)} : A \to Y$, and $\lambda_\xi^{(4)} : B \to Y$ such that

$$\xi(a, b) = \left(\lambda_\xi^{(1)}(a) + \lambda_\xi^{(2)}(b), \lambda_\xi^{(3)}(a) + \lambda_\xi^{(4)}(b)\right), \quad \textit{for all } (a, b) \in A \oplus B\,. \tag{5.15}$$

Then, by abuse of notation, we denote the expression in equation (5.15) by

$$\xi = \begin{bmatrix} \lambda_\xi^{(1)} & \lambda_\xi^{(2)} \\ \lambda_\xi^{(3)} & \lambda_\xi^{(4)} \end{bmatrix}.$$

Definition 5.73. *Let A and B be vector spaces over $\mathbb{K}$. We denote by $\iota_A : A \to A \oplus B$ the canonical inclusion $\iota_A(a) := (a, 0)$ for all $a \in A$, and by $\pi_B : A \oplus B \to B$ the canonical projection $\pi_B(a, b) := b$ for all $(a, b) \in A \oplus B$.*

The following lemma introduces a concept that will play a central role in the subsequent discussion.

Lemma 5.74. *Let A, C, X, Z be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\gamma_{\mathscr{H}} : C \to X$, and $\gamma_{\mathscr{F}} : C \to Z$ be linear maps. Then the linear map ${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}} \rangle_{\gamma_{\mathscr{F}}} : A \oplus C \to X \oplus Z$ defined by*

$${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}} \rangle_{\gamma_{\mathscr{F}}} := \begin{bmatrix} \gamma_{\mathscr{G}} & \gamma_{\mathscr{H}} \\ 0 & \gamma_{\mathscr{F}} \end{bmatrix}$$

is an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$, which we call the <u>*block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{H}}$*</u>*.*

Proof. To begin, consider the canonical inclusion and projection maps $\iota_{A \oplus C} : A \to A \oplus C$, $\iota_{X \oplus Z} : X \to X \oplus Z$, $\rho_{A \oplus C} : A \oplus C \to C$, and $\rho_{X \oplus Z} : X \oplus Z \to Z$ (see Definition (5.73)). By construction, we have that:

- $\iota_{A \oplus C}$ and $\iota_{X \oplus Z}$ are injective.
- $\rho_{A \oplus C}$ and $\rho_{X \oplus Z}$ are surjective.
- $\mathrm{im}\, \iota_{A \oplus C} = \ker \rho_{A \oplus C}$ and $\mathrm{im}\, \iota_{X \oplus Z} = \ker \rho_{X \oplus Z}$.

In addition, a direct calculation shows that

$${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}} \rangle_{\gamma_{\mathscr{F}}} \circ \iota_{A \oplus C}(a) = \iota_{X \oplus Z} \circ \gamma_{\mathscr{G}}(a)\,, \quad \text{for all } a \in A\,,$$

$$\rho_{X \oplus Z} \circ {}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}} \rangle_{\gamma_{\mathscr{F}}}(a, c) = \gamma_{\mathscr{F}} \circ \rho_{A \oplus C}(a, c)\,, \quad \text{for all } (a, c) \in A \oplus C\,.$$

Combining the above results, we observe that each square in the diagram in Figure (30) commutes and its horizontal rows are short exact sequences. Hence, building on Definition (5.70), we conclude that ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ is an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\iota_{X\oplus Z}"] & X\oplus Z \arrow[r, "\rho_{X\oplus Z}"] & Z \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[u, "\gamma_{\mathscr{G}}"'] \arrow[r, "\iota_{A\oplus C}"'] & A\oplus C \arrow[u, "{}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}"'] \arrow[r, "\rho_{A\oplus C}"'] & C \arrow[u, "\gamma_{\mathscr{F}}"'] \arrow[r] & 0
\end{tikzcd}$$

FIGURE 30. The block extension ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{H}}$.

□

The following lemma shows that every extension of linear maps is equivalent to a block extension, which will simplify the analysis of the one-degree morphism spaces in the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.

Lemma 5.75. *Let A, B, C, X, Y, Z be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\Gamma_{\mathscr{H}} : B \to Y$, and $\gamma_{\mathscr{F}} : C \to Z$ be linear maps, with $\Gamma_{\mathscr{H}}$ realizing an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.*

Then there exists a linear map $\gamma_{\mathscr{H}} : C \to X$ such that $\Gamma_{\mathscr{H}}$ and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}} : A\oplus C \to X\oplus Z$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$, where ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ denotes the block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{H}}$.

Proof. By assumption, $\Gamma_{\mathscr{H}}$ is an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$. Hence, according to Definition (5.70), there exists a diagram as in Figure (31) where the horizontal rows are short exact sequences and each square commutes. Furthermore, since A, C, X, Z are vector spaces (hence free $\mathbb{K}$-modules), we have that $\mathrm{Ext}^1(A, C) = 0$ and $\mathrm{Ext}^1(X, Z) = 0$. It then follows that there exist linear isomorphisms $\xi : B \to A \oplus C$ and $\delta : Y \to X \oplus Z$ such that the diagrams in Figures (32) and (33) commute in each square.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\alpha_2"] & Y \arrow[r, "\beta_2"] & Z \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha_1"'] \arrow[u, "\gamma_{\mathscr{G}}"'] & B \arrow[r, "\beta_1"'] \arrow[u, "\Gamma_{\mathscr{H}}"'] & C \arrow[r] \arrow[u, "\gamma_{\mathscr{F}}"'] & 0
\end{tikzcd}$$

FIGURE 31. An extension $\Gamma_{\mathscr{H}}$ of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota_{A\oplus C}"] & A \oplus C \arrow[r, "\rho_{A\oplus C}"] & C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha_1"'] \arrow[u, equal] & B \arrow[r, "\beta_1"'] \arrow[u, "\xi"'] & C \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

FIGURE 32. Two equivalent extensions B and $A \oplus C$ of A by C.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\iota_{X\oplus Z}"] & X \oplus Z \arrow[r, "\rho_{X\oplus Z}"] & Z \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r, "\alpha_2"'] \arrow[u, equal] & Y \arrow[r, "\beta_2"'] \arrow[u, "\delta"'] & Z \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

FIGURE 33. Two equivalent extensions Y and $X \oplus Z$ of X by Z.

Next, consider the diagram in Figure (34), and define the linear map $\Gamma'_{\mathscr{H}} : A \oplus C \to X \oplus Z$ by $\Gamma'_{\mathscr{H}} := \delta \circ \Gamma_{\mathscr{H}} \circ \xi^{-1}$. By the commutativity of the diagrams in Figures (31), (32), and (33), we have that

$$\begin{aligned}
\Gamma'_{\mathscr{H}} \circ \iota_{A\oplus C} &= \delta \circ \Gamma_{\mathscr{H}} \circ \xi^{-1} \circ \iota_{A\oplus C}\,, \\
&= \delta \circ \Gamma_{\mathscr{H}} \circ \xi^{-1} \circ \xi \circ \alpha_1\,, \\
&= \delta \circ \Gamma_{\mathscr{H}} \circ \alpha_1\,, \\
&= \delta \circ \alpha_2 \circ \gamma_{\mathscr{G}}\,, \\
&= \iota_{X\oplus Z} \circ \gamma_{\mathscr{G}}\,.
\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}\rho_{X\oplus Z}\circ\Gamma'_{\mathscr{H}} &= \rho_{X\oplus Z}\circ\delta\circ\Gamma_{\mathscr{H}}\circ\xi^{-1}\,,\\ &= \beta_2\circ\Gamma_{\mathscr{H}}\circ\xi^{-1}\,,\\ &= \gamma_{\mathscr{F}}\circ\beta_1\circ\xi^{-1}\,,\\ &= \gamma_{\mathscr{F}}\circ\rho_{A\oplus C}\circ\xi\circ\xi^{-1}\,,\\ &= \gamma_{\mathscr{F}}\circ\rho_{A\oplus C}\,.\end{aligned}$$

Combining the above results, we conclude that there exist linear isomorphisms $\xi : B \to A\oplus C$ and $\delta : Y \to X\oplus Z$ such that the diagram in Figure (34) commutes in each square. It then follows from Definition (5.71) that $\Gamma_{\mathscr{H}}$ and $\Gamma'_{\mathscr{H}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.

Finally, consider the linear map $\Gamma'_{\mathscr{H}} : A\oplus C \to X\oplus Z$. By the homomorphism decomposition for direct sums, there exist linear maps $\lambda^{(1)}_{\mathscr{H}} : A \to X$, $\lambda^{(2)}_{\mathscr{H}} : C \to X$, $\lambda^{(3)}_{\mathscr{H}} : A \to Z$, and $\lambda^{(4)}_{\mathscr{H}} : C \to Z$ such that

$$\Gamma'_{\mathscr{H}} = \begin{bmatrix}\lambda^{(1)}_{\mathscr{H}} & \lambda^{(2)}_{\mathscr{H}}\\ \lambda^{(3)}_{\mathscr{H}} & \lambda^{(4)}_{\mathscr{H}}\end{bmatrix}.$$

In particular, it follows from the commutativity of the diagram in Figure (34) that

$$\lambda^{(1)}_{\mathscr{H}} = \gamma_{\mathscr{G}}\,,\qquad \lambda^{(3)}_{\mathscr{H}} = 0\,,\qquad \lambda^{(4)}_{\mathscr{H}} = \gamma_{\mathscr{F}}\,.$$

Hence, by setting $\gamma_{\mathscr{H}} := \lambda^{(2)}_{\mathscr{H}} : C \to X$, the above relations ensure that $\Gamma'_{\mathscr{H}} = {}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$, where ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}} : A\oplus C \to X\oplus Z$ denotes the block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{H}}$. Bearing this in mind, we conclude that there exists a linear map $\gamma_{\mathscr{H}} : C \to X$ such that Γ_H and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$. This completes the proof.

□

The following lemma establishes necessary and sufficient conditions for the equivalence of block extensions of linear maps.

Lemma 5.76. *Let A, C, X, Z be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\gamma_{\mathscr{H}} : C \to X$, $\gamma_{\mathscr{H}'} : C \to X$, and $\gamma_{\mathscr{F}} : C \to Z$ be linear maps. In particular, consider ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}} : A\oplus C \to X\oplus Z$ and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}'}\rangle_{\gamma_{\mathscr{F}}} : A\oplus C \to X\oplus Z$ the block extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{H}}$ and $\gamma_{\mathscr{H}'}$, respectively. Then, we have that ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}'}\rangle_{\gamma_{\mathscr{F}}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ if and only if there exist linear maps $\lambda_\xi : C \to A$ and $\lambda_\delta : Z \to X$ such that*

$$\gamma_{\mathscr{H}'} = \gamma_{\mathscr{H}} + \lambda_\delta\circ\gamma_{\mathscr{F}} - \gamma_{\mathscr{G}}\circ\lambda_\xi\,.$$

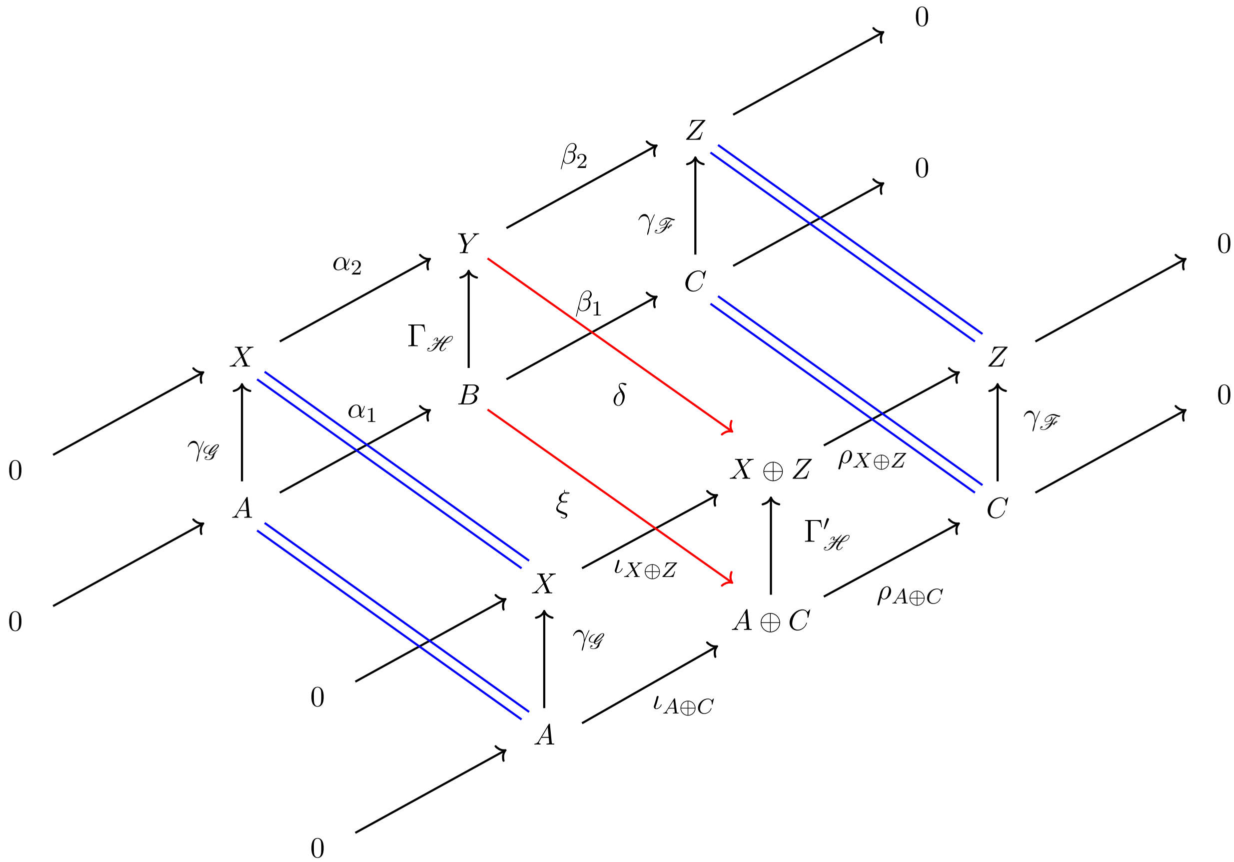


Figure 34. Two equivalent extensions $\Gamma_{\mathscr{H}}$ and $\Gamma'_{\mathscr{H}}$ of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.

Proof. To begin, suppose that ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}'}\rangle_{\gamma_{\mathscr{F}}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$. Thus, according to Definition (5.71), there exist linear isomorphisms $\xi : A \oplus C \to A \oplus C$ and $\delta : X \oplus Z \to X \oplus Z$ such that the diagram in Figure (35) commutes in each square. By the homomorphism decomposition for direct sums, there exist linear maps $\lambda_\xi^{(1)} : A \to A$, $\lambda_\xi^{(2)} : C \to A$, $\lambda_\xi^{(3)} : A \to C$, and $\lambda_\xi^{(4)} : C \to C$ such that ξ can be written as

$$\xi = \begin{bmatrix} \lambda_\xi^{(1)} & \lambda_\xi^{(2)} \\ \lambda_\xi^{(3)} & \lambda_\xi^{(4)} \end{bmatrix}.$$

Likewise, there exist linear maps $\lambda_\delta^{(1)} : X \to X$, $\lambda_\delta^{(2)} : Z \to X$, $\lambda_\delta^{(3)} : X \to Z$, and $\lambda_\delta^{(4)} : Z \to Z$ such that δ can be expressed as

$$\delta = \begin{bmatrix} \lambda_\delta^{(1)} & \lambda_\delta^{(2)} \\ \lambda_\delta^{(3)} & \lambda_\delta^{(4)} \end{bmatrix}.$$

From these decompositions, the commutativity of the diagram in Figure (35) implies that

$$\lambda_\xi^{(1)} = \mathrm{id}_A\,,$$

$$\lambda_\xi^{(3)} = 0\,,$$

$$\lambda_\xi^{(4)} = \mathrm{id}_C\,,$$

$$\lambda_\delta^{(1)} = \mathrm{id}_X\,,$$

$$\lambda_\delta^{(3)} = 0\,,$$

$$\lambda_\delta^{(4)} = \mathrm{id}_Z\,,$$

$$\gamma_{\mathscr{H}'} + \gamma_{\mathscr{G}} \circ \lambda_\xi^{(2)} = \gamma_{\mathscr{H}} + \lambda_\delta^{(2)} \circ \gamma_{\mathscr{F}}\,.$$

Thus, by setting $\lambda_\xi := \lambda_\xi^{(2)} : C \to A$ and $\lambda_\delta := \lambda_\delta^{(2)} : Z \to X$, the deduce that there exist linear maps λ_ξ and λ_δ such that

$$\gamma_{\mathscr{H}'} = \gamma_{\mathscr{H}} + \lambda_\delta \circ \gamma_{\mathscr{F}} - \gamma_{\mathscr{G}} \circ \lambda_\xi\,.$$

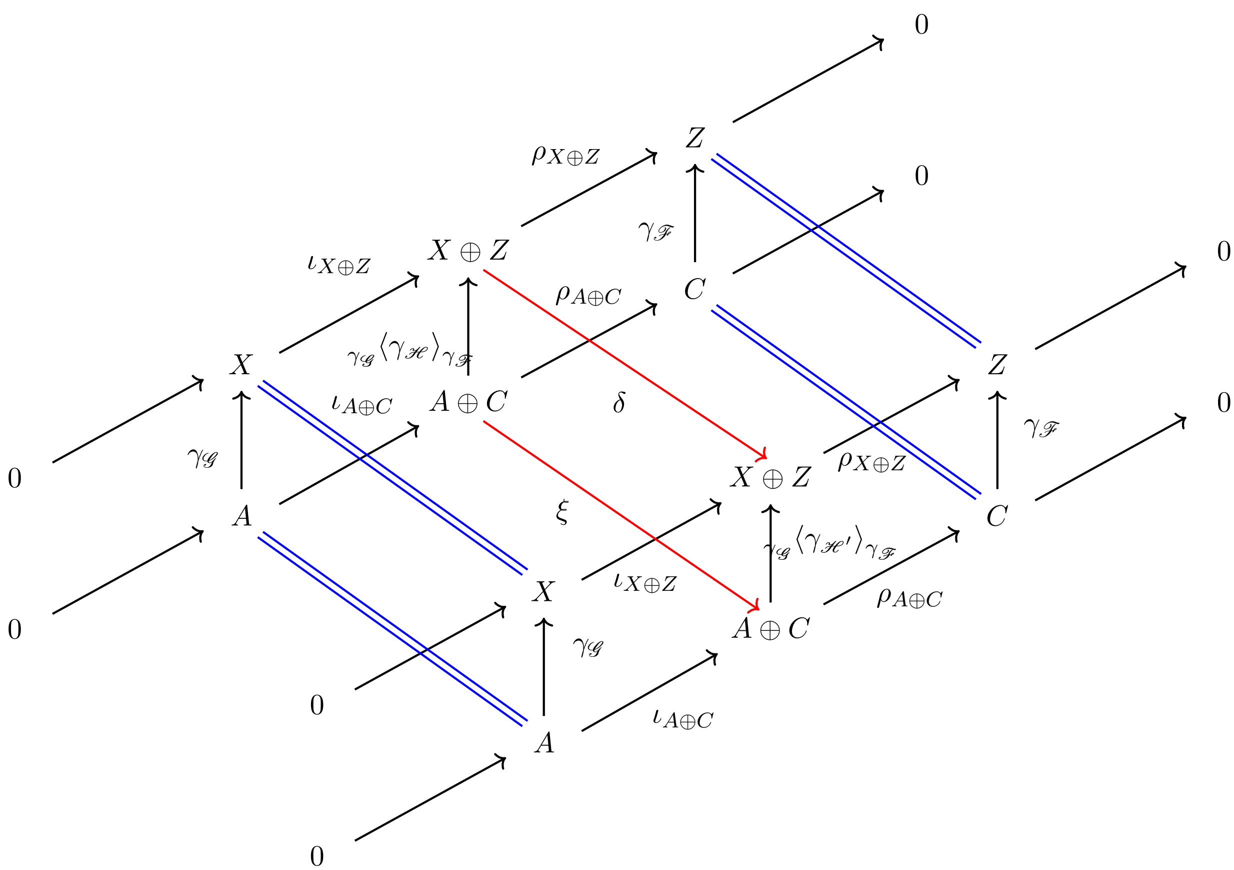


FIGURE 35. Two equivalent extensions $_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ and $_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}'}\rangle_{\gamma_{\mathscr{F}}}$ of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$.

Conversely, suppose that there exist linear maps $\lambda_\xi : C \to A$ and $\lambda_\delta : Z \to X$ such that

$$\gamma_{\mathscr{H}'} = \gamma_{\mathscr{H}} + \lambda_\delta \circ \gamma_{\mathscr{F}} - \gamma_{\mathscr{G}} \circ \lambda_\xi \,.$$

Thus, we introduce $\xi : A \oplus C \to A \oplus C$ and $\delta : X \oplus Z \to X \oplus Z$ to denote the linear isomorphisms defined by

$$\xi := \begin{bmatrix} \mathrm{id}_A & \lambda_\xi \\ 0 & \mathrm{id}_C \end{bmatrix}, \qquad \delta := \begin{bmatrix} \mathrm{id}_X & \lambda_\delta \\ 0 & \mathrm{id}_Z \end{bmatrix}.$$

In particular, a direct calculation shows the diagram in Figure (35) commutes in each square. Hence, building on Definition (5.71), we conclude that ${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}} \rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}'} \rangle_{\gamma_{\mathscr{F}}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$. This concludes the proof. $\square$

Next, we state a lemma and a related observation that make explicit how extensions of sheaves with singular support in $L(\Lambda(\beta))$ inherit the microlocal support and rank conditions, thus elucidating the structure of the one-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Lemma 5.77. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $L(\Lambda(\beta)) \subset T^*\mathbb{R}^2$ be the closed conic Lagrangian associated with $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ (see Construction (2.9)). Let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and suppose that $\mathscr{H}$ is an extension of $\mathscr{F}$ by $\mathscr{G}$. Then:*

- *The singular support of $\mathscr{H}$ is contained in $L(\Lambda(\beta))$, namely $SS(\mathscr{H}) \subset L(\Lambda(\beta))$.*
- *$\mathscr{H}$ is compactly supported.*
- *$\mathscr{H}$ has microlocal rank 2.*

Proof. By assumption, $\mathscr{H}$ is an extension of $\mathscr{F}$ by $\mathscr{G}$. Hence, $\mathscr{H}$ fits into a short exact sequence of the form

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow \mathscr{F} \longrightarrow 0\,. \tag{5.16}$$

Next, we verify that $SS(\mathscr{H}) \subset L(\Lambda(\beta))$. In particular, a direct application of the triangle inequality for the singular support to the short exact sequence in (5.16) (see [KS90]) yields that $SS(\mathscr{H}) \subset SS(\mathscr{F}) \cup SS(\mathscr{G})$. Here, recall that $\mathscr{F}$ and $\mathscr{G}$ are objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and therefore $SS(\mathscr{F}) \subset L(\Lambda(\beta))$ and $SS(\mathscr{G}) \subset L(\Lambda(\beta))$, which implies that $SS(\mathscr{H}) \subset L(\Lambda(\beta))$.

Now, we prove that $\mathscr{H}$ is compactly supported. To begin, observe that for any $x \in \mathbb{R}^2$, the short exact sequence in (5.16) yields a short exact sequence of $\mathbb{K}$-modules at the stalks:

$$0 \longrightarrow \mathscr{G}_x \longrightarrow \mathscr{H}_x \longrightarrow \mathscr{F}_x \longrightarrow 0\,.$$

Bearing this in mind, we deduce that $\mathrm{supp}(\mathscr{H}) = \mathrm{supp}(\mathscr{F}) \cup \mathrm{supp}(\mathscr{G})$. In particular, recall that, since $\mathscr{F}$ and $\mathscr{G}$ are objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, $\mathrm{supp}(\mathscr{F})$ and $\mathrm{supp}(\mathscr{G})$ are compact sets in $\mathbb{R}^2$. Then, since the finite union of compact sets is compact, we conclude that $\mathrm{supp}(\mathscr{H})$ is also a compact set in $\mathbb{R}^2$, thereby obtaining that $\mathscr{H}$ is compactly supported.

Finally, we show that $\mathscr{H}$ has microlocal rank 2. To this end, let a be an arc in $\mathcal{S}_{\Lambda(\beta)}$, the stratification of $\mathbb{R}^2$ induced by $\Lambda(\beta)$. Near a, $\mathcal{S}_{\Lambda(\beta)}$ consists of an upper 2-dimensional stratum U and a lower 2-dimensional stratum D, as illustrated in Sub-figure (4a). Given this local configuration, let $p \in U$ and $q \in D$ be two arbitrary points. Then, the short exact sequence in (5.16) yields short exact sequences of $\mathbb{K}$-modules, for the stalks at p and q, of the form

$$0 \longrightarrow \mathscr{G}_p \longrightarrow \mathscr{H}_p \longrightarrow \mathscr{F}_p \longrightarrow 0\,,$$

$$0 \longrightarrow \mathscr{G}_q \longrightarrow \mathscr{H}_q \longrightarrow \mathscr{F}_q \longrightarrow 0\,.$$

Bearing this in mind, we obtain that $\dim_{\mathbb{K}} \mathscr{H}_p = \dim_{\mathbb{K}} \mathscr{F}_p + \dim_{\mathbb{K}} \mathscr{G}_p$ and $\dim_{\mathbb{K}} \mathscr{H}_q = \dim_{\mathbb{K}} \mathscr{F}_q + \dim_{\mathbb{K}} \mathscr{G}_q$. Accordingly, we distinguish two cases:

- Suppose that a belongs to a strand with Maslov potential 0. By the microlocal rank conditions, $\dim_{\mathbb{K}} \mathscr{F}_p - \dim_{\mathbb{K}} \mathscr{F}_q = 1$ and $\dim_{\mathbb{K}} \mathscr{G}_p - \dim_{\mathbb{K}} \mathscr{G}_q = 1$. Hence, in this case $\dim_{\mathbb{K}} \mathscr{H}_p - \dim_{\mathbb{K}} \mathscr{H}_q = 2$.
- Suppose that a belongs to a strand with Maslov potential 1. By the microlocal rank conditions, $\dim_{\mathbb{K}} \mathscr{F}_q - \dim_{\mathbb{K}} \mathscr{F}_p = 1$ and $\dim_{\mathbb{K}} \mathscr{G}_q - \dim_{\mathbb{K}} \mathscr{G}_p = 1$. Hence, in this case $\dim_{\mathbb{K}} \mathscr{H}_q - \dim_{\mathbb{K}} \mathscr{H}_p = 2$.

Therefore, since a is an arbitrary arc in $\mathcal{S}_{\Lambda(\beta)}$, we conclude that $\mathscr{H}$ has microlocal rank 2. This completes the proof. □

Observation 5.78. *Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathcal{S}_{\Lambda(\beta)}$ denote the stratification of $\mathbb{R}^2$ induced by $\Lambda(\beta)$. Let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, and suppose that $\mathscr{H}$ is an extension of $\mathscr{F}$ by $\mathscr{G}$.*

By Lemma (5.77), $\mathscr{H}$ is a compactly supported sheaf of $\mathbb{K}$-modules on $\mathbb{R}^2$ of microlocal rank 2, whose singular support is contained in $L(\Lambda(\beta))$, the closed conic Lagrangian associated with $\Lambda(\beta)$. It then follows from [STZ17] that $\mathscr{H}$ is not only constructible with respect to $\mathcal{S}_{\Lambda(\beta)}$, but is also subject to the microlocal support conditions. Building on this, we now provide a detailed characterization of $\mathscr{H}$ at the key local models: arcs, cusps, and crossings.

⋆ ***Arcs****: Let a be an arc in $\mathcal{S}_{\Lambda(\beta)}$. Thus, near a, $\mathcal{S}_{\Lambda(\beta)}$ consists of a, an upper 2-dimensional stratum U, and a lower 2-dimensional stratum D, as illustrated in Sub-figure (4a). Choose arbitrary points $p \in U$ and $q \in D$, and* *denote the stalks of* $\mathscr{G}$, $\mathscr{H}$, *and* $\mathscr{F}$ *at these points by*

$$X = \mathscr{G}_p, \qquad Y = \mathscr{H}_p, \qquad Z = \mathscr{F}_p,$$

$$A = \mathscr{G}_q, \qquad B = \mathscr{H}_q, \qquad C = \mathscr{F}_q.$$

Thus, near a, *the microlocal support conditions assert that* $\mathscr{G}$, $\mathscr{H}$, *and* $\mathscr{F}$ *are specified by linear maps* $\gamma_{\mathscr{G}} : A \to X$, $\Gamma_{\mathscr{H}} : B \to Y$, *and* $\gamma_{\mathscr{F}} : C \to Z$. *Moreover, since* $\mathscr{H}$ *is an extension of* $\mathscr{F}$ *by* $\mathscr{G}$, *we deduce that* $\Gamma_{\mathscr{H}}$ *is an extension of* $\gamma_{\mathscr{F}}$ *by* $\gamma_{\mathscr{G}}$, *see Definition* (5.70).

Next, recall that (1) $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ *is identified with the set of equivalence classes of extensions of* $\mathscr{F}$ *by* $\mathscr{G}$, *and (2) every extension of linear maps is equivalent to a block extension, as established in Lemma* (5.75). *Hence, we deduce that there exists a linear map* $\gamma_{\mathscr{H}} : C \to X$ *such that, viewed as an element of* $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, $\mathscr{H}$ *is locally specified near* a *by the block extension* ${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}} \rangle_{\gamma_{\mathscr{F}}} : A \oplus C \to X \oplus Z$ *of* $\gamma_{\mathscr{F}}$ *by* $\gamma_{\mathscr{G}}$ *associated with* $\gamma_{\mathscr{H}}$ *(see Lemma* (5.74)*). In particular, we call* $\gamma_{\mathscr{H}}$ *the* <u>characteristic map</u> *of* $\mathscr{H}$ *near* a.

Let $\mathscr{H}'$ *be an extension of* $\mathscr{F}$ *by* $\mathscr{G}$ *representing the same equivalence class as* $\mathscr{H}$ *in* $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, *so that there exists a sheaf isomorphism* $\lambda : \mathscr{H} \to \mathscr{H}'$ *such that the diagram in Figure* (27) *commutes in each square. Analogously to* $\mathscr{H}$, *we have that there exists a linear map* $\gamma_{\mathscr{H}'} : C \to X$ *such that, viewed as an element of* $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, $\mathscr{H}'$ *is locally specified near* a *by the block extension* ${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}'} \rangle_{\gamma_{\mathscr{F}}} : A \oplus C \to X \oplus Z$ *of* $\gamma_{\mathscr{F}}$ *by* $\gamma_{\mathscr{G}}$ *associated with* $\gamma_{\mathscr{H}'}$ *(the characteristic map of* $\mathscr{H}'$ *near* a*). Consequently, since* $\mathscr{H}$ *and* $\mathscr{H}'$ *represent the same equivalence class in* $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, *we deduce that* ${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}} \rangle_{\gamma_{\mathscr{F}}}$ *and* ${}_{\gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}'} \rangle_{\gamma_{\mathscr{F}}}$ *must be equivalent extensions of* $\gamma_{\mathscr{F}}$ *by* $\gamma_{\mathscr{G}}$. *It then follows from Lemma* (5.76) *that there exist linear maps* $\lambda_{\xi} : C \to A$ *and* $\lambda_{\delta} : Z \to X$ *such that*

$$\gamma_{\mathscr{H}'} = \gamma_{\mathscr{H}} + \lambda_{\delta} \circ \gamma_{\mathscr{F}} - \gamma_{\mathscr{G}} \circ \lambda_{\xi}\,,$$

with the pair $(\lambda_{\xi}, \lambda_{\delta})$ *realizing the sheaf isomorphism* λ *near* a, *providing a concrete description of the relationship between two representatives of the same class in* $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ *at the level of characteristic maps near an arbitrary arc* a.

$\star$ ***<u>Cusps</u>****: Let* c *be a cusp in* $\mathcal{S}_{\Lambda(\beta)}$. *Near* c, $\mathscr{F}$ *and* $\mathscr{G}$ *are specified by two pairs of linear maps between vector spaces, say* $\alpha_{\mathscr{F}} : X_1 \to X_2$, $\beta_{\mathscr{F}} : X_2 \to X_1$, $\alpha_{\mathscr{G}} : Z_1 \to Z_2$, *and* $\beta_{\mathscr{G}} : Z_2 \to Z_1$ *such that:*

$$\beta_{\mathscr{F}} \circ \alpha_{\mathscr{F}} = \mathrm{id}_{X_1}\,, \quad \textit{and} \quad \beta_{\mathscr{G}} \circ \alpha_{\mathscr{G}} = \mathrm{id}_{Z_1}\,.$$

Hence, based on our previous discussion, an extension $\mathscr{H}$ *of* $\mathscr{F}$ *by* $\mathscr{G}$ *is specified by two characteristic maps* $\alpha_{\mathscr{H}} : X_1 \to Z_2$ *and* $\beta_{\mathscr{H}} : X_2 \to Z_1$.

Now, let ${}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}} : Z_1 \oplus X_1 \to Z_2 \oplus X_2$ *and* ${}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} : Z_2 \oplus X_2 \to Z_1 \oplus X_1$ *be the block extensions of* $\alpha_{\mathscr{F}}$ *by* $\alpha_{\mathscr{G}}$ *and of* $\beta_{\mathscr{F}}$ *by* $\beta_{\mathscr{G}}$ *associated with the characteristic maps* $\alpha_{\mathscr{H}}$ *and* $\beta_{\mathscr{H}}$, *respectively. That is, the linear* *maps describing $\mathscr{H}$ near c, which explicitly read as*

$$ {}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}} = \begin{bmatrix} \alpha_{\mathscr{G}} & \alpha_{\mathscr{H}} \\ 0 & \alpha_{\mathscr{F}} \end{bmatrix}, \quad \textit{and} \quad {}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} = \begin{bmatrix} \beta_{\mathscr{G}} & \beta_{\mathscr{H}} \\ 0 & \beta_{\mathscr{F}} \end{bmatrix}. $$

Then, by the microlocal support conditions near the cusps, we require the composition ${}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} \circ {}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}}$ to be an isomorphism. In particular, observe that

$$ {}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} \circ {}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}} = \begin{bmatrix} \beta_{\mathscr{G}} \circ \alpha_{\mathscr{G}} & \beta_{\mathscr{G}} \circ \alpha_{\mathscr{H}} + \beta_{\mathscr{H}} \circ \alpha_{\mathscr{F}} \\ 0 & \beta_{\mathscr{F}} \circ \alpha_{\mathscr{F}} \end{bmatrix}. $$

Consequently, since $\beta_{\mathscr{F}} \circ \alpha_{\mathscr{F}} = \mathrm{id}_{X_1}$ and $\beta_{\mathscr{G}} \circ \alpha_{\mathscr{G}} = \mathrm{id}_{Z_1}$, we conclude that ${}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} \circ {}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}}$ is automatically an isomorphism. Therefore, the microlocal support conditions near the cusps impose no additional constraints on extensions of $\mathscr{F}$ by $\mathscr{G}$.

$\star$ ***Crossings***: *Let x be a crossing in $\mathcal{S}_{\Lambda(\beta)}$. Near x, each of $\mathscr{F}$ and $\mathscr{G}$ is determined by four linear maps between certain vector spaces, namely $\alpha_{\mathscr{F}} : A \to B$, $\beta_{\mathscr{F}} : B \to C$, $\widetilde{\alpha}_{\mathscr{F}} : A \to D$, $\widetilde{\beta}_{\mathscr{F}} : D \to C$, and $\alpha_{\mathscr{G}} : X \to Y$, $\beta_{\mathscr{G}} : Y \to Z$, $\widetilde{\alpha}_{\mathscr{G}} : X \to W$, $\widetilde{\beta}_{\mathscr{G}} : W \to Z$, respectively. By the microlocal support conditions near the crossings, we have that:*

- $\beta_{\mathscr{F}} \circ \alpha_{\mathscr{F}} = \widetilde{\beta}_{\mathscr{F}} \circ \widetilde{\alpha}_{\mathscr{F}}$ *and* $\beta_{\mathscr{G}} \circ \alpha_{\mathscr{G}} = \widetilde{\beta}_{\mathscr{G}} \circ \widetilde{\alpha}_{\mathscr{G}}$.
- *The following sequences are short exact:*

$$ 0 \longrightarrow A \xrightarrow{\left(\alpha_{\mathscr{F}},\, \widetilde{\alpha}_{\mathscr{F}}\right)} B \oplus D \xrightarrow{\beta_{\mathscr{F}} \oplus \left(-\widetilde{\beta}_{\mathscr{F}}\right)} C \longrightarrow 0\,. $$

$$ 0 \longrightarrow X \xrightarrow{\left(\alpha_{\mathscr{G}},\, \widetilde{\alpha}_{\mathscr{G}}\right)} Y \oplus W \xrightarrow{\beta_{\mathscr{G}} \oplus \left(-\widetilde{\beta}_{\mathscr{G}}\right)} Z \longrightarrow 0\,. $$

Thus, building on our previous discussion, an extension $\mathscr{H}$ of $\mathscr{F}$ by $\mathscr{G}$ is determined by a collection of four characteristic maps $\alpha_{\mathscr{H}} : A \to Y$, $\beta_{\mathscr{H}} : B \to Z$, $\widetilde{\alpha}_{\mathscr{H}} : A \to W$, and $\widetilde{\beta}_{\mathscr{H}} : D \to Z$.

Next, consider the block extensions ${}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}} : X \oplus A \to Y \oplus B$, ${}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} : Y \oplus B \to Z \oplus C$, ${}_{\widetilde{\alpha}_{\mathscr{G}}}\langle \widetilde{\alpha}_{\mathscr{H}} \rangle_{\widetilde{\alpha}_{\mathscr{F}}} : X \oplus A \to W \oplus D$, and ${}_{\widetilde{\beta}_{\mathscr{G}}}\langle \widetilde{\beta}_{\mathscr{H}} \rangle_{\widetilde{\beta}_{\mathscr{F}}} : W \oplus D \to Z \oplus C$. That is, the linear maps determining $\mathscr{H}$ near x, which *explicitly read as*

$$ {}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}} = \begin{bmatrix} \alpha_{\mathscr{G}} & \alpha_{\mathscr{H}} \\ 0 & \alpha_{\mathscr{F}} \end{bmatrix}, \quad \textit{and} \quad {}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} = \begin{bmatrix} \beta_{\mathscr{G}} & \beta_{\mathscr{H}} \\ 0 & \beta_{\mathscr{F}} \end{bmatrix}, $$

$$ {}_{\widetilde{\alpha}_{\mathscr{G}}}\langle \widetilde{\alpha}_{\mathscr{H}} \rangle_{\widetilde{\alpha}_{\mathscr{F}}} = \begin{bmatrix} \widetilde{\alpha}_{\mathscr{G}} & \widetilde{\alpha}_{\mathscr{H}} \\ 0 & \widetilde{\alpha}_{\mathscr{F}} \end{bmatrix}, \quad \textit{and} \quad {}_{\widetilde{\beta}_{\mathscr{G}}}\langle \widetilde{\beta}_{\mathscr{H}} \rangle_{\widetilde{\beta}_{\mathscr{F}}} = \begin{bmatrix} \widetilde{\beta}_{\mathscr{G}} & \widetilde{\beta}_{\mathscr{H}} \\ 0 & \widetilde{\beta}_{\mathscr{F}} \end{bmatrix}. $$

Then, the microlocal support conditions near the crossings require:

(a) ${}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} \circ {}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}} = {}_{\widetilde{\beta}_{\mathscr{G}}}\langle \widetilde{\beta}_{\mathscr{H}} \rangle_{\widetilde{\beta}_{\mathscr{F}}} \circ {}_{\widetilde{\alpha}_{\mathscr{G}}}\langle \widetilde{\alpha}_{\mathscr{H}} \rangle_{\widetilde{\alpha}_{\mathscr{F}}}$.

(b) The following sequence must be short exact:

$$ 0 \longrightarrow X \oplus A \xrightarrow{\left({}_{\alpha_{\mathscr{G}}}\langle \alpha_{\mathscr{H}} \rangle_{\alpha_{\mathscr{F}}},\ {}_{\widetilde{\alpha}_{\mathscr{G}}}\langle \widetilde{\alpha}_{\mathscr{H}} \rangle_{\widetilde{\alpha}_{\mathscr{F}}} \right)} (Y \oplus B) \oplus (W \oplus D) \xrightarrow{{}_{\beta_{\mathscr{G}}}\langle \beta_{\mathscr{H}} \rangle_{\beta_{\mathscr{F}}} \oplus \left(- {}_{\widetilde{\beta}_{\mathscr{G}}}\langle \widetilde{\beta}_{\mathscr{H}} \rangle_{\widetilde{\beta}_{\mathscr{F}}} \right)} Z \oplus C \longrightarrow 0 . $$

In fact, once the commutativity condition (a) holds, the defining properties of the maps characterizing $\mathscr{F}$ *and* $\mathscr{G}$ *near* x *automatically guarantee that the sequence in (b) is short exact. In other words, the microlocal support conditions near the crossings only impose on* $\mathscr{H}$ *the commutativity condition (a). Concretely, a direct calculation shows that at the level of characteristic maps, the constraint on* $\mathscr{H}$ *is given by*

$$ \beta_{\mathscr{G}} \circ \alpha_{\mathscr{H}} + \beta_{\mathscr{H}} \circ \alpha_{\mathscr{F}} = \widetilde{\beta}_{\mathscr{G}} \circ \widetilde{\alpha}_{\mathscr{H}} + \widetilde{\beta}_{\mathscr{H}} \circ \widetilde{\alpha}_{\mathscr{F}} . $$

With the above observation at hand, we close this subsection by presenting several technical lemmas that will play a crucial role in the explicit computation of the one-degree morphism spaces in the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Lemma 5.79. *Fix an integer* $n \geq 2$*, let* $\left\{ X^{(i)} \in \mathrm{M}(i+1, i, \mathbb{K}) \right\}_{i=1}^{n-1}$ *be a collection of matrices, and consider the system of equations*

$$ {X'}^{(i)} = X^{(i)} + Y^{(i+1)} \cdot \iota^{(i+1,i)} - \iota^{(i+1,i)} \cdot Y^{(i)} , \qquad \textit{for all } i \in [1, n-1] , $$

with ${X'}^{(i)} \in \mathrm{M}(i+1, i, \mathbb{K})$ *and* $Y^{(j)} \in \mathrm{M}(j, \mathbb{K})$ *for all* $i \in [1, n-1]$ *and* $j \in [1, n]$*. Then:*

- ***General Case****: For any choice of* $\left\{ {X'}^{(i)} \right\}_{i=1}^{n-1}$ *there exists a collection of matrices* $\left\{ Y^{(i)} \right\}_{i=1}^{n}$ *solving the system, with the upper-triangular part of* $Y^{(n)}$ *chosen arbitrarily. More precisely, the solution space of the system is affine over the entries of the matrices* $\left\{ {X'}^{(i)} \right\}_{i=1}^{n-1}$ *and the upper-triangular part of* $Y^{(n)}$*.*
- ***Special case****: if* $X^{(i)} = {X'}^{(i)} = \mathbf{0}_{(i+1)\times i}$ *for all* $i \in [1, n-1]$*, then every solution* $\left\{ Y^{(i)} \right\}_{i=1}^{n}$ *of the system arises from an arbitrary upper-triangular matrix* $Y^{(n)}$*, with* $Y^{(i)}$ *given by the principal* $i \times i$ *submatrix of* $Y^{(n)}$*, for all* $i \in [1, n-1]$*.*

Proof. The proof follows by induction on $n-1$, keeping track of the entries of $X'^{(i)}$ and the upper-triangular part of $Y^{(n)}$. $\square$

Lemma 5.80. *Fix an integer $n \geq 2$, let $\{X^{(i)} \in \mathrm{M}(i, i+1, \mathbb{K})\}_{i=1}^{n-1}$ be a collection of matrices, and consider the system of equations*

$$X'^{(i)} = X^{(i)} + Y^{(i)} \cdot \pi^{(i,i+1)} - \pi^{(i,i+1)} \cdot Y^{(i+1)}, \qquad \text{for all } i \in [1, n-1],$$

with $X'^{(i)} \in \mathrm{M}(i, i+1, \mathbb{K})$ and $Y^{(j)} \in \mathrm{M}(j, \mathbb{K})$ for all $i \in [1, n-1]$ and $j \in [1, n]$. Then:

- ***General Case**: For any choice of $\{X'^{(i)}\}_{i=1}^{n-1}$ there exists a collection of matrices $\{Y^{(i)}\}_{i=1}^{n}$ solving the system, with the lower-triangular part of $Y^{(n)}$ chosen arbitrarily. More precisely, the solution space of the system is affine over the entries of the matrices $\{X'^{(i)}\}_{i=1}^{n-1}$ and the lower-triangular part of $Y^{(n)}$.*
- ***Special case**: if $X^{(i)} = X'^{(i)} = \mathbf{0}_{i\times(i+1)}$ for all $i \in [1, n-1]$, then every solution $\{Y^{(i)}\}_{i=1}^{n}$ of the system arises from an arbitrary lower-triangular matrix $Y^{(n)}$, with $Y^{(i)}$ given by the principal $i \times i$ submatrix of $Y^{(n)}$, for all $i \in [1, n-1]$.*

Proof. The proof follows by induction on $n-1$, keeping track of the entries of $X'^{(i)}$ and the lower-triangular part of $Y^{(n)}$. $\square$

Lemma 5.81. *Fix an integer $n \geq 2$, and let $\{\, Y^{(i)} \in \mathrm{M}(i, \mathbb{K})\,\}_{i=1}^{n}$ be a collection of matrices such that:*

$$Y^{(n)} = D(\vec{u}), \quad \text{for some } \vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}.$$

$$Y^{(i)} = \text{ principal } i \times i \text{ submatrix of } D(\vec{u}), \quad \text{for all } i \in [1, n-1].$$

Furthermore, let $\widetilde{X}^{(1)} \in \mathrm{M}(2, 1, \mathbb{K})$ be a fixed matrix, and let $z, z' \in \mathbb{K}$ be two fixed parameters. Then the system of equations

$$\widetilde{X}'^{(1)} = \widetilde{X}^{(1)} + \Big(\big(B_1^{(2)}(z')\big)^{-1} \cdot Y^{(2)} \cdot B_1^{(2)}(z) \Big) \cdot \iota^{(2,1)} - \iota^{(2,1)} \cdot \widetilde{Y}^{(1)}, \tag{5.17}$$

with $\widetilde{X}'^{(1)} \in \mathrm{M}(2, 1, \mathbb{K})$ and $\widetilde{Y}^{(1)} \in \mathrm{M}(1, \mathbb{K})$, has the following properties:

(a) ***General Solution**: The solution set $\big(\widetilde{X}'^{(1)}, \widetilde{Y}^{(1)}\big)$ of (5.17) is an affine line parametrized by the $(1,1)$-entry of $\widetilde{X}'^{(1)}$. More precisely, once the $(1,1)$-entry of $\widetilde{X}'^{(1)}$ is chosen, the system of equations (5.17) uniquely determines $\widetilde{Y}^{(1)}$ and the remaining entry of $\widetilde{X}'^{(1)}$.*

(b) ***Special case**: Consider*

$$\widetilde{X}'^{(1)} = \begin{bmatrix} 0 \\ \tilde{x}'_{2,1} \end{bmatrix}, \quad \text{and} \quad \widetilde{X}^{(1)} = \begin{bmatrix} 0 \\ \tilde{x}_{2,1} \end{bmatrix}.$$

Then the solution set $\big(\widetilde{X}'^{(1)}, \widetilde{Y}^{(1)}\big)$ *of* (5.17) *reduces to a single point:*

$$\widetilde{Y}^{(1)} = \textit{ principal } 1\times 1 \textit{ submatrix of } \left(B_1^{(n)}(z')\right)^{-1}\cdot D(\vec{u})\cdot B_1^{(n)}(z)\,,$$

$$\tilde{x}'_{2,1} = \tilde{x}_{2,1} + d(z',\vec{u},z)\,,$$

where

$$d(z',\vec{u},z) := \textit{the } (2,1)\textit{-entry of } \left(B_1^{(n)}(z')\right)^{-1}\cdot D(\vec{u})\cdot B_1^{(n)}(z)\,.$$

Proof. Set $M := \left(B_1^{(2)}(z')\right)^{-1}\cdot Y^{(2)}\cdot B_1^{(2)}(z)$. Then, since $Y^{(2)}$ is the principal 2×2 submatrix of $D(\vec{u})$, the block structure of the braid matrices guarantees that

$$M = \text{ principal } 2\times 2 \text{ submatrix of } \left(B_1^{(n)}(z')\right)^{-1}\cdot D(\vec{u})\cdot B_1^{(n)}(z)\,.$$

Next, let us write

$$\widetilde{X}'^{(1)} = \begin{bmatrix}\tilde{x}'_{1,1}\\ \tilde{x}'_{2,1}\end{bmatrix}\,, \qquad \widetilde{X}^{(1)} = \begin{bmatrix}\tilde{x}_{1,1}\\ \tilde{x}_{2,1}\end{bmatrix}\,, \qquad \widetilde{Y}^{(1)} = [\,\tilde{y}_{1,1}\,]\,.$$

Then, the system of equations in (5.17) reduces to

$$\tilde{x}'_{1,1} = \tilde{x}_{1,1} + m_{1,1} - \tilde{y}_{1,1}\,,$$

$$\tilde{x}'_{2,1} = \tilde{x}_{2,1} + m_{2,1}\,,$$

where $m_{1,1}$ and $m_{2,1}$ denote the $(1,1)$ and $(2,1)$ entries of M, respectively. This shows that once $\tilde{x}'_{1,1}$ is chosen, $\tilde{y}_{1,1}$ and $\tilde{x}'_{2,1}$ are uniquely determined. In other words, the solution set $\big(\widetilde{X}'^{(1)}, \widetilde{Y}^{(1)}\big)$ of (5.17) is an affine line parametrized by the $(1,1)$-entry of $\widetilde{X}'^{(1)}$.

Finally, in the special case $\tilde{x}_{1,1} = \tilde{x}'_{1,1} = 0$, the solution of the system of equations in (5.17) is uniquely given by

$$\tilde{y}_{1,1} = m_{1,1}\,,$$

$$\tilde{x}'_{2,1} = \tilde{x}_{2,1} + m_{2,1}\,.$$

The result follows once we identify $d(z',\vec{u},z) = m_{2,1}$. □

Lemma 5.82. *Fix an integer* $k\geq 2$. *Let* $\widetilde{X}^{(k-1)}\in \mathrm{M}(k,k-1,\mathbb{K})$ *and* $\widetilde{X}^{(k)}\in\mathrm{M}(k+1,k,\mathbb{K})$ *be fixed matrices, and let* $z'\in\mathbb{K}$ *be a fixed parameter. Consider the system of equations*

$$\left(B_k^{(k+1)}(z')\right)^{-1}\cdot\left(\iota^{(k+1,k)}\cdot\widetilde{X}^{(k-1)} + \widetilde{X}^{(k)}\cdot\iota^{(k,k-1)}\right) = \iota^{(k+1,k)}\cdot\widetilde{X}'^{(k-1)} + \widetilde{X}'^{(k)}\cdot\iota^{(k,k-1)}\,, \tag{5.18}$$

with $\widetilde{X}'^{(k-1)}\in\mathrm{M}(k,k-1,\mathbb{K})$ *and* $\widetilde{X}'^{(k)}\in\mathrm{M}(k+1,k,\mathbb{K})$. *Then the solution set* $\big(\widetilde{X}'^{(k-1)},\widetilde{X}'^{(k)}\big)$ *of* (5.18) *is an affine space parametrized by the entries of* $\widetilde{X}'^{(k-1)}$ *and the entries of the* k*-th column of* $\widetilde{X}'^{(k)}$. *More precisely,* *once $\widetilde{X}'^{(k-1)}$ and the k-th column of $\widetilde{X}'^{(k)}$ are chosen, the system of equations (5.18) uniquely determines the first $k-1$ columns of $\widetilde{X}'^{(k)}$.*

Proof. The system of equations (5.18) is linear in $\widetilde{X}'^{(k-1)}$ and $\widetilde{X}'^{(k)}$. In particular, since $\iota^{(k,k-1)}$ has only $k-1$ columns, the k-th column of $\widetilde{X}'^{(k)}$ is not involved in the system of equations, and is therefore free. The remaining columns of $\widetilde{X}'^{(k)}$ are uniquely determined once $\widetilde{X}'^{(k-1)}$ is chosen. It follows that the solution set $\big(\widetilde{X}'^{(k-1)}, \widetilde{X}'^{(k)}\big)$ of (5.18) is an affine space parametrized by the entries of $\widetilde{X}'^{(k-1)}$ and the entries in the k-th column of $\widetilde{X}'^{(k)}$. $\square$

Lemma 5.83. *Fix an integer $n \geq 2$, and let $\left\{ Y^{(i)} \in \mathrm{M}(i, \mathbb{K}) \right\}_{i=1}^{n}$ be a collection of matrices such that:*

$$Y^{(n)} = D(\vec{u}), \quad \textit{for some } \vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}.$$

$$Y^{(i)} = \textit{ principal } i \times i \textit{ submatrix of } D(\vec{u}), \quad \textit{for all } i \in [1, n-1].$$

In addition, fix an integer $k \in [2, n-1]$. Let $\widetilde{X}^{(k-1)} \in \mathrm{M}(k, k-1, \mathbb{K})$ and $\widetilde{X}^{(k)} \in \mathrm{M}(k+1, k, \mathbb{K})$ be fixed matrices, and let $z, z' \in \mathbb{K}$ be two fixed parameters. Then the system of equations

$$\widetilde{X}'^{(k-1)} = \widetilde{X}^{(k-1)} + \widetilde{Y}^{(k)} \cdot \iota^{(k,k-1)} - \iota^{(k,k-1)} \cdot Y^{(k-1)}, \tag{5.19a}$$

$$\widetilde{X}'^{(k)} = \widetilde{X}^{(k)} + \big(B_k^{(k+1)}(z')\big)^{-1} \cdot Y^{(k+1)} \cdot B_k^{(k+1)}(z) \cdot \iota^{(k+1,k)} - \iota^{(k+1,k)} \cdot \widetilde{Y}^{(k)}, \tag{5.19b}$$

with $\widetilde{X}'^{(k-1)} \in \mathrm{M}(k, k-1, \mathbb{K})$, $\widetilde{X}'^{(k)} \in \mathrm{M}(k+1, k, \mathbb{K})$, and $\widetilde{Y}^{(k)} \in \mathrm{M}(k, \mathbb{K})$, has the following properties:

(a) ***General case****: The solution set $\big(\widetilde{X}'^{(k-1)}, \widetilde{X}'^{(k)}, \widetilde{Y}^{(k)}\big)$ of (5.19) is an affine space parametrized by the entries of $\widetilde{X}'^{(k-1)}$ and the entries of the k-th column of $\widetilde{X}'^{(k)}$ except its last one. More precisely, once $\widetilde{X}'^{(k-1)}$ and the k-th column of $\widetilde{X}'^{(k)}$ except its last entry are chosen, the system of equations (5.19) uniquely determines $\widetilde{Y}^{(k)}$ and the remaining entries of $\widetilde{X}'^{(k)}$.*

(b) ***Special case****: Consider*

$$\widetilde{X}'^{(k-1)} = \mathbf{0}_{k \times (k-1)}, \quad \textit{and} \quad \widetilde{X}'^{(k)} = \begin{bmatrix} \tilde{x}'_{1,1} & \cdots & \tilde{x}'_{1,k-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{x}'_{k,1} & \cdots & \tilde{x}'_{k,k-1} & 0 \\ \tilde{x}'_{k+1,1} & \cdots & \tilde{x}'_{k+1,k-1} & \tilde{x}'_{k+1,k} \end{bmatrix},$$

$$\widetilde{X}^{(k-1)} = \mathbf{0}_{k \times (k-1)}, \quad \textit{and} \quad \widetilde{X}^{(k)} = \begin{bmatrix} x_{1,1} & \cdots & x_{1,k-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{k,1} & \cdots & x_{k,k-1} & 0 \\ x_{k+1,1} & \cdots & x_{k+1,k-1} & x_{k+1,k} \end{bmatrix}.$$

Then the solution set $\big(\widetilde{X}'^{(k-1)}, \widetilde{X}'^{(k)}, \widetilde{Y}^{(k)}\big)$ *of* (5.19) *reduces to a single point:*

$$\widetilde{Y}^{(k)} = \textit{principal } k \times k \textit{ submatrix of } \ \big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)\,.$$

$$\tilde{x}'_{i,j} = \tilde{x}_{i,j}\,, \quad \textit{for all } i \in [1, k+1] \textit{ and } j \in [1, k-1]\,,$$

$$\tilde{x}'_{k+1,k} = \tilde{x}_{k+1,k} + d(z', \vec{u}, z)\,,$$

where

$$d(z', \vec{u}, z) := \textit{the } (k+1, k)\textit{-entry of } \ \big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)\,.$$

Proof. Set $M = \big(B_k^{(k+1)}(z')\big)^{-1} \cdot Y^{(k+1)} \cdot B_k^{(k+1)}(z)$. Then, since $Y^{(k+1)}$ is the principal $(k+1) \times (k+1)$ submatrix of $D(\vec{u})$, the block structure of the braid matrices guarantees that

$$M = \ \text{principal } (k+1) \times (k+1) \text{ submatrix of } \ \big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z).$$

Next, observe that the set of equations (5.19a) allows us to solve for the first $k-1$ columns of $\widetilde{Y}^{(k)}$, while leaving its k-th column and all entries of $\widetilde{X}'^{(k-1)}$ undetermined. Then, after substituting our partial solution into the set of equations (5.19b), we can exploit the remaining freedom in the k-th column of $\widetilde{Y}^{(k)}$ to solve the system completely, while keeping all entries of $\widetilde{X}'^{(k-1)}$ and the entries of the k-th column of $\widetilde{X}'^{(k)}$, except for its very last entry, arbitrary.

Finally, in the special case where $\widetilde{X}^{(k-1)} = \widetilde{X}'^{(k-1)} = \mathbf{0}_{k \times (k-1)}$ and the constraints on the last columns of $\widetilde{X}^{(k)}$ and $\widetilde{X}'^{(k)}$ required in the lemma are imposed, the procedure described above yields a unique solution. In particular, the set of equations (5.19a) implies that the principal $(k-1) \times (k-1)$ submatrix of $\widetilde{Y}^{(k)}$ is equal to $Y^{(k-1)}$, while its remaining entries outside its last column are zero.

Furthermore, since $Y^{(k+1)}$ is the principal $(k+1) \times (k+1)$ submatrix of the diagonal matrix $D(\vec{u})$, the block structure of the braid matrices guarantee that the principal $(k-1) \times (k-1)$ sub-matrices of $Y^{(k+1)}$ and M coincide. Hence, under the assumptions of the special case, the set of equations (5.19b) further implies that $\widetilde{Y}^{(k)}$ is equal to the principal $k \times k$ submatrix of M.

Lastly, recall that M is almost diagonal, with the only potentially non-zero off-diagonal entry at the $(k+1, k)$ position. Consequently, in the set of equations (5.19b), $\widetilde{Y}^{(k)}$ cancels almost all the contribution of M, thus yielding that $\widetilde{X}'^{(k)}$ coincides with $\widetilde{X}^{(k)}$ entry-wise, except for the $(k+1, k)$-entry, which differs by the corresponding entry of M. This completes the proof. □

Having established the relevant technical lemmas and the local structure of the one-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$, we turn to their global analysis, aiming for an explicit description of these morphism spaces. As a first step, we consider the case of the Legendrian unlink on n strands, $\beta = e_n$.

5.4.2. *The Case of the Trivial Braid.* Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word. Next, we study the one-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$.

To begin, consider the open cover $\mathcal{U}_{\Lambda(e_n)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{T}}\}$ of $\mathbb{R}^2$ introduced in Construction (4.30). Let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$, and let $\mathscr{H}$ be an extension of $\mathscr{F}$ by $\mathscr{G}$. By Proposition (4.32), $\mathscr{F}$ and $\mathscr{G}$ are identically zero on U_0, and as a result, $\mathscr{H}$ is also identically zero on this region. It follows that $\mathscr{H}$ is entirely determined by its behavior on the regions U_{B} and U_{T}. Having established this, we present the following proposition.

Proposition 5.84. ***Setup***: *Let $e_n \in \mathrm{Br}_n^+$ be the trivial braid word, $\mathcal{U}_{\Lambda(e_n)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ introduced in Construction (4.30), and $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$. In this setting, Proposition (4.31) asserts that:*

- *On U_{T}, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of surjective linear maps*

$$\left\{\psi^{(i)}_{\mathscr{F}} : \mathbb{K}^{i+1} \to \mathbb{K}^i\right\}_{i=1}^{n-1}, \quad \textit{and} \quad \left\{\psi^{(i)}_{\mathscr{G}} : \mathbb{K}^{i+1} \to \mathbb{K}^i\right\}_{i=1}^{n-1},$$

  respectively (see Figure (9) for a schematic representation).

- *On U_{B}, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of injective linear maps*

$$\left\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1}, \quad \textit{and} \quad \left\{\phi^{(i)}_{\mathscr{G}} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1},$$

  respectively (see Figure (10) for a schematic representation).

- ***Compatibility conditions***: *For each $i \in [1, n-1]$,*

$$\psi^{(i)}_{\mathscr{F}} \circ \phi^{(i)}_{\mathscr{F}} = \mathrm{id}_{\mathbb{K}^i}, \quad \textit{and} \quad \psi^{(i)}_{\mathscr{G}} \circ \phi^{(i)}_{\mathscr{G}} = \mathrm{id}_{\mathbb{K}^i}.$$

$\star$ *<u>Assumption</u>: For each $i \in [1,n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}^{(i)}_j\}_{j=1}^{i}$ and $\hat{\mathbf{g}}^{(i)} := \{\hat{g}^{(i)}_j\}_{j=1}^{i}$ be bases for $\mathbb{K}^i$. Building on Definition (4.18)–(3)–(4), we assume that:*

- *$\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to both $\{\psi^{(i)}_{\mathscr{F}}\}_{i=1}^{n-1}$ and $\{\phi^{(i)}_{\mathscr{F}}\}_{i=1}^{n-1}$.*
- *$\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to both $\{\psi^{(i)}_{\mathscr{G}}\}_{i=1}^{n-1}$ and $\{\phi^{(i)}_{\mathscr{G}}\}_{i=1}^{n-1}$.*

***Main Conclusion**: Let $\mathscr{H}$ be an extension of $\mathscr{F}$ by $\mathscr{G}$. Under the given **setup**, $\mathscr{H}$ is equivalent to the trivial extension $\mathbf{0}_{\mathrm{Ext}}$ of $\mathscr{F}$ by $\mathscr{G}$. In other words, we have that*

$$\mathrm{Ext}^1(\mathscr{F},\mathscr{G}) \cong 0\,.$$

Proof. Let $\mathscr{H}'$ be an extension of $\mathscr{F}$ by $\mathscr{G}$ representing the same equivalence class as $\mathscr{H}$ in $\mathrm{Ext}^1(\mathscr{F},\mathscr{G})$, so that there exists a sheaf isomorphism $\lambda : \mathscr{H} \to \mathscr{H}'$ such that the diagram in Figure (27) commutes in each square. Building on Observation (5.78), we have that:

- On U_{T}, $\mathscr{H}$ and $\mathscr{H}'$ are determined by collections of $n-1$ characteristic maps

  $$\big\{\psi^{(i)}_{\mathscr{H}} : \mathbb{K}^{i+1} \to \mathbb{K}^i\big\}_{i=1}^{n-1}\,, \quad \text{and} \quad \big\{\psi^{(i)}_{\mathscr{H}'} : \mathbb{K}^{i+1} \to \mathbb{K}^i\big\}_{i=1}^{n-1}\,,$$

  while λ is characterized by a collection of n linear maps $\big\{S^{(i)}_{\lambda} : \mathbb{K}^i \to \mathbb{K}^i\big\}_{i=1}^{n}$ such that

  $$\psi^{(i)}_{\mathscr{H}'} = \psi^{(i)}_{\mathscr{H}} + S^{(i)}_{\lambda} \circ \psi^{(i)}_{\mathscr{F}} - \psi^{(i)}_{\mathscr{G}} \circ S^{(i+1)}_{\lambda}\,, \tag{5.20}$$

  for all $i \in [1, n-1]$.

- On U_{B}, $\mathscr{H}$ and $\mathscr{H}'$ are determined by collections of $n-1$ characteristic maps

  $$\big\{\phi^{(i)}_{\mathscr{H}} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\big\}_{i=1}^{n-1}\,, \quad \text{and} \quad \big\{\phi^{(i)}_{\mathscr{H}'} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\big\}_{i=1}^{n-1}\,,$$

  while λ is characterized by a collection of n linear maps $\big\{T^{(i)}_{\lambda} : \mathbb{K}^i \to \mathbb{K}^i\big\}_{i=1}^{n}$ such that

  $$\phi^{(i)}_{\mathscr{H}'} = \phi^{(i)}_{\mathscr{H}} + T^{(i+1)}_{\lambda} \circ \phi^{(i)}_{\mathscr{F}} - \phi^{(i)}_{\mathscr{G}} \circ T^{(i)}_{\lambda}\,, \tag{5.21}$$

  for all $i \in [1, n-1]$.

In particular, applying the sheaf axioms to the intersection $U_{\mathrm{T}} \cap U_{\mathrm{B}}$ yields that $S^{(n)}_{\lambda} = T^{(n)}_{\lambda}$.

By assumption, $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}$ and $\big\{\hat{\mathbf{g}}^{(i)}\big\}_{i=1}^{n}$ are systems of bases adapted to the pairs $\big(\big\{\psi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}, \big\{\phi^{(i)}_{\mathscr{F}}\big\}_{i=1}^{n-1}\big)$ and $\big(\big\{\psi^{(i)}_{\mathscr{G}}\big\}_{i=1}^{n-1}, \big\{\phi^{(i)}_{\mathscr{G}}\big\}_{i=1}^{n-1}\big)$, respectively. Consequently, with respect to these bases, the equivalence conditions (5.20) and (5.21) translate into the system of equations

$${}_{\hat{\mathbf{g}}^{(i)}}\big[\,\psi^{(i)}_{\mathscr{H}'}\,\big]_{\hat{\mathbf{f}}^{(i+1)}} = {}_{\hat{\mathbf{g}}^{(i)}}\big[\,\psi^{(i)}_{\mathscr{H}}\,\big]_{\hat{\mathbf{f}}^{(i+1)}} + {}_{\hat{\mathbf{g}}^{(i)}}\big[\,S^{(i)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(i)}} \cdot \pi^{(i,i+1)} - \pi^{(i,i+1)} \cdot {}_{\hat{\mathbf{g}}^{(i+1)}}\big[\,S^{(i+1)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(i+1)}}\,,$$

$${}_{\hat{\mathbf{g}}^{(i+1)}}\big[\,\phi^{(i)}_{\mathscr{H}'}\,\big]_{\hat{\mathbf{f}}^{(i)}} = {}_{\hat{\mathbf{g}}^{(i+1)}}\big[\,\phi^{(i)}_{\mathscr{H}}\,\big]_{\hat{\mathbf{f}}^{(i)}} + {}_{\hat{\mathbf{g}}^{(i+1)}}\big[\,T^{(i+1)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(i+1)}} \cdot \iota^{(i+1,i)} - \iota^{(i+1,i)} \cdot {}_{\hat{\mathbf{g}}^{(i)}}\big[\,T^{(i)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(i)}}\,,$$

for all $i \in [1, n-1]$.

Now, since ${}_{\hat{\mathbf{g}}^{(n)}}\big[\,T^{(n)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(n)}} = {}_{\hat{\mathbf{g}}^{(n)}}\big[\,S^{(n)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(n)}}$, Lemmas (5.80) and (5.79) guarantee that for any choice of matrices $\big\{{}_{\hat{\mathbf{g}}^{(i)}}\big[\,\psi^{(i)}_{\mathscr{H}'}\,\big]_{\hat{\mathbf{f}}^{(i+1)}}\big\}_{i=1}^{n-1}$ and $\big\{{}_{\hat{\mathbf{g}}^{(i+1)}}\big[\,\phi^{(i)}_{\mathscr{H}}\,\big]_{\hat{\mathbf{f}}^{(i)}}\big\}_{i=1}^{n-1}$ there exist collections of matrices $\big\{{}_{\hat{\mathbf{g}}^{(i)}}\big[\,S^{(i)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(i)}}\big\}_{i=1}^{n}$ and $\big\{{}_{\hat{\mathbf{g}}^{(i)}}\big[\,T^{(i)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(i)}}\big\}_{i=1}^{n}$ solving the system of equations, with the diagonal part of ${}_{\hat{\mathbf{g}}^{(n)}}\big[\,T^{(n)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(n)}}$ chosen arbitrarily.

Bearing this in mind, we deduce that $\mathscr{H}$ is equivalent to any extension of $\mathscr{F}$ by $\mathscr{G}$, and in particular, it is equivalent to the trivial extension $\mathbf{0}_{\mathrm{Ext}}$. Hence, since $\mathscr{H}$ is arbitrary, we conclude that $\mathrm{Ext}^1(\mathscr{F},\mathscr{G}) \cong 0$. $\square$

With the above result at hand, we conclude our study of the one-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(e_n),\mathbb{K})_0)$, and now turn to their analysis in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ for an arbitrary positive braid word $\beta \in \mathrm{Br}_n^+$.

5.4.3. *General Positive Braids.* Let $\beta := \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. Next, we analyze the one-degree morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.

Let $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ be the open cover of $\mathbb{R}^2$ from Construction (4.33), $\mathcal{R}_{\Lambda(\beta)} = \{R_j\}_{j=1}^{\ell}$ the partition of U_{B} into ℓ open vertical straps from Construction (4.36), $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$, and $\mathscr{H}$ an extension of $\mathscr{F}$ by $\mathscr{G}$. Building on Observations (5.62) and (5.78), we have that the global structure of $\mathscr{H}$ can be fully recovered from its local description on the region U_{T} and the collection of vertical straps $\mathcal{R}_{\Lambda(\beta)}$, together with the appropriate compatibility conditions. Accordingly, we present the following result.

Lemma 5.85. ***Setup****: Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ from Construction (4.33), and $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. According to Lemma (4.34), $\mathscr{F}$ and $\mathscr{G}$ have the following local descriptions:*

- *On U_{T}, $\mathscr{F}$ and $\mathscr{G}$ are specified by collections of $n-1$ surjective linear maps*

$$\{\psi^{(i)}_{\mathscr{F}} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}, \quad \textit{and} \quad \{\psi^{(i)}_{\mathscr{G}} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1},$$

  respectively. For a schematic illustration of a generic representative of one of these sheaves on U_{T}, see Figure (15).

- *On U_{L}, $\mathscr{F}$ and $\mathscr{G}$ are specified by collections of $n-1$ injective linear maps*

$$\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}, \quad \textit{and} \quad \{\phi^{(i)}_{\mathscr{G}} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\}_{i=1}^{n-1},$$

  respectively. For a schematic illustration of a generic representative of one of these sheaves on U_{L}, see Figure (13).

- ***Compatibility conditions****: For each $i \in [1, n-1]$,*

$$\psi^{(i)}_{\mathscr{F}} \circ \phi^{(i)}_{\mathscr{F}} = \mathrm{id}_{\mathbb{K}^i}, \quad \textit{and} \quad \psi^{(i)}_{\mathscr{G}} \circ \phi^{(i)}_{\mathscr{G}} = \mathrm{id}_{\mathbb{K}^i}.$$

⋆ *Assumption: For each $i \in [1,n]$, let $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_j^{(i)}\}_{j=1}^{i}$ and $\hat{\mathbf{g}}^{(i)} := \{\hat{g}_j^{(i)}\}_{j=1}^{i}$ be bases for $\mathbb{K}^i$. Building on Definition (4.18)–(3)–(4), we assume that:*

- *$\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to both $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$.*
- *$\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$ is a system of bases adapted to both $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$ and $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$.*

***Main Conclusion**: Under the given **setup**, the following statements hold:*

⋆ *Local characterization of $\operatorname{Ext}^1(\mathscr{F},\mathscr{G})$ on $U_{\mathrm{T}} \cup U_{\mathrm{L}}$: Any extension $\mathscr{H}$ of $\mathscr{F}$ by $\mathscr{G}$ is locally equivalent to the trivial extension $\mathbf{0}_{\mathrm{Ext}}$ on $U_{\mathrm{T}} \cup U_{\mathrm{L}}$.*

⋆ *Block-diagonal properties: Let $\mathscr{H}$ and $\mathscr{H}'$ be two extensions of $\mathscr{F}$ by $\mathscr{G}$ representing the same equivalence class in $\operatorname{Ext}^1(\mathscr{F},\mathscr{G})$, and suppose that both $\mathscr{H}$ and $\mathscr{H}'$ coincide with the trivial extension $\mathbf{0}_{\mathrm{Ext}}$ on $U_{\mathrm{T}} \cup U_{\mathrm{L}}$. Let $\lambda : \mathscr{H} \to \mathscr{H}'$ be a sheaf isomorphism realizing the equivalence between $\mathscr{H}$ and $\mathscr{H}'$. Then, λ is characterized by a collection of n linear maps $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^{n}$, which, with respect to the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$, satisfy the following properties:*

- *${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} \in M(n,\mathbb{K})$ is diagonal.*
- *For each $i \in [1,n-1]$, ${}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i,\mathbb{K})$ is given by the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$.*

Proof. Let $\mathscr{H}'$ be an extension of $\mathscr{F}$ by $\mathscr{G}$ representing the same equivalence class as $\mathscr{H}$ in $\operatorname{Ext}^1(\mathscr{F},\mathscr{G})$, and let $\lambda : \mathscr{H} \to \mathscr{H}'$ be a sheaf isomorphism such that the diagram in Figure (27) commutes in each square. Building on Observation (5.78), we have that:

- On U_{T}, $\mathscr{H}$ and $\mathscr{H}'$ are determined by collections of $n-1$ characteristic maps

  $$\{\psi_{\mathscr{H}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}, \quad \text{and} \quad \{\psi_{\mathscr{H}'}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1},$$

  while λ is characterized by a collection of n linear maps $\{S_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^{n}$ such that

  $$\psi_{\mathscr{H}'}^{(i)} = \psi_{\mathscr{H}}^{(i)} + S_\lambda^{(i)} \circ \psi_{\mathscr{F}}^{(i)} - \psi_{\mathscr{G}}^{(i)} \circ S_\lambda^{(i+1)}, \tag{5.22}$$

  for all $i \in [1,n-1]$.

- On U_{L}, $\mathscr{H}$ and $\mathscr{H}'$ are determined by collections of $n-1$ characteristic maps

  $$\{\phi_{\mathscr{H}}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}, \quad \text{and} \quad \{\phi_{\mathscr{H}'}^{(i)} : \mathbb{K}^{i} \to \mathbb{K}^{i+1}\}_{i=1}^{n-1},$$

  while λ is characterized by a collection of n linear maps $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^{n}$ such that

  $$\phi_{\mathscr{H}'}^{(i)} = \phi_{\mathscr{H}}^{(i)} + T_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} - \phi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i)}, \tag{5.23}$$

for all $i \in [1, n-1]$.

In particular, applying the sheaf axioms to the intersection $U_{\mathrm{T}} \cap U_{\mathrm{L}}$ yields that $S_\lambda^{(n)} = T_\lambda^{(n)}$.

By assumption, $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ are systems of bases adapted to the pairs $\big(\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}, \{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}\big)$ and $\big(\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}, \{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}\big)$, respectively. Consequently, with respect to these bases, the equivalence conditions (5.22) and (5.23) translate into the system of equations

$$\begin{aligned} {}_{\hat{\mathbf{g}}^{(i)}}\big[\psi_{\mathscr{H}'}^{(i)}\big]_{\hat{\mathbf{f}}^{(i+1)}} &= {}_{\hat{\mathbf{g}}^{(i)}}\big[\psi_{\mathscr{H}}^{(i)}\big]_{\hat{\mathbf{f}}^{(i+1)}} + {}_{\hat{\mathbf{g}}^{(i)}}\big[S_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \cdot \pi^{(i,i+1)} - \pi^{(i,i+1)} \cdot {}_{\hat{\mathbf{g}}^{(i+1)}}\big[S_\lambda^{(i+1)}\big]_{\hat{\mathbf{f}}^{(i+1)}}, \\ {}_{\hat{\mathbf{g}}^{(i+1)}}\big[\phi_{\mathscr{H}'}^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} &= {}_{\hat{\mathbf{g}}^{(i+1)}}\big[\phi_{\mathscr{H}}^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} + {}_{\hat{\mathbf{g}}^{(i+1)}}\big[T_\lambda^{(i+1)}\big]_{\hat{\mathbf{f}}^{(i+1)}} \cdot \iota^{(i+1,i)} - \iota^{(i+1,i)} \cdot {}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}}, \end{aligned} \tag{5.24}$$

for all $i \in [1, n-1]$.

Now, since ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} = {}_{\hat{\mathbf{g}}^{(n)}}\big[S_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$, Lemmas (5.80) and (5.79) guarantee that for any choice of matrices $\big\{{}_{\hat{\mathbf{g}}^{(i)}}\big[\psi_{\mathscr{H}'}^{(i)}\big]_{\hat{\mathbf{f}}^{(i+1)}}\big\}_{i=1}^{n-1}$ and $\big\{{}_{\hat{\mathbf{g}}^{(i+1)}}\big[\phi_{\mathscr{H}}^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}}\big\}_{i=1}^{n-1}$ there exist collections of matrices $\big\{{}_{\hat{\mathbf{g}}^{(i)}}\big[S_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}}\big\}_{i=1}^{n}$ and $\big\{{}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}}\big\}_{i=1}^{n}$ solving the system of equations in (5.24), with the diagonal part of ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$ chosen arbitrarily.

Bearing this in mind, we deduce that $\mathscr{H}$ is locally equivalent to any extension of $\mathscr{F}$ by $\mathscr{G}$ on $U_{\mathrm{T}} \cup U_{\mathrm{L}}$, and in particular, it is equivalent to the trivial extension $\mathbf{0}_{\mathrm{Ext}}$ on this region, that is, the extension whose characteristic maps are zero.

Next, suppose that $\mathscr{H}$ and $\mathscr{H}'$ are two equivalent extensions of $\mathscr{F}$ by $\mathscr{G}$ which coincide with the trivial extension of $U_{\mathrm{T}} \cup U_{\mathrm{L}}$. More precisely, assume that $\psi_{\mathscr{H}}^{(i)} = \psi_{\mathscr{H}'}^{(i)} = 0$ and $\phi_{\mathscr{H}}^{(i)} = \phi_{\mathscr{H}'}^{(i)} = 0$ for all $i \in [1, n-1]$. Then, the system of equations in (5.24) reduces to

$$\begin{aligned} \mathbf{0}_{i\times(i+1)} &= \mathbf{0}_{i\times(i+1)} + {}_{\hat{\mathbf{g}}^{(i)}}\big[S_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \cdot \pi^{(i,i+1)} - \pi^{(i,i+1)} \cdot {}_{\hat{\mathbf{g}}^{(i+1)}}\big[S_\lambda^{(i+1)}\big]_{\hat{\mathbf{f}}^{(i+1)}}, \\ \mathbf{0}_{(i+1)\times i} &= \mathbf{0}_{(i+1)\times i} + {}_{\hat{\mathbf{g}}^{(i+1)}}\big[T_\lambda^{(i+1)}\big]_{\hat{\mathbf{f}}^{(i+1)}} \cdot \iota^{(i+1,i)} - \iota^{(i+1,i)} \cdot {}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}}. \end{aligned} \tag{5.25}$$

Finally, recall that ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}} = {}_{\hat{\mathbf{g}}^{(n)}}\big[S_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$, and hence Lemmas (5.80) and (5.79) ensure that any solution of the reduced system of equations in (5.25) arises from a diagonal matrix ${}_{\hat{\mathbf{g}}^{(n)}}\Big[T_\lambda^{(n)}\Big]_{\hat{\mathbf{f}}^{(n)}} \in M(n, \mathbb{K})$, with ${}_{\hat{\mathbf{g}}^{(i)}}\big[S_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i, \mathbb{K})$ and ${}_{\hat{\mathbf{g}}^{(i)}}\big[T_\lambda^{(i)}\big]_{\hat{\mathbf{f}}^{(i)}} \in M(i, \mathbb{K})$ given by the principal $i \times i$ submatrix of ${}_{\hat{\mathbf{g}}^{(n)}}\big[T_\lambda^{(n)}\big]_{\hat{\mathbf{f}}^{(n)}}$ for each $i \in [1, n-1]$. It follows that $S_\lambda^{(i)} = T_\lambda^{(i)}$ for each $i \in [1, n-1]$, and hence λ is characterized by a collection of linear maps $\{T_\lambda^{(i)}\}_{i=1}^n$ satisfying the conditions stated in the lemma. □

The next lemmas will help us analyze the structure of the one-degree morphism spaces in the category of our interest on a region R_j. In order to state the lemmas, let us first introduce a setup.

Setup 5.86. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Fix $j \in [1, \ell]$, and let R_j be the vertical strap in $\mathbb{R}^2$ containing σ_{i_j}—the j-th crossing of β (see Figure (19)).*

⋆ *Assumption 1: Let $k := i_j \in [1, n-1]$ denote the index of σ_{i_j}, and suppose that $k = 1$.*

Under this assumption, on R_j, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of n injective linear maps

$$\left\{ \phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1} \right\}_{i=1}^{n-1} \cup \left\{ \widetilde{\phi}_{\mathscr{F}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2 \right\},$$

$$\left\{ \phi_{\mathscr{G}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1} \right\}_{i=1}^{n-1} \cup \left\{ \widetilde{\phi}_{\mathscr{G}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2 \right\},$$

respectively. For a schematic illustration of a generic representative of one of these sheaves, see Figure (21).

⋆ *Assumption 2: For each $i \in [1, n]$, let $\hat{\mathbf{f}}^{(i)} := \left\{ \hat{f}_j^{(i)} \right\}_{j=1}^{i}$ and $\hat{\mathbf{g}}^{(i)} := \left\{ \hat{g}_j^{(i)} \right\}_{j=1}^{i}$ be bases for $\mathbb{K}^i$. Following Definition (4.18)–(3), we assume that:*

- *$\left\{ \hat{\mathbf{f}}^{(i)} \right\}_{i=1}^{n}$ is a system of bases adapted to $\left\{ \phi_{\mathscr{F}}^{(i)} \right\}_{i=1}^{n-1}$.*
- *$\left\{ \hat{\mathbf{g}}^{(i)} \right\}_{i=1}^{n}$ is a system of bases adapted to $\left\{ \phi_{\mathscr{G}}^{(i)} \right\}_{i=1}^{n-1}$.*

Under this assumption, Lemma (4.47) ensures that $\left(\left\{ \hat{\mathbf{f}}^{(i)} \right\}_{i=1}^{n}, z \right)$ and $\left(\left\{ \hat{\mathbf{g}}^{(i)} \right\}_{i=1}^{n}, z' \right)$ are system of bases adapted to $\mathscr{F}$ and $\mathscr{G}$ on R_j, where z, $z' \in \mathbb{K}$ parameterize, relative to the bases $\hat{\mathbf{f}}^{(n)}$ and $\hat{\mathbf{g}}^{(n)}$ for $\mathbb{K}^n$, the s_1-relative position between the pair of complete flags in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$ and $\mathscr{G}$ on R_j, respectively. Following Definition (4.29), we denote by $\left\{ \hat{\mathbf{f}}^{(i)}[\sigma_1, z] \right\}_{i=1}^{n}$ and $\left\{ \hat{\mathbf{g}}^{(i)}[\sigma_1, z'] \right\}_{i=1}^{n}$ the corresponding braid-transformed bases.

Lemma 5.87. *Consider the assumptions of Setup (5.86) and fix an equivalence class $\xi \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$. Let $\mathscr{H}$ and $\mathscr{H}'$ be two equivalent extensions of $\mathscr{F}$ by $\mathscr{G}$ representing ξ, and let $\lambda : \mathscr{H} \to \mathscr{H}'$ be a sheaf isomorphism realizing their equivalence.*

On R_j, $\mathscr{H}$ and $\mathscr{H}'$ are specified by two collections of n characteristic maps

$$\begin{aligned} &\left\{ \phi_{\mathscr{H}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1} \right\}_{i=1}^{n-1} \cup \left\{ \widetilde{\phi}_{\mathscr{H}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2 \right\}, \\ &\left\{ \phi_{\mathscr{H}'}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1} \right\}_{i=1}^{n-1} \cup \left\{ \widetilde{\phi}_{\mathscr{H}'}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2 \right\}, \end{aligned} \tag{5.26}$$

while λ is characterized by a collection of $n+1$ linear maps

$$\left\{ T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i \right\}_{i=1}^{n} \cup \left\{ \widetilde{T}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^1 \right\} \tag{5.27}$$

such that:

$$\phi_{\mathscr{H}'}^{(i)} = \phi_{\mathscr{H}}^{(i)} + T_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} - \phi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i)}, \qquad \text{for all } i \in [1, n-1], \tag{5.28a}$$

$$\widetilde{\phi}^{(1)}_{\mathscr{H}'} = \widetilde{\phi}^{(1)}_{\mathscr{H}} + T^{(2)}_\lambda \circ \widetilde{\phi}^{(1)}_{\mathscr{F}} - \widetilde{\phi}^{(1)}_{\mathscr{G}} \circ \widetilde{T}^{(1)}_\lambda \,. \tag{5.28b}$$

⋆ *Assumption 1: Suppose that there are linear maps $\{\gamma^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$ that fully characterize ξ, up to equivalence, on the left of the crossing in R_j. Building on this assumption, we choose representatives $\mathscr{H}$ and $\mathscr{H}'$ such that:*

$$\phi^{(i)}_{\mathscr{H}} = \phi^{(i)}_{\mathscr{H}'} = \gamma^{(i)} \,, \qquad \textit{for all } i \in [1, n-1] \,.$$

⋆ *Assumption 2: Suppose that, with respect to the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$, the matrices representing the maps $\{T^{(i)}_\lambda\}_{i=1}^n$, which characterize λ on the left of the crossing in R_j, satisfy:*

$${}_{\hat{\mathbf{g}}^{(n)}}\big[\,T^{(n)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(n)}} \equiv D(\vec{u}), \quad \textit{for some } \vec{u} = (u_1, \dots, u_n) \in \mathbb{K}^n_{\mathrm{srd}} \,,$$

$${}_{\hat{\mathbf{g}}^{(i)}}\big[\,T^{(i)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(i)}} \equiv \textit{principal } i \times i \textit{ submatrix of } D(\vec{u}) \,, \quad \textit{for all } i \in [1, n-1] \,.$$

In particular, under the above assumptions, the equivalence conditions in (5.28a) are automatically satisfied.

⋆ *Main Conclusion: $\mathscr{H}$ is equivalent to a representative $\mathscr{H}'$ such that, with respect to the bases $\hat{\mathbf{g}}^{(2)}[\sigma_1, z']$ and $\hat{\mathbf{f}}^{(1)}[\sigma_1, z]$, the matrix representing $\widetilde{\phi}^{(1)}_{\mathscr{H}'}$ is given by*

$${}_{\hat{\mathbf{g}}^{(2)}[\sigma_1, z']}\big[\,\widetilde{\phi}^{(1)}_{\mathscr{H}'}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1, z]} = \begin{bmatrix} 0 \\ a' \end{bmatrix} \,,$$

for some $a' \in \mathbb{K}$. In other words, the parameter a', together with the data encoded in $\{\gamma^{(i)}\}_{i=1}^{n-1}$, completely determines ξ on R_j, up to equivalence.

Proof. To begin, recall that $\mathscr{H}$ and $\mathscr{H}'$ are two extensions of $\mathscr{F}$ by $\mathscr{G}$, and hence their local description on R_j via the characteristic maps in (5.26) follows from Observation (5.78). Moreover, building on Assumption 1, we suppose that $\mathscr{H}$ and $\mathscr{H}'$ are representatives with $\phi^{(i)}_{\mathscr{H}} = \phi^{(i)}_{\mathscr{H}'} = \gamma^{(i)}$, for all $i \in [1, n-1]$.

Analogously to the case of $\mathscr{H}$ and $\mathscr{H}'$, it follows from Observation (5.78) that $\lambda : \mathscr{H} \to \mathscr{H}'$, the sheaf isomorphism realizing the equivalence between $\mathscr{H}$ and $\mathscr{H}'$, is locally characterized on R_j by a collection of linear maps as in (5.27), which are subject to the equivalence conditions (5.28).

Now, given the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$, let us define

$$Y^{(n)} := {}_{\hat{\mathbf{g}}^{(n)}}\big[\,T^{(n)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(n)}} \in \mathrm{M}(n, \mathbb{K}) \,,$$

$$Y^{(i)} := {}_{\hat{\mathbf{g}}^{(i)}}\big[\,T^{(i)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(i)}} \in \mathrm{M}(i, \mathbb{K}) \,, \quad \text{for all } i \in [1, n-1].$$

By Assumption 2, we have that $Y^{(n)} = D(\vec{u})$, for some $\vec{u} = (u_1, \dots, u_n) \in \mathbb{K}^n_{\mathrm{std}}$, and that $Y^{(i)}$ is given by the principal $i \times i$ submatrix of $D(\vec{u})$, for all $i \in [1, n-1]$.

Next, given the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_1, z']\}_{i=1}^{n}$, let us define

$$\widetilde{X}^{(1)} := {}_{\hat{\mathbf{g}}^{(2)}[\sigma_1,z']}\big[\,\widetilde{\phi}^{(1)}_{\mathscr{H}}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = \begin{bmatrix} \tilde{x}_{1,1} \\ \tilde{x}_{2,1} \end{bmatrix} \in \mathrm{M}(2,1,\mathbb{K})\,,$$

$$\widetilde{X}'^{(1)} := {}_{\hat{\mathbf{g}}^{(2)}[\sigma_1,z']}\big[\,\widetilde{\phi}^{(1)}_{\mathscr{H}'}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = \begin{bmatrix} \tilde{x}'_{1,1} \\ \tilde{x}'_{2,1} \end{bmatrix} \in \mathrm{M}(2,1,\mathbb{K})\,,$$

$$\widetilde{Y}^{(1)} := {}_{\hat{\mathbf{g}}^{(1)}[\sigma_1,z']}\big[\,\widetilde{T}^{(1)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = [\tilde{y}_{1,1}] \in \mathrm{M}(1,\mathbb{K})\,.$$

Then, with respect to the braid transformed bases, relation (5.28b) translates into the system of equations

$$\widetilde{X}'^{(1)} = \widetilde{X}^{(1)} + \Big(\big(B_1^{(2)}(z')\big)^{-1} \cdot Y^{(2)} \cdot B_1^{(2)}(z)\Big) \cdot \iota^{(2,1)} - \iota^{(2,1)} \cdot \widetilde{Y}^{(1)}\,. \tag{5.29}$$

It follows from Lemma (5.81)(a) that, for any choice of $\tilde{x}'_{1,1}$, there exist unique $\tilde{y}_{1,1}$ and $\tilde{x}'_{2,1}$ solving (5.29), and hence $\mathscr{H}$ is equivalent to a representative $\mathscr{H}'$ with $\tilde{x}'_{1,1} = 0$ and $\tilde{x}'_{2,1} = a$, for some $a \in \mathbb{K}$, as claimed. □

Lemma 5.88. *Consider the assumptions of Setup (5.86) and fix an equivalence class $\xi \in \mathrm{Ext}^1(\mathscr{F},\mathscr{G})$. Let $\mathscr{H}$ and $\mathscr{H}'$ be two equivalent extensions of $\mathscr{F}$ by $\mathscr{G}$ representing ξ, and let $\lambda : \mathscr{H} \to \mathscr{H}'$ be a sheaf isomorphism realizing their equivalence.*

On R_j, $\mathscr{H}$ and $\mathscr{H}'$ are specified by two collections of n characteristic maps

$$\big\{\phi^{(i)}_{\mathscr{H}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\big\}_{i=1}^{n-1} \cup \big\{\widetilde{\phi}^{(1)}_{\mathscr{H}} : \mathbb{K}^1 \to \mathbb{K}^2\big\}\,,$$

$$\big\{\phi^{(i)}_{\mathscr{H}'} : \mathbb{K}^i \to \mathbb{K}^{i+1}\big\}_{i=1}^{n-1} \cup \big\{\widetilde{\phi}^{(1)}_{\mathscr{H}'} : \mathbb{K}^1 \to \mathbb{K}^2\big\}\,,$$

while λ is characterized by a collection of $n+1$ linear maps $\big\{T^{(i)}_{\lambda} : \mathbb{K}^i \to \mathbb{K}^i\big\} \cup \big\{\widetilde{T}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^1\big\}$ such that

$$\phi^{(i)}_{\mathscr{H}'} = \phi^{(i)}_{\mathscr{H}} + T^{(i+1)}_{\lambda} \circ \phi^{(i)}_{\mathscr{F}} - \phi^{(i)}_{\mathscr{G}} \circ T^{(i)}_{\lambda}\,, \qquad \textit{for all } i \in [1, n-1],$$

$$\widetilde{\phi}^{(1)}_{\mathscr{H}'} = \widetilde{\phi}^{(1)}_{\mathscr{H}} + T^{(2)}_{\lambda} \circ \widetilde{\phi}^{(1)}_{\mathscr{F}} - \widetilde{\phi}^{(1)}_{\mathscr{G}} \circ \widetilde{T}^{(1)}_{\lambda}\,.$$

⋆ *<u>Assumption 1</u>: Suppose that there are linear maps $\big\{\gamma^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\big\}_{i=1}^{n-1}$ that fully characterize ξ, up to equivalence, on the left of the crossing in R_j. Under this assumption, we choose representatives $\mathscr{H}$ and $\mathscr{H}'$ such that*

$$\phi^{(i)}_{\mathscr{H}} = \phi^{(i)}_{\mathscr{H}'} = \gamma^{(i)}\,, \qquad \textit{for all } i \in [1, n-1]\,.$$

⋆ *<u>Assumption 2</u>: Suppose that, with respect to the bases $\big\{\hat{\mathbf{f}}^{(i)}\big\}_{i=1}^{n}$ and $\big\{\hat{\mathbf{g}}^{(i)}\big\}_{i=1}^{n}$, the matrices representing the maps $\big\{T^{(i)}_{\lambda}\big\}_{i=1}^{n}$, which characterize λ on the left of the crossing in R_j, satisfy:*

$${}_{\hat{\mathbf{g}}^{(n)}}\big[\,T^{(n)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(n)}} \equiv D(\vec{u}), \quad \textit{for some } \vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}\,,$$

$${}_{\hat{\mathbf{g}}^{(i)}}\big[\,T^{(i)}_{\lambda}\,\big]_{\hat{\mathbf{f}}^{(i)}} \equiv \textit{principal } i \times i \textit{ submatrix of } D(\vec{u})\,, \quad \textit{for all } i \in [1, n-1]\,.$$

Under this assumption, Lemma (5.87) allows us to choose representatives $\mathscr{H}$ and $\mathscr{H}'$ such that, with respect to the bases $\hat{\mathbf{g}}^{(2)}[\sigma_1,z']$ and $\hat{\mathbf{f}}^{(1)}[\sigma_1,z]$, the matrices representing $\widetilde{\phi}^{(1)}_{\mathscr{H}}$ and $\widetilde{\phi}^{(1)}_{\mathscr{H}'}$ are given by

$$_{\hat{\mathbf{g}}^{(2)}[\sigma_1,z']}\big[\,\widetilde{\phi}^{(1)}_{\mathscr{H}}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = \begin{bmatrix} 0 \\ a \end{bmatrix}, \quad \textit{and} \quad _{\hat{\mathbf{g}}^{(2)}[\sigma_1,z']}\big[\,\widetilde{\phi}^{(1)}_{\mathscr{H}'}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = \begin{bmatrix} 0 \\ a' \end{bmatrix},$$

for some $a, a' \in \mathbb{K}$.

⋆ *Main Conclusion: Under the given assumptions, the following statements hold:*

- *Equivalence Condition: $a' = a + d(z', \vec{u}, z)$, where the residual parameter controlling the equivalence class in $\mathrm{Ext}^1(\mathscr{F},\mathscr{G})$ after the crossing in R_j is given by*

$$d(z',\vec{u},z) := (2,1)\text{-}\textit{entry of}\ \ \big(B_1^{(n)}(z')\big)^{-1}\cdot D(\vec{u})\cdot B_1^{(n)}(z)\,.$$

- *With respect to the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_1,z]\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_1,z']\}_{i=1}^n$, the matrices representing the maps $\widetilde{T}^{(1)} \cup \{T^{(i)}_\lambda\}_{i=2}^n$, which characterize λ on the right of the crossing in R_j, satisfy:*

$$_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[\,T^{(n)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]} \equiv D(\pi_1(\vec{u}))\,,$$

$$_{\hat{\mathbf{g}}^{(1)}[\sigma_1,z']}\big[\,\widetilde{T}^{(1)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} \equiv \textit{principal}\ 1\times 1\ \textit{submatrix of}\ D(\pi_1(\vec{u}))\,,$$

$$_{\hat{\mathbf{g}}^{(i)}[\sigma_1,z']}\big[\,T^{(i)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_1,z]} \equiv \textit{principal}\ i\times i\ \textit{submatrix of}\ D(\pi_1(\vec{u}))\,,\quad \textit{for all}\ i\in[2,n-1],$$

where π_1 denotes the permutation associated with σ_1.

Proof. Consider the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$, and set

$$Y^{(n)} := {}_{\hat{\mathbf{g}}^{(n)}}\big[\,T^{(n)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(n)}} \in \mathrm{M}(n,\mathbb{K})\,,$$

$$Y^{(i)} := {}_{\hat{\mathbf{g}}^{(i)}}\big[\,T^{(i)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(i)}} \in \mathrm{M}(i,\mathbb{K})\,,\qquad \text{for all } i\in[1,n-1].$$

By Assumption 2, we have that $Y^{(n)} = D(\vec{u})$, and $Y^{(i)}$ is given by the principal $i\times i$ submatrix of $D(\vec{u})$, for all $i\in[1,n-1]$.

Next, given the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_1,z]\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_1,z']\}_{i=1}^n$, let us define

$$\widetilde{X}^{(1)} := {}_{\hat{\mathbf{g}}^{(2)}[\sigma_1,z']}\big[\,\widetilde{\phi}^{(1)}_{\mathscr{H}}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = \begin{bmatrix} 0 \\ a \end{bmatrix} \in \mathrm{M}(2,1,\mathbb{K})\,,$$

$$\widetilde{X}'^{(1)} := {}_{\hat{\mathbf{g}}^{(2)}[\sigma_1,z']}\big[\,\widetilde{\phi}^{(1)}_{\mathscr{H}'}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = \begin{bmatrix} 0 \\ a' \end{bmatrix} \in \mathrm{M}(2,1,\mathbb{K})\,,$$

$$\widetilde{Y}^{(1)} := {}_{\hat{\mathbf{g}}^{(1)}[\sigma_1,z']}\big[\,\widetilde{T}^{(1)}_\lambda\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} = [\tilde{y}_{1,1}] \in \mathrm{M}(1,\mathbb{K})\,.$$

Then, with respect to the braid-transformed bases, the equivalence condition for the maps $\widetilde{\phi}^{(1)}_{\mathscr{H}}$ and $\widetilde{\phi}^{(1)}_{\mathscr{H}'}$ translates into the system of equations

$$\widetilde{X}'^{(1)} = \widetilde{X}^{(1)} + \Big(\big(B_1^{(2)}(z')\big)^{-1} \cdot Y^{(2)} \cdot B_1^{(2)}(z) \Big) \cdot \iota^{(2,1)} - \iota^{(2,1)} \cdot \widetilde{Y}^{(1)} \,. \tag{5.30}$$

It follows from Lemma (5.81)(b) that the system of equations in (5.30) has a unique solution:

$$\widetilde{Y}^{(1)} = \text{ principal } 1\times 1 \text{ submatrix of } \big(B_1^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_1^{(n)}(z)\,,$$

$$a' = a + d(z', \vec{u}, z)\,,$$

where

$$d(z', \vec{u}, z) = \text{ the } (2,1)\text{-entry of } \big(B_1^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_1^{(n)}(z)\,.$$

Specifically, we have that $d(z', \vec{u}, z)$ measures the redundancy in the choice of representatives for the equivalence classes in $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ on the right of the crossing in R_j, and since redundancies vanish when restricting to $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, we deduce that $d(z', \vec{u}, z) \equiv 0$.

In particular, observe that

$$\big(B_1^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_1^{(n)}(z) = D(\pi_1(\vec{u})) + d(z', \vec{u}, z)\, E_{2,1}\,,$$

where $E_{2,1} \in \mathrm{M}(n, \mathbb{K})$ denotes the elementary matrix with 1 in the position $(2,1)$ and zeros everywhere else. Moreover, since $Y^{(i)}$ is given by the principal $i\times i$ submatrix of $D(\vec{u})$ for all $i \in [2, n-1]$, the block structure of the braid matrices guarantees that

$$\big(B_1^{(i)}(z')\big)^{-1} \cdot Y^{(i)} \cdot B_1^{(i)}(z) = \text{ principal } i\times i \text{ submatrix of } \big(B_1^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_1^{(n)}(z)\,,$$

for all $i \in [2, n-1]$.

Finally, note that, with respect to the bases $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\big\}_{i=2}^{n}$ and $\big\{\hat{\mathbf{g}}^{(i)}[\sigma_1, z']\big\}_{i=2}^{n}$,

$${}_{\hat{\mathbf{g}}^{(i)}[\sigma_1,z']}\big[\,T_\lambda^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_1,z]} = \big(B_1^{(i)}(z')\big)^{-1} \cdot {}_{\hat{\mathbf{g}}^{(i)}}\big[\,T_\lambda^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} \cdot B_1^{(i)}(z)\,,$$

for all $i \in [2, n-1]$. Hence, when restricting to $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, we obtain that

$${}_{\hat{\mathbf{g}}^{(n)}[\sigma_1,z']}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_1,z]} \equiv D(\pi_1(\vec{u}))\,,$$

$${}_{\hat{\mathbf{g}}^{(1)}[\sigma_1,z']}\big[\,\widetilde{T}_\lambda^{(1)}\,\big]_{\hat{\mathbf{f}}^{(1)}[\sigma_1,z]} \equiv \text{ principal } 1\times 1 \text{ submatrix of } D(\pi_1(\vec{u}))\,,$$

$${}_{\hat{\mathbf{g}}^{(i)}[\sigma_1,z']}\big[\,T_\lambda^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_1,z]} \equiv \text{ principal } i\times i \text{ submatrix of } D(\pi_1(\vec{u}))\,, \quad \text{for all } i \in [2, n-1].$$

This completes the proof. $\square$

Setup 5.89. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$ and $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Fix $j \in [1,\ell]$, and let R_j be the vertical strap in $\mathbb{R}^2$ containing σ_{i_j}—the j-th crossing of β (see Figure (19)).*

⋆ *Assumption 1: Let $k := i_j \in [1, n-1]$ denote the index of σ_{i_j}, and suppose that $k \geq 2$.*

Under this assumption, on R_j, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of $n+1$ injective linear maps

$$\left\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}^{(k-1)}_{\mathscr{F}} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{F}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\right\},$$

$$\left\{\phi^{(i)}_{\mathscr{G}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}^{(k-1)}_{\mathscr{G}} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{G}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\right\},$$

respectively. For a schematic illustration of a generic representative of one of these sheaves, see Figure (23).

⋆ *Assumption 2: For each $i \in [1,n]$, let $\hat{\mathbf{f}}^{(i)} := \left\{\hat{f}^{(i)}_j\right\}_{j=1}^{i}$ and $\hat{\mathbf{g}}^{(i)} := \left\{\hat{g}^{(i)}_j\right\}_{j=1}^{i}$ be bases for $\mathbb{K}^i$. Following Definition (4.18)–(3), we assume that:*

- *$\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^{n}$ is a system of bases adapted to $\left\{\phi^{(i)}_{\mathscr{F}}\right\}_{i=1}^{n-1}$.*
- *$\left\{\hat{\mathbf{g}}^{(i)}\right\}_{i=1}^{n}$ is a system of bases adapted to $\left\{\phi^{(i)}_{\mathscr{G}}\right\}_{i=1}^{n-1}$.*

Under this assumption, Lemma (4.48) ensures that $\left(\left\{\hat{\mathbf{f}}^{(i)}\right\}_{i=1}^{n}, z\right)$ and $\left(\left\{\hat{\mathbf{g}}^{(i)}\right\}_{i=1}^{n}, z'\right)$ are system of bases adapted to $\mathscr{F}$ and $\mathscr{G}$ on R_j, where z, $z' \in \mathbb{K}$ parameterize, relative to the bases $\hat{\mathbf{f}}^{(n)}$ and $\hat{\mathbf{g}}^{(n)}$ for $\mathbb{K}^n$, the s_k-relative position between the pairs of complete flags in $\mathbb{K}^n$ that geometrically characterize $\mathscr{F}$ and $\mathscr{G}$ on R_j, respectively. Following Definition (4.29), we denote by $\left\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\right\}_{i=1}^{n}$ and $\left\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\right\}_{i=1}^{n}$ the corresponding braid-transformed bases.

Lemma 5.90. *Consider the assumptions of Setup (5.89) and fix an equivalence class $\xi \in \mathrm{Ext}^1(\mathscr{F},\mathscr{G})$. Let $\mathscr{H}$ and $\mathscr{H}'$ be two equivalent extensions of $\mathscr{F}$ by $\mathscr{G}$ representing ξ, and let $\lambda : \mathscr{H} \to \mathscr{H}'$ be a sheaf isomorphism realizing their equivalence.*

On R_j, $\mathscr{H}$ and $\mathscr{H}'$ are specified by two collections of $n+1$ characteristic maps

$$\begin{aligned}&\left\{\phi^{(i)}_{\mathscr{H}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}^{(k-1)}_{\mathscr{H}} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{H}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\right\},\\ &\left\{\phi^{(i)}_{\mathscr{H}'} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1} \cup \left\{\widetilde{\phi}^{(k-1)}_{\mathscr{H}'} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{H}'} : \mathbb{K}^k \to \mathbb{K}^{k+1}\right\},\end{aligned} \tag{5.31}$$

while λ is characterized by a collection of $n+1$ linear maps

$$\left\{T^{(i)}_\lambda : \mathbb{K}^i \to \mathbb{K}^i\right\}_{i=1}^{n} \cup \left\{\widetilde{T}^{(k)} : \mathbb{K}^k \to \mathbb{K}^k\right\} \tag{5.32}$$

such that:

$$\phi^{(i)}_{\mathscr{H}'} = \phi^{(i)}_{\mathscr{H}} + T^{(i+1)}_\lambda \circ \phi^{(i)}_{\mathscr{F}} - \phi^{(i)}_{\mathscr{G}} \circ T^{(i)}_\lambda\,, \qquad \text{for all } i \in [1, n-1]\,, \tag{5.33a}$$

$$\widetilde{\phi}^{(k)}_{\mathscr{H}'} = \widetilde{\phi}^{(k)}_{\mathscr{H}} + T^{(k+1)}_{\lambda} \circ \widetilde{\phi}^{(k)}_{\mathscr{F}} - \widetilde{\phi}^{(k)}_{\mathscr{G}} \circ \widetilde{T}^{(k)}_{\lambda}\,, \tag{5.33b}$$

$$\widetilde{\phi}^{(k-1)}_{\mathscr{H}'} = \widetilde{\phi}^{(k-1)}_{\mathscr{H}} + \widetilde{T}^{(k)}_{\lambda} \circ \widetilde{\phi}^{(k-1)}_{\mathscr{F}} - \widetilde{\phi}^{(k-1)}_{\mathscr{G}} \circ T^{(k-1)}_{\lambda}\,. \tag{5.33c}$$

$\star$ <u>*Assumption 1*</u>*: Suppose that there are linear maps* $\{\gamma^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$ *that fully characterize* ξ*, up to equivalence, on the left of the crossing in* R_j*. Building on this assumption, we choose representatives* $\mathscr{H}$ *and* $\mathscr{H}'$ *such that:*

$$\phi^{(i)}_{\mathscr{H}} = \phi^{(i)}_{\mathscr{H}'} = \gamma^{(i)}\,, \qquad \textit{for all } i \in [1, n-1]\,.$$

$\star$ <u>*Assumption 2*</u>*: Suppose that, for the above choice of representatives and with respect to the bases* $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ *and* $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$*, the matrices representing the maps* $\{T^{(i)}_{\lambda}\}_{i=1}^{n}$*, which characterize* λ *on the left of the crossing in* R_j*, satisfy:*

$$_{\hat{\mathbf{g}}^{(n)}}\big[\, T^{(n)}_{\lambda} \,\big]_{\hat{\mathbf{f}}^{(n)}} \equiv D(\vec{u}), \quad \textit{for some } \vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}\,,$$

$$_{\hat{\mathbf{g}}^{(i)}}\big[\, T^{(i)}_{\lambda} \,\big]_{\hat{\mathbf{f}}^{(i)}} \equiv \textit{principal } i \times i \textit{ submatrix of } D(\vec{u})\,, \quad \textit{for all } i \in [1, n-1]\,.$$

In particular, under the above assumptions, the equivalence conditions in (5.33a) are automatically satisfied.

$\star$ <u>*Main Conclusion*</u>*:* $\mathscr{H}$ *is equivalent to a representative* $\mathscr{H}'$ *such that, with respect to the bases* $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^{n}$ *and* $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^{n}$*, the matrices representing* $\widetilde{\phi}^{(k-1)}_{\mathscr{H}'}$ *and* $\widetilde{\phi}^{(k)}_{\mathscr{H}'}$ *are given by*

$$_{\hat{\mathbf{g}}^{(k)}[\sigma_k, z']}\big[\, \widetilde{\phi}^{(k-1)}_{\mathscr{H}'} \,\big]_{\hat{\mathbf{f}}^{(k-1)}[\sigma_k, z]} = \mathbf{0}_{k\times(k-1)}\,, \quad \textit{and} \quad _{\hat{\mathbf{g}}^{(k+1)}[\sigma_k, z']}\big[\, \widetilde{\phi}^{(k)}_{\mathscr{H}'} \,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k, z]} = \begin{bmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & 0 \\ \star & \cdots & \star & a' \end{bmatrix},$$

for some $a' \in \mathbb{K}$*, where the starred entries are uniquely determined by the data associated with* $\phi^{(k)}_{\mathscr{G}}$*,* $\gamma^{(k-1)}$*,* $\gamma^{(k)}$*, and* $\phi^{(k-1)}_{\mathscr{F}}$*. In other words, the parameter* a'*, together with the data encoded in* $\{\gamma^{(i)}\}_{i=1}^{n-1}$*, completely determines* ξ *on* R_j*, up to equivalence.*

Proof. To begin, recall that $\mathscr{H}$ and $\mathscr{H}'$ are two extensions of $\mathscr{F}$ by $\mathscr{G}$, and hence their local description on R_j via the characteristic maps in (5.31) follows from Observation (5.78). Moreover, building on <u>Assumption 1</u>, we suppose that $\mathscr{H}$ and $\mathscr{H}'$ are representatives with $\phi^{(i)}_{\mathscr{H}} = \phi^{(i)}_{\mathscr{H}'} = \gamma^{(i)}$, for all $i \in [1, n-1]$.

Analogously to the case of $\mathscr{H}$ and $\mathscr{H}'$, it follows from Observation (5.78) that $\lambda : \mathscr{H} \to \mathscr{H}'$, the sheaf isomorphism realizing the equivalence between $\mathscr{H}$ and $\mathscr{H}'$, is locally characterized on R_j by a collection of linear maps as in (5.32), which are subject to the equivalence conditions (5.33).

Moreover, building on Observation (5.78), we know that the microlocal support conditions near the crossing impose the following constraints on the characteristic maps of $\mathscr{H}$ and $\mathscr{H}'$

$$\phi_{\mathscr{G}}^{(k)} \circ \gamma^{(k)} + \gamma^{(k-1)} \circ \phi_{\mathscr{F}}^{(k-1)} = \widetilde{\phi}_{\mathscr{G}}^{(k)} \circ \widetilde{\phi}_{\mathscr{H}}^{(k-1)} + \widetilde{\phi}_{\mathscr{H}}^{(k)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k-1)} ,$$

$$\phi_{\mathscr{G}}^{(k)} \circ \gamma^{(k)} + \gamma^{(k-1)} \circ \phi_{\mathscr{F}}^{(k-1)} = \widetilde{\phi}_{\mathscr{G}}^{(k)} \circ \widetilde{\phi}_{\mathscr{H}'}^{(k-1)} + \widetilde{\phi}_{\mathscr{H}'}^{(k)} \circ \widetilde{\phi}_{\mathscr{F}}^{(k-1)} ,$$

where have used that $\phi_{\mathscr{H}}^{(i)} = \phi_{\mathscr{H}'}^{(i)} = \gamma^{(i)}$, for all $i \in [1, n-1]$.

Here, given the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$, let us define

$$Y^{(n)} := {}_{\hat{\mathbf{g}}^{(n)}}\big[\, T_{\lambda}^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}} \in \mathrm{M}(n, \mathbb{K}) ,$$

$$Y^{(i)} := {}_{\hat{\mathbf{g}}^{(i)}}\big[\, T_{\lambda}^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}} \in \mathrm{M}(i, \mathbb{K}) , \quad \text{for all } i \in [1, n-1] ,$$

$$Z^{(k-1)} := {}_{\hat{\mathbf{g}}^{(k)}}\big[\, \gamma^{(k-1)} \,\big]_{\hat{\mathbf{f}}^{(k-1)}} \in \mathrm{M}(k, k-1, \mathbb{K}) ,$$

$$Z^{(k)} := {}_{\hat{\mathbf{g}}^{(k+1)}}\big[\, \gamma^{(k)} \,\big]_{\hat{\mathbf{f}}^{(k)}} \in \mathrm{M}(k+1, k, \mathbb{K}) .$$

In particular, by Assumption 2, we have that $Y^{(n)} = D(\vec{u})$, and $Y^{(i)}$ is given by the principal $i \times i$ submatrix of $D(\vec{u})$, for all $i \in [1, n-1]$.

Also, consider the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^{n}$, and set

$$\widetilde{Y}^{(k)} := {}_{\hat{\mathbf{g}}^{(k)}[\sigma_k, z']}\big[\, \widetilde{T}_{\lambda}^{(k)} \,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k, z]} \in \mathrm{M}(k, \mathbb{K}) ,$$

$$\widetilde{X}^{(k-1)} := {}_{\hat{\mathbf{g}}^{(k)}[\sigma_k, z']}\big[\, \widetilde{\phi}_{\mathscr{H}'}^{(k-1)} \,\big]_{\hat{\mathbf{f}}^{(k-1)}[\sigma_k, z]} \in \mathrm{M}(k, k-1, \mathbb{K}) ,$$

$$\widetilde{X}^{(k)} := {}_{\hat{\mathbf{g}}^{(k+1)}[\sigma_k, z']}\big[\, \widetilde{\phi}_{\mathscr{H}'}^{(k)} \,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k, z]} \in \mathrm{M}(k+1, k, \mathbb{K}) ,$$

$$\widetilde{X'}^{(k-1)} := {}_{\hat{\mathbf{g}}^{(k)}[\sigma_k, z']}\big[\, \widetilde{\phi}_{\mathscr{H}'}^{(k-1)} \,\big]_{\hat{\mathbf{f}}^{(k-1)}[\sigma_k, z]} \in \mathrm{M}(k, k-1, \mathbb{K}) ,$$

$$\widetilde{X'}^{(k)} := {}_{\hat{\mathbf{g}}^{(k+1)}[\sigma_k, z']}\big[\, \widetilde{\phi}_{\mathscr{H}'}^{(k)} \,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k, z]} \in \mathrm{M}(k+1, k, \mathbb{K}) .$$

Next, observe that, with respect to the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^{n}$, the equivalence relations (5.33b) and (5.33b) translate into the system of equations

$$\widetilde{X'}^{(k-1)} = \widetilde{X}^{(k-1)} + \widetilde{Y}^{(k)} \cdot \iota^{(k,k-1)} - \iota^{(k,k-1)} \cdot Y^{(k-1)} ,$$

$$\widetilde{X'}^{(k)} = \widetilde{X}^{(k)} + \big(B_k^{(k+1)}(z')\big)^{-1} \cdot Y^{(k+1)} \cdot B_k^{(k+1)}(z) \cdot \iota^{(k+1,k)} - \iota^{(k+1,k)} \cdot \widetilde{Y}^{(k)} .$$

It follows from Lemma (5.83)(a) that for any choice of $\widetilde{X'}^{(k-1)}$ and the k-th column of $\widetilde{X'}^{(k)}$ expect its very last entry, the above system of equations uniquely determines $\widetilde{Y}^{(k)}$ and the remaining entries of $\widetilde{X'}^{(k)}$, and hence, $\mathscr{H}$ is equivalent to a representative $\mathscr{H}'$ such that with respect to the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^{n}$, the matrices representing $\widetilde{\phi}^{(k-1)}_{\mathscr{H}'}$ and $\widetilde{\phi}^{(k)}_{\mathscr{H}'}$ are given by

$$\widetilde{X'}^{(k-1)} = \mathbf{0}_{k\times(k-1)}\,, \quad \text{and} \quad \widetilde{X'}^{(k)} = \begin{bmatrix} \tilde{x}'_{1,1} & \cdots & \tilde{x}'_{1,k-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{x}'_{k,1} & \cdots & \tilde{x}'_{k,k-1} & 0 \\ \tilde{x}'_{k+1,1} & \cdots & \tilde{x}'_{k+1,k-1} & \tilde{x}'_{k+1,k} \end{bmatrix}\,. \tag{5.34}$$

Finally, note that, with respect to the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^{n}$, the crossing constraint for the extensions $\mathscr{H}'$ translate into the system of equations

$$\big(B_k^{(k+1)}(z')\big)^{-1} \cdot \Big(\iota^{(k+1,k)} \cdot Z^{(k-1)} + Z^{(k)} \cdot \iota^{(k,k-1)}\Big) = \iota^{(k+1,k)} \cdot \widetilde{X'}^{(k-1)} + \widetilde{X'}^{(k)} \cdot \iota^{(k,k-1)}\,. \tag{5.35}$$

In particular, given our choice of representative $\mathscr{H}'$ in (5.34), Lemma (5.82) ensures that the first $k-1$ columns of $\widetilde{X'}^{(k)}$ are uniquely determined by the system of equations in (5.35), and in this case, these columns are uniquely determined by the data associated with $\phi^{(k)}_{\mathscr{G}}$, $\gamma^{(k-1)}$, $\gamma^{(k)}$, and $\phi^{(k-1)}_{\mathscr{F}}$. Thus, the result follows once we identify $a' = \tilde{x}'_{k+1,k}$. $\square$

Lemma 5.91. *Consider the assumptions of Setup (5.89) and fix an equivalence class $\xi \in \operatorname{Ext}^1(\mathscr{F},\mathscr{G})$. Let $\mathscr{H}$ and $\mathscr{H}'$ be two equivalent extensions of $\mathscr{F}$ by $\mathscr{G}$ representing ξ, and let $\lambda : \mathscr{H} \to \mathscr{H}'$ be a sheaf isomorphism realizing their equivalence.*

On R_j, $\mathscr{H}$ and $\mathscr{H}'$ are specified by two collections of $n+1$ characteristic maps

$$\{\phi^{(i)}_{\mathscr{H}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(k-1)}_{\mathscr{H}} : \mathbb{K}^{k-1} \to \mathbb{K}^k\,,\ \widetilde{\phi}^{(k)}_{\mathscr{H}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\}\,,$$

$$\{\phi^{(i)}_{\mathscr{H}'} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(k-1)}_{\mathscr{H}'} : \mathbb{K}^{k-1} \to \mathbb{K}^k\,,\ \widetilde{\phi}^{(k)}_{\mathscr{H}'} : \mathbb{K}^k \to \mathbb{K}^{k+1}\}\,,$$

while λ is characterized by a collection of $n+1$ linear maps

$$\{T^{(i)}_{\lambda} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^{n} \cup \{\widetilde{T}^{(k)} : \mathbb{K}^k \to \mathbb{K}^k\}$$

such that:

$$\phi^{(i)}_{\mathscr{H}'} = \phi^{(i)}_{\mathscr{H}} + T^{(i+1)}_{\lambda} \circ \phi^{(i)}_{\mathscr{F}} - \phi^{(i)}_{\mathscr{G}} \circ T^{(i)}_{\lambda}\,, \qquad \textit{for all } i \in [1, n-1]\,,$$

$$\widetilde{\phi}^{(k)}_{\mathscr{H}'} = \widetilde{\phi}^{(k)}_{\mathscr{H}} + T^{(k+1)}_{\lambda} \circ \widetilde{\phi}^{(k)}_{\mathscr{F}} - \widetilde{\phi}^{(k)}_{\mathscr{G}} \circ \widetilde{T}^{(k)}_{\lambda}\,,$$

$$\widetilde{\phi}^{(k-1)}_{\mathscr{H}'} = \widetilde{\phi}^{(k-1)}_{\mathscr{H}} + \widetilde{T}^{(k)}_{\lambda} \circ \widetilde{\phi}^{(k-1)}_{\mathscr{F}} - \widetilde{\phi}^{(k-1)}_{\mathscr{G}} \circ T^{(k-1)}_{\lambda}\,.$$

⋆ *Assumption 1: Suppose that there are linear maps $\{\gamma^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}$ that fully characterize ξ, up to equivalence, on the left of the crossing in R_j. Building on this assumption, we choose representatives $\mathscr{H}$ and $\mathscr{H}'$ such that:*

$$\phi^{(i)}_{\mathscr{H}} = \phi^{(i)}_{\mathscr{H}'} = \gamma^{(i)}\,, \qquad \textit{for all } i \in [1, n-1]\,.$$

⋆ *Assumption 2: Suppose that, with respect to the bases* $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ *and* $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$*, the matrices representing the maps* $\{T_\lambda^{(i)}\}_{i=1}^n$*, which characterize* λ *on the left of the crossing in* R_j*, satisfy:*

$$
{}_{\hat{\mathbf{g}}^{(n)}}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}} \equiv D(\vec{u}), \quad \textit{for some } \vec{u} = (u_1, \dots, u_n) \in \mathbb{K}^n_{\mathrm{std}}\,,
$$

$$
{}_{\hat{\mathbf{g}}^{(i)}}\big[\,T_\lambda^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} \equiv \textit{principal } i \times i \textit{ submatrix of } D(\vec{u})\,, \quad \textit{for all } i \in [1, n-1]\,.
$$

Under this assumption, Lemma (5.90) allows us to choose representatives $\mathscr{H}$ *and* $\mathscr{H}'$ *such that with respect to the bases* $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^n$ *and* $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^n$*, the matrices representing* $\widetilde{\phi}_{\mathscr{H}}^{(k-1)}$*,* $\widetilde{\phi}_{\mathscr{H}}^{(k)}$*,* $\widetilde{\phi}_{\mathscr{H}'}^{(k-1)}$ *and* $\widetilde{\phi}_{\mathscr{H}'}^{(k)}$ *are given by*

$$
{}_{\hat{\mathbf{g}}^{(k)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{H}}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}[\sigma_k,z]} = \mathbf{0}_{k\times(k-1)}\,, \quad \textit{and} \quad {}_{\hat{\mathbf{g}}^{(k+1)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{H}}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k,z]} = \begin{bmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & 0 \\ \star & \cdots & \star & a \end{bmatrix},
$$

$$
{}_{\hat{\mathbf{g}}^{(k)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{H}'}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}[\sigma_k,z]} = \mathbf{0}_{k\times(k-1)}\,, \quad \textit{and} \quad {}_{\hat{\mathbf{g}}^{(k+1)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{H}'}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k,z]} = \begin{bmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & 0 \\ \star & \cdots & \star & a' \end{bmatrix},
$$

for some $a, a' \in \mathbb{K}$.

⋆ *Main Conclusion: Under the given assumptions, the following statements hold:*

- *Equivalence Condition:* $a' = a + d(z', \vec{u}, z)$*, where the residual parameter controlling the equivalence class in* $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ *after the crossing in* R_j *is given by*

$$
d(z', \vec{u}, z) := (k+1, k)\textit{-entry of } \left(B_k^{(n)}(z')\right)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)\,.
$$

- *With respect to the bases* $\{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]\}_{i=1}^n$ *and* $\{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']\}_{i=1}^n$*, the matrices representing the collection of maps* $\{T_\lambda^{(i)}\}_{i\in[1,n]\setminus\{k\}} \cup \{\widetilde{T}_\lambda^{(k)}\}$*, which characterize* λ *on the right of the crossing in* R_j*, satisfy:*

$$
{}_{\hat{\mathbf{g}}^{(n)}[\sigma_k,z']}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_k,z]} \equiv D(\pi_k(\vec{u}))\,,
$$

$$
{}_{\hat{\mathbf{g}}^{(k)}[\sigma_k,z']}\big[\,\widetilde{T}_\lambda^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k,z]} \equiv \textit{principal } k \times k \textit{ submatrix of } D(\pi_k(\vec{u}))\,,
$$

$$
{}_{\hat{\mathbf{g}}^{(i)}[\sigma_k,z']}\big[\,T_\lambda^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_k,z]} \equiv \textit{principal } i \times i \textit{ submatrix of } D(\pi_k(\vec{u}))\,, \quad \textit{for all } i \in [1, n-1] \setminus \{k\}\,,
$$

where π_k *denotes the permutation associated with* σ_k.

Proof. Consider the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$, and set

$$Y^{(n)} := {}_{\hat{\mathbf{g}}^{(n)}}\big[\,T_\lambda^{(n)}\,\big]_{\hat{\mathbf{f}}^{(n)}} \in \mathrm{M}(n,\mathbb{K})\,,$$

$$Y^{(i)} := {}_{\hat{\mathbf{g}}^{(i)}}\big[\,T_\lambda^{(i)}\,\big]_{\hat{\mathbf{f}}^{(i)}} \in \mathrm{M}(i,\mathbb{K})\,, \qquad \text{for all } i \in [1,n-1].$$

By Assumption 2, we have that $Y^{(n)} = D(\vec{u})$, and $Y^{(i)}$ is given by the principal $i \times i$ submatrix of $D(\vec{u})$, for all $i \in [1,n-1]$.

Next, given the bases $\{\hat{\mathbf{f}}^{(i)}[\sigma_k,z]\}_{i=1}^{n}$ and $\{\hat{\mathbf{g}}^{(i)}[\sigma_k,z']\}_{i=1}^{n}$, let us define

$$\widetilde{X}^{(k-1)} := {}_{\hat{\mathbf{g}}^{(k)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{H}}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}[\sigma_k,z]} = \mathbf{0}_{k\times(k-1)}\,, \qquad \widetilde{X}^{(k)} := {}_{\hat{\mathbf{g}}^{(k+1)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{H}}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k,z]} = \begin{bmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & 0 \\ \star & \cdots & \star & a \end{bmatrix},$$

$$\widetilde{X}'^{(k-1)} := {}_{\hat{\mathbf{g}}^{(k)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{H}'}^{(k-1)}\,\big]_{\hat{\mathbf{f}}^{(k-1)}[\sigma_k,z]} = \mathbf{0}_{k\times(k-1)}\,, \qquad \widetilde{X}'^{(k)} := {}_{\hat{\mathbf{g}}^{(k+1)}[\sigma_k,z']}\big[\,\widetilde{\phi}_{\mathscr{H}'}^{(k)}\,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k,z]} = \begin{bmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & 0 \\ \star & \cdots & \star & a' \end{bmatrix}.$$

Then, with respect to the braid-transformed bases, the equivalence conditions for the pairs of maps $\big(\widetilde{\phi}_{\mathscr{H}}^{(k-1)}, \widetilde{\phi}_{\mathscr{H}'}^{(k-1)}\big)$ and $\big(\widetilde{\phi}_{\mathscr{H}}^{(k)}, \widetilde{\phi}_{\mathscr{H}'}^{(k)}\big)$ translate into the system of equations

$$\begin{aligned} \widetilde{X}'^{(k-1)} &= \widetilde{X}^{(k-1)} + \widetilde{Y}^{(k)} \cdot \iota^{(k,k-1)} - \iota^{(k,k-1)} \cdot Y^{(k-1)}\,, \\ \widetilde{X}'^{(k)} &= \widetilde{X}^{(k)} + \big(B_k^{(k+1)}(z')\big)^{-1} \cdot Y^{(k+1)} \cdot B_k^{(k+1)}(z) \cdot \iota^{(k+1,k)} - \iota^{(k+1,k)} \cdot \widetilde{Y}^{(k)}\,. \end{aligned} \tag{5.36}$$

It follows from Lemma (5.83)(b) that the system of equations in (5.36) has a unique solution:

$$\widetilde{Y}^{(k)} = \text{ principal } k \times k \text{ submatrix of } \big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)\,,$$

$$\star = \star\,,$$

$$a' = a + d(z',\vec{u},z)\,,$$

where

$$d(z',\vec{u},z) = \text{ the } (k+1,k)\text{-entry of } \big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)\,.$$

Specifically, we have that $d(z',\vec{u},z)$ measures the redundancy in the choice of representatives for the equivalence classes in $\mathrm{Ext}^1(\mathscr{F},\mathscr{G})$ on the right of the crossing in R_j, and since redundancies vanish when restricting to $\mathrm{Ext}^1(\mathscr{F},\mathscr{G})$, we deduce that $d(z',\vec{u},z) \equiv 0$.

In particular, observe that

$$\big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z) = D(\pi_k(\vec{u})) + d(z', \vec{u}, z)\, E_{k+1,k}\,,$$

where $E_{k+1,k} \in \mathrm{M}(n, \mathbb{K})$ denotes the elementary matrix with 1 in the position $(k+1, k)$ and zeros everywhere else. Moreover, since $Y^{(i)}$ is given by the principal $i \times i$ submatrix of $D(\vec{u})$ for all $i \in [1, n-1]$, the block structure of the braid matrices guarantees that:

- For all $i \in [1, k-1]$,

$$Y^{(i)} = \text{ principal } i \times i \text{ submatrix of } \big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)\,.$$

- For all $i \in [k+1, n-1]$,

$$\big(B_k^{(i)}(z')\big)^{-1} \cdot Y^{(i)} \cdot B_k^{(i)}(z) = \text{ principal } i \times i \text{ submatrix of } \big(B_k^{(n)}(z')\big)^{-1} \cdot D(\vec{u}) \cdot B_k^{(n)}(z)\,.$$

Finally, note that, with respect to the bases $\big\{\hat{\mathbf{f}}^{(i)}[\sigma_1, z]\big\}_{i=1}^{n}$ and $\big\{\hat{\mathbf{g}}^{(i)}[\sigma_1, z']\big\}_{i=1}^{n}$,

$$\begin{aligned}
{}_{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']}\big[\, T_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]} &= {}_{\hat{\mathbf{g}}^{(i)}}\big[\, T_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}}\,, \quad \text{for all } i \in [1, k-1]\,,\\
{}_{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']}\big[\, T_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]} &= \big(B_k^{(i)}(z')\big)^{-1} \cdot {}_{\hat{\mathbf{g}}^{(i)}}\big[\, T_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}} \cdot B_k^{(i)}(z)\,, \quad \text{for all } i \in [k+1, n]\,.
\end{aligned}$$

Hence, when restricting to $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, we obtain that

$$\begin{aligned}
{}_{\hat{\mathbf{g}}^{(n)}[\sigma_k, z']}\big[\, T_\lambda^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_k, z]} &\equiv D(\pi_k(\vec{u}))\,,\\
{}_{\hat{\mathbf{g}}^{(k)}[\sigma_k, z']}\big[\, \widetilde{T}_\lambda^{(k)} \,\big]_{\hat{\mathbf{f}}^{(k)}[\sigma_k, z]} &\equiv \text{principal } k \times k \text{ submatrix of } D(\pi_k(\vec{u}))\,,\\
{}_{\hat{\mathbf{g}}^{(i)}[\sigma_k, z']}\big[\, T_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}[\sigma_k, z]} &\equiv \text{principal } i \times i \text{ submatrix of } D(\pi_k(\vec{u}))\,, \qquad \text{for all } i \in [1, n-1] \setminus \{k\}\,,
\end{aligned}$$

This completes the proof. $\square$

Having established the preceding lemmas, we are now in a position to state and prove the second part of our first main theorem.

Theorem 5.92. ***Setup****: Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, $\mathcal{U}_{\Lambda(\beta)} := \big\{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\big\}$ the open cover of $\mathbb{R}^2$ from Construction (4.33), and $\mathscr{F}$ and $\mathscr{G}$ objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.*

$\star$ *<u>Local descriptions on U_{T} and U_{L}</u>: According to Lemma (4.34), $\mathscr{F}$ and $\mathscr{G}$ have the following local descriptions:*

- *On U_{T}, $\mathscr{F}$ and $\mathscr{G}$ are specified by two collections of $n-1$ surjective linear maps*

$$\big\{\psi_{\mathscr{F}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\big\}_{i=1}^{n-1}\,, \quad \textit{and} \quad \big\{\psi_{\mathscr{G}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\big\}_{i=1}^{n-1}\,,$$

respectively. For a schematic illustration of a generic representative of one of these sheaves on U_{T}*, see Figure* (15)*.*

- *On* U_{L}*,* $\mathscr{F}$ *and* $\mathscr{G}$ *are specified by two collections of* $n-1$ *injective linear maps*

$$\left\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1}, \quad \textit{and} \quad \left\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\right\}_{i=1}^{n-1},$$

respectively. For a schematic illustration of a generic representative of one of these sheaves on U_{L}*, see Figure* (13)*.*

- ***Compatibility conditions****: For each* $i \in [1, n-1]$*,*

$$\psi_{\mathscr{F}}^{(i)} \circ \phi_{\mathscr{F}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}, \quad \textit{and} \quad \psi_{\mathscr{G}}^{(i)} \circ \phi_{\mathscr{G}}^{(i)} = \mathrm{id}_{\mathbb{K}^i}.$$

⋆ *Global flag data: By Theorem* (4.40)*,* $\mathscr{F}$ *and* $\mathscr{G}$ *are geometrically characterized by two sequence of complete flags* $\left\{\mathcal{F}_j^\bullet\right\}_{j=0}^{\ell+1}$ *and* $\left\{\mathcal{G}_j^\bullet\right\}_{j=0}^{\ell+1}$ *in* $\mathbb{K}^n$*, respectively, such that:*

- $\mathcal{F}_0^\bullet$ *is completely opposite to both* $\mathcal{F}_1^\bullet$ *and* $\mathcal{F}_{\ell+1}^\bullet$*, and for each* $j \in [1,\ell]$*,* $\mathcal{F}_j^\bullet$ *is in* s_{i_j}*-relative position with respect to* $\mathcal{F}_{j+1}^\bullet$*.*
- $\mathcal{G}_0^\bullet$ *is completely opposite to both* $\mathcal{G}_1^\bullet$ *and* $\mathcal{G}_{\ell+1}^\bullet$*, and for each* $j \in [1,\ell]$*,* $\mathcal{G}_j^\bullet$ *is in* s_{i_j}*-relative position with respect to* $\mathcal{G}_{j+1}^\bullet$*.*

In particular, by Lemma (4.35)*, we know that:*

$$\mathcal{F}_0^\bullet := {}_{\mathcal{K}}\mathcal{F}^\bullet\big(\psi_{\mathscr{F}}^{(1)}, \ldots, \psi_{\mathscr{F}}^{(n-1)}\big), \quad \textit{and} \quad \mathcal{F}_1^\bullet := {}_{\mathcal{I}}\mathcal{F}^\bullet\big(\phi_{\mathscr{F}}^{(1)}, \ldots, \phi_{\mathscr{F}}^{(n-1)}\big),$$

$$\mathcal{G}_0^\bullet := {}_{\mathcal{K}}\mathcal{F}^\bullet\big(\psi_{\mathscr{G}}^{(1)}, \ldots, \psi_{\mathscr{G}}^{(n-1)}\big), \quad \textit{and} \quad \mathcal{G}_1^\bullet := {}_{\mathcal{I}}\mathcal{F}^\bullet\big(\phi_{\mathscr{G}}^{(1)}, \ldots, \phi_{\mathscr{G}}^{(n-1)}\big),$$

are the type $\mathcal{K}$ *and type* $\mathcal{I}$ *flags in* $\mathbb{K}^n$ *associated with* $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$*,* $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$*,* $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$*, and* $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$*, respectively (cf. Definition* (4.18)–(1)–(2)*).*

⋆ *Main assumption (Adapted bases): Let* $\hat{\mathbf{f}}^{(n)}$*,* $\hat{\mathbf{g}}^{(n)}$ *be bases for* $\mathbb{K}^n$*, and let* $\vec{z} = (z_1, \ldots, z_\ell)$*,* $\vec{z}\,' = (z_1', \ldots, z_\ell') \in X(\beta, \mathbb{K})$ *be points such that the pairs* $(\hat{\mathbf{f}}^{(n)}, \vec{z})$ *and* $(\hat{\mathbf{g}}^{(n)}, \vec{z}\,')$ *algebraically characterizes* $\mathscr{F}$ *and* $\mathscr{G}$ *according to Theorem* (4.46)*, respectively. In this setting, we have that:*

- *Relative to the basis* $\hat{\mathbf{f}}^{(n)}$ *for* $\mathbb{K}^n$*:* $\mathcal{F}_0^\bullet$ *and* $\mathcal{F}_1^\bullet$ *are the anti-standard and standard flags, respectively. For each* $j \in [1,\ell]$*, the flag* $\mathcal{F}_{j+1}^\bullet$ *is represented by the path matrix* $P_{\beta_j}(\vec{z}_j) = B_{i_1}^{(n)}(z_1) \cdots B_{i_j}^{(n)}(z_j) \in \mathrm{GL}(n, \mathbb{K})$ *associated with the truncated braid word* $\beta_j = \sigma_{i_1} \cdots \sigma_{i_j} \in \mathrm{Br}_n^+$ *and the truncated tuple* $\vec{z}_j = (z_1, \ldots, z_j) \in \mathbb{K}^j_{\mathrm{std}}$*.*

- *Relative to the basis* $\hat{\mathbf{g}}^{(n)}$ *for* $\mathbb{K}^n$*:* $\mathcal{G}_0^\bullet$ *and* $\mathcal{G}_1^\bullet$ *are the anti-standard and standard flags, respectively. For each* $j \in [1,\ell]$*, the flag* $\mathcal{G}_{j+1}^\bullet$ *is represented by the path matrix* $P_{\beta_j}(\vec{z}'_j) = B_{i_1}^{(n)}(z'_1)\cdots B_{i_j}^{(n)}(z'_j) \in \mathrm{GL}(n,\mathbb{K})$ *associated with the truncated braid word* β_j *and the truncated tuple* $\vec{z}'_j = (z'_1,\ldots,z'_j) \in \mathbb{K}^j_{\mathrm{std}}$.

Furthermore, under this assumption, the ***compatibility conditions*** *and an inductive argument ensure that, for each* $i \in [1,n-1]$*, there are unique bases* $\hat{\mathbf{f}}^{(i)} := \{\hat{f}_j^{(i)}\}_{j=1}^i$ *and* $\hat{\mathbf{g}}^{(i)} := \{\hat{g}_j^{(i)}\}_{j=1}^i$ *for* $\mathbb{K}^i$ *such that (cf. Definition (4.18)–(3)–(4)):*

- *The collection* $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ *is a system of bases adapted to both* $\{\psi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$ *and* $\{\phi_{\mathscr{F}}^{(i)}\}_{i=1}^{n-1}$.
- *The collection* $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ *is a system of bases adapted to both* $\{\psi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$ *and* $\{\phi_{\mathscr{G}}^{(i)}\}_{i=1}^{n-1}$.

Building on this, for each $j \in [1,\ell]$*, we denote by* $\{\hat{\mathbf{f}}^{(i)}[\beta_j,\vec{z}_j]\}_{i=1}^n$ *and* $\{\hat{\mathbf{g}}^{(i)}[\beta_j,\vec{z}'_j]\}_{i=1}^n$ *the braid-transformed bases obtained from* $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ *and* $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$ *via the truncated braid word* β_j *and the truncated tuples* $\vec{z}_j$ *and* $\vec{z}'_j$*, respectively (cf. Definition (4.29)). Accordingly, by Theorem (4.50), we have that:*

- $\left(\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n, z_1\right)$ *is a system of bases adapted to* $\mathscr{F}$ *on* R_1.
- *For each* $j \in [1,\ell-1]$*,* $\left(\{\hat{\mathbf{f}}^{(i)}[\beta_j,\vec{z}_j]\}_{i=1}^n, z_{j+1}\right)$ *is a system of bases adapted to* $\mathscr{F}$ *on* R_{j+1}.
- $\left(\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n, z'_1\right)$ *is a system of bases adapted to* $\mathscr{G}$ *on* R_1.
- *For each* $j \in [1,\ell-1]$*,* $\left(\{\hat{\mathbf{g}}^{(i)}[\beta_j,\vec{z}'_j]\}_{i=1}^n, z'_{j+1}\right)$ *is a system of bases adapted to* $\mathscr{G}$ *on* R_{j+1}.

$\star$ *Main Conclusion: Following Definition (5.55), let* $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ *be the linear map associated with the pair* $(\mathscr{F},\mathscr{G})$*. Then, under the given* ***setup****, there is an isomorphism of vector spaces*

$$\mathrm{Ext}^1(\mathscr{F},\mathscr{G}) \cong \operatorname{coker}\delta_{\mathscr{F},\mathscr{G}}\,.$$

Proof. To begin, let $\mathcal{S}_{\Lambda(\beta)}$ be the stratification of $\mathbb{R}^2$ induced by $\Lambda(\beta) \subset (\mathbb{R}^3,\xi_{\mathrm{std}})$, $\mathcal{U}_{\Lambda(\beta)} = \{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\}$ the open cover of $\mathbb{R}^2$ from Construction (4.33), and $\mathcal{R}_{\Lambda(\beta)} = \{R_j\}_{j=1}^\ell$ the partition of U_{B} into ℓ open vertical straps from Construction (4.36).

Fix an equivalence class $\xi \in \mathrm{Ext}^1(\mathscr{F},\mathscr{G})$. Let $\mathscr{H}$ and $\mathscr{H}'$ be two equivalent extensions of $\mathscr{F}$ by $\mathscr{G}$ representing ξ, and let $\lambda : \mathscr{H} \to \mathscr{H}'$ be a sheaf isomorphism realizing their equivalence.

Then, building on Observation (5.78), we have that:

- On U_{T}, $\mathscr{H}$ and $\mathscr{H}'$ are determined by two collections of $n-1$ characteristic maps

$$\{\psi_{\mathscr{H}}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1}, \quad \text{and} \quad \{\psi_{\mathscr{H}'}^{(i)} : \mathbb{K}^{i+1} \to \mathbb{K}^i\}_{i=1}^{n-1},$$

while λ is characterized by a collection of n linear maps $\{S_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^n$ such that

$$\psi_{\mathscr{H}'}^{(i)} = \psi_{\mathscr{H}}^{(i)} + S_\lambda^{(i)} \circ \psi_{\mathscr{F}}^{(i)} - \psi_{\mathscr{G}}^{(i)} \circ S_\lambda^{(i+1)},$$

for all $i \in [1, n-1]$.

- On U_{B}, $\mathscr{H}$ and $\mathscr{H}'$ are determined by two collections of $n-1$ characteristic maps

$$\{\phi_{\mathscr{H}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1}, \quad \text{and} \quad \{\phi_{\mathscr{H}'}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1},$$

while λ is characterized by a collection of n linear maps $\{T_\lambda^{(i)} : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^n$ such that

$$\phi_{\mathscr{H}'}^{(i)} = \phi_{\mathscr{H}}^{(i)} + T_\lambda^{(i+1)} \circ \phi_{\mathscr{F}}^{(i)} - \phi_{\mathscr{G}}^{(i)} \circ T_\lambda^{(i)},$$

for all $i \in [1, n-1]$.

According to Lemma (5.85), any extension of $\mathscr{F}$ by $\mathscr{G}$ is locally equivalent to the trivial extension $\mathbf{0}_{\mathrm{Ext}}$ on $U_{\mathrm{T}} \cup U_{\mathrm{L}}$, and hence we choose representatives $\mathscr{H}$ and $\mathscr{H}'$ such that $\phi_{\mathscr{H}}^{(i)} = \phi_{\mathscr{H}'}^{(i)} = 0$ and $\psi_{\mathscr{H}}^{(i)} = \psi_{\mathscr{H}'}^{(i)} = 0$, for all $i \in [1, n-1]$. Under these conditions, Lemma (5.85) further asserts that the maps characterizing λ on $U_{\mathrm{T}} \cup U_{\mathrm{L}}$ satisfy the following properties:

- For all $i \in [1, n]$, $S_\lambda^{(i)} = T_\lambda^{(i)}$.
- With respect to the bases $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$ and $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$,

$${}_{\hat{\mathbf{g}}^{(n)}}\big[\, T_\lambda^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}} \equiv D(\vec{u}), \quad \text{for some } \vec{u} = (u_1, \dots, u_n) \in \mathbb{K}^n_{\mathrm{std}},$$

$${}_{\hat{\mathbf{g}}^{(i)}}\big[\, T_\lambda^{(i)} \,\big]_{\hat{\mathbf{f}}^{(i)}} \equiv \text{principal } i \times i \text{ submatrix of } D(\vec{u}), \quad \text{for all } i \in [1, n-1].$$

Next, consider R_1, the vertical strap in $\mathbb{R}^2$ containing the first crossing of β, namely σ_{i_1}, and denote by $k = i_1 \in [1, n-1]$ the index of σ_{i_1}. By construction, R_1 is the vertical strap immediately to the right of U_{L}, and hence, the constructibility of $\mathscr{F}$ and $\mathscr{G}$ with respect to $\mathcal{S}_{\Lambda(\beta)}$ leads to one of the following two cases:

- *Case 1*: Suppose that $k = 1$. Then, on R_1, $\mathscr{F}$ and $\mathscr{G}$ are characterized by two collections of n injective linear maps:

$$\{\phi_{\mathscr{F}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}_{\mathscr{F}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2\},$$

$$\{\phi_{\mathscr{G}}^{(i)} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}_{\mathscr{G}}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^2\}.$$

respectively. Thus, in this local configuration, Lemma (5.87) asserts that, on R_1, $\mathscr{H}$ and $\mathscr{H}'$ are specified by two collections of n characteristic maps

$$\{\phi^{(i)}_{\mathscr{H}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(1)}_{\mathscr{H}} : \mathbb{K}^1 \to \mathbb{K}^2\},$$

$$\{\phi^{(i)}_{\mathscr{H}'} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(1)}_{\mathscr{H}'} : \mathbb{K}^1 \to \mathbb{K}^2\},$$

while λ is characterized by a collection of $n+1$ linear maps $\{T^{(i)}_\lambda : \mathbb{K}^i \to \mathbb{K}^i\} \cup \{\widetilde{T}^{(1)} : \mathbb{K}^1 \to \mathbb{K}^1\}$ such that

$$\phi^{(i)}_{\mathscr{H}'} = \phi^{(i)}_{\mathscr{H}} + T^{(i+1)}_\lambda \circ \phi^{(i)}_{\mathscr{F}} - \phi^{(i)}_{\mathscr{G}} \circ T^{(i)}_\lambda\,, \quad \text{for all } i \in [1, n-1],$$

$$\widetilde{\phi}^{(1)}_{\mathscr{H}'} = \widetilde{\phi}^{(1)}_{\mathscr{H}} + T^{(2)}_\lambda \circ \widetilde{\phi}^{(1)}_{\mathscr{F}} - \widetilde{\phi}^{(1)}_{\mathscr{G}} \circ \widetilde{T}^{(1)}_\lambda\,.$$

- *Case 2*: Suppose that $k \geq 2$. Then, on R_1, $\mathscr{F}$ and $\mathscr{G}$ are characterized by two collections of injective linear maps:

$$\{\phi^{(i)}_{\mathscr{F}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(k-1)}_{\mathscr{F}} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{F}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\},$$

$$\{\phi^{(i)}_{\mathscr{G}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(k-1)}_{\mathscr{G}} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{G}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\}.$$

Thus, in this local configuration, Lemma (5.90) asserts that, on R_1, $\mathscr{H}$ and $\mathscr{H}'$ are specified by two collections of $n+1$ characteristic maps

$$\{\phi^{(i)}_{\mathscr{H}} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(k-1)}_{\mathscr{H}} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{H}} : \mathbb{K}^k \to \mathbb{K}^{k+1}\},$$

$$\{\phi^{(i)}_{\mathscr{H}'} : \mathbb{K}^i \to \mathbb{K}^{i+1}\}_{i=1}^{n-1} \cup \{\widetilde{\phi}^{(k-1)}_{\mathscr{H}'} : \mathbb{K}^{k-1} \to \mathbb{K}^k,\ \widetilde{\phi}^{(k)}_{\mathscr{H}'} : \mathbb{K}^k \to \mathbb{K}^{k+1}\},$$

while λ is characterized by a collection of $n+1$ linear maps

$$\{T^{(i)}_\lambda : \mathbb{K}^i \to \mathbb{K}^i\}_{i=1}^{n} \cup \{\widetilde{T}^{(k)} : \mathbb{K}^k \to \mathbb{K}^k\}$$

such that:

$$\phi^{(i)}_{\mathscr{H}'} = \phi^{(i)}_{\mathscr{H}} + T^{(i+1)}_\lambda \circ \phi^{(i)}_{\mathscr{F}} - \phi^{(i)}_{\mathscr{G}} \circ T^{(i)}_\lambda\,, \qquad \text{for all } i \in [1, n-1]\,,$$

$$\widetilde{\phi}^{(k)}_{\mathscr{H}'} = \widetilde{\phi}^{(k)}_{\mathscr{H}} + T^{(k+1)}_\lambda \circ \widetilde{\phi}^{(k)}_{\mathscr{F}} - \widetilde{\phi}^{(k)}_{\mathscr{G}} \circ \widetilde{T}^{(k)}_\lambda\,,$$

$$\widetilde{\phi}^{(k-1)}_{\mathscr{H}'} = \widetilde{\phi}^{(k-1)}_{\mathscr{H}} + \widetilde{T}^{(k)}_\lambda \circ \widetilde{\phi}^{(k-1)}_{\mathscr{F}} - \widetilde{\phi}^{(k-1)}_{\mathscr{G}} \circ T^{(k-1)}_\lambda\,.$$

In any of the above cases, we have that $\{\phi^{(i)}_{\mathscr{F}}\}_{i=1}^{n-1}$, $\{\phi^{(i)}_{\mathscr{G}}\}_{i=1}^{n-1}$, $\{\phi^{(i)}_{\mathscr{H}}\}_{i=1}^{n-1}$, $\{\phi^{(i)}_{\mathscr{H}'}\}_{i=1}^{n-1}$, and $\{T^{(i)}_\lambda\}_{i=1}^{n}$ are precisely the linear maps determining $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}'$, $\mathscr{H}$, and λ on U_{L}, respectively. In particular, given our choice of representatives $\mathscr{H}$ and $\mathscr{H}'$, we have that $\phi^{(i)}_{\mathscr{H}} = \phi^{(i)}_{\mathscr{H}'} = 0$, for all $i \in [1, n-1]$.

Now, let $\vec{z} = (z_1, \ldots, z_\ell)$ and $\vec{z}\,' = (z'_1, \ldots, z'_\ell)$ be the points in the braid variety $X(\beta, \mathbb{K})$ that algebraically characterize $\mathscr{F}$ and $\mathscr{G}$ in accordance with Theorem (4.46), respectively. By Theorem (4.50), we know that:

- $\left(\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}, z_1\right)$ is a system of bases adapted to $\mathscr{F}$ on R_1.
- $\left(\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}, z_1'\right)$ is a system of bases adapted to $\mathscr{G}$ on R_1.

Then, a direct application of either Lemma (5.87) or Lemma (5.65) depending on the value of k, considering that $k = i_1$, implies that:

- **Parametrization of $\mathscr{H}$ and $\mathscr{H}'$ at σ_{i_1}:** On R_1, $\mathscr{H}$ and $\mathscr{H}'$ are determined by a pair of parameters $a_1, a_1' \in \mathbb{K}$, respectively, such that $a_1' = a_1 + \delta_1(z_1', \vec{u}, z)$, where the parameter controlling the gauge freedom in $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ after the crossing in R_1 is given by

$$\delta_1(z_1', \vec{u}, z_1) := \Big[\big(B_{i_1}^{(n)}(z_1')\big)^{-1} \cdot D(\vec{u}) \cdot B_{i_1}^{(n)}(z_1) \Big]_{i_1+1, i_1}.$$

  Hence, when restricting to $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, we have that $\delta_1(z_1', \vec{u}, z_1) \equiv 0$.

- Block diagonal properties of λ at σ_{i_1}: With respect to the bases $\left\{\hat{\mathbf{f}}^{(i)}[\sigma_{i_1}, z_1]\right\}_{i=1}^{n}$ and $\left\{\hat{\mathbf{g}}^{(i)}[\sigma_{i_1}, z_1']\right\}_{i=1}^{n}$,

$${}_{\hat{\mathbf{g}}^{(n)}[\sigma_{i_1}, z_1']}\big[\, T_\lambda^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}[\sigma_{i_1}, z_1]} \equiv D(\pi_{i_1}(\vec{u})),$$

  where $\pi_{i_1}(\vec{u})$ denotes the permutation of $\vec{u}$ associated with σ_{i_1}.

Applying the same reasoning to R_2, the vertical strap in $\mathbb{R}^2$ containing σ_{i_2}—the second crossing of β—we obtain that:

- **Parametrization of $\mathscr{H}$ and $\mathscr{H}'$ at σ_{i_2}:** On R_2, $\mathscr{H}$ and $\mathscr{H}'$ are determined by a pair of tuples $\vec{a}_2 := (a_1, a_2) \in \mathbb{K}^2$ and $\vec{a}\,'_2 := (a_1', a_2') \in \mathbb{K}^2$, respectively, such that $a_2' = a_2 + \delta_2(z_2', \vec{u}, z_2)$, where:
  - a_1 and a_1' are the scalar parameterizing $\mathscr{H}$ and $\mathscr{H}'$ in R_1.
  - The additional parameter controlling the gauge freedom in $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ after the crossing in R_2 is given by

$$\delta_2(z_2', \vec{u}, z_2) := \Big[\big(B_{i_2}^{(n)}(z_2')\big)^{-1} \cdot D(\pi_1(\vec{u})) \cdot B_{i_2}^{(n)}(z_2) \Big]_{i_2+1, i_2}.$$

  Hence, when restricting to $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, we have that $\delta_2(z_2', \vec{u}, z_2) \equiv 0$.

- Block diagonal properties of λ at σ_{i_2}: With respect to $\left\{\hat{\mathbf{f}}^{(i)}[\beta_2, \vec{z}_2]\right\}_{i=1}^{n}$ and $\left\{\hat{\mathbf{g}}^{(i)}[\beta_2, \vec{z}\,'_2]\right\}_{i=1}^{n}$,

$$\begin{aligned} {}_{\hat{\mathbf{g}}^{(n)}[\beta_2, \vec{z}\,'_2]}\big[\, T_\lambda^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}[\beta_2, \vec{z}_2]} &\equiv D(\pi_{i_2}(\pi_{i_1}(\vec{u}))), \\ &\equiv D(\pi_{\beta_2}(\vec{u})), \end{aligned}$$

  where $\pi_{\beta_2}(\vec{u})$ denotes the permutation of $\vec{u}$ associated with the truncated braid word $\beta_2 = \sigma_{i_1} \cdot \sigma_{i_2}$.

Proceeding iteratively through the all vertical straps R_j, we obtain that $\mathscr{H}$ and $\mathscr{H}'$ are globally parametrized by tuples $\vec{a} = (a_1, \ldots, a_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$ and $\vec{a}\,' = (a'_1, \ldots, a'_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$ such that $a'_j = a_j + \delta_j(z'_j, \vec{u}, z_j)$, where the gauge parameters are defined by

$$\delta_j(z'_j, \vec{u}, z_j) := \Big[\big(B^{(n)}_{i_j}(z'_j)\big)^{-1} \cdot D(\pi_{\beta_{j-1}}(\vec{u})) \cdot B^{(n)}_{i_j}(z_j) \Big]_{i_j+1, i_j},$$

with $\pi_{\beta_{j-1}}(\vec{u})$ denoting the permutation of $\vec{u}$ associated with the truncated braid word $\beta_{j-1} = \sigma_{i_1} \cdots \sigma_{i_{j-1}} \in \mathrm{Br}^+_n$, for all $j \in [1, \ell]$.

Finally, building on Definition (5.55), we observe that two extensions $\mathscr{H}(\vec{a})$ and $\mathscr{H}'(\vec{a}\,')$ represent the same equivalence class in $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ if and only if the tuples $\vec{a}$, $\vec{a}\,' \in \mathbb{K}^\ell_{\mathrm{std}}$ that algebraically parametrize them satisfy $\vec{a}\,' - \vec{a} \in \operatorname{im} \delta_{\mathscr{F},\mathscr{G}} \subseteq \mathbb{K}^\ell_{\mathrm{std}}$. Hence, we conclude that

$$\mathrm{Ext}^1(\mathscr{F}, \mathscr{G}) \cong \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}}\,.$$

□

As an immediate yet striking consequence of Theorems (5.66) and (5.92), we obtain the following result, which not only provides a powerful computational tool but also offers profound insight into the rich topological information that the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ carries about the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$.

Theorem 5.93. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}^+_n$ be a positive braid word, and let $\mathscr{F}$, $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$ be bases for $\mathbb{K}^n$, and let $\vec{z}$, $\vec{z}\,' \in X(\beta, \mathbb{K})$ be points such that the pairs $\big(\hat{\mathbf{f}}^{(n)}, \vec{z}\big)$ and $\big(\hat{\mathbf{g}}^{(n)}, \vec{z}\,'\big)$ algebraically parametrize $\mathscr{F}$ and $\mathscr{G}$ according to Theorem (4.46), respectively. Then, the Poincaré–Euler characteristic of the graded vector space $\mathrm{Ext}^\bullet(\mathscr{F}, \mathscr{G})$ is given by*

$$\chi\left(\mathrm{Ext}^\bullet(\mathscr{F}, \mathscr{G})\right) = -\mathrm{tb}(\Lambda(\beta))\,,$$

where $\mathrm{tb}(\Lambda(\beta)) := \ell - n$ denotes the Thurston–Bennequin number of the Legendrian link $\Lambda(\beta) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$.

Proof. To begin, following Definition (5.55), let $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ be the linear map associated with the pair $(\mathscr{F}, \mathscr{G})$. In particular, by the rank–nullity theorem, we know that

$$\dim_{\mathbb{K}} \ker \delta_{\mathscr{F},\mathscr{G}} - \dim_{\mathbb{K}} \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}} = n - \ell\,.$$

Moreover, by Theorems (5.66) and (5.92), we have that the lower-degree morphism spaces between $\mathscr{F}$ and $\mathscr{G}$ are given by

$$\mathrm{Ext}^0(\mathscr{F}, \mathscr{G}) \cong \ker \delta_{\mathscr{F},\mathscr{G}}\,,$$

$$\mathrm{Ext}^1(\mathscr{F}, \mathscr{G}) \cong \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}}\,.$$

Hence, putting the above results together, we obtain that

$$\begin{aligned}\chi\left(\operatorname{Ext}^\bullet(\mathscr{F},\mathscr{G})\right) &= \sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{K}} \operatorname{Ext}^i(\mathscr{F},\mathscr{G})\,,\\ &= \dim_{\mathbb{K}} \operatorname{Ext}^0(\mathscr{F},\mathscr{G}) - \dim_{\mathbb{K}} \operatorname{Ext}^1(\mathscr{F},\mathscr{G})\,,\\ &= \dim_{\mathbb{K}} \ker \delta_{\mathscr{F},\mathscr{G}} - \dim_{\mathbb{K}} \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}}\,,\\ &= n-\ell\,.\end{aligned}$$

Finally, since the Thurston–Bennequin number of the Legendrian link $\Lambda(\beta)$ is given by $\operatorname{tb}(\Lambda(\beta)) = \ell - n$ [KÓ6], the result follows. □

Having analyzed the linear structure of the graded morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$, we now turn to the study of their compositions.

5.5. **Combinatorial Composition Rules in the Category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.** Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. This subsection is devoted to studying the composition of graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. More precisely, our main goal is to establish some combinatorial rules that describe how these graded morphisms compose. Bearing this in mind, we begin by introducing some preliminaries.

5.5.1. *Technical Background.* Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. Next, we now collect some preliminaries that will play a fundamental role in the explicit computation of the composition of the graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. In particular, we open the discussion with the following definition.

Definition 5.94. *Let $\mathcal{M}$ be a smooth manifold. Let $\mathscr{F}$, $\mathscr{F}'$, and $\mathscr{G}$ be sheaves of $\mathbb{K}$-modules on $\mathcal{M}$, and let $\lambda : \mathscr{F}' \to \mathscr{F}$ be a sheaf homomorphism between $\mathscr{F}'$ and $\mathscr{F}$. Suppose $\mathscr{H}$ is an extension of $\mathscr{F}$ by $\mathscr{G}$. Then, we define the pull-back $\lambda^*\mathscr{H}$ of $\mathscr{H}$ via λ to be an extension of $\mathscr{F}'$ by $\mathscr{G}$ such that there exists a sheaf homomorphism $\alpha : \lambda^*\mathscr{H} \to \mathscr{H}$ making each square in the diagram in Figure (36) commute.*

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d, equal] & \lambda^*\mathscr{H} \arrow[r] \arrow[d, "\alpha"] & \mathscr{F}' \arrow[r] \arrow[d, "\lambda"] & 0 \\ 0 \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{H} \arrow[r] & \mathscr{F} \arrow[r] & 0 \end{tikzcd}$$

FIGURE 36. The pull-back $\lambda^*\mathscr{H}$ of an extension $\mathscr{H}$ of $\mathscr{F}$ by $\mathscr{G}$ via a sheaf homomorphism λ between $\mathscr{F}'$ and $\mathscr{F}$.

Definition 5.95. *Let $\mathcal{M}$ be a smooth manifold. Let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{G}'$ be sheaves of $\mathbb{K}$-modules on $\mathcal{M}$, and let $\lambda : \mathscr{G} \to \mathscr{G}'$ be a sheaf homomorphism between $\mathscr{G}$ and $\mathscr{G}'$. Suppose $\mathscr{H}$ is an extension of $\mathscr{F}$ by $\mathscr{G}$. Then, we* *define the push-out $\lambda_* \mathscr{H}$ of $\mathscr{H}$ via λ to be an extension of $\mathscr{F}$ by $\mathscr{G}'$ such that there exists a sheaf homomorphism $\alpha : \mathscr{H} \to \lambda_* \mathscr{H}$ making each square in the diagram in Figure (37) commute.*

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d, "\lambda"] & \mathscr{H} \arrow[r] \arrow[d, "\alpha"] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{G}' \arrow[r] & \lambda_* \mathscr{H} \arrow[r] & \mathscr{F} \arrow[r] & 0 \end{tikzcd}$$

FIGURE 37. The push-out $\lambda_* \mathscr{H}$ of an extension $\mathscr{H}$ of $\mathscr{F}$ by $\mathscr{G}$ via a sheaf homomorphism λ between $\mathscr{G}$ and $\mathscr{G}'$.

With these definitions in place, we now present a formal and concrete definition of the composition of graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Definition 5.96. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. For any $p, q \geq 0$, the graded composition*

$$\circ : \mathrm{Ext}^p(\mathscr{G}, \mathscr{Q}) \times \mathrm{Ext}^q(\mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^{p+q}(\mathscr{F}, \mathscr{Q}) \,,$$

is given by the Yoneda composition for Ext *groups [HS71], which in degrees* 0 *and* 1 *is defined as follows:*

(i) Let $\lambda \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{G})$ and $\lambda' \in \mathrm{Ext}^0(\mathscr{G}, \mathscr{Q})$. Then $\lambda' \circ \lambda \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{Q})$ is given by the usual composition between sheaf homomorphisms.

(ii) Let $\lambda \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{G})$, $\xi' \in \mathrm{Ext}^1(\mathscr{G}, \mathscr{Q})$, and $\mathscr{H}'$ be an extension of $\mathscr{G}$ by $\mathscr{Q}$ representing the equivalence class ξ', namely $\left[\, \mathscr{H}' \,\right] = \xi'$. Then $\xi' \circ \lambda = \left[\, \lambda^* \mathscr{H}' \,\right] \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{Q})$.

(iii) Let $\xi \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, $\lambda' \in \mathrm{Ext}^0(\mathscr{G}, \mathscr{Q})$, and $\mathscr{H}$ be an extension of $\mathscr{F}$ by $\mathscr{G}$ representing the equivalence class ξ, namely $\left[\, \mathscr{H} \,\right] = \xi$. Then $\lambda' \circ \xi = \left[\, {\lambda'}_* \mathscr{H} \,\right] \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{Q})$.

Next, we present several technical lemmas that will enable us to explicitly describe the composition of graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. To this end, we introduce the following definitions.

Definition 5.97. *Let A, B, C, C', X, Y, Z, Z' be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\Gamma_{\mathscr{H}} : B \to Y$, $\gamma_{\mathscr{F}} : C \to Z$, and $\gamma_{\mathscr{F}'} : C' \to Z'$ be linear maps, with $\Gamma_{\mathscr{H}}$ an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ (see Definition (5.70)).*

Let $\lambda_\xi : C' \to C$ and $\lambda_\delta : Z' \to Z$ be linear maps defining a morphism $\lambda_{\xi,\delta} := (\lambda_\xi, \lambda_\delta)$ between $\gamma_{\mathscr{F}'}$ and $\gamma_{\mathscr{F}}$ (see Definition (5.57)). Then, the pull-back ${\lambda_{\xi,\delta}}^* \Gamma_{\mathscr{H}}$ of $\Gamma_{\mathscr{H}}$ via $\lambda_{\xi,\delta}$ is an extension of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$. Concretely, it is a linear map ${\lambda_{\xi,\delta}}^* \Gamma_{\mathscr{H}} : B' \to Y'$ between vector spaces B' and Y' over $\mathbb{K}$, such that there exist linear maps $W_1 : B' \to B$ and $W_2 : Y' \to Y$ making each square of the diagram in Figure (38) commute.

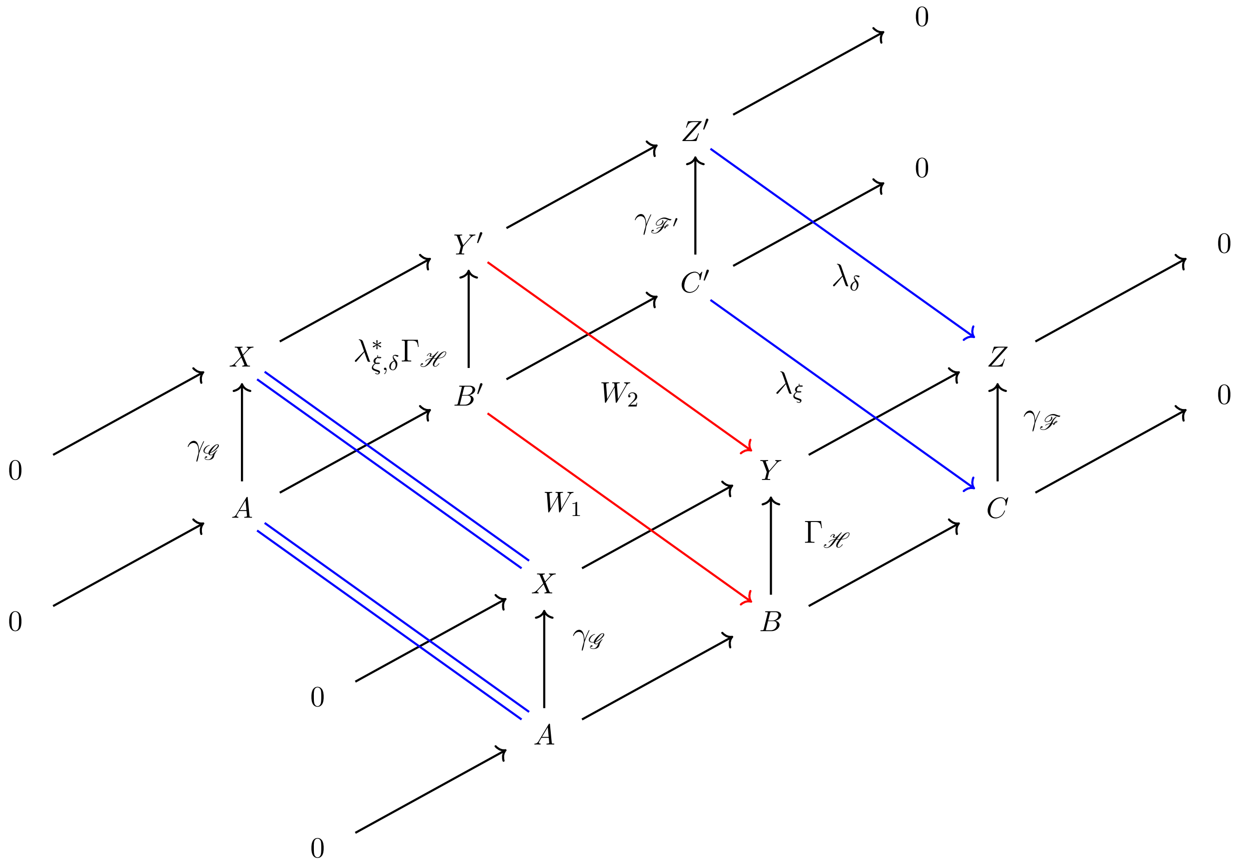


FIGURE 38. The pull-back $\lambda^*_{\xi,\delta}\Gamma_{\mathscr{H}}$ of the extension $\Gamma_{\mathscr{H}}$ of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ via the morphism $\lambda_{\xi,\delta}$ between $\gamma_{\mathscr{F}'}$ and $\gamma_{\mathscr{F}}$.

Definition 5.98. *Let A, A', B, C, X, X', Y, Z be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\gamma_{\mathscr{G}'} : A' \to X'$, $\Gamma_{\mathscr{H}} : B \to Y$, and $\gamma_{\mathscr{F}} : C \to Z$ be linear maps, with $\Gamma_{\mathscr{H}}$ an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ (see Definition (5.70)).*

Let $\lambda_\xi : A \to A'$ and $\lambda_\delta : X \to X'$ be linear maps defining a morphism $\lambda_{\xi,\delta} := (\lambda_\xi, \lambda_\delta)$ between $\gamma_{\mathscr{G}}$ and $\gamma_{\mathscr{G}'}$ (see Definition (5.57)). Then, the push-out $\lambda_{\xi,\delta_*}\Gamma_{\mathscr{H}}$ of $\Gamma_{\mathscr{H}}$ via $\lambda_{\xi,\delta}$ is an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$. Concretely, it is a linear map $\lambda_{\xi,\delta_*}\Gamma_{\mathscr{H}} : B' \to Y'$ between vector spaces B' and Y' over $\mathbb{K}$, such that there exist linear maps $W_1 : B \to B'$ and $W_2 : Y \to Y'$ making each square of the diagram in Figure (39) commute.

Lemma 5.99. *Let A, C, C', X, Z, Z' be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\gamma_{\mathscr{H}} : C \to X$, $\gamma_{\mathscr{F}} : C \to Z$, and $\gamma_{\mathscr{F}'} : C' \to Z'$ be linear maps.*

Let $\lambda_\xi : C' \to C$ and $\lambda_\delta : Z' \to Z$ be linear maps defining a morphism $\lambda_{\xi,\delta} := (\lambda_\xi, \lambda_\delta)$ between $\gamma_{\mathscr{F}'}$ and $\gamma_{\mathscr{F}}$, and let ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}} : A \oplus C \to X \oplus Z$ be the block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{H}}$ (see Lemma (5.74)).

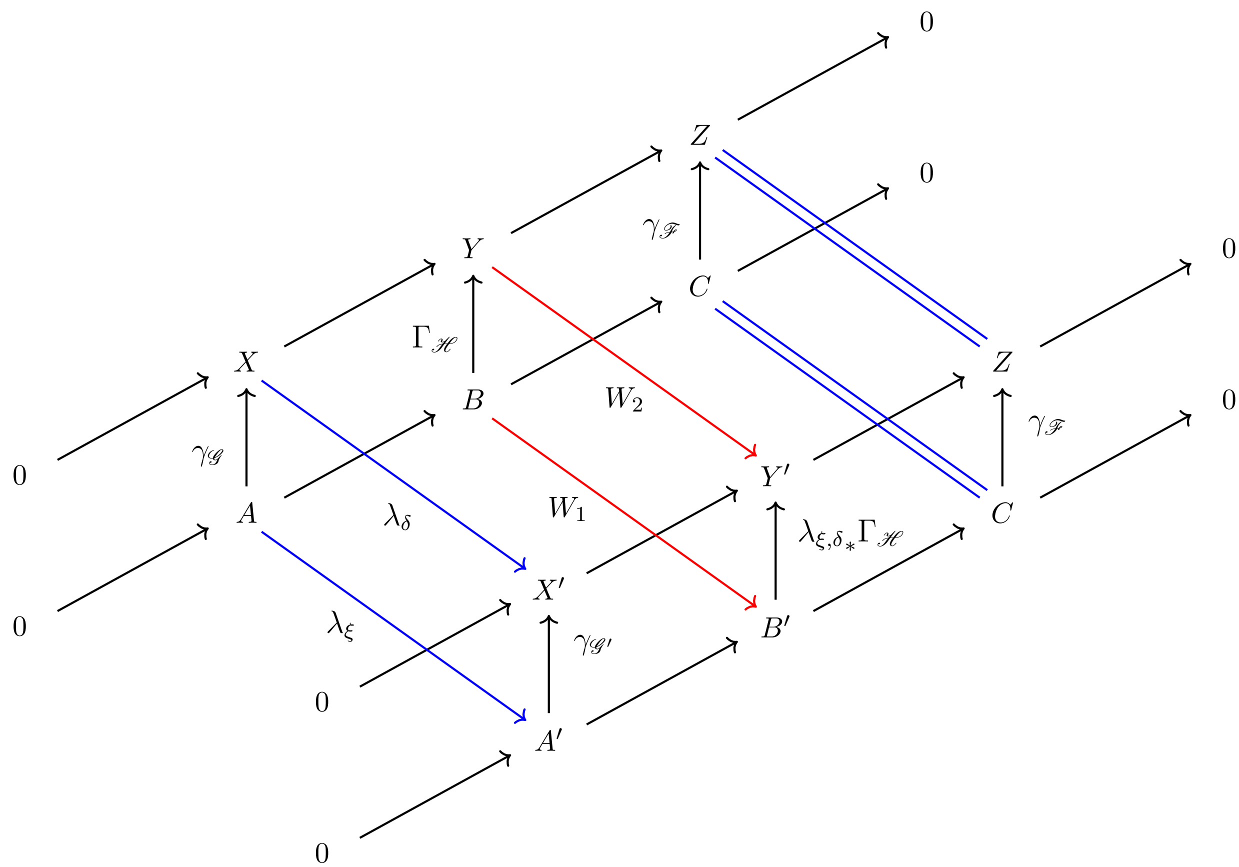


FIGURE 39. The push-out $\lambda_{\xi,\delta_*}\Gamma_{\mathscr{H}}$ of the extension $\Gamma_{\mathscr{H}}$ of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ via the morphism $\lambda_{\xi,\delta}$ between $\gamma_{\mathscr{G}}$ and $\gamma_{\mathscr{G}'}$.

Finally, let $\lambda^*_{\xi,\delta\,\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}} : B \to Y$ *be the pull-back of* ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ *via* $\lambda_{\xi,\delta}$*, see Definition* (5.97)*, and let* ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{Q}}\rangle_{\gamma_{\mathscr{F}}} : A \oplus C' \to X \oplus Z'$ *be the block extension of* $\gamma_{\mathscr{F}'}$ *by* $\gamma_{\mathscr{G}}$ *associated with the linear map* $\gamma_{\mathscr{Q}} := \gamma_{\mathscr{H}} \circ \lambda_\xi :$ $C' \to X$*. Then,* $\lambda^*_{\xi,\delta\,\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ *and* ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{Q}}\rangle_{\gamma_{\mathscr{F}}}$ *are equivalent extensions of* $\gamma_{\mathscr{F}'}$ *by* $\gamma_{\mathscr{G}}$*.*

Proof. By definition, $\lambda^*_{\xi,\delta\,\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ is an extension of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$, and hence, Lemma (5.75) asserts that there exists a linear map $\gamma_{\mathscr{R}} : C' \to X$ such that $\lambda^*_{\xi,\delta\,\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{R}}\rangle_{\gamma_{\mathscr{F}'}}$ are equivalent extensions of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$, where ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{R}}\rangle_{\gamma_{\mathscr{F}'}} : A \oplus C' \to X \oplus Z'$ denotes the block extension of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{R}}$.

By Definitions (5.71) and (5.97), there exist linear maps $N_1 : A \oplus C' \to A \oplus C$ and $N_2 : X \oplus Z' \to X \oplus Z$ making each square of the diagram in Figure (40) commute. In particular, observe that there exist linear maps $\lambda^{(1)}_{N_1} : A \to A$, $\lambda^{(2)}_{N_1} : C' \to A$, $\lambda^{(3)}_{N_1} : A \to C$, and $\lambda^{(4)}_{N_1} : C' \to C$ such that N_1 can be decomposed as

$$N_1 = \begin{bmatrix} \lambda^{(1)}_{N_1} & \lambda^{(2)}_{N_1} \\ \lambda^{(3)}_{N_1} & \lambda^{(4)}_{N_1} \end{bmatrix}.$$

Similarly, there exist linear maps $\lambda^{(1)}_{N_2} : X \to X$, $\lambda^{(2)}_{N_2} : Z' \to X$, $\lambda^{(3)}_{N_2} : X \to Z$, and $\lambda^{(4)}_{N_2} : Z' \to Z$ such that N_2 can be expressed as

$$N_2 = \begin{bmatrix} \lambda^{(1)}_{N_2} & \lambda^{(2)}_{N_2} \\ \lambda^{(3)}_{N_2} & \lambda^{(4)}_{N_2} \end{bmatrix} .$$

It then follows from the commutativity of the diagram in Figure (40) that

$$\lambda^{(1)}_{N_1} = \mathrm{id}_A \,,$$

$$\lambda^{(3)}_{N_1} = 0 \,,$$

$$\lambda^{(4)}_{N_1} = \lambda_\xi \,,$$

$$\lambda^{(1)}_{N_2} = \mathrm{id}_X \,,$$

$$\lambda^{(3)}_{N_2} = 0 \,,$$

$$\lambda^{(4)}_{N_2} = \lambda_\delta \,,$$

$$\lambda_\delta \circ \gamma_{\mathscr{F}'} = \gamma_{\mathscr{F}} \circ \lambda_\xi \,,$$

$$\gamma_{\mathscr{R}} + \lambda^{(2)}_{N_2} \circ \gamma_{\mathscr{F}'} = \gamma_{\mathscr{H}} \circ \lambda_\xi + \gamma_{\mathscr{G}} \circ \lambda^{(2)}_{N_1} \,.$$

Thus, defining $\gamma_{\mathscr{Q}} = \gamma_{\mathscr{H}} \circ \lambda_\xi : C' \to X$, the above relations show that there exist linear maps $\lambda^{(2)}_{N_1} : C' \to A$ and $\lambda^{(2)}_{N_2} : Z' \to X$ such that

$$\gamma_{\mathscr{Q}} = \gamma_{\mathscr{R}} + \lambda^{(2)}_{N_2} \circ \gamma_{\mathscr{F}'} - \gamma_{\mathscr{G}} \circ \lambda^{(2)}_{N_1}.$$

Hence, building on Lemma (5.75), we conclude that ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{R}}\rangle_{\gamma_{\mathscr{F}'}}$ and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{Q}}\rangle_{\gamma_{\mathscr{F}'}}$ are equivalent extensions of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$, where ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{Q}}\rangle_{\gamma_{\mathscr{F}'}} : A \oplus C' \to X \oplus Z'$ denotes the block extension of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{Q}}$. Finally, by transitivity of equivalence, we deduce that $\lambda^*_{\xi,\delta}\,{}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{Q}}\rangle_{\gamma_{\mathscr{F}'}}$ are equivalent extensions of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$. This completes the proof.

□

Lemma 5.100. *Let A, A', C, X, X', Z be vector spaces over $\mathbb{K}$, and let $\gamma_{\mathscr{G}} : A \to X$, $\gamma_{\mathscr{G}'} : A' \to X'$, $\gamma_{\mathscr{H}} : C \to X$, and $\gamma_{\mathscr{F}} : C \to Z$ be linear maps.*

Let $\lambda_\xi : A \to A'$ and $\lambda_\delta : X \to X'$ be linear maps defining a morphism $\lambda_{\xi,\delta} = (\lambda_\xi, \lambda_\delta)$ between $\gamma_{\mathscr{G}}$ and $\gamma_{\mathscr{G}'}$, and let ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}} : A \oplus C \to X \oplus Z$ be the block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}}$ associated with $\gamma_{\mathscr{H}}$ (see Lemma (5.74)).

Finally, let $\lambda_{\xi,\delta\,*}\,{}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}} : B \to Y$ be the push-out of ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ via $\lambda_{\xi,\delta}$, see Definition (5.98), and let ${}_{\gamma_{\mathscr{G}'}}\langle\gamma_{\mathscr{Q}}\rangle_{\gamma_{\mathscr{F}}} : A' \oplus C \to X' \oplus Z$ be the block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$ associated with the linear map $\gamma_{\mathscr{Q}} := \lambda_\delta \circ \gamma_{\mathscr{H}} : C \to X'$. Then, $\lambda_{\xi,\delta\,*}\,{}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}'}}\langle\gamma_{\mathscr{Q}}\rangle_{\gamma_{\mathscr{F}}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$.

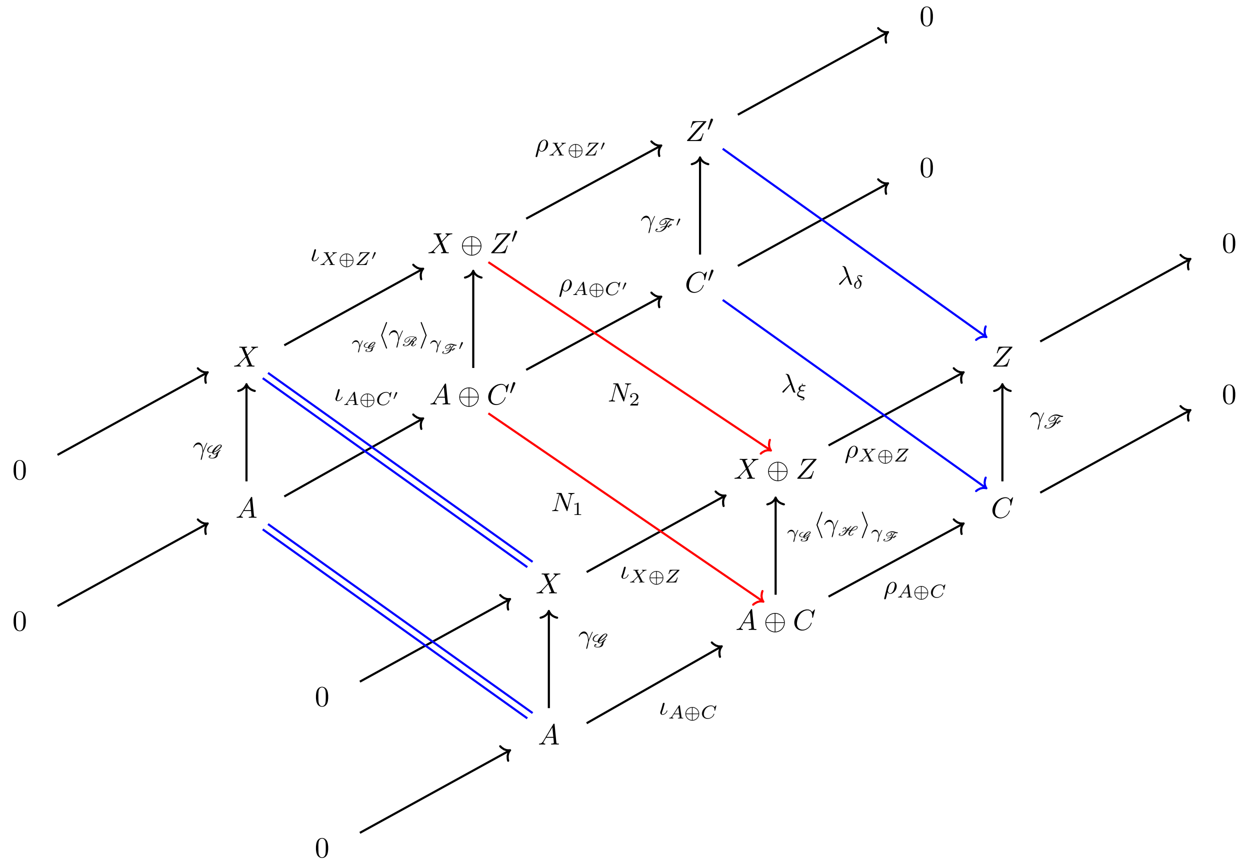


FIGURE 40. Commutative diagram between the block extensions ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{R}}\rangle_{\gamma_{\mathscr{F}'}}$ and ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$.

Proof. By definition, $\lambda_{\xi,\delta_*\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ is an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$, and hence, Lemma (5.75) asserts that there exists a linear map $\gamma_{\mathscr{R}} : C \to X'$ such that $\lambda_{\xi,\delta_*\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}'}}\langle\gamma_{\mathscr{R}}\rangle_{\gamma_{\mathscr{F}}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$, where ${}_{\gamma_{\mathscr{G}'}}\langle\gamma_{\mathscr{R}}\rangle_{\gamma_{\mathscr{F}}} : A' \oplus C \to X' \oplus Z$ denotes the block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$ associated with $\gamma_{\mathscr{R}}$.

By Definitions (5.71) and (5.98), there exist linear maps $N_1 : A \oplus C \to A' \oplus C$ and $N_2 : X \oplus Z \to X' \oplus Z$ making each square of the diagram in Figure (41) commute. In particular, observe that there exist linear maps $\lambda_{N_1}^{(1)} : A \to A'$, $\lambda_{N_1}^{(2)} : C \to A'$, $\lambda_{N_1}^{(3)} : A \to C$, and $\lambda_{N_1}^{(4)} : C \to C$ such that N_1 can be decomposed as

$$N_1 = \begin{bmatrix} \lambda_{N_1}^{(1)} & \lambda_{N_1}^{(2)} \\ \lambda_{N_1}^{(3)} & \lambda_{N_1}^{(4)} \end{bmatrix}.$$

Similarly, there exist linear maps $\lambda_{N_2}^{(1)} : X \to X'$, $\lambda_{N_2}^{(2)} : Z \to X'$, $\lambda_{N_2}^{(3)} : X \to Z$, and $\lambda_{N_2}^{(4)} : Z \to Z$ such that N_2 can be expressed as

$$N_2 = \begin{bmatrix} \lambda_{N_2}^{(1)} & \lambda_{N_2}^{(2)} \\ \lambda_{N_2}^{(3)} & \lambda_{N_2}^{(4)} \end{bmatrix}.$$

It then follows from the commutativity of the diagram in Figure (41) that

$$\lambda^{(1)}_{N_1} = \lambda_\xi \,,$$

$$\lambda^{(3)}_{N_1} = 0 \,,$$

$$\lambda^{(4)}_{N_1} = \mathrm{id}_C \,,$$

$$\lambda^{(1)}_{N_2} = \lambda_\delta \,,$$

$$\lambda^{(3)}_{N_2} = 0 \,,$$

$$\lambda^{(4)}_{N_2} = \mathrm{id}_Z \,,$$

$$\lambda_\delta \circ \gamma_{\mathscr{G}} = \gamma_{\mathscr{G}'} \circ \lambda_\xi \,,$$

$$\gamma_{\mathscr{R}} + \gamma_{\mathscr{G}'} \circ \lambda^{(2)}_{N_1} = \lambda_\delta \circ \gamma_{\mathscr{H}} + \lambda^{(2)}_{N_2} \circ \gamma_{\mathscr{F}} \,.$$

Thus, defining $\gamma_{\mathscr{Q}} := \lambda_\delta \circ \gamma_{\mathscr{H}} : C \to X'$, the above relations show that there exist linear maps $\lambda^{(2)}_{N_1} : C \to A'$ and $\lambda^{(2)}_{N_2} : Z \to X'$ such that

$$\gamma_{\mathscr{R}} = \gamma_{\mathscr{Q}} + \lambda^{(2)}_{N_2} \circ \gamma_{\mathscr{F}} - \gamma_{\mathscr{G}'} \circ \lambda^{(2)}_{N_1}.$$

Hence, building on Lemma (5.76), we conclude that ${}_{\gamma_{\mathscr{G}'}}\langle \gamma_{\mathscr{Q}} \rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}'}}\langle \gamma_{\mathscr{R}} \rangle_{\gamma_{\mathscr{F}}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$, where ${}_{\gamma_{\mathscr{G}'}}\langle \gamma_{\mathscr{Q}} \rangle_{\gamma_{\mathscr{F}}} : A' \oplus C \to X' \oplus Z$ denotes the block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$ associated with $\gamma_{\mathscr{Q}}$. Finally, by transitivity of equivalence, we deduce that $\lambda_{\xi, \delta_* \gamma_{\mathscr{G}}}\langle \gamma_{\mathscr{H}} \rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}'}}\langle \gamma_{\mathscr{Q}} \rangle_{\gamma_{\mathscr{F}}}$ are equivalent extensions of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$.

□

Having established the above technical lemmas, we now present an observation that will elucidate the structure of the composition of mixed graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Observation 5.101. *Let $\beta = \sigma_{i_1} \dots \sigma_{i_\ell} \in \mathrm{Br}^+_n$ be a positive braid word, and let $\mathcal{S}_{\Lambda(\beta)}$ denote the stratification of $\mathbb{R}^2$ induced by $\Lambda(\beta)$. Next, we analyze the composition of graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ near an arbitrary arc in $\mathcal{S}_{\Lambda(\beta)}$.*

To begin, let a be an arc in $\mathcal{S}_{\Lambda(\beta)}$. As illustrated in Sub-figure (4a), near a, the stratification $\mathcal{S}_{\Lambda(\beta)}$ consists of a, an upper 2-dimensional stratum U, and a lower 2-dimensional stratum D. In particular, given this local configuration, we choose two arbitrary points $p \in U$ and $q \in D$.

Let $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{G}$, and $\mathscr{G}'$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. In addition, fix $\lambda \in \mathrm{Ext}^0(\mathscr{F}', \mathscr{F})$, $\xi \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, and $\lambda' \in \mathrm{Ext}^0(\mathscr{G}, \mathscr{G}')$, and let $\mathscr{H}$ be an extension of $\mathscr{F}$ by $\mathscr{G}$ representing ξ, namely $\xi = \left[\, \mathscr{H} \,\right]$.

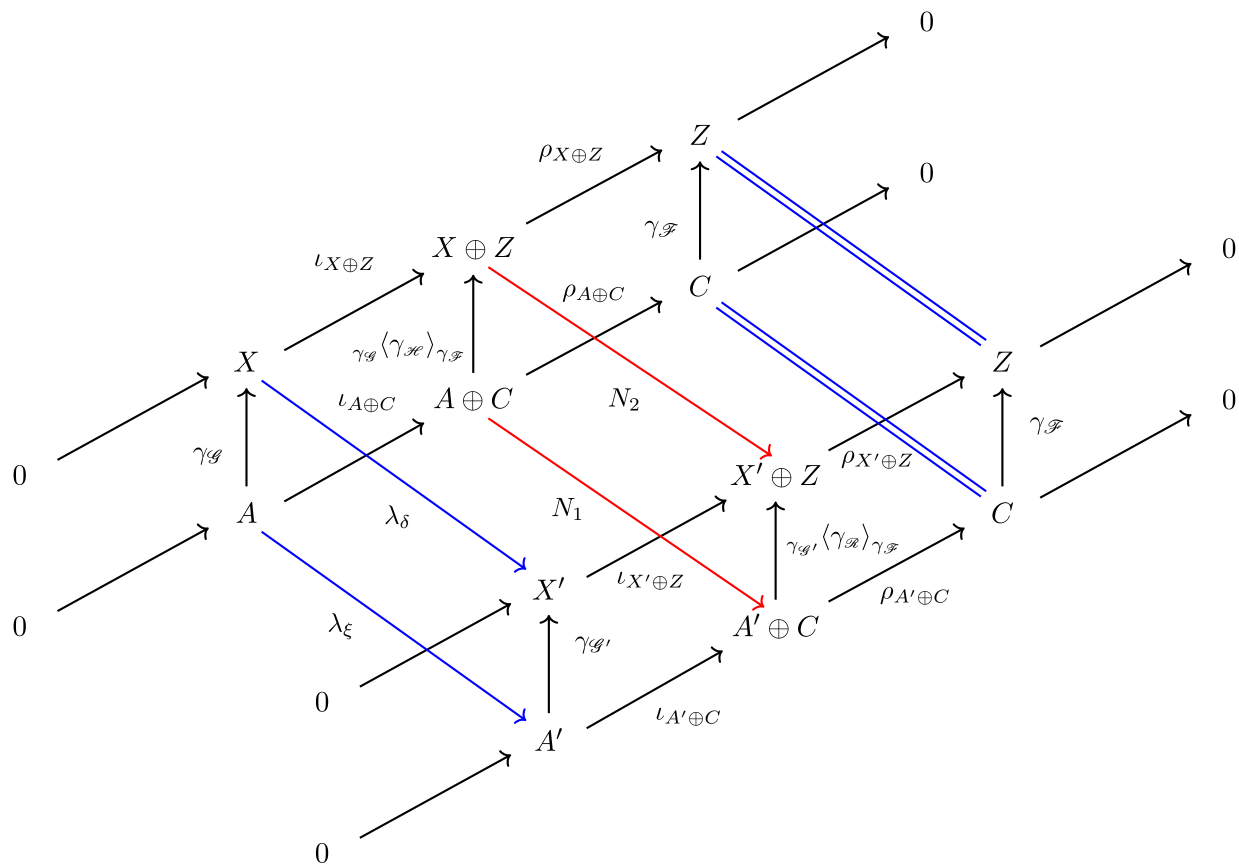


FIGURE 41. Commutative diagram between the block extensions ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ and ${}_{\gamma_{\mathscr{G}'}}\langle\gamma_{\mathscr{R}}\rangle_{\gamma_{\mathscr{F}}}$.

Bearing this in mind, our goal is to provide local models for the compositions $\lambda \circ \xi = \left[\, \lambda^* \mathscr{H} \,\right] \in \operatorname{Ext}^1(\mathscr{F}', \mathscr{G})$ *and* $\xi \circ \lambda' = \left[\, \lambda_* \mathscr{H} \,\right] \in \operatorname{Ext}^1(\mathscr{F}, \mathscr{G}')$ *near* a*, where* $\lambda^*\mathscr{H}$ *and* $\lambda'_*\mathscr{H}$ *denote the pull-back and push-out of* $\mathscr{H}$ *via* λ *and* λ'*, respectively.*

Now, consider the points p *and* q*, and denote the stalks of the sheaves* $\mathscr{G}$, $\mathscr{G}'$, $\mathscr{H}$, $\mathscr{F}$, *and* $\mathscr{F}'$ *at these points by*

$$X = \mathscr{G}_p\,, \qquad X' = \mathscr{G}'_{\,p}\,, \qquad Y = \mathscr{H}_p\,, \qquad Z = \mathscr{F}_p\,, \qquad Z' = \mathscr{F}'_p\,,$$

$$A = \mathscr{G}_q\,, \qquad A' = \mathscr{G}'_{\,q}\,, \qquad B = \mathscr{H}_q\,, \qquad C = \mathscr{F}_q\,, \qquad C' = \mathscr{F}'_q\,.$$

Then, according to the microlocal support conditions, near a*, the sheaves* $\mathscr{G}$, $\mathscr{G}'$, $\mathscr{H}$, $\mathscr{F}$, *and* $\mathscr{F}'$ *are specified by linear maps* $\gamma_{\mathscr{G}} : A \to X$, $\gamma_{\mathscr{G}'} : A' \to X'$, $\Gamma_{\mathscr{H}} : B \to Y$, $\gamma_{\mathscr{F}} : C \to Z$, *and* $\gamma_{\mathscr{F}'} : C' \to Z'$*, respectively, with* $\Gamma_{\mathscr{H}}$ *an extension of* $\gamma_{\mathscr{F}}$ *by* $\gamma_{\mathscr{G}}$ *(see Definition (5.70)). In particular, by Lemma (5.75), we know that there exists a linear map* $\gamma_{\mathscr{H}} : C \to X$ *such that* $\Gamma_{\mathscr{H}}$ *is equivalent to the block extension* ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ *of* $\gamma_{\mathscr{F}}$ *by* $\gamma_{\mathscr{G}}$ *associated with* $\gamma_{\mathscr{H}}$*. Hence, for the purpose of studying* ξ*, we identify* $\mathscr{H}$ *with the equivalent extension determined near* a *by the characteristic map* $\gamma_{\mathscr{H}}$*.*

In addition, observe that near a, the sheaf homomorphisms λ and λ' are characterized by linear maps $\mu_\lambda : C' \to C$, $\nu_\lambda : Z' \to Z$, $\mu_{\lambda'} : A \to A'$, and $\nu_{\lambda'} : X \to X'$ such that the pairs $\lambda_{\mu,\nu} := (\mu_\lambda, \nu_\lambda)$ and ${\lambda'}_{\mu,\nu} := (\mu_{\lambda'}, \nu_{\lambda'})$ define morphisms between $\gamma_{\mathscr{F}'}$ and $\gamma_{\mathscr{F}}$ and between $\gamma_{\mathscr{G}}$ and $\gamma_{\mathscr{G}'}$, respectively (see Definition (5.57)).

By definition, the pull-back $\lambda^*\mathscr{H}$ is an extension of $\mathscr{F}'$ by $\mathscr{G}$, and hence, near a, it is determined by the pull-back ${\lambda_{\mu,\nu}}^*\, {}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ of ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ via $\lambda_{\mu,\nu}$, which is an extension of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$ (see Definition (5.97)). Moreover, by Lemma (5.99), we know that ${\lambda_{\mu,\nu}}^*\, {}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ is equivalent to the block extension of $\gamma_{\mathscr{F}'}$ by $\gamma_{\mathscr{G}}$ associated with the composition $\gamma_{\mathscr{H}} \circ \mu_\lambda : C' \to X$. Consequently, we deduce that the equivalence class $\lambda \circ \xi = \left[\,\lambda^*\mathscr{H}\,\right]$ can be represented, near a, by an extension of $\mathscr{F}'$ by $\mathscr{G}$ whose characteristic map is $\gamma_{\mathscr{H}} \circ \mu_\lambda$.

Similarly, the push-out ${\lambda'}_*\mathscr{H}$ is an extension of $\mathscr{F}$ by $\mathscr{G}'$, and hence, near a, it is determined by the push-out ${{\lambda'}_{\mu,\nu}}_{*}\,{}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ of ${}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ via ${\lambda'}_{\mu,\nu}$, which is an extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$ (see Definition (5.98)). Furthermore, by Lemma (5.100), we have that ${{\lambda'}_{\mu,\nu}}_{*}\, {}_{\gamma_{\mathscr{G}}}\langle\gamma_{\mathscr{H}}\rangle_{\gamma_{\mathscr{F}}}$ is equivalent to the block extension of $\gamma_{\mathscr{F}}$ by $\gamma_{\mathscr{G}'}$ associated with the composition $\nu_{\lambda'} \circ \gamma_{\mathscr{H}} : C \to X'$. Therefore, we conclude that near a, the equivalence class $\xi \circ \lambda' = \left[\,{\lambda'}_*\mathscr{H}\,\right]$ can be represented by an extension of $\mathscr{F}$ by $\mathscr{G}'$ whose characteristic map is $\nu_{\lambda'} \circ \gamma_{\mathscr{H}}$.

With the above observation at hand, we now turn to the explicit analysis of the composition of graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

5.5.2. *Combinatorial Composition Rules.* Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. Next, we establish combinatorial rules for the composition of graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. To this end, we begin by introducing a definition and a related lemma, which ensure that our combinatorial formulas yield a well-defined composition of graded morphisms.

Definition 5.102. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. We introduce three bilinear operations associated with β: the* ***Hadamard*** $\odot : \mathbb{K}^n_{\mathrm{std}} \times \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^n_{\mathrm{std}}$, *the* ***left braided*** $\circ_{\beta_{\mathrm{L}}} : \mathbb{K}^n_{\mathrm{std}} \times \mathbb{K}^\ell_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$, *and the* ***right braided*** $\circ_{\beta_{\mathrm{R}}} : \mathbb{K}^\ell_{\mathrm{std}} \times \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ *compositions.*

Specifically, for any $\vec{u} = (u_1, \ldots, u_n)$, $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{K}^n_{\mathrm{std}}$, and any $\vec{p} = (p_1, \ldots, p_\ell)$, $\vec{q} = (q_1, \ldots, q_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$, we define:

$$
\begin{aligned}
\vec{v} \odot \vec{u} \;&:=\; (v_1 u_1, \ldots, v_n u_n) \in \mathbb{K}^n_{\mathrm{std}}\,, \\
\vec{v} \circ_{\beta_{\mathrm{L}}} \vec{p} \;&:=\; (v_{\pi_{\beta_1}(i_1+1)} p_1, \ldots, v_{\pi_{\beta_\ell}(i_\ell+1)} p_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}\,, \\
\vec{q} \circ_{\beta_{\mathrm{R}}} \vec{u} \;&:=\; (q_1 u_{\pi_{\beta_1}(i_1)}, \ldots, q_\ell u_{\pi_{\beta_\ell}(i_\ell)}) \in \mathbb{K}^\ell_{\mathrm{std}}\,.
\end{aligned}
$$

Lemma 5.103. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$, $\hat{\mathbf{q}}^{(n)}$ be bases for $\mathbb{K}^n$, and let $\vec{x}$, $\vec{y}$, $\vec{z} \in X(\beta, \mathbb{K})$ be points such that the pairs $(\,\hat{\mathbf{f}}^{(n)}, \vec{x}\,)$, $(\,\hat{\mathbf{g}}^{(n)}, \vec{y}\,)$, $(\,\hat{\mathbf{q}}^{(n)}, \vec{z}\,)$ algebraically characterize $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ according to Theorem (4.46), respectively.*

Following Definition (5.55), let $\delta_{\mathscr{F},\mathscr{G}}, \delta_{\mathscr{G},\mathscr{Q}}, \delta_{\mathscr{F},\mathscr{Q}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ *be the linear maps associated with the pairs* $(\mathscr{F},\mathscr{G})$, $(\mathscr{G},\mathscr{Q})$, $(\mathscr{F},\mathscr{Q})$, *respectively. Then the following statements hold:*

(i) Let $\vec{u} \in \ker \delta_{\mathscr{F},\mathscr{G}}$, *and* $\vec{v} \in \ker \delta_{\mathscr{G},\mathscr{Q}}$. *Then the Hadamard product* $\vec{v} \odot \vec{u} \in \mathbb{K}^n_{\mathrm{std}}$ *satisfies*

$$\vec{v} \odot \vec{u} \in \ker \delta_{\mathscr{F},\mathscr{Q}}\,.$$

(ii) Let $\vec{u} \in \ker \delta_{\mathscr{F},\mathscr{G}}$. *Then the action of* $\vec{u}$ *on* $\operatorname{im} \delta_{\mathscr{G},\mathscr{Q}}$ *induced by the right braided graded composition satisfies*

$$\operatorname{im} \delta_{\mathscr{G},\mathscr{Q}} \circ_{\beta_{\mathrm{R}}} \vec{u} \;\subseteq\; \operatorname{im} \delta_{\mathscr{F},\mathscr{Q}}\,.$$

(iii) Let $\vec{v} \in \ker \delta_{\mathscr{G},\mathscr{Q}}$. *Then the action of* $\vec{v}$ *on* $\operatorname{im} \delta_{\mathscr{F},\mathscr{G}}$ *induced by the left braided graded composition satisfies*

$$\vec{v} \circ_{\beta_{\mathrm{L}}} \operatorname{im} \delta_{\mathscr{F},\mathscr{G}} \;\subseteq\; \operatorname{im} \delta_{\mathscr{F},\mathscr{Q}}\,.$$

Proof. We prove parts (i) and (ii); part (iii) is analogous to (ii).

Part (i): By Definition (5.55), we know that $\vec{u} = (u_1, \ldots, u_n) \in \ker \delta_{\mathscr{F},\mathscr{G}} \subseteq \mathbb{K}^n_{\mathrm{std}}$ and $\vec{v} = (v_1, \ldots, v_n) \in \ker \delta_{\mathscr{G},\mathscr{Q}} \subseteq \mathbb{K}^n_{\mathrm{std}}$ if and only if

$$D(\pi_{\beta_{j-1}}(\vec{u})) \cdot B^{(n)}_{i_j}(x_j) = B^{(n)}_{i_j}(y_j) \cdot D(\pi_{\beta_j}(\vec{u}))\,,$$
$$D(\pi_{\beta_{j-1}}(\vec{v})) \cdot B^{(n)}_{i_j}(y_j) = B^{(n)}_{i_j}(z_j) \cdot D(\pi_{\beta_j}(\vec{v}))\,,$$

for all $j \in [1,\ell]$, with $\beta_0 = e_n \in \mathrm{Br}^+_n$ and $\pi_{\beta_0} = e_n \in \mathcal{S}_n$ denoting the trivial braid word on n strands and the trivial permutation on n elements, respectively. Hence, since $\vec{v} \odot \vec{u} = (v_1u_1, \ldots, v_nu_n) \in \mathbb{K}^n_{\mathrm{std}}$, we immediately observe that

$$\begin{aligned} D(\pi_{\beta_{j-1}}(\vec{v} \odot \vec{u})) \cdot B^{(n)}_{i_j}(x_j) &= D(\pi_{\beta_{j-1}}(\vec{v})) \cdot D(\pi_{\beta_{j-1}}(\vec{u})) \cdot B^{(n)}_{i_j}(x_j)\,, \\ &= D(\pi_{\beta_{j-1}}(\vec{v})) \cdot B^{(n)}_{i_j}(y_j) \cdot D(\pi_{\beta_j}(\vec{u}))\,, \\ &= B^{(n)}_{i_j}(z_j) \cdot D(\pi_{\beta_j}(\vec{v})) \cdot D(\pi_{\beta_j}(\vec{u}))\,, \\ &= B^{(n)}_{i_j}(z_j) \cdot D(\pi_{\beta_j}(\vec{v} \odot \vec{u}))\,, \end{aligned}$$

for all $j \in [1,\ell]$. It then follows from Definition (5.55) that $\vec{v} \odot \vec{u} \in \ker \delta_{\mathscr{F},\mathscr{Q}}$.

Part (ii): By Definition (5.55), we know that $\vec{u} = (u_1, \ldots, u_n) \in \ker \delta_{\mathscr{F},\mathscr{G}} \subseteq \mathbb{K}^n_{\mathrm{std}}$ if and only if

$$D(\pi_{\beta_{j-1}}(\vec{u})) \cdot B^{(n)}_{i_j}(x_j) = B^{(n)}_{i_j}(y_j) \cdot D(\pi_{\beta_j}(\vec{u}))\,, \tag{5.37}$$

for all $j \in [1,\ell]$.

Now, let $\vec{v} \in \mathbb{K}^n_{\mathrm{std}}$. By Definition (5.55), we have that $\delta_{\mathscr{G},\mathscr{Q}}(\vec{v}) = (\delta_1, \ldots, \delta_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$, where

$$\delta_j(\vec{z}, \vec{v}, \vec{y}) := \Big[\big(B^{(n)}_{i_j}(z_j)\big)^{-1} \cdot D(\pi_{\beta_{j-1}}(\vec{v})) \cdot B^{(n)}_{i_j}(y_j) \Big]_{i_j+1, i_j} ,$$

for all $j \in [1, \ell]$. Then, in light of Definition (5.102), we observe that $\delta_{\mathscr{G},\mathscr{Q}}(\vec{v}) \circ_{\beta_{\mathrm{R}}} \vec{u} = (\delta'_1, \ldots, \delta'_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$, where

$$\delta'_j(\vec{z}, \vec{v}, \vec{y}) := \Big[\big(B^{(n)}_{i_j}(z_j)\big)^{-1} \cdot D(\pi_{\beta_{j-1}}(\vec{v})) \cdot B^{(n)}_{i_j}(y_j) \cdot D(\pi_{\beta_j}(\vec{u})) \Big]_{i_j+1, i_j} ,$$

for all $j \in [1, \ell]$. Bearing this in mind, the kernel constraints (5.37) for $\vec{u}$ imply that

$$\begin{aligned} \delta'_j(\vec{z}, \vec{v}, \vec{y}) &= \Big[\big(B^{(n)}_{i_j}(z_j)\big)^{-1} \cdot D(\pi_{\beta_{j-1}}(\vec{v})) \cdot D(\pi_{\beta_{j-1}}(\vec{u})) \cdot B^{(n)}_{i_j}(x_j) \Big]_{i_j+1, i_j} , \\ &= \Big[\big(B^{(n)}_{i_j}(z_j)\big)^{-1} \cdot D(\pi_{\beta_{j-1}}(\vec{v} \odot \vec{u})) \cdot B^{(n)}_{i_j}(x_j) \Big]_{i_j+1, i_j} , \end{aligned}$$

for all $j \in [1, \ell]$. It then follows from Definition (5.55) that $\delta_{\mathscr{G},\mathscr{Q}}(\vec{v}) \circ_{\beta_{\mathrm{R}}} \vec{u} = \delta_{\mathscr{F},\mathscr{Q}}(\vec{v} \odot \vec{u})$. Finally, since $\vec{v}$ is arbitrary, we conclude that $\operatorname{im} \delta_{\mathscr{G},\mathscr{Q}} \circ_{\beta_{\mathrm{R}}} \vec{u} \subseteq \operatorname{im} \delta_{\mathscr{F},\mathscr{Q}}$. This completes the proof. ☐

With this result at hand, we now proceed to prove the first part of our second main result: a theorem providing a combinatorial rule for the composition of zero-degree morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$.

Theorem 5.104. *Let $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell} \in \mathrm{Br}^+_n$ be a positive braid word, and let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$, $\hat{\mathbf{q}}^{(n)}$ be bases for $\mathbb{K}^n$, and let $\vec{x}$, $\vec{y}$, $\vec{z} \in X(\beta, \mathbb{K})$ be points such that the pairs $(\,\hat{\mathbf{f}}\,, \vec{x}\,)$, $(\,\hat{\mathbf{g}}\,, \vec{y}\,)$, $(\,\hat{\mathbf{q}}\,, \vec{z}\,)$ algebraically characterize $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ according to Theorem (4.46), respectively.*

Following Definition (5.55), let $\delta_{\mathscr{F},\mathscr{G}}$, $\delta_{\mathscr{G},\mathscr{Q}}$, $\delta_{\mathscr{F},\mathscr{Q}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ be the linear maps associated with the pairs $(\mathscr{F}, \mathscr{G})$, $(\mathscr{G}, \mathscr{Q})$, $(\mathscr{F}, \mathscr{Q})$, respectively. In light of Theorem (5.66), let $\mu \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{G})$ and $\nu \in \mathrm{Ext}^0(\mathscr{G}, \mathscr{Q})$, and suppose that the vectors $\vec{u} \in \ker \delta_{\mathscr{F},\mathscr{G}} \subseteq \mathbb{K}^n_{\mathrm{std}}$ and $\vec{v} \in \ker \delta_{\mathscr{G},\mathscr{Q}} \subseteq \mathbb{K}^n_{\mathrm{std}}$ determine μ and ν under the isomorphisms

$$\mathrm{Ext}^0(\mathscr{F}, \mathscr{G}) \cong \ker \delta_{\mathscr{F},\mathscr{G}} , \qquad \mathrm{Ext}^0(\mathscr{G}, \mathscr{Q}) \cong \ker \delta_{\mathscr{G},\mathscr{Q}} .$$

Then, building on Definition (5.102), the composition $\nu \circ \mu \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{Q})$ is determined, under the isomorphism $\mathrm{Ext}^0(\mathscr{F}, \mathscr{Q}) \cong \ker \delta_{\mathscr{F},\mathscr{Q}}$, by the vector $\vec{v} \odot \vec{u} \in \ker \delta_{\mathscr{F},\mathscr{Q}} \subseteq \mathbb{K}^n_{\mathrm{std}}$.

Proof. To begin, observe that, given the bases $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$, and $\hat{\mathbf{q}}^{(n)}$, the sheaf homomorphisms μ and ν are fully determined by their characteristic maps $T^{(n)}_\mu : \mathbb{K}^n \to \mathbb{K}^n$ and $T^{(n)}_\nu : \mathbb{K}^n \to \mathbb{K}^n$, namely the linear maps whose matrix representations are given by

$${}_{\hat{\mathbf{g}}^{(n)}}\big[\, T^{(n)}_\mu \,\big]_{\hat{\mathbf{f}}^{(n)}} = D(\vec{u}) , \quad \text{and} \quad {}_{\hat{\mathbf{q}}^{(n)}}\big[\, T^{(n)}_\nu \,\big]_{\hat{\mathbf{g}}^{(n)}} = D(\vec{v}) .$$

By definition, $\nu \circ \mu$ corresponds to the standard composition of sheaf homomorphisms ν and μ, and hence, in our case, it is fully determined by the composition of the characteristic maps $T^{(n)}_\nu$ and $T^{(n)}_\mu$, that is, by the linear map $T_\nu^{(n)} \circ T_\mu^{(n)} : \mathbb{K}^n \to \mathbb{K}^n$ whose matrix representation with respect to the bases $\hat{\mathbf{f}}^{(n)}$ and $\hat{\mathbf{q}}^{(n)}$ is given by

$$\begin{aligned} {}_{\hat{\mathbf{q}}^{(n)}}\big[\, T_\nu^{(n)} \circ T_\mu^{(n)} \,\big]_{\hat{\mathbf{f}}^{(n)}} &= D(\vec{v}) \cdot D(\vec{u})\,, \\ &= D(\vec{v} \odot \vec{u})\,, \end{aligned}$$

where $\vec{v} \odot \vec{u} = (v_1 u_1, \ldots, v_n u_n) \in \mathbb{K}^n_{\mathrm{std}}$. Finally, observe that since $\vec{u} \in \ker \delta_{\mathscr{F},\mathscr{G}}$ and $\vec{v} \in \ker \delta_{\mathscr{G},\mathscr{Q}}$, Lemma (5.103) ensures that $\vec{v} \odot \vec{u} \in \ker \delta_{\mathscr{F},\mathscr{Q}}$, which shows that, under the isomorphism $\mathrm{Ext}^0(\mathscr{F}, \mathscr{Q}) \cong \ker \delta_{\mathscr{F},\mathscr{Q}}$, the composition $\nu \circ \mu$ is completely determined by the Hadamard product $\vec{v} \odot \vec{u}$, thereby providing a well-posed and explicit combinatorial rule for the composition of the graded morphisms under consideration. This completes the proof. $\square$

Next, we extend our combinatorial approach to give an explicit description of the composition of mixed-degree morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$, as made precise in the following theorems.

Theorem 5.105. *Let $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word, and let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$, $\hat{\mathbf{q}}^{(n)}$ be bases for $\mathbb{K}^n$, and let $\vec{x}$, $\vec{y}$, $\vec{z} \in X(\beta, \mathbb{K})$ be points such that the pairs $(\,\hat{\mathbf{f}}\,, \vec{x}\,)$, $(\,\hat{\mathbf{g}}\,, \vec{y}\,)$, $(\,\hat{\mathbf{q}}\,, \vec{z}\,)$ algebraically characterize $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ according to Theorem (4.46), respectively.*

Following Definition (5.55), let $\delta_{\mathscr{F},\mathscr{G}}$, $\delta_{\mathscr{G},\mathscr{Q}}$, $\delta_{\mathscr{F},\mathscr{Q}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ be the linear maps associated with the pairs $(\mathscr{F}, \mathscr{G})$, $(\mathscr{G}, \mathscr{Q})$, $(\mathscr{F}, \mathscr{Q})$, respectively. In light of Theorems (5.66) and (5.92), let $\mu \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{G})$ and $\Phi \in \mathrm{Ext}^1(\mathscr{G}, \mathscr{Q})$, and suppose that, for some representative $\vec{q} \in \mathbb{K}^\ell_{\mathrm{std}}$, the vector $\vec{u} \in \ker \delta_{\mathscr{F},\mathscr{G}} \subseteq \mathbb{K}^n_{\mathrm{std}}$ and the class $[\,\vec{q}\,] \in \operatorname{coker} \delta_{\mathscr{G},\mathscr{Q}}$ determine μ and Φ under the isomorphisms

$$\mathrm{Ext}^0(\mathscr{F}, \mathscr{G}) \cong \ker \delta_{\mathscr{F},\mathscr{G}}\,, \qquad \mathrm{Ext}^1(\mathscr{G}, \mathscr{Q}) \cong \operatorname{coker} \delta_{\mathscr{G},\mathscr{Q}}\,.$$

Then, building on Definition (5.102), the composition $\Phi \circ \mu \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{Q})$ is determined, under the isomorphism $\mathrm{Ext}^1(\mathscr{F}, \mathscr{Q}) \cong \operatorname{coker} \delta_{\mathscr{F},\mathscr{Q}}$, by the class $[\,\vec{q} \circ_{\beta_{\mathrm{R}}} \vec{u}\,] \in \operatorname{coker} \delta_{\mathscr{F},\mathscr{Q}}$.

Proof. To begin, let $\mathcal{U}_{\Lambda(\beta)} = \big\{U_0, U_{\mathrm{B}}, U_{\mathrm{L}}, U_{\mathrm{R}}, U_{\mathrm{T}}\big\}$ be the open cover of $\mathbb{R}^2$ from Construction (4.33). By definition, $\Phi \circ \mu = \big[\, \mu^* \mathscr{H}' \,\big]$, for any extension $\mathscr{H}'$ of $\mathscr{G}$ by $\mathscr{Q}$ representing Φ, where $\mu^* \mathscr{H}'$ denotes the pull-back of $\mathscr{H}'$ via μ. Next, consider the open set U_{B}, the region in $\mathbb{R}^2$ whose intersection with the front projection $\Pi_{x,z}(\Lambda(\beta))$ comprises the braid diagram on n strands associated with β, as illustrated in Figure (12). By Theorem (5.92), any extension of $\mathscr{G}$ by $\mathscr{Q}$ is equivalent to a simple extension—namely, an extension determined by a tuple $\vec{q} = (q_1, \ldots, q_\ell) \in \mathbb{K}^\ell_{\mathrm{std}}$ on U_{B} and equivalent to the trivial extension away from this region, where two simple extensions are equivalent if and only if their characteristic tuples, say $\vec{q}, \vec{q}\,' \in \mathbb{K}^\ell_{\mathrm{std}}$, satisfy $\vec{q}\,' - \vec{q} \in \operatorname{im} \delta_{\mathscr{G},\mathscr{Q}}$. Consequently, for the purpose of studying $\Phi \circ \mu$, it suffices to choose as representative for Φ a simple extension $\mathscr{H}'$ determined by a tuple $\vec{q} \in \mathbb{K}^\ell_{\mathrm{std}}$, compute the pull-back $\mu^* \mathscr{H}'$ on U_{B}, and then pass to its equivalence class.

Let $\mathcal{R}_{\Lambda(\beta)} = \{R_j\}_{j=1}^{\ell}$ be the partition of U_{B} into ℓ open vertical straps from Construction (4.36). Observe that, as we move from left to right through the regions R_j, the data characterizing the extension $\mathscr{H}'$ accumulates. In particular, in each region R_j, the new contribution is encoded in the characteristic map specifying the extension along the (i_j+1)-st strand immediately after the crossing σ_{i_j}, the j-th and only crossing of β contained in R_j. Thus, to analyze the pull-back $\mu^*\mathscr{H}'$, it suffices to extract, in each region R_j, the data that determines the pull-back along the (i_j+1)-st strand, for all $j \in [1,\ell]$.

Fix a region R_j for some $j \in [1,\ell]$, and denote by $k = i_j \in [1, n-1]$ the index of σ_{i_j}. Then, near the $(k+1)$-st strand immediately after the crossing σ_k, we have that:

- $\mathscr{F}$ and $\mathscr{G}$ are specified by a pair of linear maps $\widetilde{\phi}^{(k)}_{\mathscr{F}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$ and $\widetilde{\phi}^{(k)}_{\mathscr{G}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$, respectively.
- The extension $\mathscr{H}'$ is specified by a characteristic map $\widetilde{\phi}^{(k)}_{\mathscr{H}'} : \mathbb{K}^k \to \mathbb{K}^{k+1}$.
- The morphism μ is characterized by two linear maps $T^{(k+1)}_{\mu} : \mathbb{K}^{k+1} \to \mathbb{K}^{k+1}$ and $\widetilde{T}^{(k)}_{\mu} : \mathbb{K}^k \to \mathbb{K}^k$ such that the pair $(T^{(k+1)}_{\mu}, \widetilde{T}^{(k)}_{\mu})$ defines a morphism between $\widetilde{\phi}^{(k)}_{\mathscr{F}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$ and $\widetilde{\phi}^{(k)}_{\mathscr{G}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$.

Consequently, we obtain that, near the $(k+1)$-st strand immediately after the crossing σ_k, the pull-back $\mu^*\mathscr{H}'$ is determined by the characteristic map $\widetilde{\phi}^{(k)}_{\mathscr{H}'} \circ \widetilde{T}^{(k)}_{\mu} : \mathbb{K}^k \to \mathbb{K}^{k+1}$.

Now, for each $i \in [1, n-1]$, let $\hat{\mathbf{f}}^{(i)}$, $\hat{\mathbf{g}}^{(i)}$, and $\hat{\mathbf{q}}^{(i)}$ be bases for $\mathbb{K}^i$ induced by the bases $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$, and $\hat{\mathbf{q}}^{(n)}$ for $\mathbb{K}^n$, so that $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^{n}$, $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^{n}$, and $\{\hat{\mathbf{q}}^{(i)}\}_{i=1}^{n}$ are systems of bases adapted to the linear maps characterizing $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{Q}$ on the left of the crossing in R_1, respectively.

Next, consider the braid-transformed bases $\hat{\mathbf{f}}^{(i)}[\beta_j, \vec{x}_j]$, $\hat{\mathbf{g}}^{(i)}[\beta_j, \vec{y}_j]$, and $\hat{\mathbf{g}}^{(i)}[\beta_j, \vec{z}_j]$ associated with the truncated braid word β_j and the truncated tuples $\vec{x}_j$, $\vec{y}_j$, $\vec{z}_j \in \mathbb{K}^j_{\mathrm{std}}$. Then, with respect to these bases, we have that

- The matrix representing $\widetilde{\phi}^{(k)}_{\mathscr{H}'}$ is given by

$$
{}_{\hat{\mathbf{q}}^{(k+1)}[\beta_j,\,\vec{z}_j]}\big[\,\widetilde{\phi}^{(k)}_{\mathscr{H}'}\,\big]_{\hat{\mathbf{g}}^{(k)}[\beta_j,\,\vec{y}_j]} = \begin{bmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & 0 \\ \star & \cdots & \star & q_j \end{bmatrix}_{(k+1,k)} ,
$$

  where q_j denotes the j-th entry of the tuple $\vec{q} = (q_1, \dots, q_\ell) \in \mathbb{K}^{\ell}_{\mathrm{std}}$ characterizing $\mathscr{H}'$, and the starred entries depend on the data from prior crossings before σ_{i_j}.

- The matrix representing $\widetilde{T}^{(k)}_{\mu}$ is given by

$$
{}_{\hat{\mathbf{g}}^{(k)}[\beta_j,\,\vec{y}_j]}\big[\,\widetilde{T}^{(k)}_{\mu}\,\big]_{\hat{\mathbf{f}}^{(k)}[\beta_j,\,\vec{x}_j]} = \text{ principal } k \times k \text{ submatrix of } D(\pi_{\beta_j}(\vec{u}))\,.
$$

Hence, a direct calculation shows that

$$
{}_{\hat{\mathbf{q}}^{(k+1)}[\beta_j\,,\,\vec{z}_j]}\big[\,\widetilde{\phi}^{(k)}_{\mathscr{H}'}\circ\widetilde{T}^{(k)}_{\mu}\,\big]_{\hat{\mathbf{f}}^{(k)}[\beta_j\,,\,\vec{x}_j]}=\begin{bmatrix}\star & \cdots & \star & 0\\ \vdots & \ddots & \vdots & \vdots\\ \star & \cdots & \star & 0\\ \star & \cdots & \star & q_j\cdot u_{\beta_j(k)}\end{bmatrix}_{(k+1,k)}\,,
$$

where $u_{\pi_{\beta_j}(k)}$ is the (k,k)-entry of $D(\pi_{\beta_j}(\vec{u}))$, namely the k-th entry of the permutation of $\vec{u}$ by π_{β_j}, the permutation associated with the truncated braid word β_j. Moreover, since $k=i_j$, we deduce that the new contribution to the data characterizing the pull-back $\mu^*\mathscr{H}'$ after the crossing σ_{i_j} is given by the parameter $q'_j=q_j\cdot u_{\beta_j(i_j)}$.

By systematically applying the above argument to all the regions R_j, we obtain that the pull-back $\mu^*\mathscr{H}'$ is characterized by a tuple $\vec{q}\,':=(q'_1,\ldots,q'_\ell)\in\mathbb{K}^\ell_{\mathrm{std}}$, which entrywise is given by

$$
q'_j=q_j\cdot u_{\pi_{\beta_j}(i_j)}\,,
$$

for all $j\in[1,\ell]$, and in light of Definition (5.102), we immediately recognize that $\vec{q}\,'=\vec{q}\circ_{\beta_{\mathrm{R}}}\vec{u}$. Finally, observe that, since $\vec{u}\in\ker\delta_{\mathscr{F},\mathscr{G}}$, Lemma (5.103) ensures that $\operatorname{im}\delta_{\mathscr{G},\mathscr{Q}}\circ_{\beta_{\mathrm{R}}}\vec{u}\subseteq\operatorname{im}\delta_{\mathscr{F},\mathscr{Q}}$, and hence, passing to the equivalence class of the pull-back $\mu^*\mathscr{H}'$ in $\operatorname{Ext}^1(\mathscr{F},\mathscr{Q})$ allows us to conclude that the composition $\Phi\circ\mu$ is completely determined, under the isomorphism $\operatorname{Ext}^1(\mathscr{F},\mathscr{Q})\cong\operatorname{coker}\delta_{\mathscr{F},\mathscr{Q}}$, by the class $[\,\vec{q}\circ_{\beta_{\mathrm{R}}}\vec{u}\,]\in\operatorname{coker}\delta_{\mathscr{F},\mathscr{Q}}$, thereby providing a well-posed and explicit combinatorial rule for the composition of the graded morphisms under consideration. This completes the proof. ☐

Theorem 5.106. *Let $\beta=\sigma_{i_1}\ldots\sigma_{i_\ell}\in\mathrm{Br}^+_n$ be a positive braid word, and let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$, $\hat{\mathbf{q}}^{(n)}$ be bases for $\mathbb{K}^n$, and let $\vec{x}$, $\vec{y}$, $\vec{z}\in X(\beta,\mathbb{K})$ be points such that the pairs $(\,\hat{\mathbf{f}}\,,\vec{x}\,)$, $(\,\hat{\mathbf{g}}\,,\vec{y}\,)$, $(\,\hat{\mathbf{q}}\,,\vec{z}\,)$ algebraically characterize $\mathscr{F}$, $\mathscr{G}$, $\mathscr{Q}$ according to Theorem (4.46), respectively.*

Following Definition (5.55), let $\delta_{\mathscr{F},\mathscr{G}}$, $\delta_{\mathscr{G},\mathscr{Q}}$, $\delta_{\mathscr{F},\mathscr{Q}}:\mathbb{K}^n_{\mathrm{std}}\to\mathbb{K}^\ell_{\mathrm{std}}$ be the linear maps associated with the pairs $(\mathscr{F},\mathscr{G})$, $(\mathscr{G},\mathscr{Q})$, $(\mathscr{F},\mathscr{Q})$, respectively. In light of Theorems (5.66) and (5.92), let $\nu\in\operatorname{Ext}^0(\mathscr{G},\mathscr{Q})$ and $\Theta\in\operatorname{Ext}^1(\mathscr{F},\mathscr{G})$, and suppose that, for some representative $\vec{p}\in\mathbb{K}^\ell_{\mathrm{std}}$, the vector $\vec{v}\in\ker\delta_{\mathscr{G},\mathscr{Q}}\subseteq\mathbb{K}^n_{\mathrm{std}}$ and the class $[\,\vec{p}\,]\in\operatorname{coker}\delta_{\mathscr{F},\mathscr{G}}$ determine ν and Θ under the isomorphisms

$$
\operatorname{Ext}^0(\mathscr{G},\mathscr{Q})\cong\ker\delta_{\mathscr{G},\mathscr{Q}}\,,\qquad\qquad \operatorname{Ext}^1(\mathscr{F},\mathscr{G})\cong\operatorname{coker}\delta_{\mathscr{F},\mathscr{G}}\,.
$$

Then, building on Definition (5.102), the composition $\nu\circ\Theta\in\operatorname{Ext}^1(\mathscr{F},\mathscr{Q})$ is determined, under the isomorphism $\operatorname{Ext}^1(\mathscr{F},\mathscr{Q})\cong\operatorname{coker}\delta_{\mathscr{F},\mathscr{Q}}$, by the class $[\,\vec{v}\circ_{\beta_{\mathrm{L}}}\vec{p}\,]\in\operatorname{coker}\delta_{\mathscr{F},\mathscr{Q}}$.

Proof. To begin, let $\mathcal{U}_{\Lambda(\beta)}=\big\{U_0,U_{\mathrm{B}},U_{\mathrm{L}},U_{\mathrm{R}},U_{\mathrm{T}}\big\}$ be the open cover of $\mathbb{R}^2$ from Construction (4.33). By definition, $\nu\circ\Theta=\big[\,\nu_*\mathscr{H}\,\big]$, for any extension $\mathscr{H}$ of $\mathscr{F}$ by $\mathscr{G}$ representing Θ, where $\nu_*\mathscr{H}$ denotes the push-out of $\mathscr{H}$ via ν. Next, consider the open set U_{B}, the region in $\mathbb{R}^2$ whose intersection with the front projection $\Pi_{x,z}(\Lambda(\beta))$ comprises the braid diagram on n strands associated with β, as illustrated in Figure (12). By Theorem (5.92), any extension of $\mathscr{F}$ by $\mathscr{G}$ is equivalent to a simple extension—namely, an extension determined by a tuple $\vec{p} = (p_1, \dots, p_\ell) \in \mathbb{K}^\ell_{\text{std}}$ on U_{B} and equivalent to the trivial extension away from this region, where two simple extensions are equivalent if and only if their characteristic tuples, say $\vec{p}, \vec{p}\,' \in \mathbb{K}^\ell_{\text{std}}$, satisfy $\vec{p}\,' - \vec{p} \in \operatorname{im} \delta_{\mathscr{F},\mathscr{G}}$. Consequently, for the purpose of studying $\nu \circ \Theta$, it suffices to choose as representative for Θ a simple extension $\mathscr{H}$ determined by a tuple $\vec{p} \in \mathbb{K}^\ell_{\text{std}}$, compute the push-out $\nu_* \mathscr{H}$ on U_{B}, and then pass to its equivalence class.

Let $\mathcal{R}_{\Lambda(\beta)} = \{R_j\}_{j=1}^\ell$ be the partition of U_{B} into ℓ open vertical straps from Construction (4.36). Observe that, as we move from left to right through the regions R_j, the data characterizing the extension $\mathscr{H}$ accumulates. In particular, in each region R_j, the new contribution is encoded in the characteristic map specifying the extension along the (i_j+1)-st strand immediately after the crossing σ_{i_j}, the j-th and only crossing of β contained in R_j. Thus, to analyze the push-out $\nu_*\mathscr{H}$, it suffices to extract, in each region R_j, the data that determines the push-out along the (i_j+1)-st strand, for all $j \in [1, \ell]$.

Fix a region R_j for some $j \in [1, \ell]$, and denote by $k = i_j \in [1, n-1]$ the index of σ_{i_j}. Then, near the $(k+1)$-st strand immediately after the crossing σ_k, we have that:

- $\mathscr{G}$ and $\mathscr{Q}$ are specified by a pair of linear maps $\widetilde{\phi}^{(k)}_{\mathscr{G}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$ and $\widetilde{\phi}^{(k)}_{\mathscr{Q}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$, respectively.
- The extension $\mathscr{H}$ is specified by a characteristic map $\widetilde{\phi}^{(k)}_{\mathscr{H}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$.
- The morphism ν is characterized by two linear maps $T^{(k+1)}_\nu : \mathbb{K}^{k+1} \to \mathbb{K}^{k+1}$ and $\widetilde{T}^{(k)}_\nu : \mathbb{K}^k \to \mathbb{K}^k$ such that the pair $(T^{(k+1)}_\nu, \widetilde{T}^{(k)}_\nu)$ defines a morphism between $\widetilde{\phi}^{(k)}_{\mathscr{G}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$ and $\widetilde{\phi}^{(k)}_{\mathscr{Q}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$.

Consequently, we obtain that, near the $(k+1)$-st strand immediately after the crossing σ_k, the push-out $\nu_*\mathscr{H}$ is determined by the characteristic map $T^{(k+1)}_\nu \circ \widetilde{\phi}^{(k)}_{\mathscr{H}} : \mathbb{K}^k \to \mathbb{K}^{k+1}$.

Now, for each $i \in [1, n-1]$, let $\hat{\mathbf{f}}^{(i)}$, $\hat{\mathbf{g}}^{(i)}$, and $\hat{\mathbf{q}}^{(i)}$ be bases for $\mathbb{K}^i$ induced by the bases $\hat{\mathbf{f}}^{(n)}$, $\hat{\mathbf{g}}^{(n)}$, and $\hat{\mathbf{q}}^{(n)}$ for $\mathbb{K}^n$, so that $\{\hat{\mathbf{f}}^{(i)}\}_{i=1}^n$, $\{\hat{\mathbf{g}}^{(i)}\}_{i=1}^n$, and $\{\hat{\mathbf{q}}^{(i)}\}_{i=1}^n$ are systems of bases adapted to the linear maps characterizing $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{Q}$ on the left of the crossing in R_1, respectively.

Next, consider the braid-transformed bases $\hat{\mathbf{f}}^{(i)}[\beta_j, \vec{x}_j]$, $\hat{\mathbf{g}}^{(i)}[\beta_j, \vec{y}_j]$, and $\hat{\mathbf{g}}^{(i)}[\beta_j, \vec{z}_j]$ associated with the truncated braid word β_j and the truncated tuples $\vec{x}_j$, $\vec{y}_j$, $\vec{z}_j \in \mathbb{K}^j_{\text{std}}$. Then, with respect to these bases, we have that

- The matrix representing $\widetilde{\phi}^{(k)}_{\mathscr{H}}$ is given by

$$_{\hat{\mathbf{g}}^{(k+1)}[\beta_j\,,\,\vec{y}_j]}\big[\,\widetilde{\phi}^{(k)}_{\mathscr{H}}\,\big]_{\hat{\mathbf{f}}^{(k)}[\beta_j\,,\,\vec{x}_j]} = \begin{bmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & 0 \\ \star & \cdots & \star & p_j \end{bmatrix}_{(k+1,k)} ,$$

  where p_j denotes the j-th entry of the tuple $\vec{p} = (p_1, \ldots, p_\ell) \in \mathbb{K}^\ell_{\text{std}}$ characterizing $\mathscr{H}$, and the starred entries depend on the data from prior crossings before σ_{i_j}.

- The matrix representing $T^{(k+1)}_\nu$ is given by

$$_{\hat{\mathbf{q}}^{(k+1)}[\beta_j\,,\,\vec{z}_j]}\big[\,T^{(k+1)}_\nu\,\big]_{\hat{\mathbf{g}}^{(k+1)}[\beta_j\,,\,\vec{y}_j]} = \text{ principal } (k+1)\times(k+1) \text{ submatrix of } D(\pi_{\beta_j}(\vec{v}))\,.$$

Hence, a direct calculation shows that

$$_{\hat{\mathbf{q}}^{(k+1)}[\beta_j\,,\,\vec{z}_j]}\big[\,T^{(k+1)}_\nu \circ \widetilde{\phi}^{(k)}_{\mathscr{H}}\,\big]_{\hat{\mathbf{f}}^{(k)}[\beta_j\,,\,\vec{x}_j]} = \begin{bmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & 0 \\ \star & \cdots & \star & v_{\beta_j(k+1)} \cdot p_j \end{bmatrix}_{(k+1,k)} ,$$

where $v_{\pi_{\beta_j}(k+1)}$ is the $(k+1, k+1)$-entry of $D(\pi_{\beta_j}(\vec{v}))$, namely the $k+1$-th entry of the permutation of $\vec{v}$ by π_{β_j}, the permutation associated with the truncated braid word β_j. Moreover, since $k = i_j$, we deduce that the new contribution to the data characterizing the push-out $\nu_*\mathscr{H}$ after the crossing σ_{i_j} is given by the parameter $p'_j = v_{\beta_j(i_j+1)} \cdot p_j$.

By systematically applying the above argument to all the regions R_j, we obtain that the push-out $\nu_*\mathscr{H}$ is characterized by a tuple $\vec{p}\,' := (p'_1, \ldots, p'_\ell) \in \mathbb{K}^\ell_{\text{std}}$, which entrywise is given by

$$p'_j = v_{\pi_{\beta_j}(i_j+1)} \cdot p_j\,,$$

for all $j \in [1, \ell]$, and in light of Definition (5.102), we immediately recognize that $\vec{p}\,' = \vec{v} \circ_{\beta_{\text{L}}} \vec{p}$. Finally, observe that, since $\vec{v} \in \ker \delta_{\mathscr{G},\mathscr{Q}}$, Lemma (5.103) ensures that $\vec{v} \circ_{\beta_{\text{L}}} \operatorname{im} \delta_{\mathscr{F},\mathscr{G}} \subseteq \operatorname{im} \delta_{\mathscr{F},\mathscr{Q}}$, and hence, passing to the equivalence class of the push-out $\nu_*\mathscr{H}$ in $\operatorname{Ext}^1(\mathscr{F},\mathscr{Q})$ allows us to conclude that the composition $\nu\circ\Theta$ is completely determined, under the isomorphism $\operatorname{Ext}^1(\mathscr{F},\mathscr{Q}) \cong \operatorname{coker} \delta_{\mathscr{F},\mathscr{Q}}$, by the class $[\,\vec{v} \circ_{\beta_{\text{L}}} \vec{p}\,] \in \operatorname{coker} \delta_{\mathscr{F},\mathscr{Q}}$, thereby providing a well-posed and explicit combinatorial rule for the composition of the graded morphisms under consideration. This completes the proof. □

Having established the preceding results, we conclude the explicit algebraic characterization of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$. In the next section, we explore some applications of this characterization.

6. Applications

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. Next, we present some applications of the explicit description of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ that we developed in the preceding sections. More precisely, we first consider the case where $\Lambda(\beta)$ is a Legendrian knot—a single-component Legendrian link. In this setting, we establish some structural results concerning the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ over a generic field $\mathbb{K}$ and provide a simple characterization of the category in the case where $\beta = \sigma_1\sigma_2\sigma_1\sigma_2$ and $\mathbb{K} = \mathbb{Z}_2$, which corresponds to the Legendrian trefoil knot realized as the rainbow closure of a positive braid on three strands. Furthermore, to illustrate how our theoretical framework works for multi-component Legendrian links, we explicitly analyze the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ in the case where $\beta = \sigma_1\sigma_2\sigma_1$ and $\mathbb{K} = \mathbb{Z}_2$, thereby covering the case of the Legendrian Hopf link realized as the rainbow closure of a positive braid word on three strands.

6.1. **The Knot Case.**

Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word such that $\Lambda(\beta)$ is a Legendrian knot. Next, we provide some structural results concerning the linear structure of the graded morphism spaces and the algebraic properties of the graded endomorphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Moreover, we present a detailed analysis of some aspects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ in the case where $\beta = \sigma_1\sigma_2\sigma_1\sigma_2$ and $\mathbb{K} = \mathbb{Z}_2$.

6.1.1. *Some Structural Results.* Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word such that $\Lambda(\beta)$ is a Legendrian knot. In particular, recall that $\Lambda(\beta)$ defines a knot if and only if the permutation $\pi_\beta \in \mathrm{S}_n$ associated with β consists of a single cycle in its cycle decomposition in S_n [CGGS24]. Bearing this in mind, we now proceed to exploit the stated property of π_β to establish some results revealing the linear structure of the graded morphism spaces and the algebraic properties of the graded endomorphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.

Let $X(\beta,\mathbb{K})$ be the braid variety associated with β, and let $\mathcal{T}$ be the $(n-1)$-dimensional torus given by

$$\mathcal{T} := \left\{ \vec{t} \in \mathbb{K}^n_{\mathrm{std}} \mid D(\vec{t}\,) \in \mathrm{SL}(n,\mathbb{K}) \right\}.$$

As shown in [CGGS24], $\mathcal{T}$ acts naturally on $X(\beta,\mathbb{K})$. The following definition formalizes this group action.

Definition 6.107. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. The torus action on $X(\beta,\mathbb{K})$ corresponds to the map*

$$\begin{array}{rccc} \star : & \mathcal{T} \times X(\beta,\mathbb{K}) & \longrightarrow & X(\beta,\mathbb{K})\,, \\ & (\vec{t},\vec{z}\,) & \longmapsto & \vec{t}\star\vec{z}\,, \end{array}$$

where, for any $\vec{z} = (z_1,\ldots,z_\ell) \in X(\beta,\mathbb{K})$, the point $\vec{t}\star\vec{z} := (z'_1,\ldots,z'_\ell) \in X(\beta,\mathbb{K})$ is uniquely determined by the system of equations

$$D(\pi_{\beta_{j-1}}(\vec{t}\,))\cdot B^{(n)}_{i_j}(z_j) = B^{(n)}_{i_j}(z'_j)\cdot D(\pi_{\beta_j}(\vec{t}\,))\,, \qquad \textit{for all } j \in [1,\ell]\,.$$

Accordingly, we say that two points $\vec{z}_1, \vec{z}_2 \in X(\beta,\mathbb{K})$ are <u>*equivalent under the torus action*</u> *if there exists $\vec{t} \in \mathcal{T}$ such that $\vec{z}_2 = \vec{t} \star \vec{z}_1$, and we denote this relation by $\vec{z}_2 \cong \vec{z}_1$; otherwise, we write $\vec{z}_2 \ncong \vec{z}_1$.*

A notable property of the torus action on $X(\beta,\mathbb{K})$ is that, when $\Lambda(\beta)$ is a Legendrian knot, it is free [CGGS24]. In view of this property, we now introduce a definition that will play a fundamental role in the subsequent discussion.

Definition 6.108. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word. We define the subvariety $\widetilde{X}(\beta,\mathbb{K}) \subset X(\beta,\mathbb{K})$ by*

$$\widetilde{X}(\beta,\mathbb{K}) := \left\{\vec{z} \in \mathbb{K}^{\ell}_{\mathrm{std}} \,\middle|\, \textit{all leading principal minors of } P_\beta(\vec{z}) \textit{ are equal to } 1\right\}.$$

The subvariety $\widetilde{X}(\beta,\mathbb{K}) \subset X(\beta,\mathbb{K})$ plays a particularly important role in our current setting. Specifically, when $\Lambda(\beta)$ is a Legendrian knot, the defining properties of $\widetilde{X}(\beta,\mathbb{K})$ together with the freeness of the torus action on $X(\beta,\mathbb{K})$ imply that the induced map

$$\begin{array}{ccc} \mathcal{T} \times \widetilde{X}(\beta,\mathbb{K}) & \to & X(\beta,\mathbb{K}), \\ (\vec{t},\, \vec{z}\,) & \mapsto & \vec{t} \star \vec{z}, \end{array}$$

is bijective. Equivalently, for any $\vec{z} \in X(\beta,\mathbb{K})$ there exist unique $\vec{t} \in \mathcal{T}$ and $\vec{z}\,' \in \widetilde{X}(\beta,\mathbb{K})$ such that $\vec{z} = \vec{t} \star \vec{z}\,'$.

Next, we present a result that, exploiting the torus action on $X(\beta,\mathbb{K})$, provides an explicit description of the linear structure of the graded morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ in the case where $\Lambda(\beta)$ is a Legendrian knot.

Theorem 6.109. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word such that $\Lambda(\beta)$ is a Legendrian knot, and let $\mathscr{F}$, $\mathscr{G}$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Let $\hat{\mathbf{f}}^n$, $\hat{\mathbf{g}}^n$ be bases for $\mathbb{K}^n$, and let $\vec{z}_1, \vec{z}_2 \in X(\beta,\mathbb{K})$ be points such that the pairs $(\,\hat{\mathbf{f}}, \vec{z}\,)$ and $(\,\hat{\mathbf{g}}, \vec{z}\,'\,)$ algebraically characterize $\mathscr{F}$ and $\mathscr{G}$ according to Theorem (4.46), respectively. Then there is an isomorphism of graded vector spaces*

$$\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{G}) \cong \begin{cases} \mathbb{K}\,[0] \oplus \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}\,[1]\,, & \textit{if } \vec{z}\,' \cong \vec{z} \\[2ex] 0\,[0] \oplus \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))}\,[1]\,, & \textit{if } \vec{z}\,' \ncong \vec{z} \end{cases},$$

where $\mathrm{tb}(\Lambda(\beta)) := \ell - n$ corresponds to the Thurston–Bennequin number of $\Lambda(\beta)$.

Proof. To begin, let us consider the points $\vec{z}$, $\vec{z}\,' \in X(\beta,\mathbb{K})$. By our previous discussion, there exist unique $\vec{s}$, $\vec{t} \in \mathcal{T}$ and $\vec{x}$, $\vec{y} \in \widetilde{X}(\beta,\mathbb{K})$ such that $\vec{z} = \vec{s} \star \vec{x}$ and $\vec{z}\,' = \vec{t} \star \vec{y}$. More precisely, building on Definition (6.107), we have that

$$D(\pi_{\beta_{j-1}}(\,\vec{s}\,))\cdot B^{(n)}_{i_j}(x_j)=B^{(n)}_{i_j}(z_j)\cdot D(\pi_{\beta_j}(\,\vec{s}\,))\,,\quad \text{for all } j\in[1,\ell]\,,$$

$$D(\pi_{\beta_{j-1}}(\,\vec{t}\,))\cdot B^{(n)}_{i_j}(y_j)=B^{(n)}_{i_j}(z'_j)\cdot D(\pi_{\beta_j}(\,\vec{t}\,))\,,\quad \text{for all } j\in[1,\ell]\,,$$

where $\vec{z}=(z_1,\dots,z_\ell)$, $\vec{z}\,'=(z'_1,\dots,z'_\ell)$, $\vec{x}=(x_1,\dots,x_\ell)$, and $\vec{y}=(y_1,\dots,y_\ell)$.

Now, let us consider the linear map $\delta_{\mathscr{F},\mathscr{G}}:\mathbb{K}^n_{\text{std}}\to \mathrm{K}^\ell_{\text{std}}$ associated with $\mathscr{F}$ and $\mathscr{G}$, as introduced in Definition (5.55), and let $\vec{u}\in\ker\,\delta_{\mathscr{F},\mathscr{G}}$. By definition, we know that

$$D(\pi_{\beta_{j-1}}(\vec{u}))\cdot B^{(n)}_{i_j}(z_j)=B^{(n)}_{i_j}(z'_j)\cdot D(\pi_{\beta_j}(\vec{u}))\,,\quad \text{for all } j\in[1,\ell]\,.$$

Hence, putting the above relations together yields that

$$D(\pi_{\beta_{j-1}}(\vec{v}))\cdot B^{(n)}_{i_j}(x_j)=B^{(n)}_{i_j}(y_j)\cdot D(\pi_{\beta_j}(\vec{v}))\,,\quad \text{for all } j\in[1,\ell]\,,\tag{6.1}$$

where $\vec{v}=\big(\,\vec{t}\,\big)^{-1}\odot\vec{u}\odot\vec{s}$. Furthermore, implementing relations (6.1) iteratively throughout $j\in[1,\ell]$, we obtain that

$$D(\vec{v}\,)\cdot P_\beta(\,\vec{x}\,)=P_\beta(\,\vec{y}\,)\cdot D(\pi_\beta(\vec{v}\,))\,,\tag{6.2}$$

where $P_\beta(\vec{x})=B^{(n)}_{i_1}(x_1)\cdots B^{(n)}_{i_\ell}(x_\ell)$ and $P_\beta(\vec{y})=B^{(n)}_{i_1}(y_1)\cdots B^{(n)}_{i_\ell}(y_\ell)$. In particular, observe that, since $\vec{x}$, $\vec{y}\in\widetilde{X}(\beta,\mathbb{K})$, the defining properties of the subvariety $\widetilde{X}(\beta,\mathbb{K})$ and the matrix equation (6.2) imply that

$$\pi_\beta(\,\vec{v}\,)=\vec{v}\,.\tag{6.3}$$

Here, recall that, by assumption, $\Lambda(\beta)$ is a Legendrian knot, and hence the permutation π_β is a single-cycle in its cycle decomposition in S_n [CGGS24]. Bearing this in mind, identity (6.3) guarantees that there exists $u\in\mathbb{K}$ such that $\vec{v}=u\,\vec{1}_n$, where $\vec{1}_n=(1,\dots,1)\in\mathbb{K}^n_{\text{std}}$. Consequently, relations (6.1) reduce to the vector equation

$$u\,(\,\vec{y}-\vec{x}\,)=0\,,$$

which yields the following two cases:

- **Case 1**: Suppose $\vec{y}=\vec{x}$. In this case, u is a free parameter, and as a result $\vec{u}=u\,\vec{a}$, where $\vec{a}=\vec{t}\odot\big(\,\vec{s}\,\big)^{-1}\in\ker\,\delta_{\mathscr{F},\mathscr{G}}$. Hence, since since $\vec{u}$ is arbitrary, we obtain that $\ker\,\delta_{\mathscr{F},\mathscr{G}}=\mathrm{Span}_{\mathbb{K}}\{\,\vec{a}\,\}\cong\mathbb{K}$, and by Theorem (5.66), we conclude that $\mathrm{Ext}^1(\mathscr{F},\mathscr{G})\cong\mathbb{K}$. Here, it is important to emphasize that, if $\vec{z}\,'=\vec{z}$ or $\vec{z}$, $\vec{z}\,'\in\widetilde{X}(\beta,\mathbb{K})$, then $\vec{a}=\vec{1}_n\in\mathbb{K}^n_{\text{std}}$, and in any of these particular situations $\vec{u}=u\,\vec{1}_n$.
- **Case 2**: Suppose $\vec{y}\neq\vec{x}$. In this case, we have that $u=0$, and as a result $\vec{u}=u\,\vec{a}=0$. Hence, since $\vec{u}$ is arbitrary, we conclude that $\ker\,\delta_{\mathscr{F},\mathscr{G}}\cong 0$, and by Theorem (5.66), we deduce that $\mathrm{Ext}^1(\mathscr{F},\mathscr{G})\cong 0$.

Next, by Theorem (5.93), we know that

$$\begin{aligned}\chi(\mathrm{Ext}^{\bullet}(\mathscr{F},\mathscr{G})) &= \dim_{\mathbb{K}}\mathrm{Ext}^{0}(\mathscr{F},\mathscr{G}) - \dim_{\mathbb{K}}\mathrm{Ext}^{1}(\mathscr{F},\mathscr{G})\,,\\ &= -\mathrm{tb}(\Lambda(\beta))\,,\end{aligned}$$

which implies that

$$\dim_{\mathbb{K}}\mathrm{Ext}^{1}(\mathscr{F},\mathscr{G}) = \mathrm{tb}(\Lambda(\beta)) + \dim_{\mathbb{K}}\mathrm{Ext}^{0}(\mathscr{F},\mathscr{G})\,.$$

Bearing this in mind, we arrive to the following two cases:

- **Case 1**: Suppose that $\vec{y} = \vec{x}$. In this case, $\mathrm{Ext}^{1}(\mathscr{F},\mathscr{G}) \cong \mathbb{K}$, and hence $\dim_{\mathbb{K}}\mathrm{Ext}^{1}(\mathscr{F},\mathscr{G}) = \mathrm{tb}(\Lambda(\beta)) + 1$, which shows that $\mathrm{Ext}^{1}(\mathscr{F},\mathscr{G}) \cong \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}$.
- **Case 2**: Suppose that $\vec{y} \neq \vec{x}$. In this case, $\mathrm{Ext}^{1}(\mathscr{F},\mathscr{G}) \cong 0$, and therefore $\dim_{\mathbb{K}}\mathrm{Ext}^{1}(\mathscr{F},\mathscr{G}) = \mathrm{tb}(\Lambda(\beta))$, which implies that $\mathrm{Ext}^{1}(\mathscr{F},\mathscr{G}) \cong \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))}$.

Finally, observe that, if $\vec{y} = \vec{x}$, then $\vec{z}\,' \cong \vec{z}$, whereas if $\vec{y} \neq \vec{x}$, then $\vec{z}\,' \ncong \vec{z}$. The result follows. $\square$

From our point of view, Theorem (6.109) is particularly enlightening. It establishes that, in the knot case, the linear structure of the graded morphism spaces in the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ is remarkably simple: it is largely determined by intrinsic topological data associated with $\Lambda(\beta)$, namely its Thurston-Bennequin number.

Next, we present a result that, in the spirit of Theorem (6.109), provides a simple description of the unital graded ring structure of the graded endomorphism spaces in the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$.

Theorem 6.110. *Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^{+}$ be a positive braid word such that $\Lambda(\beta)$ is a Legendrian knot. Let $\mathscr{F}$ be an object of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$. Let $\hat{\mathbf{f}}^{(n)}$ a basis for $\mathbb{K}^n$, and let $\vec{z} \in X(\beta,\mathbb{K})$ be a point such that the pair $(\,\hat{\mathbf{f}}^{(n)},\,\vec{z}\,)$ algebraically characterizes $\mathscr{F}$ according to Theorem (4.46). Then:*

(1) There is an isomorphism of graded vector spaces

$$\mathrm{Ext}^{\bullet}(\mathscr{F},\mathscr{F}) \cong \mathbb{K}\,[0] \oplus \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}\,[1]\ .$$

(2) Under the isomorphism of graded vector spaces in (1), the unital graded ring structure of $\mathrm{Ext}^{\bullet}(\mathscr{F},\mathscr{F})$ induced by the graded composition is given by

$$\begin{array}{ccc}\mathrm{Ext}^{\bullet}(\mathscr{F},\mathscr{F}) \times \mathrm{Ext}^{\bullet}(\mathscr{F},\mathscr{F}) & \longrightarrow & \mathrm{End}^{\bullet}(\mathscr{F})\,,\\ \big((u,\vec{x}\,),(v,\vec{y}\,)\big) & \longmapsto & (v,\vec{y}\,)\circ(u,\vec{x}\,) \;=\; (vu,\, v\vec{x}+u\vec{y}\,)\,,\end{array}$$

for all $u,\, v \in \mathbb{K}$ and $\vec{x},\, \vec{y} \in \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}$.

Equivalently, there is an isomorphism of unital graded rings

$$\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{F}) \cong H^\bullet(\Sigma_{g,1},\mathbb{K})\,,$$

where $H^\bullet(\Sigma_{g,1},\mathbb{K})$ *denotes the unital cohomology ring over* $\mathbb{K}$ *of a genus-g surface* $\Sigma_{g,1}$ *with one puncture and* $2g = \mathrm{tb}(\Lambda(\beta)) + 1$.

Proof. First, observe that part (1) follows directly from Theorem (5.92); nevertheless, we briefly outline the main arguments of its proof, as this will naturally set the stage for the proof of part (2).

To begin, consider the linear map $\delta_{\mathscr{F},\mathscr{F}} : \mathbb{K}^n_{\mathrm{std}} \to \mathbb{K}^\ell_{\mathrm{std}}$ associated with $\mathscr{F}$, as introduced in Definition (5.55). Then, building on Theorems (5.66) and (5.92), we identify $\mathrm{Ext}^0(\mathscr{F},\mathscr{F})$ with $\ker\delta_{\mathscr{F},\mathscr{F}}$ and $\mathrm{Ext}^1(\mathscr{F},\mathscr{F})$ with $\mathrm{coker}\,\delta_{\mathscr{F},\mathscr{F}}$.

Let $\vec{u} = (u_1,\dots,u_n) \in \ker\delta_{\mathscr{F},\mathscr{F}} \subseteq \mathbb{K}^n_{\mathrm{std}}$. By an argument analogous to that in the proof of Theorem (6.110), there exists $u \in \mathbb{K}$ such that $u_i = u$ for all $i \in [1,n]$, that is, $\vec{u} = u\,\vec{1}_n$, where $\vec{1}_n = (1,\dots,1) \in \mathbb{K}^n_{\mathrm{std}}$. Hence, since $\vec{u}$ is arbitrary, we deduce that $\ker\delta_{\mathscr{F},\mathscr{F}} = \mathrm{Span}_{\mathbb{K}}\{\,\vec{1}_n\,\} \cong \mathbb{K}$. It follows from Theorem (5.93) that $\dim\mathrm{Ext}^1(\mathscr{F},\mathscr{F}) = \mathrm{tb}(\Lambda(\beta)) + \dim\mathrm{Ext}^0(\mathscr{F},\mathscr{F})$, and as a result, we conclude that $\mathrm{coker}\,\delta_{\mathscr{F},\mathscr{F}} \cong \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}$. Bearing this in mind, part *(1)* follows:

$$\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{F}) \cong \mathbb{K}\,[0] \oplus \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}\,[1]\;.$$

Next, we verify part *(2)*. For this purpose, we introduce the following notation. Let $u \in \mathbb{K}$ and $\vec{x} \in \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}$. Then, we write

$$u \simeq \vec{u} \in \ker\delta_{\mathscr{F},\mathscr{F}}\,, \quad \text{and} \quad \vec{x} \simeq [\,\vec{x}\,'\,] \in \mathrm{coker}\,\delta_{\mathscr{F},\mathscr{F}}\,, \quad \vec{x}\,' \in \mathbb{K}^\ell_{\mathrm{std}}$$

to denote that $\vec{u}$ and $[\,\vec{x}\,'\,]$ are the unique elements corresponding to u and $\vec{x}$ under the isomorphism of graded vector spaces established in part *(1)*, respectively. With this notation at hand, we proceed to analyze the algebraic structure induced by the graded composition in $\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{F})$ by considering the following three cases:

- *Case 1* $(\mathrm{Ext}^0(\mathscr{F},\mathscr{F}) \times \mathrm{Ext}^0(\mathscr{F},\mathscr{F}))$: Let $u, v \in \mathbb{K}$, and consider $\vec{u},\vec{v} \in \ker\delta_{\mathscr{F},\mathscr{F}}$ such that $u \simeq \vec{u}$ and $v \simeq \vec{v}$, namely, $\vec{u} = u\,\vec{1}_n$ and $\vec{v} = v\,\vec{1}_n$. By Theorem (5.104), we know that $\vec{v}\circ\vec{u} = \vec{v}\odot\vec{u} \in \ker\delta_{\mathscr{F},\mathscr{F}}$, which implies that $\vec{v}\circ\vec{u} = vu\,\vec{1}_n$. Hence, if we define the composition

$$\begin{array}{rccc} \circ: & \mathbb{K}\times\mathbb{K} & \to & \mathbb{K} \\ & (v,u) & \mapsto & v\circ u \end{array}$$

via the assignment $v\circ u \simeq \vec{v}\circ\vec{u}$, we deduce that $v\circ u = vu$.

• *Case 2* ($\mathrm{Ext}^1(\mathscr{F},\mathscr{F})\times\mathrm{Ext}^0(\mathscr{F},\mathscr{F})$): Let $u\in\mathbb{K}$ and $\vec{y}\in\mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}$, and consider $\vec{u}\in\ker\delta_{\mathscr{F},\mathscr{F}}$ and $[\,\vec{y}\,'\,]\in\operatorname{coker}\delta_{\mathscr{F},\mathscr{F}}$ such that $u\simeq\vec{u}$ and $\vec{y}\simeq[\,\vec{y}\,'\,]$ for some $\vec{y}\,'=(y_1',\dots,y_\ell')\in\mathbb{K}^\ell_{\mathrm{std}}$; in particular, $\vec{u}=(u_1,\dots,u_n)\in\mathbb{K}^n_{\mathrm{std}}$ with $u_i=u$ for all $i\in[1,n]$, namely $\vec{u}=u\,\vec{1}_n$. By Theorem (5.105), we know that $[\,\vec{y}\,']\circ\vec{u}=[\,\vec{y}\,'\circ_{\beta_{\mathrm{R}}}\vec{u}\,]$, where $\vec{y}\,'\circ_{\beta_{\mathrm{R}}}\vec{u}=(\tilde{y}_1,\dots,\tilde{y}_\ell)\in\mathbb{K}^\ell_{\mathrm{std}}$ is entrywise given by

$$\tilde{y}_j=y_j'\cdot u_{\pi_{\beta_j}(i_j)}\,,$$

for all $j\in[1,\ell]$. Thus, since $u_{\pi_{\beta_j}(i_j)}=u$ for all $j\in[1,\ell]$, we obtain that $\vec{y}\,'\circ_{\beta_{\mathrm{R}}}\vec{u}=u\,\vec{y}\,'$, and as a result, we also deduce that $u\,\vec{y}\simeq u\,[\,\vec{y}\,'\,]=[\,\vec{y}\,']\circ\vec{u}$. Hence, if we define the composition

$$\begin{aligned}\circ:\ \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}\times\mathbb{K}\ &\to\ \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}\\ (\,\vec{y},u\,)\qquad&\mapsto\qquad\vec{y}\circ u\end{aligned}$$

via the assignment $\vec{y}\circ u\simeq[\,\vec{y}\,']\circ\vec{u}$, we conclude that $\vec{y}\circ u=u\vec{y}$.

• *Case 3* ($\mathrm{Ext}^0(\mathscr{F},\mathscr{F})\times\mathrm{Ext}^1(\mathscr{F},\mathscr{F})$): Let $v\in\mathbb{K}$ and $\vec{x}\in\mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}$, and consider $\vec{v}\in\ker\delta_{\mathscr{F},\mathscr{F}}$ and $[\,\vec{x}\,'\,]\in\operatorname{coker}\delta_{\mathscr{F},\mathscr{F}}$ such that $v\simeq\vec{v}$ and $\vec{x}\simeq[\,\vec{x}\,'\,]$ for some $\vec{x}\,'=(x_1',\dots,x_\ell')\in\mathbb{K}^\ell_{\mathrm{std}}$; in particular, $\vec{v}=(v_1,\dots,v_n)\in\mathbb{K}^n_{\mathrm{std}}$ with $v_i=v$ for all $i\in[1,n]$, namely $\vec{v}=v\,\vec{1}_n$. By Theorem (5.106), we know that $\vec{v}\circ[\,\vec{x}\,'\,]=[\,\vec{v}\circ_{\beta_{\mathrm{L}}}\vec{x}\,'\,]$, where $\vec{v}\circ_{\beta_{\mathrm{L}}}\vec{x}\,'=(\tilde{x}_1,\dots,\tilde{x}_\ell)\in\mathbb{K}^\ell_{\mathrm{std}}$ is entrywise given by

$$\tilde{x}_j=v_{\pi_{\beta_j}(i_j+1)}\cdot x_j'\,,$$

for all $j\in[1,\ell]$. Thus, since $v_{\pi_{\beta_j}(i_j+1)}=v$ for all $j\in[1,\ell]$, we obtain that $\vec{v}\circ_{\beta_{\mathrm{L}}}\vec{x}\,'=v\,\vec{x}\,'$, and as a result, we also deduce that $v\,\vec{x}\simeq v\,[\,\vec{x}\,'\,]=\vec{v}\circ[\,\vec{x}\,']$. Hence, if we define the composition

$$\begin{aligned}\circ:\ \mathbb{K}\times\mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}\ &\to\ \mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}\\ (\,v,\vec{x}\,)\qquad&\mapsto\qquad v\circ\vec{x}\end{aligned}$$

via the assignment $v\circ\vec{x}\simeq\vec{v}\circ[\,\vec{x}\,']$, we conclude that $v\circ\vec{x}=v\,\vec{x}$.

In particular, combining the above cases, we obtain the graded composition rule

$$\begin{aligned}\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{F})\times\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{F})\quad&\longrightarrow\qquad\qquad\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{F})\,,\\ \big((u,\vec{x}\,),(v,\vec{y}\,)\big)\qquad&\longmapsto\quad(v,\vec{y}\,)\circ(u,\vec{x})\ =\ (vu,\,v\vec{x}+u\vec{y}\,)\,,\end{aligned}$$

for all u, $v\in\mathbb{K}$ and $\vec{x}$, $\vec{y}\in\mathbb{K}^{\mathrm{tb}(\Lambda(\beta))+1}$. Bearing this in mind, part *(2)* follows.

Now, let $\Sigma_{g,1}$ be a genus g-surface with one puncture. As is well known, the unital graded cohomological ring $H^\bullet(\Sigma_{g,1},\mathbb{K})$ of $\Sigma_{g,1}$ over $\mathbb{K}$ is given by

$$H^\bullet(\Sigma_{g,1},\mathbb{K})=\mathbb{K}[0]\oplus\mathbb{K}^{2g}[1]\,,$$

where the graded composition rule induced by the cup product is defined by

$$\begin{aligned} H^\bullet(\Sigma_{g,1},\mathbb{K})\times H^\bullet(\Sigma_{g,1},\mathbb{K}) &\longrightarrow H^\bullet(\Sigma_{g,1},\mathbb{K})\,,\\ \big((u,\vec{x}\,),(v,\vec{y}\,)\big) &\longmapsto (v,\vec{y}\,)\circ(u,\vec{x}) \;=\; (vu,\, v\vec{x}+u\vec{y}\,)\,, \end{aligned}$$

for all $u,\, v\in\mathbb{K}$ and $\vec{x},\ \vec{y}\in\mathbb{K}^{2g}$.

Finally, recall that since by definition $\Lambda(\beta)$ is a Legendrian knot with maximal Thurston–Bennequin number, arising as the rainbow closure of a positive braid word β, $\mathrm{tb}(\Lambda(\beta))$ is an odd integer. Thus, if we consider $\Sigma_{g,1}$ such that $2g=\mathrm{tb}(\Lambda(\beta))+1$, we conclude that there exists an isomorphism of unital graded rings

$$\mathrm{Ext}^\bullet(\mathscr{F},\mathscr{F})\cong H^\bullet(\Sigma_{g,1},\mathbb{K})\,.$$

This completes the proof. □

Conceptually, the previous result may be viewed as a sheaf-theoretic analogue of the *Seidel isomorphism theorem* [DR16]. From another perspective, and building on the insightful work of Chantraine [Cha10], Theorem (6.110) shows that, when $\Lambda(\beta)$ is a Legendrian knot, the unital graded ring structure of the graded endomorphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ is fully determined by the smooth topology of the possible exact Lagrangian fillings of $\Lambda(\beta)$. This reveals the rich algebraic, geometric, and topological structures encoded by the categorical invariant under consideration and illustrates the power of the sheaf-theoretic framework we have developed in this paper.

With the above result at hand, we conclude our discussion compiling some structural aspects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ in the Legendrian knot case. Building on this, we proceed to analyze in detail some aspects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta),\mathbb{K})_0)$ in the case where $\mathbb{K}=\mathbb{Z}_2$ and $\Lambda(\beta)$ is a concrete Legendrian knot, namely the Legendrian trefoil knot on three strands.

6.1.2. *The Case of the Trefoil Knot.* Let $\beta_{\mathrm{T}}:=\sigma_1\sigma_2\sigma_1\sigma_2\in\mathrm{Br}_3^+$. In this case, $\Lambda(\beta_{\mathrm{T}})\subset(\mathbb{R}^3,\xi_{\mathrm{std}})$ is Legendrian trefoil knot presented as the rainbow closure of a positive braid word on three strands. Next, we apply the framework we developed in the previous sections to provide a detailed description of certain aspects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{T}}),\mathbb{Z}_2)_0)$. In particular, since this category is a Legendrian isotopy invariant, our results agree with those in [STZ17], where the Legendrian trefoil knot is studied as the rainbow closure of a positive braid on two strands. What is new here is the explicit description of the compositions of graded morphisms, which is not discussed in [STZ17].

- *Setup*: To begin, observe that, for any $\vec{z} = (z_1, z_2, z_3, z_4) \in \mathbb{Z}_2^4$, the path matrix associated with β_{T} is given by

$$P_{\beta_{\mathrm{T}}}(\vec{z}) = \begin{bmatrix} z_1 z_3 + z_2 & z_1 z_4 + 1 & z_1 \\ z_3 & z_4 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \mathrm{GL}(3, \mathbb{Z}_2)\,.$$

Consequently, the braid variety associated with β_{T} is defined by

$$X(\beta_{\mathrm{T}}, \mathbb{Z}_2) = \left\{ \vec{z} = (z_1, z_2, z_3, z_4) \in \mathbb{Z}_2^4 \mid z_1 z_3 + z_2 = 1\,,\ z_2 z_4 - z_3 = 1 \right\} \subset \mathbb{Z}_2^4\,.$$

Accordingly, a straightforward calculation shows that $X(\beta_{\mathrm{H}}, \mathbb{Z}_2)$ consists of five points:

$$X(\beta_{\mathrm{T}}, \mathbb{Z}_2) = \{\vec{z}_1 = (0,1,0,1)\,, \quad \vec{z}_2 = (0,1,1,0)\,, \quad \vec{z}_3 = (1,0,1,0)\,, \quad \vec{z}_4 = (1,0,1,1)\,, \quad \vec{z}_5 = (1,1,0,1)\}\,.$$

By Theorem (4.46), the objects of the category $H^*(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{T}}), \mathbb{Z}_2)_0)$ are parametrized by a basis for $\mathbb{Z}_2^3$ together with a point in the braid variety $X(\beta_{\mathrm{T}}, \mathbb{Z}_2)$. Bearing this in mind, we write

$$\mathrm{Ob}(H^*(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{T}}), \mathbb{Z}_2)_0)) = \left\{ \left(\hat{\mathbf{f}}^{(3)}\,,\, \vec{z} \right) \,\middle|\, \hat{\mathbf{f}}^{(3)} \text{ is a basis for } \mathbb{Z}_2^3,\ \vec{z} \in X(\beta_{\mathrm{T}}, \mathbb{Z}_2) \right\}\,.$$

Under the preceding identification, the linear structure of the graded morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{T}}), \mathbb{Z}_2)_0)$ is described as follows: let $\mathscr{F} = \left(\hat{\mathbf{f}}^{(3)}\,,\, \vec{x} \right)$ and $\mathscr{G} = \left(\hat{\mathbf{g}}^{(3)}\,,\, \vec{y} \right)$ be objects of the category $H^*(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{T}}), \mathbb{Z}_2)_0)$, and let $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{Z}_2^3 \to \mathbb{Z}_2^4$ be the linear maps associated with these objects, as introduced in Definition (5.55). Specifically,

$$\delta_{\mathscr{F},\mathscr{G}}(\vec{u}) = (u_1 x_1 - y_1 u_2, u_1 x_2 - y_2 u_3, u_2 x_3 - y_3 u_3, u_2 x_4 - y_4 u_1)\,, \qquad \vec{u} = (u_1, u_2, u_3) \in \mathbb{Z}_2^3\,,$$

where $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$. Then, building on Theorems (5.66) and (5.92), we write

$$\mathrm{Ext}^0(\mathscr{F}, \mathscr{G}) = \ker \delta_{\mathscr{F},\mathscr{G}}\,, \quad \text{and} \quad \mathrm{Ext}^1(\mathscr{F}, \mathscr{G}) = \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}}\,.$$

Later, when dealing with concrete examples and in order to simplify our discussion, we will work under the identification $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G}) = W$, where W corresponds to some fixed complement of $\operatorname{im} \delta_{\mathscr{F},\mathscr{G}}$ in $\mathbb{Z}_2^4$, that is, a vector subspace such that $\mathbb{Z}_2^4 = \operatorname{im} \delta_{\mathscr{F},\mathscr{G}} \oplus W$.

With the above identifications at hand, the graded composition between the graded morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{T}}), \mathbb{Z}_2)_0)$ is characterized as follows: let $\mathscr{F} = \left(\hat{\mathbf{f}}^{(3)}\,,\, \vec{x} \right)$, $\mathscr{G} = \left(\hat{\mathbf{g}}^{(3)}\,,\, \vec{y} \right)$, and $\mathscr{Q} = \left(\hat{\mathbf{q}}^{(3)}\,,\, \vec{z} \right)$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{T}}), \mathbb{Z}_2)_0)$. Fix $\vec{u} = (u_1, u_2, u_3) \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{G})$, $\vec{v} = (v_1, v_2, v_3) \in \mathrm{Ext}^0(\mathscr{G}, \mathscr{Q})$, $\Sigma \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, and $\Gamma \in \mathrm{Ext}^1(\mathscr{G}, \mathscr{Q})$, and let $\vec{p} = (p_1, p_2, p_3, p_4) \in \mathbb{Z}_2^4$ and $\vec{q} = (q_1, q_2, q_3, q_4) \in \mathbb{Z}_2^4$ be representatives such that $\Sigma = [\vec{p}]$ and $\Gamma = [\vec{q}]$. Then, by Theorems (5.104), (5.106), and (5.105), the graded compositions between the morphisms under consideration are given by:

$$\vec{v} \circ \vec{u} = \vec{v} \odot \vec{u} \in \mathrm{Ext}^0(\mathscr{F}, \mathscr{Q}), \qquad \vec{v} \odot \vec{u} = (v_1 u_1, v_2 u_2, v_3 u_3, v_4 u_4) \in \mathbb{Z}_2^3,$$

$$\Gamma \circ \vec{u} = [\, \vec{q} \circ_{\beta_{\mathrm{R}}} \vec{u}\,] \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{Q}), \qquad \vec{q} \circ_{\beta_{\mathrm{R}}} \vec{u} = (q_1 u_2, q_2 u_3, q_3 u_3, q_4 u_1) \in \mathbb{Z}_2^4,$$

$$\vec{v} \circ \Sigma = [\, \vec{v} \circ_{\beta_{\mathrm{L}}} \vec{p}\,] \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{Q}), \qquad \vec{v} \circ_{\beta_{\mathrm{L}}} \vec{p} = (v_1 p_1, v_1 p_2, v_2 p_3, u_2 p_4) \in \mathbb{Z}_2^4.$$

Later, when working with concrete examples and in order to simplify our discussion, we identify $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$, $\mathrm{Ext}^1(\mathscr{G}, \mathscr{Q})$, $\mathrm{Ext}^1(\mathscr{F}, \mathscr{Q})$ with some fixed complements W_1, W_2, W_3 of the images of the corresponding linear maps $\delta_{\mathscr{F},\mathscr{G}}$, $\delta_{\mathscr{G},\mathscr{Q}}$, $\delta_{\mathscr{F},\mathscr{Q}} : \mathbb{Z}_2^3 \to \mathbb{Z}_2^4$ in $\mathbb{Z}_2^4$. In this setting, the mixed degree graded compositions are computed via the left $\circ_{\beta_{\mathrm{L}}} : \mathbb{Z}_2^3 \times \mathbb{Z}_2^4 \to \mathbb{Z}_2^4$ and right $\circ_{\beta_{\mathrm{R}}} : \mathbb{Z}_2^4 \times \mathbb{Z}_2^3 \to \mathbb{Z}_2^4$ braided compositions, and the resulting vectors in the ambient space $\mathbb{Z}_2^4$ are then projected onto W_3 for consistency.

• *Some Concrete Computations*: To illustrate how to apply the setup we established above, we consider five explicit objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{T}}), \mathbb{Z}_2)_0)$:

$$\left\{ \mathscr{F}_i = \left(\hat{\mathbf{f}}_i^{(3)}\,,\, \vec{z}_i \right) \;\middle|\; \begin{array}{c} \mathbf{f}_i^{(3)} \text{ is a fixed basis for } \mathbb{Z}_2^3\,, \\ \vec{z}_i \text{ is the } i\text{-th distinct point in } X(\beta_{\mathrm{T}}, \mathbb{Z}_2)\,, \\ i \in [1,5] \end{array} \right\}.$$

Next, we analyze in detail the algebraic properties of some of the graded morphism spaces associated with the objects $\{\mathscr{F}_i\}_{i=1}^n$. In particular, we begin by examining the ring structure of the graded endomorphism space $\mathrm{End}^\bullet(\mathscr{F}_1)$, summarized for clarity in Table 1. From this table, we see that $\mathrm{End}^\bullet(\mathscr{F}_1)$ has a particularly simple ring structure, namely that of the cohomology ring of a genus-one surface with one puncture. This is not a coincidence: since $\mathrm{tb}(\Lambda(\beta_{\mathrm{T}})) = \ell - n = 1$, Theorem (6.110) asserts that, for any $i \in [1,5]$, there is an isomorphism of unital graded rings

$$\mathrm{End}^\bullet(\mathscr{F}_i) \cong H^\bullet(\Sigma_{1,1}, \mathbb{Z}_2),$$

where $H^\bullet(\Sigma_{1,1}, \mathbb{Z}_2)$ denotes the unital cohomology ring over $\mathbb{Z}_2$ of a genus-one surface with one puncture. It follows that the algebraic structure of the graded endomorphism spaces of the objects $\{\mathscr{F}_i\}_{i=1}^5$ is particularly simple and depends primarily on intrinsic data associated with the Legendrian knot $\Lambda(\beta_{\mathrm{T}})$.

We now extend our discussion by explicitly analyzing the graded morphism spaces and their compositions for the ordered triples $(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$ and $(\mathscr{F}_1, \mathscr{F}_1, \mathscr{F}_2)$. As summarized in Tables 2 and 3, these examples illustrate two contrasting behaviors: when all objects are distinct, compositions are trivial, whereas when some objects coincide, nontrivial mixed-degree compositions appear.

Finally, we conclude our discussion by examining the linear algebraic structure of the graded morphism spaces associated with the objects $\{\mathscr{F}_i\}_{i=1}^5$. For this purpose, recall that $\mathrm{tb}(\Lambda(\beta_{\mathrm{T}})) = 1$. Then, since the points

Object: $\mathscr{F}_1$
Graded Endomorphism Spaces
$\mathrm{End}^0(\mathscr{F}_1) = \mathrm{Span}_{\mathbb{Z}_2}\{\hat{u}_1 = \langle 1,1,1\rangle\}$
$\mathrm{End}^1(\mathscr{F}_1) = \mathrm{Span}_{\mathbb{Z}_2}\{\hat{\alpha}_1 = \langle 1,0,0,0\rangle\,,\ \hat{\alpha}_2 = \langle 0,0,1,0\rangle\}$
Graded Composition
$\circ:\ \mathrm{End}^0(\mathscr{F}_1)\times\mathrm{End}^0(\mathscr{F}_1) \to \mathrm{End}^0(\mathscr{F}_1)\,,\quad (b_1\hat{u}_1\,, a_1\hat{u}_1) \mapsto b_1a_1\hat{u}_1\,.$
$\circ:\ \mathrm{End}^1(\mathscr{F}_1)\times\mathrm{End}^0(\mathscr{F}_1) \to \mathrm{End}^1(\mathscr{F}_1)\,,\quad (q_1\hat{\alpha}_1 + q_2\hat{\alpha}_2\,, a_1\hat{u}_1) \mapsto q_1a_1\hat{\alpha}_1 + q_2a_1\hat{\alpha}_2\,.$
$\circ:\ \mathrm{End}^0(\mathscr{F}_1)\times\mathrm{End}^1(\mathscr{F}_1) \to \mathrm{End}^1(\mathscr{F}_1)\,,\quad (b_1\hat{u}_1\,, p_1\hat{\alpha}_1 + p_2\hat{\alpha}_2) \mapsto b_1p_1\hat{\alpha}_1 + b_1p_2\hat{\alpha}_2\,.$

TABLE 1. Unital ring structure of the graded endomorphism space $\mathrm{End}^\bullet(\mathscr{F}_1)$.

Ordered Triple: $(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$	
Graded morphism Spaces	
$\mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) = 0$	$\mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) = \mathrm{Span}_{\mathbb{Z}_2}\{\hat{\alpha}_1 = \langle 1,0,0,0\rangle\}$
$\mathrm{Ext}^0(\mathscr{F}_2, \mathscr{F}_3) = 0$	$\mathrm{Ext}^1(\mathscr{F}_2, \mathscr{F}_3) = \mathrm{Span}_{\mathbb{Z}_2}\{\hat{\beta}_1 = \langle 0,0,0,1\rangle\}$
$\mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_3) = 0$	$\mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_3) = \mathrm{Span}_{\mathbb{Z}_2}\{\hat{\gamma}_1 = \langle 1,0,0,1\rangle\}$
Graded Composition	
$\circ:\ \mathrm{Ext}^0(\mathscr{F}_2, \mathscr{F}_3)\times\mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) \to \mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_3)\,,\quad (0\,,0) \mapsto 0\,.$	
$\circ:\ \mathrm{Ext}^1(\mathscr{F}_2, \mathscr{F}_3)\times\mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) \to \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_3)\,,\quad (q_1\hat{\beta}_1\,,0) \mapsto 0\,.$	
$\circ:\ \mathrm{Ext}^0(\mathscr{F}_2, \mathscr{F}_3)\times\mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) \to \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_3)\,,\quad (0\,,p_1\hat{\alpha}_1) \mapsto 0\,.$	

TABLE 2. Graded morphism spaces and their compositions for the ordered triple $(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$.

$\vec{z}_i$ are all distinct and the torus action on $X(\beta_{\mathrm{T}}, \mathbb{Z}_2)$ is trivial, Theorem (6.109) yields an isomorphism of graded vector spaces

$$\mathrm{Ext}^\bullet(\mathscr{F}_i, \mathscr{F}_j) \cong \begin{cases} \mathbb{Z}_2[0]\oplus\mathbb{Z}_2^2[1], & \text{if } i = j, \\ 0[0]\oplus\mathbb{Z}_2[1], & \text{if } i \neq j, \end{cases}$$

which shows that the linear structure of the graded morphism spaces associated with the objects $\{\mathscr{F}_i\}_{i=1}^5$ is particularly simple and largely determined by intrinsic data associated with the Legendrian knot $\Lambda(\beta_{\mathrm{T}})$.

<table>
<tr><td colspan="2">Ordered Triple: $(\mathscr{F}_1, \mathscr{F}_1, \mathscr{F}_2)$</td></tr>
<tr><td colspan="2">Graded morphism Spaces</td></tr>
<tr><td>$\mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_1) = \mathrm{Span}_{\mathbb{Z}_2}\{\hat{u}_1 = \langle 1,1,1\rangle\}$</td><td>$\mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_1) = \mathrm{Span}_{\mathbb{Z}_2}\{\hat{\alpha}_1 = \langle 1,0,0,0\rangle\,,\ \hat{\alpha}_2 = \langle 0,0,1,0\rangle\}$</td></tr>
<tr><td>$\mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) = 0$</td><td>$\mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) = \mathrm{Span}_{\mathbb{Z}_2}\{\hat{\beta}_1 = \langle 1,0,0,0\rangle\}$</td></tr>
<tr><td colspan="2">Graded Composition</td></tr>
<tr><td colspan="2">$$\begin{array}{rccc} \circ: & \mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) \times \mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_1) & \to & \mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_2)\,, \\ & \big(0\,, a_1\hat{u}_1\big) & \mapsto & 0\,. \end{array}$$</td></tr>
<tr><td colspan="2">$$\begin{array}{rccc} \circ: & \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) \times \mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_1) & \to & \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2)\,, \\ & \big(q_1\hat{\beta}_1\,, a_1\hat{u}_1\big) & \mapsto & q_1 a_1\hat{\beta}_1\,. \end{array}$$</td></tr>
<tr><td colspan="2">$$\begin{array}{rccc} \circ: & \mathrm{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) \times \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_1) & \to & \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2)\,, \\ & \big(0\,, p_1\hat{\alpha}_1 + p_2\hat{\alpha}_2\big) & \mapsto & 0\,. \end{array}$$</td></tr>
</table>

Table 3. Graded morphism spaces and their compositions for the ordered triple $(\mathscr{F}_1, \mathscr{F}_1, \mathscr{F}_2)$.

Having established this, we conclude our study of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ in the case where $\Lambda(\beta)$ is a Legendrian knot. The general case of Legendrian links is richer in structure and depends more heavily on the choice of the positive braid word β. In the next subsection, we apply the machinery we developed in the preceding sections to a simple yet illuminating example that offers significant insight into the structure of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta), \mathbb{K})_0)$ in the case where $\mathbb{K} = \mathbb{Z}_2$ and $\Lambda(\beta)$ is a Legendrian link, namely the Legendrian Hopf link on three strands.

6.2. **The Case of the Hopf Link.** Let $\beta_{\mathrm{H}} := \sigma_1\sigma_2\sigma_1 \in \mathrm{Br}_3^+$. In this case, $\Lambda(\beta_{\mathrm{H}}) \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ is the Legendrian Hopf link presented as the rainbow closure of a positive braid word on three strands. Next, we apply the framework we developed in the previous sections to provide a detailed description of certain aspects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{H}}), \mathbb{Z}_2)_0)$.

- *Setup*: To begin, observe that, for any $\vec{z} = (z_1, z_2, z_3) \in \mathbb{Z}_2^3$, the path matrix associated with β_{H} is given by

$$P_{\beta_{\mathrm{H}}}(\vec{z}\,) = \begin{bmatrix} z_1z_3 + z_2 & z_1 & 1 \\ z_3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathrm{GL}(3, \mathbb{Z}_2)\,.$$

Consequently, the braid variety associated with β_{H} is defined by

$$X(\beta_{\mathrm{H}}, \mathbb{Z}_2) = \left\{\vec{z} = (z_1, z_2, z_3) \in \mathbb{Z}_2^3 \mid z_1z_3 + z_2 = 1\,,\ z_2 = 1\right\} \subset \mathbb{Z}_2^3\,.$$

Accordingly, a straightforward calculation shows that $X(\beta_{\mathrm{H}}, \mathbb{Z}_2)$ consists of three points:

$$X(\beta_{\mathrm{H}}, \mathbb{Z}_2) = \{\vec{z}_1 = (0,1,0)\,, \quad \vec{z}_2 = (0,1,1)\,, \quad \vec{z}_3 = (1,1,0)\}\,.$$

By Theorem (4.46), the objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{H}}),\mathbb{Z}_2)_0)$ are parametrized by a choice of basis for $\mathbb{Z}_2^3$ together with a point in the braid variety $X(\beta_{\mathrm{H}},\mathbb{Z}_2)$. Bearing this in mind, we write

$$\mathrm{Ob}(H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{H}}),\mathbb{Z}_2)_0)) = \left\{ \left(\hat{\mathbf{f}}^{(3)} \,,\, \vec{z} \right) \,\middle|\, \hat{\mathbf{f}}^{(3)} \text{ is a basis for } \mathbb{Z}_2^3,\ \vec{z} \in X(\beta_{\mathrm{H}},\mathbb{Z}_2) \right\} .$$

Under the preceding identification, the linear structure of the graded morphism spaces in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{H}}),\mathbb{Z}_2)_0)$ is described as follows: let $\mathscr{F} = \left(\hat{\mathbf{f}}^{(3)} \,,\, \vec{x} \right)$ and $\mathscr{G} = \left(\hat{\mathbf{g}}^{(3)} \,,\, \vec{y} \right)$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{H}}),\mathbb{Z}_2)_0)$, and let $\delta_{\mathscr{F},\mathscr{G}} : \mathbb{Z}_2^3 \to \mathbb{Z}_2^3$ be the linear maps associated with these objects, as introduced in Definition (5.55). Specifically,

$$\delta_{\mathscr{F},\mathscr{G}}(\vec{u}) = (u_1 x_1 - y_1 u_2, u_1 x_2 - y_2 u_3, u_2 x_3 - y_3 u_3)\,, \qquad \vec{u} = (u_1,u_2,u_3) \in \mathbb{Z}_2^3\,,$$

where $\vec{x} = (x_1,x_2,x_3)$ and $\vec{y} = (y_1,y_2,y_3)$. Then, building on Theorems (5.66) and (5.92), we write

$$\mathrm{Ext}^0(\mathscr{F},\mathscr{G}) = \ker \delta_{\mathscr{F},\mathscr{G}}\,, \quad \text{and} \quad \mathrm{Ext}^1(\mathscr{F},\mathscr{G}) = \operatorname{coker} \delta_{\mathscr{F},\mathscr{G}}\,.$$

Later, when dealing with concrete examples and in order to simplify our discussion, we will work under the identification $\mathrm{Ext}^1(\mathscr{F},\mathscr{G}) = W$, where W corresponds to some fixed complement of $\operatorname{im} \delta_{\mathscr{F},\mathscr{G}}$ in $\mathbb{Z}_2^3$, that is, a vector subspace such that $\mathbb{Z}_2^3 = \operatorname{im} \delta_{\mathscr{F},\mathscr{G}} \oplus W$.

In particular, with the preceding identifications at hand, the composition of graded morphisms in the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{H}}),\mathbb{Z}_2)_0)$ is characterized as follows: let $\mathscr{F} = \left(\hat{\mathbf{f}}^{(3)} \,,\, \vec{x} \right)$, $\mathscr{G} = \left(\hat{\mathbf{g}}^{(3)} \,,\, \vec{y} \right)$, and $\mathscr{Q} = \left(\hat{\mathbf{q}}^{(3)} \,,\, \vec{z} \right)$ be objects of the category $H^\bullet(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{H}}),\mathbb{Z}_2)_0)$. Fix $\vec{u} = (u_1,u_2,u_3) \in \mathrm{Ext}^0(\mathscr{F},\mathscr{G})$, $\vec{v} = (v_1,v_2,v_3) \in \mathrm{Ext}^0(\mathscr{G},\mathscr{Q})$, $\Sigma \in \mathrm{Ext}^1(\mathscr{F},\mathscr{G})$, and $\Gamma \in \mathrm{Ext}^1(\mathscr{G},\mathscr{Q})$, and let $\vec{p} = (p_1,p_2,p_3) \in \mathbb{Z}_2^3$ and $\vec{q} = (q_1,q_2,q_3) \in \mathbb{Z}_2^3$ be representatives such that $\Sigma = [\vec{p}]$ and $\Gamma = [\vec{q}]$. Then, by Theorems (5.104), (5.106), and (5.105), the graded compositions between the morphisms under consideration are given by:

$$\vec{v} \circ \vec{u} = \vec{v} \odot \vec{u} \in \mathrm{Ext}^0(\mathscr{F},\mathscr{Q})\,, \qquad \vec{v} \odot \vec{u} = (v_1u_1, v_2u_2, v_3u_3) \in \mathbb{Z}_2^3\,,$$

$$\Gamma \circ \vec{u} = [\, \vec{q} \circ_{\beta_{\mathrm{R}}} \vec{u}\,] \in \mathrm{Ext}^1(\mathscr{F},\mathscr{Q})\,, \qquad \vec{q} \circ_{\beta_{\mathrm{R}}} \vec{u} = (q_1u_2, q_2u_3, q_3u_3) \in \mathbb{Z}_2^3\,,$$

$$\vec{v} \circ \Sigma = [\, \vec{v} \circ_{\beta_{\mathrm{L}}} \vec{p}\,] \in \mathrm{Ext}^1(\mathscr{F},\mathscr{Q})\,, \qquad \vec{v} \circ_{\beta_{\mathrm{L}}} \vec{p} = (v_1p_1, v_1p_2, v_2p_3) \in \mathbb{Z}_2^3\,.$$

Later, when working with concrete examples and in order to simplify our discussion, we identify $\mathrm{Ext}^1(\mathscr{F},\mathscr{G})$, $\mathrm{Ext}^1(\mathscr{G},\mathscr{Q})$, $\mathrm{Ext}^1(\mathscr{F},\mathscr{Q})$ with some fixed complements W_1, W_2, W_3 of the corresponding linear maps $\delta_{\mathscr{F},\mathscr{G}}$, $\delta_{\mathscr{G},\mathscr{Q}}$, $\delta_{\mathscr{F},\mathscr{Q}} : \mathbb{Z}_2^3 \to \mathbb{Z}_2^3$ in $\mathbb{Z}_2^3$. In this setting, the mixed degree graded compositions are computed via the left $\circ_{\beta_{\mathrm{L}}} : \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \to \mathbb{Z}_2^3$ and right $\circ_{\beta_{\mathrm{R}}} : \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \to \mathbb{Z}_2^3$ braided compositions, and the resulting vectors in the ambient space $\mathbb{Z}_2^3$ are then projected onto W_3 for consistency.

• *Some Concrete Computations*: To illustrate how to apply the framework we developed in the previous sections, we consider three concrete objects of the category $H^{\bullet}(\mathcal{S}h_1(\Lambda(\beta_{\mathrm{H}}), \mathbb{Z}_2)_0)$:

$$\mathscr{F}_1 = \left(\hat{\mathbf{f}}^{(3)} , \vec{z}_1 \right) , \qquad \mathscr{F}_2 = \left(\hat{\mathbf{g}}^{(3)} , \vec{z}_2 \right) , \qquad \mathscr{F}_3 = \left(\hat{\mathbf{q}}^{(3)} , \vec{z}_3 \right) ,$$

where $\hat{\mathbf{f}}^{(3)}$, $\hat{\mathbf{g}}^{(3)}$, $\hat{\mathbf{q}}^{(3)}$ are fixed bases for $\mathbb{Z}_2^3$, and $\vec{z}_1$, $\vec{z}_2$, $\vec{z}_3$ are the three distinct points in the braid variety $X(\beta_{\mathrm{H}}, \mathbb{Z}_2)$.

Next, we explicitly describe the algebraic properties of some of the graded morphism spaces associated with $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$. To begin, note that the ring structure of the graded endomorphism space $\mathrm{End}^{\bullet}(\mathscr{F}_1)$ is summarized in Table 4, which reveals that $\mathrm{End}^0(\mathscr{F}_1)$ and $\mathrm{End}^1(\mathscr{F}_1)$ are 2-dimensional over $\mathbb{Z}_2$, with the graded composition exhibiting a non-trivial, rich algebraic structure.

Object: $\mathscr{F}_1$
Graded Endomorphism Spaces
$\mathrm{End}^0(\mathscr{F}_1) = \mathrm{Span}_{\mathbb{Z}_2}\left\{ \hat{u}_1 = \langle 0,1,0\rangle \,,\ \hat{u}_2 = \langle 1,0,1\rangle \right\}$
$\mathrm{End}^1(\mathscr{F}_1) = \mathrm{Span}_{\mathbb{Z}_2}\left\{ \hat{\alpha}_1 = \langle 1,0,0\rangle \,,\ \hat{\alpha}_2 = \langle 0,0,1\rangle \right\}$
Graded Composition
$\begin{array}{rccc} \circ : & \mathrm{End}^0(\mathscr{F}_1) \times \mathrm{End}^0(\mathscr{F}_1) & \to & \mathrm{End}^0(\mathscr{F}_1)\,, \\ & \left(b_1\hat{u}_1 + b_2\hat{u}_2 \,, a_1\hat{u}_1 + a_2\hat{u}_2 \right) & \mapsto & b_1a_1\hat{u}_1 + b_2a_2\hat{u}_2 \,. \end{array}$
$\begin{array}{rccc} \circ : & \mathrm{End}^1(\mathscr{F}_1) \times \mathrm{End}^0(\mathscr{F}_1) & \to & \mathrm{End}^1(\mathscr{F}_1)\,, \\ & \left(q_1\hat{\alpha}_1 + q_2\hat{\alpha}_2 \,, a_1\hat{u}_1 + a_2\hat{u}_2 \right) & \mapsto & q_1a_1\hat{\alpha}_1 + q_2a_2\hat{\alpha}_2 \,. \end{array}$
$\begin{array}{rccc} \circ : & \mathrm{End}^0(\mathscr{F}_1) \times \mathrm{End}^1(\mathscr{F}_1) & \to & \mathrm{End}^1(\mathscr{F}_1)\,, \\ & \left(b_1\hat{u}_1 + b_2\hat{u}_2 \,, p_1\hat{\alpha}_1 + p_2\hat{\alpha}_2 \right) & \mapsto & b_2p_1\hat{\alpha}_1 + b_1p_2\hat{\alpha}_2 \,. \end{array}$

TABLE 4. Unital ring structure of the graded endomorphism space $\mathrm{End}^{\bullet}(\mathscr{F}_1)$.

Similarly, the ring structure of the graded endomorphism spaces $\mathrm{End}^{\bullet}(\mathscr{F}_2)$ and $\mathrm{End}^{\bullet}(\mathscr{F}_3)$ is summarized in Tables (5) and (6). In particular, these tables reveal that, compared to $\mathrm{End}^{\bullet}(\mathscr{F}_1)$, the graded endomorphism spaces associated with $\mathscr{F}_2$ and $\mathscr{F}_3$ have a slightly simpler ring structure, namely that of the cohomology ring of an annulus.

Next, we extend our discussion by explicitly analyzing the graded morphism spaces and their compositions for the ordered triple $(\mathscr{F}_3, \mathscr{F}_1, \mathscr{F}_2)$, which for clarity is summarized in Table (7). In this concrete example, the graded composition is largely trivial, with the only non-trivial contributions arising from the degree zero morphism spaces.

Object: $\mathscr{F}_2$
Graded Endomorphism Spaces
$\mathrm{End}^0(\mathscr{F}_2) = \mathrm{Span}_{\mathbb{Z}_2}\{\, \hat{u}_1 = \langle 1,1,1\rangle \,\}$
$\mathrm{End}^1(\mathscr{F}_2) = \mathrm{Span}_{\mathbb{Z}_2}\{\, \hat{\alpha}_1 = \langle 1,0,0\rangle \,\}$
Graded Composition
$\circ : \mathrm{End}^0(\mathscr{F}_2) \times \mathrm{End}^0(\mathscr{F}_2) \to \mathrm{End}^0(\mathscr{F}_2)$, $\big(b_1\hat{u}_1, a_1\hat{u}_1\big) \mapsto b_1a_1\hat{u}_1$.
$\circ : \mathrm{End}^1(\mathscr{F}_2) \times \mathrm{End}^0(\mathscr{F}_2) \to \mathrm{End}^1(\mathscr{F}_2)$, $\big(q_1\hat{\alpha}_1, a_1\hat{u}_1\big) \mapsto q_1a_1\hat{\alpha}_1$.
$\circ : \mathrm{End}^0(\mathscr{F}_2) \times \mathrm{End}^1(\mathscr{F}_2) \to \mathrm{End}^1(\mathscr{F}_2)$, $\big(b_1\hat{u}_1, p_1\hat{\alpha}_1\big) \mapsto b_1p_1\hat{\alpha}_1$.

TABLE 5. Unital ring structure of the graded endomorphism space $\mathrm{End}^\bullet(\mathscr{F}_2)$.

Object: $\mathscr{F}_3$
Graded Endomorphism Spaces
$\mathrm{End}^0(\mathscr{F}_3) = \mathrm{Span}_{\mathbb{Z}_2}\{\, \hat{u}_1 = \langle 1,1,1\rangle \,\}$
$\mathrm{End}^1(\mathscr{F}_3) = \mathrm{Span}_{\mathbb{Z}_2}\{\, \hat{\alpha}_1 = \langle 0,0,1\rangle \,\}$
Graded Composition
$\circ : \mathrm{End}^0(\mathscr{F}_3) \times \mathrm{End}^0(\mathscr{F}_3) \to \mathrm{End}^0(\mathscr{F}_3)$, $\big(b_1\hat{u}_1, a_1\hat{u}_1\big) \mapsto b_1a_1\hat{u}_1$.
$\circ : \mathrm{End}^1(\mathscr{F}_3) \times \mathrm{End}^0(\mathscr{F}_3) \to \mathrm{End}^1(\mathscr{F}_3)$, $\big(q_1\hat{\alpha}_1, a_1\hat{u}_1\big) \mapsto q_1a_1\hat{\alpha}_1$.
$\circ : \mathrm{End}^0(\mathscr{F}_3) \times \mathrm{End}^1(\mathscr{F}_3) \to \mathrm{End}^1(\mathscr{F}_3)$, $\big(b_1\hat{u}_1, p_1\hat{\alpha}_1\big) \mapsto b_1p_1\hat{\alpha}_1$.

TABLE 6. Unital ring structure of the graded endomorphism space $\mathrm{End}^\bullet(\mathscr{F}_3)$.

Finally, we close our discussion by summarizing in Table (8) the linear algebraic structure of the graded morphism spaces associated with $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$. From this table, we observe that $\mathscr{F}_1$ stands out, as its graded morphism spaces exhibit a richer structure compared to the others.

7. Auxiliary Results on Braid Matrices

In this appendix, we record several auxiliary results concerning the braid matrices (see Definition (4.27)). Let $n \geq 2$ be an integer. Given $M \in \mathrm{M}(n, \mathbb{K})$, $k \in [1, n-1]$, and $z \in \mathbb{K}$, the following claims provide explicit formulas for the products $MB_k^{(n)}(z)$, $B_k^{(n)}(z)M$, $M(B_k^{(n)}(z))^{-1}$, and $(B_k^{(n)}(z))^{-1}M$.

Ordered Triple: $(\mathscr{F}_3, \mathscr{F}_1, \mathscr{F}_2)$	
Graded morphism Spaces	
$\operatorname{Ext}^0(\mathscr{F}_3, \mathscr{F}_1) = \operatorname{Span}_{\mathbb{Z}_2}\{\, \hat{u}_1 = \langle 0,1,0\rangle \,\}$	$\operatorname{Ext}^1(\mathscr{F}_3, \mathscr{F}_1) = \operatorname{Span}_{\mathbb{Z}_2}\{\, \hat{\alpha}_1 = \langle 0,0,1\rangle \,\}$
$\operatorname{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) = \operatorname{Span}_{\mathbb{Z}_2}\{\, \hat{v}_1 = \langle 0,1,0\rangle \,\}$	$\operatorname{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) = \operatorname{Span}_{\mathbb{Z}_2}\{\, \hat{\beta}_1 = \langle 1,0,0\rangle \,\}$
$\operatorname{Ext}^0(\mathscr{F}_3, \mathscr{F}_2) = \operatorname{Span}_{\mathbb{Z}_2}\{\, \hat{w}_1 = \langle 0,1,0\rangle \,\}$	$\operatorname{Ext}^1(\mathscr{F}_3, \mathscr{F}_2) = \operatorname{Span}_{\mathbb{Z}_2}\{\, \hat{\gamma}_1 = \langle 1,1,1\rangle \,\}$
Graded Composition	
$\begin{array}{rcl} \circ :\ \operatorname{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) \times \operatorname{Ext}^0(\mathscr{F}_3, \mathscr{F}_1) & \to & \operatorname{Ext}^0(\mathscr{F}_3, \mathscr{F}_2)\,, \\ \big(b_1\hat{v}_1\,, a_1\hat{u}_1\big) & \mapsto & b_1 a_1 \hat{w}_1\,. \end{array}$	
$\begin{array}{rcl} \circ :\ \operatorname{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) \times \operatorname{Ext}^0(\mathscr{F}_3, \mathscr{F}_1) & \to & \operatorname{Ext}^1(\mathscr{F}_3, \mathscr{F}_2)\,, \\ \big(q_1\hat{\beta}_1\,, a_1\hat{u}_1\big) & \mapsto & 0\,. \end{array}$	
$\begin{array}{rcl} \circ :\ \operatorname{Ext}^0(\mathscr{F}_1, \mathscr{F}_2) \times \operatorname{Ext}^1(\mathscr{F}_3, \mathscr{F}_1) & \to & \operatorname{Ext}^1(\mathscr{F}_3, \mathscr{F}_2)\,, \\ \big(b_1\hat{v}_1\,, p_1\hat{\alpha}_1\big) & \mapsto & 0\,. \end{array}$	

Table 7. Graded morphism spaces and their compositions for the ordered triple $(\mathscr{F}_3, \mathscr{F}_1, \mathscr{F}_2)$.

$\operatorname{Ext}^\bullet(\mathscr{F}_i, \mathscr{F}_j)$	$\mathscr{F}_1$	$\mathscr{F}_2$	$\mathscr{F}_3$
$\mathscr{F}_1$	$\mathbb{Z}_2^2[0] \oplus \mathbb{Z}_2^2[1]$	$\mathbb{Z}_2[0] \oplus \mathbb{Z}_2[1]$	$\mathbb{Z}_2[0] \oplus \mathbb{Z}_2[1]$
$\mathscr{F}_2$	$\mathbb{Z}_2[0] \oplus \mathbb{Z}_2[1]$	$\mathbb{Z}_2[0] \oplus \mathbb{Z}_2[1]$	$\mathbb{Z}_2[0] \oplus \mathbb{Z}_2[1]$
$\mathscr{F}_3$	$\mathbb{Z}_2[0] \oplus \mathbb{Z}_2[1]$	$\mathbb{Z}_2[0] \oplus \mathbb{Z}_2[1]$	$\mathbb{Z}_2[0] \oplus \mathbb{Z}_2[1]$

Table 8. Linear structure of the graded morphism spaces associated with $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$.

Claim 7.111. *Let $n \geq 2$ be an integer, and let $M \in \mathrm{M}(n, \mathbb{K})$ be a matrix given by*

$$M = \big[\vec{c}_1, \dots, \vec{c}_n\big]\,,$$

where $\vec{c}_i$ denotes the i-th column vector of M for each $i \in [1, n]$. Then, for any $k \in [1, n-1]$ and $z \in \mathbb{K}$, we have that

$$MB_k^{(n)}(z) = \big[\vec{c}_1, \dots, \vec{c}_{k-1}, \vec{c}_{k+1} + \vec{c}_k \cdot z, \vec{c}_k, \vec{c}_{k+2}, \dots, \vec{c}_n\big]\,.$$

Proof. Multiplying the matrix M on the right by the braid matrix $B_k^{(n)}(z)$ only changes the k-th and $(k+1)$-th columns of M. Specifically,

$$\vec{c}_k \mapsto \vec{c}_k \cdot z + \vec{c}_{k+1}, \qquad \vec{c}_{k+1} \mapsto \vec{c}_k,$$

while all other columns of M remain unchanged. The claim follows. $\square$

Claim 7.112. *Let $n \geq 2$ be an integer, and let $M \in \mathrm{M}(n, \mathbb{K})$ be a matrix given by*

$$M = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{bmatrix},$$

where $\vec{r}_i$ denotes the i-th row vector of M for each $i \in [1, n]$. Then, for any $k \in [1, n-1]$ and $z \in \mathbb{K}$, we have that

$$B_k^{(n)}(z)M = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_{k-1} \\ \vec{r}_{k+1} + z \cdot \vec{r}_k \\ \vec{r}_k \\ \vec{r}_{k+2} \\ \vdots \\ \vec{r}_n \end{bmatrix}.$$

Proof. Multiplying the matrix M on the left by the braid matrix $B_k^{(n)}(z)$ only changes the k-th and $(k+1)$-th rows of M. Explicitly,

$$\vec{r}_k \mapsto \vec{r}_{k+1} + z \cdot \vec{r}_k, \qquad \vec{r}_{k+1} \mapsto \vec{r}_k,$$

while all other rows of M remain unchanged. The claim follows. □

Claim 7.113. *Let $n \geq 2$ be an integer, and let $M \in \mathrm{M}(n, \mathbb{K})$ be a matrix given by*

$$M = \left[\vec{c}_1, \dots, \vec{c}_n\right],$$

where $\vec{c}_i$ denotes the i-th column vector of M for each $i \in [1, n]$. Then, for any $k \in [1, n-1]$ and $z \in \mathbb{K}$, we have that

$$M(B_k^{(n)}(z))^{-1} = \left[\vec{c}_1, \dots, \vec{c}_{k-1}, \vec{c}_{k+1}, \vec{c}_k - \vec{c}_{k+1} \cdot z, \vec{c}_{k+2}, \dots, \vec{c}_n\right].$$

Proof. Multiplying the matrix M on the right by the inverse braid matrix $\left(B_k^{(n)}(z)\right)^{-1}$ only changes the k-th and $(k+1)$-th columns of M. Specifically,

$$\vec{c}_k \mapsto \vec{c}_{k+1}, \qquad \vec{c}_{k+1} \mapsto \vec{c}_k - \vec{c}_{k+1} \cdot z,$$

while all other columns of M remain unchanged. The claim follows. □

Claim 7.114. *Let $n \geq 2$ be an integer, and let $M \in \mathrm{M}(n, \mathbb{K})$ be a matrix given by*

$$M = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{bmatrix},$$

where $\vec{r}_i$ denotes the i-th row vector of M for each $i \in [1, n]$. Then, for any $k \in [1, n-1]$ and $z \in \mathbb{K}$, we have that

$$(B_k^{(n)}(z))^{-1} M = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_{k-1} \\ \vec{r}_{k+1} \\ \vec{r}_k - z \cdot \vec{r}_{k+1} \\ \vec{r}_{k+2} \\ \vdots \\ \vec{r}_n \end{bmatrix}.$$

Proof. Multiplying the matrix M on the left by the inverse braid matrix $\left(B_k^{(n)}(z)\right)^{-1}$ only changes the k-th and $(k+1)$-th rows of M. Explicitly,

$$\vec{r}_k \mapsto \vec{r}_{k+1}, \qquad \vec{r}_{k+1} \mapsto \vec{r}_k - z \cdot \vec{r}_{k+1},$$

while all other rows of M remain unchanged. The claim follows. □

We conclude this appendix with the following lemma, which describes the effect of a generalized conjugation of a diagonal matrix by braid matrices.

Lemma 7.115. *Let $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{K}^n_{\mathrm{std}}$ be a fixed tuple. Then, for any $k \in [1, n-1]$ and any z, $z' \in \mathbb{K}$, we have that*

$$\left(B_k^{(n)}(z')\right)^{-1} D(\vec{u})\, B_k^{(n)}(z) = \begin{bmatrix} u_1 & & & & & & & \\ & \ddots & & & & & & \\ & & u_{k-1} & & & & & \\ & & & u_{k+1} & 0 & & & \\ & & & u_k\, z - z'\, u_{k+1} & u_k & & & \\ & & & & & u_{k+2} & & \\ & & & & & & \ddots & \\ & & & & & & & u_n \end{bmatrix}.$$

Proof. The result follows from a straightforward application of claims (7.111) and (7.114). □

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