ON FAIR AND TOLERANT COLORINGS OF GRAPHS

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ABSTRACT. A (not necessarily proper) vertex coloring of a graph G with color classes $V_1,\ V_2,\ \ldots,\ V_k$, is said to be a *Fair And Tolerant vertex coloring of G with k colors*, whenever V_1, V_2, …, V_k are nonempty and there exist two real numbers α and β such that $\alpha \in [0,1]$ and $\beta \in [0,1]$ and the following condition holds for each arbitrary vertex v and every arbitrary color class V_i:

$$\big|V_i \cap N(v)\big| = \begin{cases} \alpha \deg(v) & \text{if} \quad v \notin V_i \\ \beta \deg(v) & \text{if} \quad v \in V_i. \end{cases}$$

The *FAT chromatic number* of G, denoted by $\chi^{\rm FAT}(G)$, is defined as the maximum positive integer k for which G admits a Fair And Tolerant vertex coloring with k colors. The concept of the FAT chromatic number of graphs was introduced and studied by Beers and Mulas, where they asked for the existence of a function $f: \mathbb{N} \to \mathbb{R}$ in such a way that the inequality $\chi^{\rm FAT}(G) \ \leq \ f\big(\chi(G)\big)$ holds for all graphs G. Another similar interesting question concerns the existence of some function $g\colon \mathbb{N} \to \mathbb{R}$ such that the inequality $\chi(G) \ \leq \ g\left(\chi^{\rm FAT}(G)\right)$ holds for every graph G. In this paper, we establish that both questions admit negative resolutions.

1 Introduction

In this paper, we investigate coloring parameters of graphs. Throughout the text, except when $[\,\cdot\,]$ indicates bibliographic references, the shorthand $[n]$ will denote the set $\{1, 2, 3, \ldots, n\}$.

Our focus is on simple graphs whose vertex sets are nonempty and finite. Standard graph-theoretic notation is employed, following classical references such as [2], [3], [4], and [5].

For a graph G and a vertex v of G, the *neighborhood* of v, which is denoted by $N_G(v)$, is defined as the set of all vertices that are adjacent to v. Each member of $N_G(v)$ is called a *neighbor* of v. The size of $N_G(v)$ is called the *degree* of v, and it is denoted by $\deg_G(v)$. Whenever there is no ambiguity on the graph G, and the graph G is clear from

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context, we just write $N(v)$ and $\deg(v)$ instead of $N_G(v)$ and $\deg_G(v)$, respectively.
By a *vertex coloring* of a graph G, we mean a function $c\colon V(G) \to C$, where C is called the *set of colors*, and its elements are referred to as *colors*. If $c(v) = r$, we say that the vertex v is assigned the color r.
For each color r in C, the set of all vertices v for which $c(v) = r$, which is equal to $c^{-1}(r)$, is said to be the *color class of the color* r.
The set of nonempty color classes of a coloring $c\colon V(G) \to C$ provides a partition of $V(G)$. Regardless of the specific names of the colors, this partition divides $V(G)$ into pairwise disjoint blocks, each consisting of all vertices assigned the same color. Thus, a partition of $V(G)$ may be naturally regarded as the set of nonempty color classes induced by some vertex coloring of G.

Let G be a graph, v be an arbitrary vertex of G, and S be an arbitrary subset of $V(G)$. The symbol $e(v, S)$ stands for the number of neighbors of v that are also included in S. More precisely, $e(v, S)$ is equal to

$$e(v, S) := \big|S \cap N(v)\big|.$$

A coloring $c\colon V(G) \to C$ is called *proper* if no two adjacent vertices are assigned the same color. The *chromatic number* of G, denoted by $\chi(G)$, is defined as the minimum size of a set C that a proper coloring $c\colon V(G) \to C$ exists.

Let G be a graph, and C be a set of size k. Also, let $c\colon V(G) \to C$ be a (not necessarily proper) vertex coloring of G with color classes V_1, V_2, $\ldots$, V_k. The coloring $c\colon V(G) \to C$ is called a *Fair And Tolerant vertex coloring of G with k colors*, whenever V_1, V_2, $\ldots$, V_k are nonempty and there exist two real numbers α and β such that $\alpha \in [0, 1]$ and $\beta \in [0, 1]$ and the following condition holds for each arbitrary vertex v and every arbitrary color class V_i:

$$e\left(v, V_i\right) = \begin{cases} \alpha \deg(v) & \text{if} \quad v \notin V_i \\ \beta \deg(v) & \text{if} \quad v \in V_i. \end{cases}$$

The concept of Fair And Tolerant vertex colorings was introduced and investigated by Beers and Mulas in [1], and for brevity they used the abbreviated phrase "*FAT Colorings*". Also, the abbreviated phrase "*FAT k-coloring*" was used in [1] instead of "Fair And Tolerant vertex coloring with k colors".

Let G be a graph with at least one vertex v of nonzero degree. Suppose that G admits a FAT k-coloring with color classes V_1, V_2, $\ldots$, V_k

and corresponding parameters α and β. Then we have:

$$\deg(v) = \sum_{i=1}^{k} e\big(v, V_i\big) = \beta \deg(v) + (k-1)\alpha \deg(v);$$

and therefore, the equality

$$\beta + (k-1)\alpha = 1$$

holds [1] as a relation between k and α and β.

For every graph G, a FAT 1-coloring is obtained by setting $\alpha = 0$ and $\beta = 1$, and taking the entire vertex set $V(G)$ as the sole color class. Therefore, the set

$$\big\{k\colon\ \text{The graph } G \text{ admits a FAT } k\text{-coloring}\big\}$$

is nonempty. Also, this set is bounded above, because the number of color classes in every FAT coloring of G never exceeds $|V(G)|$. Therefore, this set has a maximum value, which is called the *FAT chromatic number* of G, and it is denoted by $\chi^{\mathrm{FAT}}(G)$. More precisely,

$$\chi^{\mathrm{FAT}}(G) \ := \ \max\big\{k\colon\ \text{The graph } G \text{ admits a FAT } k\text{-coloring}\big\}.$$

The concept of the FAT chromatic number of graphs was introduced and studied by Beers and Mulas [1] in 2025.

Remark 1. In the definition of FAT colorings, it is required that all color classes be nonempty to ensure that the FAT chromatic number is well-defined. Otherwise, if empty color classes were permitted in FAT colorings, then for any arbitrary graph G and any positive integer k, the monochromatic coloring $c\colon V(G) \to [k]$ that assigns color 1 to every vertex of G would be a FAT k-coloring with corresponding parameters $\alpha = 0$ and $\beta = 1$. Hence, a FAT k-coloring would exist for every k, and the notion of FAT chromatic number would be rendered meaningless.

It was shown by Beers and Mulas [1] that there exist some graphs G_1, G_2, and G_3 in such a way that the following three conditions hold:

$$\chi^{\mathrm{FAT}}\left(G_1\right) < \chi\left(G_1\right), \qquad \chi^{\mathrm{FAT}}\left(G_2\right) = \chi\left(G_2\right), \qquad \chi^{\mathrm{FAT}}\left(G_3\right) > \chi\left(G_3\right).$$

Therefore, the parameters $\chi^{\mathrm{FAT}}(G)$ and $\chi(G)$ may coincide, or one may be strictly greater or strictly smaller than the other. So, the following interesting open questions were raised in [1] by Beers and Mulas.

Question 1. What is the maximum possible gap between χ^{FAT} and χ?

Question 2. Is χ^{FAT} bounded in terms of χ? More precisely, does there exist a function $f\colon \mathbb{N} \to \mathbb{R}$ in such a way that the inequality

$$\chi^{\mathrm{FAT}}(G) \;\leq\; f\big(\chi(G)\big)$$

holds for all graphs G?

A similar question could also be raised as follows.

Question 3. Is χ bounded in terms of χ^{FAT}? In other words, does there exist a function $g\colon \mathbb{N} \to \mathbb{R}$ such that the inequality

$$\chi(G) \;\leq\; g\Bigg(\chi^{\mathrm{FAT}}(G)\Bigg)$$

holds for every graph G?

Question 4. Does there exist a class of graphs $\mathcal{G}$ for which $\chi(G)$ is bounded over $\mathcal{G}$ while $\chi^{\mathrm{FAT}}(G)$ diverges when G ranges over $\mathcal{G}$?

Question 5. Does there exist a class of graphs $\mathcal{G}$ such that $\chi^{\mathrm{FAT}}(G)$ is bounded over $\mathcal{G}$ whereas $\chi(G)$ diverges when G ranges over $\mathcal{G}$?

In this paper, we aim to answer these 5 questions.

2 Main Results

This section is devoted to addressing Questions 1, 2, 3, 4, and 5. To this end, we divide the discussion into two cases: disconnected graphs and connected graphs.

2.1 The Case of Disconnected Graphs

In this subsection, we aim to answer Questions 1, 2, 3, 4, and 5 for the class of disconnected graphs. In this regard, we provide the following two theorems.

Theorem 2.1. *For every two positive integers L_1 and L_2 with $L_1 < L_2$, there exists a disconnected graph G for which $\chi(G) = L_1$ and $\chi^{\mathrm{FAT}}(G) = L_2$.*

Proof. First, we prove the theorem for the case where $L_1 = 1$.
Let us regard the edgeless graph $\bar{K}_{L_2}$ with L_2 vertices whose vertex set equals $\{x_1, x_2, \ldots, x_{L_2}\}$. This graph is the complement of the complete graph K_{L_2}.
The chromatic number of $\bar{K}_{L_2}$ is equal to 1. Now the singletons $\{x_1\}$, $\{x_2\}$, $\ldots$, $\{x_{L_2}\}$ constitute the color classes of a FAT L_2-coloring of $\bar{K}_{L_2}$ with corresponding parameters $\alpha = 0$ and $\beta = 1$. Therefore, $\chi^{\mathrm{FAT}}\big(\bar{K}_{L_2}\big) \geq L_2$.

Since the definition of the notion of FAT coloring stipulates that every color class is nonempty, it follows that the number of color classes in every FAT coloring of $\bar{K}_{L_2}$ is less than or equal to the number of vertices of $\bar{K}_{L_2}$. Therefore, $\chi^{\mathrm{FAT}}\big(\bar{K}_{L_2}\big) \leq \left|V\left(\bar{K}_{L_2}\right)\right| = L_2$. Accordingly, $\chi\big(\bar{K}_{L_2}\big) = 1$ and $\chi^{\mathrm{FAT}}\big(\bar{K}_{L_2}\big) = L_2$ for each positive integer L_2 which is greater than 1.

Having treated the case $L_1 = 1$, we now assume $L_1 \geq 2$ for the rest of the proof.

Let us consider L_2 pairwise disjoint sets S_1, S_2, …, S_{L_2}, each of size L_1, as follows:

$$S_i := \left\{x_1^{(i)}, x_2^{(i)}, \ldots, x_{L_1}^{(i)}\right\} \qquad \text{for} \quad i = 1, 2, \ldots, L_2.$$

Now, define a graph G with $L_1 L_2$ vertices whose vertex set equals

$$V(G) := S_1 \cup S_2 \cup \cdots \cup S_{L_2};$$

and its edge set equals

$$E(G) := \left\{x_j^{(i)} x_k^{(i)} : \quad i \in [L_2] \text{ and } 1 \leq j < k \leq L_1\right\}.$$

Indeed, the graph G is a union of L_2 vertex disjoint cliques, each of order L_1.
Since G is a union of L_2 vertex disjoint cliques, each with chromatic number L_1, it follows that the chromatic number of G is also L_1.

Now, we concern the FAT chromatic number of G. If we assign the color i to all vertices of S_i for $i = 1, 2, \ldots, L_2$, then S_1, S_2, …, S_{L_2} constitute the color classes of a FAT L_2-coloring of G with corresponding parameters $\alpha = 0$ and $\beta = 1$. Accordingly,

$$\chi^{\mathrm{FAT}}(G) \geq L_2.$$

Our goal is to prove that $\chi^{\mathrm{FAT}}(G) = L_2$.
Put $\chi^{\mathrm{FAT}}(G) = k$, and consider some FAT k-coloring of G with k nonempty color classes T_1, T_2, …, T_k, together with some corresponding parameters $\alpha' \in [0, 1]$ and $\beta' \in [0, 1]$. Since $\chi^{\mathrm{FAT}}(G) \geq L_2$, it suffices to show that $k \leq L_2$. We proceed by considering two cases.

Case I: The case where $S_1 \cap T_b \neq \varnothing$ for every $b \in [k]$.
In this case, each of T_1, T_2, …, T_k intersects S_1. Hence, the number of color classes must be less than or equal to the number of vertices in S_1. Thus, $k \leq |S_1| = L_1$. Now, the condition $L_1 < L_2$ implies $k < L_2$; and we are done in this case.

Case II: The case where $S_1 \cap T_b = \varnothing$ for some $b \in [k]$.

In this case, since $x_1^{(1)} \in S_1$, all neighbors of $x_1^{(1)}$ also belong to the clique S_1. Therefore, the condition $S_1 \cap T_b = \varnothing$ implies

$$e\left(x_1^{(1)}, T_b\right) = 0.$$

On the other hand, because of $x_1^{(1)} \notin T_b$, it follows that

$$e\left(x_1^{(1)}, T_b\right) = \alpha'.\deg_G\left(x_1^{(1)}\right) = \alpha'.\left(L_1 - 1\right).$$

Therefore,

$$\alpha'.\left(L_1 - 1\right) = 0.$$

Now, since $L_1 \geq 2$, we conclude that

$$\alpha' = 0.$$

Also, by the condition $\beta' + (k-1).\alpha' = 1$, one finds that

$$\beta' = 1.$$

For each arbitrary $S_i \in \{S_1, S_2, \ldots, S_{L_2}\}$, define

$$a_i := \min\{j: \quad j \in [k] \text{ and } S_i \cap T_j \neq \varnothing\};$$

and consider some vertex z in $S_i \cap T_{a_i}$. Since $z \in T_{a_i}$, it follows that

$$e\left(z, T_{a_i}\right) = \beta'.\deg_G(z) = \deg_G(z).$$

Thus, all neighbors of z must also belong to the T_{a_i}. Accordingly,

$$S_i \subseteq T_{a_i}.$$

This shows that for each arbitrary clique $S_i \in \{S_1, S_2, \ldots, S_{L_2}\}$, there is a unique color class in $\{T_1, T_2, \ldots, T_k\}$ that intersects S_i; and besides, the clique S_i is entirely contained in that unique color class. Moreover, every arbitrary color class T_j is equal to the union of all cliques S_ℓ for which $S_\ell \cap T_j \neq \varnothing$. Therefore, because T_1, T_2, $\ldots$, T_k are nonempty, the function

$$\begin{array}{rccc} f: & \{S_1, S_2, \ldots, S_{L_2}\} & \longrightarrow & \{T_1, T_2, \ldots, T_k\} \\ & S_i & \longmapsto & T_{a_i} \end{array}$$

is well-defined and surjective. Accordingly, $k \leq L_2$, which completes the proof.

$\square$

Theorem 2.2. *For every two positive integers L_1 and L_2 with $1 < L_1 < L_2$, there exists a disconnected graph G such that $\chi^{\mathrm{FAT}}(G) = L_1$ and $\chi(G) = L_2$.*

Proof. Let S_1, S_2, $\ldots$, S_{L_1-1} denote $L_1 - 1$ pairwise disjoint sets of size L_1, as follows:

$$S_i := \left\{x_1^{(i)}, x_2^{(i)}, \ldots, x_{L_1}^{(i)}\right\} \qquad \text{for} \quad i = 1, 2, \ldots, L_1 - 1.$$

In addition, consider another set S_{L_1}, disjoint from all of S_1, S_2, $\ldots$, S_{L_1-1}, whose cardinality is L_2, given by:

$$S_{L_1} := \left\{x_1^{(L_1)}, x_2^{(L_1)}, \ldots, x_{L_2}^{(L_1)}\right\}.$$

Now, regard a graph G with vertex set

$$V(G) := S_1 \cup S_2 \cup \cdots \cup S_{L_1};$$

and the edge set $E(G) := E_1 \cup E_2$, where E_1 and E_2 are defined as the following:

- $E_1 := \left\{x_j^{(i)} x_k^{(i)} : \quad i \in [L_1 - 1] \text{ and } 1 \leq j < k \leq L_1\right\}$,
- $E_2 := \left\{x_j^{(L_1)} x_k^{(L_1)} : \quad 1 \leq j < k \leq L_2\right\}$.

The graph G is defined as the union of L_1 pairwise disjoint complete graphs. Among these, the first $L_1 - 1$ complete graphs have chromatic number L_1, whereas the L_1-th complete graph has chromatic number L_2. Therefore, the chromatic number of G is equal to

$$\chi(G) = \max\{L_1, L_2\} = L_2.$$

By assigning color i to all vertices of S_i for $i = 1, 2, \ldots, L_1$, we obtain a FAT L_1-coloring of G with parameters $\alpha = 0$ and $\beta = 1$. This construction immediately implies

$$\chi^{\text{FAT}}(G) \geq L_1.$$

We now demonstrate that this inequality is in fact an equality, that is, $\chi^{\text{FAT}}(G) = L_1$.
Set $\chi^{\text{FAT}}(G) = k$, and suppose that G admits a FAT k-coloring with k nonempty color classes T_1, T_2, $\ldots$, T_k, accompanied by parameters $\alpha' \in [0, 1]$ and $\beta' \in [0, 1]$. Given that $\chi^{\text{FAT}}(G) \geq L_1$, our task reduces to proving that $k \leq L_1$.
The proof is carried forward by a case analysis, divided into two parts.

Case I: Suppose that $S_1 \cap T_b \neq \varnothing$ for every $b \in [k]$.
In this situation, each of the color classes T_1, T_2, $\ldots$, T_k intersects S_1. Consequently, the number of color classes cannot exceed the number of vertices contained in S_1. Therefore, $k \leq |S_1| = L_1$.

Case II: Assume that $S_1 \cap T_b = \varnothing$ for some $b \in [k]$.

The vertex $x_1^{(1)}$ and all of its neighbors lie within S_1. Hence, the condition $S_1 \cap T_b = \varnothing$ forces

$$e\left(x_1^{(1)}, T_b\right) = 0.$$

On the other hand, because $x_1^{(1)} \notin T_b$, the definition of a FAT coloring yields

$$e\left(x_1^{(1)}, T_b\right) = \alpha'.\deg_G\left(x_1^{(1)}\right) = \alpha'.\left(L_1 - 1\right).$$

Thus,

$$\alpha'.\left(L_1 - 1\right) = 0.$$

Since $L_1 \geq 2$, we infer that

$$\alpha' = 0.$$

Moreover, the relation $\beta' + (k-1)\alpha' = 1$ immediately gives

$$\beta' = 1.$$

Now, fix any clique $S_i \in \{S_1, S_2, \dots, S_{L_1}\}$ and define

$$a_i := \min\left\{j \in [k] \ : \quad S_i \cap T_j \neq \varnothing\right\}.$$

Choose a vertex $z \in S_i \cap T_{a_i}$. Since $z \in T_{a_i}$, we obtain

$$e\left(z, T_{a_i}\right) = \beta'.\deg_G(z) = \deg_G(z).$$

Consequently, all neighbors of z must also lie in T_{a_i}, and therefore,

$$S_i \subseteq T_{a_i}.$$

This shows that each clique S_i is entirely contained in a unique color class T_{a_i}. Conversely, every color class T_j is the union of those cliques S_ℓ intersecting it. In other words, every T_j equals the union of all cliques S_ℓ with $S_\ell \cap T_j \neq \varnothing$. Since $T_1, T_2, \dots, T_k$ are nonempty, the mapping

$$\begin{array}{rccc} f\colon & \{S_1, S_2, \dots, S_{L_1}\} & \longrightarrow & \{T_1, T_2, \dots, T_k\} \\ & S_i & \longmapsto & T_{a_i} \end{array}$$

is well-defined and surjective. Hence, $k \leq L_1$, completing the argument. $\square$

Now we turn to Questions 1, 2, 3, 4, and 5, considered within the class of disconnected graphs. The conclusions are summarized in the following Remark.

Remark 2. Theorem 2.1 shows that for every arbitrary positive integer L, there exists a sequence $\{G_n\}_{n=1}^{\infty}$ of disconnected graphs for which $\chi(G_n) = L$ for each positive integer n, while the sequence $\chi^{\mathrm{FAT}}(G_n)$ diverges to $+\infty$ as n tends to infinity. Thus, Question 4 is affirmed, while Question 2 receives a negative answer. In addition, Theorem 2.2 implies that for every arbitrary positive integer $L > 1$, there exists a sequence $\{H_n\}_{n=1}^{\infty}$ of disconnected graphs such that $\chi^{\mathrm{FAT}}(H_n) = L$ for all positive integers n, whereas the sequence $\chi(H_n)$ diverges to $+\infty$ as n tends to infinity. Consequently, Question 5 admits an affirmative resolution, whereas Question 3 is negated. Moreover, in answer to Question 1, the sequences $\{G_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ demonstrate that both differences $\chi^{\mathrm{FAT}}(G) - \chi(G)$ and $\chi(H) - \chi^{\mathrm{FAT}}(H)$ remain unbounded, even within the class of disconnected graphs.

Let L_1 and L_2 be two arbitrary positive integers with $L_1 < L_2$. Theorem 2.1 asserts that there exists a disconnected graph G_1 satisfying $\chi(G_1) = L_1$ and $\chi^{\mathrm{FAT}}(G_1) = L_2$. Moreover, Theorem 2.2 guarantees the existence of some disconnected graph G_2 with $\chi^{\mathrm{FAT}}(G_2) = L_1$ and $\chi(G_2) = L_2$, provided that $L_1 > 1$. Now, the following natural Question arises.

Question 6. Does Theorem 2.2 remain valid without the condition $L_1 > 1$?

Question 6 is answered in the negative in the following Remark.

Remark 3. Let G be any arbitrary disconnected graph consisting of the components $G_1, G_2, \ldots, G_\ell$. If we assign the color i to all vertices of G_i for $i = 1, 2, \ldots, \ell$, then $G_1, G_2, \ldots, G_\ell$ constitute the color classes of a FAT ℓ-coloring of G with corresponding parameters $\alpha = 0$ and $\beta = 1$. Consequently, $\chi^{\mathrm{FAT}}(G) \geq \ell$. We conclude that the FAT chromatic number of every disconnected graph is strictly greater than 1. This shows that the condition $L_1 > 1$ is necessary in Theorem 2.2.

2.2 The Case of Connected Graphs

In this subsection we address Questions 1, 2, 3, 4, and 5, for the class of connected graphs. To this end, we state the following theorem.

Theorem 2.3. *For each odd positive integer $n \geq 5$, there exist some connected graphs G_1 and G_2 for which*

$$\chi(G_1) = 2, \quad \chi^{\mathrm{FAT}}(G_1) = n, \quad \chi(G_2) = n, \quad \chi^{\mathrm{FAT}}(G_2) = 2.$$

Proof. Let $A := \{x_1, x_2, \ldots, x_n\}$ and $B := \{y_1, y_2, \ldots, y_n\}$. Also, let G_1 be a graph with vertex set $V(G_1) := A \cup B$ whose edge set equals

$$E(G_1) := \big\{x_i y_j : \quad i, j \in [n] \text{ and } i \neq j\big\}.$$

Indeed, G_1 is a connected bipartite graph which is obtained by removing the edges of the perfect matching $\{x_1y_1, x_2y_2, \dots, x_ny_n\}$ from the complete bipartite graph $K_{n,n}$ with one part A and the other part B.

The bipartite graph G_1 is an $(n-1)$-regular graph with chromatic number $\chi(G_1) = 2$.
It was shown in [1] that the FAT chromatic number of every connected graph with minimum degree δ never exceeds $\delta+1$. Now, since $\delta(G_1) = n-1$, it follows that

$$\chi^{\mathrm{FAT}}(G_1) \leq \delta(G_1) + 1 = n.$$

Now, in order to show that $\chi^{\mathrm{FAT}}(G_1) = n$, it suffices to find a FAT coloring of G_1 with exactly n color classes. For each i in $[n]$, we assign the color i to the vertices x_i and y_i; forming the i-th color class

$$C_i := \{x_i, y_i\}.$$

For each vertex $z \in V(G_1)$ and every color class $C_i \in \{C_1, C_2, \dots, C_n\}$ we have:

$$e(z, C_i) = \begin{cases} 1 & \text{if} \quad z \notin C_i \\ 0 & \text{if} \quad z \in C_i. \end{cases}$$

Hence, by setting

$$\alpha_{G_1} := \tfrac{1}{n-1} \qquad \text{and} \qquad \beta_{G_1} := 0,$$

and the fact that $\deg_{G_1}(z) = n-1$, one finds that

$$\alpha_{G_1}.\deg_{G_1}(z) = 1 \qquad \text{and} \qquad \beta_{G_1}.\deg_{G_1}(z) = 0.$$

Accordingly, $\beta_{G_1} + (n-1)\alpha_{G_1} = 1$ and for each vertex $z \in V(G_1)$ and every color class $C_i \in \{C_1, C_2, \dots, C_n\}$ the condition

$$e(z, C_i) = \begin{cases} \alpha_{G_1}.\deg_{G_1}(z) & \text{if} \quad z \notin C_i \\ \beta_{G_1}.\deg_{G_1}(z) & \text{if} \quad z \in C_i \end{cases}$$

holds. Thus, C_1, C_2, …, C_n are the color classes of a FAT n-coloring of G_1. We conclude that $\chi^{\mathrm{FAT}}(G_1) = n$.

Thus far, we have constructed the connected graph G_1 in such a way that

$$\chi(G_1) = 2 \qquad \text{and} \qquad \chi^{\mathrm{FAT}}(G_1) = n.$$

Following the completion of the earlier stage, we are prepared to undertake the construction of graph G_2.
We define three sets V_1, V_2, and V_3, as follows:

- $V_1 := \{w_1, w_2, \dots, w_n\}$;

- $V_2 := \left\{ u_j^{(i)} : \quad i \in [n] \text{ and } j \in \left[\frac{n-1}{2}\right] \right\}$;
- $V_3 := \left\{ v_j^{(i)} : \quad i \in [n] \text{ and } j \in \left[\frac{n-1}{2}\right] \right\}$.

Now, the vertex set of G_2 is defined as $V(G_2) := V_1 \cup V_2 \cup V_3$. In order to define the edge set of G_2, first we consider some matchings M_1, M_2, $\ldots$, M_n, each of size $\frac{n-1}{2}$.
For each $i \in [n]$, the matching M_i equals

$$M_i := \left\{ u_1^{(i)} v_1^{(i)}, u_2^{(i)} v_2^{(i)}, \ldots, u_{\frac{n-1}{2}}^{(i)} v_{\frac{n-1}{2}}^{(i)} \right\}.$$

Now, the edge set of G_2 equals $E(G_2) := E_1 \cup E_2 \cup E_3 \cup E_4$, where E_1, E_2, E_3, and E_4 are defined as follows:

- $E_1 := \{ w_i w_j : \quad 1 \leq i < j \leq n \}$;
- $E_2 := \left\{ w_i u_j^{(i)} : \quad i \in [n] \text{ and } j \in \left[\frac{n-1}{2}\right] \right\}$;
- $E_3 := \left\{ w_i v_j^{(i)} : \quad i \in [n] \text{ and } j \in \left[\frac{n-1}{2}\right] \right\}$;
- $E_4 := M_1 \cup M_2 \cup \cdots \cup M_n$.

Formally, G_2 is obtained from the complete graph K_n on vertices $w_1, w_2, \ldots, w_n$ by attaching to each vertex w_i exactly $\frac{n-1}{2}$ pendant triangles

$$w_i u_j^{(i)} v_j^{(i)} \qquad \text{for all } j = 1, 2, \ldots, \tfrac{n-1}{2};$$

each such triangle consisting of the edges $w_i u_j^{(i)}$, $w_i v_j^{(i)}$, and $u_j^{(i)} v_j^{(i)}$.

For each $i \in [n]$ and every $j \in \left[\frac{n-1}{2}\right]$, we have

$$\deg_{G_2}(w_i) = 2(n-1) \qquad \text{and} \qquad \deg_{G_2}\left(u_j^{(i)}\right) = \deg_{G_2}\left(v_j^{(i)}\right) = 2.$$

Let us put

$$S_1 := V_1, \qquad S_2 := V_2 \cup V_3, \qquad \alpha_{G_2} := \tfrac{1}{2}, \qquad \text{and} \qquad \beta_{G_2} := \tfrac{1}{2}.$$

Now, for each vertex $z \in S_1$, we have

- $e(z, S_1) = n - 1 = \frac{1}{2} \deg_{G_2}(z) = \beta_{G_2}. \deg_{G_2}(z)$,
- $e(z, S_2) = n - 1 = \frac{1}{2} \deg_{G_2}(z) = \alpha_{G_2}. \deg_{G_2}(z)$.

Also, for each vertex $z \in S_2$, we have

- $e(z, S_1) = 1 = \frac{1}{2}\deg_{G_2}(z) = \alpha_{G_2}.\deg_{G_2}(z)$,
- $e(z, S_2) = 1 = \frac{1}{2}\deg_{G_2}(z) = \beta_{G_2}.\deg_{G_2}(z)$.

We conclude that for each vertex $z \in V(G_2)$ and every $S_i \in \{S_1, S_2\}$, we have

$$e(z, S_i) = \begin{cases} \alpha_{G_2}.\deg_{G_2}(z) & \text{if} \quad z \notin S_i \\ \beta_{G_2}.\deg_{G_2}(z) & \text{if} \quad z \in S_i; \end{cases}$$

and therefore, S_1 and S_2 provide the color classes of a FAT 2-coloring of G_2. Accordingly,

$$\chi^{\mathrm{FAT}}(G_2) \geq 2.$$

Since G_2 is connected, its FAT chromatic number is less than or equal to $\delta(G_2) + 1$. Therefore,

$$\chi^{\mathrm{FAT}}(G_2) \leq \delta(G_2) + 1 = 2 + 1 = 3.$$

Now, we claim that $\chi^{\mathrm{FAT}}(G_2) = 2$. Suppose, contrary to our claim, that $\chi^{\mathrm{FAT}}(G_2) \neq 2$. So, $\chi^{\mathrm{FAT}}(G_2) = 3$, and G_2 admits some FAT 3-coloring with three color classes, say T_1, T_2, and T_3, together with corresponding parameters α and β. On account of the definition of FAT colorings, the color classes T_1, T_2, and T_3 must be nonempty. Hence, the connectedness of G_2 implies that for two distinct color classes, say T_1 and T_2, there exists some edge of G_2, say t_1t_2, in such a way that $t_1 \in T_1$ and $t_2 \in T_2$. Therefore,

$$e(t_1, T_2) \neq 0, \qquad \text{and} \qquad e(t_1, T_2) = \alpha.\deg_{G_2}(t_1).$$

Hence, α must be positive.
Let i be an arbitrary element of $[n]$ and j be an arbitrary element of $\left[\frac{n-1}{2}\right]$. We may assume, without loss of generality, that $u_j^{(i)} \in T_1$. Since the degree of $u_j^{(i)}$ is 2, one finds that

$$e\left(u_j^{(i)}, T_2\right) = e\left(u_j^{(i)}, T_3\right) = \alpha.\deg_{G_2}\left(u_j^{(i)}\right) = 2\alpha > 0.$$

Thus, the vertex $u_j^{(i)}$ is adjacent to precisely one vertex of T_2 and precisely one vertex of T_3. Therefore, $2\alpha = 1$; which implies $\alpha = \frac{1}{2}$. Since $\beta + (3-1)\alpha = 1$, we find that $\beta = 0$. This shows that no two distinct vertices in $\{w_1, w_2, \ldots, w_n\}$ belong to the same color class. Since all vertices lie in the union of the three color classes T_1 and T_2 and T_3, it would follow that $n \leq 3$; which contradicts the fact that $n \geq 5$. So, we conclude that $\chi^{\mathrm{FAT}}(G_2) = 2$; as desired. $\square$

We now proceed to examine Questions 1, 2, 3, 4, and 5, under the restriction to connected graphs. The conclusions of this analysis are condensed in the subsequent Remark.

Remark 4. Theorem 2.3 furnishes a sequence $\{G_n\}_{n=1}^{\infty}$ of connected graphs for which $\chi(G_n) = 2$ for each positive integer n, while the sequence $\chi^{\mathrm{FAT}}(G_n)$ diverges to $+\infty$ as n tends to infinity. Thus Question 4 is affirmed, whereas Question 2 is answered negatively. Also, Theorem 2.3 produces a sequence $\{H_n\}_{n=1}^{\infty}$ of connected graphs such that $\chi^{\mathrm{FAT}}(H_n) = 2$ for all positive integers n, whereas the sequence $\chi(H_n)$ diverges to $+\infty$ as n tends to infinity. Hence Question 5 is settled affirmatively and Question 3 negatively. Moreover, with regard to Question 1, the sequences $\{G_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ exhibit that $\chi^{\mathrm{FAT}}(G) - \chi(G)$ and $\chi(H) - \chi^{\mathrm{FAT}}(H)$ are unbounded, already among connected graphs.

3 Concluding Remarks

In the preceding section of this paper, in two distinct subsections, we examined Questions 1–5 separately for connected and for disconnected graphs. In the present section, we dispense with that dichotomy and address these Questions in their full generality. The following theorem and remark furnish concise answers to these Questions without any restriction on whether the graphs are connected.

Theorem 3.1. *For each integer $n \geq 3$, there exist graphs G_1 and G_2 satisfying*

$$\chi(G_1) = 1, \quad \chi^{\mathrm{FAT}}(G_1) = n, \quad \chi(G_2) = n, \quad \chi^{\mathrm{FAT}}(G_2) = 1.$$

Proof. Let G_1 be $\bar{K}_n$, which is the edgeless graph on n vertices. In view of the Proof of Theorem 2.1, the graph G_1 satisfies

$$\chi(G_1) = 1 \qquad \text{and} \qquad \chi^{\mathrm{FAT}}(G_1) = n.$$

Let G_2 be a graph with $n+1$ vertices, whose vertex set equals

$$V(G_2) := \{a_1, a_2, \ldots, a_{n+1}\}.$$

Also, let $E(G_2) := E_1 \cup E_2$, where E_1 and E_2 are defined as the following:

$$E_1 := \{a_i a_j : \ 1 \leq i < j \leq n\} \qquad \text{and} \qquad E_2 := \{a_1 a_{n+1}\}.$$

Thus G_2 is obtained from the complete graph on n vertices a_1, a_2, $\ldots$, a_n, by adjoining a vertex a_{n+1} that is pendant to a_1.

The graph G_2 satisfies $\chi(G_2) = n$. Since G_2 is a connected graph and $\delta(G_2) = 1$, it follows that

$$\chi^{\mathrm{FAT}}(G_2) \leq \delta(G_2) + 1 = 2.$$

We claim that $\chi^{\mathrm{FAT}}(G_2) = 1$. Suppose, contrary to the claim, that $\chi^{\mathrm{FAT}}(G_2) \neq 1$. Therefore, $\chi^{\mathrm{FAT}}(G_2) = 2$, and G_2 admits some FAT 2-coloring with nonempty color classes V_1 and V_2 together with corresponding parameters α and β. Without loss of generality, one may assume that $a_{n+1} \in V_2$. Since G_2 is connected and V_1 and V_2 are nonempty, we must have $\alpha > 0$. This shows that

$$\alpha = \alpha \times 1 = \alpha. \deg_{G_2}(a_{n+1}) = e(a_{n+1}, V_1) \in \{0, 1\}.$$

Accordingly,

$$\alpha = e(a_{n+1}, V_1) = 1 \qquad \text{and} \qquad a_1 \in V_1.$$

Thus, the fact that $a_1 \in V_1$ implies

$$e(a_1, V_2) = \alpha. \deg_{G_2}(a_1) = \deg_{G_2}(a_1) = |V(G_2)| - 1.$$

Hence,

$$V_1 = \{a_1\} \qquad \text{and} \qquad V_2 = V(G_2) \setminus \{a_1\}.$$

We conclude that $a_3 \in V_2$. On the other hand, since $a_2 \in V_2$, it follows that

$$e(a_2, V_1) = \alpha. \deg_{G_2}(a_2) = \deg_{G_2}(a_2).$$

Accordingly, all neighbors of a_2 must belong to V_1. In particular, $a_3 \in V_1$, which contradicts the former fact that $a_3 \in V_2$. The contradiction shows that the assumption was false, and therefore the claim holds. This completes the proof. $\square$

We conclude with the following remark.

Remark 5. According to Theorem 3.1, Question 1 is answered negatively with respect to boundedness: each of $\chi^{\mathrm{FAT}}(G) - \chi(G)$ and $\chi(H) - \chi^{\mathrm{FAT}}(H)$ is unbounded both above and below. Moreover, Theorem 3.1 implies that Questions 2 and 3 are answered negatively, whereas Questions 4 and 5 are settled affirmatively.

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