

Intrinsic Quantum Codes: One Code to Rule Them All*

Eric Kubischta¹ and Ian Teixeira²

¹*Department of Mathematics, Florida State University, Tallahassee, FL 32306*

²*Department of Mathematics, University of California, San Diego, CA 92093*

Drawing on the analogy between intrinsic and extrinsic geometry, we define an intrinsic quantum code as a subspace of a group representation. We show how this abstract code rules over any physical (extrinsic) realization by dictating its error-correction and covariant properties. By finding one intrinsic code, we simultaneously uncover properties of all its realizations. This approach brings whole families of fault-tolerant schemes across disparate platforms into a unified framework and binds them through a single underlying symmetry.

INTRODUCTION

Quantum error correction is usually formulated extrinsically: one specifies a code subspace inside a particular physical Hilbert space and analyzes its robustness against a concrete set of physical errors. Yet many recent developments suggest that error-protection properties reflect deeper algebraic and symmetry structures [2–6]. The central premise of this Letter is that quantum error correction can instead be formulated as an *intrinsic* property of a representation, independent of any specific physical realization.

This viewpoint is directly analogous to the distinction between *intrinsic* and *extrinsic* geometry in differential geometry. A Riemannian manifold (M, g) possesses curvature, geodesic distance, and other invariants determined entirely by its internal metric data, independent of any embedding into a higher-dimensional Euclidean space [7]. For example, Gauss’s *Theorema Egregium* [8] famously established that the Gaussian curvature of a surface is an intrinsic invariant: it does not change when the surface is bent or deformed isometrically in $\mathbb{R}^3$.

Our framework establishes an analogous principle for quantum codes. An *intrinsic quantum code* $(V, \mathcal{I})$ is simply a subspace $\mathcal{I} \subset V$ where V is a representation of a group G . Then any G -equivariant isometric embedding of V into a physical Hilbert space $\mathcal{H}$ plays the role of an isometric embedding of a manifold into an ambient Euclidean space. And our main result, the Schur-Bootstrap, serves as the quantum analog of the Theorema Egregium: it guarantees that the “curvature of the code”—the set of symmetry-labeled Knill-Laflamme error protection relations—remains invariant under every such embedding.

This intrinsic–extrinsic correspondence provides a unifying language for quantum codes. Once a code is defined at the representation level, all of its error-detecting and covariance properties automatically persist in any physical system carrying the same group action. Rep-

resentation theory thereby furnishes the intrinsic geometry of quantum error correction, while the hardware determines only the embedding. We formalize this principle and demonstrate it through distinct physical realizations—qubit, qudit, heterogeneous, bosonic, spin, $\mathbb{A}_E$, rigid rotor, molecular, and Landau Level systems—each representing a different embedding of the same intrinsic code with identical protection and covariant (or transversal) gate structure. This intrinsic formulation unifies a wide range of quantum codes within a single representation-theoretic framework, suggesting new routes to hardware-independent fault-tolerance.

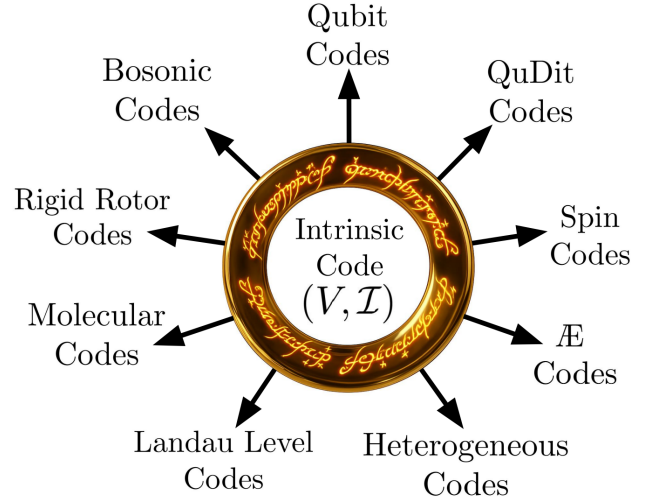


FIG. 1: “One Code to rule them all, One Code to find them, One Code to bring them all, and in the Symmetry bind them,” adapted from J.R.R. Tolkien, *The Fellowship of the Ring* (1954) [1]. Figure generated with ChatGPT-5 [9].

MAIN RESULT

Intrinsic Codes.— Let V be a unitary representation of a group G . We call V the *intrinsic* Hilbert space and define *intrinsic errors* as operators in $\mathcal{L}(V) \simeq V \otimes V^* = \bigoplus_{\xi} W_{\xi}$ where $W_{\xi} \simeq \xi \otimes \mathbb{C}^{m_{\xi}} \simeq (m_{\xi})\xi$ is the ξ -isotypic subspace of $\mathcal{L}(V)$ corresponding to the irreducible rep-

* These authors contributed equally to this work.

ekubischta@fsu.edu, iteixeira@ucsd.edu

Title inspired by J.R.R. Tolkien, *The Fellowship of the Ring* [1].

representation (irrep) ξ of G that occurs in $\mathcal{L}(V)$ with multiplicity m_ξ . See [10, 11] for a background on representation theory.

Now let $\mathcal{I} \subset V$ be a subspace of V and let $\Omega : V \rightarrow V$ be the orthogonal projector onto $\mathcal{I}$. Let $\mathcal{E}$ be a set of irreps of G . Then we call $\mathcal{I}$ an *intrinsic quantum code* for the error set $\mathcal{E}$ if $\mathcal{I}$ satisfies the Knill-Laflamme (KL) condition for all ξ -isotypic errors for irreps $\xi \in \mathcal{E}$, that is, if $\Omega F \Omega = c \Omega$ for all $F \in W_\xi$ and all $\xi \in \mathcal{E}$ (and c is a constant that only depends on the error F).

Extrinsic Codes.— Now let $\mathcal{H}$ be a Hilbert space, also carrying a unitary action of G . We call $\mathcal{H}$ the *extrinsic* or *physical* Hilbert space and define *extrinsic errors* as operators in $\mathcal{L}(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H}^*$. This also splits into symmetry sectors as $\mathcal{H} \otimes \mathcal{H}^* = \bigoplus_\xi K_\xi$ where now K_ξ is the ξ -isotypic subspace of $\mathcal{L}(\mathcal{H})$.

Let $\mathcal{U} : V \hookrightarrow \mathcal{H}$ be any G -equivariant, isometric embedding of V into $\mathcal{H}$. The isometric property means distances are preserved and is formally equivalent to the condition $\mathcal{U}^\dagger \mathcal{U} = \mathbb{1}_V$. We will denote the image of an intrinsic code $\mathcal{I}$ under such an embedding by $\mathcal{C} := \mathcal{U}(\mathcal{I}) \subset \mathcal{H}$. We call $\mathcal{C}$ an *extrinsic quantum code* (or a *physical realization* of $\mathcal{I}$). Note that if $\Pi : \mathcal{H} \rightarrow \mathcal{H}$ is the orthogonal projector onto $\mathcal{C}$ then we can write $\Pi = \mathcal{U} \Omega \mathcal{U}^\dagger$.

The Schur-Bootstrap.— We will now show that extrinsic quantum codes automatically inherit error protection properties from the intrinsic code (in much the same way that an embedded Riemannian manifold inherits curvature from the original intrinsic Riemannian manifold).

Define the compression map $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(V)$ by $E \mapsto \mathcal{U}^\dagger E \mathcal{U}$. This map is the pullback of an extrinsic error $E \in \mathcal{L}(\mathcal{H})$ to an intrinsic error $\mathcal{U}^\dagger E \mathcal{U} \in \mathcal{L}(V)$. The map Φ is G -equivariant since $\mathcal{U}$ is an intertwiner between the G action on V and the G action on $\mathcal{H}$. And G -equivariance means that Schur's lemma applies: so Φ maps the ξ sector of $\mathcal{L}(\mathcal{H})$, denoted K_ξ , into the ξ sector of $\mathcal{L}(V)$, denoted W_ξ .

In particular, suppose an extrinsic error $E \in \mathcal{L}(\mathcal{H})$ transforms in the irrep ξ of G . Then, by Schur's lemma, the corresponding intrinsic error $\Phi(E) = \mathcal{U}^\dagger E \mathcal{U}$ either transforms as ξ , or $\Phi(E) = 0$. It follows that if $\mathcal{I}$ is an intrinsic quantum code for the (intrinsic) error set $\mathcal{E}$ then the extrinsic quantum code $\mathcal{C}$ is also a quantum code for the (extrinsic) error set $\mathcal{E}$.

We can formally prove this for any extrinsic error $E \in K_\xi$ for $\xi \in \mathcal{E}$ by recalling that $\Pi = \mathcal{U} \Omega \mathcal{U}^\dagger$ so that $\Pi E \Pi = \mathcal{U} \Omega \Phi(E) \Omega \mathcal{U}^\dagger = c \mathcal{U} \Omega \mathcal{U}^\dagger = c \Pi$. In words, if the intrinsic quantum code $\mathcal{I}$ satisfies the KL condition for all errors in some isotypic subspace of $\mathcal{L}(V)$, then the extrinsic quantum code $\mathcal{C}$ will also satisfy the KL condition for all errors in the corresponding isotypic subspace of $\mathcal{L}(\mathcal{H})$.

Now let $\mathcal{G}$ be the subgroup of G preserving $\mathcal{I}$ as a subspace, and let $\mathcal{G}_0$ be the subgroup of G fixing $\mathcal{I}$ pointwise. That is, $\mathcal{G}$ preserves the codespace $\mathcal{I}$ while $\mathcal{G}_0$ acts like the identity on the codespace. Then we say that $\mathcal{I}$ is

G -covariant for the quotient group $\mathbf{G} := \mathcal{G}/\mathcal{G}_0$ [12, 13]. In words, the covariant group $\mathbf{G}$ contains those logical operations that can be implemented by the symmetry group G (analogous to the transversal gate group for multi-qudit codes). Because the embedding $\mathcal{U}$ is G -equivariant it follows that $\mathcal{C}$ is also $\mathbf{G}$ -covariant. In particular if some element $g \in G$ enacts a logical operation $\hat{g} \in \mathbf{G}$ on $\mathcal{I}$ (via the G -action of V) then $g \in G$ will also enact the logical operation $\hat{g} \in \mathbf{G}$ on $\mathcal{C}$ (via the G -action of $\mathcal{H}$).

We have thus proven the following theorem, our main result of this Letter. We have dubbed it the ‘‘Schur-Bootstrap’’ because the preceding proof was a direct application of Schur's lemma and it provides a mechanism for ‘‘bootstrapping’’ intrinsic error protection to extrinsic error protection.

Theorem (Schur-Bootstrap). If $\mathcal{I} \subset V$ satisfies the Knill-Laflamme condition for all errors in some isotypic subspace of $\mathcal{L}(V)$, then *any* physical realization $\mathcal{U}(\mathcal{I}) = \mathcal{C} \subset \mathcal{H}$ will satisfy the Knill-Laflamme condition for all errors in the corresponding isotypic subspace of $\mathcal{L}(\mathcal{H})$. Moreover, if $\mathcal{I}$ is $\mathbf{G}$ -covariant then $\mathcal{C}$ is also $\mathbf{G}$ -covariant.

Specializing to the case that V is an irrep of $G = \text{SU}(q)$, we define the *intrinsic distance* d of an intrinsic code $\mathcal{I} \subset V$ as follows. For each irrep ξ of $\text{SU}(q)$, let

$$\text{ord}(\xi) := \min\{t \geq 0 : \xi \subset \text{Ad}^{\otimes t}\} \quad (1)$$

be the smallest tensor power of the adjoint representation in which ξ appears. We say that $\mathcal{I}$ has intrinsic distance d if it satisfies the KL condition for every irrep sector ξ with $\text{ord}(\xi) < d$, i.e. for all irreps that occur in $\text{Ad}^{\otimes t}$ with $t < d$.

Corollary (Multi-qudit Schur-Bootstrap). If $\mathcal{I}$ has intrinsic distance d , then for any $\text{SU}(q)$ -equivariant isometric embedding $\mathcal{U} : V \hookrightarrow \mathcal{H}$ into a multi-qudit Hilbert space $\mathcal{H} \cong (\mathbb{C}^q)^{\otimes n}$, the subspace $\mathcal{C} := \mathcal{U}(\mathcal{I})$ is a multi-qudit code of distance d . Moreover, if $\mathcal{I}$ is $\mathbf{G}$ -covariant, then $\mathcal{C}$ implements $\mathbf{G}$ transversally.

This corollary dramatically generalizes our previous ‘‘Dicke-Bootstrap’’ result [14] and complements the recent work of [15]. For fundamental $\text{SU}(q)$ qudits, single-site errors transform in $\mathbf{q} \otimes \bar{\mathbf{q}} = \mathbf{1} \oplus \text{Ad}$, and weight- t errors live in $\text{Ad}^{\otimes t}$ up to trivial factors. Thus the intrinsic distance coincides with the usual Hamming distance when V is realized inside $(\mathbb{C}^q)^{\otimes n}$.

APPLICATION

The simplicity of the Schur-Bootstrap belies its broad and powerful reach. To illustrate this, we examine a minimal intrinsic code and show how it manifests in a variety

of physical (extrinsic) realizations. More examples of intrinsic codes are provided in the Supplemental Material [16].

Consider the 5-dimensional irrep $V = \mathbf{5}$ of $G = \text{SU}(2)$. Label the basis for this space as you would a ququint: $|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle$. Intrinsic errors on this space live in $\mathcal{L}(V) = \mathbf{5} \otimes \mathbf{5} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \oplus \mathbf{7} \oplus \mathbf{9}$. There is a 2-dimensional intrinsic quantum code $\mathcal{I} \subset V$ for the error set $\mathcal{E} = \{\mathbf{1}, \mathbf{3}\}$ given by the codewords:

$$|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |4\rangle), \quad |\bar{1}\rangle = |2\rangle. \quad (2)$$

Indeed this code satisfies the intrinsic KL condition for $\mathbf{1}$ (spanned by the identity matrix $\mathbb{1}$) and $\mathbf{3}$ (spanned by the 5-dimensional representations of J_+, J_- , and J_z). And so it has intrinsic distance $d = 2$. We will refer to this intrinsic code by its parameters as $\{\{V, \dim(\mathcal{I}), d\}\} = \{\{\mathbf{5}, 2, 2\}\}$ where the group G being used is implicit in V .

This intrinsic code is covariant for $G = S_3$, where S_3 is the symmetric group on 3-letters. In particular, the 5-dimensional representation of the determinant-1 Hadamard gate $H = -iH$ enacts the logical gate $R := \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$. And the 5-dimensional representation of the determinant-1 phase gate $S = e^{-i\pi/4}S$ implements the logical Pauli gate $-Z$.

Example 1: Qubit Codes

Four-Qubit Realization.— In the physical Hilbert space of four qubits, $\mathcal{H} = \mathbf{2}^{\otimes 4}$, the $\mathbf{5}$ irrep of $\text{SU}(2)$ appears exactly once as the permutation-invariant subspace $\text{Sym}^4(\mathbb{C}^2) \cong \mathbf{5}$. The unique $\text{SU}(2)$ -intertwiner $\mathcal{U} : \mathbf{5} \hookrightarrow \mathcal{H}$ identifies $|w\rangle$ with the Dicke state [17–20] of Hamming weight w ,

$$\begin{aligned} |0\rangle &\mapsto |0000\rangle \\ |1\rangle &\mapsto \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \\ |2\rangle &\mapsto \frac{|0011\rangle}{\sqrt{6}} + \frac{|0101\rangle}{\sqrt{6}} + \frac{|1001\rangle}{\sqrt{6}} + \frac{|1100\rangle}{\sqrt{6}} + \frac{|1010\rangle}{\sqrt{6}} + \frac{|0110\rangle}{\sqrt{6}} \\ |3\rangle &\mapsto \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle) \\ |4\rangle &\mapsto |1111\rangle. \end{aligned} \quad (3)$$

So the intrinsic $\{\{\mathbf{5}, 2, 2\}\}$ code gets mapped under $\mathcal{U}$ to a particular permutation invariant multi-qubit code [21–23] which previously appeared in [24–27]:

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \\ |\bar{1}\rangle &= \frac{|0011\rangle}{\sqrt{6}} + \frac{|0101\rangle}{\sqrt{6}} + \frac{|1001\rangle}{\sqrt{6}} + \frac{|1100\rangle}{\sqrt{6}} + \frac{|1010\rangle}{\sqrt{6}} + \frac{|0110\rangle}{\sqrt{6}}. \end{aligned} \quad (4)$$

Since the intrinsic code satisfies the KL condition for the $\mathbf{1}$ and $\mathbf{3}$ sectors of $\mathcal{L}(V)$, the $\text{SU}(2)$ -equivariant embedding Schur–Bootstraps this to $\mathcal{H}$: any physical error in the $\mathbf{1}$ or $\mathbf{3}$ symmetry sectors of $\mathcal{L}(\mathcal{H})$ is detected.

In particular, weight-0 and weight-1 errors (transforming as $\mathbf{1}$ and $\mathbf{3}$) are detected, so the code has distance $d = 2$. Indeed, this extrinsic code is precisely the permutation-invariant subcode of the $[[4, 2, 2]]$ stabilizer code.

What went previously unnoticed is that this protection extends to *all* errors transforming as $\mathbf{1}$ or $\mathbf{3}$, irrespective of weight. For example, a weight-2 error on a fixed pair of qubits transforms as $\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$, but only the $\mathbf{5}$ sector is “bad” (i.e., doesn’t automatically satisfy the KL condition). So intrinsic codes offer protection even beyond the low weight errors, a phenomenon similar to what happens for stabilizer codes, where high weight errors that anticommute with a stabilizer generator are still detectable.

Because the intrinsic code $\mathcal{I}$ is $G = S_3$ covariant then we can implement S_3 on $\mathcal{C}$ transversally. For example, $S^{\otimes 4}$ implements logical Z and $H^{\otimes 4}$ implements logical R .

Six-Qubit Realizations.— For six qubits, $\mathcal{H} = \mathbf{2}^{\otimes 6}$, the embedding $\mathcal{U} : \mathbf{5} \hookrightarrow \mathcal{H}$ is no longer unique: the irrep $\mathbf{5}$ appears with multiplicity five. Each embedding therefore defines a distinct $((6, 2, 2))$ realization of the intrinsic code $\{\{\mathbf{5}, 2, 2\}\}$. Because any normalized linear combination of these five equivalent subspaces again transforms as $\mathbf{5}$, the set of embeddings forms the complex projective manifold $\mathbb{C}P^4$.

Equivalently, every point $z = [z_1 : \dots : z_5] \in \mathbb{C}P^4$ specifies coefficients z_i defining an embedding $\mathcal{U}_z = \sum_{i=1}^5 z_i \mathcal{U}_i$ and hence a different extrinsic $((6, 2, 2))$ code. For explicit codewords, see the Supplemental Material [16].

Every code obtained in this way has distance $d = 2$ and is $G = S_3$ -covariant, for example, $S^{\otimes 6}$ and $H^{\otimes 6}$, enact logical $-Z$ and R , respectively. The same construction extends to all even $n \geq 6$: for example, the multiplicity of $\mathbf{5}$ in $\mathbf{2}^{\otimes 8}$ is 20, giving a $\mathbb{C}P^{19}$ family of $((8, 2, 2))$ codes, while in ten qubits it is 75, yielding a $\mathbb{C}P^{74}$ family. Thus, for each even $n \geq 6$, the intrinsic code $\{\{\mathbf{5}, 2, 2\}\}$ admits a continuous moduli space of extrinsic realizations parameterized by $\mathbb{C}P^{m-1}$, where m is the multiplicity of $\mathbf{5}$ in $\mathbf{2}^{\otimes n}$.

Example 2: Spin Codes

Under the embedding $\mathcal{U} : |w\rangle \mapsto |_{w-2}^2\rangle$ where $|_m^j\rangle$ is the angular-momentum basis, we can realize the intrinsic code $\mathcal{I}$ as the $j = 2$ spin code

$$|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|_{-2}^2\rangle + |_2^2\rangle), \quad |\bar{1}\rangle = |_0^2\rangle. \quad (5)$$

This realization employs a single spin-2 particle, illustrating that even an individual quantum system can host a nontrivial error-detecting code. The code detects infinitesimal $\text{SU}(2)$ rotations to first order in perturbation theory and thus possesses spin distance $d = 2$ in the sense of [2, 14]. The logical operations $-Z$ and R correspond

respectively to the physical spin rotations $D^{(j)}(\mathbf{S})$ and $D^{(j)}(\mathbf{H})$, both of which can be implemented experimentally via controlled magnetic-field pulses generating the appropriate Zeeman Hamiltonian [2].

Example 3: Qudit Codes

The intrinsic code $\{\{5, 2, 2\}\}$ can also be embedded into multiple qutrits rather than qubits, so long as the physical system realizes each qutrit as the irrep $\mathbf{3}$ of $\text{SU}(2)$ (for example, spin-1 atoms [28], Nitrogen-Vacancy centers [29, 30], NMR [31], and certain trapped ions [32]). For two qutrits, $\mathcal{H} = \mathbf{3}^{\otimes 2} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$. The unique $\text{SU}(2)$ -equivariant isometric embedding of $\mathbf{5}$ into $\mathcal{H}$ is

$$\begin{aligned} |0\rangle &\mapsto |00\rangle \\ |1\rangle &\mapsto \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |2\rangle &\mapsto \frac{1}{\sqrt{6}}(|02\rangle + 2|11\rangle + |20\rangle) \\ |3\rangle &\mapsto \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle) \\ |4\rangle &\mapsto |22\rangle. \end{aligned} \quad (6)$$

The resulting extrinsic code is

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |22\rangle), \\ |\bar{1}\rangle &= \frac{1}{\sqrt{6}}(|02\rangle + 2|11\rangle + |20\rangle). \end{aligned} \quad (7)$$

Extrinsic errors on $\mathcal{H}$ transform as

$$\mathbf{3}^{\otimes 2} \otimes \mathbf{3}^{\otimes 2} = (3)\mathbf{1} \oplus (6)\mathbf{3} \oplus (6)\mathbf{5} \oplus (3)\mathbf{7} \oplus \mathbf{9}. \quad (8)$$

The KL condition is indeed satisfied for all errors in the $\mathbf{1}$ and $\mathbf{3}$ sectors, but not for errors in $\mathbf{5}$. Hence this code is not distance-2 in the conventional multi-qudit sense: a single-qutrit error transforms as $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$, and full distance-2 protection would require detecting errors in the $\mathbf{5}$ sector as well. This illustrates that the Schur-Bootstrap preserves *representation-theoretic* error protection but not always conventional weight-based error-protection. For example, in order to get a qudit code with distance 2 you would have had to start with an intrinsic code $\mathcal{I}$ with intrinsic distance 3.

Nevertheless, we motivate and define a perturbative notion of distance for qudit (and heterogeneous) quantum codes in the Supplemental Material [16], called *depth* d . We show that if $\mathcal{I}$ has intrinsic distance d then the extrinsic code has depth $\mathbf{d} = d$ as well. For example, the depth of the above two-qutrit code is $\mathbf{d} = 2$. We argue that in certain cases, the traditional weight-based notion of distance is too strong and depth is more appropriate.

The physical G -action on $\mathcal{H}$ is by $D^{j=1}(g) \otimes D^{j=1}(g)$, so logical Z and R are implemented transversally by $D^{j=1}(\mathbf{S}) \otimes D^{j=1}(\mathbf{S})$ and $D^{j=1}(\mathbf{H}) \otimes D^{j=1}(\mathbf{H})$.

Example 4: Heterogeneous Codes

Our framework naturally produces heterogeneous codes—also called mixed-alphabet or composite codes [3, 33–36]—in which the individual subsystems are qudits with differing local dimensions. In the present context, each qudit in the composite system may be regarded as residing in an irrep of $\text{SU}(2)$, with its local dimension determined by the spin of that representation. These codes are less explored but arise naturally from the Schur-Bootstrap.

Consider the Hilbert space of a qubit-ququart system $\mathcal{H} = \mathbf{2} \otimes \mathbf{4}$, both realized using particles that transform under $\text{SU}(2)$. The unique $\text{SU}(2)$ -equivariant embedding $\mathcal{U} : \mathbf{5} \hookrightarrow \mathbf{2} \otimes \mathbf{4}$ is

$$\begin{aligned} |0\rangle &\mapsto |00\rangle, \\ |1\rangle &\mapsto \frac{\sqrt{3}}{2}|01\rangle + \frac{1}{2}|10\rangle, \\ |2\rangle &\mapsto \frac{1}{\sqrt{2}}(|02\rangle + |11\rangle), \\ |3\rangle &\mapsto \frac{1}{2}|03\rangle + \frac{\sqrt{3}}{2}|12\rangle \\ |4\rangle &\mapsto |13\rangle. \end{aligned} \quad (9)$$

The resulting extrinsic code under the Schur-Bootstrap is

$$|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |13\rangle), \quad |\bar{1}\rangle = \frac{1}{\sqrt{2}}(|02\rangle + |11\rangle). \quad (10)$$

Extrinsic errors transform as

$$(\mathbf{2} \otimes \mathbf{4}) \otimes (\mathbf{2} \otimes \mathbf{4}) = (2)\mathbf{1} \oplus (4)\mathbf{3} \oplus (4)\mathbf{5} \oplus (3)\mathbf{7} \oplus \mathbf{9}. \quad (11)$$

The code satisfies the Knill–Laflamme conditions for all $\mathbf{1}$ and $\mathbf{3}$ components but not for $\mathbf{5}$ or $\mathbf{7}$, so it is again not distance-2 in the conventional weight-based sense. But again, the depth is $\mathbf{d} = 2$ (see the Supplemental Material [16]), and in particular, it detects any error on the qubit, and any $\mathbf{1}$ or $\mathbf{3}$ error on the ququart. Logical operators are implemented transversally as

$$Z_L : \mathbf{S} \otimes D^{j=3/2}(\mathbf{S}), \quad R_L : \mathbf{H} \otimes D^{j=3/2}(\mathbf{H}). \quad (12)$$

Similar to this example, there is also a heterogeneous realization of $\{\{5, 2, 2\}\}$ in a qubit-qubit-qutrit system $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{3}$ and a $\mathbb{C}P^1$ worth of embeddings in a qubit-qutrit-ququart system $\mathbf{2} \otimes \mathbf{3} \otimes \mathbf{4}$.

Example 5: Bosonic Codes

A single bosonic mode has Fock states $|n\rangle$ ($n \geq 0$) with ladder operators $a, a^\dagger$ and number operator $\hat{n} = a^\dagger a$ [37–39]. For two modes (a, b) , the subspace with fixed total excitation $n_1 + n_2 = N$ carries the spin- $j = N/2$ irrep of $\text{SU}(2)$ under the Schwinger mapping [40–42].

The intrinsic code $\{\{5, 2, 2\}\}$ is realized in the two-mode manifold with total excitation number $N = 4$ via the embedding $\mathcal{U} : |k\rangle \mapsto |k, 4 - k\rangle$. The logical states are

$$|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|0, 4\rangle + |4, 0\rangle), \quad |\bar{1}\rangle = |2, 2\rangle. \quad (13)$$

This is precisely the Chuang–Leung–Yamamoto (CLY) code (see equation 3.1 of [43]). Because it is an extrinsic realization of the intrinsic code, it satisfies the KL condition for *any* error transforming as **1** or **3**, including the dephasing errors J_+ , J_- , J_z , as well as many other errors discussed in the Supplemental Material [16].

This provides a continuous-variable realization of the intrinsic $\{\{5, 2, 2\}\}$ code in the two-mode Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2)$. Within the fixed- N manifold, logical gates R_L and Z_L correspond to $SU(2)$ rotations, which can be implemented experimentally using linear-optical networks of beam splitters and phase shifters [44–46].

Example 6: $\mathcal{A}\mathcal{E}$ Codes

A heterogeneous diatomic molecule (a linear rotor) has rotational Hilbert space $\mathcal{H} = L^2(S^2)$ with decomposition $L^2(S^2) = \bigoplus_{\ell \geq 0} (2\ell + 1)$ under $SO(3)$. The intrinsic code $\{\{5, 2, 2\}\}$ appears in the $\ell = 2$ sector, with logical states

$$|\bar{0}\rangle = \frac{1}{\sqrt{2}}(Y_{-2}^2 + Y_2^2), \quad |\bar{1}\rangle = Y_0^2. \quad (14)$$

Because this subspace transforms as the spin-2 irrep of $SU(2)$, it detects all first-order errors transforming as **3**, including first-order rotational errors ($J_\pm$, J_z) and first-order momentum-kicks $\hat{\mathcal{Y}}_M^{L=1}$ (the spherical tensor operators [47]). Thus the intrinsic code structure immediately determines which perturbations the rotor embedding suppresses. This code is an example of an absorption-emission $\mathcal{A}\mathcal{E}$ code [48], a special case of a linear rotor code [47]. A similar idea to the Schur-Bootstrap in this case was examined in [49].

Example 7: Rigid Rotor Codes

A rigid rotor (asymmetric polyatomic molecule) has rotational Hilbert space $\mathcal{H} = L^2(SO(3))$, which decomposes into angular-momentum sectors labeled by ℓ under the Peter–Weyl theorem. Each sector carries independent left and right $SU(2)$ actions, corresponding to lab-fixed and body-fixed rotations, and thus supports two natural classes of errors: left/right rotational perturbations (J_i^L, J_i^R) and orientation (momentum-kick) errors generated by rank- L Wigner- D operators.

The intrinsic code $\{\{5, 2, 2\}\}$ embeds naturally into the $\ell = 2$ manifold, yielding a generalized absorption-emission ($\mathcal{A}\mathcal{E}$) code. A left-embedded realization protects against all first-order right rotations, and by the Schur-bootstrap also detects all first-order left rotations

and rank-1 orientation perturbations, since these transform in **3** of $SU(2)$. Analogous right-embedded and diagonal embeddings exist, each implementing the logical S_3 symmetry by physical rotations. Further details appear in the Supplemental Material [16].

Example 8: Molecular Codes

Molecules with rotational symmetry $K \subset SO(3)$ have rotational Hilbert space $\mathcal{H} = L^2(SO(3)/K)$, which decomposes into angular-momentum sectors labelled by ℓ . The intrinsic code $\{\{5, 2, 2\}\}$ embeds whenever the $\ell = 2$ spin-2 irrep appears with multiplicity, yielding a family of molecular realizations.

As a concrete example, sulfur dioxide SO_2 has symmetry group $K = C_2$, and the $\ell = 2$ sector contains three copies of the spin-2 irrep. Thus the intrinsic code gives rise to a CP^2 family of extrinsic molecular codes. Because these embeddings lie entirely in the left rotational sector, they protect against all first-order left rotations, all right rotations, and all rank-1 orientation (momentum-kick) errors, which transform in the **3** of $SU(2)$. The logical S_3 action can be implemented covariantly using left rotations. Detailed analysis and further examples, such as encoding a qubit in a $^{12}C_{60}$ buckyball, are given in the Supplement [16].

Example 9: Landau Level Codes

A charged particle constrained to the surface of a sphere threaded by a magnetic monopole has a rotational Hilbert space organized into Landau levels, each of which forms an $SU(2)$ irrep. The intrinsic code $\{\{5, 2, 2\}\}$ appears whenever the spin-2 irrep occurs, which happens for integral monopole charge j with $j \leq 2$. In these cases the extrinsic codewords take the form

$$|\bar{0}\rangle = \frac{1}{\sqrt{2}}({}_jY_{-2}^2 + {}_jY_2^2), \quad |\bar{1}\rangle = {}_jY_0^2, \quad (15)$$

where ${}_jY_m^\ell$ are the monopole harmonics. Since the spin-2 manifold transforms as the **3** under small rotations, these embeddings detect all first-order rotational errors and all first-order momentum-kick perturbations.

In the lowest Landau level limit the physical Hilbert space collapses to a single spin- j multiplet, so only rotational errors remain; the corresponding intrinsic code appears for $j = 2$. Further details are given in the Supplemental Material [16].

CONCLUSION

We have shown that quantum error correction admits an intrinsically representation-theoretic formulation: a code is simply a subspace of a representation V that satisfies the Knill–Laflamme conditions for symmetry sectors of $\mathcal{L}(V)$. The Schur–Bootstrap then guarantees that any G -equivariant embedding of V into a physical Hilbert space $\mathcal{H}$ produces a code with exactly the same symmetry-resolved error protection and covariant gate structure. The essential properties of these codes are fixed already at the intrinsic level, independent of hardware or platform.

The minimal intrinsic code $\{\{5, 2, 2\}\}$ of $SU(2)$ illustrates this unifying mechanism: qubit, qudit, spin, bosonic, absorption–emission, rigid-rotor, molecular, and Landau-level codes are all revealed as different embeddings of a single intrinsic structure.

More broadly, intrinsic codes offer a kind of *compactification* of fault-tolerant schemes. The same order- ϵ error suppression achieved by a four-qubit code can be “compressed” into two higher-dimensional sites—or even a single five-level system—without altering its logical gate group or perturbative distance. This interpolation between architectures suggests that the design of covariant quantum codes can be reduced to a tractable representation-theoretic problem on low-dimensional spaces, opening a systematic path for the discovery of new codes and transversal gate structures across a variety of platforms.

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- [1] J. R. R. Tolkien, *The Fellowship of the Ring*, The Lord of the Rings, Vol. 1 (George Allen & Unwin, 1954).
 - [2] J. A. Gross, Designing codes around interactions: The case of a spin, *Phys. Rev. Lett.* **127**, 010504 (2021).
 - [3] A. Denys and A. Leverrier, Quantum error-correcting codes with a covariant encoding, *Phys. Rev. Lett.* **133**, 240603 (2024).
 - [4] Y. Wang, Y. Xu, and Z.-W. Liu, Tessellation codes: Encoded quantum gates by geometric rotation, *Phys. Rev. Lett.* **135**, 140602 (2025).
 - [5] E. Kubischta and I. Teixeira, Family of quantum codes with exotic transversal gates, *Phys. Rev. Lett.* **131**, 240601 (2023).
 - [6] E. Kubischta and I. Teixeira, Quantum codes from twisted unitary t -groups, *Phys. Rev. Lett.* **133**, 030602 (2024).
 - [7] J. M. Lee, *Introduction to Riemannian manifolds*, 2nd ed., Graduate texts in mathematics (Springer International Publishing, Cham, Switzerland, 2019).
 - [8] C. F. Gauss, *Disquisitiones Generales Circa Superficies Curvas* (Dieterich, Göttingen, 1828) göttingen: Dieterich, 1828; translated in *General Investigations of Curved Surfaces*, Princeton Univ. Press (1902).
 - [9] OpenAI, Chatgpt (GPT-5) model (2025), large language model, accessed via <https://chat.openai.com> Nov 2025.
 - [10] W. Fulton and J. Harris, *Representation theory*, 1st ed., Graduate texts in mathematics (Springer, New York, NY, 1999).
 - [11] H. Georgi, *Lie algebras in particle physics*, 2nd ed., Vol. 54 (Perseus Books, Reading, MA, 1999).
 - [12] P. Faist, S. Nezami, V. V. Albert, G. Salton, F. Pastawski, P. Hayden, and J. Preskill, Continuous symmetries and approximate quantum error correction, *Physical Review X* **10**, 10.1103/physrevx.10.041018 (2020).
 - [13] P. Hayden, S. Nezami, S. Popescu, and G. Salton, Error correction of quantum reference frame information, *PRX Quantum* **2**, 10.1103/prxquantum.2.010326 (2021).
 - [14] E. Kubischta and I. Teixeira, Permutation-invariant quantum codes with transversal generalized phase gates, *IEEE Transactions on Information Theory* **71**, 485 (2025).
 - [15] A. Aydin, V. V. Albert, and A. Barg, Quantum error correction beyond $SU(2)$: spin, bosonic, and permutation-invariant codes from convex geometry (2025), [arXiv:2509.20545](https://arxiv.org/abs/2509.20545) [quant-ph].
 - [16] See supplemental material which includes references [14, 15, 21, 50–52, 56, 58? ? ? –60].
 - [17] R. H. Dicke, Coherence in spontaneous radiation processes, *Phys. Rev.* **93**, 99 (1954).
 - [18] A. Bärttschi and S. Eidenbenz, Deterministic preparation of Dicke states, in *Fundamentals of Computation Theory* (Springer International Publishing, 2019) pp. 126–139.
 - [19] B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, G. Tóth, and C. Klempt, Detecting multiparticle entanglement of Dicke states, *Phys. Rev. Lett.* **112**, 155304 (2014).
 - [20] D. B. Hume, C. W. Chou, T. Rosenband, and D. J. Wineland, Preparation of Dicke states in an ion chain, *Phys. Rev. A* **80**, 052302 (2009).
 - [21] M. B. Ruskai, Pauli exchange errors in quantum computation, *Phys. Rev. Lett.* **85**, 194 (2000).
 - [22] H. Pollatsek and M. B. Ruskai, Permutationally invariant codes for quantum error correction, *Linear Algebra and its Applications* **392**, 255 (2004).
 - [23] Y. Ouyang, Permutation-invariant quantum codes, *Physical Review A* **90**, 10.1103/physreva.90.062317 (2014).
 - [24] M. Hagiwara and A. Nakayama, A four-qubits code that is a quantum deletion error-correcting code with the optimal length, in *2020 International Symposium on Information Theory (ISIT)* (2020) pp. 917–921.
 - [25] A. Nakayama and M. Hagiwara, Single quantum deletion error-correcting codes, in *2021 International Symposium on Information Theory (ISIT)* (2021) pp. 1500–1504.
 - [26] Y. Ouyang, Permutation-invariant quantum coding for quantum deletion channels, in *2021 IEEE International Symposium on Information Theory (ISIT)* (2021) pp. 1499–1504, [arXiv:2102.02494](https://arxiv.org/abs/2102.02494) [quant-ph].

- [27] S. Bravyi, D. Lee, Z. Li, and B. Yoshida, How much entanglement is needed for quantum error correction?, *Phys. Rev. Lett.* **134**, 210602 (2025).
- [28] J. Lindon, A. Tashchilina, L. W. Cooke, and L. J. LeBlanc, Complete unitary qutrit control in ultracold atoms, *Phys. Rev. Appl.* **19**, 034089 (2023).
- [29] Y. Fu, W. Liu, X. Ye, Y. Wang, C. Zhang, C.-K. Duan, X. Rong, and J. Du, Experimental investigation of quantum correlations in a two-qutrit spin system, *Phys. Rev. Lett.* **129**, 100501 (2022).
- [30] Y. Guo, W. Ji, X. Kong, M. Wang, H. Sun, J. Zhou, Z. Chai, X. Rong, F. Shi, Y. Wang, and J. Du, Single-shot readout of a solid-state electron spin qutrit, *Phys. Rev. Lett.* **132**, 060601 (2024).
- [31] D. Suter and T. S. Mahesh, Spins as qubits: Quantum information processing by nuclear magnetic resonance, *The Journal of Chemical Physics* **128**, 052206 (2008).
- [32] C. Senko, P. Richerme, J. Smith, A. Lee, I. Cohen, A. Retzker, and C. Monroe, Realization of a quantum integer-spin chain with controllable interactions, *Phys. Rev. X* **5**, 021026 (2015).
- [33] Z. Wang, S. Yu, H. Fan, and C. H. Oh, Quantum error-correcting codes over mixed alphabets, *Phys. Rev. A* **88**, 022328 (2013).
- [34] R. Yan, S. Pang, M. Chen, and F. Yang, Quantum error-correcting codes based on orthogonal arrays, *Entropy* **25**, 10.3390/e25040680 (2023).
- [35] Z. Tian, *Quantum stabilizer formalism for any composite system* (2024), [arXiv:2311.04255 \[quant-ph\]](https://arxiv.org/abs/2311.04255).
- [36] G. Nebe, E. M. Rains, and N. J. A. Sloane, *Self-Dual Codes and Invariant Theory*, 2006th ed., Algorithms and Computation in Mathematics (Springer, Berlin, Germany, 2006).
- [37] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997).
- [38] D. F. Walls and G. J. Milburn, *Quantum Optics*, 2nd ed. (Springer, 2008).
- [39] V. V. Albert, Bosonic coding: introduction and use cases, in *Quantum Fluids of Light and Matter*, International School of Physics “Enrico Fermi”, Vol. 209, edited by A. Bramati, I. Carusotto, and C. Ciuti (IOS Press, NL, 2025) p. 79–107.
- [40] J. Schwinger, On angular momentum, in *Quantum Theory of Angular Momentum*, edited by L. C. Biedenharn and H. van Dam (Academic Press, 1965) pp. 229–279.
- [41] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Atomic coherent states in quantum optics, *Physical Review A* **6**, 2211 (1972).
- [42] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed. (Pearson, 2017).
- [43] I. L. Chuang, D. W. Leung, and Y. Yamamoto, Bosonic quantum codes for amplitude damping, *Phys. Rev. A* **56**, 1114 (1997).
- [44] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Experimental realization of any discrete unitary operator, *Physical Review Letters* **73**, 58 (1994).
- [45] W. R. Clements, P. C. Humphreys, B. J. Metcalf, W. S. Kolthammer, and I. A. Walmsley, Optimal design for universal multiport interferometers, *Optica* **3**, 1460 (2016).
- [46] A. Zeilinger, General properties of lossless beam splitters in interferometry, *American Journal of Physics* **49**, 882 (1981).
- [47] V. V. Albert, J. P. Covey, and J. Preskill, Robust encoding of a qubit in a molecule, *Physical Review X* **10**, 10.1103/physrevx.10.031050 (2020).
- [48] S. P. Jain, E. R. Hudson, W. C. Campbell, and V. V. Albert, Absorption-emission codes for atomic and molecular quantum information platforms, *Phys. Rev. Lett.* **133**, 260601 (2024).
- [49] A. Aydin and A. Barg, Class of codes correcting absorptions and emissions, *Physical Review A* **111**, 10.1103/physreva.111.022415 (2025).
- [50] R. M. Fonseca, Groupmath: A mathematica package for group theory calculations, *Computer Physics Communications* **267**, 108085 (2021).
- [51] Wolfram Research, Inc., *Mathematica* 14.
- [52] E. Rains, Quantum weight enumerators, *IEEE Transactions on Information Theory* **44**, 1388 (1998).
- [53] R. Andrade e Silva and T. Jacobson, Particle on the sphere: group-theoretic quantization in the presence of a magnetic monopole, *Journal of Physics A: Mathematical and Theoretical* **54**, 235303 (2021).
- [54] T. Dray, The relationship between monopole harmonics and spin-weighted spherical harmonics, *J. Math. Phys.* **26**, 1030 (1985).
- [55] Y. Fan, W. Fischler, and E. Kubischta, Quantum error correction in the lowest landau level, *Physical Review A* **107**, 10.1103/physreva.107.032411 (2023).
- [56] S. Omanakuttan and J. A. Gross, Multispin clifford codes for angular momentum errors in spin systems (2023), [arXiv:2304.08611 \[quant-ph\]](https://arxiv.org/abs/2304.08611).
- [57] P. R. Bunker and P. Jensen, *Molecular symmetry and spectroscopy 2nd ed.* (NRC Research Press, 1998).
- [58] Y. Ouyang, Permutation-invariant quantum codes, *Physical Review A* **90**, 10.1103/physreva.90.062317 (2014).
- [59] A. Aydin, M. A. Alekseyev, and A. Barg, A family of permutationally invariant quantum codes, *Quantum* **8**, 1321 (2024).
- [60] Y. Ouyang, Permutation-invariant qudit codes from polynomials, *Linear Algebra and its Applications* **532**, 43–59 (2017).

Supplemental Material

Parameterizing 6 qubit realizations of the $\{(5, 2, 2)\}$ code

The basis for the moduli space of extrinsic $((6, 2, 2))$ codes can be found using the GroupMath package [50] in MATHEMATICA [51]. In the computational basis, the logical state $|\bar{0}\rangle$ may be written as

$$\begin{aligned} |\bar{0}\rangle = & c_1(|100000\rangle - |011111\rangle) + c_2(|010000\rangle - |101111\rangle) \\ & + c_3(|001000\rangle - |110111\rangle) + c_4(|000100\rangle - |111011\rangle) \\ & + c_5(|000010\rangle - |111101\rangle) + c_6(|000001\rangle - |111110\rangle), \end{aligned} \quad (1)$$

where the six coefficients c_k are linear functions of the parameters $z_1, \dots, z_5$:

$$\begin{aligned} c_1 &= \frac{1}{2} \sqrt{\frac{5}{3}} z_1, \\ c_2 &= -\frac{z_1}{2\sqrt{15}} + \sqrt{\frac{2}{5}} z_2, \\ c_3 &= \frac{1}{60} (-2\sqrt{15} z_1 - 3\sqrt{10} z_2 + 15\sqrt{6} z_3), \\ c_4 &= \frac{1}{60} (-2\sqrt{15} z_1 - 3\sqrt{10} z_2 - 5\sqrt{6} z_3 + 20\sqrt{3} z_4), \\ c_5 &= \frac{1}{60} (-2\sqrt{15} z_1 - 3\sqrt{10} z_2 - 5\sqrt{6} z_3 - 10\sqrt{3} z_4 - 30 z_5), \\ c_6 &= \frac{1}{60} (-2\sqrt{15} z_1 - 3\sqrt{10} z_2 - 5\sqrt{6} z_3 - 10\sqrt{3} z_4 + 30 z_5). \end{aligned} \quad (2)$$

We have not written $|\bar{1}\rangle$ for brevity but it also depends on a choice of $z = (z_1, \dots, z_5)$. We can check the weight enumerators [52] to find

$$\begin{aligned} A &= (1 \ 0 \ f_1^A(z) \ 0 \ f_2^A(z) \ 0 \ f_3^A(z)) \\ B &= (1 \ 0 \ f_1^B(z) \ 8 \ f_2^B(z) \ 24 \ f_3^B(z)). \end{aligned} \quad (3)$$

Here f_i^A and f_i^B are polynomial functions of $z = [z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{CP}^4$. So we see that for any choice of z , indeed we get a $((6, 2, 2))$ code. And one can check that there are no choice of z such that $f_1^A(z) = f_1^B(z)$ (i.e., there isn't some singular $d = 3$ code hidden within these).

Perturbative Order and Depth

This appendix formalizes the notion of *perturbative order* used in the main text and explains how intrinsic distance d of an intrinsic code maps, via the Schur-Bootstrap, to a physical error-suppression exponent $\mathbf{d}$ for any G -equivariant realization. Let G be the physical

symmetry group and let Ad denote its adjoint representation, describing the leading (infinitesimal) perturbations. We assign $\text{ord}(\mathbf{1}) = 0$ and $\text{ord}(\text{Ad}) = 1$, and for any irrep ξ of G define

$$\text{ord}(\xi) := \min\{t \geq 0 : \xi \subset \text{Ad}^{\otimes t}\}. \quad (4)$$

If each qudit transforms as the fundamental irrep $\mathbf{q}$ of $\text{SU}(q)$, then $\mathbf{q} \otimes \bar{\mathbf{q}} = \mathbf{1} \oplus \text{Ad}$ so any small single-site error can schematically be written $\mathbf{1} + \epsilon \text{Ad}$. Two sites give

$$(\mathbf{1} + \epsilon \text{Ad})^{\otimes 2} = \mathbf{11} + \epsilon(\text{Ad} \otimes \mathbf{1} + \mathbf{1} \otimes \text{Ad}) + \epsilon^2(\text{Ad} \otimes \text{Ad}) \quad (5)$$

which coincides exactly with weight-0, weight-1, and weight-2 errors. Hence for multi-qudit systems built from multiple fundamental representations of $\text{SU}(q)$ the perturbative order coincides with error weight and a distance- d code has logical errors of order $\mathcal{O}(\epsilon^d)$.

For systems not built from fundamental representations of $\text{SU}(q)$, such as spin-1 atoms ($\mathbf{3}$ of $\text{SU}(2)$), then the space of errors contains multiple irreps with *different* perturbative orders inside the same weight class. For a spin-1 system, $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$ where $\text{ord}(\mathbf{1}) = 0$, $\text{ord}(\mathbf{3}) = 1$, and $\text{ord}(\mathbf{5}) = 2$ corresponding to scalar shifts, linear type errors, and quadratic (tensor) perturbations. Thus a small local error can schematically be written $\mathbf{1} + \epsilon \mathbf{3} + \epsilon^2 \mathbf{5}$. And for two sites:

$$\begin{aligned} (\mathbf{1} + \epsilon \mathbf{3} + \epsilon^2 \mathbf{5})^{\otimes 2} &= \mathbf{11} + \epsilon(\mathbf{31} + \mathbf{13}) \\ &+ \epsilon^2(\mathbf{51} + \mathbf{15} + \mathbf{33}) + \epsilon^3(\mathbf{35} + \mathbf{53}) + \epsilon^4(\mathbf{55}). \end{aligned} \quad (6)$$

The conventional weight-1 sector contains $\mathbf{31}$, $\mathbf{13}$, $\mathbf{51}$, and $\mathbf{15}$. However, these are order- ϵ and order- ϵ^2 errors. So a code may fail to protect against all weight-1 errors yet still suppress every order- ϵ error. In this way conventional distance d is too coarse for heterogeneous or higher-spin platforms. We therefore say a code has *depth* $\mathbf{d}$ if the KL condition holds for all ξ with $\text{ord}(\xi) < \mathbf{d}$. Equivalently, the leading logical error scales as $\mathcal{O}(\epsilon^{\mathbf{d}})$. Depth coincides with distance for fundamental $\text{SU}(q)$ qudits but is strictly finer for higher-spin and heterogeneous systems.

If an intrinsic code has intrinsic distance d , then any G -equivariant realization obtained by the Schur-Bootstrap has depth $\mathbf{d} = d$ so intrinsic distance fixes the physical perturbative suppression exponent universally across all hardware embeddings. The two spin-1 example in the main text illustrates this: the extrinsic realization inside $\mathbf{3} \otimes \bar{\mathbf{3}}$ has conventional distance $d = 1$ but detects all order- ϵ error sectors ($\mathbf{31}$ and $\mathbf{13}$), so its logical errors scale as ϵ^2 and the depth is $\mathbf{d} = 2$. The qubit-quart embedding $\mathbf{2} \otimes \bar{\mathbf{4}}$ has $d = 1$ but depth $\mathbf{d} = 2$ for the same reason.

Depth also clarifies the “compactification” phenomenon discussed in the main text: the intrinsic $\{\{5, 2, 2\}\}$ code has $d = 2$ and thus all of its extrinsic realizations—four-qubit, two-qutrit, qubit-quart, or single-spin-2—achieve order- $\mathcal{O}(\epsilon^2)$ suppression. What changes is only the number of sites and the local Hilbert-space dimension. Depth therefore provides a representation-theoretic measure of physical performance that is intrinsic to the code and independent of the hardware.

Bosonic Codes (Detailed Analysis)

For two modes (a, b) , physical noise separates into two natural classes. *Loss/gain errors* (amplitude-damping-type processes) are generated by $a, a^\dagger, b, b^\dagger$, while *number-preserving (dephasing) errors* are generated by the Schwinger operators

$$J_+ = a^\dagger b, \quad J_- = ab^\dagger, \quad J_z = \frac{1}{2}(a^\dagger a - b^\dagger b). \quad (7)$$

Consider first the number-preserving sector. The operator $\hat{n}$ transforms as the trivial irrep $\mathbf{1}$, and the set $\{J_+, J_-, J_z\}$ transforms as $\mathbf{3}$, so each satisfies the KL condition for the intrinsic $\{\{5, 2, 2\}\}$ code. The same argument applies to all higher-order dephasing terms that transform in $\mathbf{1}$ or $\mathbf{3}$. For example, the second-order commutator

$$[J_+, J_z] = J_+ J_z - J_z J_+ \in \mathbf{3} \quad (8)$$

is also detectable.

Next consider loss/gain errors—exactly the sort of amplitude damping errors originally addressed by the Chuang–Leung–Yamamoto construction. Single-loss operators $\{a, b\}$ transform as $\bar{\mathbf{2}} \simeq \mathbf{2}$ and single-gain operators $\{a^\dagger, b^\dagger\}$ as $\mathbf{2}$. However, $\mathcal{L}(V)$ on this code space decomposes as

$$\mathcal{L}(V) = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \oplus \mathbf{7} \oplus \mathbf{9}, \quad (9)$$

so no $\mathbf{2}$ component acts nontrivially on $\mathcal{L}$. Thus all single-loss and single-gain errors automatically satisfy the KL condition.

Second-order loss/gain processes such as $a^\dagger a, b^\dagger b, a^\dagger b$, and $b^\dagger a$ transform as

$$\mathbf{2} \otimes \bar{\mathbf{2}} = \mathbf{1} \oplus \mathbf{3}, \quad (10)$$

and are therefore also detectable. Finally, any cubic (or more generally any odd-order) product of $\{a, b, a^\dagger, b^\dagger\}$ decomposes entirely into even-dimensional irreps of $\text{SU}(2)$, none of which appear in $\mathcal{L}(V)$, so such operators act trivially on the code.

Together these observations show how the Schur-bootstrap and representation-theoretic analysis reveals substantially broader error-detecting capabilities than previously recognized.

Rigid Rotor Codes (Detailed Analysis)

Consider an asymmetric polyatomic molecule (or a rigid rotor) which has rotational Hilbert space $\mathcal{H} = L^2(\text{SO}(3))$ [47]. From the Peter-Weyl theorem this can be decomposed as

$$L^2(\text{SO}(3)) = \bigoplus_{\ell \geq 0} (\mathbf{2\ell + 1}) \otimes (\mathbf{2\ell + 1}). \quad (11)$$

Here ℓ is an integer representing the total angular momentum. And the left quantum number $-\ell \leq m \leq \ell$ is the space-fixed projection (onto the lab’s z -axis) while the right quantum number $-\ell \leq k \leq \ell$ is the body-fixed projection (onto the molecule’s z -axis). There is an orthonormal angular momentum basis for this space denoted $|\ell_{mk}\rangle$.

There are two independent $\text{SU}(2)$ rotations on this space: a left $\text{SU}(2)_L$ rotation and a right $\text{SU}(2)_R$ rotation. The left action rotates the molecule about a lab-fixed axis (and affects the quantum number m). The right action rotates the molecule about a body-fixed axis (and affects the quantum number k). It follows that there are both left and right rotational errors generated by the angular momentum operators J_i^L and J_i^R respectively. Moreover, there are momentum-kick errors $\hat{D}_{MK}^L$ corresponding to the Wigner-D matrices (that Lie generate the “position” or “orientation” operators).

Unlike the linear rotor, there are many different embeddings of the intrinsic $\{\{5, 2, 2\}\}$ code into this space. For example, The intrinsic code $\{\{5, 2, 2\}\}$ can be realized inside the $\ell = 2$ sector two natural ways:

$$\mathbf{5}_L \otimes \mathbf{5}_R \cong (5) \mathbf{5}_L \quad \text{or} \quad \mathbf{5}_L \otimes \mathbf{5}_R \cong (5) \mathbf{5}_R. \quad (12)$$

Each embedding is parameterized by a $\mathbb{C}P^4$ family of possible choices. A left-embedded code automatically protects against all right-rotational errors J_i^R , and vice-versa. In addition, by the Schur-Bootstrap, left-embedded codes also detect all first-order left rotations J_i^L and first-order momentum kicks $\hat{D}_{MK}^{L=1}$, since these transform in $\mathbf{3}$ of $\text{SU}(2)$. Analogous statements hold for the right embeddings.

There is also a diagonal embedding. For any $\ell \geq 1$, the tensor product $(\mathbf{2\ell + 1}) \otimes (\mathbf{2\ell + 1})$ contains a *unique* copy of $\mathbf{5}$. For example, when $\ell = 1$ the embedding $|k\rangle \mapsto |(\begin{smallmatrix} 1 \\ mk \end{smallmatrix})\rangle$ is

$$\begin{aligned} |0\rangle &\mapsto |(\begin{smallmatrix} 1 \\ -1, -1 \end{smallmatrix})\rangle, \\ |1\rangle &\mapsto \frac{1}{\sqrt{2}} \left(|(\begin{smallmatrix} 1 \\ 0, -1 \end{smallmatrix})\rangle + |(\begin{smallmatrix} 1 \\ -1, 0 \end{smallmatrix})\rangle \right), \\ |2\rangle &\mapsto \frac{1}{\sqrt{6}} \left(|(\begin{smallmatrix} 1 \\ 1, -1 \end{smallmatrix})\rangle + 2 |(\begin{smallmatrix} 1 \\ 0, 0 \end{smallmatrix})\rangle + |(\begin{smallmatrix} 1 \\ -1, 1 \end{smallmatrix})\rangle \right), \\ |3\rangle &\mapsto \frac{1}{\sqrt{2}} \left(|(\begin{smallmatrix} 1 \\ 0, 1 \end{smallmatrix})\rangle + |(\begin{smallmatrix} 1 \\ 1, 0 \end{smallmatrix})\rangle \right), \\ |4\rangle &\mapsto |(\begin{smallmatrix} 1 \\ 1, 1 \end{smallmatrix})\rangle. \end{aligned} \quad (13)$$

The resulting extrinsic codewords are

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{2}} \left(|(-1, -1)\rangle + |(1, 1)\rangle \right), \\ |\bar{1}\rangle &= \frac{1}{\sqrt{6}} \left(|(1, -1)\rangle + 2|(0, 0)\rangle + |(-1, 1)\rangle \right). \end{aligned} \quad (14)$$

This diagonal embedding protects against both first-order left and right rotations. Momentum-kick operators $\hat{D}_{MK}^L$ transform in the reducible representation $(2\ell + 1) \otimes (2\ell + 1)$, and the code detects any irreducible component that appears in **3**. For each $\ell \geq 1$, exactly one such **3** appears.

All three realizations—left, right, and diagonal—constitute generalized absorption-emission (AE) rotor codes. Each implements the logical group S_3 covariantly through physical rotations of the molecule.

Molecular Codes (Detailed Analysis)

Let $K \subset \text{SO}(3)$ be the rotational symmetry group of a molecule. The rotational Hilbert space is

$$\mathcal{H} = L^2(\text{SO}(3)/K), \quad (15)$$

which decomposes, by the Peter–Weyl theorem, as

$$L^2(\text{SO}(3)/K) = \bigoplus_{\ell \geq 0} (2\ell + 1) \otimes M_\ell^K. \quad (16)$$

Here M_ℓ^K is the multiplicity space consisting of the K -fixed vectors in the body-fixed irrep $(2\ell + 1)$. The basis states are written $|_{m\kappa}^\ell\rangle$, where κ indexes an orthonormal basis of M_ℓ^K .

For sulfur dioxide SO_2 , the symmetry group is $K = C_2$, the cyclic group of order 2 (and each oxygen can be taken to have spin-0 so that nuclear degrees of freedom can be neglected). A computation shows

$$M_\ell^{C_2} = \{-\ell, -\ell + 2, \dots, \ell - 2, \ell\}, \quad (17)$$

so $\kappa = k$ must be even. Thus the multiplicity in the ℓ sector is $\ell + 1$ for even ℓ and ℓ for odd ℓ . So the multiplicity of **5** within $L^2(\text{SO}(3)/C_2)$ is 3 and thus the intrinsic code corresponds under the Schur-Bootstrap to a $\mathbb{CP}^2$ worth of extrinsic codes within this molecule

$$|w\rangle \mapsto |_{w-2, \kappa}^{-2}\rangle, \quad \kappa \in \{-2, 0, 2\}. \quad (18)$$

The corresponding extrinsic codewords are

$$\begin{aligned} |\bar{0}\rangle &= \sum_{\kappa \in \{-2, 0, 2\}} \frac{z_\kappa}{\sqrt{2}} \left(|_{-2, \kappa}^{-2}\rangle + |_{2, \kappa}^2\rangle \right), \\ |\bar{1}\rangle &= \sum_{\kappa \in \{-2, 0, 2\}} z_\kappa |_{0, \kappa}^2\rangle, \end{aligned} \quad (19)$$

where $[z_{-2} : z_0 : z_2] \in \mathbb{CP}^2$.

All embeddings detect any first-order left rotation, since these transform in the **3** of $\text{SU}(2)$. Because the code lies entirely in the left sector, it also detects all right rotations. $\hat{D}_{MK}^{L=1}$ errors likewise transform in **3** and are detectable. The logical S_3 action is implemented covariantly using left rotations.

Buckyball Codes

The most naturally occurring Buckminsterfullerene (buckyball) is composed of 60 ^{12}C atoms (each with nuclear spin-0), denoted $^{12}\text{C}_{60}$. Its rotational symmetry group is the icosahedral group $I \subset \text{SO}(3)$ (order 60). So the rotational Hilbert space is $\mathcal{H} = L^2(\text{SO}(3)/I)$ which decomposes into irreps under $\text{SO}(3)$ (or equivalently $\text{SU}(2)$) as

$$L^2(\text{SO}(3)/I) = \mathbf{1} \oplus \mathbf{13} \oplus \mathbf{21} \oplus \mathbf{25} \oplus \mathbf{31} \oplus \dots \quad (20)$$

The smallest non-trivial irrep is **13** (corresponding to the lowest non-trivial rotational energy). So if we can find intrinsic codes within the irrep **13** of $\text{SU}(2)$, then we will have found, for the first time, quantum codes within the buckyball's rotational symmetry.

Qutrit Code.— Using a computerized search, we have found a $\{\{\mathbf{13}, 3, 2\}\}$ intrinsic $\text{SU}(2)$ code as follows:

$$\begin{aligned} |\bar{0}\rangle &= -\sqrt{\frac{21}{80}}|1\rangle + \frac{|3\rangle}{\sqrt{10}} + \sqrt{\frac{63}{440}}|5\rangle - \sqrt{\frac{63}{176}}|9\rangle - \sqrt{\frac{3}{22}}|11\rangle \\ |\bar{1}\rangle &= \frac{|0\rangle}{4\sqrt{11}} + \sqrt{\frac{21}{55}}|2\rangle - \frac{3|4\rangle}{4\sqrt{5}} + \frac{3|8\rangle}{4\sqrt{5}} - \sqrt{\frac{21}{55}}|10\rangle - \frac{|12\rangle}{4\sqrt{11}} \\ |\bar{2}\rangle &= \sqrt{\frac{3}{22}}|1\rangle + \sqrt{\frac{63}{176}}|3\rangle - \sqrt{\frac{63}{440}}|7\rangle - \frac{|9\rangle}{\sqrt{10}} + \sqrt{\frac{21}{80}}|11\rangle. \end{aligned} \quad (21)$$

This code is T -covariant where T is the order 12 tetrahedral group.

Then via the Schur-Bootstrap, this corresponds to an extrinsic qutrit code *within* the lowest non-trivial energy level of a buckyball. It can detect any 1st-order left rotation since those transform in **3** of $\text{SU}(2)$ and it can protect against any right rotation (since the code lives entirely within the left sector). Interestingly, the momentum-kick operators $\hat{D}_{MK}^L$ are already protected up to and including 5th-order because of the symmetry of the Hilbert space (which is not a property of the code but of the Hilbert space itself). Lastly, any gate from T can be implemented by left rotations of the molecule.

Qubit Code.— Using a computerized search, we have found also found a $\{\{\mathbf{13}, 2, 3\}\}$ intrinsic $\text{SU}(2)$ code as follows. Let

$$\begin{aligned} v &= \frac{\sqrt[3]{611808967 + 9900\sqrt{3807314905}}}{1050}, \\ w &= v + \frac{104929}{1102500v} - \frac{11}{150}, \\ D &= \frac{49w^3}{66} + \frac{49w^2}{120} + w + 1. \end{aligned} \quad (22)$$

Then the intrinsic codewords are

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{D}} \left(\frac{7w^{3/2}}{\sqrt{66}} |1\rangle - \frac{7w}{2\sqrt{30}} |4\rangle + \sqrt{w} |7\rangle + |10\rangle \right), \\ |\bar{1}\rangle &= \frac{1}{\sqrt{D}} \left(|2\rangle + \sqrt{w} |5\rangle - \frac{7w}{2\sqrt{30}} |8\rangle + \frac{7w^{3/2}}{\sqrt{66}} |11\rangle \right). \end{aligned} \quad (23)$$

This code is S_3 covariant where S_3 is the symmetric group on 3 letters.

Then via the Schur-Bootstrap, this corresponds to an extrinsic qubit code within the lowest non-trivial energy level of a buckyball that can *correct* any first order error (whether it is rotational or a momentum kick). Any gate from S_3 can be implemented by rotations of the molecule.

Remarks.— Although we have only demonstrated qubit and qutrit codes with distance 3 and 2 respectively, you can create codes of any dimension and with any distance (that can correct arbitrary order errors) by finding intrinsic codes within higher irreps (energy levels), for example **21**, **25**, **31**, etc.

Landau Level Codes (Detailed Analysis)

Consider a particle of electric charge q constrained to the surface of a sphere of radius R threaded by a magnetic monopole of charge BR^2 , with $qB > 0$. Defining

$$j = \frac{qBR^2}{\hbar}, \quad (24)$$

the Dirac quantization condition ensures that j is integer or half-integer. Let χ_j denote the one-dimensional representation of $U(1)$ of weight $2j$. The physical Hilbert space is the induced representation $\mathcal{H} = \text{Ind}_{U(1)}^{\text{SU}(2)} \chi_j$ [53]. The resulting $\text{SU}(2)$ decomposition is

$$\mathcal{H} = \bigoplus_{\ell \geq j} (\mathbf{2\ell + 1}), \quad (25)$$

with ℓ having the same parity as j . Each $(\mathbf{2\ell + 1})$ is a Landau level of energy ℓ and is spanned by the monopole spherical harmonics ${}_jY_m^\ell$ for $\ell \geq j$ and $|m| \leq \ell$ [54]. The case $j = 0$ reproduces the ordinary spherical harmonics $L^2(S^2)$.

Two classes of physical errors act on this system: (i) rotational errors generated by J_i , and (ii) momentum-kick errors generated by ${}_j\hat{Y}_M^L$, which transform as the $(\mathbf{2L + 1})$ of $\text{SU}(2)$.

The intrinsic code $\{\{\mathbf{5}, 2, 2\}\}$ occupies the spin-2 irrep. Thus an extrinsic realization exists exactly when j is integral and $j \leq 2$, so that **5** appears once in $\mathcal{H}$. The codewords are

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{2}} ({}_jY_{-2}^2 + {}_jY_2^2), \\ |\bar{1}\rangle &= {}_jY_0^2, \end{aligned} \quad (26)$$

for $j = 0, 1, 2$. These detect all first-order rotations and rank-1 momentum kicks because both classes of operators transform in the **3** under $\text{SU}(2)$.

As noted in [55], taking $B \rightarrow \infty$ and $R \rightarrow 0$ while fixing $j = \frac{qBR^2}{\hbar}$ isolates the lowest Landau level. In this limit the Hilbert space reduces to the single spin- j multiplet spanned by ${}_jY_m^j$, and the only physical errors are rotational. For $j = 2$, the intrinsic code appears in the LLL via

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{2}} ({}_2Y_{-2}^2 + {}_2Y_2^2), \\ |\bar{1}\rangle &= {}_2Y_0^2. \end{aligned} \quad (27)$$

This code detects all first-order rotational perturbations from the **3** sector.

Intrinsic $\{\{\mathbf{14}, 2, 3\}\}$ Code

There is an $\text{SU}(2)$ intrinsic $\{\{\mathbf{14}, 2, 3\}\}$ code:

$$\begin{aligned} |\bar{0}\rangle &= \frac{\sqrt{5}}{112} (13\sqrt{2} + 2\sqrt{11}) |0\rangle + \frac{\sqrt{13}}{112} (10 - 3\sqrt{22}) |4\rangle \\ &\quad - \frac{\sqrt{65}}{112} (6 + \sqrt{22}) |8\rangle + \frac{\sqrt{65}}{112} (\sqrt{2} - 2\sqrt{11}) |12\rangle \\ |\bar{1}\rangle &= \frac{\sqrt{5}}{112} (13\sqrt{2} + 2\sqrt{11}) |13\rangle + \frac{\sqrt{13}}{112} (10 - 3\sqrt{22}) |9\rangle \\ &\quad - \frac{\sqrt{65}}{112} (6 + \sqrt{22}) |5\rangle + \frac{\sqrt{65}}{112} (\sqrt{2} - 2\sqrt{11}) |1\rangle. \end{aligned} \quad (28)$$

Indeed this code has intrinsic distance $d = 3$ meaning it can correct any linear combination of the 14-dimensional representations of J_+ , J_- , and J_z . Equivalently, it can detect errors from **1**, **3**, and **5**.

The 14 dim representations of the special Pauli **X** and **Z** gates implement logical special **X** and **Z** respectively. Moreover, the 14 dim gate $S^{3\dagger}$ implements logical special **S** and the 14 dim inverse Hadamard $H^\dagger$ implements logical special Hadamard **H**. Thus this code is covariant for the single qubit Clifford group **2O**.

- *Multi-qubit:* There is a unique embedding of **14** into $\mathbf{2}^{\otimes 13}$. So this corresponds to a unique $((13, 2, 3))$ code that can implement the single qubit Clifford gates transversally. This code is permutation invariant and originally found in [56].
- *Higher multi-qubit:* The irrep **14** appears with multiplicity in $\mathbf{2}^{\otimes n}$ for all odd n for $n \geq 15$. For example, when $n = 15$ the multiplicity is 14 so there is a $\mathbb{CP}^{13}$ moduli space of $((15, 2, 3))$ codes, when $n = 17$ the multiplicity is 119 so there is a $\mathbb{CP}^{118}$ moduli space of $((17, 2, 3))$ codes. In fact it can be shown that the multiplicity μ of **14** in $\mathbf{2}^{\otimes n}$ for odd n is

$$\mu = \frac{28}{n+15} \binom{n}{\frac{n-13}{2}} \sim \mathcal{O}(2^n). \quad (29)$$

So in general there is a $\mathbb{CP}^{\mu-1}$ moduli space of these $((n, 2, 3))$ Clifford transversal codes and the moduli space is growing exponentially in n .

- *Multi-qudit*: There is a $\mathbb{CP}^3$ moduli space of codes in $\mathbf{4}^{\otimes 5}$ and a $\mathbb{CP}^1$ worth of codes in $\mathbf{6}^{\otimes 3}$. Each has depth $d = 3$ and is Clifford transversal.
- *Heterogeneous*: There are many different heterogeneous realizations. For example, each of the following is an example of a physical Hilbert space $\mathcal{H}$ that has a single copy of **14**:

- (2 sites): **78**
- (3 sites): **466**
- (4 sites): **4445**
- (5 sites): **24444**
- (7 sites): **2333333**

Again each of the extrinsic codes with these Hilbert spaces has depth $d = 3$ and is Clifford transversal. In the $\mathcal{H} = \mathbf{2333333}$, the extrinsic code even has conventional distance $d = 2$.

- *Spin version*: We can always treat an $SU(2)$ irrep as a large spin, in this case $j = 13/2$, using the embedding $|k\rangle \mapsto |j, j - k\rangle$.
- *Bosonic*: There is an extrinsic embedding into the two mode subspace with fixed excitation $N = 13$.

The code detects both 1st and 2nd order dephasing errors and up to 5th order gain/loss errors.

- *Landau Levels*: There is a Landau-Level extrinsic code that can detect either: (1) 2nd order rotations, (2) 2nd order momentum kicks, or (3) a 1st order rotation together with a 1st order momentum kick.

Note there are no $\mathbb{A}_E$, rigid rotor, or molecular embeddings for this code since it is an even-dimensional irrep of $SU(2)$. However, if electronic or nuclear spin is present, chemists often consider the “rotational double group” in which case the even dimensional irreps of $SU(2)$ would show up and an embedding would be possible [57].

Related Work

Every permutation invariant (PI) quantum code can be thought of as an extrinsic realization of some intrinsic code (that is, the converse of the Schur-Bootstrap holds in this case). A proof is given in [15]. But even in this case the benefit of the intrinsic picture is that one can then realize that same code in a variety of other settings, and it also proves that permutation invariant codes can always detect many errors of higher weight, vastly generalizing the analysis in [22] of higher weight errors detected by 9-qubit PI codes. See also [14, 21, 58–60] for many different examples of PI codes.