RESTRICTION OF EIGENFUNCTIONS ON PRODUCTS OF SPHERES TO SUBMANIFOLDS OF MAXIMAL FLATS

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Abstract. Let M be a product of rank-one symmetric spaces of compact type, each of dimension at least 3. We establish sharp L^p bounds for the restriction of Laplace–Beltrami eigenfunctions on M to arbitrary submanifolds contained in a maximal flat, for all $p \geq 2$. The proof combines precise asymptotics of Jacobi polynomials and positivity of Fourier coefficients of spherical functions.

1. Introduction

Let M be a compact Riemannian manifold of dimension d. Let Δ be the Laplace–Beltrami operator on M, and let f be its eigenfunctions such that $\Delta f = -N^2 f$, $N > 1$. The study of concentration of f has become a popular subject [14], and one of the approaches is to bound its restriction to submanifolds of M, as initiated by Tataru [12] and Reznikov [9]. Let S be a submanifold of dimension k. Let $L^p(M)$, $L^p(S)$ be the Lebesgue spaces associated to the Riemannian volume measure on M and S respectively. The following fundamental restriction bounds for general M were established by Burq–Gérard–Tzvetkov [1] and Hu [6]: it holds

$$\|f\|_{L^p(S)} \lesssim N^{\rho(k,d,p)} \|f\|_{L^2(M)}, \tag{1.1}$$

where

$$\rho(d-1,d,p) = \begin{cases} \frac{d-1}{2} - \frac{d-1}{p}, & \text{if } \frac{2d}{d-1} \leq p \leq \infty, \\ \frac{d-1}{4} - \frac{d-2}{2p}, & \text{if } 2 \leq p \leq \frac{2d}{d-1}, \end{cases}$$

$$\rho(d-2,d,p) = \frac{d-1}{2} - \frac{d-2}{p}, \qquad \text{if } 2 < p \leq \infty,$$

$$\rho(k,d,p) = \frac{d-1}{2} - \frac{k}{p}, \qquad \text{if } 1 \leq k \leq d-3 \text{ and } 2 \leq p \leq \infty;$$

and for the case $p = 2$ and $k = d-2$, it holds

$$\|f\|_{L^2(S)} \lesssim N^{\frac{1}{2}} (\log N)^{\frac{1}{2}} \|f\|_{L^2(M)}. \tag{1.2}$$

All the above estimates are sharp on spheres, except for the logarithmic loss, which however has been eliminated by Chen–Sogge [2] and Wang–Zhang [13] for totally geodesic submanifolds.

The goal of this note is to record an instance of *sharp polynomial improvement* over the above results of Burq–Gérard–Tzvetkov and Hu, on products of compact rank-one symmetric spaces, for all $p \geq 2$: namely, the estimate (1.4) below. For each $i = 1, \ldots, r$ ($r \geq 2$), let M_i be a compact rank-one symmetric space (CROSS)

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(of compact type) of dimension d_i; this family includes the standard spheres. Let $M = M_1 \times \cdots \times M_r$ be their product. A maximal flat of M is a maximally embedded flat, totally geodesic submanifold, which is isomorphic to $\mathbb{T}^r$. We establish:

Theorem 1.1. *Let S be a k-dimensional submanifold of a maximal flat of M, $k = 0, 1, \ldots, r$. Suppose without loss of generality that $d_1 \le d_2 \le \cdots \le d_r$. Let*

$$\tau(d,p) := \begin{cases} -\frac{1}{p}, & p \ge \frac{4}{d-1}, \\ -\frac{d-1}{4}, & 0 < p \le \frac{4}{d-1}. \end{cases}$$

Let f be an eigenfunction of Δ such that $\Delta f = -N^2 f$, $N > 1$. Then

$$\|f\|_{L^p(S)} \lesssim_\varepsilon N^{\frac{d-2}{2} + \sum_{i=1}^k \tau(d_i,p) + \varepsilon} \|f\|_{L^2(M)}, \qquad \text{for all } p \ge 2. \tag{1.3}$$

Moreover, when $r \ge 5$, the ε-loss in (1.3) *can be removed. In particular, if* $\dim M_i \ge 3$ *for all i and $r \ge 5$, then*

$$\|f\|_{L^p(S)} \lesssim N^{\frac{d-2}{2} - \frac{k}{p}} \|f\|_{L^2(M)}, \qquad \text{for all } p \ge 2, \tag{1.4}$$

and the above estimate is sharp.

Remark 1.2. In the setting of compact Lie groups, the author established the same sharp estimate as (1.4) for eigenfunctions restricted to submanifolds of maximal flats [16].

Remark 1.3. In the setting of standard tori, Huang–Zhang [7] proved that for totally geodesic submanifolds defined by linear equations with rational coefficients, the same eigenfunction restriction estimate as (1.4) holds for all $p \ge 2$, up to an ε-loss.

Remark 1.4. In particular, our estimates apply to geodesics, since every geodesic lies in a maximal flat of M. It remains an interesting question to get sharp estimates for restriction of eigenfunctions to submanifolds which are not contained in any maximal flat. A possible approach, which we leave for future work, is to adapt the argument of Burq–Gérard–Tzvetkov [1] to the product manifold setting.

Remark 1.5. When two-dimensional factors such as $\mathbb{S}^2$ or $\mathbb{RP}^2$ are present, the bound (1.3) is not expected to be sharp even without the ε-loss, as can be seen through the case of products of two-spheres, for which the estimate (1.3) when specialized to the case $p = 2$ is worse than the general bounds (1.3) of Burq–Gérard–Tzvetkov and Hu. It is thus an interesting question to get sharp bounds for this bad case.

Remark 1.6. Suppose S is a submanifold of M of product type, namely, $S = S_1 \times \cdots \times S_r$ where S_i is a submanifold of M_i for each i. Then by applying to each (M_i, S_i) the estimate (1.1), along with the lossless version of (1.2) in the cases where it is available, one can recover Theorem 1.1 in this product-type setting. Thus the purpose of Theorem 1.1 is to treat submanifolds which are not of product type.

Remark 1.7. For global L^p estimates of eigenfunctions, the author established on general symmetric spaces of compact type that [15]

$$\|f\|_{L^p(M)} \lesssim N^{\frac{d-2}{2} - \frac{d}{p}} \|f\|_{L^2(M)},$$

for $r \ge 5$ and $p > 2 + 8/(r-4)$. The above sharp estimate also holds on products of CROSSs for $r \ge 5$ and $p \ge 2(d_i + 1)/(d_i - 1)$ for all i, by applying Sogge's classical spectral projector estimates [10] to each factor. It remains open to establish sharp L^p estimates for all $p \geq 2$.

Our approach to proving Theorem 1.1 is to extract information about general eigenfunctions from a special class of eigenfunctions, namely the spherical functions. A key ingredient is the mapping property of convolution with Jacobi polynomials, which express spherical functions on CROSSs. Let $P_n^{(\alpha,\beta)}(x)$ ($n \in \mathbb{Z}_{\geq 0}$) denote the Jacobi polynomials, defined as the orthogonal polynomials on the interval $[-1,1]$ with the weight function $(1-x)^\alpha(1+x)^\beta$. We follow the classical treatment in Szegö's book [11]. In particular, we use the same normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

It will be convenient to denote, for $\delta \geq 0$, $p > 0$, and $n \geq 0$,

$$A(\delta,p,n) := \begin{cases} (n+1)^{\delta-\frac{1}{p}}, & p > \frac{1}{\delta+\frac{1}{2}}, \\ (n+1)^{-\frac{1}{2}}, & 0 < p \leq \frac{1}{\delta+\frac{1}{2}}. \end{cases} \tag{1.5}$$

We establish:

Theorem 1.8. *Assume that $0 \leq \beta \leq \alpha$. For $n \in \mathbb{Z}_{\geq 0}$, let $T_n^{\alpha,\beta}$ denote the convolution operator on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with kernel $P_n^{(\alpha,\beta)}(\cos\theta)$, where $\theta \in \mathbb{T}$. Then for the following cases:*

(1) $\alpha > \frac{1}{2}$,
(2) $0 \leq \alpha < \frac{1}{2}$,
(3) $\alpha = \beta = \frac{1}{2}$,

the operator $T_n^{\alpha,\beta}$ satisfies

$$\|T_n^{\alpha,\beta}\|_{L^{p'}(\mathbb{T})\to L^p(\mathbb{T})} \lesssim A(\alpha,p/2,n), \quad \textit{for all } p \geq 2. \tag{1.6}$$

In particular, the case $\alpha = \beta = \frac{1}{2}$ corresponds to the three-dimensional factors $\mathbb{S}^3$ or $\mathbb{RP}^3$ in the product manifold M. For this case, the sharp estimate (1.6) at the endpoint $p = 2$ mirrors the log-loss removal of (1.2) for geodesics on three-dimensional compact manifolds established by Chen–Sogge [2]. In our setting of product manifolds containing three-dimensional factors, another key ingredient that enables the use of (1.6) for eigenfunction restriction to arbitrary submanifolds of maximal flats is the positivity of Fourier coefficients of spherical functions, as in Lemma 2.1. This positivity helps preserve oscillation in the convolution kernel and effectively reduces the study of arbitrary submanifolds to those of product type; see (2.9).

The rest of the paper is organized as follows. In Section 2, we prove the estimates in Theorem 1.1 by first establishing restriction estimates for the joint eigenfunctions of the Laplace–Beltrami operators on each factor M_i, using Theorem 1.8. In Section 3, we prove Theorem 1.8. In Section 4, we show the sharpness of (1.4) in Theorem 1.1.

Notation. Throughout the paper, we employ the standard notation $A \lesssim B$ to indicate that $A \leq CB$ for some constant $C > 0$. If $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$. We write $A \lesssim_\delta B$ when $A \leq CB$ for some constant $C > 0$ depending on δ. $A \ll B$ means there is a sufficiently small constant $c > 0$ such that $A \leq cB$. We also write $A \sim B$ to mean that $A/B \to 1$.

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2. Proof of Theorem 1.1

2.1. **Spherical functions.** Let M be a CROSS of dimension d. The spherical functions Φ_n ($n \in \mathbb{Z}_{\geq 0}$) on M, when restricted to a maximal torus (large circle) $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$ of M, can be expressed by Jacobi polynomials as follows:

$$\Phi_n(\theta) = \binom{n+\alpha}{n}^{-1} P_n^{(\alpha,\beta)}(\cos\theta), \qquad \theta \in \mathbb{T}, \tag{2.1}$$

where $\alpha, \beta \in \frac{1}{2} \cdot \mathbb{Z}$ such that

$$0 \leq \beta \leq \alpha = \frac{d-2}{2}. \tag{2.2}$$

In particular, we have picked the standard normalization such that $\Phi_n(0) = 1$. For the above formula, we refer to Theorem 4.5 in Chapter V of [4], where the Φ_n are expressed in terms of hypergeometric functions, and to formula (4.21.2) in [11], where these hypergeometric functions are rewritten as Jacobi polynomials. A specific case is when $d = 3 \Leftrightarrow \alpha = 1/2$, corresponding to $\mathbb{S}^3$ or $\mathbb{RP}^3$, for which we also have $\beta = 1/2$.

We will also use the following convenient formula, which expresses spherical functions as positive linear combinations of exponentials.

Lemma 2.1 (Proposition 9.4 of Chapter III of [5])**.** *For $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\Phi_n(\theta) = \sum_{j=0}^{q_n} c_{n,j} e^{i m_{n,j} \theta}, \qquad \theta \in \mathbb{T},$$

where $c_{n,j} \geq 0$ and $m_{n,j} \in \mathbb{Z}$. In particular, $\sum_{j=0}^{q_n} c_{n,j} = \Phi_n(0) = 1$.

Let $k(n)$ denote the dimension of the spherical representation of the isometry group U of M associated to Φ_n. We need the following standard fact about $k(n)$:

Lemma 2.2. *$k(n)$ is a polynomial in n of degree $d-1$, so that*

$$k(n) \simeq (n+1)^{d-1}, \qquad n \in \mathbb{Z}_{\geq 0}. \tag{2.3}$$

Proof. This is a consequence of the Weyl dimension formula combined with the characterization of the highest weight of spherical representations (Theorem 4.1 of Chapter V of [4]). □

2.2. **Eigenfunctions.** Let $M = M_1 \times \cdots \times M_r$ be a product of CROSSs. Each M_i, being a symmetric space of compact type, may be written as a homogeneous quotient $M_i \cong U_i/K_i$. Using the Haar measure du_i on the compact group U_i, we have the convolution of functions f, g on M_i by

$$(f * g)(x_i) = \int_{U_i} f(u_i K_i)\, g(u_i^{-1} x_i)\, du_i, \qquad x_i \in M_i.$$

These convolutions on each M_i together induce the convolution on the product M.

For $f \in L^2(M)$, following Proposition 9.1 of Chapter III of [4], we have the L^2 decomposition

$$f = \sum_{n_i \in \mathbb{Z}_{\geq 0},\ i=1,\ldots,r} P_{n_1,\ldots,n_r} f, \tag{2.4}$$

where $P_{n_1,\dots,n_r}$ is the projection onto the space of *joint eigenfunctions* of the Laplace–Beltrami operators Δ_i on M_i, such that

$$\Delta_i P_{n_1,\dots,n_r} f = -(n_i^2 + a_i n_i) P_{n_1,\dots,n_r} f.$$

Here each $a_i \in \mathbb{Z}_{>0}$ is a constant associated to M_i. Importantly, spherical functions are such joint eigenfunctions, and these projection operators are simply convolution with spherical functions:

$$P_{n_1,\dots,n_r} f = f * \prod_{i=1}^{r} k_i(n_i)\Phi_{i,n_i}, \tag{2.5}$$

where Φ_{i,n_i} ($n_i \in \mathbb{Z}_{\geq 0}$) are the spherical functions on M_i, and $k_i(n_i)$ denotes the dimension of the corresponding spherical representation. In particular, we have

$$\prod_{i=1}^{r} k_i(n_i)\Phi_{i,n_i} * \prod_{i=1}^{r} k_i(n_i)\Phi_{i,n_i} = \prod_{i=1}^{r} k_i(n_i)\Phi_{i,n_i}. \tag{2.6}$$

We establish the following restriction bounds of joint eigenfunctions on products of CROSSs.

Theorem 2.3. *Let S be a k-dimensional submanifold of a maximal flat of M, $k = 0, 1, \dots, r$. Suppose that $d_1 \leq d_2 \leq \cdots \leq d_r$. Let*

$$\tau(d,p) := \begin{cases} -\frac{1}{p}, & p \geq \frac{4}{d-1}, \\ -\frac{d-1}{4}, & 0 < p \leq \frac{4}{d-1}. \end{cases}$$

Let f be a joint eigenfunction of all the Δ_i, $i = 1, \dots, r$, such that

$$\Delta_i f = -(n_i + a_i n_i)^2 f, \qquad \text{for some } n_i \in \mathbb{Z}_{\geq 0}.$$

Let $N^2 = \sum_{i=1}^{r} n_i^2 + a_i n_i > 1$. Then

$$\|f\|_{L^p(S)} \lesssim N^{\frac{d-r}{2} + \sum_{i=1}^{k} \tau(d_i,p)} \|f\|_{L^2(M)}, \qquad \text{for all } p \geq 2. \tag{2.7}$$

Proof. A maximal flat E in M is of the form

$$E = \mathbb{T}_1 \times \cdots \times \mathbb{T}_r,$$

where $\mathbb{T}_i$ is a maximal torus of M_i. By taking a finite open cover of S, we may assume there exist $1 \leq i_1 < \cdots < i_k \leq r$ such that S is parametrized by the map

$$\mathbb{T}^k \supset V \ni \underline{\theta} = (\theta_{i_1}, \dots, \theta_{i_k}) \mapsto (\theta_1(\underline{\theta}), \dots, \theta_r(\underline{\theta})) \in S \subset E.$$

Proving the bound (2.7) for joint eigenfunctions is equivalent to proving the $L^2(M) \to L^p(S)$ bound for the projection operators $P_{n_1,\dots,n_r}$. By a TT^* argument, and using (2.5), (2.6) and the fact that the spherical functions are real-valued (as ensured by (2.1)), it suffices to demonstrate the $N^{d-r+2\sum_{i=1}^{k}\tau(d_i,p)}$ bound for the $L^{p'}(V, J(\underline{\theta})\, d\underline{\theta}) \to L^p(V, J(\underline{\theta})\, d\underline{\theta})$ norm of the operator

$$P_{n_1,\dots,n_r} P^*_{n_1,\dots,n_r} g(\underline{\theta}) = \int_V g(\underline{\theta}') \mathscr{K}(\underline{\theta}', \underline{\theta}) J(\underline{\theta}')\, d\underline{\theta}',$$

where

$$\mathscr{K}(\underline{\theta}', \underline{\theta}) = \prod_{i=1}^{r} k_i(n_i)\Phi_{i,n_i}(\theta_i(\underline{\theta}) - \theta_i(\underline{\theta}')),$$

and $J(\underline{\theta}')\, d\underline{\theta}'$ is the Riemannian measure on S, with $|J(\underline{\theta}')| \simeq 1$.

For $i \neq i_l$, $l = 1, \ldots, k$, we use Lemma 2.1 to write

$$\Phi_{i,n_i}(\theta_i) = \sum_{j_i=0}^{q_{i,n_i}} c_{i,n_i,j_i} e^{im_{i,n_i,j_i}\theta_i}, \qquad \theta_i \in \mathbb{T}_i,$$

where $m_{i,n_i,j_i} \in \mathbb{Z}$, and

$$c_{i,n_i,j_i} \geq 0, \qquad \sum_{j=0}^{q_{i,n_i}} c_{i,n_i,j_i} = 1. \tag{2.8}$$

Then we may rewrite the kernel $\mathscr{K}(\underline{\theta}', \underline{\theta})$ and obtain

$$\mathscr{K}(\underline{\theta}', \underline{\theta}) = \left(\prod_{i \notin \{i_1, \ldots, i_k\}} \sum_{j_i=0}^{q_{i,n_i}} c_{i,n_i,j_i} e^{im_{i,n_i,j_i}(\theta_i(\underline{\theta}) - \theta_i(\underline{\theta}'))} \right) \widetilde{\mathscr{K}}(\underline{\theta}', \underline{\theta}), \tag{2.9}$$

where

$$\begin{aligned} \widetilde{\mathscr{K}}(\underline{\theta}', \underline{\theta}) &= \prod_{i=1}^{r} k_i(n_i) \cdot \prod_{l=1}^{k} \Phi_{i_l, n_{i_l}}(\theta_{i_l} - \theta'_{i_l}) \\ &= \prod_{i=1}^{r} k_i(n_i) \cdot \prod_{l=1}^{k} \binom{n + \alpha_{i_l}}{n}^{-1} P_{n_{i_l}}^{(\alpha_{i_l}, \beta_{i_l})}(\cos(\theta_{i_l} - \theta'_{i_l})), \end{aligned}$$

with $0 \leq \beta_{i_l} \leq \alpha_{i_l} = (d_{i_l} - 2)/2$.

Applying (2.8), (2.9), and $|J(\underline{\theta}')| \simeq 1$, it suffices to get the same bound for the $L^{p'}(V,\ d\underline{\theta}) \to L^p(V,\ d\underline{\theta})$ norm of the operator

$$\mathcal{A} : g \mapsto \int_V g(\underline{\theta}') \widetilde{\mathscr{K}}(\underline{\theta}', \underline{\theta})\ d\underline{\theta}'. \tag{2.10}$$

Furthermore, we may now assume $V = \mathbb{T}^k$ without loss of generality, and then the above operator is convolution on $\mathbb{T}^k$ with a product of Jacobi polynomials. We may apply Theorem 1.8 along with the Minkowski inequality (using $p \geq 2 \Leftrightarrow p \geq p'$) to bound such a convolution operator, as well as apply the standard asymptotics

$$\binom{n + \alpha}{n} \sim \frac{n^\alpha}{\Gamma(\alpha + 1)}, \qquad n \to \infty, \tag{2.11}$$

and Lemma 2.2 to bound the coefficients, to obtain

$$\|\mathcal{A}\|_{L^{p'}(S) \to L^p(S)} \lesssim N^{d - r + 2\sum_{l=1}^{k} \tau(d_{i_l}, p)} \leq N^{d - r + 2\sum_{i=1}^{k} \tau(d_i, p)}.$$

Here in the last inequality we applied $d_1 \leq \cdots \leq d_r$. The proof is now complete. □

We are ready to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose $\Delta f = -N^2 f$, $N > 1$. By (2.4), and a standard counting estimate [3], we get

$$\begin{aligned} |f| &\leq \left| \left\{ (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r : \sum_{i=1}^{r} n_i^2 + a_i n_i = N^2 \right\} \right|^{\frac{1}{2}} \|P_{n_1, \ldots, n_r} f\|_{l^2_{n_1, \ldots, n_r}} \\ &\lesssim N^{\frac{r}{2} - 1 + \varepsilon} \|P_{n_1, \ldots, n_r} f\|_{l^2_{n_1, \ldots, n_r}}, \end{aligned}$$

which implies by an application of the Minkowski inequality, that for $p \geq 2$,

$$\|f\|_{L^p(M)} \lesssim N^{\frac{r}{2}-1+\varepsilon} \| \|P_{n_1,\ldots,n_r} f\|_{L^p(M)} \|_{l^2_{n_1,\ldots,n_r}},$$

and the ε-loss may be removed when $r \geq 5$.

The estimates (1.3) and (1.4) of Theorem 1.1 are now consequences of Theorem 2.3. We leave the proof of sharpness of (1.4) to Section 4. $\square$

3. Proof of Theorem 1.8

In addition to (1.5), it will also be convenient to denote

$$\widetilde{A}(\delta,p,n) := \begin{cases} (n+1)^{\delta-\frac{1}{p}}, & p > \frac{1}{\delta+\frac{1}{2}}, \\ (n+1)^{-\frac{1}{2}} \log^{\delta+\frac{1}{2}}(n+2), & p = \frac{1}{\delta+\frac{1}{2}}, \\ (n+1)^{-\frac{1}{2}}, & 0 < p < \frac{1}{\delta+\frac{1}{2}}. \end{cases}$$

We first demonstrate Theorem 1.8 except for the kink point:

Proposition 3.1. *For all the three cases in Theorem 1.8, we have*

$$\|T_n^{\alpha,\beta}\|_{L^{p'}(\mathbb{T})\to L^p(\mathbb{T})} \lesssim \widetilde{A}(\alpha, p/2, n), \quad \textit{for all } p \geq 2.$$

Proof. We decompose the convolution kernel $P_n^{\alpha,\beta}(\cos\theta)$ according to the value of θ, and it suffices to restrict attention to $\theta \in [0,\pi]$. Let c be a fixed positive constant. Let $\widetilde{n} := n + (\alpha+\beta+1)/2$, $\gamma := -(\alpha+\frac{1}{2})\pi/2$.

Case 1. $c(n+1)^{-1} \leq \theta \leq \pi - c(n+1)^{-1}$. We use (8.21.18) of [11] as follows:

$$P_n^{(\alpha,\beta)}(\cos\theta) =$$
$$\pi^{-\frac{1}{2}} n^{-\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}} [\cos(\widetilde{n}\theta+\gamma) + (n\sin\theta)^{-1}O(1)], \qquad n\to\infty.$$

This implies

$$|P_n^{(\alpha,\beta)}(\cos\theta)| \lesssim (n+1)^{-\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}. \tag{3.1}$$

An elementary calculation then gives

$$\|P_n^{(\alpha,\beta)}(\cos\theta)\|_{L^p_\theta([c(n+1)^{-1},\pi/2])} \lesssim \widetilde{A}(\alpha,p,n),$$

and

$$\|P_n^{(\alpha,\beta)}(\cos\theta)\|_{L^p_\theta([\pi/2,\pi-c(n+1)^{-1}])} \lesssim \widetilde{A}(\beta,p,n). \tag{3.2}$$

Noting $\widetilde{A}(\beta,p,n) \leq \widetilde{A}(\alpha,p,n)$, the above two estimates yield the desired bound of the convolution operator with the kernel $P_n^{\alpha,\beta}(\cos\theta)$ restricted to the interval $[c(n+1)^{-1}, \pi - c(n+1)^{-1}]$, via an application of Young's convolution inequality.

Case 2. $0 < \theta \leq c(n+1)^{-1}$ or $\pi - c(n+1)^{-1} \leq \theta < \pi$. For the former, we use (8.21.17) of [11] as follows:

$$\begin{aligned} &\left(\sin\frac{\theta}{2}\right)^{\alpha} \left(\cos\frac{\theta}{2}\right)^{\beta} P_n^{(\alpha,\beta)}(\cos\theta) \\ &= \widetilde{n}^{-\alpha} \frac{\Gamma(n+\alpha+1)}{n!} \left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} J_\alpha(\widetilde{n}\theta) + \theta^{\alpha+2}O(n^\alpha), \qquad n\to\infty. \end{aligned} \tag{3.3}$$

Here J_α stands for the Bessel function of the first kind of order α, as defined in (1.71.1) of [11]. By (1.71.10) of [11], we have the asymptotics,

$$J_\alpha(x) \sim x^\alpha, \qquad x \to 0+.$$

We also need the classical asymptotics for Gamma functions:

$$\frac{\Gamma(n+\alpha+1)}{n!} \sim n^\alpha, \qquad n \to \infty.$$

Apply the above two asymptotics to (3.3), we get

$$|P_n^{(\alpha,\beta)}(\cos\theta)| \lesssim (n+1)^\alpha, \qquad 0 < \theta < c(n+1)^{-1}.$$

This implies

$$\|P_n^{(\alpha,\beta)}(\cos\theta)\|_{L^p_\theta((0,c(n+1)^{-1}])} \lesssim (n+1)^{\alpha-\frac{1}{p}}, \qquad \text{for all } p > 0. \tag{3.4}$$

A similar approach works for the other case $\pi - c(n+1)^{-1} \le \theta < \pi$: combining (8.21.17) and (4.1.3) of [11], we get

$$\begin{aligned}&\left(\sin\frac{\theta}{2}\right)^\alpha \left(\cos\frac{\theta}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos\theta)\\ =&(-1)^n\widetilde{n}^{-\beta}\frac{\Gamma(n+\beta+1)}{n!}\left(\frac{\pi-\theta}{\sin\theta}\right)^{\frac{1}{2}} J_\beta(\widetilde{n}(\pi-\theta)) + (\pi-\theta)^{\beta+2}O(n^\beta), \qquad n\to\infty,\end{aligned}$$

which implies, in a similar manner as above, that

$$\|P_n^{(\alpha,\beta)}(\cos\theta)\|_{L^p_\theta([\pi-c(n+1)^{-1},\pi))} \lesssim (n+1)^{\beta-\frac{1}{p}} \le (n+1)^{\alpha-\frac{1}{p}}, \qquad \text{for all } p > 0. \tag{3.5}$$

Again by an application of Young's inequality, (3.4) and (3.5) together yield the desired bound of the convolution operator with the kernel $P_n^{(\alpha,\beta)}(\cos\theta)$ restricted to the intervals $(0, c(n+1)^{-1}]$ and $[\pi - c(n+1)^{-1}, \pi)$. This completes the proof. □

Remark 3.2. The above proof yields the sharp L^p estimates of Jacobi polynomials: for $0 \le \beta \le \alpha$, we have

$$\|P_n^{(\alpha,\beta)}(\cos\theta)\|_{L^p_\theta([0,2\pi])} \lesssim \widetilde{A}(\alpha, p, n).$$

In particular,

$$\|P_n^{(\alpha,\beta)}\|_{L^\infty([-1,1])} \lesssim (n+1)^\alpha. \tag{3.6}$$

Finally, we eliminate the logarithmic factor present in Proposition 3.1 for the cases relevant to Theorem 1.8.

Proof of Theorem 1.8. Note that Proposition 3.1 already covers the case $\alpha > 1/2$ of Theorem 1.8. For the case $0 \le \beta \le \alpha < 1/2$, it suffices to prove the desired bound

$$\|T_n^{(\alpha,\beta)}\|_{L^{p'}(\mathbb{T})\to L^p(\mathbb{T})} \lesssim (n+1)^{\frac{1}{2}}$$

at the kink point $p = 4/(2\alpha+1) > 2$. To accomplish this, we adjust the proof of Proposition 3.1: the only change needed is to apply the Hardy–Littlewood–Sobolev inequality in place of Young's convolution inequality to (3.1) over the interval $[c(n+1)^{-1}, \pi/2]$, as well as over the interval $[\pi/2, \pi - c(n+1)^{-1}]$ if $\beta = \alpha$; if $\beta < \alpha$, the original application of Young's inequality over the interval $[\pi/2, \pi - c(n+1)^{-1}]$ via

(3.2) remains sufficient. For the remaining case $\alpha = \beta = 1/2$, we use the explicit formula (4.1.7) of [11],

$$P_n^{(\frac{1}{2},\frac{1}{2})}(\cos\theta) = \binom{n+\frac{1}{2}}{n} \frac{\sin(n+1)\theta}{(n+1)\sin\theta}.$$

Noting (2.11), the desired bound for $\|T_n^{(\frac{1}{2},\frac{1}{2})}\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})}$ results from the L^2-boundedness of the convolution operator with the Dirichlet kernel. The proof is now complete. □

4. Sharpness of (1.4)

Fix an origin $e = (e_1, \dots, e_r)$ of $M = M_1 \times \cdots \times M_r$, and take S to be any open subset of $\mathbb{T}_1 \times \cdots \times \mathbb{T}_k \times \{e_{k+1}\} \times \cdots \times \{e_r\}$ containing e. For $N > 1$, take

$$\Lambda := \{(n_1, \dots, n_r) \in \mathbb{Z}_{\geq 0}^r : n_1 \geq \cdots \geq n_r \geq \frac{n_1}{2},\ \sum_{i=1}^r n_i^2 + a_i n_i = N^2\},$$

and then take

$$f(\theta_1, \dots, \theta_r) = \sum_{(n_1,\dots,n_r)\in\Lambda} \prod_{i=1}^r \sqrt{k_i(n_i)}\, \Phi_{i,n_i}(\theta_i). \tag{4.1}$$

As the spherical functions Φ_{i,n_i} are matrix entries of the associated irreducible spherical representations expressed via unit vectors (see Theorem 4.3 of Chapter V of [4]), the Schur orthogonality relations yield orthogonality between Φ_{i,n_i} for different n_i, and

$$\|\sqrt{k_i(n_i)}\, \Phi_{i,n_i}\|_{L^2(M_i)} = 1,$$

which together imply

$$\|f\|_{L^2(M)} = |\Lambda|^{\frac{1}{2}}. \tag{4.2}$$

We need the following derivative estimate for spherical functions.

Lemma 4.1. *For spherical functions $\Phi_n(\theta)$ ($n \in \mathbb{Z}_{\geq 0}$) on a CROSS, we have the pointwise estimate*

$$|\Phi_n'(\theta)| \lesssim (n+1)^2 \sin\theta, \qquad \theta \in \mathbb{T}.$$

In particular, this implies that

$$|\Phi_n(\theta) - 1| \ll 1, \qquad \text{if } |\theta| \ll \frac{1}{n+1}. \tag{4.3}$$

Proof. By (4.21.7) of [11] and (2.1), we have

$$\Phi_n'(\theta) = -\frac{\sin\theta}{2}(n+\alpha+\beta+1)\binom{n+\alpha}{n}^{-1} P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta).$$

Noting (2.11) and (3.6), the result follows. □

Note that $n_i \sim N$ for all i whenever $(n_1, \dots, n_r) \in \Lambda$. By (4.3) and Lemma 2.2, if $|\theta_i| \ll 1/N$ for all i, then for the chosen f in (4.1), we have

$$|f(\theta_1, \dots, \theta_r)| \gtrsim \sum_{(n_1,\dots,n_r)\in\Lambda} \prod_{i=1}^r \sqrt{k_i(n_i)} \gtrsim N^{\frac{d-r}{2}} |\Lambda|.$$

The above estimate further implies that

$$\|f\|_{L^p(S)} \gtrsim N^{\frac{d-r}{2} - \frac{k}{p}} |\Lambda|,$$

which together with (4.2) yields

$$\frac{\|f\|_{L^p(S)}}{\|f\|_{L^2(M)}} \gtrsim N^{\frac{d-r}{2}-\frac{k}{p}}|\Lambda|^{\frac{1}{2}}.$$

When $r \geq 5$, by the equidistribution of lattice points on r-dimensional spheres as established by Pommerenke [8], it holds $|\Lambda| \gtrsim N^{r-2}$ whenever $\Lambda \neq \varnothing$, which yields the sharpness of (1.4).

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