SEGREGATED SOLUTIONS TO CRITICAL ELLIPTIC SYSTEMS IN HIGH DIMENSIONS ($N \geq 5$)

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ABSTRACT. We study the existence of multiple segregated solutions to the critical coupled Schrödinger system

$$\begin{cases} -\Delta u_1 = K_1(|y|)|u_1|^{2^*-2}u_1 + \beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1, & y \in \mathbb{R}^N, \\ -\Delta u_2 = K_2(|y|)|u_2|^{2^*-2}u_2 + \beta|u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2, & y \in \mathbb{R}^N, \\ u_1, u_2 \geq 0,\ u_1, u_2 \in C_0(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N), \end{cases}$$

with $N \geq 5$, $2^* = \frac{2N}{N-2}$, radial potentials $K_1, K_2 > 0$, and repulsive coupling $\beta < 0$. Under the assumption that K_1 and K_2 attain local maxima at distinct radii $r_0 \neq \rho_0$ with precise asymptotic expansions near these points, we prove the existence of infinitely many non-radial segregated solutions $(u_{1,k}, u_{2,k})$ for all sufficiently large integers k. These solutions exhibit multiple bumps concentrating on two separate circles of radius r_0 and ρ_0 respectively. Moreover, each component develops a "dead core" near the concentration points of the other. The proof overcomes the sublinear and non-smooth nature of the coupling term ($2^*/2 - 1 < 1$) by constructing a tailored complete metric space and combining a finite-dimensional reduction with a novel tail minimization argument.

Keywords: Critical systems; Sublinear coupling; Segregated solutions; Reduction method; Infinitely many solutions.

1. INTRODUCTION

We study multiple segregated solutions to the following nonlinear critical Schrödinger system

$$\begin{cases} -\Delta u_1 = K_1(|y|)|u_1|^{2^*-2}u_1 + \beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1, & y \in \mathbb{R}^N, \\ -\Delta u_2 = K_2(|y|)|u_2|^{2^*-2}u_2 + \beta|u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2, & y \in \mathbb{R}^N, \\ u_1, u_2 \geq 0,\ u_1, u_2 \in C_0(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 5$, $2^* = \frac{2N}{N-2}$, $K_i(r) > 0$ $(i = 1, 2)$ are radial potentials, and $\beta \in \mathbb{R}$ is a coupling constant. System (1.1) arises when studying the solitary wave solutions of the following time-dependent coupled nonlinear Schrödinger equations

$$\begin{cases} -\mathrm{i}\frac{\partial \Phi_j}{\partial t} = \Delta\Phi_j + \mu_j|\Phi_j|^{p-2}\Phi_j + \sum_{i\neq j}\beta_{ij}|\Phi_i|^{\frac{p}{2}}|\Phi_j|^{\frac{p}{2}-2}\Phi_j, & y \in \mathbb{R}^N,\ t > 0, \\ \Phi_j(y, 0) = \phi_j(y), & j = 1, \dots, n. \end{cases} \tag{1.2}$$

where $\mu_j(y) > 0$, β_{ij} are coupling constants and the exponent $p \in (2, 2^*]$. The system (1.2) arises in many physical problems, particularly in nonlinear optics. Here, Φ_j represents the jth component of a beam in Kerr-like photorefractive media. The condition $\mu_j > 0$ shows that there is an attractive intraspecies interaction

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between the atoms inside each component. The interaction between two states is determined by the sign of the scattering length $\beta = \beta_{ij}$, see [1] and references and therein. A positive β implies an attractive interaction, causing the components of a vector solution to come together. Conversely, a negative β indicates the repulsive interaction, causing the components to repel each other and forming the phase separations.

To obtain the solitary wave solutions of system (1.1), one sets $\Phi_j(y,t) = e^{\mathrm{i}\lambda_j t}u_j(y)$, reducing the system to the to the following n-coupled nonlinear Schrödinger equations

$$\begin{cases} \Delta u_j - \lambda_j u_j + \mu_j |u_j|^{p-2}u + \sum_{i\neq j}\beta_{ij}|u_i|^{\frac{p}{2}}|u_j|^{\frac{p}{2}-2}u_j = 0, & y \in \mathbb{R}^N, \\ u_j(y) \to 0, \text{ as } |y| \to \infty, & j = 1, \dots, n. \end{cases} \tag{1.3}$$

In recent years, significant mathematical studies have been conducted on nonlinear Schrödinger equations with critical exponents, for example, [2, 13, 22–24, 36–38, 41]. Del Pino, Musso, Pacard and Pistoia [15, 16] developed sequences of sign-changing solutions with high energy that concentrate along certain special submanifolds of $\mathbb{R}^N$. In [20], Guo, Li and Wei demonstrated the existence of infinitely many positive non-radial solutions to the coupled elliptic system. In [33], an unbounded sequence of non-radial positive vector solutions of segregated type was constructed when $p < \frac{2N}{N-2}$. In [34], Pistoia and Vaira found positive non-radial solutions for N-coupled systems with weak interspecies forces in $N = 2$ and $N = 3$. For more information on the subcritical case of the system, we refer to [5, 13, 22, 24, 32, 38]. Recently, Chen, Medina and Pistoia [10] studied the existence of segregated solutions for a critical elliptic system with a small interspecies repulsive force in $\mathbb{R}^4$. When $n = 1$ and $p = 2^*$, system (1.3) reduces to the scalar equation

$$\begin{cases} -\Delta u = K(y)u^{\frac{N+2}{N-2}}, \ y \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.4}$$

This equation also arises via stereographic projection from the prescribed curvature problem on the sphere S^N:

$$-\Delta_{S^N} u + \frac{N(N-2)}{2}u = \tilde{K}u^{\frac{N+2}{N-2}} \text{ on } S^N, \tag{1.5}$$

where $\tilde{K}$ is positive and rotationally symmetric. Equations (1.4) and (1.5) are deeply rooted in conformal geometry and has attracted extensive research; see [3, 4, 8, 9, 11, 17, 21, 26–29, 31]. For (1.4), Wei and Yan [40] proved the existence of infinitely many solutions to the prescribed scalar curvature problem (1.4) using the reduction argument. On the other hand, Y. Li proved in [25] that the system (1.4) has infinitely many solutions if $K(y)$ is periodic, while similar result was obtained in [42] if $K(y)$ has a sequence of strict local maximum points approaching infinitely.

The goal of this paper is to investigate the high-dimensional critical case with $p = \frac{2N}{N-2}$ and the coupling coefficient $\beta_{ij} < 0$ (representing a repulsive interaction) in system (1.3). To present our main results, we establish the following assumptions on K_1, K_2 are bounded:

($\mathbb{K}$): There exist constants $r_0 > 0$, $\rho_0 > 0$ with $\rho_0 \neq r_0$, exponents $m \in [2, N-2)$, and positive constants $c_{0,1}, c_{0,2}, \theta_1, \theta_2, \delta > 0$ such that for $r \in (r_0 - \delta,\ r_0 + \delta)$ and $\rho \in (\rho_0 - \delta,\ \rho_0 + \delta)$,

$$K_1(r) = 1 - c_{0,1}|r - r_0|^m + O\big(|r - r_0|^{m+\theta_1}\big),$$
$$K_2(\rho) = 1 - c_{0,2}|\rho - \rho_0|^m + O\big(|\rho - \rho_0|^{m+\theta_2}\big).$$

The primary results are as follows.

Theorem 1.1. *Suppose $N \geq 5$, assumption $(\mathbb{K})$ holds, and $\beta < 0$. Then there exists an integer $k_0 > 0$ such that for every integer $k \geq k_0$, the system (1.1) admits a non-radial segregated solution $(u_{1,k}, u_{2,k})$ with*

$$\lim_{k\to\infty} \max_{\mathbb{R}^N} u_{1,k} = +\infty, \qquad \lim_{k\to\infty} \max_{\mathbb{R}^N} u_{2,k} = +\infty.$$

Consequently, the system has infinitely many segregated solutions. Let us outline the main idea in the proof of Theorem 1.1. We additionally establish an alternative form of segregated solutions. In other words, segregated solutions display multiple bumps near infinity, with the bumps of u_1 and u_2 occurring on different circles. To illustrate this phenomenon, we first introduce some notations.

Fix a positive integer $k \geq k_0$, where k_0 is a large integer to be determined. Set

$$\mu = k^{\frac{1}{\tau_1}}$$

to be the scaling parameter, where $\tau_1 = \frac{N-2-m}{N-2}$. Using the transformation $u(y) \mapsto \mu^{-\frac{N-2}{2}} u(\frac{y}{\mu})$, the system (1.1) becomes

$$\begin{cases} -\Delta u_1 = K_1\big(\frac{|y|}{\mu}\big)|u_1|^{2^*-2}u_1 + \beta |u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1, \quad y \in \mathbb{R}^N, \\ -\Delta u_2 = K_2\big(\frac{|y|}{\mu}\big)|u_2|^{2^*-2}u_2 + \beta |u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2, \quad y \in \mathbb{R}^N, \\ u_1, u_2 \geq 0, \ u_1, u_2 \in C_0(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.6}$$

It is well known that all positive solutions of

$$-\Delta u = u^{\frac{N+2}{N-2}}, \ u > 0 \text{ in } \mathbb{R}^N$$

are given by the family

$$\left\{ (N(N-2))^{\frac{N-2}{4}} \Big(\frac{\lambda}{1+\lambda^2|y-x|^2}\Big)^{\frac{N-2}{2}} \ \Big| \ \lambda > 0, \ x \in \mathbb{R}^N \right\}.$$

Write $y = (y', y'')$ with $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. Define

$$\begin{aligned} H_s = \Big\{ & u \text{ is a measurable function on } \mathbb{R}^N \ | \ u \text{ is even in } y_d, \ d = 2, \dots, N, \\ & u(r\cos\theta, r\sin\theta, y'') = u\Big(r\cos\Big(\theta + \frac{2\pi j}{k}\Big), r\sin\Big(\theta + \frac{2\pi j}{k}\Big), y''\Big)\Big\} \end{aligned}$$

and set

$$H_{s_0} = H_s \cap C_0(\mathbb{R}^N).$$

Define

$$\mathbb{H} = H_{s_0} \times H_{s_0},$$

equipped with the norm $\|(u_1, u_2)\|_\infty = \|u_1\|_{L^\infty(\mathbb{R}^N)} + \|u_2\|_{L^\infty(\mathbb{R}^N)}$.

For $i = 1, \dots, k$, denote

$$x^i = r\Big(\cos\frac{2(i-1)\pi}{k}, \ \sin\frac{2(i-1)\pi}{k}, 0\Big), \quad y^i = \rho\Big(\cos\frac{2(i-1)\pi}{k}, \ \sin\frac{2(i-1)\pi}{k}, 0\Big) \tag{1.7}$$

where $r \in \Big[r_0\mu - 1, r_0\mu + 1\Big]$, $\rho \in \Big[\rho_0\mu - 1, \rho_0\mu + 1\Big]$ and 0 is the zero in $\mathbb{R}^{N-2}$.

We set

$$U_{r,\lambda}(y) = \sum_{i=1}^k U_{x^i,\lambda}(y), \quad V_{\rho,\nu}(y) = \sum_{i=1}^k V_{y^i,\nu}(y),$$

where

$$U_{x^i,\lambda}(y) = (N(N-2))^{\frac{N-2}{4}} \Big(\frac{\lambda}{1+\lambda^2|y-x^i|^2}\Big)^{\frac{N-2}{2}},$$

$$V_{y^i,\nu}(y) = (N(N-2))^{\frac{N-2}{4}} \Big(\frac{\nu}{1+\nu^2|y-y^i|^2}\Big)^{\frac{N-2}{2}},$$

for x^i and y^i are defined in (1.7).

We will prove Theorem 1.1 by verifying the following result.

Theorem 1.2. *Suppose $N \geq 5$, assumption $(\mathbb{K})$ holds, and $\beta < 0$. Then there exist some small $\bar{\theta} > 0$, some constants $\lambda_0 > 0, \nu_0 > 0$ and an integer $k_0 > 0$ such that for any integer $k \geq k_0$, the system (1.6) has a solution $(u_{1,k}, u_{2,k})$ of the form*

$$\big(u_{1,k}(y), u_{2,k}(y)\big) = \big(U_{r_k,\lambda_k}(y) + \varphi_k, V_{\rho_k,\nu_k}(y) + \psi_k\big),$$

where $(\varphi_k, \psi_k) \in \mathbb{H}$, $\lambda_k \in \Big[\lambda_0 - \frac{1}{\mu^{\frac{3}{2}\bar{\theta}}}, \lambda_0 + \frac{1}{\mu^{\frac{3}{2}\bar{\theta}}}\Big]$, $\nu_k \in \Big[\nu_0 - \frac{1}{\mu^{\frac{3}{2}\bar{\theta}}}, \nu_0 + \frac{1}{\mu^{\frac{3}{2}\bar{\theta}}}\Big]$, $r_k \in \Big[r_0\mu - \dfrac{1}{\mu^{\bar{\theta}}}, r_0\mu + \dfrac{1}{\mu^{\bar{\theta}}}\Big]$, $\rho_k \in \Big[\rho_0\mu - \dfrac{1}{\mu^{\bar{\theta}}}, \rho_0\mu + \dfrac{1}{\mu^{\bar{\theta}}}\Big]$ and $\|(\varphi_k, \psi_k)\|_\infty \to 0$ as $k \to \infty$.

Moreover, for $N \geq 5$, these segregated solutions exhibit a "dead core" phenomenon [35]: each component vanishes in neighborhoods of the other's concentration points.

Theorem 1.3. *For any $\vartheta \in (0, 1-\tau_1] \cap \big(0, \frac{(N-4)(N-2-\tau_1)}{(N-2)^2}\big)$, there exists k_ϑ such that for all $k \geq k_\vartheta$, the solution $(u_{1,k}, u_{2,k})$ from Theorem 1.2 satisfies*

$$u_{1,k} = 0 \text{ in } \bigcup_{j=1}^{k} B_{\mu^\vartheta}(y^j), \quad u_{2,k} = 0 \text{ in } \bigcup_{j=1}^{k} B_{\mu^\vartheta}(x^j).$$

When $N \geq 5$, the coupling exponent $\frac{2^*}{2} - 1 = \frac{2}{N-2} < 1$ introduces significant challenges for classical reduction methods. These difficulties arise from the non-smooth and sublinear nature of the coupling nonlinearities. A critical obstacle emerges when applying the fixed-point theorem in the complement of the kernel space to demonstrate the existence of solutions, as this requires reducing the problem to a finite-dimensional framework. To address this, we adopt a strategy which generalizes and simplifies the approach in [39].

The sublinearity-induced singularities occur where the solution vanishes. This motivates analyzing the behavior of functions away from concentration points first. We partition the space into inner regions (near concentration points) and an outer region (far from concentration points), based on the distribution of these points (see Section 2 for details).

To apply the fixed-point theorem, we introduce a set of functions with favorable decay properties. In prior works [14, 40], weighted spaces were employed to ensure sufficient regularity for contraction mapping. However, in the sublinear regime, constructing such spaces becomes problematic due to the inherent non-smoothness. Our key innovation lies in designing a tailored functional space specifically suited for the contraction mapping theorem.

We construct a complete metric space using a tail minimization technique, inspired by that used in variational gluing methods (e.g., [7, 12]), to derive sharp decay estimates for approximate solutions. In this space, each function is defined through solutions to boundary value problems in the outer region, with boundary conditions determined by prescribed inner functions. Unlike the weighted spaces or tailoring techniques in [7, 12], we rigorously establish a priori decay estimates not only for the functions themselves but also for their deviations from the limiting profiles $U_{r,\lambda}, V_{\rho,\nu}$, and the differences between distinct functions within the space. We emphasize that the "dead core" estimates are crucial for establishing the decay of differences between distinct functions, as they provide quantitative separation bounds in the interaction regions.

By restricting the fixed-point problem to this metric space and leveraging the derived decay estimates, we successfully reduce the analysis to a finite-dimensional setting.

The structure of this paper is organized as follows. In Section 2, we introduce some basic estimates necessary for proving Theorem 1.2. In Section 3, we formulate a minimization problem for the outer region and establish various a priori estimates for the solution to the exterior problem. In Section 4, we demonstrate the existence of solutions using the fixed-point theorem in the orthogonal complement of the kernel space, reducing the problem to a finite-dimensional one. In Section 5 concludes the proof of the theorems by solving the finite-dimensional problem. In the Appendix, we present some basic estimates used in the article.

2. Preliminaries

Let

$$\sigma_k := k^{\frac{1}{N-2}}.$$

Set

$$P_l = \bigcup_{i=1}^{k} B_{\sigma_k^l}(x^i),\ Q_l = \bigcup_{i=1}^{k} B_{\sigma_k^l}(y^i),\ l = 1, 2.$$

Let $\chi_0 \in C_0^1(\mathbb{R}^N)$ be a fixed truncation function such that $\chi_0 = 1$ in $B_1(0)$ and $\chi_0 = 0$ in $\mathbb{R}^N \setminus B_2(0)$. Denote

$$Y_1 = \sum_{i=1}^{k} \chi_0(\cdot - x^i)\frac{\partial U_{x^i,\lambda}}{\partial r}, \quad Y_2 = \sum_{i=1}^{k} \chi_0(\cdot - x^i)\frac{\partial U_{x^i,\lambda}}{\partial \lambda},\ i = 1, \dots, k,$$

$$Z_1 = \sum_{i=1}^{k} \chi_0(\cdot - y^i)\frac{\partial V_{y^i,\nu}}{\partial \rho}, \quad Z_2 = \sum_{i=1}^{k} \chi_0(\cdot - y^i)\frac{\partial V_{y^i,\nu}}{\partial \nu},\ i = 1, \dots, k,$$

where x^i and y^i are defined in (1.7).

Define

$$\mathbb{E} = \mathbb{E}_{r,\rho,\lambda,\nu} = E_{r,\lambda} \times E_{\rho,\nu},$$

where

$$E_{r,\lambda} = \Big\{u \in H_{s_0} \mid \int_{\mathbb{R}^N} Y_1 u = 0 \text{ and } \int_{\mathbb{R}^N} Y_2 u = 0\Big\},$$
$$E_{\rho,\nu} = \Big\{v \in H_{s_0} \mid \int_{\mathbb{R}^N} Z_1 v = 0 \text{ and } \int_{\mathbb{R}^N} Z_2 v = 0\Big\}.$$

Let

$$\lambda \in [\gamma_1, \gamma_2] \text{ and } \nu \in [\gamma_3, \gamma_4],$$

where $\gamma_2 > \gamma_1 > 0$ and $\gamma_4 > \gamma_3 > 0$ are some constants.

For $i = 1, \dots, k$, denote

$$\Omega_i = \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \;\middle|\; \left\langle \frac{z'}{|z'|}, \frac{(x^i)'}{|(x^i)'|} \right\rangle \geq \cos\frac{\pi}{k} \right\} = \left\{ z \;\middle|\; \left\langle \frac{z'}{|z'|}, \frac{(y^i)'}{|(y^i)'|} \right\rangle \geq \cos\frac{\pi}{k} \right\}.$$

Let $h_1, h_2 \in L^\infty(\mathbb{R}^N) \cap H_s$, and denote by 1_{P_2} and 1_{Q_2} the characteristic functions of the sets P_2 and Q_2, respectively. Consider the following linear problems:

$$\begin{cases} -\Delta\varphi - (2^* - 1)K_1\big(\frac{|y|}{\mu}\big)U_{r,\lambda}^{2^*-2}\varphi = 1_{P_2}h_1 + \sum_{j=1}^{2} c_j Y_j, \text{ in } \mathbb{R}^N, \\ \varphi \in E_{r,\lambda}, \end{cases} \tag{2.1}$$

and

$$\begin{cases} -\Delta\psi - (2^* - 1)K_2\big(\frac{|y|}{\mu}\big)V_{\rho,\nu}^{2^*-2}\psi = 1_{Q_2}h_2 + \sum_{j=1}^{2} d_j Z_j, \text{ in } \mathbb{R}^N, \\ \psi \in E_{\rho,\nu}, \end{cases} \tag{2.2}$$

for some real numbers c_j and d_j. In what follows, we use notation $\langle u_1, u_2\rangle = \int_{\mathbb{R}^N} u_1 u_2$.

Lemma 2.1. *Let $h_{1,k}, h_{2,k} \in L^\infty(\mathbb{R}^N) \cap H_s$. Suppose $(\varphi_k, c_{1,k}, c_{2,k})$ and $(\psi_k, d_{1,k}, d_{2,k})$ are the solutions to (2.1) and (2.2) with $h_1 = h_{1,k}$ and $h_2 = h_{2,k}$, respectively. Then the following hold:*

(i) *If* $\sigma_k^4\|h_{1,k}\|_{L^\infty(P_2)} \to 0$ *as* $k\to\infty$, *then*

$$\|\varphi_k\|_{L^\infty(\mathbb{R}^N)} + \sum_{j=1}^{2}|c_{j,k}| \ \to\ 0 \quad \textit{as } k\to\infty.$$

(ii) *If* $\sigma_k^4\|h_{2,k}\|_{L^\infty(Q_2)} \to 0$ *as* $k\to\infty$, *then*

$$\|\psi_k\|_{L^\infty(\mathbb{R}^N)} + \sum_{j=1}^{2}|d_{j,k}| \ \to\ 0 \quad \textit{as } k\to\infty.$$

Proof. We only prove (i). By contradiction, we assume that there exist a constant $c'>0$ such that as $k\to\infty$, for $h_1=h_{1,k}$, $\lambda=\lambda_k\in[\gamma_1,\gamma_2]$, $r=r_k\in\Big[r_0\mu-1, r_0\mu+1\Big]$, with $\sigma_k^4\|h_{1,k}\|_{L^\infty(P_2)}\to 0$ there holds

$$\sum_{j=1}^{2}|c_{j,k}| + \|\varphi_k\|_{L^\infty(\mathbb{R}^N)} \geq 2c' > 0. \tag{2.3}$$

For simplicity, we drop the subscript k.

We first estimate $c_l, l=1,2$. Multiplying (2.1) by $\frac{\partial U_{x^1,\lambda}}{\partial y_1}$ and integrating, we see that

$$-c_1\int \chi_0\left|\frac{\partial U_{0,\lambda}}{\partial y_1}\right|^2 = \Big\langle -\Delta\varphi - (2^*-1)K_1\Big(\frac{|y|}{\mu}\Big)U_{r,\lambda}^{2^*-2}\varphi, \frac{\partial U_{x^1,\lambda}}{\partial y_1}\Big\rangle - \langle 1_{P_2}h_1, \frac{\partial U_{x^1,\lambda}}{\partial y_1}\rangle. \tag{2.4}$$

From Lemma A.1, we have

$$\begin{aligned}
&|\langle 1_{P_2}h_1, \frac{\partial U_{x^1,\lambda}}{\partial y_1}\rangle| \leq C\|h_1\|_{L^\infty(P_2)}\sum_{i=1}^{k}\int_{B_{\sigma_k^2}(x^i)}\frac{1}{(1+|y-x^1|)^{N-2}}dy\\
\leq& C\|h_1\|_{L^\infty(P_2)}\int_{B_{\sigma_k^2}(x^1)}\frac{1}{(1+|y-x^1|)^{N-2}}dy + C\|h_1\|_{L^\infty(P_2)}\sum_{i=2}^{k}\int_{B_{\sigma_k^2}(x^i)}\frac{1}{(1+|y-x^1|)^{N-2}}dy\\
\leq& C\sigma_k^4\|h_1\|_{L^\infty(P_2)}.
\end{aligned} \tag{2.5}$$

On the other hand, we also have

$$\begin{aligned}
&\Big\langle -\Delta\varphi - (2^*-1)K_1\Big(\frac{|y|}{\mu}\Big)U_{r,\lambda}^{2^*-2}\varphi, \frac{\partial U_{x^1,\lambda}}{\partial y_1}\Big\rangle\\
=&\Big\langle -\Delta\frac{\partial U_{x^1,\lambda}}{\partial y_1} - (2^*-1)K_1\Big(\frac{|y|}{\mu}\Big)U_{r,\lambda}^{2^*-2}\frac{\partial U_{x^1,\lambda}}{\partial y_1}, \varphi\Big\rangle\\
=&(2^*-1)\Big\langle\Big(1-K_1\Big(\frac{|y|}{\mu}\Big)\Big)U_{r,\lambda}^{2^*-2}\frac{\partial U_{x^1,\lambda}}{\partial y_1}, \varphi\Big\rangle - (2^*-1)\Big\langle(U_{r,\lambda}^{2^*-2} - U_{x^1,\lambda}^{2^*-2})\frac{\partial U_{x^1,\lambda}}{\partial y_1}, \varphi\Big\rangle\\
=:&\, G_1 + G_2.
\end{aligned}$$

When $||y|-\mu r_0|\leq\sqrt{\mu}$, we get

$$\int_{||y|-\mu r_0|\leq\sqrt{\mu}}\Big|K_1\Big(\frac{|y|}{\mu}\Big)-1\Big|U_{r,\lambda}^{2^*-2}\frac{1}{(1+|y-x^1|)^{N-2}}dy \leq \frac{C}{\sqrt{\mu}}\int_{\mathbb{R}^N}U_{r,\lambda}^{2^*-2}\frac{1}{(1+|y-x^1|)^{N-2}}dy \leq \frac{C}{\sqrt{\mu}}.$$

If $||y|-\mu r_0|\geq\sqrt{\mu}$, then

$$|y-x^j| \geq ||y|-\mu r_0| - ||x^j|-\mu r_0| \geq \sqrt{\mu}-1 \geq \frac{1}{2}\sqrt{\mu},$$

which implies that

$$\int_{||y|-\mu r_0|\geq\sqrt{\mu}}\Big|K_1\Big(\frac{|y|}{\mu}\Big)-1\Big|U_{r,\lambda}^{2^*-2}\frac{1}{(1+|y-x^1|)^{N-2}}dy=o(1).$$

Then we obtain

$$G_1=\|\varphi\|_{L^\infty(\mathbb{R}^N)}O\Big(\int_{\mathbb{R}^N}\Big|K_1\Big(\frac{|y|}{\mu}\Big)-1\Big|U_{r,\lambda}^{2^*-2}\frac{1}{(1+|y-x^1|)^{N-2}}dy\Big)=o(\|\varphi\|_{L^\infty(\mathbb{R}^N)}). \tag{2.6}$$

For G_2, by Lemma A.1, (A.4), and (A.1), we have

$$\begin{aligned}G_2=&\|\varphi\|_{L^\infty(\mathbb{R}^N)}O\Big(\int_{\mathbb{R}^N}U_{x^1,\lambda}^{2^*-3}\sum_{i=2}^{k}U_{x^i,\lambda}\frac{\partial U_{x^1,\lambda}}{\partial y_1}+\int_{\mathbb{R}^N}(\sum_{i=2}^{k}U_{x^i,\lambda})^{2^*-2}\frac{\partial U_{x^1,\lambda}}{\partial y_1}\Big)\\=&\|\varphi\|_{L^\infty(\mathbb{R}^N)}O\Big(\sum_{i=2}^{k}\frac{1}{|x^i-x^1|^{1+\frac{2\tau_1}{3}}}\Big(\int_{\mathbb{R}^N}\frac{1}{(1+|y-x^1|)^{N+1-\tau_1}}+\int_{\mathbb{R}^N}\frac{1}{(1+|y-x^i|)^{N+1-\tau_1}}\Big)\Big)\\=&o(\|\varphi\|_{L^\infty(\mathbb{R}^N)}).\end{aligned} \tag{2.7}$$

Then, from (2.4), (2.5), (2.6), and (2.7), we obtain

$$c_1=o(1+\|\varphi\|_{L^\infty(\mathbb{R}^N)}). \tag{2.8}$$

Similarly, multiplying (2.1) by $\frac{\partial U_{x^1,\lambda}}{\partial\lambda}$ and integrating, we can obtain

$$c_2=o(1+\|\varphi\|_{L^\infty(\mathbb{R}^N)}). \tag{2.9}$$

From (2.3), (2.8) and (2.9), we know $\|\varphi\|_{L^\infty(\mathbb{R}^N)}\geq c'>0$.

Next we write (2.1) as

$$\begin{aligned}\varphi(y)=&(2^*-1)\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}K_1\Big(\frac{|z|}{\mu}\Big)U_{r,\lambda}^{2^*-2}(z)\varphi(z)dz\\&+\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\Big(1_{P_2}(z)h_1(z)+\sum_{j=1}^{2}c_jY_j(z)\Big)dz.\end{aligned}$$

For $z\in\Omega_1$, we get $|z-x^i|\geq|z-x^1|,\ i=2,\dots,k$. Using Lemma A.1 and (A.2), we obtain

$$\sum_{i=2}^{k}\frac{1}{(1+|z-x^i|)^{N-2}}\leq\frac{1}{(1+|z-x^1|)^{N-2-\tau_1}}\sum_{i=2}^{k}\frac{1}{|x^i-x^1|^{\tau_1}}\leq C\frac{1}{(1+|z-x^1|)^{N-2-\tau_1}},$$

where $\tau_1=\frac{N-2-m}{N-2}$.

Using (A.4) and Lemma A.2, there holds

$$\begin{aligned}&\Big|(2^*-1)\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}K_1\Big(\frac{|z|}{\mu}\Big)U_{r,\lambda}^{2^*-2}(z)\varphi(z)dz\Big|\\\leq&C\|\varphi\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\Big(\sum_{i=1}^{k}\frac{1}{(1+|z-x^i|)^{N-2}}\Big)^{\frac{4}{N-2}}dz\\\leq&C\|\varphi\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\sum_{i=1}^{k}\frac{1}{(1+|z-x^i|)^{4-\frac{(6-N)^+}{N-2}\tau_1}}dz\\\leq&C\|\varphi\|_{L^\infty(\mathbb{R}^N)}\sum_{i=1}^{k}\frac{1}{(1+|y-x^i|)^{2-\frac{\tau_1}{N-2}}},\end{aligned}$$

where $\tau_1 = \frac{N-2-m}{N-2}$.

For $y \in \Omega_1 \setminus B_{2\sigma_k^2}(x^1)$, we get

$$\Big| \int_{B_{\sigma_k^2}(x^1)} \frac{1}{|z-y|^{N-2}} h_1(z)dz \Big| \le C\|h_1\|_{L^\infty(P_2)} \int_{B_{\sigma_k^2}(x^1)} \frac{1}{\sigma_k^{2(N-2)}} dz \le C\sigma_k^4 \|h_1\|_{L^\infty(P_2)}.$$

For $y \in B_{2\sigma_k^2}(x^1)$, we also obtain

$$\Big| \int_{B_{\sigma_k^2}(x^1)} \frac{1}{|z-y|^{N-2}} h_1(z)dz \Big| \le \|h_1\|_{L^\infty(P_2)} \Big| \int_{B_{3\sigma_k^2}(y)} \frac{1}{|z-y|^{N-2}} dz \Big| \le C\sigma_k^4 \|h_1\|_{L^\infty(P_2)}.$$

For $y \in \Omega_1$ and $z \in B_{\sigma_k^2}(x^i)$, there holds $|z-y| \ge \frac{1}{2}|x^i - x^1|, i = 2, \ldots, k$. By (A.1), we get

$$\Big| \sum_{i=2}^k \int_{B_{\sigma_k^2}(x^i)} \frac{1}{|z-y|^{N-2}} h_1(z)dz \Big| \le C \sum_{i=2}^k \frac{1}{|x^i - x^1|^{N-2}} \sigma_k^{2N} \|h_1\|_{L^\infty(P_2)} \le C\frac{1}{\mu^m} \sigma_k^{2N} \|h_1\|_{L^\infty(P_2)}.$$

Therefore,

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} 1_{P_2}(z) h_1(z)dz \Big| = \Big| \sum_{i=1}^k \int_{B_{\sigma_k^2}(x^i)} \frac{1}{|z-y|^{N-2}} h_1(z)dz \Big| \\
\le& \Big| \int_{B_{\sigma_k^2}(x^1)} \frac{1}{|z-y|^{N-2}} h_1(z)dz \Big| + \Big| \sum_{i=2}^k \int_{B_{\sigma_k^2}(x^i)} \frac{1}{|z-y|^{N-2}} h_1(z)dz \Big| \\
\le& C\sigma_k^4 \|h_1\|_{L^\infty(P_2)} + C\frac{1}{\mu^m} \sigma_k^{2N} \|h_1\|_{L^\infty(P_2)} = o(1).
\end{aligned}$$

By Lemma A.2, we also obtain

$$\begin{aligned}
\Big| \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} Y_j(z)dz \Big| \le& C \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \frac{1}{(1+|z-x^i|)^{N-2}} dz \\
\le& C \sum_{i=1}^k \frac{1}{(1+|y-x^i|)^{N-4}} \le C.
\end{aligned} \tag{2.10}$$

Combining (2.8), (2.9) and (2.10), we obtain

$$\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^2 c_j Y_j(z)dz = o(1 + \|\varphi\|_{L^\infty(\mathbb{R}^N)}).$$

Therefore, for $y \in \Omega_1$,

$$\frac{|\varphi(y)|}{\|\varphi\|_{L^\infty(\mathbb{R}^N)}} \le C \sum_{i=1}^k \frac{1}{(1+|y-x^i|)^{2-\frac{\tau_1}{3}}} + o(1) \le C \frac{1}{(1+|y-x^i|)^{2-\frac{4}{3}\tau_1}} + o(1). \tag{2.11}$$

We obtain from (2.11) that there is $R > 0$ independent of k such that

$$\|\varphi(y)\|_{L^\infty(B_R(x^1))} / \|\varphi\|_{L^\infty(\mathbb{R}^N)} = 1. \tag{2.12}$$

But $\bar{\varphi}(y) = \varphi(y + x^1)/\|\varphi\|_{L^\infty(\mathbb{R}^N)}$ converges uniformly in any compact set to a solution u of

$$-\Delta u - (2^* - 1) U_{0,\lambda}^{2^*-2} u = 0 \text{ in } \mathbb{R}^N,$$

for some $\lambda \in [\gamma_1, \gamma_2]$. Since $\varphi \in H_{s_0}$, u is even in $y_j, j = 2, \dots, N$. Therefore, we have $u \in \text{span}\{\frac{\partial U_{0,\lambda}}{\partial y_1}, \frac{\partial U_{0,\lambda}}{\partial \lambda}\}$. However, since $\int_{\mathbb{R}^N} u\chi_0 \frac{\partial U_{0,\lambda}}{\partial y_1} = \int_{\mathbb{R}^N} u\chi_0 \frac{\partial U_{0,\lambda}}{\partial \lambda} = 0$, we conclude $u = 0$. This contradicts the uniform lower bound in (2.12), completing the proof. □

From Lemma 2.1, using the same argument as in the proof of Proposition 4.1 in [14], we can prove the following result:

Proposition 2.2. *There exists $k_0 > 0$ such that for all $k \geq k_0$, the following hold for some constant $C > 0$ independent of k:*

(i) *For every $h_1 \in L^\infty(\mathbb{R}^N)$, the linear problem (2.1) admits a unique solution $\varphi = L_1(h_1) \in E_{r,\lambda} \cap D^{1,2}(\mathbb{R}^N)$ together with unique constants $c_j = c_j(h_1) \in \mathbb{R}$, $j = 1, 2$, such that*

$$\|L_1(h_1)\|_{L^\infty(\mathbb{R}^N)} \leq C\,\sigma_k^4\,\|h_1\|_{L^\infty(P_2)}, \qquad |c_j| \leq C\,\sigma_k^4\,\|h_1\|_{L^\infty(P_2)}, \quad j = 1, 2.$$

(ii) *For every $h_2 \in L^\infty(\mathbb{R}^N)$, the linear problem (2.2) admits a unique solution $\psi = L_2(h_2) \in E_{\rho,\nu} \cap D^{1,2}(\mathbb{R}^N)$ together with unique constants $d_j = d_j(h_2) \in \mathbb{R}$, $j = 1, 2$, such that*

$$\|L_2(h_2)\|_{L^\infty(\mathbb{R}^N)} \leq C\,\sigma_k^4\,\|h_2\|_{L^\infty(Q_2)}, \qquad |d_j| \leq C\,\sigma_k^4\,\|h_2\|_{L^\infty(Q_2)}, \quad j = 1, 2.$$

To express the continuity of the operators with respect to the parameters, we denote by $\widehat{P}_i$, $\widehat{Q}_i$, $\widehat{Y}_i$, and $\widehat{Z}_i$ $(i = 1, 2)$ the objects obtained from P_i, Q_i, Y_i, and Z_i by replacing the parameters (r, ρ, λ, ν) with $(\widehat{r}, \widehat{\rho}, \widehat{\lambda}, \widehat{\nu})$. For $h_1 \in C_0(P_2 \cap \widehat{P}_2)$, let $\varphi = L_1(h_1)$ and $\widehat{\varphi} = \widehat{L}_1(h_1)$ be the unique solutions to the linear problem (2.1) corresponding to the parameter pairs (r, λ) and $(\widehat{r}, \widehat{\lambda})$, respectively. Similarly, for $h_2 \in C_0(Q_2 \cap \widehat{Q}_2)$, we denote by $\psi = L_2(h_2)$ and $\widehat{\psi} = \widehat{L}_2(h_2)$ the unique solutions to (2.2) corresponding to the parameter pairs (ρ, ν) and $(\widehat{\rho}, \widehat{\nu})$, respectively.

Then we have the following result.

Lemma 2.3. *The operators L_1 and L_2 defined in Proposition 2.2 are continuous with respect to the parameters (r, λ) and (ρ, ν) in the following sense:*

$$\begin{aligned} &\sup\{\, \|(L_1 - \widehat{L}_1)(h_1)\|_{L^\infty(\mathbb{R}^N)} \mid h_1 \in M_1 \,\} \to 0, \quad \text{as } (r,\lambda) \to (\widehat{r}, \widehat{\lambda}), \\ &\sup\{\, \|(L_2 - \widehat{L}_2)(h_2)\|_{L^\infty(\mathbb{R}^N)} \mid h_2 \in M_2 \,\} \to 0, \quad \text{as } (\rho,\nu) \to (\widehat{\rho}, \widehat{\nu}), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} M_1 &= \{\, h_1 \in L^\infty(\mathbb{R}^N) \mid h_1 = 0 \text{ a.e. in } \mathbb{R}^N \setminus (P_2 \cap \widehat{P}_2),\ \widehat{L}_1(h_1) \in C_0(P_2),\ \|h_1\|_{L^\infty(\mathbb{R}^N)} = 1 \,\}, \\ M_2 &= \{\, h_2 \in L^\infty(\mathbb{R}^N) \mid h_2 = 0 \text{ a.e. in } \mathbb{R}^N \setminus (Q_2 \cap \widehat{Q}_2),\ \widehat{L}_2(h_2) \in C_0(Q_2),\ \|h_2\|_{L^\infty(\mathbb{R}^N)} = 1 \,\}. \end{aligned}$$

Remark 2.4. *The sets M_i, $i = 1, 2$ are nonempty, ensuring that the suprema under consideration are well-defined. To verify this for M_1, let $\varphi_0 \in C_0^\infty(B) \setminus \{0\}$ be arbitrary, where B is a closed domain contained in $\widehat{P}_1 \setminus \cup_{i=1}^k B_3(\widehat{x}^j)$. Define*

$$h = -\Delta\varphi_0 - (2^* - 1)K_1\Big(\frac{|y|}{\mu}\Big)U_{\widehat{r},\widehat{\lambda}}^{2^*-2}\varphi_0.$$

Then $h \in L^\infty(\mathbb{R}^N)$ satisfies $h = 0$ a.e. in $\mathbb{R}^N \setminus (P_2 \cap \widehat{P}_2)$ and $\widehat{L}_1(h) = \varphi_0 \in C_0(P_2)$. By Proposition 2.2, we have $\|h\|_{L^\infty(\mathbb{R}^N)} \neq 0$. Then $h/\|h\|_{L^\infty(\mathbb{R}^N)} \in M_1$. An analogous construction gives the nonemptiness of M_2.

Proof of Lemma 2.3. We only prove (2.13). For $h_1 \in C_0(P_2 \cap \widehat{P}_2)$ satisfying $\widehat{\varphi} = \widehat{L}_1(h_1) \in C_0(P_2)$, $\|h_1\|_{L^\infty(\mathbb{R}^N)} = 1$ from system (2.1), we get for some c_j and $\hat{c}_j$, $j = 1, 2$,

$$-\Delta\varphi - (2^* - 1)K_1\Big(\frac{|y|}{\mu}\Big)U_{r,\lambda}^{2^*-2}\varphi = h_1 + \sum_{j=1}^{2} c_j Y_j$$

and

$$-\Delta\widehat{\varphi}-(2^*-1)K_1\big(\frac{|y|}{\mu}\big)U_{\widehat{r},\widehat{\lambda}}^{2^*-2}\widehat{\varphi}=h_1+\sum_{j=1}^{2}\widehat{c}_j\widehat{Y}_j.$$

Subtracting the two equations, we obtain

$$\varphi-\widehat{\varphi}=L_1[(2^*-1)K_1\big(\frac{|y|}{\mu}\big)(U_{r,\lambda}^{2^*-2}-U_{\widehat{r},\widehat{\lambda}}^{2^*-2})\widehat{\varphi}+\sum_{j=1}^{2}\widehat{c}_j(Y_j-\widehat{Y}_j)].$$

Then the desired continuity follows from the boundedness of $\widehat{\varphi}$, $\widehat{c}_j$ and the continuity of $U_{r,\lambda}^{2^*-2}$ and Y_j with respect to (r,ρ) in the $L^\infty(P)$ norm on any compact set P. □

To obtain decay estimates, we need the following comparison principle in $\Omega\subset\mathbb{R}^N$. For this purpose, we first recall the classical Sobolev embedding theorem. For every $N\geq 3$, there exists an optimal constant $\mathcal{S}>0$ depending only on N such that

$$\mathcal{S}\|u\|_{L^{2^*}(\mathbb{R}^N)}^2\leq\|\nabla u\|_{L^2(\mathbb{R}^N)}^2,\ \text{for all } u\in D^{1,2}(\mathbb{R}^N),$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|=\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^{\frac{1}{2}}$.

Lemma 2.5. *Let $\Omega\subset\mathbb{R}^N$ ($N\geq 3$) be an open set with C^1 boundary. Suppose $\phi\in L^{N/2}(\Omega)$ satisfies $\|\phi\|_{L^{N/2}(\Omega)}<\mathcal{S}$. Let $v_1\in D_{loc}^{1,2}(\Omega)\cap C(\overline{\Omega})$ with $v_1\geq 0$, and $v_2\in D^{1,2}(\Omega)\cap C(\overline{\Omega})$. If*

$$-\Delta(v_2-v_1)-\phi(v_2-v_1)\leq 0\quad \text{in } \Omega$$

in the weak sense, and $v_2\leq v_1$ on $\partial\Omega$, then $v_2\leq v_1$ in Ω.

Proof. By assumption, $\omega=(v_2-v_1)^+\in L^{2^*}(\Omega)$. For any $R>1$, consider cut off function $\eta_R\in C_0^\infty(\mathbb{R}^N;[0,1])$ with $\eta_R=1$ in $B_R(0)$, $\eta_R=0$ in $\mathbb{R}^N\setminus B_{2R}(0)$, $|\nabla\eta_R|\leq 2/R$. Since $v_2\leq v_1$ on $\partial\Omega$, by [6, Theorem 9.17] we have $\eta_R^2\omega\in H_0^1(\Omega\cap B_{2R}(0))$. Using $\eta_R^2\omega$ as a test function in the weak formulation of the inequality $-\Delta(v_2-v_1)-\phi(v_2-v_1)\leq 0$, we obtain

$$\begin{aligned}\mathcal{S}\|\eta_R\omega\|_{L^{2^*}(\Omega)}^2&\leq\int_\Omega|\nabla(\eta_R\omega)|^2=\int_\Omega\nabla(v_2-v_1)\nabla(\eta_R^2\omega)+\int_\Omega|\nabla\eta_R|^2\omega^2\\&\leq\int_\Omega\phi\eta_R^2\omega^2+\frac{4}{R^2}\int_{\Omega\cap B_{2R}(0)\setminus B_R(0)}\omega^2\\&\leq\|\phi\|_{L^{\frac{N}{2}}(\Omega)}\|\eta_R\omega\|_{L^{2^*}(\Omega)}^2+C\|\omega\|_{L^{2^*}(\Omega\setminus B_R(0))}^2,\end{aligned}$$

where $C>0$ depends only on N. Taking limits as $R\to\infty$, we obtain $\mathcal{S}\|\omega\|_{L^{2^*}(\Omega)}^2\leq\|\phi\|_{L^{\frac{N}{2}}(\Omega)}\|\omega\|_{L^{2^*}(\Omega)}^2$. Since $\|\phi\|_{L^{N/2}(\Omega)}<\mathcal{S}$, this implies $\|\omega\|_{L^{2^*}(\Omega)}=0$, so $\omega=0$ in Ω. Thus, $v_2\leq v_1$ in Ω. □

3. Minimization Problem

Recall

$$\sigma_k:=k^{\frac{1}{N-2}}$$

and

$$P_l=\bigcup_{i=1}^{k}B_{\sigma_k^l}(x^i),\ Q_l=\bigcup_{i=1}^{k}B_{\sigma_k^l}(y^i),\ l=1,2.$$

Define

$$\Lambda_k=\Big\{(\varphi,\psi)\in\mathbb{E}\mid\|(\varphi,\psi)\|_\infty\leq\frac{1}{\mu^{\frac{m}{2}}}\frac{1}{\sigma_k^{N-2}}\Big\},$$

where $\|(\varphi,\psi)\|_\infty=\|\varphi\|_{L^\infty(\mathbb{R}^N)}+\|\psi\|_{L^\infty(\mathbb{R}^N)}$.

For any $(\varphi_0, \psi_0) \in \Lambda_k$, we will solve the following problem

$$\begin{cases} -\Delta u_1 = K_1\big(\frac{|y|}{\mu}\big)|u_1|^{2^*-2}u_1 + \beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1, & \text{in } \mathbb{R}^N \setminus P_1, \\ -\Delta u_2 = K_2\big(\frac{|y|}{\mu}\big)|u_2|^{2^*-2}u_2 + \beta|u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2, & \text{in } \mathbb{R}^N \setminus Q_1 \end{cases} \tag{3.1}$$

with

$$u_1 = \varphi_0 + U_{r,\lambda} \text{ in } P_1,\ u_2 = \psi_0 + V_{\rho,\nu} \text{ in } Q_1, \tag{3.2}$$

satisfying

$$(2^* - 1)K_\infty\|u_1\|^{2^*-2}_{L^{2^*}(\mathbb{R}^N\setminus P_1)} < \frac{\mathcal{S}}{2} \text{ and } (2^* - 1)K_\infty\|u_2\|^{2^*-2}_{L^{2^*}(\mathbb{R}^N\setminus Q_1)} < \frac{\mathcal{S}}{2}, \tag{3.3}$$

where $K_\infty = \|(K_1, K_2)\|_\infty$.

3.1. **Decay Estimates for Solutions and Deviations from the Limit Profile.** Since $\beta < 0$, by Kato's inequality, any solution (u_1, u_2) satisfying (3.1) also satisfies

$$\begin{cases} -\Delta|u_1| \le K_\infty|u_1|^{2^*-1}, & \text{in } \mathbb{R}^N \setminus P_1, \\ -\Delta|u_2| \le K_\infty|u_2|^{2^*-1}, & \text{in } \mathbb{R}^N \setminus Q_1. \end{cases} \tag{3.4}$$

In the remainder of this section, let $(\varphi_0, \psi_0) \in \Lambda_k$. We will use the notation $\varpi_{r,\alpha}$, $\varpi_{\rho,\alpha}$, and ϖ_α as defined in (A.5), (A.8), and (A.9), respectively.

Lemma 3.1. *Suppose (u_1, u_2) is a solution to the problem (3.4) satisfying (3.2) and (3.3). Then there exists a constant $A > 0$, independent of k and the choice of (φ_0, ψ_0), such that*

$$|u_1(y)| \le A\sum_{i=1}^{k}\frac{1}{(1 + |y - x^i|)^{N-2}}, \quad |u_2(y)| \le A\sum_{i=1}^{k}\frac{1}{(1 + |y - y^i|)^{N-2}}, \quad \forall y \in \mathbb{R}^N.$$

Proof. From (A.10), and $1 - \frac{4\tau_1}{N-2} > 0$, there holds

$$-\Delta\varpi_{r,N-2} \ge K_\infty\varpi^{2^*-1}_{r,N-2}, \text{ in } \mathbb{R}^N \setminus P_1.$$

For $(\varphi_0, \psi_0) \in \Lambda_k$, there exists $C > 0$ independent of k such that

$$|u_1(y)| = |\varphi_0 + U_{r,\lambda}| \le U_{r,\lambda} + \frac{1}{\mu^{\frac{m}{2}}}\frac{1}{\sigma_k^{N-2}} \le C\varpi_{r,N-2}, \quad y \in \partial P_1. \tag{3.5}$$

By the mean value theorem, there holds

$$-\Delta(|u_1| - C\varpi_{r,N-2}) \le K_\infty(|u_1|^{2^*-1} - (C\varpi_{r,N-2})^{2^*-1}) = \phi(|u_1| - C\varpi_{r,N-2}), \text{ in } \mathbb{R}^N \setminus P_1,$$

where $\phi = K_\infty(|u_1|^{2^*-1} - (C\varpi_{r,N-2})^{2^*-1})/(|u_1| - C\varpi_{r,N-2})$. By direct calculation, it follows that

$$\|C\varpi_{r,N-2}\|^{2^*-2}_{L^{2^*}(\mathbb{R}^N\setminus P_1)} \to 0 \quad \text{as } k \to \infty.$$

Therefore, from (3.3), there holds

$$\|\phi\|_{L^{\frac{N}{2}}(\mathbb{R}^N\setminus P_1)} \le (2^* - 1)K_\infty(\|u_1\|_{L^{2^*}(\mathbb{R}^N\setminus P_1)} + \|C\varpi_{r,N-2}\|_{L^{2^*}(\mathbb{R}^N\setminus P_1)})^{2^*-2} < \mathcal{S}.$$

By Lemma 2.5, we get

$$|u_1(y)| \le C\varpi_{r,N-2}(y), \text{ in } \mathbb{R}^N \setminus P_1.$$

The inequality on P_1 is already established by (3.5). Similarly, we can obtain the corresponding estimate for u_2. □

We also get the following estimates.

Lemma 3.2. *For any $(\varphi_0, \psi_0) \in \Lambda_k$, let (u_1, u_2) be a solution to the problem (3.1) satisfying (3.2) and (3.3). There exist some large $C > 0$ and $\theta > 0$ small enough independent of k such that*

$$|u_1(y) - U_{r,\lambda}(y)| \leq C\frac{1}{\mu^{\frac{m}{2}+\theta}}\sum_{i=1}^{k}\frac{1}{(1+|y-x^i|)^{\frac{N}{2}}} + C\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\sum_{j=1}^{k}\frac{1}{(1+|y-y^j|)^{N-\frac{2\tau_1}{N-2}-2}}$$

and

$$|u_2(y) - V_{\rho,\nu}(y)| \leq C\frac{1}{\mu^{\frac{m}{2}+\theta}}\sum_{i=1}^{k}\frac{1}{(1+|y-y^i|)^{\frac{N}{2}}} + C\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\sum_{j=1}^{k}\frac{1}{(1+|y-x^j|)^{N-\frac{2\tau_1}{N-2}-2}},$$

for $y \in \mathbb{R}^N$. Moreover,

$$\begin{aligned}&|\nabla(u_1(y) - U_{r,\lambda}(y))|\\ \leq& C\frac{1}{\mu^{\frac{m}{2}+\theta}}\sum_{i=1}^{k}\frac{1}{(1+|y-x^i|)^{\frac{N}{2}}} + C\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\sum_{j=1}^{k}\frac{1}{(1+|y-y^j|)^{N-\frac{2\tau_1}{N-2}-2}},\ y \in \mathbb{R}^N \setminus \cup_{i=1}^k B_{\sigma_k+1}(x^i)\end{aligned}$$

and

$$\begin{aligned}&|\nabla(u_2(y) - V_{\rho,\nu}(y))|\\ \leq& C\frac{1}{\mu^{\frac{m}{2}+\theta}}\sum_{i=1}^{k}\frac{1}{(1+|y-y^i|)^{\frac{N}{2}}} + C\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\sum_{j=1}^{k}\frac{1}{(1+|y-x^j|)^{N-\frac{2\tau_1}{N-2}-2}},\ y \in \mathbb{R}^N \setminus \cup_{i=1}^k B_{\sigma_k+1}(y^i).\end{aligned}$$

Proof. Let $(\varphi, \psi) = (u_1, u_2) - (U_{r,\lambda}, V_{\rho,\nu})$. By (3.1) and (3.2), we have

$$\begin{cases}-\Delta\varphi = K_1\big(\frac{|y|}{\mu}\big)|u_1|^{2^*-2}u_1 - \sum_{i=1}^k U_{x^i,\lambda}^{2^*-1} + \beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1, & \text{in } \mathbb{R}^N \setminus P_1,\\ -\Delta\psi = K_2\big(\frac{|y|}{\mu}\big)|u_2|^{2^*-2}u_2 - \sum_{i=1}^k V_{y^i,\nu}^{2^*-1} + \beta|u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2, & \text{in } \mathbb{R}^N \setminus Q_1,\\ \varphi = \varphi_0 \text{ in } P_1,\ \psi = \psi_0 \text{ in } Q_1.\end{cases} \tag{3.6}$$

(i) From Lemmas A.3 and 3.1, there holds

$$\begin{aligned}K_1(\frac{y}{\mu})|u_1|^{2^*-2}u_1 - \sum_{i=1}^{k}U_{x^i,\lambda}^{2^*-1} &= O\Big(\varpi_{r,N-2}^{2^*-2}\Big)\varphi + K_1(\frac{y}{\mu})|U_{r,\lambda}|^{2^*-2}U_{r,\lambda} - \sum_{i=1}^{k}U_{x^i,\lambda}^{2^*-1}\\ &= O\Big(\varpi_{r,N-2}^{2^*-2}\Big)\varphi + O\Big(\frac{1}{\mu^{\frac{m}{2}+\theta}}\Big)\varpi_{r,\frac{N}{2}+2}.\end{aligned} \tag{3.7}$$

For $y \in \mathbb{R}^N \setminus B_{\frac{r+\rho}{2}}(0)$, we have $|y - x^j| \geq \frac{\rho - r}{2} = \frac{|x^j - y^j|}{2}$. Hence

$$|y-y^j| \leq |y-x^j| + |x^j-y^j| \leq 3|y-x^j|, \quad j = 1,\dots,k, \quad y \in \mathbb{R}^N \setminus B_{\frac{r+\rho}{2}}(0). \tag{3.8}$$

Similarly, we have

$$|y-y^j| \geq \frac{\rho-r}{2}, \quad |y-y^j| \geq \frac{1}{3}|y-x^j|, \quad j = 1,\dots,k, \quad \text{for } y \in B_{\frac{r+\rho}{2}}(0). \tag{3.9}$$

Note that

$$\frac{N}{2} - \frac{m}{2} - \frac{2N-2}{N-2}\tau_1 > \frac{N-4-m}{2} + \frac{2(m-1)}{N-2} = \begin{cases}\frac{m-1}{6} > 0, & N = 5,\\ \frac{(N-4)(N-2)-m(N-6)-4}{2(N-2)} > \frac{2N-8}{2(N-2)} > 0, & N \geq 6.\end{cases}$$

In $\Omega_1 \cap B_{\frac{r+\rho}{2}}(0)$, by Lemmas 3.1, A.1 and (A.3), (A.4), (3.9), we obtain

$$\begin{aligned}
&|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1\\
\le& C\Big(\sum_{i=1}^{k}\frac{1}{(1+|y-x^i|)^{N-2}}\Big)^{\frac{2^*}{2}-1}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-y^j|)^{N-2}}\Big)^{\frac{2^*}{2}}\\
\le& C\sum_{i=1}^{k}\frac{1}{(1+|y-x^i|)^{2}}\sum_{j=1}^{k}\frac{1}{(1+|y-y^j|)^{N-\frac{2}{N-2}\tau_1}}\\
\le& C\frac{1}{(1+|y-x^1|)^{2-\tau_1}}\frac{1}{(1+|y-y^1|)^{N-\frac{N}{N-2}\tau_1}}\\
\le& C\frac{1}{\mu^{\frac{N}{2}-\frac{2N-2}{N-2}\tau_1}}\frac{1}{(1+|y-x^1|)^{\frac{N}{2}+2}}\le C\frac{1}{\mu^{\frac{m}{2}+\theta}}\frac{1}{(1+|y-x^1|)^{\frac{N}{2}+2}},
\end{aligned}\tag{3.10}$$

where $\theta \in \left(0, \frac{N}{2}-\frac{m}{2}-\frac{2N-2}{N-2}\tau_1\right)$ is chosen sufficiently small.

For $y \in \mathbb{R}^N \setminus B_{\frac{r+\rho}{2}}(0)$, we know that $|y-x^j| \ge \frac{\rho-r}{2} \ge C\mu$, $j=1,\dots,k$. by Lemma 3.1 and (A.4), it holds

$$\begin{aligned}
|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1 \le& C\Big(\sum_{i=1}^{k}\frac{1}{(1+|y-x^i|)^{N-2}}\Big)^{\frac{2^*}{2}-1}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-y^j|)^{N-2}}\Big)^{\frac{2^*}{2}}\\
\le& C\Big(\frac{k}{\mu^{N-2}}\Big)^{\frac{2^*}{2}-1}\sum_{j=1}^{k}\frac{1}{(1+|y-y^j|)^{N-\frac{2}{N-2}\tau_1}}\\
\le& C\frac{1}{\mu^{2-\frac{2}{N-2}\tau_1}}\sum_{j=1}^{k}\frac{1}{(1+|y-y^j|)^{N-\frac{2}{N-2}\tau_1}}.
\end{aligned}\tag{3.11}$$

Combining (3.7)-(3.11), for the first equation in (3.6), we find that $|\varphi|$ satisfies

$$-\Delta|\varphi| \le C\varpi_{r,N-2}^{2^*-2}|\varphi| + C\frac{1}{\mu^{\frac{m}{2}+\theta}}\varpi_{r,\frac{N}{2}+2} + C\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\varpi_{\rho,N-\frac{2\tau_1}{N-2}},\ y\in\mathbb{R}^N\setminus(P_1\cup Q_1).\tag{3.12}$$

Let

$$\omega := \frac{1}{\mu^{\frac{m}{2}+\theta}}\varpi_{r,\frac{N}{2}} + \frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\varpi_{\rho,N-\frac{2\tau_1}{N-2}-2}\quad \text{in } \mathbb{R}^N.$$

From Lemma A.4, in $\mathbb{R}^N\setminus(P_1\cup Q_1)$, there holds

$$\begin{aligned}
-\Delta\omega \ge& C\sigma_k^{4-\frac{4\alpha}{N-2}}\left(\frac{\varpi_{r,N-2}^{2^*-2}}{\mu^{\frac{m}{2}+\theta}}\varpi_{r,\frac{N}{2}} + \frac{\varpi_{\rho,N-2}^{2^*-2}}{\mu^{2-\frac{2\tau_1}{N-2}}}\varpi_{\rho,N-\frac{2\tau_1}{N-2}-2}\right)\\
&+\frac{N(N-4)}{8}\frac{1}{\mu^{\frac{m}{2}+\theta}}\varpi_{r,\frac{N}{2}+2} + \frac{((N-2)^2-2\tau_1)\tau_1}{(N-2)^2}\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\varpi_{\rho,N-\frac{2\tau_1}{N-2}}.
\end{aligned}\tag{3.13}$$

In $\mathbb{R}^N\setminus B_{\frac{r+\rho}{2}}(0)$, by (3.8), we have

$$\varpi_{\rho,N-2}^{2^*-2} \ge \frac{1}{3^4}\varpi_{r,N-2}^{2^*-2}.$$

In $B_{\frac{r+\rho}{2}}(0)$, by (3.9), we have

$$\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\varpi_{\rho,N-\frac{2\tau_1}{N-2}-2} \le \frac{C}{\mu^{\frac{N}{2}-\frac{4\tau_1}{N-2}}}\varpi_{\rho,\frac{N}{2}} \le \frac{1}{\mu^{\frac{m}{2}+\theta}}\varpi_{r,\frac{N}{2}}.$$

In either case, we have

$$\frac{\varpi_{r,N-2}^{2^*-2}}{\mu^{\frac{m}{2}+\theta}}\varpi_{r,\frac{N}{2}}+\frac{\varpi_{\rho,N-2}^{2^*-2}}{\mu^{2-\frac{2}{N-2}\tau_1}}\varpi_{\rho,N-\frac{2}{N-2}\tau_1-2}\geq\frac{1}{3^4}\varpi_{r,N-2}^{2^*-2}\omega.$$

Therefore, together with (3.12) and (3.13), there is $M_0>0$ large such that

$$-\Delta(|\varphi|-M_0\omega)\leq C\varpi_{r,N-2}^{2^*-2}(|\varphi|-M_0\omega),\quad \text{in } \mathbb{R}^N\setminus(P_1\cup Q_1).$$

In P_1, if we assume further that $\theta<\frac{(N-4)\tau_1}{2(N-2)}$, then there holds

$$|\varphi|\leq\frac{1}{\mu^{\frac{m}{2}}}\frac{1}{\sigma_k^{N-2}}=\frac{1}{\mu^{\frac{m}{2}+\frac{(N-4)\tau_1}{2(N-2)}}}\frac{1}{\sigma_k^{\frac{N}{2}}}=o\left(\frac{1}{\mu^{\frac{m}{2}+\theta}\sigma_k^{\frac{N}{2}}}\right)<M_0\omega.$$

In Q_1,

$$|\varphi|\leq|u_1|+|U_{r,\lambda}|\leq C\frac{k}{\mu^{N-2}}=o\left(\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}\sigma_k^{N-2-\frac{2\tau_1}{N-2}}}\right)\leq M_0\omega.$$

By Lemma 2.5, we conclude that

$$|\varphi|\leq M_0\omega\quad\text{in } \mathbb{R}^N\setminus(P_1\cup Q_1).$$

Since the inequality already holds in $P_1\cup Q_1$, it follows that

$$|\varphi|\leq M_0\omega\quad\text{in } \mathbb{R}^N.$$

A similar argument applies to obtain the corresponding estimate for $|\psi|$.

(ii) By the gradient estimates in [18, Theorem 3.9], it follows that for $y\in\mathbb{R}^N\setminus\cup_{i=1}^kB_{\sigma_k+1}(x^i)$,

$$|\nabla\varphi(y)|\leq C\Big(\sup_{B_1(y)}|\varphi|+\sup_{B_1(y)}\Big|K_1\big(\frac{|\cdot|}{\mu}\big)|u_1|^{2^*-2}u_1-\sum_{i=1}^kU_{x^i,\lambda}^{2^*-1}+\beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1\Big|\Big).$$

By (3.7), (3.10) and (3.11), we obtain

$$\begin{aligned}&\sup_{B_1(y)}\Big|K_1\big(\frac{|\cdot|}{\mu}\big)|u_1|^{2^*-2}u_1-\sum_{i=1}^kU_{x^i,\lambda}^{2^*-1}+\beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1\Big|\\
\leq&\sup_{B_1(y)}\Big|K_1\big(\frac{|\cdot|}{\mu}\big)|u_1|^{2^*-2}u_1-\sum_{i=1}^kU_{x^i,\lambda}^{2^*-1}\Big|+\sup_{B_1(y)}\Big|\beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1\Big|\\
\leq&C\sup_{B_1(y)}\varpi_{r,N-2}^{2^*-2}|\varphi|+C\sup_{z\in B_1(y)}\frac{1}{\mu^{\frac{m}{2}+\theta}}\sum_{i=1}^k\frac{1}{(1+|z-x^i|)^{\frac{N}{2}+2}}\\
&+C\sup_{z\in B_1(y)}\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\sum_{j=1}^k\frac{1}{(1+|z-y^j|)^{N-\frac{2\tau_1}{N-2}}}.\end{aligned}$$

Therefore, we get the estimate of $|\nabla(u_1(y)-U_{r,\lambda}(y))|$. A similar argument can be applied to obtain the estimate of $|\nabla(u_2(y)-V_{\rho,\nu}(y))|$. □

Corollary 3.3. *Under the assumption of Lemma 3.2, there hold*

$$|u_1(y)-U_{r,\lambda}(y)|\leq C\frac{1}{\mu^{\frac{m}{2}+\theta}}\sum_{i=1}^k\frac{1}{(1+|y-x^i|)^{\frac{N}{2}}}\quad \textit{for } y\in\cup_{i=1}^kB_{\frac{\rho-r}{2}}(x^i)$$

and

$$|u_2(y) - V_{\rho,\nu}(y)| \le C\frac{1}{\mu^{\frac{m}{2}+\theta}}\sum_{i=1}^{k}\frac{1}{(1+|y-y^i|)^{\frac{N}{2}}} \quad \textit{for } y \in \cup_{i=1}^{k} B_{\frac{\rho-r}{2}}(y^i).$$

Proof. For $y \in \cup_{i=1}^{k} B_{\frac{\rho-r}{2}}(x^i)$, we have $|y-y^j| \ge |y-x^j|$ and $|y-y^j| \ge C\mu$, $j = 1,\dots,k$. Therefore,

$$\frac{1}{\mu^{2-\frac{2\tau_1}{N-2}}}\frac{1}{(1+|y-y^j|)^{N-\frac{2\tau_1}{N-2}-2}} \le \frac{1}{\mu^{\frac{N}{2}-\frac{4\tau_1}{N-2}}}\frac{1}{(1+|y-y^j|)^{\frac{N}{2}}} = o(\frac{1}{\mu^{\frac{m}{2}+\theta}})\frac{1}{(1+|y-x^j|)^{\frac{N}{2}}}.$$

Then we conclude the proof. □

3.2. **Estimates for Dead Core Formation.** We estimate the size of dead cores for a solution (u_1, u_2) to the problem (3.4) with (3.2) and (3.3). To achieve this result, we rely on the following lemma, which can be derived from [35, Theorems 8.4.2–8.4.7]. A direct proof can also be found in our previous work [19].

Lemma 3.4. *Suppose $N \ge 5$. For each $M > 0$, there exists $\delta > 0$ such that*

$$\begin{cases} -\Delta\omega + M\omega^{\frac{2}{N-2}} = 0, & \textit{in } B_{3/2}(0), \\ \omega = \delta, & \textit{on } \partial B_{3/2}(0) \end{cases}$$

has a unique distribution non-negative radial solution $\omega(y) = \omega(|y|)$, satisfying $\omega \in C^1(B_1(0))$, $\omega = 0$ in $B_1(0)$ and $\omega'(|y|) > 0$ if $\omega > 0$.

By Corollary 3.3, we obtain in $B_{\mu^{1-\tau_1}}(x^1) \cap \Omega_1$

$$\frac{|\varphi|}{U_{r,\lambda}} = O(\frac{1}{\mu^{\frac{m}{2}+\theta}})(1+|y-x^1|)^{\frac{N}{2}-2+\tau_1} = O(\frac{1}{\mu^{\theta}}).$$

Hence

$$u_1 > \frac{1}{2}U_{x^1,\lambda} \quad in \quad B_{\mu^{1-\tau_1}}(x^1) \cap \Omega_1.$$

Lemma 3.5. *For any $\vartheta \in (0, 1-\tau_1] \cap (0, \frac{(N-4)(N-2-\tau_1)}{(N-2)^2})$, there exists a large k_ϑ such that for any $k \ge k_\vartheta$,*

$$u_1 = 0 \textit{ in } \cup_{j=1}^{k} B_{\mu^{\vartheta}}(y^j), \quad u_2 = 0 \textit{ in } \cup_{j=1}^{k} B_{\mu^{\vartheta}}(x^j).$$

Proof. Fixing any $\vartheta \in (0, 1-\tau_1]$, we know that

$$u_1 > \frac{1}{2}U_{x^1,\lambda} > C\mu^{-\vartheta(N-2)} \gg |u_2| = O(\frac{k}{\mu^{N-2}}) = O(\frac{1}{\mu^{N-2-\tau_1}}) \quad \text{in } B_{\frac{3}{2}\mu^{\vartheta}}(x^1).$$

From system (1.6), we have

$$-\Delta|u_2| + C\mu^{-\vartheta N}|u_2|^{\frac{2}{N-2}} \le -\Delta|u_2| - \Big(K_2(\frac{|y|}{\mu})|u_2|^{\frac{2^*}{2}} + \beta|u_1|^{\frac{2^*}{2}}\Big)|u_2|^{\frac{2^*}{2}-1} \le 0, \text{ in } B_{\frac{3}{2}\mu^{\vartheta}}(x^1).$$

Setting $M = C$ in Lemma 3.4, consider

$$\bar\omega(y) = \mu^{-\vartheta\frac{(N-2)^2}{N-4}}\omega\Big(\frac{y-x^1}{\mu^{\vartheta}}\Big).$$

then we can get

$$\begin{cases} -\Delta\bar\omega + C\mu^{-\vartheta N}\bar\omega^{\frac{2}{N-2}} = 0, & \text{in } B_{\frac{3}{2}\mu^{\vartheta}}(x^1), \\ \bar\omega = \delta\mu^{-\vartheta\frac{(N-2)^2}{N-4}}, & \text{on } \partial B_{\frac{3}{2}\mu^{\vartheta}}(x^1), \\ \bar\omega = 0, & \text{in } B_{\frac{\pi r}{k}}(x^1). \end{cases}$$

If $\vartheta < \frac{(N-4)(N-2-\tau_1)}{(N-2)^2}$, then for large k there holds, on $\partial B_{\frac{3}{2}\mu^\vartheta}(x^1)$,

$$\bar{\omega} \geq C\frac{1}{\mu^{\vartheta\frac{(N-2)^2}{N-4}}} > |u_2| = O(\frac{1}{\mu^{N-2-\tau_1}}).$$

Therefore, we have

$$u_2 = 0 \text{ in } \cup_{i=1}^k B_{\mu^\vartheta}(x^i).$$

The proof is concluded by applying similar arguments to u_1. □

Remark 3.6. *1. If $1-\tau_1 < \frac{(N-4)(N-2-\tau_1)}{(N-2)^2}$, we can prove*

$$u_1 = 0 \text{ in } \cup_{j=1}^k B_{|y^1-y^2|}(y^j), \quad u_2 = 0 \text{ in } \cup_{j=1}^k B_{|x^1-x^2|}(x^j).$$

2. We can check that $\frac{2\tau_1}{N-2} < \min\{1-\tau_1, \frac{(N-4)(N-2-\tau_1)}{(N-2)^2}\}$. Therefore, we can choose a proper ϑ such that $\mu^\vartheta > \sigma_k^2 = \mu^{\frac{2\tau_1}{N-2}}$. Hence, for large k, $u_1 = 0$ in Q_2 and $u_2 = 0$ in P_2.

3.3. **Decay of Solution Differences with Different Boundary Conditions.** For any $(\varphi_0, \psi_0) \in \Lambda_k$, we assume that (u_1, u_2) is a solution to the problem (3.1) satisfying (3.2) and (3.3). Let $(\bar{\varphi}_0, \bar{\psi}_0) \in \Lambda_k$ be another point and $(\bar{u}_1, \bar{u}_2)$ be a solution to the corresponding problem (3.1) satisfying the corresponding (3.2) and (3.3). Denote

$$(w, v) := (u_1, u_2) - (\bar{u}_1, \bar{u}_2),\ (w_0, v_0) := (\varphi_0, \psi_0) - (\bar{\varphi}_0, \bar{\psi}_0).$$

We give some estimates on (w, v).

Lemma 3.7. *There are some $C > 0$ independent of k such that*

$$|w(y)| \leq C\sigma_k^{N-2}(\|w_0\|_{L^\infty(P_1)} + \|v_0\|_{L^\infty(Q_1)})\sum_{i=1}^k\left(\frac{1}{(1+|y-x^i|)^{N-2}} + \frac{1}{(1+|y-y^i|)^{N-2}}\right). \tag{3.14}$$

and

$$|v(y)| \leq C\sigma_k^{N-2}(\|w_0\|_{L^\infty(P_1)} + \|v_0\|_{L^\infty(Q_1)})\sum_{i=1}^k\left(\frac{1}{(1+|y-x^i|)^{N-2}} + \frac{1}{(1+|y-y^i|)^{N-2}}\right). \tag{3.15}$$

Moreover,

$$|\nabla w(y)| \leq C\sigma_k^{N-2}(\|w_0\|_{L^\infty(P_1)} + \|v_0\|_{L^\infty(Q_1)})\sum_{i=1}^k\frac{1}{(1+|y-x^i|)^{N-2}},\ y \in P_2 \setminus \cup_{i=1}^k B_{\sigma_k+1}(x^i)$$

and

$$|\nabla v(y)| \leq C\sigma_k^{N-2}(\|w_0\|_{L^\infty(P_1)} + \|v_0\|_{L^\infty(Q_1)})\sum_{i=1}^k\frac{1}{(1+|y-x^i|)^{N-2}},\ y \in Q_2 \setminus \cup_{i=1}^k B_{\sigma_k+1}(y^i).$$

Proof. By (3.1) and (3.2), we have

$$\begin{cases} -\Delta w = K_1\big(\frac{|y|}{\mu}\big)\big(|u_1|^{2^*-2}u_1 - |\bar{u}_1|^{2^*-2}\bar{u}_1\big) + \beta\big(|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1 - |\bar{u}_2|^{\frac{2^*}{2}}|\bar{u}_1|^{\frac{2^*}{2}-2}\bar{u}_1\big), & \text{in } \mathbb{R}^N \setminus P_1, \\ -\Delta v = K_2\big(\frac{|y|}{\mu}\big)\big(|u_2|^{2^*-2}u_2 - |\bar{u}_2|^{2^*-2}\bar{u}_2\big) + \beta\big(|u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2 - |\bar{u}_1|^{\frac{2^*}{2}}|\bar{u}_2|^{\frac{2^*}{2}-2}\bar{u}_2\big), & \text{in } \mathbb{R}^N \setminus Q_1, \\ w = w_0 \text{ in } P_1,\ v = v_0 \text{ in } Q_1. \end{cases}$$

(i) For $y \in \mathbb{R}^N \setminus (P_1 \cup Q_1)$, there holds

$$|u_1|^{2^*-2}u_1 - |\bar{u}_1|^{2^*-2}\bar{u}_1 = O\big(\varpi_{N-2}^{2^*-2}\big)w \tag{3.16}$$

and

$$\begin{aligned}&|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1-|\bar u_2|^{\frac{2^*}{2}}|\bar u_1|^{\frac{2^*}{2}-2}\bar u_1\\=&|u_2|^{\frac{2^*}{2}}\big(|u_1|^{\frac{2^*}{2}-2}u_1-|\bar u_1|^{\frac{2^*}{2}-2}\bar u_1\big)+(|u_2|^{\frac{2^*}{2}}-|\bar u_2|^{\frac{2^*}{2}})|\bar u_1|^{\frac{2^*}{2}-2}\bar u_1\\=&|u_2|^{\frac{2^*}{2}}\big(|u_1|^{\frac{2^*}{2}-2}u_1-|\bar u_1|^{\frac{2^*}{2}-2}\bar u_1\big)+O\Big(\varpi_{N-2}^{\frac{2^*-2}{2}}\Big)v,\end{aligned}$$

where ϖ_{N-2} is defined in (A.9).

Then by Kato's inequality, and noting that

$$\beta|u_2|^{\frac{2^*}{2}}\big(|u_1|^{\frac{2^*}{2}-2}u_1-|\bar u_1|^{\frac{2^*}{2}-2}\bar u_1\big)\mathrm{sgn}(w)\le 0,$$

with sgn denoting the signum function, we find that $|w|$ satisfies

$$-\Delta|w|\le C\varpi_{N-2}^{\frac{2^*-2}{2}}(|w|+|v|),\quad \text{in } \mathbb{R}^N\setminus(P_1\cup Q_1).$$

Similarly,

$$-\Delta|v|\le C\varpi_{N-2}^{\frac{2^*-2}{2}}(|w|+|v|),\quad \text{in } \mathbb{R}^N\setminus(P_1\cup Q_1).$$

Set

$$\tilde\omega=\sigma_k^{N-2}(\|w\|_{L^\infty(P_1\cup Q_1)}+\|v\|_{L^\infty(P_1\cup Q_1)})\varpi_{N-2}.$$

We have

$$\tilde\omega>|w|+|v| \text{ in } P_1\cup Q_1,$$

and by (A.10),

$$-\Delta\tilde\omega\ge C\varpi_{N-2}^{\frac{2^*-2}{2}}\tilde\omega\quad \text{in } \mathbb{R}^N\setminus(P_1\cup Q_1).$$

Therefore,

$$|w|+|v|\le\sigma_k^{N-2}(\|w\|_{L^\infty(P_1\cup Q_1)}+\|v\|_{L^\infty(P_1\cup Q_1)})\varpi_{N-2},\quad \text{in } \mathbb{R}^N\setminus(P_1\cup Q_1).$$

Since $w=0$ in Q_1, $v=0$ in P_1, we get the conclusion.

(ii) By the gradient estimate in [18, Theorem 3.9], for $y\in P_2\setminus\cup_{i=1}^k B_{\sigma_k+1}(x^i)$, we know that

$$\begin{aligned}|\nabla w(y)|\le&C\Big(\sup_{B_1(y)}|w(x)|+\sup_{B_1(y)}\Big|K_1\big(\frac{|\cdot|}{\mu}\big)\big(|u_1|^{2^*-2}u_1-|\bar u_1|^{2^*-2}\bar u_1\big)\\&+\beta\big(|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1-|\bar u_2|^{\frac{2^*}{2}}|\bar u_1|^{\frac{2^*}{2}-2}\bar u_1\big)\Big|\Big).\end{aligned}$$

From Remark 3.6, we know that $u_2=0$ and $\bar u_2=0$ in P_2. By (3.16), we obtain

$$\sup_{B_1(y)}\Big|K_1\big(\frac{|\cdot|}{\mu}\big)\big(|u_1|^{2^*-2}u_1-|\bar u_1|^{2^*-2}\bar u_1\big)\Big|\le\quad C\sup_{B_1(y)}\big(|u_1|^{2^*-2}+|\bar u_1|^{2^*-2}\big)|w|.$$

Thus,

$$|\nabla w(y)|\le C\sup_{B_1(y)}|w(x)|.$$

We can obtain the estimate for ∇v in a similar way. □

Remark 3.8. *From* (3.14) *and* (3.15)*, we know that there is at most one solution* (u_1,u_2) *to the problem* (3.1) *satisfying* (3.2) *and* (3.3).

Next, we will proof the existence of a solution (u_1,u_2) to problem (3.1) satisfying (3.2) and (3.3).

3.4. **The Existence of Minimizer.** Let $A > 0$ be the constant given in Lemma 3.1. We consider solutions (u_1, u_2) to problem (3.1) satisfying (3.2) and

$$|u_1(y)| \le 2A\sum_{i=1}^{k} \frac{1}{(1+|y-x^i|)^{N-2}}, \quad |u_2(y)| \le 2A\sum_{i=1}^{k} \frac{1}{(1+|y-y^i|)^{N-2}}, \quad \forall y \in \mathbb{R}^N. \tag{3.17}$$

We note that condition (3.17) implies (3.3).

For any $(\varphi_0, \psi_0) \in \Lambda_k \cap (D^{1,2}(\mathbb{R}^N))^2$, denote

$$\mathbb{S}(\varphi_0, \psi_0) := \Big\{(u_1, u_2) \in (D^{1,2}(\mathbb{R}^N) \cap H_s)^2 \mid (u_1, u_2) \text{ satisfies (3.2) and (3.17)}\Big\}.$$

This set is a closed subset of $(D^{1,2}(\mathbb{R}^N))^2$ under the induced product norm

$$\|(u_1, u_2)\|^2_{(D^{1,2})^2} = \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2.$$

Define a functional $\Gamma : \mathbb{S}(\varphi_0, \psi_0) \to \mathbb{R}$ as

$$\begin{aligned}\Gamma(u_1, u_2) =& \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) - \frac{1}{2^*}\int_{\mathbb{R}^N} \Big(K_1\Big(\frac{|y|}{\mu}\Big)|u_1|^{2^*} + K_2\Big(\frac{|y|}{\mu}\Big)|u_2|^{2^*}\Big) \\ &- \frac{2\beta}{2^*}\int_{\mathbb{R}^N} |u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}},\ (u_1, u_2) \in \mathbb{S}(\varphi_0, \psi_0).\end{aligned}$$

The solution to (3.1) can be obtianed by solving the following minimizing problem

$$\min\big\{\Gamma(u_1, u_2) \mid (u_1, u_2) \in \mathbb{S}(\varphi_0, \psi_0)\big\}. \tag{3.18}$$

Lemma 3.9. *The minimization problem* (3.18) *is attained by a solution to* (3.1) *in* $\mathbb{S}(\varphi_0, \psi_0)$.

Proof. First note that $\mathbb{S}(\varphi_0, \psi_0)$ is closed and convex in $\mathbb{E}$. Hence, it is weakly closed. To show the minimization problem (3.18) is attained, we show that the functional Γ is coercive and lower semicontinuous on $\mathbb{S}(\varphi_0, \psi_0)$.

(i) Coercivity. By (3.17), there exists $C > 0$ such that

$$-\frac{1}{2^*}\int_{\mathbb{R}^N\setminus P_1} K_1\Big(\frac{|y|}{\mu}\Big)|u_1|^{2^*} - \frac{1}{2^*}\int_{\mathbb{R}^N\setminus Q_1} K_2\Big(\frac{|y|}{\mu}\Big)|u_2|^{2^*} > -C.$$

By $\beta < 0$, there holds

$$\Gamma(u_1, u_2) > \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) - C.$$

Then, $\Gamma(u_1, u_2) \to +\infty$ as $\|(u_1, u_2)\|_{(D^{1,2})^2} \to \infty$.

(ii) Γ is weakly lower semi-continuous on $\mathbb{S}(\varphi_0, \psi_0)$.

Since $\beta < 0$, Fatou's Lemma implies that $-\frac{2\beta}{2^*}\int_{\mathbb{R}^N} |u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}}$ is weakly lower semi-continuous on $\mathbb{S}(\varphi_0, \psi_0)$. We now consider the functional $I(u_1, u_2) = \Gamma(u_1, u_2) + \frac{2\beta}{2^*}\int_{\mathbb{R}^N} |u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}}$, which is of class C^2. For $(\xi, \eta) \in D_0^{1,2}(\mathbb{R}^N \setminus P_1) \times D_0^{1,2}(\mathbb{R}^N \setminus Q_1) \cap (H_s)^2$, we have

$$\begin{aligned}I''(u_1, u_2)[(\xi, \eta), (\xi, \eta)] \ge& \|\nabla\xi\|_2^2 + \|\nabla\eta\|_2^2 - C\int_{\mathbb{R}^N\setminus P_1} |u_1|^{2^*-2}\xi^2 - C\int_{\mathbb{R}^N\setminus Q_1} |u_2|^{2^*-2}\eta^2 \\ \ge& (1 - C\|u_1\|^{2^*-2}_{L^{2^*}(\mathbb{R}^N\setminus P_1)} - C\|u_2\|^{2^*-2}_{L^{2^*}(\mathbb{R}^N\setminus Q_1)})\|(\xi, \eta)\|^2_{(D^{1,2})^2}\end{aligned}$$

By (3.17), we know that $I(u_1, u_2)$ is convex for large k. Therefore, it is weakly lower semicontinuous.

(iii) Suppose the minimization problem (3.18) is achieved by some $(u_1, u_2) \in \mathbb{S}(\varphi_0, \psi_0)$. We now show that this minimizer is in fact a solution to (3.1).

Since $(|u_1|, |u_2|) \in \mathbb{S}(\varphi_0, \psi_0)$ and $\Gamma(u_1, u_2) = \Gamma(|u_1|, |u_2|)$, it follows that $(|u_1|, |u_2|)$ is also a minimizer of Γ over $\mathbb{S}(\varphi_0, \psi_0)$.

Let $\xi \in C_0^\infty(\mathbb{R}^N \setminus P_1) \cap H_s$ with $\xi \geq 0$. Then for sufficiently small $t > 0$, we have $(|u_1| - t\xi, |u_2|) \in \mathbb{S}(\varphi_0, \psi_0)$. Therefore,

$$\Gamma'(|u_1|, |u_2|)(\xi, 0) = \lim_{t \to 0^+} \frac{\Gamma(|u_1|, |u_2|) - \Gamma(|u_1| - t\xi, |u_2|)}{t} \leq 0.$$

This implies that

$$-\Delta |u_1| \leq K_1 \left(\frac{|y|}{\mu} \right) |u_1|^{2^*-1} + \beta |u_2|^{\frac{2^*}{2}} |u_1|^{\frac{2^*}{2}-1}, \quad \text{in } \mathbb{R}^N \setminus P_1.$$

Since $\beta < 0$, the second term on the right-hand side is non-positive, so the inequality above implies that $|u_1|$ satisfies the first inequality in (3.4). Similarly, $|u_2|$ satisfies the second inequality.

Now, applying Lemma 3.1, we deduce that the bounds in (3.17) are strictly satisfied. Therefore, (u_1, u_2) is a solution to (3.1). □

Since $\Lambda_k \cap (D^{1,2}(\mathbb{R}^N))^2$ is dense in Λ_k with respect to the norm $\|\cdot\|_\infty$, and the a priori estimates from Lemma 3.7 hold uniformly, we may extend the solvability of (3.1) from the dense subspace to all of Λ_k. Consequently, for every $(\varphi_0, \psi_0) \in \Lambda_k$, there exists a unique solution (u_1, u_2) to problem (3.1) satisfying (3.2) and (3.3).

This allows us to introduce the operator S as follows:

Definition 3.10. *Let (u_1, u_2) denote the unique solution to (3.1) corresponding to $(\varphi_0, \psi_0) \in \Lambda_k$, satisfying (3.2) and (3.3). We define the operator $S : \Lambda_k \to \mathbb{E}$ by*

$$S(\varphi_0, \psi_0) := (u_1, u_2) - (U_{r,\lambda}, V_{\rho,\nu}).$$

Remark 3.11. *When $(\varphi_0, \psi_0) \in \Lambda_k \cap (D^{1,2}(\mathbb{R}^N))^2$, by Lemma 3.9, $(u_1, u_2) \in (D^{1,2}(\mathbb{R}^N))^2$, and hence, $S(\varphi_0, \psi_0) \in (D^{1,2}(\mathbb{R}^N))^2$.*

4. Reduction

In this section, we are now ready to carry out the reduction for system (1.6). We consider the following problem

$$\begin{cases} -\Delta\varphi - (2^*-1)K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-2}\varphi = l_1 + R_1(\varphi) + D_1(\varphi,\psi) + \sum_{j=1}^2 c_j Y_j, \\ -\Delta\psi - (2^*-1)K_2(\frac{|y|}{\mu})V_{\rho,\nu}^{2^*-2}\psi = l_2 + R_2(\psi) + D_2(\varphi,\psi) + \sum_{j=1}^2 d_j Z_j, \\ (\varphi,\psi) \in \mathbb{E}. \end{cases} \tag{4.1}$$

where

$$l_1 = K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-1} - \sum_{i=1}^k U_{x^i,\lambda}^{2^*-1}, \quad l_2 = K_2(\frac{|y|}{\mu})V_{\rho,\nu}^{2^*-1} - \sum_{i=1}^k V_{y^i,\nu}^{2^*-1}$$

$$R_1(\varphi) = K_1(\frac{|y|}{\mu})|U_{r,\lambda} + \varphi|^{2^*-2}(U_{r,\lambda} + \varphi) - K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-1} - (2^*-1)K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-2}\varphi,$$

$$R_2(\psi) = K_2(\frac{|y|}{\mu})|V_{\rho,\nu} + \psi|^{2^*-2}(V_{\rho,\nu} + \psi) - K_2(\frac{|y|}{\mu})V_{\rho,\lambda}^{2^*-1} - (2^*-1)K_2(\frac{|y|}{\mu})V_{\rho,\nu}^{2^*-2}\psi,$$

$$D_1(\varphi,\psi) = \beta|\psi + V_{\rho,\nu}|^{\frac{2^*}{2}}|\varphi + U_{r,\lambda}|^{\frac{2^*}{2}-2}(\varphi + U_{r,\lambda}),$$

$$D_2(\varphi,\psi) = \beta|\varphi + U_{r,\lambda}|^{\frac{2^*}{2}}|\psi + V_{\rho,\nu}|^{\frac{2^*}{2}-2}(\psi + V_{\rho,\nu}).$$

By the definition of S (Definition 3.10), if (φ_0, ψ_0) in Λ_k solves system (4.1), then $(\varphi_0, \psi_0) \in S(\Lambda_k)$. Next we are sufficed to solve system (4.1) in $S(\Lambda_k)$.

Let $(\chi_1,\chi_2)\in C_0^1(\mathbb{R}^N)\times C_0^1(\mathbb{R}^N)$ be the fixed truncation functions such that

$$\begin{cases}\chi_1=1, \text{ in } \mathbb{R}^N\setminus\cup_{i=1}^k B_{\frac{3}{4}\sigma_k^2}(x^i),\\ \chi_1=0, \text{ in } \cup_{i=1}^k B_{\frac{1}{4}\sigma_k^2}(x^i),\\ |\nabla\chi_1|\le\frac{4}{\sigma_k^2},\ |\Delta\chi_1|\le\frac{8N}{\sigma_k^4}.\end{cases} \quad\text{and}\quad \begin{cases}\chi_2=1, \text{ in } \mathbb{R}^N\setminus\cup_{i=1}^k B_{\frac{3}{4}\sigma_k^2}(y^i),\\ \chi_2=0, \text{ in } \cup_{i=1}^k B_{\frac{1}{4}\sigma_k^2}(y^i),\\ |\nabla\chi_2|\le\frac{4}{\sigma_k^2},\ |\Delta\chi_2|\le\frac{8N}{\sigma_k^4}.\end{cases} \tag{4.2}$$

For any $(\varphi_0,\psi_0)\in S(\Lambda_k)$, $(u_1,u_2)=(\varphi_0,\psi_0)+(U_{r,\lambda},V_{\rho,\nu})$ is the solution of system (3.1). Therefore,

$$\begin{cases}-\Delta\varphi_0-(2^*-1)K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-2}\varphi_0=l_1+R_1(\varphi_0)+D_1(\varphi_0,\psi_0), \text{ in } \mathbb{R}^N\setminus P_1,\\ -\Delta\psi_0-(2^*-1)K_2(\frac{|y|}{\mu})V_{\rho,\nu}^{2^*-2}\psi_0=l_2+R_2(\psi_0)+D_2(\varphi_0,\psi_0), \text{ in } \mathbb{R}^N\setminus Q_1.\end{cases} \tag{4.3}$$

Then we have, in $\mathbb{R}^N$,

$$-\Delta(\chi_1\varphi_0)-(2^*-1)K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-2}(\chi_1\varphi_0)=\chi_1\Big(l_1+R_1(\varphi_0)+D_1(\varphi_0,\psi_0)\Big)-\varphi_0\Delta\chi_1-2\nabla\chi_1\nabla\varphi_0 \tag{4.4}$$

and

$$-\Delta(\chi_2\psi_0)-(2^*-1)K_2(\frac{|y|}{\mu})V_{\rho,\nu}^{2^*-2}(\chi_2\psi_0)=\chi_2\Big(l_2+R_2(\psi_0)+D_2(\varphi_0,\psi_0)\Big)-\psi_0\Delta\chi_2-2\nabla\chi_2\nabla\psi_0. \tag{4.5}$$

If (φ_0,ψ_0) further satisfies the systems (4.1), by Remark 3.6, (4.4) and (4.5), we get

$$\begin{aligned}&-\Delta(\varphi_0-\chi_1\varphi_0)-(2^*-1)K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-2}(\varphi_0-\chi_1\varphi_0)\\ =&(1-\chi_1)\Big(l_1+R_1(\varphi_0)+D_1(\varphi_0,\psi_0)\Big)+\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+\sum_{j=1}^2 c_jY_j\\ =&(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)+\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+\sum_{j=1}^2 c_jY_j\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}&-\Delta(\psi_0-\chi_2\psi_0)-(2^*-1)K_2(\frac{|y|}{\mu})V_{\rho,\nu}^{2^*-2}(\psi_0-\chi_2\psi_0)\\ =&(1-\chi_2)\Big(l_2+R_2(\psi_0)+D_2(\varphi_0,\psi_0)\Big)+\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+\sum_{j=1}^2 d_jZ_j\\ =&(1-\chi_2)\big(l_2+R_2(\psi_0)\big)+\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+\sum_{j=1}^2 d_jZ_j.\end{aligned} \tag{4.7}$$

This motivates us to solve the fixed point for the operator T defined as follows:

Definition 4.1. *Let $S(\Lambda_k)$ be the complete metric space equipped with the distance*

$$d\big((\varphi_0,\psi_0),(\bar\varphi_0,\bar\psi_0)\big)=\|\varphi_0-\bar\varphi_0\|_{L^\infty(P_1)}+\|\psi_0-\bar\psi_0\|_{L^\infty(Q_1)}.$$

For any $(\varphi_0,\psi_0)\in S(\Lambda_k)$, define the map $T: S(\Lambda_k)\to\mathbb{E}$ by

$$T(\varphi_0,\psi_0):=S\begin{pmatrix}L_1\Big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\Big)\\ L_2\Big(\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+(1-\chi_2)\big(l_2+R_2(\psi_0)\big)\Big)\end{pmatrix}^\top.$$

The following Lemmas 4.2, 4.3 and Corollary 4.4 establish the well-definedness of T.

Lemma 4.2. *Assume that* $||x^1|-\mu r_0|\le 1$ *and* $||y^1|-\mu\rho_0|\le 1$. *If* $N\ge 5$, *then*

$$\|l_1\|_{L^\infty(P_2)}\le C\Big(\frac{1}{\mu}\Big)^m,\ \|l_2\|_{L^\infty(Q_2)}\le C\Big(\frac{1}{\mu}\Big)^m.$$

Proof. We are sufficed to deal with $\|l_1\|_{L^\infty(P_2)}$, since for $\|l_2\|_{L^\infty(Q_2)}$, one can obtain similar estimates just in the same way. Recalling the definition of l_1, we have

$$l_1=K_1\big(\frac{|y|}{\mu}\big)U_{r,\lambda}^{2^*-1}-\sum_{i=1}^k U_{x^i,\lambda}^{2^*-1}=J_1+J_2$$

where

$$J_1:=K_1\big(\frac{|y|}{\mu}\big)\Big(U_{r,\lambda}^{2^*-1}-\sum_{i=1}^k U_{x^i,\lambda}^{2^*-1}\Big),\quad J_2:=\Big(K_1\big(\frac{|y|}{\mu}\big)-1\Big)\sum_{i=1}^k U_{x^i,\lambda}^{2^*-1}.$$

By symmetry, we might as well assume that $y\in B_{\sigma_k^2}(x^1)$, then $|y-x^i|\ge|y-x^1|,\ i=2,\dots,k$. Therefore,

$$|J_1|\le C\frac{1}{(1+|y-x^1|)^4}\sum_{i=2}^k\frac{1}{(1+|y-x^i|)^{N-2}}+C\Big(\sum_{i=2}^k\frac{1}{(1+|y-x^i|)^{N-2}}\Big)^{2^*-1}.$$

By (A.4), we obtain

$$\Big(\sum_{i=2}^k\frac{1}{(1+|y-x^i|)^{N-2}}\Big)^{2^*-1}\le C\sum_{i=2}^k\frac{1}{(1+|y-x^i|)^{N+2-\frac{4}{N-2}\tau_1}}.$$

Thus, by (A.1),

$$\|J_1\|_{L^\infty(P_2)}\le C\sum_{i=2}^k\frac{1}{|x^i-x^1|^{N-2}}+C\sum_{i=2}^k\frac{1}{|x^i-x^1|^{N+2-\frac{4}{N-2}\tau_1}}\le C\Big(\frac{1}{\mu}\Big)^m.$$

Next we estimate J_2. For $y\in B_{\sigma_k^2}(x^1)$, we have

$$\Big|\Big(K_1\big(\frac{|y|}{\mu}\big)-1\Big)\sum_{i=2}^k U_{x^i,\lambda}^{2^*-1}\Big|\le C\sum_{i=2}^k\frac{1}{(1+|y-x^i|)^{N+2}}\le C\sum_{i=2}^k\frac{1}{|x^i-x^1|^{N+2}}\le C\Big(\frac{1}{\mu}\Big)^m.\tag{4.8}$$

If $y\in B_{\sigma_k^2}(x^1)$, there holds

$$\Big|K_1\big(\frac{|y|}{\mu}\big)-1\Big|\le C\Big|\frac{|y|}{\mu}-r_0\Big|^m\le\frac{C}{\mu^m}\Big(||y|-|x^1||^m+||x^1|-r_0\mu|^m\Big)\le\frac{C}{\mu^m}|y-x^1|^m.$$

Thus,

$$\Big|K_1\big(\frac{|y|}{\mu}\big)-1\Big|U_{x^1,\lambda}^{2^*-1}\le\frac{C}{\mu^m}\frac{|y-x^1|^m}{(1+|y-x^1|)^{N+2}}\le\frac{C}{\mu^m}\frac{1}{(1+|y-x^1|)^{N+2-m}}\le C\Big(\frac{1}{\mu}\Big)^m.\tag{4.9}$$

Hence, (4.8) and (4.9) imply that

$$\|J_2\|_{L^\infty(P_2)}\le C\Big(\frac{1}{\mu}\Big)^m.$$

□

Lemma 4.3. *Suppose* $N\ge 5$. *For* $(\varphi_0,\psi_0)\in S(\Lambda_k)$, *we obtain*

$$\|R_1(\varphi_0)\|_{L^\infty(P_2)}\le C\Big(\frac{1}{\mu}\Big)^m,\ \textit{and}\ \|R_2(\psi_0)\|_{L^\infty(Q_2)}\le C\Big(\frac{1}{\mu}\Big)^m.$$

Proof. If $N=5$, we get $\tau_1=\frac{N-2-m}{N-2}=\frac{3-m}{3}\le\frac{1}{3}$. By Lemma (A.2), there holds

$$U_{r,\lambda}^{\frac{1}{3}}\le C\sum_{i=1}^{k}\frac{1}{(1+|y-x^i|)}<C\sum_{i=1}^{k}\frac{1}{(1+|y-x^i|)^{\tau_1}}\le C.$$

Then we have

$$|R_1(\varphi_0)|\le CU_{r,\lambda}^{\frac{1}{3}}|\varphi_0|^2+C|\varphi_0|^{\frac{7}{3}}\le C|\varphi_0|^2,\ N=5.$$

For $N\ge 6$ and $\delta_2>0$ small enough, we obtain $\min_{P_2}\{U_{r,\lambda}\}>\frac{1}{\sigma_k^{2N-4+\delta_2}}$. Then there exists $C>0$ independent of k such that $\frac{\varphi_0}{U_{r,\lambda}}\le C$ in P_2. Thus,

$$|R_1(\varphi_0)|\le CU_{r,\lambda}^{2^*-3}|\varphi_0|^2,\ N\ge 6.$$

By Corollary 3.3, we obtain

$$\frac{|\varphi_0|^2}{U_{r,\lambda}^{3-2^*}}=O\Big(\frac{1}{\mu^{m+2\theta}}\Big)(1+|y-x^1|)^{-6+\tau_1},\quad y\in B_{\sigma_k^2}(x^1).$$

Therefore,

$$\|R_1(\varphi_0)\|_{L^\infty(P_2)}\le C\Big(\frac{1}{\mu}\Big)^m.$$

Similarly, we can get the estimate of $\|R_2(\psi_0)\|_{L^\infty(Q_2)}$. □

We can show the map $T:S(\Lambda_k)\to S(\Lambda_k)$ is well-defined; that is,

Corollary 4.4. *For large k, there holds*

$$\begin{pmatrix}L_1\big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\big)\\ L_2\big(\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+(1-\chi_2)\big(l_2+R_2(\psi_0)\big)\big)\end{pmatrix}^\top\in\Lambda_k.$$

Proof. Note that $\operatorname{supp}(1-\chi_1)\subset P_2$ and $\operatorname{supp}\nabla\chi_1\subset\big(\cup_{i=1}^k B_{\frac{3}{4}\sigma_k^2}(x^i)\setminus\cup_{i=1}^k B_{\frac{1}{4}\sigma_k^2}(x^i)\big)$. By Lemma 3.2, we get

$$\|\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0\|_{L^\infty(P_2)}\le C\frac{1}{\mu^{\frac{m}{2}+\theta}}\frac{1}{\sigma_k^{N+2}}.$$

From Lemmas 4.2 and 4.3, using Proposition 2.2, we obtain

$$\begin{aligned}&\|L_1\big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\big)\|_{L^\infty(\mathbb{R}^N)}\\ \le&\sigma_k^4\|\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0\|_{L^\infty(P_2)}+\sigma_k^4\|l_1+R_1(\varphi_0)\|_{L^\infty(P_2)}\le\frac{1}{2}\frac{1}{\mu^{\frac{m}{2}}}\frac{1}{\sigma_k^{N-2}}.\end{aligned}$$

Similarly,

$$\|L_2\big(\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+(1-\chi_2)\big(l_2+R_2(\psi_0)\big)\big)\|_{L^\infty(\mathbb{R}^N)}\le\frac{1}{2}\frac{1}{\mu^{\frac{m}{2}}}\frac{1}{\sigma_k^{N-2}}.$$

The proof is complete. □

Lemma 4.5. *The operator $T:S(\Lambda_k)\to S(\Lambda_k)$ is a contraction:*

$$d(T(\varphi_0,\psi_0),T(\bar\varphi_0,\bar\psi_0))\le\frac{1}{2}d((\varphi_0,\psi_0),(\bar\varphi_0,\bar\psi_0)).$$

Proof. Take $(\varphi_0,\psi_0)\in S(\Lambda_k)$, $(\bar\varphi_0,\bar\psi_0)\in S(\Lambda_k)$. Set

$$(u_1,u_2)=(\varphi_0,\psi_0)+(U_{r,\lambda},V_{\rho,\nu}),\ (\bar u_1,\bar u_2)=(\bar\varphi_0,\bar\psi_0)+(U_{r,\lambda},V_{\rho,\nu})$$

$$(w_0,v_0)=(\varphi_0,\psi_0)-(\bar\varphi_0,\bar\psi_0)=(u_1,u_2)-(\bar u_1,\bar u_2),$$

$$W=L_1\Big((1-\chi_1)\big(R_1(\varphi_0)-R_1(\bar\varphi_0)\big)+w_0\Delta\chi_1+2\nabla\chi_1\nabla w_0\Big)$$

and

$$V=L_2\Big((1-\chi_2)\big(R_2(\psi_0)-R_2(\bar\psi_0)\big)+v_0\Delta\chi_2+2\nabla\chi_2\nabla v_0\Big).$$

Recalling the definition of S, we only need to show

$$\|W\|_{L^\infty(P_1)}+\|V\|_{L^\infty(Q_1)}\le\frac{1}{2}(\|w_0\|_{L^\infty(P_1)}+\|v_0\|_{L^\infty(Q_1)}).$$

By Proposition 2.2, we know

$$\|W\|_{L^\infty(P_1)}\le C\sigma_k^4\left(\|R_1(\varphi_0)-R_1(\bar\varphi_0)\|_{L^\infty(P_2)}+\|w_0\Delta\chi_1+2\nabla\chi_1\nabla w_0\|_{L^\infty(P_2)}\right). \tag{4.10}$$

By Lemma 3.7, when $N=5$, it follows that

$$\begin{aligned}\|R_1(\varphi_0)-R_1(\bar\varphi_0)\|_{L^\infty(P_2)}\le&C\|(|\varphi_0|+|\bar\varphi_0|)w_0\|_{L^\infty(P_2)}\\ \le&\frac{C}{k\mu^{\frac{m}{2}}}(\|w_0\|_{L^\infty(P_1)}+\|v_0\|_{L^\infty(Q_1)}).\end{aligned}$$

When $N\ge 6$, we have

$$\begin{aligned}|R_1(\varphi_0)-R_1(\bar\varphi_0)|=&\Big|K_1(\frac{|y|}{\mu})\Big(|u_1|^{2^*-2}u_1-|\bar u_1|^{2^*-2}\bar u_1-(2^*-1)U_{r,\lambda}^{2^*-2}w_0\Big)\Big|\\ \le&C\Big|\int_0^1\Big(|t\varphi_0+(1-t)\bar\varphi_0+U_{r,\lambda}|^{2^*-2}-U_{r,\lambda}^{2^*-2}\Big)w_0\mathrm{d}t\Big|\\ \le&C|(|\varphi_0|^{2^*-2}+|\bar\varphi_0|^{2^*-2})w_0|.\end{aligned}$$

Therefore,

$$\|R_1(\varphi_0)-R_1(\bar\varphi_0)\|_{L^\infty(P_2)}\le C\frac{1}{\mu^{\frac{2m}{N-2}}\sigma_k^4}(\|w_0\|_{L^\infty(P_1)}+\|v_0\|_{L^\infty(Q_1)}). \tag{4.11}$$

Since supp $\Delta\chi_1\cup$ supp $\nabla\chi_1\subset\cup_{i=1}^kB_{\frac34\sigma_k^2}(x^i)\setminus\cup_{i=1}^kB_{\frac14\sigma_k^2}(x^i)$, by Lemma 3.7, and (4.2), we obtain

$$\|w_0\Delta\chi_1+2\nabla\chi_1\nabla w_0\|_{L^\infty(P_2)}=O\Big(\frac{1}{\sigma_k^N}\Big)(\|w_0\|_{L^\infty(P_1)}+\|v_0\|_{L^\infty(Q_1)}). \tag{4.12}$$

Combining (4.10), (4.11), and (4.12), we obtain

$$\|W\|_{L^\infty(Q_1)}\le\frac{1}{4}(\|w_0\|_{L^\infty(P_1)}+\|v_0\|_{L^\infty(Q_1)}). \tag{4.13}$$

A similar argument establishes the estimate for V, which completes the proof. □

Lemma 4.5 implies that T has a unique fixed point in $S(\Lambda_k)$. To see it is a solution, we show

Lemma 4.6. *(φ_0,ψ_0) is a fixed point of T in $S(\Lambda_k)$ if and only if it is a solution of (4.1) in Λ_k satisfying*

$$(2^*-1)K_\infty\|\varphi_0+U_{r,\lambda}\|^{2^*-2}_{L^{2^*}(\mathbb{R}^N\setminus P_1)}<\frac{\mathcal{S}}{2}\ \textit{and}\ (2^*-1)K_\infty\|\psi_0+V_{\rho,\nu}\|^{2^*-2}_{L^{2^*}(\mathbb{R}^N\setminus Q_1)}<\frac{\mathcal{S}}{2}.$$

Furthermore, whenever either of the two equivalent conditions is satisfied, it necessarily follows that

$$(\varphi_0,\psi_0)\in S(\Lambda_k)\cap\Lambda_k\cap(D^{1,2}(\mathbb{R}^N))^2.$$

Proof. (i) Assume (φ_0,ψ_0) is a fixed point of T, then

$$(\varphi_0,\psi_0)=S\begin{pmatrix}L_1\big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\big)\\ L_2\big(\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+(1-\chi_2)\big(l_2+R_2(\psi_0)\big)\big)\end{pmatrix}^{\top}.$$

We denote

$$\zeta_1:=L_1\Big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\Big)+\chi_1\varphi_0$$
$$\zeta_2:=L_2\Big(\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+(1-\chi_2)\big(l_2+R_2(\psi_0)\big)\Big)+\chi_2\psi_0.$$

By Proposition 2.2 and Remark 3.11, we know $(\varphi_0,\psi_0)\in(D^{1,2}(\mathbb{R}^N))^2$ and $(\zeta_1,\zeta_2)\in(D^{1,2}(\mathbb{R}^N))^2$. Recalling the cut off functions (χ_1,χ_2) defined in (4.2), and the definition of S, we obtain

$$\zeta_1=L_1\Big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\Big)=\varphi_0 \text{ in } P_1,$$
$$\zeta_2=L_2\Big(\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+(1-\chi_2)\big(l_2+R_2(\psi_0)\big)\Big)=\psi_0 \text{ in } Q_1.$$

By the definition of L_1 and (4.4), we know that

$$-\Delta\zeta_1-(2^*-1)K_1\big(\frac{|y|}{\mu}\big)U_{r,\lambda}^{2^*-2}\zeta_1=l_1+R_1(\varphi_0)+D_1(\varphi_0,\psi_0)+\sum_{j=1}^{2}c_jY_j. \tag{4.14}$$

Subtracting the first equation of system (4.3) and (4.14), we obtain

$$-\Delta(\zeta_1-\varphi_0)-(2^*-1)K_1\big(\frac{|y|}{\mu}\big)U_{r,\lambda}^{2^*-2}(\zeta_1-\varphi_0)=0,\ \text{in } \mathbb{R}^N\setminus P_1,$$

and $\zeta_1-\varphi_0\in D_0^{1,2}(\mathbb{R}^N\setminus P_1)$. Therefore,

$$\|\nabla(\zeta_1-\varphi_0)\|_{L^2}^2\le C\|U_{r,\lambda}^{2^*-2}\|_{L^{N/2}(\mathbb{R}^N\setminus P_1)}\|\zeta_1-\varphi_0\|_{L^{2^*}}^2=o(1)\|\nabla(\zeta_1-\varphi_0)\|_{L^2}^2.$$

Hence $\zeta_1=\varphi_0$. Similarly, $\zeta_2=\psi_0$.

(ii) If $(\varphi_0,\psi_0)\in\Lambda_k$ solves (4.1), then it follows that $(\varphi_0,\psi_0)\in S(\Lambda_k)$ and the conditions (4.6) and (4.7) hold. Therefore,

$$\varphi_0=L_1\Big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\Big)+\chi_1\varphi_0,$$
$$\psi_0=L_2\Big(\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+(1-\chi_2)\big(l_2+R_2(\psi_0)\big)\Big)+\chi_2\psi_0.$$

Then

$$(\varphi_0,\psi_0)=S(\varphi_0,\psi_0)=S\begin{pmatrix}L_1\big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\big)\\ L_2\big(\psi_0\Delta\chi_2+2\nabla\chi_2\nabla\psi_0+(1-\chi_2)\big(l_2+R_2(\psi_0)\big)\big)\end{pmatrix}^{\top}.$$

□

Proposition 4.7. *For every $(r,\rho,\lambda,\nu)\in\big[r_0\mu-1,\ r_0\mu+1\big]\times\big[\rho_0\mu-1,\ \rho_0\mu+1\big]\times[\gamma_1,\gamma_2]\times[\gamma_3,\gamma_4]$, there exists a unique pair $(\varphi_0(r,\rho,\lambda,\nu),\ \psi_0(r,\rho,\lambda,\nu))\in\Lambda_k$ that solves (4.1). Moreover, the mapping*

$$(r,\rho,\lambda,\nu)\mapsto(\varphi_0(r,\rho,\lambda,\nu),\ \psi_0(r,\rho,\lambda,\nu))$$

is continuous.

Proof. The existence and uniqueness has been verified in Lemmas 4.5 and 4.6. We consider

$$(u_1,u_2)=(\varphi_0+U_{r,\lambda},\psi_0+V_{\rho,\nu}),\quad(\varphi_0,\psi_0)=(\varphi_0(r,\rho,\lambda,\nu),\psi_0(r,\rho,\lambda,\nu))$$

and

$$(\widehat{u}_1,\widehat{u}_2)=(\widehat{\varphi}_0+U_{\widehat{r},\widehat{\lambda}},\widehat{\psi}_0+V_{\widehat{\rho},\widehat{\nu}}),\quad(\widehat{\varphi}_0,\widehat{\psi}_0)=(\varphi_0(\widehat{r},\widehat{\rho},\widehat{\lambda},\widehat{\nu}),\psi_0(\widehat{r},\widehat{\rho},\widehat{\lambda},\widehat{\nu}))$$

corresponding to parameters (r,ρ,λ,ν), and $(\widehat{r},\widehat{\rho},\widehat{\lambda},\widehat{\nu})$, respectively.

From systems (4.6) and (4.7), using Proposition 2.2, we have

$$(1-\chi_1)\varphi_0 =L_1\Big(\varphi_0\Delta\chi_1+2\nabla\chi_1\nabla\varphi_0+(1-\chi_1)\big(l_1+R_1(\varphi_0)\big)\Big)$$

and

$$(1-\chi_1)\widehat{\varphi}_0 =\widehat{L}_1\Big(\widehat{\varphi}_0\Delta\chi_1+2\nabla\chi_1\nabla\widehat{\varphi}_0+(1-\chi_1)\big(\widehat{l}_1+\widehat{R}_1(\widehat{\varphi}_0)\big)\Big),$$

where

$$\widehat{l}_1=K_1\big(\frac{|y|}{\mu}\big)U_{\widehat{r},\widehat{\lambda}}^{2^*-1}-\sum_{i=1}^{k}U_{\widehat{x}^i,\widehat{\lambda}}^{2^*-1}$$

and

$$\widehat{R}_1(\widehat{\varphi}_0)=K_1\big(\frac{|y|}{\mu}\big)|U_{\widehat{r},\widehat{\lambda}}+\widehat{\varphi}_0|^{2^*-2}(U_{\widehat{r},\widehat{\lambda}}+\widehat{\varphi}_0)-K_1\big(\frac{|y|}{\mu}\big)U_{\widehat{r},\widehat{\lambda}}^{2^*-1}-(2^*-1)K_1\big(\frac{|y|}{\mu}\big)U_{\widehat{r},\widehat{\lambda}}^{2^*-2}\widehat{\varphi}_0.$$

Note that

$$\|l_1-\widehat{l}_1\|_{L^\infty(P_2)}+\|R_1(\widehat{\varphi}_0)-\widehat{R}_1(\widehat{\varphi}_0)\|_{L^\infty(P_2)}\to 0\quad \text{as } (r,\rho,\lambda,\nu)\to(\widehat{r},\widehat{\rho},\widehat{\lambda},\widehat{\nu}).$$

By Lemma 2.3, as $(r,\rho,\lambda,\nu)\to(\widehat{r},\widehat{\rho},\widehat{\lambda},\widehat{\nu})$, we obtain

$$\begin{aligned}&(1-\chi_1)(\varphi_0-\widehat{\varphi}_0)\\ =&(L_1-\widehat{L}_1)\Big(\widehat{\varphi}_0\Delta\chi_1+2\nabla\chi_1\nabla\widehat{\varphi}_0+(1-\chi_1)\big(\widehat{l}_1+\widehat{R}_1(\widehat{\varphi}_0)\big)\Big)\\ &+L_1\Big((\varphi_0-\widehat{\varphi}_0)\Delta\chi_1+2\nabla\chi_1\nabla(\varphi_0-\widehat{\varphi}_0)+(1-\chi_1)\big(R_1(\varphi_0)-\widehat{R}_1(\widehat{\varphi}_0)\big)\Big)\\ =&L_1\Big((\varphi_0-\widehat{\varphi}_0)\Delta\chi_1+2\nabla\chi_1\nabla(\varphi_0-\widehat{\varphi}_0)+(1-\chi_1)\big(R_1(\varphi_0)-R_1(\widehat{\varphi}_0)\big)\Big)+o(1).\end{aligned}$$

Therefore, by (4.13), we get

$$\|(\varphi_0-\widehat{\varphi}_0)\|_{L^\infty(P_1)}\le\frac{1}{4}(\|(\varphi_0-\widehat{\varphi}_0)\|_{L^\infty(P_1)}+\|\psi_0-\widehat{\psi}_0\|_{L^\infty(Q_1)})+o(1).$$

Similarly, we have

$$\|\psi_0-\widehat{\psi}_0\|_{L^\infty(Q_1)}\le\frac{1}{4}(\|(\varphi_0-\widehat{\varphi}_0)\|_{L^\infty(P_1)}+\|\psi_0-\widehat{\psi}_0\|_{L^\infty(Q_1)})+o(1).$$

Therefore, as $(r,\rho,\lambda,\nu)\to(\widehat{r},\widehat{\rho},\widehat{\lambda},\widehat{\nu})$,

$$\|(\varphi_0-\widehat{\varphi}_0)\|_{L^\infty(P_1)}+\|\psi_0-\widehat{\psi}_0\|_{L^\infty(Q_1)}\to 0.$$

Using Lemma 3.7 again we obtain

$$\begin{aligned}&\|(\varphi_0,\psi_0)-(\widehat{\varphi}_0,\widehat{\psi}_0)\|_\infty\\ \le&C_k\Big(\|\varphi_0-\widehat{\varphi}_0\|_{L^\infty(P_1)}+\|\psi_0-\widehat{\psi}_0\|_{L^\infty(Q_1)}+\|(U_{r,\lambda}-U_{\widehat{r},\widehat{\lambda}})\|_{L^\infty(P_1)}+\|V_{\rho,\nu}-V_{\widehat{\rho},\widehat{\nu}}\|_{L^\infty(Q_1)}\Big)\\ \to&0,\quad \text{as } (r,\rho,\lambda,\nu)\to(\widehat{r},\widehat{\rho},\widehat{\lambda},\widehat{\nu}).\end{aligned}$$

□

5. Proof of Main Theorems

Section 4 implies that

$$\begin{cases} -\Delta u_{1,k} - K_1\big(\frac{|y|}{\mu}\big)|u_{1,k}|^{2^*-2}u_{1,k} - \beta|u_{2,k}|^{\frac{2^*}{2}}|u_{1,k}|^{\frac{2^*}{2}-2}u_{1,k} = \sum_{j=1}^2 c_jY_j,\\ -\Delta u_{2,k} - K_2\big(\frac{|y|}{\mu}\big)|u_{2,k}|^{2^*-2}u_{2,k} - \beta|u_{1,k}|^{\frac{2^*}{2}}|u_{2,k}|^{\frac{2^*}{2}-2}u_{2,k} = \sum_{j=1}^2 d_jZ_j,\end{cases} \tag{5.1}$$

for some constants c_j and d_j. It remains to find suitable (r,ρ,λ,ν) such that $c_j,d_j=0,\ j=1,2$.

Therefore, if the left hand side of (5.1) belongs to $\mathbb{E}$, then the function on the right hand side of (5.1) must be zero. Recall that

$$\big(u_{1,k}(y),u_{2,k}(y)\big) = \big(U_{r_k,\lambda_k}(y)+\varphi_k, V_{\rho_k,\nu_k}(y)+\psi_k\big),$$

where

$$U_{r_k,\lambda_k}(y)=\sum_{i=1}^k U_{x^i,\lambda_k}(y),\ V_{\rho_k,\nu_k}(y)=\sum_{i=1}^k V_{y^i,\nu_k}(y).$$

For simplicity, we drop the subscript k. Denote

$$\big(u_1,u_2\big) := \big(u_{1,k}(y),u_{2,k}(y)\big).$$

Let ζ be a smooth cut-off function such that $\zeta(y)=1$ if $|y-x^1|\le \mu^{\frac{m}{N-3/2}}$, $\zeta(y)=0$ if $|y-x^1|\ge 2\mu^{\frac{m}{N-3/2}}$, $0\le\zeta(y)\le 1$, $|\nabla\zeta(y)|\le 2/\mu^{\frac{m}{N-3/2}}$ for $y\in\mathbb{R}^N$. For $\bar\alpha>1$ and $|y-x^1|\le 2\mu^{\frac{m}{N-3/2}}$, there holds

$$\sum_{j=2}^k \frac{1}{(1+|y-x^j|)^{\bar\alpha}} = O(\mu^{\frac{m\bar\alpha}{N-3/2}-\frac{m\bar\alpha}{N-2}})\frac{1}{(1+|y-x^1|)^{\bar\alpha}}.$$

Let χ be a smooth cut-off function such that $\chi(y)=1$ if $|y-y^1|\le\mu^{\frac{m}{N-3/2}}$, $\chi(y)=0$ if $|y-y^1|\ge 2\mu^{\frac{m}{N-3/2}}$, $0\le\chi(y)\le1$, $|\nabla\chi(y)|\le 2/\mu^{\frac{m}{N-3/2}}$ for $y\in\mathbb{R}^N$.

Define

$$\begin{aligned}
\widetilde{c}_1 &= \int_{\mathbb{R}^N}\Big(-\Delta u_1 - K_1\big(\frac{|y|}{\mu}\big)|u_1|^{2^*-2}u_1 - \beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1\Big)\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1},\\
\widetilde{c}_2 &= \int_{\mathbb{R}^N}\Big(-\Delta u_1 - K_1\big(\frac{|y|}{\mu}\big)|u_1|^{2^*-2}u_1 - \beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1\Big)\zeta\frac{\partial U_{x^1,\lambda}}{\partial \lambda},\\
\widetilde{d}_1 &= \int_{\mathbb{R}^N}\Big(-\Delta u_2 - K_2\big(\frac{|y|}{\mu}\big)|u_2|^{2^*-2}u_2 - \beta|u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2\Big)\chi\frac{\partial V_{y^1,\nu}}{\partial y_1},\\
\widetilde{d}_2 &= \int_{\mathbb{R}^N}\Big(-\Delta u_2 - K_2\big(\frac{|y|}{\mu}\big)|u_2|^{2^*-2}u_2 - \beta|u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2\Big)\chi\frac{\partial V_{y^1,\nu}}{\partial \nu}.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^N} Y_1\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1} = -\int\chi_0\Big|\frac{\partial U_{0,\lambda}}{\partial y_1}\Big|^2,\quad \int_{\mathbb{R}^N} Y_2\zeta\frac{\partial U_{x^1,\lambda}}{\partial \lambda} = \int\chi_0\Big|\frac{\partial U_{0,\lambda}}{\partial \lambda}\Big|^2,$$

we know $c_i=0$ if and only if $\widetilde{c}_i=0$, $i=1,2$. Similarly, $d_i=0$ if and only if $\widetilde{d}_i=0$, $i=1,2$.

To find (r,ρ,λ,ν) such that $c_i=0$ and $d_i=0$ for $i=1,2$, we first estimate the quantities.

Lemma 5.1. *There exists $\theta>0$ such that*

$$\widetilde{c}_2 = -\frac{B_1}{\lambda^{m+1}\mu^m} + \sum_{i=2}^k\frac{B_2}{\lambda^{N-1}|x^i-x^1|^{N-2}} + O\Big(\frac{1}{\mu^{m+\theta}}+\frac{1}{\mu^m}|\mu r_0-|x^1||^2\Big),$$

$$\widetilde{d}_2 = -\frac{B_3}{\nu^{m+1}\mu^m} + \sum_{i=2}^k\frac{B_4}{\nu^{N-1}|y^i-y^1|^{N-2}} + O\Big(\frac{1}{\mu^{m+\theta}}+\frac{1}{\mu^m}|\mu\rho_0-|y^1||^2\Big),$$

where $B_i > 0, i = 1, 2, 3, 4$ *are some constants.*

Proof. From Lemmas 3.1 and A.1, there holds

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N} |u_2|^{\frac{2^*}{2}} |u_1|^{\frac{2^*}{2}-2} u_1 \zeta \frac{\partial U_{x^1,\lambda}}{\partial \lambda}\Big| \leq \int_{B_{2\mu^{\frac{m}{N-3/2}}}(x^1)} |u_2|^{\frac{2^*}{2}} |u_1|^{\frac{2^*}{2}-1} \Big|\frac{\partial U_{x^1,\lambda}}{\partial \lambda}\Big| \\
\leq & \int_{|y-x^1|\leq 2\mu^{\frac{m}{N-3/2}}} \sum_{j=1}^{k} \frac{1}{(1+|y-y^j|)^{2^*(N-2)/2}} \frac{1}{(1+|y-x^1|)^{2^*(N-2)/2}} \\
\leq & \sum_{j=1}^{k} \frac{1}{|y^j - x^1|^{N-1}} \int_{\mathbb{R}^N} \frac{1}{(1+|y-y^j|)^{N+1}} + \frac{1}{(1+|y-x^1|)^{N+1}} = O\Big(\frac{1}{\mu^{N-2}}\Big).
\end{aligned} \tag{5.2}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Big(-\Delta u_1 - K_1(\frac{|y|}{\mu})|u_1|^{2^*-2}u_1\Big)\zeta \frac{\partial U_{x^1,\lambda}}{\partial \lambda} \\
=&\int_{\mathbb{R}^N} \Big(-\Delta U_{r,\lambda} - K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-1}\Big)\frac{\partial U_{x^1,\lambda}}{\partial \lambda} + \int_{\mathbb{R}^N} \Big(\Delta U_{r,\lambda} + K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-1}\Big)(1-\zeta)\frac{\partial U_{x^1,\lambda}}{\partial \lambda} \\
&- \int_{\mathbb{R}^N} K_1(\frac{|y|}{\mu}) \left(|u_1|^{2^*-2}u_1 - U_{r,\lambda}^{2^*-1}\right)\zeta \frac{\partial U_{x^1,\lambda}}{\partial \lambda} - \int_{\mathbb{R}^N} \Delta \left(\zeta \frac{\partial U_{x^1,\lambda}}{\partial \lambda}\right)\varphi \\
=&\frac{1}{k}\frac{\partial}{\partial \lambda} F(U_{r,\lambda}) + M_1 + M_2 + M_3 + M_4,
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= \int_{\mathbb{R}^N} \left(\Delta U_{r,\lambda} + K_1\left(\frac{|y|}{\mu}\right) U_{r,\lambda}^{2^*-1}\right)(1-\zeta)\frac{\partial U_{x^1,\lambda}}{\partial \lambda}, \\
M_2 &= -\int_{\mathbb{R}^N} K_1\left(\frac{|y|}{\mu}\right)\left(|u_1|^{2^*-2}u_1 - U_{r,\lambda}^{2^*-1} - (2^*-1)U_{x^1,\lambda}^{2^*-2}\varphi\right)\zeta \frac{\partial U_{x^1,\lambda}}{\partial \lambda}, \\
M_3 &= -\int_{\mathbb{R}^N} (\Delta\zeta)\frac{\partial U_{x^1,\lambda}}{\partial \lambda}\varphi - 2\int_{\mathbb{R}^N} \nabla\zeta\nabla \frac{\partial U_{x^1,\lambda}}{\partial \lambda}\varphi, \\
M_4 &= -(2^*-1)\int_{\mathbb{R}^N}\left(K_1\left(\frac{|y|}{\mu}\right) - 1\right) U_{x^1,\lambda}^{2^*-2}\varphi\zeta\frac{\partial U_{x^1,\lambda}}{\partial \lambda}.
\end{aligned}
$$

and the functional $F : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ is defined for any $u \in D^{1,2}(\mathbb{R}^N)$

$$
F(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K_1(\frac{y}{\mu})|u|^{2^*}.
$$

By Lemmas A.1 and A.3, we have

$$
\begin{aligned}
|M_1| \leq & \int_{|y-x^1|\geq \mu^{\frac{m}{N-1}}} \left|\left(K_1(\frac{|y|}{\mu})U_{r,\lambda}^{2^*-1} - \sum_{j=1}^{k} U_{x^j,\lambda}^{2^*-1}\right)\frac{\partial U_{x^1,\lambda}}{\partial \lambda}\right| \\
\leq & \frac{C}{\mu^{\frac{m}{2}+\theta}} \int_{|y-x^1|\geq \mu^{\frac{m}{N-1}}} \sum_{j=1}^{k} \frac{1}{(1+|y-x^j|)^{\frac{N}{2}+2}} \frac{1}{(1+|y-x^1|)^{N-2}} \\
\leq & \frac{C}{\mu^{\frac{m}{2}+\theta}} \sum_{j=2}^{k} \frac{1}{|x^1 - x^j|^{\frac{N-2}{2}}} \int_{\mathbb{R}^N} \left(\frac{1}{(1+|y-x^j|)^{N+1}} + \frac{1}{(1+|y-x^1|)^{N+1}}\right) \\
&+ \frac{C}{\mu^{\frac{m}{2}+\theta}} \int_{|y-x^1|\geq \mu^{\frac{m}{N-1}}} \frac{1}{(1+|y-x^1|)^{\frac{3N}{2}}} = O\left(\frac{1}{\mu^{m+\theta}}\right).
\end{aligned}
$$

To estimate M_2, we first note that for $y \in B_{2\mu^{\frac{m}{N-3/2}}}(x^1)$,

$$\frac{\varphi}{U_{x^1,\lambda}} = O(\frac{1}{\mu^\theta}), \quad \frac{\sum_{j=2}^k U_{x^j,\lambda}}{U_{x^1,\lambda}} = O(\frac{1}{\mu^{\frac{m}{2N-3}}}).$$

Then for $|y-x^1| \leq 2\mu^{\frac{m}{N-3/2}}$,

$$\begin{aligned}&|u_1|^{2^*-2}u_1 - U_{r,\lambda}^{2^*-1} - (2^*-1)U_{x^1,\lambda}^{2^*-2}\varphi \\ =&(2^*-1)\left((U_{r,\lambda}+t_y\varphi)^{2^*-2} - U_{x^1,\lambda}^{2^*-2}\right)\varphi = O(U_{x^1,\lambda}^{2^*-3}\sum_{j=2}^k U_{x^j,\lambda}|\varphi| + U_{x^1,\lambda}^{2^*-3}\varphi^2),\end{aligned}$$

where $t_y \in (0,1)$.

From Corollary 3.3, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} U_{x^1,\lambda}^{2^*-3}\sum_{j=2}^k U_{x^j,\lambda}\zeta\left|\varphi\frac{\partial U_{x^1,\lambda}}{\partial\lambda}\right| = \frac{1}{2^*-2}\int_{\mathbb{R}^N}\sum_{j=2}^k U_{x^j,\lambda}\zeta\left|\varphi\frac{\partial U_{x^1,\lambda}^{2^*-2}}{\partial\lambda}\right| \\ \leq& C\frac{1}{\mu^{\frac{m}{2}+\theta}}\int_{B_{2\mu^{\frac{m}{N-3/2}}}(x^1)}\frac{1}{(1+|y-x^1|)^{\frac{N}{2}+4}}\sum_{j=2}^k\frac{1}{(1+|y-x^j|)^{N-2}} \\ \leq& C\frac{1}{\mu^{\frac{m}{2}+\theta}}\sum_{j=2}^k\frac{1}{|x^1-x^j|^{\frac{N}{2}}}\int_{B_{2\mu^{\frac{m}{N-3/2}}}(x^1)}\frac{1}{(1+|y-x^1|)^{N+2}} \\ =&O\Big(\frac{1}{\mu^{m+\theta}}\Big)\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\zeta\Big|U_{x^1,\lambda}^{2^*-3}\frac{\partial U_{x^1,\lambda}}{\partial\lambda}\varphi^2\Big| = \frac{1}{2^*-2}\int_{\mathbb{R}^N}\zeta\Big|\frac{\partial U_{x^1,\lambda}^{2^*-2}}{\partial\lambda}\varphi^2\Big| \\ \leq& C\frac{1}{\mu^{m+2\theta}}\int_{B_{2\mu^{\frac{m}{N-3/2}}}(x^1)}\frac{1}{(1+|y-x^1|)^{N+4}} = O\Big(\frac{1}{\mu^{m+\theta}}\Big).\end{aligned}\tag{5.4}$$

From (5.3) and (5.4), we get

$$M_2 = O\Big(\frac{1}{\mu^{m+\theta}}\Big).$$

We now estimate M_3. By Corollary 3.3, there holds

$$\begin{aligned}&\int_{\mathbb{R}^N}|\,(\Delta\zeta)\,\frac{\partial U_{x^1,\lambda}}{\partial\lambda}\varphi| \leq \frac{C}{\mu^{\frac{4m}{2N-3}}}\int_{B_{2\mu^{\frac{2m}{2N-3}}}(x^1)\backslash B_{\mu^{\frac{2m}{2N-3}}}(x^1)}|\frac{\partial U_{x^1,\lambda}}{\partial\lambda}\varphi| \\ \leq&\frac{C}{\mu^{\frac{4m}{2N-3}+\frac{m}{2}+\theta}}\int_{B_{2\mu^{\frac{2m}{2N-3}}}(x^1)\backslash B_{\mu^{\frac{2m}{2N-3}}}(x^1)}\frac{1}{(1+|y-x^1|)^{\frac{3N}{2}-2}} \leq \frac{C}{\mu^{\frac{Nm}{2N-3}+\frac{m}{2}+\theta}} = O\Big(\frac{1}{\mu^{m+\theta}}\Big).\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla\zeta\nabla\frac{\partial U_{x^1,\lambda}}{\partial\lambda}\varphi| \leq \frac{C}{\mu^{\frac{2m}{2N-3}}}\int_{B_{2\mu^{\frac{2m}{2N-3}}}(x^1)\backslash B_{\mu^{\frac{2m}{2N-3}}}(x^1)}|\nabla\frac{\partial U_{x^1,\lambda}}{\partial\lambda}\varphi| \\ \leq&\frac{C}{\mu^{\frac{2m}{2N-3}+\frac{m}{2}+\theta}}\int_{B_{2\mu^{\frac{2m}{2N-3}}}(x^1)\backslash B_{\mu^{\frac{2m}{2N-3}}}(x^1)}\frac{1}{(1+|y-x^1|)^{\frac{3N}{2}-1}} = O\Big(\frac{1}{\mu^{m+\theta}}\Big).\end{aligned}$$

Then

$$|M_3| = O\Big(\frac{1}{\mu^{m+\theta}}\Big).$$

At last

$$\begin{aligned}|M_4| \leq & \frac{C}{\mu^{m+\frac{m}{2}+\theta}} \int_{\mathbb{R}^N} \zeta(|y-x^1|^m+1)\frac{1}{(1+|y-x^1|)^{\frac{3N}{2}+2}} \\ \leq & \frac{C}{\mu^{m+\frac{m}{2}+\theta-\frac{m(2m-N-4)^+}{2N-3}}} + \frac{Ck}{\mu^{m+1}} = O(\frac{1}{\mu^{m+\theta}}).\end{aligned}$$

Therefore, by the estimates for M_i, $i=1,2,3,4$, and [40, Proposition A.2], we obtain the first estimate. We can also prove the other result by similar arguments. □

Lemma 5.2. *There exists $\theta > 0$ such that*

$$\begin{aligned}\widetilde{c}_1 =& -A_1\frac{|x^1|-\mu r_0}{\lambda^{m-2}\mu^m} + O\Big(\frac{1}{\mu^{m+\theta}} + \frac{(m-2)^+}{\mu^m}\left(||x^1|-\mu r_0||^{m-1} + ||x^1|-\mu r_0||^2\right)\Big),\\ \widetilde{d}_1 =& -A_2\frac{|y^1|-\mu\rho_0}{\nu^{m-1}\mu^m} + O\Big(\frac{1}{\mu^{m+\theta}} + \frac{(m-2)^+}{\mu^m}\left(||y^1|-\mu r_0||^{m-1} + ||y^1|-\mu\rho_0||^2\right)\Big),\end{aligned}$$

where $A_i > 0, i = 1,2$ are some constants.

Proof. Similar to (5.2), we obtain

$$\Big|\int_{\mathbb{R}^N} |u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1}\Big| = O\Big(\frac{1}{\mu^{N-2}}\Big). \tag{5.5}$$

It follows that

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big(-\Delta u_1 - K_1(\frac{|y|}{\mu})|u_1|^{2^*-2}u_1\Big)\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1}\\ =& -\int_{\mathbb{R}^N} K_1(\frac{|y|}{\mu})|u_1|^{2^*-2}u_1\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1} + (2^*-1)\int_{\mathbb{R}^N}\varphi U_{x^1,\lambda}^{2^*-2}\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1} + \sum_{i=1}^k U_{x^i,\lambda}^{2^*-1}\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1}\\ &-\int_{\mathbb{R}^N}\varphi\left(\frac{\partial U_{x^1,\lambda}}{\partial y_1}\Delta\zeta + 2\nabla\zeta\nabla\frac{\partial U_{x^1,\lambda}}{\partial y_1}\right)\\ =&\int_{\mathbb{R}^N}\left(1-K_1(\frac{|y|}{\mu})\right)U_{r,\lambda}^{2^*-1}\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1} + N_1+N_2+N_3+N_4,\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}N_1 =& -\int_{\mathbb{R}^N}\left(U_{r,\lambda}^{2^*-1} - \sum_{i=1}^k U_{x^i,\lambda}^{2^*-1}\right)\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1},\\ N_2 =& -\int_{\mathbb{R}^N} K_1(\frac{|y|}{\mu})\left(|u_1|^{2^*-2}u_1 - U_{r,\lambda}^{2^*-1} - (2^*-1)U_{x^1,\lambda}^{2^*-2}\varphi\right)\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1},\\ N_3 =& -\int_{\mathbb{R}^N}\varphi\left(\frac{\partial U_{x^1,\lambda}}{\partial y_1}\Delta\zeta + 2\nabla\zeta\nabla\frac{\partial U_{x^1,\lambda}}{\partial y_1}\right),\\ N_4 =& -(2^*-1)\int_{\mathbb{R}^N}\left(K_1\left(\frac{|y|}{\mu}\right)-1\right)U_{x^1,\lambda}^{2^*-2}\varphi\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1}.\end{aligned}$$

Similar to M_i, $i=2,3,4$ in the proof of Lemma 5.1, we also get $|N_i| = O\Big(\frac{1}{\mu^{m+\theta}}\Big)$, $i=2,3,4$.

Next, we estimate N_1. For $y \in B_{2\mu^{\frac{m}{N-3/2}}}(x^1)$, there holds

$$\frac{\sum_{j=2}^k U_{x^j,\lambda}}{U_{x^1,\lambda}} = O\Big(\frac{1}{\mu^{\frac{m}{2N-3}}}\Big).$$

Then

$$U_{r,\lambda}^{2^*-1} - U_{x^1,\lambda}^{2^*-1} = (2^*-1)U_{x^1,\lambda}^{2^*-2}\sum_{j=2}^k U_{x^j,\lambda} + O\left(U_{x^1,\lambda}^{2^*-3}\left(\sum_{j=2}^k U_{x^j,\lambda}\right)^2\right).$$

By (A.4), we have

$$\begin{aligned}\int_{\mathbb{R}^N} U_{x^1,\lambda}^{2^*-3}\left(\sum_{j=2}^k U_{x^j,\lambda}\right)^2 \zeta\left|\frac{\partial U_{x^1,\lambda}}{\partial y_1}\right| =& \frac{1}{2^*-2}\int_{\mathbb{R}^N}\left(\sum_{j=2}^k U_{x^j,\lambda}\right)^2 \zeta\left|\frac{\partial U_{x^1,\lambda}^{2^*-2}}{\partial y_1}\right| \\ \le& C\int_{\Omega_1}\sum_{j=2}^k \frac{1}{(1+|y-x^j|)^{2N-4-\tau_1}}\frac{1}{(1+|y-x^1|)^5} \\ \le& C\sum_{j=2}^k \frac{1}{|x^1-x^j|^N}\int_{\Omega_1}\frac{1}{(1+|y-x^1|)^{N+1-\tau_1}} = O(\frac{1}{\mu^{m+\theta}}).\end{aligned}$$

We also have

$$\int_{\mathbb{R}^N}\sum_{j=2}^k U_{x^j,\lambda}^{2^*-1}\zeta\left|\frac{\partial U_{x^1,\lambda}}{\partial y_1}\right| \le C\sum_{j=2}^k\int_{\Omega_1}\frac{1}{(1+|y-x^j|)^{N+2}}\frac{1}{(1+|y-x^1|)^{N-1}} = O(\frac{1}{\mu^{m+\theta}}).$$

Thus,

$$N_1 = -\sum_{j=2}^k\int_{\mathbb{R}^N} U_{x^j,\lambda}\zeta\frac{\partial U_{x^1,\lambda}^{2^*-1}}{\partial y_1} + O(\frac{1}{\mu^{m+\theta}}).$$

Integration by parts, there holds for $j \neq 1$,

$$\left|\int_{\mathbb{R}^N} U_{x^j,\lambda}\zeta\frac{\partial U_{x^1,\lambda}^{2^*-1}}{\partial y_1}\right| \le \int_{\mathbb{R}^N}|\nabla U_{x^j,\lambda}|\zeta U_{x^1,\lambda}^{2^*-1} + \int_{\mathbb{R}^N} U_{x^j,\lambda}|\nabla\zeta| U_{x^1,\lambda}^{2^*-1}.$$

On one hand we have

$$\int_{\mathbb{R}^N}|\nabla U_{x^j,\lambda}|\zeta U_{x^1,\lambda}^{2^*-1} \le C\int_{\Omega_1}\frac{1}{(1+|y-x^j|)^{N-1}}\frac{1}{(1+|y-x^1|)^{N+2}} \le \frac{C}{|x^1-x^j|^{N-1}}.$$

On the other hand we have

$$\int_{\mathbb{R}^N} U_{x^j,\lambda}|\nabla\zeta| U_{x^1,\lambda}^{2^*-1} \le \frac{C}{\mu^{\frac{2m}{2N-3}}|x^1-x^j|^{N-2}}\int_{\mathbb{R}^N\setminus B_{\mu^{\frac{2m}{2N-3}}}(x^1)} U_{x^1,\lambda}^{2^*-1} \le \frac{C}{\mu^{\frac{6m}{2N-3}}|x^1-x^j|^{N-2}}.$$

Therefore,

$$|N_1| = O(\sum_{j=2}^k\frac{1}{|x^1-x^j|^{N-1}} + \sum_{j=2}^k\frac{1}{\mu^{\frac{6m}{2N-3}}|x^1-x^j|^{N-2}}) + O(\frac{1}{\mu^{m+\theta}}) = O(\frac{1}{\mu^{m+\theta}}).$$

Then, we estimate the main term. In supp ζ, we have

$$U_{r,\lambda}^{2^*-1} = \big(1+O(\mu^{-\frac{m}{2N-3}})\big)U_{x^1,\lambda}^{2^*-1},$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(1-K_1\big(\frac{|y|}{\mu}\big)\right)U_{r,\lambda}^{2^*-1}\zeta\frac{\partial U_{x^1,\lambda}}{\partial y_1}\\
=&\frac{1}{2^*}\frac{c_{0,1}}{\mu^m}\int_{\mathbb{R}^N}||y|-\mu r_0|^m\zeta\frac{\partial U_{x^1,\lambda}^{2^*}}{\partial y_1}+O\left(\frac{1}{\mu^{m+\theta}}\int_{\mathbb{R}^N}||y|-\mu r_0|^{m+\theta_1}\zeta\left|\frac{\partial U_{x^1,\lambda}^{2^*}}{\partial y_1}\right|\right),
\end{aligned}$$

for some $\theta\in(0,\min\{\theta_1,\frac{m}{2N-3}\})$. We can check that

$$\int_{\mathbb{R}^N}||y|-\mu r_0|^{m+\theta_1}\zeta\left|\frac{\partial U_{x^1,\lambda}^{2^*}}{\partial y_1}\right|\le C\int_{\mathbb{R}^N}\frac{1+|y-x^1|^{m+\theta_1}}{(1+|y-x^1|)^{2N+1}}\zeta\le C. \tag{5.7}$$

Integrating by parts, we have

$$\int_{\mathbb{R}^N}||y|-\mu r_0|^m\zeta\frac{\partial U_{x^1,\lambda}^{2^*}}{\partial y_1}=-m\int_{\mathbb{R}^N}||y|-\mu r_0|^{m-2}(|y|-\mu r_0)\frac{y_1}{|y|}\zeta U_{x^1,\lambda}^{2^*}-\int_{\mathbb{R}^N}||y|-\mu r_0|^mU_{x^1,\lambda}^{2^*}\frac{\partial\zeta}{\partial y_1}.$$

Similar to (5.7), there holds

$$\left|\int_{\mathbb{R}^N}||y|-\mu r_0|^mU_{x^1,\lambda}^{2^*}\frac{\partial\zeta}{\partial y_1}\right|\le\frac{C}{\mu^{\frac{2m}{2m-3}}}\int_{\mathbb{R}^N}\frac{|y-x^1|^m}{(1+|y-x^1|)^{2N}}=O(\frac{1}{\mu^\theta}).$$

We then estimate

$$\begin{aligned}
\mathcal{I}:=&\int_{\mathbb{R}^N}||y|-\mu r_0|^{m-2}(|y|-\mu r_0)\frac{y_1}{|y|}\zeta U_{x^1,\lambda}^{2^*}\\
=&\int_{\mathbb{R}^N}||z+x^1|-\mu r_0|^{m-2}(|z+x^1|-\mu r_0)\frac{z_1+r}{|z+x^1|}\tilde\zeta U_{0,\lambda}^{2^*},
\end{aligned}$$

where $z=(z_1,z_*)\in\mathbb{R}\times\mathbb{R}^{N-1}$ and $\tilde\zeta(z)=\zeta(z+x^1)$. For $z\in\operatorname{supp}\tilde\zeta=B_{2\mu^{\frac{2m}{2N-3}}}(0)$, we have

$$|z+x^1|=\sqrt{r^2+2z_1r+|z|^2}=r+z_1+O(\frac{|z|^2}{r}),$$

where $r=|x^1|$. Therefore,

$$\begin{aligned}
\mathcal{I}=&\int_{\mathbb{R}^N}||z+x^1|-\mu r_0|^{m-2}(|z+x^1|-\mu r_0)(1+O(\frac{|z|^2}{r^2}))\tilde\zeta U_{0,\lambda}^{2^*}\\
=&\int_{\mathbb{R}^N}||z+x^1|-\mu r_0|^{m-2}(|z+x^1|-\mu r_0)\tilde\zeta U_{0,\lambda}^{2^*}+O(\frac{1}{r^2}\int_{\mathbb{R}^N}\frac{1}{(1+|z|)^{2N-m-1}})\\
=&\int_{\mathbb{R}^N}||z+x^1|-\mu r_0|^{m-2}(|z+x^1|-\mu r_0)\tilde\zeta U_{0,\lambda}^{2^*}+O(\frac{1}{\mu^\theta}).
\end{aligned}$$

Without loss of generality, we can assume that $\tilde\zeta$ is radially symmetric. Then

$$\int_{\mathbb{R}^N}|z_1|^{m-2}z_1\tilde\zeta U_{0,\lambda}^{2^*}=0.$$

Note also that

$$\int_{\mathbb{R}^N}|z_1|^{m-2}(1-\tilde\zeta)U_{0,\lambda}^{2^*}=O(\int_{|z|\ge\mu^{\frac{2m}{2N-3}}}\frac{1}{(1+|z|)^{2N-m+2}})=O(\frac{1}{\mu^\theta}).$$

If $m=2$, then

$$\mathcal{I}=\int_{\mathbb{R}^N}(r-\mu r_0)\tilde\zeta U_{0,\lambda}^{2^*}+O(\frac{1}{\mu^\theta})=(r-\mu r_0)\int_{\mathbb{R}^N}U_{0,1}^{2^*}+O(\frac{1}{\mu^\theta}). \tag{5.8}$$

If $m>2$, by Lemma A.5, we obtain

$$\begin{aligned}&||z+x^1|-\mu r_0|^{m-2}(|z+x^1|-\mu r_0)=|z_1|^{m-2}z_1+(m-1)|z_1|^{m-2}(r-\mu r_0+O(\frac{|z|^2}{r}))\\&+O(|r-\mu r_0|^{m-1})+O(\frac{|z|^{2m-2}}{r^{m-1}})+O((m-3)^+|z_1|^{m-3}(\frac{|z|^4}{r^2}+|r-\mu r_0|^2))\\=&|z_1|^{m-2}z_1+(m-1)|z_1|^{m-2}(r-\mu r_0)\\&+O(\frac{|z|^m}{r})+O((1+|z|)^{m-3}(|r-\mu r_0|^{m-1}+|r-\mu r_0|^2)).\end{aligned}$$

We can also check that

$$\int_{\mathbb{R}^N}\frac{|z|^m}{r}\tilde{\zeta}U_{0,\lambda}^{2^*}=O(\frac{1}{\mu^\theta}),\quad \int_{\mathbb{R}^N}(1+|z|)^{m-3}\tilde{\zeta}U_{0,\lambda}^{2^*}=O(1).$$

Therefore, for the case $m>2$, there holds

$$\begin{aligned}\mathcal{I}=&(m-1)(r-\mu r_0)\int_{\mathbb{R}^N}|z_1|^{m-2}U_{0,\lambda}^{2^*}+O(\frac{1}{\mu^\theta}+|r-\mu r_0|^{m-1}+|r-\mu r_0|^2)\\=&\frac{(m-1)(r-\mu r_0)}{\lambda^{m-2}}\int_{\mathbb{R}^N}|z_1|^{m-2}U_{0,1}^{2^*}+O(\frac{1}{\mu^\theta}+|r-\mu r_0|^{m-1}+|r-\mu r_0|^2).\end{aligned}\tag{5.9}$$

Combining (5.5), (5.6), (5.8) and (5.9), the conclusion follows. □

Proof of Theorem 1.2 and Theorem 1.1. From [40], there is a constant $B_5>0$ such that

$$\sum_{i=2}^k\frac{1}{|x^i-x^1|^{N-2}}=\frac{B_5k^{N-2}}{|x^1|^{N-2}}+O\Big(\frac{k}{|x^1|^{N-2}}\Big).$$

Similarly,

$$\sum_{i=2}^k\frac{1}{|y^i-y^1|^{N-2}}=\frac{B_6k^{N-2}}{|y^1|^{N-2}}+O\Big(\frac{k}{|y^1|^{N-2}}\Big),$$

where $B_6>0$ is a constant. Therefore, by Lemmas 5.1 and 5.2,

$$\begin{aligned}\widetilde{c}_2=&-\frac{B_1}{\lambda^{m+1}\mu^m}+\frac{B_5k^{N-2}}{\lambda^{N-1}r^{N-2}}+O\Big(\frac{1}{\mu^{m+\theta}}+\frac{1}{\mu^m}|\mu r_0-|x^1||^2+\frac{k}{r^{N-2}}\Big),\\\widetilde{d}_2=&-\frac{B_3}{\nu^{m+1}\mu^m}+\frac{B_6k^{N-2}}{\nu^{N-1}\rho^{N-2}}+O\Big(\frac{1}{\mu^{m+\theta}}+\frac{1}{\mu^m}|\mu\rho_0-|y^1||^2+\frac{k}{\rho^{N-2}}\Big),\\\widetilde{c}_1=&-A_1\frac{|x^1|-\mu r_0}{\lambda^{m-2}\mu^m}+O\Big(\frac{1}{\mu^{m+\theta}}+\frac{(m-2)^+}{\mu^m}\left(||x^1|-\mu r_0||^{m-1}+||x^1|-\mu r_0||^2\right)\Big),\\\widetilde{d}_1=&-A_2\frac{|y^1|-\mu\rho_0}{\nu^{m-1}\mu^m}+O\Big(\frac{1}{\mu^{m+\theta}}+\frac{(m-2)^+}{\mu^m}\left(||y^1|-\mu r_0||^{m-1}+||y^1|-\mu\rho_0||^2\right)\Big).\end{aligned}$$

Let

$$\lambda_0=\Big(\frac{B_5}{B_1r_0^{N-2}}\Big)^{\frac{1}{N-2-m}},\quad \nu_0=\Big(\frac{B_6}{B_3\rho_0^{N-2}}\Big)^{\frac{1}{N-2-m}}.$$

by the solution of

$$-\frac{B_1}{\lambda^{m+1}}+\frac{B_5}{\lambda^{N-1}r_0^{N-2}}=0,\quad -\frac{B_3}{\nu^{m+1}}+\frac{B_6}{\nu^{N-1}\rho_0^{N-2}}=0.$$

Denote

$$\mathbb{M}:=\Big[r_0\mu-\frac{1}{\mu^{\bar\theta}},r_0\mu+\frac{1}{\mu^{\bar\theta}}\Big]\times\Big[\rho_0\mu-\frac{1}{\mu^{\bar\theta}},\rho_0\mu+\frac{1}{\mu^{\bar\theta}}\Big]\times\Big[\lambda_0-\frac{1}{\mu^{\frac32\bar\theta}},\lambda_0+\frac{1}{\mu^{\frac32\bar\theta}}\Big]\times\Big[\nu_0-\frac{1}{\mu^{\frac32\bar\theta}},\nu_0+\frac{1}{\mu^{\frac32\bar\theta}}\Big].$$

where $0 < \bar\theta < \min\{\theta, N-2-m-\tau_1\}/\max\{4, 2m-2\}$. We find a zero of the map $(r,\lambda,\rho,\nu) \mapsto (\widetilde{c}_1, \widetilde{c}_2, \widetilde{d}_1, \widetilde{d}_2)$ on $\mathbb{M}$.

For $(r,\lambda,\rho,\nu) \in \mathbb{M}$, there holds

$$\begin{aligned}\widetilde{c}_2 =& -\frac{B_1}{\lambda^{m+1}\mu^m} + \frac{B_5}{\lambda^{N-1} r_0^{N-2}} \frac{1}{\mu^m} + O\Big(\frac{1}{\mu^{m+2\bar\theta}}\Big),\\ \widetilde{c}_1 =& - A_1 \frac{r-\mu r_0}{\mu^m} + O\Big(\frac{1}{\mu^{m+2\bar\theta}}\Big).\end{aligned}$$

If $\lambda = \lambda_0 - \frac{1}{\mu^{\frac{3}{2}\bar\theta}}$, then

$$\begin{aligned}\widetilde{c}_2 =& -\frac{B_1}{(\lambda_0 - \frac{1}{\mu^{\frac{3}{2}\bar\theta}})^{m+1}\mu^m} + \frac{B_5}{(\lambda_0 - \frac{1}{\mu^{\frac{3}{2}\bar\theta}})^{N-1} r_0^{N-2}} \frac{1}{\mu^m} + O\Big(\frac{1}{\mu^{m+2\bar\theta}}\Big)\\ =& -\frac{B_1}{\lambda_0^{m+1}\mu^m} + \frac{B_5}{\lambda_0^{N-1} r_0^{N-2}} \frac{1}{\mu^m} + (N-m-2)\frac{B_1}{\lambda_0^{m+2}\mu^{m+\frac{3}{2}\bar\theta}} + O\Big(\frac{1}{\mu^{m+2\bar\theta}}\Big)\\ =&(N-2-m)\frac{B_1}{\lambda_0^{m+2}\mu^{m+\frac{3}{2}\bar\theta}} + O\Big(\frac{1}{\mu^{m+2\bar\theta}}\Big) > 0.\end{aligned}$$

If $r = r_0\mu - \dfrac{1}{\mu^{\bar\theta}}$, then

$$\widetilde{c}_1 = -A_1 \frac{1}{\mu^m}\Big(r_0\mu - \frac{1}{\mu^{\bar\theta}} - \mu r_0\Big) + O\Big(\frac{1}{\mu^{m+2\bar\theta}}\Big) = A_1 \frac{1}{\mu^{m+\bar\theta}} + O\Big(\frac{1}{\mu^{m+2\bar\theta}}\Big) > 0,$$

Similarly, we can get $\widetilde{c}_2 < 0$ if $\lambda = \lambda_0 + \frac{1}{\mu^{\frac{3}{2}\bar\theta}}$; $\widetilde{d}_2 > 0$ if $\nu = \nu_0 - \frac{1}{\mu^{\frac{3}{2}\bar\theta}}$; $\widetilde{d}_2 < 0$ if $\nu = \nu_0 + \frac{1}{\mu^{\frac{3}{2}\bar\theta}}$. $\widetilde{c}_1 < 0$ if $r = r_0\mu + \dfrac{1}{\mu^{\bar\theta}}$; $\widetilde{d}_1 > 0$ if $\rho = \rho_0\mu - \dfrac{1}{\mu^{\bar\theta}}$ $\widetilde{d}_1 < 0$ if $\rho = \rho_0\mu + \dfrac{1}{\mu^{\bar\theta}}$.

Then by Miranda's theorem [30, Theorem 3.1], the map $(r,\lambda,\rho,\nu) \mapsto (\widetilde{c}_1, \widetilde{c}_2, \widetilde{d}_1, \widetilde{d}_2)$ has at least one zero point in $\mathbb{M}$.

Finally, to see $u_{1,k}, u_{2,k}$ are non-negative, by $(u_{1,k}, u_{2,k}) \in \Lambda_k$, we know $\operatorname{supp} u_{1,k}^- \subset \mathbb{R}^N \setminus P_2$ and $\operatorname{supp} u_{2,k}^- \subset \mathbb{R}^N \setminus Q_2$. Testing the first equation in (3.4) by $u_{1,k}^-$, we have

$$\|\nabla u_{1,k}^-\|_{L^2}^2 \le C\|u_{1,k}^-\|_{L^{2^*}}^{2^*} \le \|u_{1,k}\|_{L^{2^*}(\mathbb{R}^N\setminus P_2)}^{2^*-2} \|\nabla u_{1,k}^-\|_{L^2}^2.$$

By Lemma 3.1, $\|u_{1,k}\|_{L^{2^*}(\mathbb{R}^N\setminus P_2)}^{2^*-2} \to 0$ as $k \to +\infty$. So $u_{1,k}^- = 0$. Similarly, we also get $u_{2,k}^- = 0$, concluding the proof. □

Proof of Theorem 1.3. In the outer region, from Lemma 3.5, we can obtain Theorem 1.3. Using the operator S, in the inner region, we also obtain the solution $(u_{1,k}, u_{2,k})$ defined in Theorem 1.2 through reduction method. Thus, we complete the proof. □

Appendix A. Basic Estimates

In this section, we collect several fundamental and useful estimates that will be used repeatedly throughout the paper. The Lemmas A.1–A.3 stated below are adapted from [40].

Lemma A.1. *For any constant $0 < \tau \le \min\{\alpha_1, \alpha_2\}$, there is a constant $C > 0$ such that*

$$\frac{1}{(1+|y-x^i|)^{\alpha_1}} \frac{1}{(1+|y-y^i|)^{\alpha_2}} \le \frac{C}{|x^i - y^i|^\tau}\Big(\frac{1}{(1+|y-x^i|)^{\alpha_1+\alpha_2-\tau}} + \frac{1}{(1+|y-y^i|)^{\alpha_1+\alpha_2-\tau}}\Big).$$

Lemma A.2. *For any constant $\sigma > 0$ with $\sigma \neq N-2$, there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} dz \leq \frac{C}{(1+|y|)^{\min\{\sigma, N-2\}}}.$$

Lemma A.3. *Let $r \in [r_0\mu - 1, r_0\mu + 1]$, $\rho \in [\rho_0\mu - 1, \rho_0\mu + 1]$. Then there exists $\theta > 0$ small enough such that as $k \to +\infty$,*

$$K_1(\frac{y}{\mu}) U_{r,\lambda}^{2^*-1} - \sum_{i=1}^{k} U_{x^i,\lambda}^{2^*-1} = O\Big(\frac{1}{\mu^{\frac{m}{2}+\theta}}\Big) \sum_{i=1}^{k} \frac{1}{(1+|y-x^i|)^{\frac{N}{2}+2}}.$$

From [40], we also obtain that for $\tau_1 = \frac{N-2-m}{N-2}$ and $\tau > 1$, there is some $C > 0$ independent of k such that

$$\sum_{i=2}^{k} \frac{1}{|x^i - x^1|^\tau} + \sum_{i=2}^{k} \frac{1}{|y^i - y^1|^\tau} \leq C \left(\frac{k}{\mu}\right)^\tau \leq C\mu^{-\frac{m\tau}{N-2}}. \tag{A.1}$$

$$\sum_{i=2}^{k} \frac{1}{|x^i - x^1|^{\tau_1}} \leq C \text{ and } \sum_{i=1}^{k} \frac{1}{(1+|y-x^i|)^{\tau_1}} \leq C. \tag{A.2}$$

Moreover,

$$\sum_{i=2}^{k} \frac{1}{|y^i - y^1|^{\tau_1}} \leq C \text{ and } \sum_{i=1}^{k} \frac{1}{(1+|y-y^i|)^{\tau_1}} \leq C. \tag{A.3}$$

We also use the following important basic estimates in this article. By Hölder's inequality, for $\alpha \geq \tau_1$, $\beta_1 > 1$, there holds

$$\begin{aligned}
\left(\sum_{i=i_0}^{k} \frac{1}{(1+|y-x^i|)^\alpha}\right)^{\beta_1} &= \left(\sum_{i=i_0}^{k} \frac{1}{(1+|y-x^i|)^{\alpha - \frac{\beta_1 - 1}{\beta_1}\tau_1}} \frac{1}{(1+|y-x^i|)^{\frac{\beta_1-1}{\beta_1}\tau_1}}\right)^{\beta_1} \\
&\leq \sum_{i=i_0}^{k} \frac{1}{(1+|y-x^i|)^{\alpha\beta_1 - (\beta_1-1)\tau_1}} \left(\sum_{i=i_0}^{k} \frac{1}{(1+|y-x^i|)^{\tau_1}}\right)^{\beta_1 - 1} \\
&\leq C \sum_{i=i_0}^{k} \frac{1}{(1+|y-x^i|)^{\alpha\beta_1 - (\beta_1-1)\tau_1}},
\end{aligned} \tag{A.4}$$

where $i_0 = 1$ or 2. For $\alpha > 0$, define

$$\varpi_{r,\alpha}(y) := \sum_{j=1}^{k} \frac{1}{(1+|y-x^j|)^\alpha}. \tag{A.5}$$

With this notation, the inequality (A.4) implies

$$\varpi_{r,\alpha}^{\beta_1} \leq \varpi_{r,\alpha\beta_1 - (\beta_1-1)\tau_1} \quad \text{for } \alpha \geq \tau_1,\ \beta_1 > 1. \tag{A.6}$$

Note that $\varpi_{r,\alpha}$ is decreasing with respect to α. Moreover, since $y \in \mathbb{R}^N \setminus P_1$, there holds

$$\varpi_{r,\alpha} \leq \sigma_k^{\alpha'-\alpha} \varpi_{r,\alpha'}, \quad \text{for } 0 < \alpha' < \alpha. \tag{A.7}$$

We define

$$\varpi_{\rho,\alpha}(y) := \sum_{j=1}^{k} \frac{1}{(1+|y-y^j|)^\alpha}, \tag{A.8}$$

and

$$\varpi_\alpha(y) := \varpi_{r,\alpha}(y) + \varpi_{\rho,\alpha}(y). \tag{A.9}$$

The functions $\varpi_{\rho,\alpha}$ and ϖ_α inherit the same properties as $\varpi_{r,\alpha}$—namely, (A.6) and (A.7)—in $\mathbb{R}^N \setminus Q_1$ and $\mathbb{R}^N \setminus (P_1 \cup Q_1)$, respectively.

Lemma A.4. *There holds*

$$\begin{aligned} &-\Delta\varpi_{r,N-2} \geq C\sigma_k^{1-\frac{4\tau_1}{N-2}}\varpi_{r,N-2}^{2^*-1}, && \text{in } \mathbb{R}^N \setminus P_1,\\ &-\Delta\varpi_{\rho,N-2} \geq C\sigma_k^{1-\frac{4\tau_1}{N-2}}\varpi_{r,N-2}^{2^*-1}, && \text{in } \mathbb{R}^N \setminus Q_1,\\ &-\Delta\varpi_{N-2} \geq C\sigma_k^{1-\frac{4\tau_1}{N-2}}\varpi_{N-2}^{2^*-1}, && \text{in } \mathbb{R}^N \setminus (P_1 \cup Q_1). \end{aligned} \tag{A.10}$$

For $\alpha \in [\frac{N}{2} - 1 + \tau_1, N-2)$,

$$\begin{aligned} -\Delta\varpi_{r,\alpha} &\geq C\sigma_k^{4-\frac{4\alpha}{N-2}}\varpi_{r,N-2}^{2^*-2}\varpi_{r,\alpha} + \frac{1}{2}\alpha(N-2-\alpha)\varpi_{r,\alpha+2}, && \text{in } \mathbb{R}^N \setminus P_1,\\ -\Delta\varpi_{\rho,\alpha} &\geq C\sigma_k^{4-\frac{4\alpha}{N-2}}\varpi_{\rho,N-2}^{2^*-2}\varpi_{\rho,\alpha} + \frac{1}{2}\alpha(N-2-\alpha)\varpi_{\rho,\alpha+2}, && \text{in } \mathbb{R}^N \setminus Q_1. \end{aligned} \tag{A.11}$$

Proof. For $\alpha \in (0, N-2]$, $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$-\Delta(1+|y|)^{-\alpha} = \alpha(1+|y|)^{-\alpha-2}\left(\frac{N-1}{|y|} + N - 2 - \alpha\right). \tag{A.12}$$

Therefore,

$$-\Delta\varpi_{r,N-2} \geq (N-1)(N-2)\varpi_{r,N+1}.$$

On the other hand, by (A.6) and (A.7),

$$\varpi_{r,N-2}^{2^*-1} \leq C\varpi_{r,N+2-\frac{4\tau_1}{N-2}} \leq C\sigma_k^{\frac{4\tau_1}{N-2}-1}\varpi_{r,N+1}.$$

Hence, we obtain the first inequality of (A.10).

Now let $\alpha \in [\frac{N}{2} - 1 + \tau_1, N-2)$. Then by (A.12),

$$-\Delta\varpi_{r,\alpha} \geq \alpha(N-2-\alpha)\varpi_{r,\alpha+2}.$$

By (A.6) and (A.7) again and since $\frac{4(\alpha-\tau_1)}{N-2} \geq 2$,

$$\varpi_{r,N-2}^{2^*-2}\varpi_{r,\alpha} \leq \sigma_k^{\frac{4\alpha}{N-2}-4}\varpi_{r,\alpha}^{2^*-1} \leq C\sigma_k^{\frac{4\alpha}{N-2}-4}\varpi_{r,\alpha+\frac{4(\alpha-\tau_1)}{N-2}} \leq C\sigma_k^{\frac{4\alpha}{N-2}-4}\varpi_{r,\alpha+2}.$$

Therefore, we get the first inequality of (A.11). As our arguments only depends on the properties (A.6) and (A.7), other inequalities in (A.10), (A.11) follow in a same way. □

The following inequality will be used.

Lemma A.5. *Let* $m_0 > 2$. *Then there is* $C > 0$ *such that for* $a, b \in \mathbb{R}$,

$$\left||a+b|^{m_0-2}(a+b) - |a|^{m_0-2}a - (m_0-1)|a|^{m_0-2}b\right| \leq C\begin{cases} |b|^{m_0-1} & \text{if } 2 < m_0 < 3,\\ |a|^{m_0-3}b^2 + |b|^{m_0-1} & \text{if } m_0 \geq 3. \end{cases}$$

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