UNIVERSAL KERNEL MODELS FOR ITERATED COMPLETELY POSITIVE MAPS

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Abstract. We study how iterated and composed completely positive maps act on operator-valued kernels. Each kernel is realized inside a single Hilbert space where composition corresponds to applying bounded creation operators to feature vectors. This model yields a direct formula for every iterated kernel and allows pointwise limits, contractive behavior, and kernel domination to be read as standard operator facts. The main results include an explicit limit kernel for unital maps, a Stein-type decomposition, a Radon-Nikodym representation under subunitality, and an almost-sure growth law for random compositions. The construction keeps all iterates in one space, making their comparison and asymptotic analysis transparent.

Contents

1. Introduction

Positive definite (p.d.) kernels and completely positive (CP) maps are usually studied in separate settings. Kernels describe geometric relationships through inner products in reproducing kernel Hilbert spaces, while CP maps describe structure and dynamics in operator algebras and quantum channels. This paper connects these ideas by showing how iterated CP maps act naturally on operator-valued kernels and how all such iterates can be represented within a single Hilbert space built from the initial kernel.

A p.d. kernel assigns to each pair of points an operator or scalar in a way that guarantees positivity of all finite Gram matrices. Such kernels generate Hilbert spaces where functions are evaluated by inner products, a construction now standard in functional analysis and machine learning. (See, e.g,. [Aro50, BTA04,

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CdVTU10, ARL12].) Operator-valued kernels extend this picture by allowing values in $B(H)$, which has been developed in the last two decades in work on vector-valued learning, operator spaces, and noncommutative function theory [Aro50, BBF07, BMV16]. On the other hand, completely positive maps are central objects in operator theory and quantum information. They preserve positivity at all matrix levels, and Stinespring's theorem describes them by isometries or Kraus operators acting on larger Hilbert spaces [Sti55, Arv69, Cho75, EHeK78]. When such maps are iterated, they produce linear but generally non-unitary dynamics on operator algebras and are used to model repeated quantum evolutions and open-system channels [KT13, BCG$^+$13, BJM14, ABM24].

The main motivation of the present work is to understand what happens when a family of completely positive maps acts repeatedly on an operator-valued p.d. kernel. Each map modifies the kernel by pushing its operator entries forward through its Kraus operators. Composing several such maps corresponds to following different sequences of these transformations. Rather than studying every composition in isolation, we construct a single Hilbert space in which all such iterates can be represented at once. In this space, each map acts by a bounded linear operator, formally a "creation" operator, on labelled copies of the original kernel sections. Any finite composition of maps then corresponds to multiplying the associated creation operators, and the resulting composed kernel appears as the inner product of the transformed feature vectors. Within this unified setting, contraction and invariance become operator-norm properties, and limits or asymptotic projections arise as standard geometric facts about these shifts.

This geometric view connects to several strands of current research. In operator theory, it parallels recent analyses of kernel dilations and Radon-Nikodym representations for completely positive maps [BBLS04, HJ10, BRS12, BDT13, SS14, BS15, MKP17]. In kernel theory, it complements the growing work on operator-valued and noncommutative kernels, which appear in harmonic analysis, control, and data-driven systems [BBF08, BDT13, BMV16]. In quantum information, the iteration of CP maps corresponds to repeated noisy evolution or to discrete semigroups of quantum channels. Descriptions of asymptotic fixed points, invariant subspaces, and convergence rates play a major role there. Our construction expresses these asymptotic properties directly in kernel form, using standard Hilbert space operations instead of algebraic manipulations.

The model also provides a clear link with ergodic results for random compositions of channels. When each step applies a random CP map, the product of the associated creation operators forms a subadditive process. Kingman's subadditive ergodic theorem ([Kin68, Kin73, BCG$^+$13, Kre85]) then gives a deterministic Lyapunov exponent controlling the exponential rate of growth or decay of the iterated kernels. This matches ideas already present in random matrix products and random quantum channels. By formulating the problem at the kernel level, we make the comparison between deterministic and random iteration more transparent.

The paper proceeds by first constructing the universal model and proving that each iterated kernel has the stated realization (Theorem 2.1). We then describe the convergence and Stein-type decomposition for unital maps (Theorem 3.1 and Theorem 3.4), the Radon-Nikodym representation for subunital cases (Theorem 4.2, with Lemma 4.1 as the kernel RN tool), and the random iteration results (Theorem 5.1 together with Corollaries 5.2–5.5). Throughout, the emphasis is on keeping all iterates inside one fixed space so that asymptotic behaviour and comparison become standard geometric facts (see also Corollaries 3.2-3.3 for contraction bounds).

2. Model Space and Realization

Our goal here is to build a single Hilbert space where every "composed kernel" lives and can be compared directly. We start from the given operator-valued kernel and pass to its scalar lift; this lets us view each data point (x, a) as a concrete feature vector. Next, for each completely positive map, we add a simple "creation" step that pushes those feature vectors forward along a labeled edge. Chaining steps corresponds to composing maps, where different chains record different "Kraus choices" along the way. The resulting model space is the orthogonal sum of all such chains, so inner products separate cleanly by where the chains land.

The payoff is a transparent realization, where every iterated kernel is the inner product of pushed-forward feature vectors in this space. Two immediate benefits follow. First, norm control of the CP maps translates into operator-norm bounds for the created shifts, hence for all iterates of the kernel. Second, we obtain a single, universal stage where limits (deterministic or random), Radon-Nikodym compressions, and maximality statements can be read off as standard facts about orthogonal projections and contractions. The formal statement and proof appear in Theorem 2.1 below. The rest of the paper keeps returning to this model whenever convergence or comparison is needed.

Let X be a set and H a complex Hilbert space. Throughout, all inner products are linear in the second variable. Let $K : X \times X \to B(H)$ be a positive definite (p.d.) kernel, i.e., for any finite family $\{(x_i, h_i)\}_{i=1}^{m} \subset X \times H$,

$$\sum_{i,j=1}^{m} \langle h_i, K(x_i, x_j) h_j \rangle_H \geq 0.$$

Define the scalar kernel $\tilde{K}$ on $X \times H$ by

$$\tilde{K}((x, a), (y, b)) := \langle a, K(x, y) b \rangle_H .$$

Indeed, $\tilde{K} : (X \times H)^2 \to \mathbb{C}$ is p.d., since

$$\begin{aligned}\sum_{i,j} \overline{c_i} c_j \tilde{K}((x_i, a_i), (x_j, a_j)) &= \sum_{i,j} \langle c_i a_i, K(x_i, x_j) c_j a_j \rangle_H \\ &= \sum_{i,j} \langle h_i, K(x_i, x_j) h_j \rangle_H , \quad h_j := c_j a_j,\end{aligned}$$

which holds for any finite family $\{(x_i, a_i)\}$ in $X \times H$, and $\{c_i\}$ in $\mathbb{C}$. The corresponding kernel sections are

$$\tilde{K}_{(x,a)}(\cdot) = \tilde{K}(\cdot, (x, a))$$

with

$$\left\langle \tilde{K}_{(x,a)}, \tilde{K}_{(y,b)} \right\rangle_{\mathcal{H}_{\tilde{K}}} = \langle a, K(x, y) b \rangle_H . \tag{2.1}$$

Let $\mathcal{H}_{\tilde{K}}$ be its reproducing kernel Hilbert space (RKHS), i.e.,

$$H_{\tilde{K}} = \overline{span} \left\{ \tilde{K}_{(x,a)} : x \in X, a \in H \right\} .$$

This is a space of functions $F : X \times H \to \mathbb{C}$, with the reproducing property

$$F(x,a) = \langle \tilde{K}_{(x,a)}, F\rangle_{H_{\tilde{K}}}, \qquad (x,a) \in X \times H.$$

Moreover, for all $x \in X$, setting $V_x : H \to H_{\tilde{K}}$ by $V_x a = \tilde{K}_{(x,a)}$ through the kernel sections, then (2.1) gives the Kolmogorov decomposition

$$K(x,y) = V_x^* V_y, \qquad x, y \in X.$$

For details of this scalar 'trick', we refer to [Ped58] and the recent survey [Sza21].

Fix a nonempty index set S. Let S^* denote the free monoid on S (finite words, including the empty word $\emptyset$), and use juxtaposition ww' for concatenation.

For each $s \in S$, let $\Phi_s : B(H) \to B(H)$ be a normal completely positive (CP) map, with a (possibly countable) Kraus family

$$\Phi_s(T) = \sum_{r \in I(s)} A_{s,r}^* T A_{s,r}$$

(converging in the strong operator topology) If Φ_s is unital, then $\sum_{r\in I(s)} A_{s,r}^* A_{s,r} = I$ (strongly).

For $w = s_1 s_2 \cdots s_n \in S^*$, define the iterated kernel by

$$K_w := \Phi_{s_1} \circ \Phi_{s_2} \circ \cdots \circ \Phi_{s_n}(K), \qquad K_\emptyset := K.$$

Define the index set

$$R := \bigsqcup_{w = s_1\cdots s_n \in S^*} I(s_1) \times \cdots \times I(s_n),$$

with the convention that for the empty word $\emptyset$, the corresponding factor is the singleton $\{\emptyset\}$. Thus, elements of R are finite index strings $\rho = (r_1, \ldots, r_n)$ with $r_j \in I(s_j)$ for some $w = s_1 \cdots s_n \in S^*$, and include the empty string $\emptyset$. For $r \in I(s)$ and an index string ρ, write $r\rho$ for left-concatenation (so $r\emptyset = r$).

Define the model space

$$M := l^2(R) \otimes \mathcal{H}_{\tilde{K}} \simeq \bigoplus_{\rho \in R} \mathcal{H}_{\tilde{K}}.$$

Let $\{e_\rho\}_{\rho\in R}$ denote the canonical orthonormal basis (ONB) for $l^2(R)$. We will use the notation $e_\rho \cdot F$ to denote the elementary tensor $e_\rho \otimes F$, which corresponds to the element F in the ρ-th summand.

Define $J_x : H \to M$ by

$$J_x(a) := e_\emptyset \cdot \tilde{K}_{(x,a)}.$$

For each $s \in S$, define a bounded operator $C_s : M \to M$ on the algebraic span by

$$C_s\left(e_\rho \cdot \tilde{K}_{(x,a)}\right) = \sum_{r\in I(s)} e_{r\rho} \cdot \tilde{K}_{(x, A_{s,r} a)}.$$

For any $w = s_1 \cdots s_n \in S^*$, set the right-regular product

$$C_w := C_{s_n} \cdots C_{s_1}, \qquad C_\emptyset := I_M. \tag{2.2}$$

Theorem 2.1. *Let K, $\{\Phi_s\}$, and $\{K_w\}$ be as defined above. Then the following hold:*

(1) *K_w is a $B(H)$-valued p.d. kernel.*

(2) *Each* C_s *is bounded with* $\|C_s\| \leq \|\Phi_s\|_{\mathrm{cb}}^{1/2}$*, and* $\|C_s\| = 1$ *if* Φ_s *is unital. Consequently,*

$$\|C_w\| \leq \prod_{j=1}^{n} \left\|\Phi_{s_j}\right\|_{\mathrm{cb}}^{1/2}, \quad \forall w = s_1 \cdots s_n \in S^*. \tag{2.3}$$

(3) K_w *admits the representation*

$$K_w(x, y) = J_x^* C_w^* C_w J_y. \tag{2.4}$$

Equivalently,

$$\langle a, K_w(x, y)\, b\rangle_H = \langle C_w J_x(a), C_w J_y(b)\rangle_M. \tag{2.5}$$

Proof. (1) Fix $w = s_1 \cdots s_n \in S^*$ and a finite family $\{(x_i, h_i)\}_{i=1}^m$ in $X \times H$. The block matrix $[K(x_i, x_j)] \in M_m(B(H))$ is positive. Each Φ_{s_k} is completely positive, so $(id_{M_m} \otimes \Phi_{s_k})$ preserves positivity. Iterating for $k = n, \ldots, 1$ yields $[K_w(x_i, x_j)] \geq 0$.

For (2), take a finite vector $u = \sum_i e_{\rho_i} f_i \in M$, with

$$f_i = \sum_j c_{ij} \tilde{K}_{(x_{ij}, a_{ij})} \in \mathcal{H}_{\tilde{K}}.$$

Using orthogonality of the direct-sum summands and the reproducing property,

$$\begin{aligned}
\|C_s u\|_M^2 &= \sum_i \sum_{r \in I(s)} \Big\| \sum_j c_{ij} \tilde{K}_{(x_{ij}, A_{s,r} a_{ij})} \Big\|_{\mathcal{H}_{\tilde{K}}}^2 \\
&= \sum_i \sum_{r \in I(s)} \sum_{j,k} \overline{c_{ij}} c_{ik} \left\langle \tilde{K}_{(x_{ij}, A_{s,r} a_{ij})}, \tilde{K}_{(x_{ik}, A_{s,r} a_{ik})} \right\rangle_{\mathcal{H}_{\tilde{K}}} \\
&= \sum_i \sum_{r \in I(s)} \sum_{j,k} \overline{c_{ij}} c_{ik} \langle A_{s,r} a_{ij}, K(x_{ij}, x_{ik})\, A_{s,r} a_{ik}\rangle_H \\
&= \sum_i \sum_{j,k} \overline{c_{ij}} c_{ik} \langle a_{ij}, \Phi_s(K(x_{ij}, x_{ik}))\, a_{ik}\rangle_H.
\end{aligned}$$

Recall that, for any positive block $[T_{jk}] \in M_N(B(H))$ and $\alpha \in H^N$, one has

$$\langle \alpha, [\Phi_s(T_{jk})]\, \alpha\rangle \leq \|\Phi_s\|_{\mathrm{cb}} \langle \alpha, [T_{jk}]\, \alpha\rangle.$$

Applying this to $T_{jk} = K(x_{ij}, x_{ik})$ gives

$$\|C_s u\|_M^2 \leq \|\Phi_s\|_{\mathrm{cb}} \|u\|_M^2,$$

hence $\|C_s\|_{M \to M} \leq \|\Phi\|_{\mathrm{cb}}^{1/2}$. If Φ_s is unital, then $\|\Phi_s\|_{\mathrm{cb}} = 1$ and $\|C_s\|_{M \to M} = 1$. The bound for C_w as in (2.3) then follows from submultiplicativity.

For (3), let $w = s_1 \cdots s_n \in S^*$. From $J_x(a) = e_\emptyset \cdot \tilde{K}_{(x,a)}$ and the right-regular product $C_w = C_{s_n} \cdots C_{s_1}$, we have

$$C_w J_x(a) = \sum_{(r_1, \ldots, r_n)} e_{(r_n, \ldots, r_1)} \cdot \tilde{K}_{\left(x, A_{s_n, r_n} \cdots A_{s_1, r_1} a\right)}.$$

By orthogonality of the direct-sum summands and the RKHS inner product,

$$\begin{aligned}\left\langle C_w J_x(a), C_w J_y(b)\right\rangle_M &= \sum_{(r_1,\dots,r_n)} \left\langle \tilde{K}_{(x, A_{s_n,r_n}\cdots A_{s_1,r_1}a)}, \tilde{K}_{(y, A_{s_n,r_n}\cdots A_{s_1,r_1}b)}\right\rangle_{\mathcal{H}_{\tilde{K}}} \\ &= \sum_{(r_1,\dots,r_n)} \left\langle A_{s_n,r_n}\cdots A_{s_1,r_1}a, K(x,y) A_{s_n,r_n}\cdots A_{s_1,r_1}b\right\rangle_H \\ &= \sum_{(r_1,\dots,r_n)} \left\langle a, A^*_{s_1,r_1}\cdots A^*_{s_n,r_n} K(x,y) A_{s_n,r_n}\cdots A_{s_1,r_1}b\right\rangle_H \\ &= \left\langle a, (\Phi_{s_1}\circ\cdots\circ\Phi_{s_n})(K(x,y))\, b\right\rangle_H \\ &= \left\langle a, K_w(x,y)\, b\right\rangle_H .\end{aligned}$$

The stated identities (2.4)-(2.5) follow. □

Let $\mathcal{D}$ denote the algebraic span of the vectors $\{C_w J_x a : w\in S^*, x\in X, a\in H\}$ inside the ambient model M. Equip $\mathcal{D}$ with the sesquilinear form

$$\left\langle C_w J_x a, C_v J_y b\right\rangle_{\mathcal{D}} := \left\langle C_w J_x a, C_v J_y b\right\rangle_M . \tag{2.6}$$

A triple $(\tilde{M}, \{\tilde{C}_s\}, \{\tilde{J}_x\})$ is called an admissible realization if the linear map

$$\mathcal{D}\longrightarrow \tilde{M}, \qquad C_w J_x a \longmapsto \tilde{C}_w \tilde{J}_x a$$

is isometric with respect to the form (2.6). It is minimal if

$$\tilde{M} = \overline{span}\left\{\tilde{C}_w \tilde{J}_x a : w\in S^*, x\in X, a\in H\right\}.$$

Proposition 2.2. *Let $(M, \{C_s\}, \{J_x\})$ be the model constructed in Theorem 2.1, and set*

$$M_0 := \overline{span}\{C_w J_x a : w\in S^*, x\in X, a\in H\}\subset M.$$

Then M_0 is invariant under each C_s and provides a minimal admissible realization. If $(\tilde{M}, \{\tilde{C}_s\}, \{\tilde{J}_x\})$ is another minimal admissible realization, there exists a unique unitary operator

$$U : M_0 \longrightarrow \tilde{M}$$

such that $UJ_x = \tilde{J}_x$ and $UC_s = \tilde{C}_s U$ for all $x\in X$, $s\in S$.

Proof. Since $C_s(C_w J_x a) = C_{ws} J_x a\in\mathcal{D}$ and $J_x a = C_\emptyset J_x a\in\mathcal{D}$, we have $M_0 = \overline{D}$ and M_0 is C_s-invariant.

Define $U_0 : \mathcal{D}\to\tilde{M}$ by $U_0(C_w J_x a) := \tilde{C}_w\tilde{J}_x a$ and extend linearly. For $u = \sum_i \alpha_i C_{w_i} J_{x_i} a_i$ and $v = \sum_j \beta_j C_{v_j} J_{y_j} b_j$ in $\mathcal{D}$,

$$\begin{aligned}\langle u, v\rangle_M &= \sum_{i,j} \overline{\alpha_i}\beta_j \left\langle C_{w_i} J_{x_i} a_i, C_{v_j} J_{y_j} b_j\right\rangle_M \\ &= \sum_{i,j} \overline{\alpha_i}\beta_j \left\langle \tilde{C}_{w_i}\tilde{J}_{x_i} a_i, \tilde{C}_{v_j}\tilde{J}_{y_j} b_j\right\rangle_M \qquad \text{(admissibility)} \\ &= \langle U_0 u, U_0 v\rangle_{\tilde{M}} .\end{aligned}$$

Since $\mathcal{D}$ is dense in M_0, U_0 extends uniquely by continuity to an isometry $U : M_0 \to \overline{U_0(\mathcal{D})} = \tilde{M}$, by minimality of $\tilde{M}$.

For each $x\in X$ and $a\in H$,

$$UJ_x a = U(C_\emptyset J_x a) = \tilde{C}_\emptyset \tilde{J}_x a = \tilde{J}_x a,$$

and for $a \in S$,

$$UC_s\left(C_w J_x a\right) = U\left(C_{ws} J_x a\right) = \tilde{C}_{ws}\tilde{J}_x a = \tilde{C}_s\tilde{C}_w\tilde{J}_x a = \tilde{C}_s U\left(C_w J_x a\right).$$

By linearity and continuity these relations hold on all of M_0. Uniqueness of U follows from the density of $\mathcal{D}$. □

3. Iteration and Limit Kernel

This section deals with what repeated application of the same completely positive map does to the kernel. In the model from Section 2, the process is simply pushing feature vectors forward by a single contraction, and the geometry forces a clear outcome, where the non-decaying component survives while the transient part fades. Passing back to kernels, the iterates converge pointwise to a limit obtained by projecting feature vectors onto the non-decaying subspace and taking their inner products there. This yields an explicit limit formula and a canonical Stein decomposition of the original kernel into its invariant part and a positive defect series.

Fix a single normal completely positive map $\Phi : B(H) \to B(H)$ with Kraus family $\Phi(T) = \sum_r A_r^* T A_r$ (strongly convergent). Set $K_n := \Phi^n(K)$ and define $C := C_\Phi$ by

$$C\left(e_\rho \cdot \tilde{K}_{(x,a)}\right) = \sum_r e_{r\rho} \cdot \tilde{K}_{(x,A_r a)}.$$

By Theorem 2.1, $\|C\|_{M\to M} \leq \|\Phi\|_{\mathrm{cb}}^{1/2}$ and

$$\langle a, K_n(x,y)\, b\rangle_H = \langle C^n J_x(a), C^n J_y(b)\rangle_M.$$

Theorem 3.1. *Assume $\Phi : B(H) \to B(H)$ is unital completely positive. Then for all $x, y \in X$ and $a, b \in H$,*

$$\lim_{n\to\infty} \langle a, K_n(x,y)\, b\rangle_H = \left\langle a, \overline{K}(x,y)\, b\right\rangle_H, \tag{3.1}$$

where

$$\overline{K}(x,y) = J_x^* P J_y, \tag{3.2}$$

and

$$P = s\text{-}\lim_{n\to\infty} C^{*n} C^n, \tag{3.3}$$

which is the orthogonal projection onto the isometric subspace

$$M_{iso} := \{v \in M : \|C^n v\| = \|v\|,\ \forall n\}. \tag{3.4}$$

In particular, $\overline{K}$ is a $B(H)$-valued p.d. kernel.

Proof. **Step 1.** By Theorem 2.1, $K_n(x,y) = J_x^* C^{*n} C^n J_y$. Unital CP implies $\|C\|_{M\to M} \leq 1$. For $n \geq 0$, set

$$T_n := C^{*n} C^n \in B(M).$$

Each T_n is a positive contraction, and

$$T_{n+1} = C^{*n}\left(C^* C\right) C^n \leq C^{*n} I C^n = T_n,$$

so $(T_n)_{n\geq 0}$ is monotone decreasing in the operator order, with $0 \leq T_n \leq I_M$.

For any $v \in M$, the scalar sequence $\langle v, T_n v\rangle$ is decreasing and bounded below by 0, hence convergent. Define a quadratic form

$$q(v) := \lim_{n\to\infty} \langle v, T_n v\rangle, \quad v \in M$$

Since $0 \le T_n \le I_M$, then $0 \le q(v) \le \|v\|^2$ and so q is bounded. By the representation theorem for bounded positive quadratic forms on a Hilbert space, there exists a unique bounded positive operator P with $0 \le P \le I$ such that

$$q(v) = \langle v, Pv \rangle, \quad v \in M.$$

Claim: $T_n \xrightarrow{s} P$ on M.

Indeed, set $S_n := T_n - P$. Then $S_n \ge 0$ and $S_{n+1} \le S_n$ (since $T_n \downarrow P$), hence $\|S_n\| \le \|S_0\| = \|I_M - P\|$. For any $v \in M$,

$$\|S_n v\|^2 = \left\langle v, S_n^2 v \right\rangle \le \|S_n\| \langle v, S_n v \rangle \le \|S_0\| \langle v, S_n v \rangle,$$

where we used the operator inequality $A^2 \le \|A\| A$ for any positive operator A.

By the definition of P, we have $\langle v, S_n v \rangle \to 0$ for every $v \in M$, hence $\|S_n v\| \to 0$, for all $v \in M$. Therefore $S_n \to 0$ in the strong operator topology, i.e., $T_n \to P$ strongly. This is (3.3).

Step 2. Let M_{iso} be as in (3.4). For $v \in M_{iso}$, we have $\langle v, T_n v \rangle = \|C^n v\|^2 = \|v\|^2$, hence $q(v) = \langle v, Pv \rangle = \|v\|^2$ and therefore $Pv = v$ (using the fact that P is a positive contraction). Thus $M_{iso} \subset ran(P)$. Conversely, suppose $Pv = v$. Then $\|C^n v\|^2 = \langle v, T_n v \rangle \to \langle v, Pv \rangle = \|v\|^2$. But $\|C^n v\|$ is a decreasing sequence (since $\|C\| \le 1$), so $\|C^n v\| = \|v\|$ for all n. Hence $v \in M_{iso}$. Thus $ran(P) = M_{iso}$, which shows that P is the orthogonal projection onto M_{iso}.

Step 3. Using the realization $K_n(x,y) = J_x^* C^{*n} C^n J_y$, we have

$$\langle a, K_n(x,y) b \rangle_H = \langle J_x a, T_n J_y b \rangle_M \longrightarrow \langle J_x(a), P J_y(b) \rangle_M = \left\langle a, \overline{K}(x,y) b \right\rangle_H.$$

This is (3.2). □

Corollary 3.2 (strict contraction)**.** *Let $\Phi : B(H) \to B(H)$ be normal completely positive with $\|\Phi\|_{\mathrm{cb}} < 1$. Set $K_n := \Phi^n(K)$ and let $C := C_\Phi$ be as above. Then:*

1. *For all $x, y \in X$ and $a, b \in H$,*
$$|\langle a, K_n(x,y) b \rangle_H| \le \|\Phi\|_{\mathrm{cb}}^n \|J_x(a)\|_M \|J_y(b)\|_M. \tag{3.5}$$
2. *$C^{*n} C^n \xrightarrow{s} 0$ on M.*
3. *The following norm estimate holds:*
$$\|K_n(x,y)\|_{B(H)} \le \|\Phi\|_{\mathrm{cb}}^n \sqrt{\|K(x,x)\| \|K(y,y)\|}. \tag{3.6}$$

Proof. By (2.1), for all x, y, a, b,

$$\langle a, K_n(x,y) b \rangle_H = \langle C^n J_x(a), C^n J_y(b) \rangle_M.$$

Let $r := \|C\|$. Part (2) of (2.1) gives $r \le \|\Phi\|_{\mathrm{cb}}^{1/2} < 1$.

By Cauchy-Schwarz,

$$\begin{aligned} |\langle a, K_n(x,y) b \rangle_H| &\le \|C^n J_x(a)\|_M \|C^n J_y(b)\|_M \\ &\le \|C^n\|_M^2 \|J_x(a)\|_M \|J_y(b)\|_M, \end{aligned}$$

where $\|C^n\|_M^2 \le r^{2n} \le \|\Phi\|_{\mathrm{cb}}^n$. This is (3.5).

Since $r < 1$, $\|C^n v\|_M \le r^n \|v\|_M \to 0$ for ever $v \in M$. Hence $C^n \to 0$ strongly. Then for any $v \in M$,

$$\|C^{n*} C^n v\|_M \le r^{2n} \|v\| \to 0,$$

so $C^{*n} C^n \xrightarrow{s} 0$.

Finally, it follows from (3.5) that

$$\|K_n(x,y)\| \le \|\Phi\|_{\mathrm{cb}}^n \|J_x\|_{H\to M} \|J_y\|_{H\to M}. \tag{3.7}$$

By the RKHS construction,

$$\begin{aligned}\|J_x\|_{H\to M}^2 &= \sup_{\|a\|_H=1} \|J_x(a)\|_M^2 \\ &= \sup_{\|a\|_H=1} \langle a, K(x,x)a\rangle_H = \|K(x,x)\|_{B(H)},\end{aligned}$$

and similarly $\|J_y\|_M^2 = \|K(y,y)\|_{B(H)}$. Substituting this into (3.7) gives the stated bound in (3.6). ☐

Corollary 3.3. *Fix S, S^* as in Section 2, and let $\{\Phi_s\}_{s\in S}$ be normal completely positive maps with*

$$\sup_{s\in S} \|\Phi_s\|_{\mathrm{cb}} \le L < 1.$$

For any $w = s_1\cdots s_n \in S^*$, set $K_w := \Phi_{s_1}\circ\cdots\circ\Phi_{s_n}(K)$ and $C_w := C_{s_n}\cdots C_{s_1}$. Then:

1. $\|C_w\|_M \le \prod_{j=1}^n \left\|\Phi_{s_j}\right\|_{\mathrm{cb}}^{1/2} \le L^{n/2}$.
2. *For all $x,y\in X$ and $a,b\in H$,*
$$|\langle a, K_w(x,y)b\rangle_H| \le L^n \|J_x(a)\|_M \|J_y(b)\|_M.$$
3. *Moreover,*
$$\|K_w(x,y)\| \le L^{|w|}\sqrt{\|K(x,x)\|\,\|K(y,y)\|}.$$

Proof. Immediate from the above discussion. ☐

We now refine the asymptotic picture from Theorem 3.1 by separating the initial kernel K into a Φ-harmonic component and a transient part. Writing $Q := K - \Phi(K)$, the kernel Q measures the one-step "defect" of K under Φ. Iterating this defect yields a canonical series that accounts for all non-harmonic contributions of K to the orbit $K_n = \Phi^n(K)$. The next result shows that $K - \overline{K}$ is the monotone sum of these iterates, that $\overline{K}$ is Φ-harmonic, and that it is maximal among Φ-harmonic kernels dominated by K.

This discrete-time decomposition parallels the classical Stein (or discrete Lyapunov) equation and its Neumann-series solution, standard in operator and matrix theory (see e.g., [Bha07, Sim16, LR95]). Related notions of Φ-harmonic elements and noncommutative Poisson boundaries appear in e.g., [EHeK78, Izu02].

Theorem 3.4 (Stein decomposition)**.** *Let X be a set, H a complex Hilbert space, $K : X\times X \to B(H)$ a p.d. kernel, and $\Phi : B(H) \to B(H)$ a normal unital completely positive map. Set $K_n := \Phi^n(K)$ and let $\overline{K}$ be the limit kernel given by Theorem 3.1, so $K_n(x,y) \xrightarrow{w} \overline{K}(x,y)$ for each $x,y\in X$ (in the weak operator topology). Define*

$$Q(x,y) := K(x,y) - \Phi(K)(x,y).$$

Then:

1. *For all $x,y\in X$,*
$$K(x,y) - \overline{K}(x,y) = \sum_{j=0}^{\infty} \Phi^j(Q)(x,y), \tag{3.8}$$

where the series converges pointwise in the weak operator topology and, moreover, in the p.d. order as a monotone increasing sequence of kernels.

(2) *The limit kernel* $\overline{K}$ *is* Φ*-harmonic, i.e.,*

$$\Phi\left(\overline{K}\right)=\overline{K}. \tag{3.9}$$

(3) *If* $L: X\times X\to B(H)$ *is* Φ*-harmonic and* $0\preceq L\preceq K$*, then* $L\preceq\overline{K}$*. Equivalently,* $\overline{K}$ *is the largest* Φ*-harmonic kernel dominated by* K*.*

Proof. Set $S_N:=\sum_{j=0}^{N-1}\Phi^j\left(Q\right)$ for $N>1$. Then

$$K-\Phi^N\left(K\right)=\sum_{j=0}^{N-1}\left(\Phi^j\left(K\right)-\Phi^{j+1}\left(K\right)\right)=\sum_{j=0}^{N-1}\Phi^j\left(Q\right)=S_N.$$

Each $\Phi^j\left(Q\right)$ is p.d., hence S_N is p.d. and $S_N\preceq S_{N+1}$.

By Theorem 3.1, $\Phi^N\left(K\right)=K_N\xrightarrow{w}\overline{K}$ pointwise in the weak operator topology, hence

$$S_N\left(x,y\right)=K\left(x,y\right)-\Phi^N\left(K\right)\left(x,y\right)\xrightarrow{w}K\left(x,y\right)-\overline{K}\left(x,y\right).$$

Because $\left(S_N\right)_{N\geq1}$ is monotone increasing in the p.d. order and is bounded above by K, the limit exists also in the p.d. order and equals $K-\overline{K}$. This proves (3.8).

Apply Φ to the identity (3.8), so that

$$\Phi\left(K\right)-\Phi\left(\overline{K}\right)=\sum_{j=1}^{\infty}\Phi^j\left(Q\right).$$

Adding $Q=K-\Phi\left(K\right)$ to both sides gives

$$K-\Phi\left(\overline{K}\right)=Q+\sum_{j=1}^{\infty}\Phi^j\left(Q\right)=\sum_{j=0}^{\infty}\Phi^j\left(Q\right).$$

This implies that $K-\Phi\left(\overline{K}\right)=K-\overline{K}$, hence $\Phi\left(\overline{K}\right)=\overline{K}$, which is (3.9).

Suppose L is Φ-harmonic and $0\preceq L\preceq K$. Then $K-L$ is p.d., and complete positivity gives $\Phi^N\left(K-L\right)\succeq0$ for all N. Iterating the identity

$$K-L=\Phi\left(K-L\right)+Q$$

gives

$$K-L=\Phi^N\left(K-L\right)+S_N.$$

Since $\Phi^N\left(K-L\right)\succeq0$ we obtain the monotone lower bound

$$K-L\succeq S_N,\quad\forall N.$$

Taking $N\to\infty$ and using $S_N\xrightarrow{w}K-\overline{K}$ gives

$$K-L\succeq K-\overline{K},$$

which is equivalent to $\overline{K}-L\succeq0$, i.e., $L\preceq\overline{K}$. This proves the final part. ∎

4. Kernel Domination and RN Representation

This section moves from the universal model back down to the base RKHS of the original kernel. When each composing map is subunital, every iterate sits below the starting kernel in the positive definite order. On the Kolmogorov space this means there is a single positive contraction that "explains" the iterate as a compression of the original features. Concretely, we identify each composed kernel with a bounded operator on the base space, so comparison, bounds, and limits reduce to ordinary operator inequalities there. The picture is minimal, no large ambient sum is needed, and it makes the domination $K_w \preceq K$ and the associated bounds more transparent.

Fix the data from the setting in Section 2. Let $K : X \times X \to B(H)$ be p.d., and let $\mathcal{H}_{\tilde{K}}$ be the canonical Kolmogorov space for K with maps

$$V_x : H \to \mathcal{H}_{\tilde{K}}, \qquad V_x a = \tilde{K}_{(x,a)},$$

so that

$$K(x,y) = V_x^* V_y.$$

Recall that $J_x : H \to M$ is given by $J_x(a) = e_\emptyset \cdot \tilde{K}_{(x,a)}$. That is, we identify $\mathcal{H}_{\tilde{K}}$ with the $\emptyset$-summand of M via the isometric embedding

$$\iota : \mathcal{H}_{\tilde{K}} \to M, \qquad \iota(f) = e_\emptyset \cdot f,$$

so that $J_x = \iota \circ V_x$.

We consider normal completely positive maps $\{\Phi_s\}_{s\in S}$ with Kraus forms as in Section 2, and the iterated kernels

$$K_w := \Phi_{s_1} \circ \cdots \circ \Phi_{s_n}(K), \qquad w = s_1 \cdots s_n \in S^*,$$

with $K_\emptyset = K$.

Lemma 4.1 (Radon-Nikodym for p.d. kernels)**.** *Let $K, L : X \times X \to B(H)$ be p.d. kernels with $L \preceq K$ (i.e., $K - L$ is p.d.) Let $\mathcal{H}_{\tilde{K}}$ be the canonical Kolmogorov space of K with maps $V_x : H \to \mathcal{H}_{\tilde{K}}$ so that $K(x,y) = V_x^* V_y$. Then there exists a unique operator $A \in B(\mathcal{H}_{\tilde{K}})$ with $0 \leq A \leq I_{\mathcal{H}_{\tilde{K}}}$ such that*

$$L(x,y) = V_x^* A V_y, \qquad x, y \in X.$$

Proof sketch. On the algebraic span of $\{V_y b\}$, define

$$\langle V_x a, V_y b \rangle_L := \langle a, L(x,y) b \rangle_H .$$

Since $L \preceq K$, Cauchy-Schwarz gives

$$|\langle \xi, \eta \rangle_L| \leq \|\xi\|_{\mathcal{H}_{\tilde{K}}} \|\eta\|_{\mathcal{H}_{\tilde{K}}} ,$$

so this form is bounded by the $\mathcal{H}_{\tilde{K}}$ norm and extends uniquely to a bounded positive operator A with $0 \leq A \leq I$ via $\langle \xi, \eta \rangle_L = \langle \xi, A\eta \rangle_{\mathcal{H}_{\tilde{K}}}$. Uniqueness follows from the minimality of $\mathcal{H}_{\tilde{K}}$ (the span of $\{V_y b\}$ is dense).

For full details and variants, see e.g., [Ped58, Pau02, Arv69, Sza21]. ☐

Remark. The operator A in Lemma 4.1 is called the Radon-Nikodym derivative of L with respect to K (on the Kolmogorov space $\mathcal{H}_{\tilde{K}}$), denoted by $A = dL/dK$.

Theorem 4.2 (RN representation under subunitality)**.** *Assume each Φ_s is subunital, i.e., $\Phi_s(I) \leq I$. Then for every $w \in S^*$, there exists a unique bounded positive operator $A_w \in B(\mathcal{H}_{\tilde{K}})$ with $0 \leq A_w \leq I_{\mathcal{H}_{\tilde{K}}}$ such that*

$$K_w(x,y) = V_x^* A_w V_y, \qquad x, y \in X.$$

Equivalently, for all $x, y \in X$ and $a, b \in H$,

$$\langle a, K_w(x,y)\, b\rangle_H = \langle V_x a, A_w V_y b\rangle_{\mathcal{H}_{\tilde{K}}}.$$

Moreover, with $C_w := C_{s_n} \cdots C_{s_1}$ from (2.2), one has the explicit formula

$$A_w = \iota^* C_w^* C_w \iota,$$

and hence the kernel domination $K_w \preceq K$ (in the usual p.d. order) holds for all $w \in S^*$.

Proof. Since each Φ_s is subunital, part (2) of Theorem 2.1 gives $\|C_s\| \leq 1$. Hence every C_w is a contraction and $0 \leq C_w^* C_w \leq I_M$. Thus,

$$A_w := \iota^* C_w^* C_w \iota \in B(\mathcal{H}_{\tilde{K}})$$

is a positive contraction, $0 \leq A_w \leq \iota^* I_M \iota = I_{\mathcal{H}_{\tilde{K}}}$.

For $x, y \in X$ and $a, b \in H$,

$$\begin{aligned}\langle a, K_w(x,y)\, b\rangle_H &= \langle C_w J_x(a), C_w J_y(b)\rangle_M \\ &= \langle C_w \iota V_x a, C_w \iota V_y b\rangle_M \\ &= \langle V_x a, \iota^* C_w^* C_w \iota V_y b\rangle_{\mathcal{H}_{\tilde{K}}}.\end{aligned}$$

This proves the representation $K_w(x,y) = V_x^* A_w V_y$ and shows that $0 \leq A_w \leq I$. Kernel domination follows immediately:

$$\langle a, (K - K_w)(x,y)\, b\rangle_H = \langle V_x a, (I - A_w) V_y b\rangle \geq 0,$$

so $K_w \preceq K$.

Uniqueness: by minimality of the Kolmogorov decomposition, $\mathcal{H}_{\tilde{K}}$ is the closed linear span of $\{V_y b : y \in X, b \in H\}$. If B is another bounded operator on $\mathcal{H}_{\tilde{K}}$ with

$$V_x^* B V_y = V_x^* A_w V_y$$

for all $x, y \in X$, then

$$\langle V_x a, (B - A_w) V_y b\rangle = 0$$

for all x, y, a, b, hence $B = A_w$. □

Remark 4.3 (Relation to the M-model)*.* The operator A_w is exactly the compression of $C_w^* C_w$ to the $\emptyset$-layer of M. Thus the RN picture on $\mathcal{H}_{\tilde{K}}$ is a compressed form of the universal model from Theorem 2.1. In particular,

$$\|A_w\| \leq \|C_w\|^2 \leq \prod_{j=1}^{n} \left\|\Phi_{s_j}\right\|_{\mathrm{cb}},$$

and if $\sup_s \|\Phi_s\|_{\mathrm{cb}} \leq L < 1$ then $\|A_w\| \leq L^{|w|}$.

Remark 4.4. If some Φ_s is not subunital, the contraction property of C_s can fail, and so can the kernel domination $K_w \preceq K$. In that case, there need not exist a global positive contraction $0 \leq A_w \leq I$ on $\mathcal{H}_{\tilde{K}}$ with $K_w(x,y) = V_x^* A_w V_y$, but the universal model still exists (Theorem 2.1).

Lemma 4.1 established the RN representation for positive definite kernels: if $L \preceq K$, then there exists a unique $0 \leq A \leq I_{\mathcal{H}_{\tilde{K}}}$ such that $L(x,y) = V_x^* A V_y$. This operator A acts on the canonical Kolmogorov space of K and compresses K to L. The same argument extends to the iterated kernels generated by the family $\{\Phi_s\}_{s\in S}$. Through the universal model constructed in Section 2, the Radon-Nikodym operator A acts diagonally across the graded space M, producing a single compression that intertwines all iterates. The following result makes this extension precise.

Theorem 4.5 (order-monotone compression across models)**.** *Let $K^{(1)}, K^{(2)} : X \times X \to B(H)$ be p.d. kernels with $K^{(1)} \preceq K^{(2)}$. Fix the same CP family $\{\Phi_s\}_{s\in S}$ and Kraus data as in Section 2, and form the universal models*

$$\left(M_i, \{C_s^{(i)}\}_{s\in S}, \{J_x^{(i)}\}_{x\in X}\right), \quad i = 1, 2,$$

for $\left(K^{(i)}, \{\Phi_s\}\right)$ via Theorem 2.1.

Then there exists a unique positive contraction $A \in B\left(\mathcal{H}_{\tilde{K}_2}\right)$ (i.e., $A = d\tilde{K}^{(1)}/d\tilde{K}^{(2)}$) such that, for every word $w \in S^*$ and all $x, y \in X$,

$$K_w^{(1)}(x,y) = J_x^{(2)*} C_w^{(2)*} \left(I_{l^2(R)} \otimes A\right) C_w^{(2)} J_y^{(2)}. \tag{4.1}$$

Proof. Consider the scalar lifts

$$\tilde{K}^{(i)}\left((x,a),(y,b)\right) := \langle a, K^{(i)}(x,y)\, b\rangle_H$$

on $X \times H$ and their RKHSs $\mathcal{H}_{\tilde{K}^{(i)}}$ with kernel sections $\tilde{K}^{(i)}_{(x,a)}$. Since $K^{(1)} \preceq K^{(2)}$, we have $\tilde{K}^{(1)} \preceq \tilde{K}^{(2)}$ as scalar kernels. By Lemma 4.1, there exists a unique RN derivative $A = d\tilde{K}^{(1)}/d\tilde{K}^{(2)}$. In particular, $A \in B\left(\mathcal{H}_{\tilde{K}_2}\right)$ with $0 \leq A \leq I$.

Both models M_i are constructed as

$$M_i = l^2(R) \otimes \mathcal{H}_{\tilde{K}^{(i)}} \simeq \bigoplus_{\rho\in R} \mathcal{H}_{\tilde{K}^{(i)}},$$

with the same grading set R (built only from S and the Kraus index sets). Write elements of M_i as $\sum_\rho e_\rho \cdot F_\rho^{(i)}$, where $F_\rho^{(i)} \in \mathcal{H}_{\tilde{K}^{(i)}}$ and $\{e_\rho\}$ is the canonical ONB of $l^2(R)$.

By Theorem 2.1, for each $i = 1, 2$,

$$K_w^{(i)}(x,y) = J_x^{(i)*} C_w^{(i)*} C_w^{(i)} J_y^{(i)}.$$

Fix $w = s_1 \cdots s_n \in S^*$. Expanding by the definition of $C_w^{(2)}$ and the orthogonality of the R-summands,

$$C_w^{(2)} J_x^{(2)} a = \sum_{(r_1,\ldots,r_n)} e_{(r_1,\ldots,r_n)} \cdot \tilde{K}^{(2)}_{\left(x, A_{s_n,r_n} \cdots A_{s_1,r_1} a\right)}.$$

Therefore,

$$\begin{aligned}
& \left\langle C_w^{(2)} J_x^{(2)} a, (I \otimes A)\, C_w^{(2)} J_y^{(2)} b \right\rangle_{M_2} \\
= & \sum_{(r_1,\ldots,r_n)} \left\langle \tilde{K}^{(2)}_{\left(x, A_{s_n,r_n} \cdots A_{s_1,r_1} a\right)}, A \tilde{K}^{(2)}_{\left(y, A_{s_n,r_n} \cdots A_{s_1,r_1} b\right)} \right\rangle_{\mathcal{H}_{\tilde{K}^{(2)}}} \\
= & \sum_{(r_1,\ldots,r_n)} \left\langle \tilde{K}^{(1)}_{\left(x, A_{s_n,r_n} \cdots A_{s_1,r_1} a\right)}, \tilde{K}^{(1)}_{\left(y, A_{s_n,r_n} \cdots A_{s_1,r_1} b\right)} \right\rangle_{\mathcal{H}_{\tilde{K}^{(1)}}} \\
= & \left\langle a, (\Phi_{s_1} \circ \cdots \circ \Phi_{s_n}) \left(K^{(1)}(x,y) \right) b \right\rangle_H \\
= & \left\langle a, K_w^{(1)}(x,y)\, b \right\rangle_H
\end{aligned}$$

and so (4.1) follows. Uniqueness is from that of RN derivative on $\mathcal{H}_{\tilde{K}^{(2)}}$. □

5. Random Iterations

This section lets the composing map vary randomly at each step and asks for the typical exponential growth (or decay) rate of the pushed-forward features. On the universal model from Section 2, a random word becomes a random product of the creation operators, subadditivity then forces an almost-sure limit for the average log-norm (the Lyapunov exponent). Translating back to kernels, this single number controls how fast the composed kernels amplify or contract on any fixed pair of points, and it yields clean almost-sure bounds for their entries and operator norms. In the deterministic unital case the exponent collapses to zero, matching the non-decaying limit kernel identified in Section 3.

Fix the data from the setting in Section 2. In particular, for each $z \in S$ we have the bounded operator $C_z : M \to M$ and, for a word $w = z_1 \cdots z_n \in S^*$, the product $C_w := C_{z_n} \cdots C_{z_1}$ and the kernel realization

$$K_w(x,y) = J_x^* C_w^* C_w J_y.$$

Let μ be a probability measure on S. Work on the canonical path space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega := S^{\mathbb{N}}, \quad \mathbb{P} := \mu^{\otimes \mathbb{N}}$$

with the product σ-algebra $\mathcal{F}$. Let $z_n : \Omega \to S$ be the coordinate maps, $z_n(\omega) = \omega_n$ (i.i.d. with law μ) For $\omega \in \Omega$, set the random word of length n

$$\xi_n(\omega) := z_1(\omega) \cdots z_n(\omega),$$

and

$$C_{\xi_n(\omega)} = C_{z_n(\omega)} \cdots C_{z_1(\omega)}.$$

We assume the integrability condition

$$\mathbb{E}\left[\log^+ \|C_{z_1}\|\right] < \infty \tag{5.1}$$

where $\log^+ t = \max\{0, \log t\}$.

Define the left shift $\theta : \Omega \to \Omega$,

$$\theta(\omega_1, \omega_2, \omega_3, \ldots) = (\omega_2, \omega_3, \ldots).$$

Then θ is measure-preserving and $z_{n+1} = z_n \circ \theta$.

Theorem 5.1. *Under (5.1), there exists a deterministic constant $\lambda \in [-\infty, \infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| C_{\xi_n(\omega)} \right\| = \lambda \qquad \mathbb{P}\text{-a.e. } \omega.$$

Moreover,

$$\lambda = \inf_{n\geq 1} \frac{1}{n} \mathbb{E}\left[\log \left\| C_{\xi_n} \right\|\right] = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\log \left\| C_{\xi_n} \right\|\right].$$

Proof. For $n \geq 1$, set

$$X_n(\omega) := \log \left\| C_{\xi_n(\omega)} \right\| \in [-\infty, \infty).$$

By submultiplicativity of the operator norm and the definition of ξ_n,

$$\begin{aligned} X_{m+n}(\omega) &= \log \left\| C_{\xi_{m+n}(\omega)} \right\| \\ &= \log \left\| C_{z_{m+n}(\omega)} \cdots C_{z_1(\omega)} \right\| \\ &\leq \log \left\| C_{z_{m+n}(\omega)} \cdots C_{z_{m+1}(\omega)} \right\| + \log \left\| C_{z_m(\omega)} \cdots C_{z_1(\omega)} \right\| \\ &= X_n(\theta^m \omega) + X_m(\omega). \end{aligned}$$

Thus $(X_n)_{n\geq 1}$ is a subadditive, stationary process over the measure-preserving transformation θ. The integrability hypothesis (5.1) yields $\mathbb{E}\left[X_1^+\right] < \infty$. Here, stationary with respect to θ means

$$X_n \circ \theta^m \overset{d}{=} X_n$$

for all $m \geq 0$. Equivalently,

$$\mathbb{E}\left[g\left(X_n \circ \theta^m\right)\right] = \mathbb{E}\left[g\left(X_n\right)\right]$$

for all bounded measurable g.

By Kingman's subadditive ergodic theorem (see e.g., [Kin68, Kin73, Kre85, Dur19]), there exists a deterministic $\lambda \in [-\infty, \infty)$ such that

$$\frac{1}{n} X_n(\omega) \to \lambda \text{ for } \mathbb{P}\text{-a.e. } \omega,$$

and moreover,

$$\lambda = \inf_{n\geq 1} \frac{1}{n} \mathbb{E}\left[X_n\right] = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[X_n\right].$$

∎

Corollary 5.2 (kernel scalar growth)**.** *Let λ be as in Theorem 5.1. For all $x, y \in X$ and $a, b \in H$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \left| \left\langle a, K_{\xi_n(\omega)}(x, y)\, b \right\rangle_H \right| \leq 2\lambda, \quad \mathbb{P}\text{-a.e. } \omega.$$

Proof. By the realization $K_w(x, y) = J_x^* C_w^* C_w J_y$ and Cauchy-Schwarz,

$$\begin{aligned} \left| \left\langle a, K_{\xi_n(\omega)}(x, y)\, b \right\rangle_H \right| &= \left| \left\langle C_{\xi_n} J_x(a), C_{\xi_n} J_y(b) \right\rangle_M \right| \\ &\leq \left\| C_{\xi_n} \right\|_M^2 \left\| J_x(a) \right\|_M \left\| J_y(b) \right\|_M. \end{aligned}$$

Take logs, divide by n, and use Theorem 5.1. ∎

Corollary 5.3 (operator-norm bound)**.** *For any random word ξ_n,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| K_{\xi_n(\omega)}(x, y) \right\| \leq 2\lambda, \quad \mathbb{P}\text{-a.e. } \omega.$$

Proof. Recall that, for any word $w = s_1 \cdots s_n \in S^*$ and $x, y \in X$,

$$\begin{aligned}\|K_w(x,y)\|_{B(H)} &\leq \|C_w\|_M^2 \|J_x\|_M \|J_y\|_M \\ &= \|C_w\|_M^2 \|K(x,x)\|_{B(H)}^{1/2} \|K(y,y)\|_{B(H)}^{1/2}.\end{aligned}$$

Use the random word ξ_n, and apply Theorem 5.1 to $\|C_{\xi_n}\|_M$ to get the a.s. $\limsup$. ☐

Corollary 5.4 (RN formulation under subunitality)**.** *Assume each Φ_z is subunital. Then for any $w \in S^*$ there exists a positive contraction $A_w \in B(\mathcal{H}_{\tilde{K}})$ with*

$$K_w(x,y) = V_x^* A_w V_y$$

with $0 \leq A_w \leq I$ (see Theorem 4.2). Hence, for any $x, y \in X$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\|K_{\xi_n(\omega)}(x,y)\right\| \leq \limsup_{n\to\infty} \frac{1}{n} \log \left\|A_{\xi_n(\omega)}\right\| \leq 2\lambda \quad a.s.$$

In particular, if $\lambda < 0$, then $\left\|K_{\xi_n(\omega)}(x,y)\right\| \to 0$ exponentially fast almost surely.

Proof. This follows from Section 4, where $A_w = \iota^* C_w^* C_w \iota$, so $0 \leq A_w \leq I$ and $\|A_w\| \leq \|C_w\|^2$, combined with Theorem 5.1. ☐

Corollary 5.5 (uniform cb-bound)**.** *If $\sup_{z\in S} \|\Phi_z\|_{\mathrm{cb}} \leq L < \infty$, then $\|C_z\| \leq L^{1/2}$ for all z, and for any word w of length $|w|$,*

$$\|K_w(x,y)\| \leq L^{|w|} \|K(x,x)\|^{1/2} \|K(y,y)\|^{1/2}.$$

In particular, if $L < 1$, then for the random words ξ_n,

$$\left\|K_{\xi_n(\omega)}(x,y)\right\| \leq L^n \|K(x,x)\|^{1/2} \|K(y,y)\|^{1/2}, \qquad \forall n,$$

and thus $\lambda \leq \frac{1}{2}\log L < 0$ and exponential decay holds almost surely.

Proof. The bound $\|C_z\| \leq \|\Phi_z\|_{\mathrm{cb}}^{1/2}$ is Theorem 2.1, part (2). The rest is immediate. ☐

Remark 5.6 (UCP, spectral radius). Let $\Phi_z \equiv \Phi$ be unital CP, so $C_z = C$ and $C_{\xi_n} = C^n$. Then

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \|C^n\| = \log r(C) \leq 0,$$

where $r(C)$ is the spectral radius.

By Theorem 3.1, $C^{*n}C^n \xrightarrow{s} P$, where P is the orthogonal projection onto $M_{iso} = \{v : \|C^n v\| = \|v\|, \forall n\}$.

(1) If $P \neq 0$, then $r(C) = 1$ and $\lambda = 0$. In this case,

$$K_n(x,y) = J_x^* C^{*n} C^n J_y \to J_x^* P J_y = \overline{K}(x,y).$$

(2) If $P = 0$, then $K_n(x,y) \to 0$ for each $x, y \in X$. Hence $\lambda = \log r(C) \leq 0$, with $\lambda \leq 0$ giving exponential decay and $\lambda = 0$ allowing subexponential decay.

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