Any fully graphic region of degree sequences can be sampled rapidly

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Abstract

Let $n > c_1 \geq c_2$ and Σ be positive integers with $n \cdot c_1 \geq \Sigma \geq n \cdot c_2$. Let $\mathbb{D} = \mathbb{D}(n, \Sigma, c_1, c_2)$ denote the set of all degree sequences of length n with the even sum Σ and satisfying $c_1 \geq d_i \geq c_2$. We show that if all degree sequences in $\mathbb{D}$ are graphic, then $\mathbb{D}$ is $3n^{13}$-stable. (The concept of P-stability was introduced by Jerrum and Sinclair in 1990.) In particular, this implies that the switch Markov-chain mixes rapidly on all such degree sequences.

In this paper we also study the inverse direction. We show the following: if all graphic sequences of a degree sequence region satisfy the $p(n)$-stability condition then the overwhelming majority of the sequences in the region is graphic. This answers affirmatively a question raised in the paper DOI:10.1016/j.aam.2024.102805.

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1. Introduction

Generating uniformly random graphs with a given degree sequence is a fundamental, and long researched, task in graph theory. There are two standard approaches to it: the configuration model and MCMC methods. On one hand, the configuration model has known limitations when the degree sequence is heterogeneous. On the other hand, while MCMC methods work well in practice, rigorous results on convergence rates are difficult to obtain.

The simplest MCMC method, proposed by Rao, Jana and Bandyopadhyay in [9], is the so-called swap Markov-chain: one step consists of replacing a pair of edges $(v_1w_1), (v_2w_2)$ with $(v_1w_2), (v_2, w_1)$. This is allowed only when such a change does not produce multi-edges. It was conjectured by Kannan, Tetali, and Vempala in 1997 [8] that this Markov-chain mixes in polynomial time for all graphic degree sequences. However, the conjecture is open to this day. In fact, it has been verified only for a few families of degree sequences.

Jerrum and Sinclair [7] introduced a notion - called P-stability - for degree sequences, in order to study a different Markov-chain for graph generation. However, it turned out to be a useful concept for the swap Markov-chain as well. So useful, in fact, that in [1] not only was it shown that P-stability implies fast mixing of the swap Markov-chain, but also that all previously known cases can be proved this way.

However, verifying P-stability of degree sequence families is still a challenging task. There are only a handful of known results in this direction - see Section 1.2. **The main contribution**

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of this paper is a novel method to show P-stability, and in turn polynomial-time mixing, for a broad new class of degree sequences. Essentially, we show that when P-stability fails for a sequence, then there is another degree sequence with the same total number of edges and the same maximum and minimum degree that is not graphic. **We also show a partial converse to this:** if a degree sequence is not graphic, there is almost always an other (graphic) sequence with the same total number of edges and same maximum and minimum degree that is not P-stable. This positively answers a question raised in [2].

The outline of the paper is as follows. We introduce the key definitions in Section 1.1. In Section 1.2 we list the most important known results about P-stability. Our new results are described in Section 1.3. The proofs of the main Theorems 1.3 and 1.6 are given in Sections 2 and 3, respectively.

1.1. Preliminaries

Here we recall (and slightly expand) the notions of various degree sequence regions from [2] that are central to stating our results, as well as the concept of P-stability.

Degree sequence regions. A **degree sequence** is a sequence $\mathbf{d} = (d_1, \dots, d_n)$ of positive integers, where no d_i can exceed $n-1$, with even sum denoted by Σ. (In this paper, the parameter Σ is always even.) We will often use $c_1 = c_1(\mathbf{d}) := \max(\mathbf{d})$ and $c_2 = c_2(\mathbf{d}) := \min(\mathbf{d})$) to denote the largest and smallest degrees of the sequence. We say $\mathbf{d}$ is *graphical* if there exists a simple graph on the vertex set $[n]$ such that vertex i has degree d_i for every $i \in [n]$.

We refer to a subset of all degree sequences given by a formula $\varphi = \varphi(\mathbf{d})$ as **degree sequence region** (or just region, for short) and denote it as

$$\mathbb{D}[\varphi] = \{\mathbf{d} : \varphi(\mathbf{d})\}. \tag{1}$$

We say that a region is **simple** if φ depends only on n, Σ, c_1, c_2. In particular, for positive integers $n > a \geq b$ we denote

$$\mathbb{D}(n, \Sigma, a, b) = \mathbb{D}\left[\,|\mathbf{d}| = n, a \geq c_1 \geq c_2 \geq b, \text{ and } \textstyle\sum_{i=1}^{n} d_i = \Sigma\,\right]. \tag{2}$$

When φ doesn't even depend on Σ, we call the region **very simple**. In particular, we write

$$\mathbb{D}(n, a, b) = \mathbb{D}[|\mathbf{d}| = n, a \geq c_1 \geq c_2 \geq b] = \bigcup \{\mathbb{D}(n, \Sigma, a, b) \,:\, n \cdot a \geq \Sigma \geq n \cdot b\}. \tag{3}$$

Typically, a region $\mathbb{D}$ may contain both graphic and non-graphic elements. We will be specifically interested in regions in which every sequence is graphic. Such regions are called **fully graphic**.

P-stability. The notion of P-stability of an infinite set of graphic degree sequences was introduced by Jerrum and Sinclair in [7]. The notion plays a crucial role in complexity theory and graph theory.

Given a graphic degree sequence D of length $|D| = n$, let $\mathcal{G}(D)$ denote the set of all realizations of D and let

$$\partial^{++}(D) = \sum_{1 \leq i < j \leq n} |\mathcal{G}(D + 1^{+i}_{+j})| / |\mathcal{G}(D)|, \tag{4}$$

where the vector 1^{+i}_{+j} consists of all zeros, except at the ith and jth coordinates, where the values are $+1$. The operation $D \mapsto D + 1^{+i}_{+j}$ is called a **perturbation operation** on the degree sequences. We define the analogous operation $D + 1^{-i}_{-j}$ similarly. Let us emphasize that we do not assume that i and j are different, so for example the operation $D \mapsto D + 1^{+j}_{+j}$ is also defined and it adds 2 to the j^{th} coordinate.

An infinite region $\mathbb{D}$ is called P**-stable** if there is a polynomial $p(n)$ such that

$$\forall D \in \mathbb{D}: \; D \text{ is graphic} \Rightarrow \partial^{++}(D) \leq p(|D|) \tag{5}$$

The notion can be extended meaningfully for finite regions by fixing the polynomial $p(n)$ instead of just assuming its existence. Thus, we will say that a region is p-stable when it satisfies (5) for a concrete polynomial $p = p(n)$. Let us emphasize that we do not require that every element of a P-stable family to be graphic. P-stability has alternative, but equivalent, definitions that use different perturbation operations (see the Appendix in [2] a for detailed explanation).

1.2. Previous results

Since the introduction of P-stability, only the following four infinite P-stable degree sequence regions were found:

(P1) (Jerrum, McKay, Sinclair [5]) The very simple degree sequence region $\mathbb{D}[\varphi_{JMS}]$ is P-stable, where $\varphi_{JMS} \equiv (c_1 - c_2 + 1)^2 \leq 4c_2(n - c_1 - 1)$.

(P2) (Jerrum, McKay, Sinclair [5]) The simple degree sequence region $\mathbb{D}[\varphi^*_{JMS}]$ is P-stable, where $\varphi^*_{JMS} \equiv (\Sigma - nc_2)(nc_1 - \Sigma) \leq (c_1 - c_2)\left\{(\Sigma - nc_2)(n - c_1 - 1) + (nc_1 - \Sigma)c_2\right\}$.

(P3) (Greenhill, Sfragara [4]) The simple degree sequence region $\mathbb{D}[\varphi_{GS}]$ is P-stable, where $\varphi_{GS} \equiv (2 \leq c_2 \text{ and } 3 \leq c_1 \leq \sqrt{\Sigma/9})$. (This result was not announced explicitly, but [4, Lemma 2.5] clearly proved this fact.)

(P4) (Gao, Greenhill [3]) $\mathbb{D}[\varphi_{GG}]$ is P-stable, where $\varphi_{GG} \equiv (\sum_{j=1}^{c_1} d_j + 6c_1 + 2 \leq \sum_{j=c_1+1}^{n} d_j)$. Here the degrees are understood to be in non-increasing order.

Note that region (P1) is very simple, regions (P2) and (P3) are simple, while (P4) is not a simple region.

The paper [2] was inspired by the observation that all of these regions are fully graphic. This is basically obvious for (P4) and was proved in [2] for the others. This prompted a more thorough investigation of a potential connection between P-stability and the fully graphic property of regions. In [2] it was shown that in the case of very simple regions the two notions are nearly equivalent. We summarize their main findings briefly, and without some of the technical details, in the following statements. We refer the reader to [2] for the exact formulation of Theorem 1.2 below.

Theorem 1.1 ([2, Theorems 4.2, 4.4])**.** *Every fully graphic very simple region is* (n^9)*-stable. Thus the maximal fully graphic very simple region*

$$\mathbb{D}_{\max} := \bigcup \{\mathbb{D}(n, c_2, c_1) : \; \mathbb{D}(n, c_2, c_1) \textit{ is fully graphic}\}$$

is (n^9)*-stable, and the switch Markov chain is rapidly mixing on* $\mathbb{D}_{\max}$ *with* $O(n^{11})$ *mixing time.*

Theorem 1.2 ([2, Theorem 5.8])**.** *If the very simple region* $\mathbb{D}(n, c_1, c_2)$ *is not fully graphic then it is contained in a slightly larger very simple region* $\mathbb{D}(n, c_1 \cdot (1 + \varepsilon), c_2 \cdot (1 - \varepsilon))$ *that is not* P*-stable, for some suitable* $\varepsilon = o(1)$ *function.*

1.3. New results

The goal of this paper is to extend the results of [2] to simple regions. For the generalization of Part 1 of Theorem 1.1 we show the following:

Theorem 1.3. *If $\mathbb{D}(n,\Sigma,c_1,c_2)$ is fully graphic, then $\partial^{++}(D) \le 3{\cdot}n^{13}$ for each $D \in \mathbb{D}(n,\Sigma,c_1,c_2)$.*

Corollary 1.4. *The largest fully graphic simple region*

$$\mathbb{D} := \bigcup \{\mathbb{D}(n,\Sigma,c_1,c_2) : \mathbb{D}(n,\Sigma,c_1,c_2) \text{ is fully graphic}\}$$

is P-stable.

To state the converse results, let us first recall from [2] when a simple region $\mathbb{D}(n,\Sigma,c_1,c_2)$ is not fully graphic.

Lemma 1.5. *If $\mathbb{D}(n,\Sigma,c_1,c_2)$ is not fully graphic then*

$$Q := (c_1-c_2+1)^2 - 4c_2(n-1-c_1) > 0 \tag{6}$$

and

$$\left|\frac{\Sigma - nc_2}{(c_1-c_2)/2} - (1+c_1+c_2)\right| \le \sqrt{Q}+2. \tag{7}$$

That is, for fixed n, c_1, c_2, the range of Σ for which $\mathbb{D}(n,\Sigma,c_1,c_2)$ is not fully graphic is contained an interval of length roughly equal to $2\sqrt{Q}$. We will show, under mild extra conditions, that for most of these Σ values the region is also not P-stable. More precisely, it contains a sequence D with a large value of $\partial^{++}(D)$.

Theorem 1.6. *Suppose $\beta > 0$, $r \in \mathbb{N}$ and $\varepsilon = \frac{3(r+3)}{\beta(c_1-c_2)}$. If*

$$(1-\beta)(c_1-c_2+1)^2 \ge 4c_2(n-1-c_1). \tag{8}$$

and

$$\left|\frac{\Sigma - nc_2}{(c_1-c_2)/2} - (1+c_1+c_2)\right| \le (1-\varepsilon)\sqrt{Q}, \tag{9}$$

then the simple region $\mathbb{D}(n,\Sigma,c_1,c_2)$ contains a sequence D such that

$$\partial^{++}(D) \ge 2^{r/2}. \tag{10}$$

Remark 1.7. *Let us set $r = \log^2 n$. The $Q > 0$ requirement already implies that $c_1 - c_2 \ge \Omega(\sqrt{n})$, hence the ε obtained is $o(1)$ as $n \to \infty$, as long as $\beta > 0$ is fixed. Thus, when the non-fully graphic Σ range is not negligible, then almost all of these regions have a sequence D for which* (10) *implies*

$$\partial^{++}(D) \ge 2^{r/2} \ge n^{c\log n},$$

so they are not P-stable.

2. Fully graphic simple regions are P-stable

The main goal of this section is to prove Theorem 1.3. This will be done in many steps. We begin by recalling alternating trails and their relevance for proving P-stability below. Section 2.1 contains preliminary reductions, while Section 2.2 demonstrates the main method of our proof on a warm-up case. The actual proof is done in Sections 2.3 through 2.5. Finally, Section 2.6 contains an application of Theorem 1.3.

Let us fix the convention that the nodes of an n-vertex graph G will be denoted by $V(G) = \{v_1,\dots,v_n\}$.

Definition 2.1. Let G be a graph, and let v and v' be (not necessarily distinct) vertices of G. An alternating trail of edges and non-edges between v and v' of odd length is called **edge-deficient** if its first element is a non-edge (hence the trail contains fewer edges than non-edges). Similarly, it is called **edge-abundant** if its first element is an edge, (hence the trail contains more edges than non-edges). An edge-abundant trail of length at most k is called **k-witness trail**. When the length is implicitly understood, we may simply call it a witness trail.

The following well-known proposition shows why witness trails are relevant for comparing realizations of degree sequences.

Proposition 2.2. *Assume that D_0 is a degree sequence of length n, and let $D_1 = D_0 + 1^{+p}_{+q}$. Suppose that the graphs H_0 and H_1 are realizations of D_0 and D_1, respectively. Then there exists a witness trail between v_p and v_q that alternates between the edges and non-edges of H_1.*

Proof. Observe that for each vertex $v \in V \setminus \{v_p, v_q\}$, the numbers of neighbors of v in $E(H_0) \setminus E(H_1)$ and in $E(H_1) \setminus E(H_0)$ are the same. In contrast, both v_p and v_q have one more neighbor in $E(H_1) \setminus E(H_0)$ than in $E(H_0) \setminus E(H_1)$.

Consider the symmetric difference $\Delta = E(H_0) \triangle E(H_1)$, and take a maximal trail in Δ such that (i) the trail alternates between edges from $E(H_1) \setminus E(H_0)$ and from $E(H_0) \setminus E(H_1)$, (ii) the first vertex of the trail is v_q, and (iii) the first edge of the trail belongs to $E(H_1) \setminus E(H_0)$. Then the last vertex of the trail must be v_p, and the last edge must also belong to $E(H_1) \setminus E(H_0)$. Hence, this maximal trail satisfies the required conditions. $\square$

A key observation, already made in [5], is that the existence of short witness trails implies P-stability. This is also the first step in our approach.

Theorem 2.3. *Assume that $D \in \mathbb{D}(n, \Sigma, c_1, c_2)$, $1 \le p, q \le n$ and let $D^\diamond = D + 1^{+p}_{+q}$. Suppose G is a graph with degree sequence $D^\diamond$, moreover, $\Gamma_G(v_p) = \Gamma_G(v_q)$. If $\mathbb{D}(n, \Sigma, c_1, c_2)$ is fully graphic, then there exists an 11-witness trail between v_p and v_q in G.*

Proof of Theorem 1.3 from Theorem 2.3. Assume that G' is a graph whose degree sequence is $D' = D + 1^{+i}_{+j}$ for some $1 \le i \le j \le n$.

If $\Gamma_{G'}(v_i) = \Gamma_{G'}(v_j)$, then we can apply Theorem 2.3 with $G = G'$, $p = i$ and $q = j$ to obtain a witness trail P between v_i and v_j. By flipping edges and non-edges along P, we obtain a graph $G^\dagger$ which is a realization of D.

If $\Gamma_{G'}(v_i) \ne \Gamma_{G'}(v_j)$, then there exists an alternating trail Q of length 2 between v_i and v_j. Assume that $Q = v_i v_m v_j$, where (v_i, v_m) is an edge, and (v_m, v_j) is a non-edge. Flipping edges along Q yields a graph G^* with degree sequence

$$D^* = D' + 1^{+j}_{-i} = D + 1^{+i}_{+j} + 1^{-i}_{+j} = D + 1^{+j}_{+j}.$$

Now, applying Theorem 2.3 to $G = G^*$ with $p = q = j$, we obtain a witness trail P from v_j to v_j. Flipping edges and non-edges P results a graph $G^\dagger$, which is a realization of D.

To reconstruct G' from $G^\dagger$, we need to determine whether we were in case where $\Gamma_{G'}(v_i) = \Gamma_{G'}(v_j)$ or in the case $\Gamma_{G'}(v_i) \ne \Gamma_{G'}(v_j)$.

If $\Gamma_{G'}(v_i) = \Gamma_{G'}(v_j)$, we should know P. The trail P contains at most 12 vertices, so there are at most n^{12} possibilities.

If $\Gamma_{G'}(v_i) \ne \Gamma_{G'}(v_j)$, we need to know both P and Q. Since P starts and ends at the same vertex, there are at most n^{11} possibilities for P. Given P, we can compute G^* and identify either v_i or v_j. To determine Q, we need to know its two further vertices, which gives n^2 possibilities for Q. Knowing Q we can compute G'. We should know which endpoint of Q is v_i and which is v_j. In this case we have at most $2 \cdot n^2 \cdot n^{11} = 2 \cdot n^{13}$ possibilities.

Putting together, for a given $G^\dagger$ we have at most $n^{12} + 2 \cdot n^{13} \le 3 \cdot n^{13}$ possibilities for G', i.e. the operation $G' \mapsto G^\dagger$ from $\mathcal{G}(D + 1^{+i}_{+j})$ to $\mathcal{G}(D)$ is "$3 \cdot n^{13}$-to-1". Hence, $\partial^{++}(D) \le 3 \cdot n^{13}$. $\square$

2.1. Hostile configurations

Before continuing our analysis, we introduce the notion of a **hostile configuration**, which enables us to show that certain degree sequences are not graphic.

Definition 2.4. Let $D'' \in \mathbb{D}(n, \Sigma, c_1, c_2)$, $1 \le p \le q \le n$, and let $G' \in \mathcal{G}(D'' + 1^{+p}_{+q})$,. We say that G has a **hostile configuration** iff $V(G')$ can be partitioned into four subsets $V(G') = S \uplus K' \uplus Y' \uplus R'$ satisfying the following properties:

(a) $G'[K']$ is a complete subgraph;

(b) $G'[K'; R']$ is a complete bipartite subgraph;

(c) $G'[Y']$ is an empty subgraph;

(d) $G'[Y'; R']$ is an empty bipartite subgraph;

(e) $S = \{v_p, v_q\}$ and $\Gamma(v_p) \cup \Gamma(v_q) \subseteq K'$.

Lemma 2.5. *Let $D'' \in \mathbb{D}(n, \Sigma, c_1, c_2)$, and $1 \le p \le q \le n$. If $G' \in \mathcal{G}(D'' + 1^{+p}_{+q})$, is a hostile configuration, then the degree sequence D'' is not graphic.*

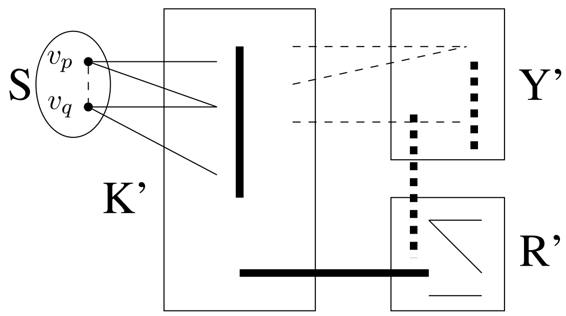


Figure 1: The thick dashed lines indicate no edge at all, the solid thick lines indicate all edges.

Proof. Assume on the contrary that D'' is graphic. Then, by Proposition 2.2, there is an edge-abundant trail $v_p = x_0, x_1, \ldots, x_{2m}, x_{2m+1} = v_q$ of odd length that alternates between the edges and non-edges of G'.

Then, by induction on $0 \le j \le 2m+1$, we can show that $x_j \in S \cup Y'$ whenever j is even, and $x_j \in K'$ whenever j is odd. Indeed, if $x_{2i} \in S \cup Y'$ and (x_{2i}, x_{2i+1}) is an edge, then $x_{2i+1} \in K'$ by (c), (d) and (e). Similarly, if $x_{2i+1} \in K'$ and (x_{2i+1}, x_{2i+2}) is a non-edge, then $x_{2i+2} \in S \cup Y'$ by (a) and (b).

Hence, $x_{2m+1} \neq v_q$, which a contradiction. $\square$

The usefulness of hostile configurations becomes apparent from the following statement, which is at the heart of our argument.

Lemma 2.6. *Assume that $D \in \mathbb{D}(n, \Sigma, c_1, c_2)$, $1 \le p, q \le n$ and let $D^\diamond = D + 1^{+p}_{+q}$. Let G be a realization of $D^\diamond$ where $\Gamma_G(v_p) = \Gamma_G(v_q)$. Suppose there is no 11-witness trail between v_p and v_q in G. Then there is a degree sequence $D'' \in \mathbb{D}(n, \Sigma, c_1, c_2)$ for which $D'' + 1^{+p}_{+q}$ has a realization G' having a hostile configuration.*

Theorem 2.3 is now a trivial consequence of Lemmas 2.5 and 2.6. The rest of this section is devoted to the proof of Lemma 2.6.

2.2. The initial structure and a warm-up case

From here on we use the notation and assumptions of Lemma 2.6. We start by recalling a result from [5] showing that the graph G under consideration has a certain structure. To do so, we define several pairwise disjoint subsets of the vertex set (introduced originally in [5]): Let

$$S = \{v_p, v_q\}, \qquad X = \Gamma(v_p) = \Gamma(v_q), \qquad Y = \{y \in V : |X \setminus \Gamma_G(y)| \geq 2\},$$

furthermore let

$$Z = \Gamma_G(Y) \setminus X, \qquad R = V \setminus (S \cup X \cup Y \cup Z), \qquad K = X \cup Z.$$

The following statements were already explicitly formulated and proved by Jerrum, McKay and Sinclair in [5, Proof of Lemma 1] and were also used in [2, Proof of Lemma 4.5].

Lemma 2.7. *Using the notation and terminology of Lemma 2.6, assume the weaker condition that there is no 7-witness trail between v_p and v_q in G (*in Lemma 2.6 we assume the stronger condition that there is no 11-witness trail*). Then we have:*

(i) v_pv_q *is not an edge;*

(ii) $G[Y]$ *is independent;*

(iii) $G[K]$ *is a clique;*

(iv) $G[Y, R]$ *is empty bipartite graph;*

(v) $|X \setminus \Gamma_G(r)| \leq 1$ *for each* $r \in R$.

At this point we are in the position to give a short, calculation free proof for [2, Lemma 4.5]. While this will not be directly used in the proof of Lemma 2.6, the method serves as an illustration of our approach.

Lemma. *Assume that $D \in \mathbb{D}(n, c_1, c_2)$, $1 \leq p, q \leq n$ and let $D^\diamond = D + 1^{+p}_{+q}$. Suppose G is a graph with vertex set $V = \{v_1, \ldots v_n\}$ and degree sequence $D^\diamond$, moreover, $\Gamma_G(v_p) = \Gamma_G(v_q)$. If $\mathbb{D}(n, c_1, c_2)$ is fully graphic, then there exists a 7-witness trail between v_p and v_q in G.*

Proof. The proof is contrapositive: we assume that there is no 7-witness trail between v_p and v_q in G, and we will derive that $\mathbb{D}(n, c_1, c_2)$ is not fully graphic.

Fix Σ with $D \in \mathbb{D}(n, \Sigma, c_1, c_2)$. Consider the sets (S, X, Y, Z, R) defined in Lemma 2.7. Consider the graph G' obtained from G by removing all the edges from R. Let D' be the degree sequence of G'.

Observe that G' has a hostile configuration with $S = \{v_p, v_q\}$, $K' = X \cup Z$, $Y' = Y \cup R$ and $R' = \emptyset$. Thus $D' - 1^{+p}_{+1}$ is not graphic.

Since $\deg_{G'}(v) = \deg_G(v)$ for $v \in V \setminus R$, and $\deg_{G'}(r) \geq |X| - 1 \geq n_2$ by (v), we have $D' - 1^{+p}_{+p} \in \mathbb{D}(n, c_1, c_2)$. Thus $\mathbb{D}(n, c_1, c_2)$ is not fully graphic. □

Theorem 1.1 now follows from the combination of this Lemma with Lemma 2.5 in the same way as Theorem 1.3 is obtained from Theorem 2.3.

Although the proof of Lemma 2.6 is substantially more intricate, its underlying structure closely parallels that of the above proof: First, assuming the non-existence of an 11-witness trail, we refine the structural analysis by further partitioning the set R into smaller subsets. Then, using hinge-flip operations, we modify the graph G in such a way that (1) the degree sequence of the resulting graph remains in $\mathbb{D}(n, \Sigma, c_1, c_2)$, and (2) the final graph exhibits a hostile configuration. The core difficulty of the argument lies in the fact that it is not sufficient to remain within $\mathbb{D}(n, c_1, c_2)$; we must ensure that all constructions stay within the more restrictive class $\mathbb{D}(n, \Sigma, c_1, c_2)$.

2.3. Proof of Lemma 2.6: the refined structure

We will use the following notation: if x, y, z, u, v, w form an alternating edge-deficient trail in G starting with a non-edge, then we write:

$$x \cdots y - z \cdots u - v \cdots w.$$

Any edge-abundant alternating trail can be described analogously.

Definition 2.8. An edge-deficient trail with length at most five is called **pre-witness trail**.

Lemma 2.9. *(1) There is no pre-witness trail in $G[K \cup R]$ between two different vertices of K.*
(2) There is no pre-witness trail in $G[K \cup R]$ from a vertex $w \in K$ to itself provided $|\Gamma_G(w) \cap Y| \geq 2$.

Proof. (1) Assume on the contrary that $w_1 \cdots a_1 - a_2 \cdots a_3 - a_4 \cdots w_2$ is a pre-witness trail in $G[K \cup R]$ between two different points of K. Pick $y_i \in Y \cap \Gamma_G(w_i)$ for $i = 1, 2$. Since $|X \setminus \Gamma_G(y_i)| \geq 2$ we can pick $\{x_1, x_2\} \in [X]^2$ such that (x_i, y_i) is non-edge for $i = 1, 2$. Then $v_p - x_1 \cdots y_1 - w_1 \cdots a_1 - a_2 \cdots a_3 - a_4 \cdots w_2 - y_2 \cdots x_2 - v_q$ is a witness trail because there is no edge repetition in it because $w_1 \neq w_2$ and $x_1 \neq x_2$.
(2). Assume on the contrary that $w \cdots a_1 - a_2 \cdots a_3 - a_4 \cdots w$ is a pre-witness trail in $G[K \cup R]$ with $w \in K$. Pick $\{y_1, y_2\} \in [Y \cap \Gamma_G(w)]^2$. Pick $\{x_1, x_2\} \in [X]^2$ such that (x_i, y_i) for $i = 1, 2$ are non-edges. Then $v_p - x_1 \cdots y_1 - w \cdots a_1 - a_2 \cdots a_3 - a_4 \cdots w - y_2 \cdots x_2 - v_q$ is a witness trail because there is no edge repetition in it because $y_1 \neq y_2$ and $x_1 \neq x_2$. $\square$

Recall that $K = X \cup Z = \{v_1, \ldots, v_k\}$. Consider the following partition of R:

(r1) $R_0 = \{r \in R : K \subset \Gamma_H(r)\}$,

(r2) $R_i = \{r \in R : K \setminus \Gamma_H(r) = \{v_i\}\}$ for $1 \leq i \leq k$,

(r3) $R_\infty = \{r \in R : |K \setminus \Gamma_H(r)| \geq 2\}$.

(See Figure 2.)

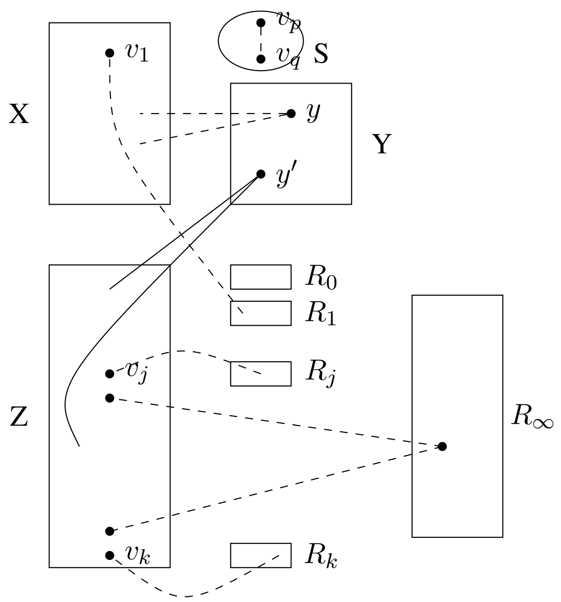


Figure 2: The dashed lines indicate non-edges, the solid lines indicate edges.

Lemma 2.10. *R_∞ does not contain edges.*

Proof. Assume on the contrary that (x, y) in an edge in R_∞. Since $|K \setminus \Gamma_G(x)| \geq 2$ and $|K \setminus \Gamma_G(y)| \geq 2$, we can choose two distinct vertices, $v_i \neq v_j$ from K such that (x, v_i) and (y, v_j) are non-edges. Then $v_i \cdots x - y \cdots v_j$ is a forbidden pre-witness trail of length 3. Contradiction. □

Lemma 2.11. *There is no edge between R_i and R_j for $i \neq j \in (\{1, 2, \ldots, k\} \cup \{\infty\})$*

Proof. Assume on the contrary that $a \in R_i$, $b \in R_j$ and (a, b) is an edge. We can assume that $i \neq \infty$. Then (a, v_i) is a non-edge by the definition of R_i.

If $j = \infty$ pick $k \neq i$ such that (b, v_k) is a non-edge. If, however, $j < \infty$, then let $k = j$. In both cases $k \neq i$ and (v_k, b) is a non-edge. Hence, $v_i \cdots a - b \cdots v_j$ is a forbidden pre-witness trail of length 3. □

Lemma 2.12. *Assume (a, b) is an edge in R_i for some $1 \leq i \leq k$. Then $|\Gamma_G(v_i) \cap Y| = 1$.*

Proof. By Lemma 2.9(2), we have $|\Gamma_G(v_i) \cap Y| \leq 1$ because $x_i \cdots a - b \cdots x_i$ is a pre-witness trail. □

Lemma 2.13. *If there exists an edge (a, b) in R_i for some $1 \leq i \leq k$ then $R_j = \emptyset$ provided $1 \leq j \leq k$ and $j \neq i$.*

Proof. Assume on the contrary that there exists $c \in R_j$ for some $j \neq i, 1 \leq j \leq k$. Observe that (x_i, c) is an edge because $c \in R_j$ and $j \neq i$. Hence, we have a pre-witness trail between x_i and x_j of length 5: $x_i \cdots a - b \cdots x_i - c \cdots x_j$. (see Fig. 3). Contradiction. □

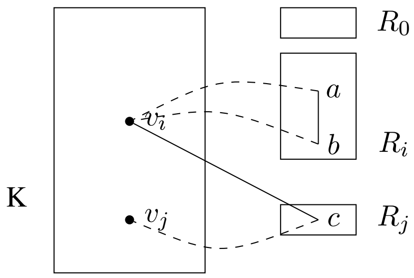


Figure 3: We provide the complete witness trail.

After this preparation we can see that we can distinguish two cases concerning the structure of G.

Case I: $R_N := \bigcup\{R_i : 1 \leq i \leq k\} \cup R_\infty$ is an independent subset,

Case II: There exists $1 \leq i \leq k$ such that R_i contains edges, and so $R = R_0 \cup R_i \cup R_\infty$.

Our goal in the remaining part of the proof is to transform our degree sequence D into a new degree sequence $D'' \in \mathbb{D}(n, \Sigma, c_1, c_2)$ and our graph G into a realization $G' \in \mathcal{G}(D'' + 1^{+p}_{+q})$ which is a hostile configuration.

Definition 2.14. Assume that G is a graph with degree sequence D, and let x, y, z be distinct vertices such that (x, y) is an edge and (x, z) is a non-edge. The **hinge-flip** operation $(x, y) \Rightarrow (x, z)$ deletes the edge (x, y) from H and adds the edge (x, z) to obtain a new graph H'. The degree sequence of H' is $D' = D + 1^{-i}_{+j}$. The operation was already introduced in [6] as **Type 0 transition**.

The hinge-flip operation clearly changes the degree sequence of H, since $\deg_{H'}(y) = \deg_H(y) - 1$ and $\deg_{H'}(z) = \deg_H(z) + 1$. The other degrees are not changed.

2.4. Proof of Lemma 2.6: Case I. There is no edge in R_N.

We generate a sequence of graphs $G_0, G_1, \ldots, G_m$ and their degree sequences $D_0, D_1, \ldots, D_m$ with consecutive hinge-flip operations as follows.

Let $G_0 = G$, and so $D_0 = D^\diamond = D + 1^{+p}_{+q}$. Assume that G_ℓ is already produced. Pick $x_\ell \neq z_\ell \in R_0$ and $y_\ell \in R_N$ such that (x_ℓ, y_ℓ) is an edge and (x_ℓ, z_ℓ) is a non-edge in G_ℓ. Apply the corresponding hinge-flip operation $(x_\ell, y_\ell) \Rightarrow (x_\ell, z_\ell)$ for G_ℓ to obtain $G_{\ell+1}$. (We will call this operation a **downward twist**.) We stop the process if there are no suitable x_ℓ, y_ℓ and z_ℓ.

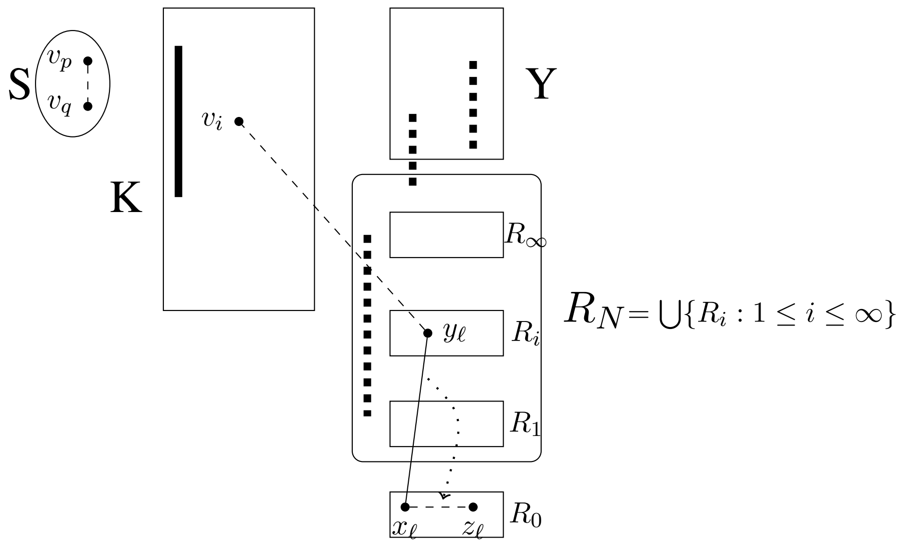


Figure 4: In R_N there is no edge. The vertices $x_j, z_j \in R_0$ and $y_j \in R_N$. The curved, loosely dotted arrow indicates a hinge-flip operation. As earlier: the thick dashed lines indicate no edge at all, the solid thick lines indicate all edges.

Lemma 2.15. $D_m + 1^{-p}_{-q} \in \mathbb{D}(n, \Sigma, c_1, c_2)$.

Proof. The hinge-flip operations keep the sum of the degrees, so $\Sigma(D_m) = \Sigma(D_0) = \Sigma(D) + 2$. Since we added and removed edges only inside $R_N \cup R_0$, we have to show that

$$c_1 \geq \deg_{G_m}(v) \geq c_2$$

whenever $v \in R_N \cup R_0$. Since $\Gamma_{G_m}(v) \cap X = \Gamma_G(v) \cap X$ and $|X \setminus \Gamma_G(v)| \leq 1$ by (v), we have $\deg_{G_m}(v) \geq |X| - 1 \geq c_2$.

For $v \in R_N$, the applied hinge-flip operations can not increase the degree in v. So $\deg_{G_m}(v) \leq \deg_G(v) \leq c_1$. So, to complete the proof of the Lemma, we should show $\deg_{G_{\ell+1}}(v) \leq c_1$ for each $v \in R_0$. Assume on the contrary that $\deg_{G_m}(z) > c_1$ for some $z \in R_0$. Since $\deg_{G_0}(z) \leq c_1$, we increased the degree of z by some hinge-flip operation $(x_\ell, y_\ell) \Rightarrow (x_\ell, z_\ell)$, and so $z = z_\ell$.

Since $y_\ell \in R_N$, we can pick $1 \leq i \neq j \leq k$ such that $v_i y_\ell$ is an edge and $v_j y_\ell$ is a non-edge in G_ℓ. Moreover, the hinge-flip operations $(x_s, y_s) \Rightarrow (x_s, z_s)$ modify the edges only inside R, so

$$v_i y_\ell \text{ is an edge and } v_j y_\ell \text{ is a non-edge in } G$$

as well. We know that the bipartite graph $G_\ell[R_0, \{v_j\}]$ is complete. Hence, we have the following inequality chain:

$$\deg_{G \setminus R_0}(z_\ell) + |R_0| \geq \deg_{G_m}(z_\ell) > c_1 \geq \deg_G(v_j) = \deg_{G \setminus R_0}(v_j) + |R_0|.$$

Thus, there exists a vertex d such that (z_ℓ, d) is an edge, and (v_j, d) is a non-edge in G. Then $d \notin Y \cup S$ because (z_ℓ, d) is an edge and $z_\ell \in R_0$. Moreover, $d \notin K \cup R_0$ because $v_j d$ is a non-edge. Thus, the only possibility is that $d \in R_N$.

Hence, $v_j \cdots d - z_\ell \cdots x_\ell - y_\ell \cdots v_i$ is a forbidden pre-witness trail of length 5 which does not exist by Lemma 2.9(1). $\square$

Define
$$R_0^0 := \{r \in R_0 : \Gamma_m(r) \cap R_N \text{ is not empty}\} \quad \text{and} \quad R_0^1 = R_0 \setminus R_0^0.$$

Lemma 2.16. $R_0 \subseteq \Gamma_m(r)$ *for each* $r \in R_0^0$.

Proof. Assume on the contrary that $z \in R_0$ and (z, r) is a non-edge in G_m. Since $r \in R_0^0$, there is $y \in R_N$ such that (r, y) is an edge in G_m. So we have a further downward twist operation $(r, y) \Rightarrow (r, z)$ in G_m. Contradiction. $\square$

Lemma 2.17. G_m *has a hostile configuration.*

Proof. We have $S = \{v_p, v_q\}$. Define
$$K' := K \cup R_0^0, \quad Y' := Y \cup R_N, \quad R' := R_0^1.$$

We should check that all conditions in Definition 2.4. (a)-(e) hold.

(a) $G_m[K']$ **is a complete subgraph.** Indeed, K is a clique by Lemma 2.7, $[K, R_0]$ is complete by the definition of R_0, and R_0^0 is a clique by Lemma 2.16.

(b) $G_m[K'; R']$ **is a complete bipartite subgraph.** Indeed, $R' = R_0^1 \subset R_0$ and $[K, R_0]$ is complete by the definition of R_0, so $[R', K]$ is complete. Moreover, $[R_0^0, R_0^1]$ is complete by Lemma 2.16.

(c) $G_m[Y']$ **is an empty subgraph.** Indeed, Y is independent by Lemma 2.7. $[R, Y]$ is empty by Lemma 2.7. R_N is independent because we are in **Case I**.

(d) $G_m[Y'; R']$ **is an empty bipartite subgraph.** Indeed, $[Y, R]$ is empty by Lemma 2.7. $[R_N, R_0^1]$ is empty by the definition of R_0^1.

(e) $S = \{v_p, v_q\}$ **and** $\Gamma_{G_m}(v_p) \cup \Gamma_{G_m}(v_q) \subseteq K'$**.** This is trivial from the construction.

$\square$

Therefore, we arrived at a hostile configuration, which completes the proof of Lemma 2.6 in *Case I.*

2.5. *Proof of Lemma 2.6: Case II. There exists* $1 \le i \le k$ *such that* R_i *contains edges.*

Then,
$$R = R_0 \cup R_i \cup R_\infty \text{ and } R_i \text{ contains an edge } ab.$$

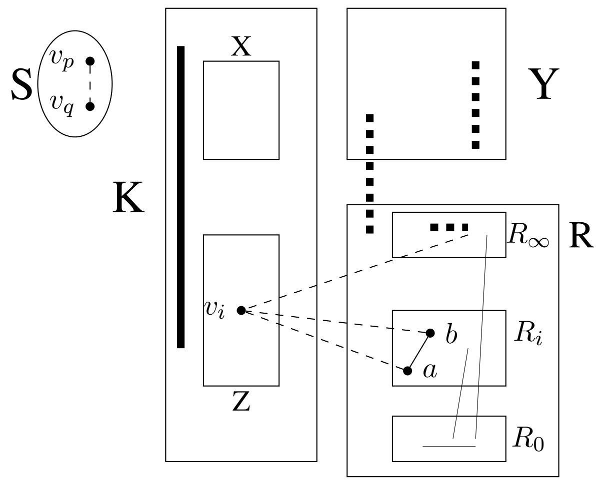


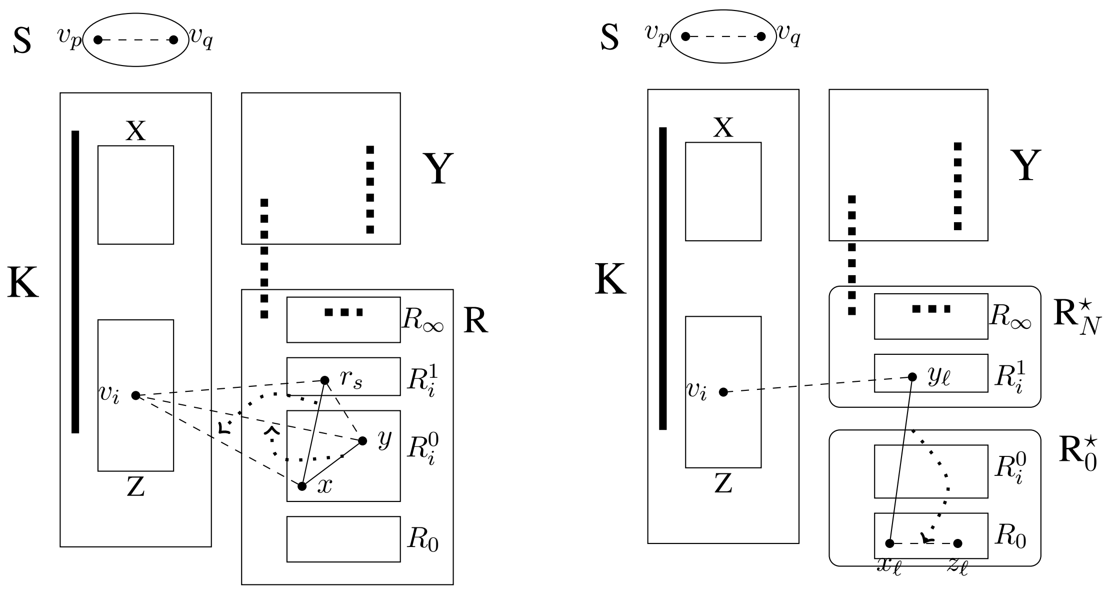
Figure 5: In $Y \cup R_\infty$ there is no edge. The solid lines are existing edges, the dashed line are missing edges The thick lines note all edges / non-edges in the subgraph.

In Case II it will be more challenging to transform our graph G into a graph G_m which has a hostile configuration. We will do it in two steps.

Step 1.

In the first step we will "uplift" edges from R_i into the bipartite graph $G[R_i, K]$ as follows. Let $H_1, \ldots, H_t$ be an enumeration of the connected components of the subgraph $G[R_i]$. In each component H_s fix a vertex r_s and choose a spanning tree T_s with root r_s such that $\Gamma_{T_s}(r_s) = \Gamma_{G[R_i]}(r_s)$. Let define

$$R_i^1 := \{r_\beta : 1 \leq \beta \leq t\} \text{ and } R_i^0 := (R_i \setminus R_i^1). \tag{11}$$



(a) Step 1: Two upward twist operations

(b) Step 2: One downward twist operation

Figure 6: Step 1 and 2

We define the **upward twist** operation of this graph as follows. In each spanning tree T_s let orient the edges outward from the root r_s. Then, for each oriented edge $\overrightarrow{xy}$ execute the hinge-flip operation $(y, x) \Rightarrow (y, v_i)$.

In other words, in each component H_s, delete the edges of the spanning tree T_s, and connect each vertex but r_s in T_s to the vertex v_i. (See Figure 6a.)

As a result of the upward twist operation, we obtain the graph $G^\star$ with degree sequence $D^\star$. Since the upward twist operation was obtained as a sequence of hinge-flip operation, $\sum D^\star = \sum D^\diamond = \sum D + 2$.

Lemma 2.18.

$$\left(D^\star + 1_{-q}^{-p}\right) \in \mathbb{D}(n, \Sigma, c_1, c_2). \tag{12}$$

Proof. If $\Gamma_{G^\star}(v) \neq \Gamma_G(v)$, then $v = v_i$ or $v \in R$. So we should show $c_2 \leq \deg_{G^\star}(v) \leq c_1$ for $v \in \{v_i\} \cup R$.

If $v \in R$, then $\deg_{G^\star}(v) \leq \deg_G(v) \leq c_1$ and $|X \setminus \Gamma_{G^\star}(v)| \leq |X \setminus \Gamma_G(v)| \leq 1$ by Lemma 2.7, so $\deg_{G^\star}(v) \geq |X| - 1 \geq c_2$. Since $\Gamma_{G^\star}(v_i) \cap K = \Gamma_G(v_i) \cap K = K \setminus \{v_i\}$, we have $\deg_{G^\star}(v_i) \geq |K| - 1 \geq c_2$.

Finally, we should prove that $\deg_{G^\star}(v_i)$, does not exceed c_1. To show that fix $1 \leq j \leq k$ such that $j \neq i$ and $v_j \in \Gamma_G(v_p) \cap \Gamma_G(v_q)$. Assume on the contrary that

$$\deg_{G^\star}(v_i) > c_1 \geq |\Gamma_G(v_j)| = |\Gamma_{G^\star}(v_j)|.$$

Since $R_i \setminus \Gamma_{G^*}(v_i) = \{r_1, \ldots, r_t\}$, we have $|R_i \cap \Gamma_{G^\star}(v_i)| \leq |R_i| - |\{r_1, \ldots, r_t\}| \leq |R_i| - 1$. Moreover, $|K \cap \Gamma_{G^\star}(v_i)| = |K \cap \Gamma_{G^\star}(v_j)| = |K| - 1$. Hence,

$$\begin{aligned}(|R_i| - 1) + (|K| - 1) + |\Gamma_{G^\star}(v_i) \setminus (R_i \cup K)| \geq \deg_{G^\star}(v_i) > c_1 \geq \\ \geq \deg_G(v_j) = |\Gamma_G(v_j) \setminus (R_i \cup K)| + |R_i| + |K| - 1.\end{aligned}$$

Since $\Gamma_{G^\star}(v_i) \setminus R_i = \Gamma_G(v_i) \setminus R_i$, we have

$$|\Gamma_G(v_i) \setminus (R_i \cup K)| > |\Gamma_G(v_j) \setminus (R_i \cup K)| + 1.$$

Consequently, we can choose

$$\{d_0, d_1\} \in [\Gamma_G(v_i) \setminus \Gamma_G(v_j) \setminus (R_i \cup K)]^2.$$

Then, $d_\alpha \notin \Gamma_G(v_j)$ implies $d_\alpha \notin R_0 \cup S$. Hence, $\{d_0, d_1\} \subset Y \cup R_\infty$. Since $|\Gamma_G(v_i) \cap Y| \leq 1$ by Lemma 2.12, we can assume that $d_0 \in R_\infty$. Then in $G[K \cup R]$ we have the following pre-witness trail of length 5:

$$v_j \cdots d_0 - v_i \cdots a - b \cdots v_i.$$

Contradiction, we have $\deg_{G^\star}(v_i) \leq c_1$. $\square$

Step 2.

Unfortunately, the graph $G^\star$ may contain witness trails, so we cannot simply say: "apply Case I for $G^\star$". Instead of that, in the second step we just imitate the "downward twist" construction of **Case I** in the graph $G^\star$. Recalling that R_i^1 and R_i^0 were defined in (11), observe that

$$E(G^\star) \setminus E(G) = G^\star[\{v_i\}, R_i^0], \qquad E(G) \setminus E(G^\star) \subset R_i^0 \times (R_i^1 \cup R_i^0).$$

Moreover,

$$E(G^\star) \cap [(R_i^0 \cup R_i^1) \times R_i^1] = \emptyset. \tag{13}$$

We also have

$$\Gamma_{G^\star}(w) \supset K \text{ for } w \in R_i^0. \tag{14}$$

Write

$$R_0^\star = R_0 \cup R_i^0, \qquad R_N^\star = R_\infty \cup R_i^1.$$

Observe that

$$E(G^\star) \cap [R_N^*]^2 = \emptyset \tag{15}$$

by Lemma 2.11 and by $E(G^\star) \cap \left[R_i^1\right]^2 = \emptyset$.

After this preparation, generate a sequence of graphs $G_0, G_1, \ldots, G_m$ and their degree sequences $D_0, D_1, \ldots, D_m$ with consecutive hinge-flip operations as follows.

Let $G_0 = G^\star$. Assume that G_ℓ is already produced. Pick $x_\ell \neq z_\ell \in R_0^\star$ and $y_\ell \in R_N^\star$ such that (x_ℓ, y_ℓ) is an edge and (x_ℓ, z_ℓ) is a non-edge in G_ℓ. Apply the corresponding hinge-flip operation $(x_\ell, y_\ell) \Rightarrow (x_\ell, z_\ell)$ for G_ℓ to obtain $G_{\ell+1}$ (They are just the downward twist operations from **Case I**, see Figure 6b.) We stop the construction if there are no suitable x_ℓ, y_ℓ and z_ℓ.

Observe that

$$\{x_\ell : \ell < m\} \subset R_0. \tag{16}$$

Indeed, (x_ℓ, y_ℓ) is an edge in G_ℓ, and there is no edge between R_i^0 and $R_N^\star = R_i^1 \cup R_\infty$ in G_ℓ by (13) and by Lemma 2.11.

Lemma 2.19. $D_m + 1_{-q}^{-p} \in \mathbb{D}$.

Proof. Since we carried out hinge flip operations, $\sum D_m = \sum D^\star = \sum D^\diamond = \sum (D + 1_{+q}^{+p}) = \sum D + 2$, furthermore we changed degrees only in $R_0^\star \cup R_N^\star$.

In $R_N^\star$ we decreased the degrees. Moreover, for each $r \in R_N^\star$, we have $|X \setminus \Gamma_{G_m}(r)| = |X \setminus \Gamma_{G^\star}(r)| \leq |X \setminus \Gamma_G(r)| \leq 1$ by Lemma 2.7, and so $\deg_{G_m}(r) \geq |\Gamma_G(r) \cap X| - 1 \geq c_2$. So $c_2 \leq \Gamma_{G_m}(r) \leq c_1$ for $r \in R_N^\star$.

In $R_0^\star$ we increased some degrees. So, to complete the proof of the Lemma, we should show $\deg_{G_m}(z) \leq c_1$ for each $z \in R_0^\star$.

Assume on the contrary, that $d_{G_m}(z) > c_1$ for some $z \in R_0^\star$. Since $\deg_{G^\star}(z) \leq c_1$, we increased the degree of z by some hinge flip operation, i.e., there is $1 \leq \ell \leq m$ such that $z_\ell = z$. Then $x_\ell, z_\ell \in R_0^\star$ and $y_\ell \in R_N^\star$, furthermore (x_ℓ, y_ℓ) is an edge and (x_ℓ, z_ℓ) is a non-edge in G_ℓ. By (16) we know $x_\ell \in R_0$. Hence,

$$(x_\ell, y_\ell) \text{ is an edge in } G, \text{ and } (x_\ell, z_\ell) \text{ is a non-edge in } G. \tag{17}$$

Since $y_\ell \in R_N^*$, we can pick $1 \leq j \leq k$ such that $j \neq i$ and $v_j \in X \cap \Gamma_G(z_\ell)$. We know that the bipartite graph $G_\ell[R_0^\star, \{v_j\}]$ is complete by (14) and by the definition of R_0. Hence, we have the following inequality chain:

$$\deg_{G^\star \setminus R_0^\star}(z_\ell) + (|R_0^\star| - 1) \geq \deg_{G_m}(z_\ell) > c_1 \geq \deg_G(v_j) = \deg_{G \setminus R_0}(v_j) + |R_0^\star|.$$

Moreover, $\Gamma_{G^\star \setminus R_0^\star}(z_\ell) \setminus \Gamma_{G \setminus R_0^\star}(z_\ell) \subset \{v_i\}$, and so $\deg_{G^\star \setminus R_0^\star}(z_\ell) \leq \deg_{G \setminus R_0^\star}(z_\ell) + 1$.

Putting together, we obtain

$$\deg_{G \setminus R_0^\star}(z_\ell) > \deg_{G \setminus R_0}(v_j).$$

Thus, there exists a vertex d such that (z_ℓ, d) is an edge and (v_j, z_ℓ) is a non-edge in G.

Then $d \notin Y \cup S$ because (z_ℓ, d) is an edge and $z_\ell \in R_0^\star$. Moreover, $d \notin K \cup R_0$ because (v_j, d) is a non-edge. Thus, the only possibility is that $d \in R_N$.

- If $y \in R_i$, then let $i^\star = i$.

- If $y \in R_\infty$, let $i^\star \neq j$ such that $(v_{i^\star}, y_\ell)$ is a non-edge.

Thus $i^\star \neq j$, and

$$v_{i^\star} \cdots y_\ell - x_\ell \cdots z_\ell - d \cdots v_j$$

is a prohibited pre-witness trail of length 5 in $G[K \cup R]$.

The obtained contradiction proves that $d_{G_m}(z) \leq c_1$ for all $z \in R_0^\star$. □

We carried out the construction, and we verified that the degree sequence D_m can be obtained as $D'_m + 1_{+q}^{+p}$ for some $D'_m \in \mathbb{D}$. Finally, we will find a hostile configuration for G_m. Let

$$R_K = \{r \in R_0^\star : \text{there is } G_m\text{-edge between } r \text{ and } R_N^\star\} = \Gamma_{G_m}(R_N^*) \cap R_0^*.$$

Lemma 2.20. *$R_K \subset R_0$ and $[R_K, R_0^\star] \subset E(G_m)$.*

Proof. Since

$$E(G_m) \cap [R_i^0, R_i^1] \subset E(G^*) \cap [R_i^0, R_i^1] = \emptyset \tag{18}$$

by (13), and

$$E(G_m) \cap [R_i^0, R_\infty] \subset E(G) \cap [R_i, R_\infty] = \emptyset \tag{19}$$

by Lemma 2.11, we have

$$E(G_m) \cap [R_i^0, R_N^*] = (E(G_m) \cap [R_i^0, R_i^1]) \cup (E(G_m) \cap [R_i^0, R_\infty]) = \emptyset. \tag{20}$$

Hence, $R_K \cap R_i^0 = \emptyset$, and so $R_K \subset R_0$.

To show $[R_K, R_0^\star] \subset E(G_m)$ assume on the contrary that $x \in R_K$ and $z \in R_0^\star$ such that $(x, z) \notin E(G_m)$. Since $x \in R_K$, there is $y \in R_N^*$ such that (x, y) is an edge in G_m. So we have a further downward twist operation $(x, y) \Rightarrow (x, z)$ in G_m. Contradiction. □

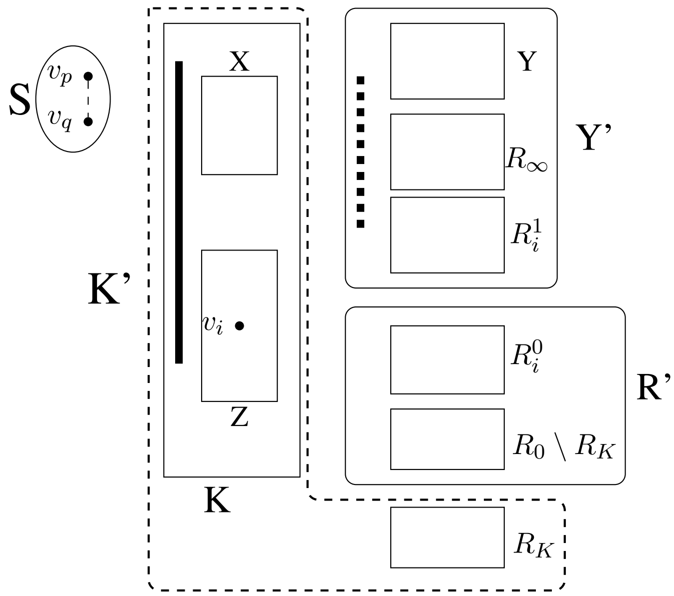


Figure 7: The hostile configuration after **Step 2**

Lemma 2.21. *G_m has a hostile configuration.*

Proof. We have the set $S = \{v_p, v_q\}$, and let

$$K' = K \cup R_K, \qquad Y' = Y \cup R_N^*, \qquad R' = R_0^* \setminus R_K.$$

We should check that conditions 2.4.(a)-(e) hold for the partition (S, K', Y', R').

(a) $G_m[K']$ **is a complete subgraph.** Indeed, K is a clique by Lemma 2.7(iii). R_K is a clique by Lemma 2.20. $[R_K, K]$ is complete because $R_K \subset R_0$ by Lemma 2.20, and $[R_0, K]$ is complete by the definition of R_0.

(b) $G_,[K'; R']$ **is a complete bipartite subgraph.** Indeed, $[K, R_0^*]$ is complete by (14). Moreover, $R' \subset R_0^*$ and $[R_K, R_0^*] \subset E(G_m)$ by Lemma 2.20.

(c) $G_m[Y']$ **is an empty subgraph.** Indeed $[Y, Y \cup R]$ is empty in G by Lemma 2.7(ii,iv) , and R_N^* is empty by (15).

(d) $G_m[Y'; R']$ **is an empty bipartite subgraph.** Indeed, $[Y, R]$ is empty by Lemma 2.7. $[R_N^*, R']$ is empty by the definition of R_K.

(e) $S = \{v_p, v_q\}$ **and** $\Gamma_{G_m}(v_p) \cup \Gamma_{G_m}(v_q) \subseteq K'$**.** Indeed, $\Gamma_{G_m}(v_p) \cup \Gamma_{G_m}(v_q) = \Gamma_G(v_p) \cup \Gamma_G(v_q) \subseteq K \subseteq K'$.

Hence, the partition really witnesses a hostile configuration. $\square$

This completes the proof of Lemma 2.6 $\square$

2.6. An application

Theorem 1.3 provides a new, general, method to verify the P-stability of certain regions. As a consequence we obtain the following strengthening of the statement of condition (P3):

Theorem 2.22. *For any constant $\varepsilon > 0$, the simple region $\mathbb{D}[\varphi_{GS+}]$ defined by the inequality*

$$\varphi_{GS+} \equiv \left(n \geq \frac{1}{2\varepsilon^2}, 2 \leq c_2 \text{ and } 3 \leq c_1 \leq \sqrt{(1-\varepsilon)\Sigma} \right).$$

is fully graphic and hence P-stable.

Proof. In [2, Theorem 3.4] it was proved that if

$$D \in \mathbb{D}[2 \leq c_2, 3 \leq c_1 \leq \sqrt{(1-\varepsilon)\Sigma}]$$

is not fully graphic, then

$$n < \frac{1}{8(1-\sqrt{1-\varepsilon})^2} \leq \frac{1}{2\varepsilon^2} \tag{21}$$

So for all ε there exists an n_ε that for all $n > n_\varepsilon$ the region $\mathbb{D}[\varphi_{GS+}]$ is fully graphic.

$\square$

3. Almost all not fully graphic simple regions are also not P-stable

We begin by recalling a useful characterization of graphicality from [2] and use it to prove Lemma 1.5. For the simple degree sequence region $\mathbb{D}(n, \Sigma, c_1, c_2)$ we define the degree sequence $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ as follows:

$$\mathrm{LEG}(n, \Sigma, c_1, c_2) = \underbrace{c_1, \ldots, c_1}_{\lfloor \alpha \rfloor}, a, \underbrace{c_2, \ldots, c_2}_{(n-1)-\lfloor \alpha \rfloor}, \tag{22}$$

where

$$\alpha = \frac{\Sigma - n \cdot c_2}{c_1 - c_2} \quad \text{and} \quad a = c_2 + (c_1 - c_2) \cdot \{\alpha\}. \tag{23}$$

The following simple statement characterizes the fully graphic simple degree sequence regions in terms of its associated LEG sequence:

Lemma 3.1 ([2, Theorem 2.5])**.** *The simple region $\mathbb{D}(n, \Sigma, c_1, c_2)$ is fully graphic if and only if the degree sequence $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ is graphic.* □

Proof of Lemma 1.5. Assume that a simple degree sequence region $\mathbb{D}(n, \Sigma, c_1, c_2)$ is not fully graphic. Then Lemma 3.1 implies that, in particular, the sequence $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ is not graphic.

Tripathi and Vijay showed in [10] that for this particular type of sequence non-graphicality implies $\alpha \geq c_2$ and that the Erdős–Gallai inequality fails for either $k = \lfloor \alpha \rfloor$ or $k = \lceil \alpha \rceil$. That is,

$$\begin{aligned} c_1 k &> k(k-1) + \min(c_2, k)(n-k-1) + \min(a, k) &&\geq k(k-1) + c_2(n-k) \text{ for } k = \lfloor \alpha \rfloor \text{ or} \\ c_1 k &\geq a + c_1(k-1) > k(k-1) + \min(c_2, k)(n-k) &&\geq k(k-1) + c_2(n-k) \text{ for } k = \lceil \alpha \rceil. \end{aligned}$$

Therefore, we have for $k = \lfloor \alpha \rfloor$ or $k = \lceil \alpha \rceil$ that

$$c_1 k > k(k-1) + c_2(n-k), \tag{24}$$

or, equivalently, that

$$0 > k^2 - (c_2 + c_1 + 1)k + nc_2. \tag{25}$$

A simple algebraic manipulation shows that the discriminant of the right-hand side, $(c_2 + c_1 + 1)^2 - 4nc_2$, is equal to Q from (6). Since the discriminant has to be positive, this implies (6). Furthermore, $\lfloor \alpha \rfloor$ or $\lceil \alpha \rceil$ has to lie between the two roots of the equation

$$k^2 - (c_2 + c_1 + 1)k + nc_2 = 0,$$

which implies (7) by the identity (23). □

The proof strategy of Theorem 1.6 is rather straightforward. There is a known construction that yields a degree sequence with an exponential boundary: the degree sequence of the Tyskevich product of a half-graph and a bipartite graph. Our aim is then to show that there is a such a sequence in $D(n, \Sigma, c_2, c_1)$ whenever Σ satisfies (9).

3.1. Half-graphs and split degree sequences

The unique realization of the following degree sequence is called the **half-graph**. (The name was originally coined by Paul Erdős and A. Hajnal.)

$$\mathbf{h}_{r/2} = (1, 2, \ldots, r/2, r/2, \ldots, r-1).$$

Its relevance for us comes from the following estimate from [5]

$$\partial^{++}(\mathbf{h}_{r/2}) = \Theta\left(3^{r/2}\right) \tag{26}$$

We will consider a graph G where $V(G)$ can be partitioned into three disjoint subsets $X \uplus Y \uplus R$ such that:

- $G[X]$ is a complete subgraph,
- $G[Y]$ is an empty subgraph,
- $G[X, R]$ is a complete bipartite graph,
- $G[Y, R]$ is an empty bipartite graph,
- The degrees sequence of $G[R]$ is $(\mathbf{h}_{r/2})$.

According to the proof of [5, Theorem 8.2], the degree sequence of this graph has boundary quotient at least $\partial^{++}(\mathbf{h}_{r/2})$.

Let us fix the size of R to an even value $r = |R|$. For any $c_2 \leq x \leq c_1 - r + 1$ we can consider a partition with $x = |X|$ and $n - r - x = y = |Y|$. We are going to determine, what values of Σ can a graph have that is constructed in the above manner, and with the given sizes for X, Y, R. Everything in the graph is fixed except the bipartite portion between the sets X and Y, so

$$\Sigma = r^2/2 + 2xr + x(x-1) + 2|E(X, Y)|.$$

For that subgraph the only constraints are that the nodes in Y need to have degree $\geq c_2$ while nodes in X must have degree $\leq c_1 - x - r + 1$.

Lemma 3.2. *For any e satisfying*

$$c_2(n - x - r) \leq e \leq x(c_1 - x - r + 1) \tag{27}$$

there is a bipartite graph $G'(X, Y)$ with e edges that satisfies the degree constraints described above.

Proof. Pick a bi-degree sequence that takes only 2 adjacent values on the X side as well as on the Y side, with the sum being equal to e on both sides. This sequence clearly satisfies the lower and upper constraints, and it is easy to verify that there is a bipartite graph with this given bi-degree sequence. □

Definition 3.3. Let I^x denote the interval given by

$$\begin{aligned}
&I^x = [I_0^x, I_1^x] \text{ where} \\
&I_0^x := r^2/2 + x(x + 2r - 1) + 2c_2(n - x - r) \text{ and} \\
&I_1^x := r^2/2 + x(x + 2r - 1) + 2x(c_1 - x - r + 1).
\end{aligned}$$

Corollary 3.4. *If $\Sigma \in I^x$ for some $c_2 \leq x \leq c_1 - r + 1$ then $D(n, \Sigma, c_2, c_1)$ is not P-stable.*

From now on our goal is to show that $\cup I^x$ covers the interval described in (9). Upon closer inspection, it is evident that both I_0^x and I_1^x are monotone increasing in x. The key to showing that their union covers a large interval, then, is to show that they overlap for consecutive values of x.

Claim 3.5. *$I^x \cap I^{x+1} \neq \emptyset$ if and only if $I_0^{x+1} \leq I_1^x$, that is when*

$$(x+1)(x+2r) + 2c_2(n - x - 1 - r) \leq x(x + 2r - 1) + 2x(c_1 - x - r - +1),$$

or, equivalently,

$$x^2 - x(c_1 + c_2 - r) + r + c_2(n - 1 - r) \leq 0 \tag{28}$$

Let $x_{\min}(r)$ and $x_{\max}(r)$ denote the smallest and largest integers satisfying (28) and let

$$\Sigma_{\min}(r) = I_0^{x_{\min}(r)} = \frac{r^2}{2} + x_{\min}(r) \cdot (x_{\min}(r) + 2r - 1) + 2c_2(n - x_{\min}(r) - r)$$
$$\Sigma_{\max}(r) = I_1^{x_{\max}(r)} = \frac{r^2}{2} + x_{\max}(r) \cdot (x_{\max}(r) + 2r - 1) + 2x_{\max}(r)(c_1 - x_{\max}(r) - r + 1).$$

Corollary 3.6. *The simple region $D(n, \Sigma, c_2, c_1)$ is not P-stable for*

$$\Sigma_{\min}(r) \leq \Sigma \leq \Sigma_{\max}(r)$$

3.2. Bounding the interval

Proof of Theorem 1.6. All we need to establish that the interval defined by (9) is contained in $[\Sigma_{\min}(r), \Sigma_{\max}(r)]$. Or rather, we need to determine a value of ε that allows us to establish this, and then we need to prove that this $\varepsilon \leq \frac{3(r+3)}{\beta(c_1-c_2)}$.

Using that $x_{\min}(r)$ and $x_{\max}(r)$ satisfy (28) we can further bound the endpoints of the $[\Sigma_{\min}(r), \Sigma_{\max}(r)]$ interval:

$$\begin{aligned}
\Sigma_{\min}(r) = I_0^{x_{\min}(r)} &\leq I_0^{x_{\min}(r)} + x_{\min}(r)(c_1 + c_2 - r) - x_{\min}(r)^2 - c_2(n - 1 - r) \\
&= \frac{r^2}{2} + x_{\min}(r)(c_1 - c_2 + r - 1) + c_2(n - r - 1) - r \\
&\leq x_{\min}(r)(c_1 - c_2) + c_2 n + r\left(x_{\min}(r) - c_2 + \frac{r}{2}\right) \\
&\leq (x_{\min}(r) + r)(c_1 - c_2) + nc_2 \qquad (29) \\
\Sigma_{\max}(r) = I_1^{x_{\max}(r)} &\geq I_1^{x_{\max}(r)} + x_{\max}(r)^2 - x_{\max}(r)(c_1 + c_2 - r) + c_2(n - 1 - r) \\
&= \frac{r^2}{2} + x_{\max}(r)(c_1 - c_2 + r + 1) + c_2(n - 1 - r) + r \\
&\geq x_{\max}(r)(c_1 - c_2) + c_2 n + r\left(x_{\max}(r) - c_2 + \frac{r}{2}\right) \\
&\geq x_{\max}(r)(c_1 - c_2) + c_2 n \qquad (30)
\end{aligned}$$

Thus, to show $\Sigma_{\min}(r) \leq \Sigma \leq \Sigma_{\max}(r)$ it suffices to prove

$$x_{\min}(r) + r \leq \frac{\Sigma - nc_2}{(c_1 - c_2)} \leq x_{\max}(r)$$

From (9) we know that

$$(c_1 + c_2 + 1) - (1 - \varepsilon)\sqrt{Q} \leq \frac{\Sigma - nc_2}{(c_1 - c_2)/2} \leq (c_1 + c_2 + 1) + (1 - \varepsilon)\sqrt{Q},$$

hence we will be done if we choose ε such that

$$2r + 2x_{\min}(r) \leq (c_1 + c_2 + 1) - (1 - \varepsilon)\sqrt{Q} \text{ and}$$
$$2x_{\max}(r) \geq (c_1 + c_2 + 1) + (1 - \varepsilon)\sqrt{Q}$$

We know that $x_{\max}(r)$ is the largest integer solution of (28) hence

$$x_{\max}(r) \geq \frac{(c_2 + c_1 - r) + \sqrt{(c_2 + c_1 - r)^2 - 4c_2(n - r - 1) - 4r}}{2} - 1$$

so

$$\begin{aligned}
2x_{\max}(r) - (c_1 + c_2 + 1) &\geq \sqrt{(c_2 + c_1 - r)^2 - 4(c_2(n - r - 1) + r)} - (r + 3) \\
&= \sqrt{(c_1 - c_2 - r)^2 - 4c_2(n - c_1 - 1) - 4r} - (r + 3)
\end{aligned}$$

Similarly, $x_{\min}(r)$ is the smallest integer solution of (28), so

$$x_{\min}(r) \leq \frac{(c_2 + c_1 - r) - \sqrt{(c_2 + c_1 - r)^2 - 4c_2(n - r - 1) - 4r}}{2} + 1$$

so

$$\begin{aligned}
2r + 2x_{\min}(r) - (c_1 + c_2 + 1) &\leq -\sqrt{(c_2 + c_1 - r)^2 - 4(c_2(n - r - 1) + r)} + (r + 1) \\
&= -\sqrt{(c_1 - c_2 - r)^2 - 4c_2(n - c_1 - 1) - 4r} + (r + 1)
\end{aligned}$$

To simplify notation, it will be convenient to introduce:

$$Q(r) := (c_1 - c_2 - r)^2 - 4c_2(n - 1 - c_1) + 4r.$$

Now we simply set

$$\varepsilon = 1 - \frac{\sqrt{Q(r)} - r - 3}{\sqrt{Q}}. \tag{31}$$

This is equivalent to

$$(1 - \varepsilon)\sqrt{Q} = \sqrt{Q(r)} - (r + 3).$$

Putting everything together, for Σ satisfying (9) we can combine all the above steps and get

$$\begin{aligned}
\frac{\Sigma_{\min}(r) - c_2 n}{(c_1 - c_2)/2} - (c_1 + c_2 + 1) &\leq 2r + x_{\min}(r) - (c_1 + c_2 + 1) \\
&\leq r + 1 - \sqrt{Q(r)} \\
&\leq -(1 - \varepsilon)\sqrt{Q} \\
&\leq \frac{\Sigma - nc_2}{(c_1 - c_2)/2} - (c_1 + c_2 + 1) \\
&\leq (1 - \varepsilon) \cdot \sqrt{Q} \\
&= \sqrt{Q(r)} - (r + 3) \\
&\leq 2x_{\max}(r) - (c_1 + c_2 + 1) \leq \frac{\Sigma_{\max}(r) - nc_2}{(c_1 - c_2)/2} - (c_1 + c_2 + 1)
\end{aligned}$$

and hence $\Sigma_{\min}(r) \leq \Sigma \leq \Sigma_{\max}(r)$

The only task left is to bound ε. To that end, let us calculate

$$\begin{aligned}
\sqrt{Q} - (\sqrt{Q(r)} - r - 3) &= r + 3 + \frac{Q - Q(r)}{\sqrt{Q} + \sqrt{Q(r)}} \\
&= r + 3 + \frac{(r + 1)(2c_1 - 2c_2 + 1 - r) - 4r}{\sqrt{Q} + \sqrt{Q(r)}} \\
&\leq (r + 3) \cdot \left(\frac{\sqrt{Q} + 2c_1 - 2c_2 + 1 - r}{\sqrt{Q}} \right) \\
&\leq (r + 3) \cdot \left(\frac{3c_1 - 3c_2 + 2 - r}{\sqrt{Q}} \right) \\
&\leq 3(r + 3) \cdot \frac{(c_1 - c_2)}{\sqrt{Q}},
\end{aligned}$$

where we used $\sqrt{Q} \le c_1 - c_2 + 1$ along with some rather crude estimates. Recall that our assumption was $(1-\beta)(c_1 - c_2 + 1)^2 \ge 4c_2(n-1-c_1)$, equivalently $Q > \beta(c_1 - c_2 + 1)^2$. Using this assumption on Q we see that

$$\varepsilon = \frac{\sqrt{Q} - \sqrt{Q(r)} + r + 3}{\sqrt{Q}} \le 3(r+3)\frac{c_1 - c_2}{Q} \le 3(r+3)\frac{c_1 - c_2}{\beta(c_1 - c_2 + 1)^2} \le \frac{3(r+3)}{\beta(c_1 - c_2)}$$

as claimed. □

4. Future directions

Infinite P-stable degree sequence sets are known to exist for both bipartite and directed graphs. Moreover, for these classes, the associated switch Markov chains are known to be rapidly mixing (see [3]). Nevertheless, our understanding of such infinite sets remains considerably less complete than in the simple (undirected) case, although the known examples display notable structural analogies. Consequently, we intend to extend our investigation to the bipartite and directed settings, where we expect to find analogous relationships between P-stable sets and "fully graphic" sets.

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