SAMPLING DENSITY FOR GABOR PHASE RETRIEVAL

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Abstract. Gabor phase retrieval stands for recovering a square integrable function up to a global phase from absolute values of its Gabor transform. In this paper, we study Gabor phase retrieval from discrete samples. We consider three types of sampling sequences, which include square root lattices, square root sequences on two intersecting lines and on three parallel lines respectively. In all cases we give the optimal sampling density for a sequence to do Gabor phase retrieval.

1. Introduction and main results

Phase retrieval arises in recovering a signal from its intensity measurements, which is widely applied in X-ray crystallography, optical imaging, electron microscopy and many other fields [44]. Recently, phase retrieval has been studied in various aspects, which include phaseless sampling in shift invariant spaces [10–13, 16, 20, 39, 40, 42, 45, 46], phase retrieval for quaternion signals [41, 52], affine linear measurements [17, 18, 32], masked Fourier measurements [37, 38] and sparse signals [50], and the stability of recovery [2, 3, 5, 9, 21, 27, 30, 31, 51, 55].

For the short time Fourier transform

$$V_g f(t,\omega) = \int_{\mathbb{R}} f(x)\bar{g}(x-t)e^{-2\pi i x\omega}\mathrm{d}x, \tag{1.1}$$

where $g \in L^2(\mathbb{R})$ is a window function and $f \in L^2(\mathbb{R})$ is a signal, it is asked if it is possible to recover f up to a global phase from the absolute values of its short time Fourier transform on some set $\Lambda \subset \mathbb{R}^n$. If it is the case, i.e., $|V_g f_1(t,\omega)| = |V_g f_2(t,\omega)|$ for all $(t,\omega) \in \Lambda$ implies that there is some constant phase α such that $f_1 = e^{i\alpha} f_2$, we say that Λ does phase retrieval for the window function g.

Whenever the window function is the Gaussian

$$\varphi(x) := e^{-\pi x^2},$$

the short time Fourier transform is also known as the Gabor transform. And we say that Λ does Gabor phase retrieval if it does phase retrieval for the Gaussian window φ.

For general window functions and bandlimited signals, Alaifari and Wellershoff [6] showed that an equally spaced sequence in certain line does phase retrieval for real-valued bandlimited functions. Zhang, Guo, Liu and Li [54] extended this result for vector-valued functions. Matthias [47] showed that two sequences taken respectively from two lines do Gabor phase retrieval for bandlimited functions. See also Grohs and Liehr [22] and Wellershoff [48] for the phase retrieval of complex-valued compactly supported functions, and [1, 4, 24, 28, 29] for the stability of Gabor phase retrieval.

Key words and phrases. Phase retrieval, Gabor transform, sampling density, square root lattice.

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In this paper, we focus on density conditions of Gabor phase retrieval for square integrable functions.

We see from the frame theory that for a well-chosen window function, any function in $L^2(\mathbb{R})$ is uniquely determined by its sampled values on a lattice with sufficiently large density [14, 15, 19]. However, things become quite different for phase retrieval. Alaifari and Wellershoff [7] proved that any lattice does not do Gabor phase retrieval. And in [23], Grohs and Liehr showed that for any window function, any lattice does not do phase retrieval. Recently, Grohs, Liehr and Rathmair [26] gave a positive result which shows that for well chosen four window functions, a lattice with certain density does phase retrieval.

An interesting problem arises. Is it possible for a sequence of points to do phase retrieval with only one window function?

The answer is positive. In [25], Grohs and Liehr studied Gabor phase retrieval for d-dimensional signals. Here we cite a 1-dimensional version. For any $0 < \nu < 1$, denote

$$\mathbb{Z}^\nu := \{\pm n^\nu : n \geq 0, n \in \mathbb{Z}\}.$$

Grohs and Liehr showed that for a large class of window functions, the square root lattice

$$\Lambda := a\mathbb{Z}^{1/2} \times b\mathbb{Z}^{1/2}$$

does phase retrieval when a and b are small enough. As a consequence, they obtained that if

$$0 < a < \frac{1}{\sqrt{2\pi e}} \quad \text{and} \quad 0 < b < \frac{1}{\sqrt{2\pi e}},$$

then Λ does Gabor phase retrieval.

In this paper, we improve this result. We show that for the square root lattice to do Gabor phase retrieval, only one of a and b needs to be small enough. Moreover, we obtain the optimal sampling density for Gabor phase retrieval. Specifically, we have the following result.

Theorem 1.1. *Let $\Lambda = a\mathbb{Z}^{1/2} \times b\mathbb{Z}^{1/2}$, where $a, b > 0$. If $a < 1$ or $b < 1$, then Λ does Gabor phase retrieval.*

Moreover, if $a = b > 1$, then Λ does not do Gabor phase retrieval.

In [49], Wellershoff studied the phase retrieval for entire functions. It was shown that an entire function with finite order is uniquely determined up to a global phase by its absolute values on three parallel lines for which the ratio of distances between these lines are irrational numbers. With help of this result, we show that certain sampling set on such three lines does Gabor phase retrieval. Moreover, we obtain the optimal sampling density.

Let $\mathbb{Q}$ be the set consisting of all rational numbers. For any $a, \nu > 0$ and $\theta \in [0, 2\pi]$, denote

$$a\mathbb{Z}^\nu(\sin\theta, \cos\theta) := \{(\lambda \sin\theta, \lambda\cos\theta) : \lambda \in a\mathbb{Z}^\nu\}.$$

When $\theta = \theta_0$ is fixed, the distance between two parallel lines $\{(t_l, \omega_l) + x(\sin\theta_0, \cos\theta_0) : x \in \mathbb{R}\}$, $l = 1, 2$ is

$$|(\omega_1 - \omega_2)\sin\theta_0 - (t_1 - t_2)\cos\theta_0|.$$

Theorem 1.2. *Suppose that $\Lambda = \cup_{l=1}^3 \big((t_l, \omega_l) + a\mathbb{Z}^\nu(\sin\theta_0, \cos\theta_0)\big)$, where a and ν are positive numbers, $\theta_0 \in [0, 2\pi]$, (t_1, ω_1), (t_2, ω_2) and (t_3, ω_3) are three points and*

$$d_l := |(\omega_l - \omega_3)\sin\theta_0 - (t_l - t_3)\cos\theta_0| > 0, \quad l = 1, 2.$$

Then Λ does Gabor phase retrieval if $d_1/d_2 \notin \mathbb{Q}$ and either $\nu < 1/2$ or $\nu = 1/2$ with $a < 1$.

Moreover, Λ does not do Gabor phase retrieval if one of the following conditions is satisfied,

(i) *$d_1/d_2 \in \mathbb{Q}$;*
(ii) *$\nu > 1/2$;*
(iii) *$\nu = 1/2$, $a > 1$, (t_1, ω_1), (t_2, ω_2) and (t_3, ω_3) are in the same line whose direction is $(\cos\theta_0, -\sin\theta_0)$, and $d_l = a n_l^{1/2}$ for some integers n_l, $l = 1, 2$. See Figure 1.*

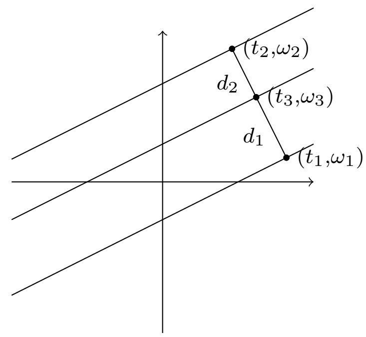

FIGURE 1. Gabor phase retrieval over three parallel lines

On the other hand, Jaming [33] proved that two lines with an angle which is not a rational multiple of π does phase retrieval for entire functions of finite orders. For Gabor phase retrieval on intersecting lines, we also get the optimal sampling density.

Theorem 1.3. *Suppose that $\Lambda = \cup_{l=1}^{2}\big((t_0, \omega_0) + a\mathbb{Z}^{\nu}(\sin\theta_l, \cos\theta_l)\big)$, where a and ν are positive numbers, $(t_0, \omega_0) \in \mathbb{R}^2$. If $(\theta_1 - \theta_2)/\pi \notin \mathbb{Q}$ and either $\nu < 1/2$ or $\nu = 1/2$ with $a < 1$, then Λ does Gabor phase retrieval.*

Moreover, Λ does not do Gabor phase retrieval if one of the following conditions is satisfied,

(i) *$(\theta_1 - \theta_2)/\pi \in \mathbb{Q}$;*
(ii) *$\nu > 1/2$;*
(iii) *$\nu = 1/2$, $a > 1$ and $|\cos(\theta_1 - \theta_2)| > 1/a^2$;*
(iv) *$\nu = 1/2$, $a > 1$ and $|\sin(\theta_1 - \theta_2)| > 1/a^2$;*
(v) *$\nu = 1/2$, $a > 1$ and $|\sin(2\theta_1 - 2\theta_2)| > 1/a^2$.*

To prove the above results, the main ingredients include the Bargmann transform and the distribution of zeros of entire functions. We generalize a classical result of Carlson, which was originally applied to entire functions of exponential type, to entire functions of high orders. In Section 2, we give some necessary preliminaries to be used in the proofs of the main results. In Section 3, we give proofs for Theorems 1.1, 1.2 and 1.3. In Section 4, we show that the sequence $a\mathbb{Z}^{1/2}$ in the main results can be replaced by irregular sequences satisfying certain density conditions. And in Section 5, we show that our method also works for multivariate signals and more general window functions introduced in [25].

2. Preliminaries

In this section, we collect some preliminaries to be used in the proofs.

For two functions f and g, $f \sim g$ means that there is some $\alpha \in [0, 2\pi]$ such that $f(x) = e^{i\alpha} g(x)$ for almost all $x \in \mathbb{R}$.

$\mathcal{F}f$ and $\hat{f}$ denote the Fourier transform of a function f, i.e.,

$$\mathcal{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-2\pi i x\omega} \mathrm{d}x.$$

Given a sequence $\Lambda \subset \mathbb{R}^2$, define the associated sequence in $\mathbb{C}$ by

$$\Gamma^*_\Lambda = \{x + iy : (y, x) \in \Lambda\}. \tag{2.1}$$

2.1. **The Bargmann transform.** Let $\mathrm{d}\mu(z) = e^{-\pi|z|^2} \mathrm{d}x\, \mathrm{d}y$ be the Gaussian measure. Recall that the Bargmann-Fock space $F^2(\mathbb{C})$ consists of all entire functions f for which

$$\|f\|_{F^2(\mathbb{C})} := \Big(\int_{\mathbb{C}} |f(z)|^2 \mathrm{d}\mu(z) \Big)^{1/2} < \infty.$$

For any $f \in L^2(\mathbb{R})$, the Bargmann transform of f is defined by

$$Bf(z) = 2^{1/4} \int_{\mathbb{R}} f(s) e^{2\pi s z - \pi s^2 - \pi z^2/2} \mathrm{d}s, \quad z \in \mathbb{C}. \tag{2.2}$$

The Bargmann transform is a unitary operator from $L^2(\mathbb{R})$ onto $F^2(\mathbb{C})$. We see from (2.2) that the Bargmann and Gabor transforms are related by

$$V_\varphi f(t, -\omega) = 2^{-1/4} e^{\pi i t\omega - \pi|z|^2/2} Bf(z),$$

where $z = t + i\omega \in \mathbb{C}$, $t, \omega \in \mathbb{R}$. For an introduction on Bargmann transforms, we refer to [19, Chapter 3].

2.2. **Density of a sequence of points.** Let Λ be a set in $\mathbb{R}^2$ with no accumulation points. Define the counting function $n_\Lambda(r)$ for Λ by

$$n_\Lambda(r) := \#(\Lambda \cap \{\lambda \in \mathbb{R}^2 : |\lambda| \le r\}),$$

where # stands for the cardinality of a sequence or a set. The density of Λ is defined by

$$L_\Lambda := \limsup_{r \to \infty} \frac{\log n_\Lambda(r)}{\log r}. \tag{2.3}$$

For a set Γ in the complex plane $\mathbb{C}$ with no accumulation points, $n_\Gamma(r)$ and L_Γ are defined similarly.

2.3. **The canonical product.** Here we introduce some results for entire functions. Define the elementary factors by $E_0(z) = 1 - z$ and

$$E_k(z) = (1 - z) \exp\{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\}, \quad k \ge 1.$$

Let $\Gamma = \{z_n : n \ge 1, z_n \ne 0\}$ be a sequence in $\mathbb{C}$. Define the convergence exponent of Γ by

$$\rho_\Gamma = \inf\Big\{\alpha > 0 : \sum_{n=1}^{\infty} \frac{1}{|z_n|^\alpha} < \infty\Big\}.$$

Let α be the minimal positive integer such that the above series converge. We call $p := \alpha - 1$ the genus of Γ.

Whenever Γ is the zero set of an entire function f, z_n must be listed according to their multiplicities. In this case, we also call p the genus of zero set of f.

Let Γ be the zero set of f. If $\Gamma \neq \emptyset$, then

$$\rho_\Gamma = L_\Gamma. \tag{2.4}$$

We refer to [8, Theorem 2.5.8] for a proof.

The order of a nonzero entire function f is defined by

$$\varrho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r},$$

where $M(r)$ denotes the maximum of $|f(z)|$ on $\{z : |z| \le r\}$.

The type τ of an entire function f with positive order $\varrho < \infty$ is defined by

$$\tau := \limsup_{r\to\infty} \frac{\log M(r)}{r^\varrho}.$$

Let $\Gamma := \{z_n : n \ge 1\}$ be a complex sequence, $z_n \neq 0$ and $\lim\limits_{n\to\infty} |z_n| = \infty$. Suppose that the genus of Γ equals p. Then

$$P(z) := \prod_{n=1}^{\infty} E_p(\frac{z}{z_n})$$

defines an entire function P, for which p is the genus of its zero set and Γ is the zero set [43, Theorem 15.9]. We call P a canonical product of genus p.

Proposition 2.1 ([8, Theorem 2.6.5]). *A canonical product of genus p is an entire function of order equal to the convergence exponent of its zeros.*

2.4. **Indicators and supporting functions.** Let f be an entire function of order ϱ. We call

$$H_f(\theta) := \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^\varrho}, \quad \theta \in [-\pi, \pi]$$

the indicator function of $f(z)$ with respect to the order ϱ [35, Page 53].

The supporting function $k(\theta)$ of a set $K \subset \mathbb{C}$ is defined by

$$k(\theta) = \sup_{z\in K}\{x\cos\theta + y\sin\theta\} = \sup_{z\in K}\{\Re(ze^{-i\theta})\}, \quad \theta \in [0, 2\pi].$$

For each $\theta \in [0, 2\pi]$, $l_\theta := \{z : \Re(ze^{-i\theta}) = k(\theta)\}$ is called a supporting line of K [35, Page 63].

Let f be an entire function of exponential type. Then its indicator H_f is the supporting function of a convex compact set, which is called the indicator diagram of the function f [35, Page 65].

Proposition 2.2 ([35, Theorem 12.3, Page 83]). *Let $f(z)$ be an entire function of exponential type. If f vanishes at an increasing sequence of positive numbers $\{\lambda_n : n \ge 1\}$ having the density*

$$\triangle := \lim_{n\to\infty} \frac{n}{\lambda_n},$$

then the supporting line of the indicator diagram of $f(z)$, which is orthogonal to the direction $\arg z = 0$, *and the indicator diagram itself have a common segment of length at least $2\pi\triangle$.*

3. Regular sampling for Gabor phase retrieval

In this section, we give proofs of Theorems 1.1, 1.2 and 1.3 respectively. The following proposition states conditions for a function to be entire, which can be proved with Morera's theorem and Fubini's theorem.

Proposition 3.1 ([25, Lemma 3.2]). *Let $F : \mathbb{R}^m \times \mathbb{C}^n \to \mathbb{C}$ be a function meets the followings,*

(i) *for each $z \in \mathbb{C}^n$, $F(\cdot, z)$ is a Lebesgue measurable function,*
(ii) *for each $s \in \mathbb{R}^m$, $F(s, \cdot)$ is an entire function,*
(iii) *the function $z \mapsto \int_{\mathbb{R}^m} |F(s,z)|\mathrm{d}s$ is locally bounded, that is, for any $z_0 \in \mathbb{C}^n$, there is some constant $\delta > 0$ such that*

$$\sup_{\substack{z\in\mathbb{C}^n \\ |z-z_0|\le\delta}} \int_{\mathbb{R}^m} |F(s,z)|\mathrm{d}s < \infty,$$

then $z \mapsto \int_{\mathbb{R}^m} F(s,z)\mathrm{d}s$ defines an entire function with n variables.

For any $f \in L^2(\mathbb{R})$, define

$$h_f(x) = f(x)e^{-\pi x^2}. \tag{3.1}$$

Then $h_f \in L^1 \cap L^2$. By Proposition 3.1, $\hat{h}_f$ extends to an entire function on $\mathbb{C}$ with

$$\hat{h}_f(z) = \int_{\mathbb{R}} e^{-\pi s^2} f(s) e^{-2\pi i s(x+iy)}\mathrm{d}s, \qquad z = x + iy. \tag{3.2}$$

It is easy to check that

$$\begin{aligned} V_\varphi f(t,\omega) &= e^{-\pi t^2} \int_{\mathbb{R}} e^{-\pi s^2} f(s) e^{-2\pi i(\omega+it)s}\mathrm{d}s \\ &= e^{-\pi t^2}\hat{h}_f(z), \quad z = \omega + it. \end{aligned} \tag{3.3}$$

To study phase retrieval with samples taken from three parallel lines, we need the following characterization of entire functions.

Proposition 3.2 ([49, Theorem 1]). *Suppose that ℓ_1, ℓ_2, $\ell_3 \subset \mathbb{C}$ are three parallel lines. Let d_k be the distance between ℓ_k and ℓ_3, $k = 1, 2$. If f and g are entire functions, $d_1 > 0$, $d_2 > 0$, $d_1/d_2 \notin \mathbb{Q}$, and $|f| = |g|$ on $\cup_{k=1}^3 \ell_k$, then $f \sim g$.*

And the following is another characterization of entire functions on two intersecting lines.

Proposition 3.3 ([33, Theorem 3.3]). *Let $\theta_1, \theta_2 \in \mathbb{R}$ be such that $\theta_1 - \theta_2 \notin \mathbb{Q}\pi$. Suppose that f, g are entire functions of finite order and $z_0 \in \mathbb{C}$. If*

$$|f(z_0 + xe^{i\theta_l})| = |g(z_0 + xe^{i\theta_l})|, \quad \forall x \in \mathbb{R}, l = 1, 2,$$

then $f \sim g$.

To prove the main result, we need the following density result of zeros of entire functions.

Proposition 3.4 ([8, Theorem 2.5.13]). *Let f be a nonzero entire function of order ϱ and type τ. Denote by $n(r)$ the number of zeros of f in the area $\{z : |z| \le r\}$. Then*

$$\liminf_{r\to\infty} \frac{n(r)}{r^\varrho} \le \varrho\tau.$$

For entire functions of exponential type, Carlson's theorem gives a density condition for the zero set.

Proposition 3.5 ([8, Theorem 9.2.1]). *Suppose that f is analytic in the half plane $\{x+iy : x \geq 0, y \in \mathbb{R}\}$ and is of exponential type. If $f(n) = 0$ for all nonnegative integers n and $H_f(\pi/2) + H_f(-\pi/2) < 2\pi$, then $f(z) = 0$ for all $z \in \mathbb{C}$.*

The hypothesis that $f(n) = 0$ for all nonnegative integers can be replaced by $f(\lambda_n) = 0$ for a sequence of positive numbers $\{\lambda_n : n \geq 1\}$ for which the limit

$$a := \lim_{n\to\infty} \frac{\lambda_n}{n}$$

exists and is equal to or less than 1. For a proof, see [34, Theorem 8 in Chapter 4]. See also [36, 53] for more results on the distribution of zeros of entire functions. Here we present a generalized version for entire functions of high orders.

Lemma 3.6. *Let $m \geq 1$ be an integer and f be an entire function of order m with finite type. Let $\{\lambda_n : n \geq 1\}$ be an increasing sequence of positive numbers such that the limit*

$$a := \lim_{n\to\infty} \frac{\lambda_n}{n^{1/m}}$$

exists. If $f(\lambda_n e^{2l\pi i/m}) = 0$ for all $n \geq 1$ and $0 \leq l \leq m-1$, and

$$\max_{0\leq l\leq m-1} \left\{ H_f\Big(\frac{(4l+1)\pi}{2m}\Big)\right\} + \max_{0\leq l\leq m-1} \left\{ H_f\Big(\frac{(4l-1)\pi}{2m}\Big)\right\} < \frac{2\pi}{a^m},$$

then $f(z) = 0$ for all $z \in \mathbb{C}$.

Proof. Denote $\omega = e^{2\pi i/m}$. For any $0 \leq k \leq m-1$, set

$$f_k(z) = \sum_{l=0}^{m-1} \omega^{(m-k)l} f(\omega^l z).$$

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We have

$$f_k(z) = \sum_{n=0}^{\infty} \sum_{l=0}^{m-1} a_n \omega^{(m-k+n)l} z^n.$$

Observe that

$$\sum_{l=0}^{m-1} \omega^{(m-k+n)l} = \begin{cases} m, & \text{if } n-k=0 \mod m, \\ 0, & \text{if } n-k \neq 0 \mod m. \end{cases}$$

We have

$$f_k(z) = m z^k \sum_{n=0}^{\infty} a_{mn+k} z^{mn}.$$

Let

$$\tilde{f}_k(z) := z^{m-k} f_k(z) = m \sum_{n=0}^{\infty} a_{mn+k} z^{m(n+1)}$$

and

$$g_k(z) = m \sum_{n=0}^{\infty} a_{mn+k} z^{n+1}.$$

Then g_k is an entire function of exponential type and

$$g_k(z^m) = \tilde{f}_k(z), \quad 0 \leq k \leq m-1.$$

Assume that g_k is not identical to zero for some $0 \le k \le m-1$. Fix some $\theta \in [-\pi, \pi]$. For any $\varepsilon > 0$, there is some $r_0 > 0$ such that

$$|f(re^{i\theta}\omega^l)| \le e^{r^m(H_f(\theta+2l\pi/m)+\varepsilon)}, \quad \forall r \ge r_0,\, 0 \le l \le m-1.$$

Hence

$$|f_k(re^{i\theta})| \le m \max_{0\le l\le m-1} \{e^{r^m(H_f(\theta+2l\pi/m)+\varepsilon)}\}.$$

Therefore,

$$H_{f_k}(\theta) \le \max_{0\le l\le m-1} \Big\{H_f\Big(\theta + \frac{2l\pi}{m}\Big)\Big\}.$$

Consequently,

$$\begin{aligned}
H_{g_k}\Big(\frac{\pi}{2}\Big) + H_{g_k}\Big(-\frac{\pi}{2}\Big) &= H_{f_k}\Big(\frac{\pi}{2m}\Big) + H_{f_k}\Big(-\frac{\pi}{2m}\Big) \\
&\le \max_{0\le l\le m-1} \Big\{H_f\Big(\frac{(4l+1)\pi}{2m}\Big)\Big\} + \max_{0\le l\le m-1} \Big\{H_f\Big(\frac{(4l-1)\pi}{2m}\Big)\Big\} \\
&< \frac{2\pi}{a^m}.
\end{aligned} \tag{3.4}$$

On the other hand, observe that

$$g_k(\lambda_n^m) = \tilde{f}_k(\lambda_n) = \lambda_n^{m-k} f_k(\lambda_n) = 0, \quad \forall n \ge 1.$$

By Proposition 2.2, the indicator diagram of g_k contains a segment of the form $\{x_0 + iy : y_1 \le y \le y_2\}$, where $y_2 - y_1 \ge 2\pi/a^m$. Hence $H_{g_k}(\pi/2) \ge y_2$ and $H_{g_k}(-\pi/2) \ge -y_1$. Consequently, $H_{g_k}(\pi/2) + H_{g_k}(-\pi/2) \ge 2\pi/a^m$, which contradicts (3.4). It follows that $g_k(z) = 0$ for all $z \in \mathbb{C}$ and $0 \le k \le m-1$. Consequently, $a_{mn+k} = 0$ for all $n \ge 0$ and $0 \le k \le m-1$. Therefore, f is identical to 0. □

Setting $m = 2$ and $\lambda_n = an^{1/2}$ in Lemma 3.6, we get the following consequence.

Corollary 3.7. *Let $\Lambda = a\mathbb{Z}^{1/2}$. Suppose that f is an entire function of order 2 with finite type such that f vanishes on Λ. If*

$$\max\Big\{H_f\Big(\frac{\pi}{4}\Big), H_f\Big(\frac{5\pi}{4}\Big)\Big\} + \max\Big\{H_f\Big(-\frac{\pi}{4}\Big), H_f\Big(-\frac{5\pi}{4}\Big)\Big\} < \frac{2\pi}{a^2}, \tag{3.5}$$

then $f(z) = 0$ for all $z \in \mathbb{C}$.

In the following two lemmas we present a method to construct functions whose Gabor transforms have the same absolute values while they do not differ by a global phase.

Lemma 3.8. *Let c_1 and c_2 be complex numbers such that $|c_1| = |c_2| = 1$ and $c_1/c_2 \notin \mathbb{R}$. Let p_1 and p_2 be two functions in $L^2(\mathbb{R})$ which are linearly independent. Set*

$$\begin{cases} f_1(s) = c_1 p_1(s) + c_2 p_2(s), \\ f_2(s) = \bar{c}_1 p_1(s) + \bar{c}_2 p_2(s). \end{cases} \tag{3.6}$$

Then $f_1 \nsim f_2$.

Proof. Assume on the contrary that $f_1 \sim f_2$. Then there is some number λ with $|\lambda| = 1$ such that $f_1 = \lambda f_2$. It follows from (3.6) that

$$(c_1 - \lambda\bar{c}_1)p_1(s) + (c_2 - \lambda\bar{c}_2)p_2(s) = 0.$$

Since p_1 and p_2 are linearly independent, we have $c_1 = \lambda\bar{c}_1$ and $c_2 = \lambda\bar{c}_2$. Hence $c_1^2 = c_2^2 = \lambda$. Consequently, $c_1 = \pm c_2$, which contradicts the hypothesis $c_1/c_2 \notin \mathbb{R}$. So $f_1 \nsim f_2$. □

Lemma 3.9. *Let $\Lambda \subset \mathbb{R}^2$ be a sequence. If there exist $p_1, p_2 \in L^2$ and an entire function $Q(z)$ which is not a constant such that $Q(z)$ takes real values on Γ_Λ^* and*

$$\hat{h}_{p_2}(z) = Q(z)\hat{h}_{p_1}(z), \quad \forall z \in \mathbb{C},$$

then Λ does not do Gabor phase retrieval.

Proof. Let f_1, f_2 be defined as in (3.6). We have

$$\hat{h}_{f_1}(z) = c_1\hat{h}_{p_1}(z) + c_2\hat{h}_{p_2}(z) = (c_1 + c_2Q(z))\hat{h}_{p_1}(z),$$
$$\hat{h}_{f_2}(z) = \bar{c}_1\hat{h}_{p_1}(z) + \bar{c}_2\hat{h}_{p_2}(z) = (\bar{c}_1 + \bar{c}_2Q(z))\hat{h}_{p_1}(z).$$

Since $Q(z)$ takes real values on Γ_Λ^*, we have $|\hat{h}_{f_1}(z)| = |\hat{h}_{f_2}(z)|$, $\forall z \in \Gamma_\Lambda^*$. It follows from (3.3) that $|V_\varphi f_1(t, \omega)| = |V_\varphi f_2(t, \omega)|$ for all $(t, \omega) \in \Lambda$. Moreover, we see from Lemma 3.8 that $f_1 \nsim f_2$. This completes the proof. $\square$

With the help of Bargmann transform, we obtain functions with certain properties, which are used in the construction of counterexamples.

Lemma 3.10. *Let $G(z)$ be a non-constant entire function which meets one of the following conditions,*

(i) *the order ϱ of G is less than 2;*
(ii) *the order ϱ is equal to 2 and the type τ is less than $\pi/2$;*
(iii) *$G(z) = e^{cz_1(z+z_0)^2}$, where $0 < c < \pi$, $z_0, z_1 \in \mathbb{C}$ and $|z_1| = 1$.*

Then there exist two functions p_1 and $p_2 \in L^2(\mathbb{R})$ which are linearly independent such that

$$\hat{h}_{p_2}(z) = G(z)\hat{h}_{p_1}(z). \tag{3.7}$$

Proof. Define the operator $T : L^2(\mathbb{R}) \mapsto e^{-\pi z^2/2}F^2(\mathbb{C})$ by

$$Tf(z) = \mathcal{F}(\check{f} \cdot \varphi)(z),$$

here $\check{f}(s)$ stands for the inverse Fourier transform of f. Observe that for any $x \in \mathbb{R}$,

$$Tf(x) = (f * \hat{\varphi})(x) = \int_{\mathbb{R}} f(s)e^{-\pi(x-s)^2}\mathrm{d}s.$$

Since both sides extend to entire functions, we have

$$\begin{aligned} Tf(z) &= \int_{\mathbb{R}} f(s)e^{-\pi(z-s)^2}\mathrm{d}s \\ &= e^{-\pi z^2}\int_{\mathbb{R}} f(s)e^{2\pi sz-\pi s^2}\mathrm{d}s \\ &= 2^{-1/4}e^{-\pi z^2/2}Bf(z), \quad z \in \mathbb{C}. \end{aligned} \tag{3.8}$$

There are two cases.

(C1) G meets (i) or (ii).

In this case, for $\varepsilon > 0$ small enough, there exists some constant C_ε such that

$$|G(z)| \leq C_\varepsilon e^{(\pi/2-\varepsilon)|z|^2}.$$

Fix some $\beta \in (0, \varepsilon)$. Since $e^{-\beta z^2} \in F^2(\mathbb{C})$, there is some $p_1 \in L^2(\mathbb{R})$ such that $B\hat{p}_1(z) = e^{-\beta z^2}$, where B stands for the Bargmann transform. Note that

$$\begin{aligned} |G(z)B\hat{p}_1(z)|^2e^{-\pi|z|^2} &\lesssim e^{2(\pi/2-\varepsilon)|z|^2}e^{-2\beta(x^2-y^2)}e^{-\pi(x^2+y^2)} \\ &= e^{-(2\varepsilon+2\beta)x^2}e^{-(2\varepsilon-2\beta)y^2}. \end{aligned}$$

We have $G(z)B\hat{p}_1(z) \in F^2(\mathbb{C})$. Hence there is some $p_2 \in L^2(\mathbb{R})$ such that $B\hat{p}_2(z) = G(z)B\hat{p}_1(z)$. Now we obtain

$$2^{-1/4}e^{-\pi z^2/2}B\hat{p}_2(z) = 2^{-1/4}e^{-\pi z^2/2}G(z)B\hat{p}_1(z).$$

That is,

$$T\hat{p}_2(z) = G(z)T\hat{p}_1(z).$$

It follows from the definition of the operator T that

$$\hat{h}_{p_2}(z) = G(z)\hat{h}_{p_1}(z).$$

Since $G(z)$ is not a constant, p_1 and p_2 are linearly independent.

(C2) $G(z) = e^{cz_1(z+z_0)^2}$ with $0 < c < \pi$. Fix some $\beta \in (c-\pi/2, \pi/2)$. Then $|\beta - c| < \pi/2$. Since $e^{-\beta z_1 z^2} \in F^2(\mathbb{C})$, there is some $p_1 \in L^2(\mathbb{R})$ such that $B\hat{p}_1(z) = e^{-\beta z_1 z^2}$. Note that

$$|G(z)B\hat{p}_1(z)|^2 e^{-\pi|z|^2} = |e^{(c-\beta)z_1 z^2} e^{2cz_1 z_0 z + cz_1 z_0^2}|^2 e^{-\pi|z|^2} \le e^{2c|z_0|^2 + 4c|z_0|\cdot|z|} e^{(2|c-\beta|-\pi)|z|^2}.$$

We have $G(z)B\hat{p}_1(z) \in F^2(\mathbb{C})$. The rest can be proved similarly to Case (C1). $\square$

The second part of Theorem 1.1 is a consequence of the following more general result.

Lemma 3.11. *Let $\Lambda = a\mathbb{Z}^{1/2} \times b\mathbb{Z}^{1/2}$, where a, b and c are positive numbers satisfying $a^2c \in \pi\mathbb{Z}$, $b^2c \in \pi\mathbb{Z}$, and $0 < c < \pi$. Then Λ does not do Gabor phase retrieval.*

Proof. Setting $G(z) = e^{-icx^2}$ in Lemma 3.10(iii) yields that there exist functions $p_1, p_2 \in L^2(\mathbb{R})$ such that

$$\hat{h}_{p_2}(z) = e^{-icz^2}\hat{h}_{p_1}(z), \quad z \in \mathbb{C}. \tag{3.9}$$

Take some $c_1, c_2 \in \mathbb{C}$ such that $|c_1| = |c_2| = 1$ and $c_1/c_2 \notin \mathbb{R}$. Let f_1, f_2 be defined as in (3.6). Then $f_1 \nsim f_2$, thanks to Lemma 3.8.

It remains to show that $|V_\varphi f_1(t,\omega)| = |V_\varphi f_2(t,\omega)|$ for all $(t,\omega) \in \Lambda$.

We see from (3.6) and (3.9) that

$$\begin{aligned} \hat{h}_{f_1}(z) &= c_1\hat{h}_{p_1}(z) + c_2\hat{h}_{p_2}(z) = (c_1 + c_2 e^{-icz^2})\hat{h}_{p_1}(z), \\ \hat{h}_{f_2}(z) &= (\bar{c}_1 + \bar{c}_2 e^{-icz^2})\hat{h}_{p_1}(z). \end{aligned}$$

For any $z \in \Gamma_\Lambda^*$, we have

$$e^{-icz^2} = e^{-ic(x^2-y^2)}e^{2cxy} = e^{-ic(b^2m - a^2n)}e^{2cxy}.$$

It follows from $a^2c, b^2c \in \pi\mathbb{Z}$ that $e^{-icz^2} \in \mathbb{R}$. Hence $|\hat{h}_{f_1}(z)| = |\hat{h}_{f_2}(z)|$, $\forall z \in \Gamma_\Lambda^*$. By (3.3), $|V_\varphi f_1(t,\omega)| = |V_\varphi f_2(t,\omega)|$, $\forall (t,\omega) \in \Lambda$. This completes the proof. $\square$

To prove the negative part of Theorem 1.3, we need the following lemma.

Lemma 3.12. *Let $\Lambda = \cup_{l=1}^2\big((t_0,\omega_0) + \mathbb{R}(\sin\theta_l, \cos\theta_l)\big)$, where $(t_0,\omega_0) \in \mathbb{R}^2$ and $\theta_1 - \theta_2 \in \mathbb{Q}\pi$. Then there is an entire function Q with order less than 2 such that Q is not a constant and takes real values on Γ_Λ^*.*

Proof. Note that $\Gamma_\Lambda^* = \cup_{l=1}^2\big(z_0 + \mathbb{R}e^{i\theta_l}\big)$, where $z_0 = \omega_0 + it_0$. Suppose that $\theta_1 - \theta_2 = p\pi/q \in [-\pi, \pi]$, where p, q are integers and $q > 0$.

For all $z = x + iy$, set

$$Q(z) = \sum_{n=0}^{2q-1} \exp\{(z - z_0)e^{-i(\theta_2 + pn\pi/q)}\}.$$

Then $Q(z)$ is not a constant. To see this, take the $2q$-th derivative of $Q(z)$. We obtain that

$$\frac{d^{2q}Q}{dz^{2q}}(z) = \sum_{n=0}^{2q-1} e^{-2q(\theta_2+pn\pi/q)i} \exp\{(z-z_0)e^{-i(\theta_2+pn\pi/q)}\}$$
$$= \sum_{n=0}^{2q-1} e^{-2q\theta_2 i} \exp\{(z-z_0)e^{-i(\theta_2+pn\pi/q)}\}.$$

Hence

$$\frac{d^{2q}Q}{dz^{2q}}(z_0) = \sum_{n=0}^{2q-1} e^{-2q\theta_2 i} = 2qe^{-2q\theta_2 i} \neq 0.$$

Consequently, $Q(z)$ is not a constant. Moreover, $Q(z)$ is an entire function of order 1.

It remains to show that Q takes real values on Γ^*_Λ. First, we consider the case $z \in z_0 + \mathbb{R}e^{i\theta_2}$. Suppose that $z = z_0 + re^{i\theta_2}$, where $r \in \mathbb{R}$. We have

$$Q(z) = \sum_{n=0}^{2q-1} \exp\{re^{-ipn\pi/q}\}, \quad r \in \mathbb{R}.$$

For $n = 0$ or q, we have $\exp\{re^{-ipn\pi/q}\} \in \mathbb{R}$. For $1 \leq n \leq q-1$, denote $n^* = 2q - n$. Since

$$\exp\{re^{-ipn\pi/q}\} + \exp\{re^{-ipn^*\pi/q}\} = \exp\{re^{-ipn\pi/q}\} + \exp\{re^{ipn\pi/q}\} \in \mathbb{R},$$

we have

$$\sum_{n=0}^{2q-1} \exp\{re^{-ipn\pi/q}\} = 1 + \exp\{re^{-ip\pi}\} + \sum_{n=1}^{q-1} \Big(\exp\{re^{-ipn\pi/q}\} + \exp\{re^{-ipn^*\pi/q}\}\Big) \in \mathbb{R}. \tag{3.10}$$

Hence $Q(z) \in \mathbb{R}$ for all $z \in z_0 + \mathbb{R}e^{i\theta_2}$.

Next, we consider the case $z \in z_0 + \mathbb{R}e^{i\theta_1}$. Suppose that $z = z_0 + re^{i\theta_1}$, where $r \in \mathbb{R}$. Notice that

$$Q(z) = \sum_{n=0}^{2q-1} \exp\{re^{-i\big((\theta_2-\theta_1)+pn\pi/q\big)}\}$$
$$= \sum_{n=0}^{2q-1} \exp\{re^{-ip(n-1)\pi/q}\}$$
$$= \sum_{m=-1}^{2q-2} \exp\{re^{-ipm\pi/q}\}$$
$$= \sum_{m=0}^{2q-1} \exp\{re^{-ipm\pi/q}\}.$$

Now we see from (3.10) that $Q(z) \in \mathbb{R}$ for all $z \in z_0 + \mathbb{R}e^{i\theta_1}$. This completes the proof. □

We are now ready to give a proof of Theorem 1.3.

Proof of Theorem 1.3. Fix some $f \in L^2(\mathbb{R})$ and $\theta \in [0, 2\pi]$. Let $\tilde{f}(s) = e^{-2\pi i s(\omega_0+t_0 i)} f(s)$. We have

$$|\hat{h}_f(\omega_0 + t_0 i + xe^{i\theta})|^2 = \Big| \int_{\mathbb{R}} f(s) e^{-\pi s^2} e^{-2\pi i s(\omega_0+t_0 i+xe^{i\theta})} \mathrm{d}s \Big|^2$$

$$
\begin{aligned}
&= \Big| \int_{\mathbb{R}} \tilde{f}(s) e^{-\pi s^2} e^{-2\pi i s x e^{i\theta}} \mathrm{d}s \Big|^2 \\
&= \iint_{\mathbb{R}^2} \tilde{f}(s) \bar{\tilde{f}}(t) e^{-\pi(s^2+t^2)} e^{-2\pi i s x e^{i\theta}} e^{2\pi i t x e^{-i\theta}} \mathrm{d}s\, \mathrm{d}t \\
&= \iint_{\mathbb{R}^2} \tilde{f}(s) \bar{\tilde{f}}(t) e^{-\pi(s^2+t^2)} e^{-2\pi i x (s e^{i\theta} - t e^{-i\theta})} \mathrm{d}s\, \mathrm{d}t.
\end{aligned}
$$

Substitute $z \in \mathbb{C}$ for x in the last equation, we obtain a function

$$
F_f(z) := \iint_{\mathbb{R}^2} \tilde{f}(s) \bar{\tilde{f}}(t) e^{-\pi(s^2+t^2)} e^{-2\pi i z (s e^{i\theta} - t e^{-i\theta})} \mathrm{d}s\, \mathrm{d}t.
$$

Now we see from Proposition 3.1 that $F_f(z)$ is an entire function. Moreover, for $z = re^{i\alpha}$, we have

$$
\begin{aligned}
|F_f(z)| &= \Big| \iint_{\mathbb{R}^2} \tilde{f}(s) \bar{\tilde{f}}(t) e^{-\pi(s^2+t^2)} e^{-2\pi i r (s e^{i(\alpha+\theta)} - t e^{i(\alpha-\theta)})} \mathrm{d}s\, \mathrm{d}t \Big| \\
&\le \iint_{\mathbb{R}^2} |f(s) f(t)| e^{2\pi(s+t)t_0} e^{-\pi(s^2+t^2)} e^{2\pi r (s \sin(\alpha+\theta) - t \sin(\alpha-\theta))} \mathrm{d}s\, \mathrm{d}t \\
&= e^{\pi(t_0 + r\sin(\alpha+\theta))^2} e^{\pi(t_0 - r\sin(\alpha-\theta))^2} \int_{\mathbb{R}} |f(s)| e^{-\pi(s - (t_0 + r\sin(\alpha+\theta)))^2} \mathrm{d}s \\
&\qquad \times \int_{\mathbb{R}} |f(t)| e^{-\pi(t - (t_0 - r\sin(\alpha-\theta)))^2} \mathrm{d}t \\
&\le C e^{\pi(t_0 + r\sin(\alpha+\theta))^2} e^{\pi(t_0 - r\sin(\alpha-\theta))^2} \|f\|_2^2. \qquad (3.11)
\end{aligned}
$$

Hence F_f is of order 2. Note that

$$
\limsup_{r\to\infty} \frac{\pi(t_0 + r\sin(\alpha+\theta))^2 + \pi(t_0 - r\sin(\alpha-\theta))^2}{r^2} = \pi\Big(\sin^2(\alpha+\theta) + \sin^2(\alpha-\theta) \Big).
$$

We have

$$
H_{F_f}\Big(\pm \frac{\pi}{4} \Big) = H_{F_f}\Big(\pm \frac{5\pi}{4} \Big) = \pi.
$$

Setting $\theta = \theta_1, \theta_2$ in above arguments respectively, we obtain functions $F_{f,l}$ of order 2 for which

$$
H_{F_{f,l}}\Big(\pm \frac{\pi}{4} \Big) = H_{F_{f,l}}\Big(\pm \frac{5\pi}{4} \Big) = \pi, \qquad l = 1, 2
$$

and

$$
|\hat{h}_f(\omega_0 + it_0 + xe^{i\theta_l})|^2 = F_{f,l}(x), \quad x \in \mathbb{R}, l = 1, 2.
$$

Let $F_{g,l}$ be defined similarly, where $g \in L^2(\mathbb{R})$. By (3.3), if $|V_\varphi f| = |V_\varphi g|$ on Λ, then

$$
|\hat{h}_f(\omega_0 + it_0 + \lambda e^{i\theta_l})| = |\hat{h}_g(\omega_0 + it_0 + \lambda e^{i\theta_l})|, \quad \forall \lambda \in a\mathbb{Z}^\nu, l = 1, 2.
$$

Hence

$$
F_{f,l}(\lambda) = F_{g,l}(\lambda), \quad \forall \lambda \in a\mathbb{Z}^\nu, l = 1, 2.
$$

That is, $F_{f,l}$ and $F_{g,l}$ agree on $a\mathbb{Z}^\nu$.

Denote by $n_l(r)$ the number of zeros of $F_{f,l} - F_{g,l}$ on the interval $[-r, r]$. If $\nu < 1/2$, then we have

$$
\lim_{r\to\infty} \frac{n_l(r)}{r^2} = \infty.
$$

By Proposition 3.4, we obtain that $F_{f,l} - F_{g,l} \equiv 0$.

If $\nu = 1/2$ and $a < 1$, then Corollary 3.7 yields that both functions $F_{f,l}$ and $F_{g,l}$ are uniquely determined by their values on $a\mathbb{Z}^{1/2}$. Hence $F_{f,l}(x) = F_{g,l}(x)$, $\forall x \in \mathbb{R}$.

In both cases, we obtain that

$$|\hat{h}_f(\omega_0 + it_0 + xe^{i\theta_l})| = |\hat{h}_g(\omega_0 + it_0 + xe^{i\theta_l})|, \quad \forall x \in \mathbb{R},\, l = 1, 2.$$

Since both $\hat{h}_f$ and $\hat{h}_g$ are entire functions of finite order, by Proposition 3.3, we obtain $\hat{h}_f \sim \hat{h}_g$. Hence $f \sim g$.

Next we prove the negative part. There are five cases.

(i) $(\theta_1 - \theta_2)/\pi \in \mathbb{Q}$.

By Lemma 3.12, there is some entire function Q with order less than 2 which takes real values on Γ_Λ^*. And by Lemma 3.10, there exist functions $p_1, p_2 \in L^2(\mathbb{R})$ such that

$$\hat{h}_{p_2}(z) = Q(z)\hat{h}_{p_1}(z), \quad z \in \mathbb{C}.$$

We see from Lemma 3.9 that Λ does not do Gabor phase retrieval.

(ii) $\nu > 1/2$.

See Remark 4.2.

(iii) $\nu = 1/2$, $a > 1$ and $|\cos(\theta_1 - \theta_2)| > 1/a^2$.

By Lemma 3.10, for any $0 < c < \pi$, there exist functions $p_1, p_2 \in L^2(\mathbb{R})$ such that

$$\hat{h}_{p_2}(z) = e^{ce^{i\alpha}(z - z_0)^2}\hat{h}_{p_1}(z), \quad \forall z \in \mathbb{C}.$$

By Lemma 3.9, it suffices to show that for some c, $Q(z) := e^{ce^{i\alpha}(z-z_0)^2}$ takes real values on Γ_Λ^*.

Since $|\cos(\theta_1 - \theta_2)| > 1/a^2$, there is some $0 < c < \pi$ such that

$$ca^2|\cos(\theta_1 - \theta_2)| = \pi.$$

Set $\alpha = \pi/2 - \theta_1 - \theta_2$. For $z = z_0 \pm ak^{1/2}e^{i\theta_1}$, we have

$$\begin{aligned} Q(z) = \exp\{ce^{i\alpha}a^2ke^{2i\theta_1}\} &= \exp\{ca^2k(\cos(\alpha + 2\theta_1) + i\sin(\alpha + 2\theta_1))\} \\ &= \exp\{ca^2k(\cos(\alpha + 2\theta_1) + i\cos(\theta_1 - \theta_2))\}. \end{aligned}$$

And for $z = z_0 \pm ak^{1/2}e^{i\theta_2}$, we have

$$Q(z) = \exp\{ce^{i\alpha}a^2ke^{2i\theta_2}\} = \exp\{ca^2k(\cos(\alpha + 2\theta_2) + i\cos(\theta_2 - \theta_1))\}.$$

In both cases, we have $Q(z) \in \mathbb{R}$.

(iv) $\nu = 1/2$, $a > 1$ and $|\sin(\theta_1 - \theta_2)| > 1/a^2$.

Let p_1, p_2 and Q be defined as in Case (iii). Since $|\sin(\theta_1 - \theta_2)| > 1/a^2$, there is some $0 < c < \pi$ such that

$$ca^2|\sin(\theta_1 - \theta_2)| = \pi.$$

Set $\alpha = -\theta_1 - \theta_2$. For $z = z_0 \pm ak^{1/2}e^{i\theta_1}$, we have

$$\begin{aligned} Q(z) = \exp\{ce^{i\alpha}a^2ke^{2i\theta_1}\} &= \exp\{ca^2k(\cos(\alpha + 2\theta_1) + i\sin(\alpha + 2\theta_1))\} \\ &= \exp\{ca^2k(\cos(\alpha + 2\theta_1) + i\sin(\theta_1 - \theta_2))\}. \end{aligned}$$

And for $z = z_0 \pm ak^{1/2}e^{i\theta_2}$, we have

$$Q(z) = \exp\{ce^{i\alpha}a^2ke^{2i\theta_2}\} = \exp\{ca^2k(\cos(\alpha + 2\theta_2) + i\sin(\theta_2 - \theta_1))\}.$$

In both cases, we have $Q(z) \in \mathbb{R}$.

(v) $\nu = 1/2$, $a > 1$ and $|\sin(2\theta_1 - 2\theta_2)| > 1/a^2$.

Let p_1, p_2 and Q be defined as in Case (iii). Since $|\sin(2\theta_1 - 2\theta_2)| > 1/a^2$, there is some $0 < c < \pi$ such that

$$ca^2|\sin(2\theta_1 - 2\theta_2)| = \pi.$$

Set $\alpha = -2\theta_2$. For $z = z_0 \pm ak^{1/2}e^{i\theta_1}$, we have

$$\begin{aligned} Q(z) = \exp\{ce^{i\alpha}a^2ke^{2i\theta_1}\} &= \exp\{ca^2k(\cos(\alpha + 2\theta_1) + i\sin(\alpha + 2\theta_1))\} \\ &= \exp\{ca^2k(\cos(\alpha + 2\theta_1) + i\sin(2\theta_1 - 2\theta_2))\}. \end{aligned}$$

And for $z = z_0 \pm ak^{1/2}e^{i\theta_2}$, we have

$$Q(z) = \exp\{ce^{i\alpha}a^2ke^{2i\theta_2}\} = \exp\{ca^2k\cos(\alpha + 2\theta_2)\}.$$

In both cases, we have $Q(z) \in \mathbb{R}$. This completes the proof. $\square$

The following lemma is used in the proof of Theorem 1.2.

Lemma 3.13. *Let $\Lambda = \cup_{l=1}^3\big((t_l, \omega_l) + \mathbb{R}(\sin\theta_0, \cos\theta_0)\big)$, where $\theta_0 \in [0, 2\pi]$, $(t_l, \omega_l) \in \mathbb{R}^2$, $l = 1, 2, 3$, $d_l := |(\omega_l - \omega_3)\sin\theta_0 - (t_l - t_3)\cos\theta_0| > 0$, $l = 1, 2$ and $d_1/d_2 \in \mathbb{Q}$. Then there is an entire function Q with order less than 2 such that Q is not a constant and takes real values on Γ_Λ^*.*

Proof. Denote $z_l = \omega_l + it_l$, $l = 1, 2, 3$. We have $\Gamma_\Lambda^* = \cup_{l=1}^3(z_l + \mathbb{R}e^{i\theta_0})$. Suppose that $d_1/d_2 = n/m$, where m, n are positive integers.

Let

$$Q(z) := \exp\Big\{\frac{n\pi(z - z_3)e^{-i\theta_0}}{d_1}\Big\}.$$

It is easy to see that $Q(z)$ is an entire function with order 1. Let us prove that Q takes real values on Γ_Λ^*. There are three cases.

(i) $z = z_1 + re^{i\theta_0}$, $r \in \mathbb{R}$.

In this case, we have

$$\frac{(z_1 - z_3)e^{-i\theta_0}}{d_1} = \frac{(\omega_1 - \omega_3)\cos\theta_0 + (t_1 - t_3)\sin\theta_0 + i(t_1 - t_3)\cos\theta_0 - i(\omega_1 - \omega_3)\sin\theta_0}{d_1}.$$

Note that

$$\frac{i(t_1 - t_3)\cos\theta_0 - i(\omega_1 - \omega_3)\sin\theta_0}{d_1} = \pm i.$$

We have

$$Q(z) = \exp\{\frac{n\pi r}{d_1}\}\exp\{\frac{n\pi(z_1 - z_3)e^{-i\theta_0}}{d_1}\} \in \mathbb{R},\ \forall r \in \mathbb{R}.$$

(ii) $z \in z_2 + re^{i\theta_0}$, $r \in \mathbb{R}$.

We have

$$\frac{(z_2 - z_3)e^{-i\theta_0}}{d_1} = \frac{(\omega_2 - \omega_3)\cos\theta_0 + (t_2 - t_3)\sin\theta_0 + i(t_2 - t_3)\cos\theta_0 - i(\omega_2 - \omega_3)\sin\theta_0}{d_1}.$$

Note that

$$\frac{i(t_2 - t_3)\cos\theta_0 - i(\omega_2 - \omega_3)\sin\theta_0}{d_1} = \pm\frac{m}{n}i.$$

We have

$$Q(z) = \exp\{\frac{n\pi r}{d_1}\}\exp\{\frac{n\pi(z_2 - z_3)e^{-i\theta_0}}{d_1}\} \in \mathbb{R},\ \forall r \in \mathbb{R}.$$

(iii) $z \in z_3 + re^{i\theta_0}$, $r \in \mathbb{R}$.

In this case,

$$Q(z) = \exp\{\frac{n\pi r}{d_1}\} \in \mathbb{R}, \ \forall r \in \mathbb{R}.$$

This completes the proof. $\square$

Proof of Theorem 1.2. Applying Propsotion 3.2 instead of Proposition 3.3, with similar arguments as in the proof of Theorem 1.3 we obtain that Λ does Gabor phase retrieval if $d_1/d_2 \notin \mathbb{Q}$ and either $\nu < 1/2$ or $\nu = 1/2$ with $a < 1$.

For the negative part, there are three cases.

First, we see from Lemmas 3.9, 3.10 and 3.13 that Λ does not do Gabor phase retrieval if $d_1/d_2 \in \mathbb{Q}$. This proves (i).

The proof of (ii) is postponed to Remark 4.2. It remains to show that Λ does not do Gabor phase retrieval if it meets (iii).

Set $Q(z) = e^{ice^{-2i\theta_0}(z-z_3)^2}$. By Lemma 3.10, for $c = \pi/a^2$, there exist functions $p_1, p_2 \in L^2(\mathbb{R})$ such that

$$\hat{h}_{p_2}(z) = Q(z)\hat{h}_{p_1}(z), \quad \forall z \in \mathbb{C}.$$

By Lemma 3.9, it suffices to show that $Q(z)$ takes real values on Γ_Λ^*.

For $z = z_1 \pm ak^{1/2}e^{i\theta_0}$, we have $z - z_3 = d_1 e^{i(\theta_0 \pm \pi/2)} \pm ak^{1/2}e^{i\theta_0}$. Hence

$$\begin{aligned} Q(z) &= \exp\{ice^{-2i\theta_0}(d_1 e^{i(\theta_0 \pm \pi/2)} \pm ak^{1/2}e^{i\theta_0})^2\} \\ &= \exp\{ic(a^2k - d_1^2) \pm 2acd_1k^{1/2}\} \\ &\in \mathbb{R}. \end{aligned}$$

Similarly we obtain that $Q(z) \in \mathbb{R}$ for $z = z_2 \pm ak^{1/2}e^{i\theta_0}$ or $z = z_3 \pm ak^{1/2}e^{i\theta_0}$. This completes the proof. $\square$

Proof of Theorem 1.1. If $a < 1$, then Theorem 1.2 shows that $a\mathbb{Z}^{1/2} \times \{b, b\sqrt{2}, b\sqrt{3}\}$ does Gabor phase retrieval. So does Λ. And the case $b < 1$ can be proved similarly.

For the negative part, since $a = b > 1$, setting $c = \pi/a^2$ in Lemma 3.11 yields that Λ does not do Gabor phase retrieval. $\square$

4. Irregular sampling for Gabor phase retrieval

In this section, we consider Gabor phase retrieval with irregular sampling sequence.

First, we give a necessary condition for a sequence to do Gabor phase retrieval. Then we present some sufficient conditions. We consider sampling sequences of the form $\{z_0 \pm \lambda_n e^{i\theta} : n \geq 1\}$, where $z_0 \in \mathbb{C}$, $\theta \in [0, 2\pi]$ and $\{\lambda_n : n \geq 1\}$ is a sequence of increasing positive numbers which has a uniform density, that is, the limit

$$a := \lim_{n\to\infty} \frac{\lambda_n}{n^{1/2}}$$

exists.

4.1. A necessary condition.

For a sequence Λ defined as in Theorem 1.2 or 1.3, we see from (2.3) that its density is

$$L_\Lambda = 2.$$

We show that $L_\Lambda \geq 2$ is necessary for Λ to do Gabor phase retrieval.

Theorem 4.1. *Let Λ be a set in $\mathbb{R}^2$ with no accumulation points. If Λ does Gabor phase retrieval, then its density L_Λ satisfies that*

$$L_\Lambda \geq 2.$$

Proof. It suffices to show that Λ does not do Gabor phase retrieval if $L_\Lambda < 2$. Suppose that the genus of Γ_Λ^* is p. Set $P(z) = \prod_{n=1}^{\infty} E_p(z/z_n)$ if $0 \notin \Gamma_\Lambda^*$, and $P(z) = z\prod_{n:\, z_n\neq 0} E_p(z/z_n)$ if $0 \in \Gamma_\Lambda^*$. In both cases, P is an entire function with order ϱ equal to the convergence exponent of Γ, thanks to Proposition 2.1. By (2.4), $\varrho = L_\Lambda < 2$. Moreover, $P(z) = 0$ for all $z \in \Gamma_\Lambda^*$.

By Lemma 3.10, there exist $p_1, p_2 \in L^2(\mathbb{R})$ which are linearly independent such that

$$\hat{h}_{p_2}(z) = P(z)\hat{h}_{p_1}(z).$$

Now Lemma 3.9 yields that Λ does not do Gabor phase retrieval. $\square$

Remark 4.2. *Let Λ be defined as in Theorem 1.2 or 1.3. When $\nu > 1/2$, then we have*

$$L_\Lambda = \frac{1}{\nu} < 2.$$

It follows from Theorem 4.1 that Λ does not do Gabor phase retrieval.

4.2. **Phase retrieval with sampling sequences of uniform density.** Next we consider Gabor phase retrieval with irregular sampling sequences of the form $\Lambda = \{\pm\lambda_n^{1/2} : n \geq 1\}$.

We begin with some lemmas. Setting $m = 2$ in Lemma 3.6, we obtain the following uniqueness result.

Lemma 4.3. *Let $\{\lambda_n : n \geq 1\}$ be an increasing sequence of positive numbers such that the limit*

$$a := \lim_{n\to\infty} \frac{\lambda_n}{n^{1/2}}$$

exists. Let $\Lambda = \{\pm\lambda_n^{1/2} : n \geq 1\}$. Suppose that f is an entire function of order 2 with finite type such that f vanishes on Λ. If

$$\max\Big\{H_f\Big(\frac{\pi}{4}\Big), H_f\Big(\frac{5\pi}{4}\Big)\Big\} + \max\Big\{H_f\Big(-\frac{\pi}{4}\Big), H_f\Big(-\frac{5\pi}{4}\Big)\Big\} < \frac{2\pi}{a^2}, \tag{4.1}$$

then $f(z) = 0$ for all $z \in \mathbb{C}$.

Next we show that for a sequence Λ satisfying certain density condition, there exists an entire function of order 2 which vanishes on Γ_Λ^*.

Lemma 4.4. *Let $\{\lambda_{n,l} : n \geq 1\}$, $l = 1, 2$ be increasing sequences of positive numbers such that the limits*

$$a_l := \lim_{n\to\infty} \frac{\lambda_{n,l}}{n^{1/2}}, \quad l = 1, 2$$

exist. Suppose that $\Lambda = \cup_{l=1}^2 \big\{(t_0, \omega_0) \pm \lambda_{n,l}(\sin\theta_l, \cos\theta_l) : n \geq 1\big\}$, where $(t_0, \omega_0) \in \mathbb{R}^2$. Then there is an entire function $U_1(z)$ with order 2 and type $\tau \leq \pi(1/a_1^2 + 1/a_2^2)$ such that U_1 vanishes on Γ_Λ^*.

Proof. Set

$$f_l(z) := \prod_{n=1}^{\infty} E_2\Big(\frac{z}{\lambda_{n,l}}\Big)E_2\Big(-\frac{z}{\lambda_{n,l}}\Big)E_2\Big(\frac{z}{i\lambda_{n,l}}\Big)E_2\Big(-\frac{z}{i\lambda_{n,l}}\Big).$$

By Proposition 2.1, $f_l(z)$ is an entire function with order $\varrho = 2$. Moreover,

$$f_l(z) = \prod_{n=1}^{\infty}\Big(1 - \frac{z^4}{\lambda_{n,l}^4}\Big).$$

Since $\lim_{n\to\infty} \lambda_{n,l}/n^{1/2} = a_l$, for any $\varepsilon > 0$ small enough, there exists some integer N such that for any $n > N$,

$$\lambda_{n,l} > (a_l - \varepsilon)n^{1/2}.$$

Hence

$$\begin{aligned}|f_l(z)| &\le \prod_{n=1}^{\infty}(1 + \frac{|z|^4}{\lambda_{n,l}^4}) \\ &\le \prod_{n=1}^{N}(1 + \frac{|z|^4}{\lambda_{n,l}^4}) \prod_{n=N+1}^{\infty} (1 + \frac{|z|^4}{n^2(a_l - \varepsilon)^4}).\end{aligned}$$

Applying the identity $(\sinh \pi x)/(\pi x) = \prod_{n=1}^{\infty}(1 + x^2/n^2)$ yields that

$$|f_l(z)| \le C_\varepsilon(1 + |z|^{4N})\frac{\sinh(\pi|z|^2/(a_l - \varepsilon)^2)}{\pi|z|^2/(a_l - \varepsilon)^2} \le C'_\varepsilon(1 + |z|^{4N})e^{\pi|z|^2/(a_l-\varepsilon)^2}.$$

Hence f_l is of type π/a_l^2. Let

$$U_1(z) = (z - (\omega_0 + it_0)) \prod_{l=1}^{2} f_l((z - \omega_0 - it_0)e^{-i\theta_l}).$$

Since the order and type of an entire function are translation invariant, U_1 is of order 2 and type $\pi(1/a_1^2 + 1/a_2^2)$. Moreover, U_1 vanishes on Γ^*_Λ. □

We are now ready to give a result on irregular sampling of Gabor phase retrieval over two intersecting lines.

Theorem 4.5. *Let $\{\lambda_{n,l} : n \ge 1\}$, $l = 1, 2$ be increasing sequences of positive numbers such that the limits*

$$a_l := \lim_{n\to\infty} \frac{\lambda_{n,l}}{n^{1/2}}, \quad l = 1, 2$$

exist. Suppose that $\Lambda = \cup_{l=1}^{2}\big\{(t_0, \omega_0) \pm \lambda_{n,l}(\sin\theta_l, \cos\theta_l) : n \ge 1\big\}$, where $(t_0, \omega_0) \in \mathbb{R}^2$. If

$$\max\{a_1, a_2\} < 1$$

and $(\theta_1 - \theta_2)/\pi \notin \mathbb{Q}$, then Λ does Gabor phase retrieval.

Moreover, if $(\theta_1 - \theta_2)/\pi \in \mathbb{Q}$ or $1/a_1^2 + 1/a_2^2 < 1/2$, then Λ does not do Gabor phase retrieval.

Proof. Applying Lemma 4.3 instead of Corollary 3.7, with almost the same arguments as that in the proof of Theorem 1.2 we obtain the positive part.

Next we prove the negative part. For the case $(\theta_1 - \theta_2)/\pi \in \mathbb{Q}$, with verbatim arguments as that in the proof of Theorem 1.3 we obtain that Λ does not do Gabor phase retrieval.

Finally we consider the case $1/a_1^2 + 1/a_2^2 < 1/2$.

By Lemma 4.4, there is an entire function $U_1(z)$ with order 2 and type

$$\tau \le \pi\Big(\frac{1}{a_1^2} + \frac{1}{a_2^2}\Big) < \frac{\pi}{2}$$

such that U_1 vanishes on $\Gamma^*_{\Lambda_1\cup\Lambda_2}$. Applying Lemma 3.10 again we obtain two functions $p_1, p_2 \in L^2(\mathbb{R})$ which are linearly independent such that

$$\hat{h}_{p_2}(z) = U_1(z)\hat{h}_{p_1}(z).$$

We see from Lemma 3.9 that Λ does not do Gabor phase retrieval. □

For Gabor phase retrieval over three parallel lines with irregular sampling sequences, we have the following results, which can be proved similarly to Theorem 4.5.

Theorem 4.6. *Let $\{\lambda_{n,l} : n \geq 1\}$, $l = 1, 2, 3$ be increasing sequences of positive numbers such that the limits*

$$a_l := \lim_{n\to\infty} \frac{\lambda_{n,l}}{n^{1/2}}, \quad l = 1, 2, 3 \tag{4.2}$$

exist.

Suppose that $\Lambda = \cup_{l=1}^{3} \big\{(t_l, \omega_l) \pm \lambda_{n,l}(\sin\theta_0, \cos\theta_0) : n \geq 1\big\}$, where $\theta_0 \in [0, 2\pi]$, (t_1, ω_1), (t_2, ω_2) and (t_3, ω_3) are three points for which

$$d_l := |(\omega_l - \omega_3)\sin\theta_0 - (t_l - t_3)\cos\theta_0| > 0, \quad l = 1, 2.$$

If $d_1/d_2 \notin \mathbb{Q}$ and

$$\max\{a_1, a_2, a_3\} < 1,$$

then Λ does Gabor phase retrieval.

Moreover, Λ does not do Gabor phase retrieval if $d_1/d_2 \in \mathbb{Q}$ or $1/a_1^2+1/a_2^2+1/a_3^2 < 1/2$.

5. Phase retrieval for multivariate functions

In this section, we show that our method also works for high-dimensional signals and a class of window functions $\mathcal{O}^{\tau}_{\tau'}(\mathbb{C}^d)$ introduced in [25].

Recall that for $\tau, \tau' \in (0, \infty)^d$, $\mathcal{O}^{\tau}_{\tau'}(\mathbb{C}^d)$ consists of all entire functions f for which

$$|f(x + iy)| \leq C \prod_{l=1}^{d} e^{-\tau_l' x_l^2} e^{\tau_l y_l^2}, \quad \forall x + iy \in \mathbb{C}^d.$$

And $\mathcal{O}^{\tau}(\mathbb{C}^d)$ stands for the set of all entire functions f for which

$$|f(x + iy)| \leq C \prod_{l=1}^{d} e^{\tau_l |z_l|^2}, \quad \forall x + iy \in \mathbb{C}^d.$$

Grohs and Liehr [25, Theorem 1.1] proved that if a window function g is in $\mathcal{O}^{\tau}_{\tau'}(\mathbb{C}^d)$ and

$$a_l < \sqrt{\frac{1}{2\tau_l e}} \quad \text{and} \quad b_l < \sqrt{\frac{\tau_l'}{2\pi^2 e}}, \quad l = 1, \dots, d,$$

then $\Lambda := \prod_{l=1}^{d} a_l \mathbb{Z}^{1/2} \times \prod_{l=1}^{d} b_l \mathbb{Z}^{1/2}$ does phase retrieval with respect to the window function g.

With Lemma 4.3, we show that a lower sampling density is sufficient for phase retrieval. To this end, we need the following uniqueness result from [25].

Proposition 5.1 ([25, Corollary 3.3])**.** *Suppose that $\tau, \tau' \in (0, \infty)^d$ and $g \in \mathcal{O}^{\tau}_{\tau'}(\mathbb{C}^d)$. If $f_1, f_2 \in L^2(\mathbb{R}^d)$ and $|V_g f_1(t, \omega)| = |V_g f_2(t, \omega)|$ for all $(t, \omega) \in \mathbb{R}^{2d}$, then $f_1 \sim f_2$.*

Theorem 5.2. *Suppose that $\tau, \tau' \in (0, \infty)^d$, $g \in \mathcal{O}^{\tau}_{\tau'}(\mathbb{C}^d)$, $\Lambda_l := \{\pm\lambda_{n,l} : n \geq 1\}$ and $\tilde{\Lambda}_l := \{\pm\tilde{\lambda}_{n,l} : n \geq 1\}$, where $\{\lambda_{n,l} : n \geq 1\}$ and $\{\tilde{\lambda}_{n,l} : n \geq 1\}$ are increasing sequences of positive numbers, $1 \leq l \leq d$, and*

$$\Lambda = \prod_{l=1}^{d} \Lambda_l \times \prod_{l=1}^{d} \tilde{\Lambda}_l.$$

If the limits $a_l := \lim_{n\to\infty} \frac{\lambda_{n,l}}{n^{1/2}}$ and $b_l := \lim_{n\to\infty} \frac{\tilde{\lambda}_{n,l}}{n^{1/2}}$ exist for all $1 \leq l \leq d$ and

$$a_l < \sqrt{\frac{\pi}{\tau_l}}, \quad b_l < \sqrt{\frac{\tau_l'}{\pi}}, \quad 1 \leq l \leq d,$$

then Λ does phase retrieval with respect to the window function g.

Proof. The conclusion can be proved with Lemma 4.3 and similar arguments as that of [25, Theorem 1.1]. Here we give only a sketch.

Fix some $f \in L^2(\mathbb{R}^d)$. It was show in [25] that $|V_g f(t,\omega)|^2$ extends to an entire function $F(z,z')$ in $\mathcal{O}^c(\mathbb{C}^{2d})$, where

$$c = \Big(2\tau_1, \dots, 2\tau_d, \frac{2\pi^2}{\tau_1'}, \dots, \frac{2\pi^2}{\tau_d'}\Big).$$

We need to estimate the indicator function of F with respect to each variable.

Note that

$$\begin{aligned}
|V_g f(t,\omega)|^2 &= \Big|\int_{\mathbb{R}^d} f(u)\bar{g}(u-t)e^{-2\pi i\langle u,\omega\rangle}\mathrm{d}u\Big|^2 \\
&= \iint_{\mathbb{R}^{2d}} f(u)\bar{f}(v)\bar{g}(u-t)g(v-t)e^{-2\pi i\langle u,\omega\rangle}e^{2\pi i\langle v,\omega\rangle}\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

By Proposition 3.1, the last integral defines an entire function $F_f(z,z')$ on $\mathbb{C}^{2d}$ if we replace $(\bar{z},z')$ for (t,ω). That is, $|V_g f(t,\omega)|^2$ extends to an entire function

$$F_f(z,z') = \iint_{\mathbb{R}^{2d}} f(u)\bar{f}(v)\overline{g(u-\bar{z})}g(v-z)\exp\Big\{\sum_{l=1}^{n}(-2\pi i u_l z_l' + 2\pi i v_l z_l')\Big\}\mathrm{d}u\mathrm{d}v.$$

Denote $z = x+iy$ and $z' = x'+iy'$. We have

$$\begin{aligned}
|F_f(z,z')| &\le \iint_{\mathbb{R}^{2d}} |f(u)f(v)|\cdot|g(u-\bar{z})g(v-z)e^{-2\pi i\langle u,z'\rangle}e^{2\pi i\langle v,z'\rangle}|\mathrm{d}u\mathrm{d}v \\
&\le \iint_{\mathbb{R}^{2d}} |f(u)f(v)|\cdot C\exp\Big\{\sum_{l=1}^{d}\Big(-\tau_l'(u_l-x_l)^2 - \tau_l'(v_l-x_l)^2 \\
&\qquad + 2\tau_l y_l^2 + 2\pi u_l y_l' - 2\pi v_l y_l'\Big)\Big\}\mathrm{d}u\mathrm{d}v \\
&= \iint_{\mathbb{R}^{2d}} |f(u)f(v)|\cdot C\exp\Big\{\sum_{l=1}^{d}\Big(-\tau_l'\Big(u_l - x_l - \frac{\pi y_l'}{\tau_l'}\Big)^2 \\
&\qquad - \tau_l'\Big(v_l - x_l + \frac{\pi y_l'}{\tau_l'}\Big)^2 + 2\tau_l y_l^2 + \frac{2\pi^2}{\tau_l'}{y_l'}^2\Big)\Big\}\mathrm{d}u\mathrm{d}v \\
&\le C'\|f\|_2^2\exp\Big\{2\tau_l y_l^2 + \frac{2\pi^2}{\tau_l'}{y_l'}^2\Big\}.
\end{aligned}$$

Hence, when other variables are fixed, the indicator function of F_f with respect to the variable z_l satisfies

$$H_{F_f,z_l}\Big(\pm\frac{\pi}{4}\Big) = H_{F_f,z_l}\Big(\pm\frac{5\pi}{4}\Big) = \tau_l. \tag{5.1}$$

And the indicator function with respect to the variable z_l' satisfies

$$H_{F_f,z_l'}\Big(\pm\frac{\pi}{4}\Big) = H_{F_f,z_l'}\Big(\pm\frac{5\pi}{4}\Big) = \frac{\pi^2}{\tau_l'}. \tag{5.2}$$

Suppose that $|V_g f_1|$ and $|V_g f_2|$ agree on Λ. Then the corresponding entire functions F_{f_1} and F_{f_2} also agree on Λ. Fix some $\lambda_l \in \Lambda_l$ for $1 \le l \le d$ and $\tilde{\lambda}_l \in \tilde{\Lambda}_l$ for $1 \le l \le d-1$. We have for any $z'_d \in \tilde{\Lambda}_d$,

$$F_{f_1}(\lambda_1, \ldots, \lambda_d, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{d-1}, z'_d) = F_{f_2}(\lambda_1, \ldots, \lambda_d, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{d-1}, z'_d).$$

It follows from (5.2) and Lemma 4.3 that the above equation holds for all $z'_d \in \mathbb{C}$.

Similar arguments yield that for all $z'_{d-1}, z'_d \in \mathbb{C}$

$$F_{f_1}(\lambda_1, \ldots, \lambda_d, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{d-2}, z'_{d-1}, z'_d) = F_{f_2}(\lambda_1, \ldots, \lambda_d, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{d-2}, z'_{d-1}, z'_d).$$

By induction, we obtain that

$$F_{f_1}(z, z') = F_{f_2}(z, z'), \quad \forall z, z' \in \mathbb{C}^{2d}.$$

By Proposition 5.1, $f_1 \sim f_2$. That is, Λ does phase retrieval with respect to the window function g. □

For the multivariate Gaussian window function, we have a multivariate version of Theorem 1.1. Let

$$\varphi_d(x) = e^{-\pi|x|^2}, \quad x \in \mathbb{R}^d$$

be the d-dimensional Gaussian window function and $V_{\varphi_d} f$ be the d-dimensional Gabor transform.

Theorem 5.3. *Let $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in (0, \infty)^d$ and $\Lambda = \prod_{l=1}^d a_l \mathbb{Z}^{1/2} \times \prod_{l=1}^d b_l \mathbb{Z}^{1/2}$. If*

$$\max_{1 \le l \le d} \min\{a_l, b_l\} < 1,$$

then Λ does Gabor phase retrieval .

Moreover, If $a_{l_0} = b_{l_0} > 1$ for some $1 \le l_0 \le d$, then Λ does not do Gabor phase retrieval.

Proof. We prove the conclusion by induction on the dimension d. First, we see from Theorem 1.1 that it is true for $d = 1$.

Assume that the conclusion holds for the dimension $d-1$. Let us consider the dimension d. Denote $t = (t_1, \tilde{t})$, where $t_1 \in \mathbb{R}$ and $\tilde{t} \in \mathbb{R}^{d-1}$. And $s = (s_1, \tilde{s})$, $\omega = (\omega_1, \tilde{\omega})$ are defined similarly.

For any $f \in L^2(\mathbb{R}^d)$ and $s_1 \in \mathbb{R}$, let $V^{(d-1)}_{\varphi_{d-1}} f$ be the Gabor transform of $f(s_1, \cdot)$, that is,

$$V^{(d-1)}_{\varphi_{d-1}} f(s_1; \tilde{t}, \tilde{\omega}) = \int_{\mathbb{R}^{d-1}} f(s_1, \tilde{s}) \varphi_{d-1}(\tilde{s} - \tilde{t}) e^{-2\pi i \langle \tilde{s}, \tilde{\omega} \rangle} \mathrm{d}\tilde{s}.$$

Applying Hölder's inequality yields that

$$|V^{(d-1)}_{\varphi_{d-1}} f(s_1; \tilde{t}, \tilde{\omega})| \le \|f(s_1, \cdot)\|_2, \quad \forall (\tilde{t}, \tilde{\omega}) \in \mathbb{R}^{2(d-1)}.$$

Hence for any $(\tilde{t}, \tilde{\omega}) \in \mathbb{R}^{2(d-1)}$, the function $s_1 \mapsto V^{(d-1)}_{\varphi_{d-1}} f(s_1; \tilde{t}, \tilde{\omega})$ is in $L^2(\mathbb{R})$. Denote by $V^{(1)}_{\varphi_1}$ the Gabor transform on $L^2(\mathbb{R})$. For any $(t, \omega) \in \mathbb{R}^{2d}$, we have

$$V_{\varphi_d} f(t, \omega) = V^{(1)}_{\varphi_1} \left(V^{(d-1)}_{\varphi_{d-1}} f(\cdot; \tilde{t}, \tilde{\omega}) \right) (t_1, \omega_1) = V^{(d-1)}_{\varphi_{d-1}} \left(V^{(1)}_{\varphi_1} f(\cdot; t_1, \omega_1) \right) (\tilde{t}, \tilde{\omega}).$$

Suppose that $|V_{\varphi_d} f_1|$ and $|V_{\varphi_d} f_2|$ agree on Λ. Fix some $(\tilde{t}, \tilde{\omega})$ in $\prod_{l=2}^d a_l \mathbb{Z}^{1/2} \times \prod_{l=2}^d b_l \mathbb{Z}^{1/2}$. Then for any $(t_1, \omega_1) \in a_1 \mathbb{Z}^{1/2} \times b_1 \mathbb{Z}^{1/2}$,

$$\left| V^{(1)}_{\varphi_1} \left(V^{(d-1)}_{\varphi_{d-1}} f_1(\cdot; \tilde{t}, \tilde{\omega}) \right) (t_1, \omega_1) \right| = \left| V^{(1)}_{\varphi_1} \left(V^{(d-1)}_{\varphi_{d-1}} f_2(\cdot; \tilde{t}, \tilde{\omega}) \right) (t_1, \omega_1) \right|.$$

We see from Theorem 1.1 and the hypothesis $\min\{a_1, b_1\} < 1$ that the above equation holds for all (t_1, ω_1) in $\mathbb{R}^2$. That is, for all (t_1, ω_1) in $\mathbb{R}^2$ and $(\tilde{t}, \tilde{\omega})$ in $\prod_{l=2}^d a_l \mathbb{Z}^{1/2} \times \prod_{2=1}^d b_l \mathbb{Z}^{1/2}$,

$$\left| V_{\varphi_{d-1}}^{(d-1)} \left(V_{\varphi_1}^{(1)} f_1(\cdot; t_1, \omega_1) \right) (\tilde{t}, \tilde{\omega}) \right| = \left| V_{\varphi_{d-1}}^{(d-1)} \left(V_{\varphi_1}^{(1)} f_2(\cdot; t_1, \omega_1) \right) (\tilde{t}, \tilde{\omega}) \right|.$$

By the inductive hypothesis, we obtain that the above equation holds for all $(t_1, \tilde{t})$ and $(\omega_1, \tilde{\omega})$. Hence

$$|V_{\varphi_d} f_1(t, \omega)| = |V_{\varphi_d} f_2(t, \omega)|, \quad \forall (t, \omega) \in \mathbb{R}^{2d}.$$

Applying Proposition 5.1 yields that $f_1 \sim f_2$.

Now suppose that $a_{l_0} = b_{l_0} > 1$ for some l_0. Without loss of generality, we assume that $l_0 = 1$. We see from the proof of Theorem 1.1 that there exist two functions $f_1, f_2 \in L^2(\mathbb{R})$ such that $|V_{\varphi_1}^{(1)} f_1| = |V_{\varphi_1}^{(1)} f_2|$ on $\mathbb{R}^2$ while $f_1 \not\sim f_2$. Take some $\tilde{f} \in L^2(\mathbb{R}^{d-1}) \setminus \{0\}$. Let $F_1(s) = f_1(s_1)\tilde{f}(\tilde{s})$ and $F_2(s) = f_2(s_1)\tilde{f}(\tilde{s})$. Then $|V_{\varphi_d} F_1| = |V_{\varphi_d} F_2|$ and $F_1 \not\sim F_2$. $\square$

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