DIMENSION-FREE ESTIMATES FOR NONCOMMUTATIVE DISCRETE MAXIMAL SPHERICAL MEANS

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Abstract. In this paper, we establish dimension-free L_p-estimates for operator-valued maximal spherical means on cyclic groups $\mathbb{Z}_{m+1}^d$ for all $p > 1$ and $m \geq 1$. The key ingredient is a noncommutative extension of the spectral technique developed by Nevo and Stein. As an application, we obtain a noncommutative spherical maximal inequality for automorphism actions of von Neumann algebras.

1. Introduction

Let B be a symmetric convex body in d-dimensional Euclidean space $\mathbb{R}^d$. For a locally integrable function f, the Hardy-Littlewood maximal function is defined by

$$M_B f(v) = \sup_{t>0} \frac{1}{|B|} \int_B |f(v + tu)| du,$$

where $|B|$ denotes the volume of B. For $1 < p < \infty$, let $C_p(B)$ be the optimal constant such that

$$\|M_B f\|_p \leq C_p(B)\|f\|_p, \ \forall f \in L_p(\mathbb{R}^d);$$

while $p = 1$, $C_1(B)$ is taken to satisfy the weak type $(1, 1)$ inequality,

$$\|M_B f\|_{1,\infty} \leq C_1(B)\|f\|_1, \ \forall f \in L_1(\mathbb{R}^d).$$

By applying the theory of spherical maximal functions, Stein [25] proved that for the d-dimensional Euclidean ball $B = B_2^d$ and for all $p > 1$, the constant $C_p(B_2^d)$ is bounded and independent of the dimension d. For $p = 1$, Stein and Strömberg [26] later established the bound $C_1(B) \lesssim d \log d$. Subsequent work by Bourgain [2] and Carbery [4] extended these results, proving dimension-free bounds for $C_p(B)$ for general symmetric convex B for $p > \frac{3}{2}$. For specific classes of symmetric convex bodies, such as the d-dimensional ℓ_q-ball B_q^d, Müller [21] proved the dimension-free bounds for $C_p(B_q^d)$ for all $p > 1$ and $1 \leq q < \infty$. In the case of the cube $B_\infty^d = [-1, 1]^d$, Bourgain [3] showed that the dimension-free bounds of $C_p(B_\infty^d)$ hold for all $1 < p < \infty$. Furthermore, Aldaz [1] proved that $C_1(B_\infty^d)$ grows to infinity as $d \to \infty$.

In recent years, dimension-free estimates have been extensively studied for discrete Hardy-Littlewood maximal operators. For $m \geq 1$, let $\mathbb{Z}_{m+1}^d$ denote the d-dimensional cyclic group equipped with the ℓ_0 metric (Hamming distance),

$$|u| = |\{1 \leq i \leq d : u(i) \neq 0\}|$$

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for $u=(u(1),\cdots,u(d))\in\mathbb{Z}_{m+1}^d$. For each $1\le k\le d$, let

$$\sigma_k := \frac{1}{|\{|u|=k\}|}\chi_{\{|u|=k\}} = \frac{1}{\binom{d}{k}m^k}\chi_{\{|u|=k\}}$$

denote the L_1-normalized indicator function of the k-sphere. The associated spherical maximal operator M is given by

$$M(f)(s)=\sup_{1\le k\le d}|\sigma_k * f(s)|, s\in\mathbb{Z}_{m+1}^d$$

for $f:\mathbb{Z}_{m+1}^d\to\mathbb{C}$, where the convolution is defined as

$$\sigma_k * f(s)=\sum_{u\in\mathbb{Z}_{m+1}^d}\sigma_k(u)f(s-u).$$

The dimension-free boundedness of the maximal operator M was first investigated by Harrow, Kolla and Schulman [8], who established its dimension-free L_2-boundedness on the hypercube $\mathbb{Z}_2^d$. Krause [18] later extended this result to all $p>1$. Subsequently, the dimension-free L_p-boundedness of M was obtained for all cyclic groups $\mathbb{Z}_{m+1}^d$ with $m\ge 1$ in [14]. More precisely, Greenblatt, Kolla and Krause [14] proved that for any $p>1$ and any $m\ge1$, there is a constant $C_{p,m}$ depends only on p and m such that

$$\|M(f)\|_{L_p(\mathbb{Z}_{m+1}^d)}\le C_{p,m}\|f\|_{L_p(\mathbb{Z}_{m+1}^d)},\ \forall f\in L_p(\mathbb{Z}_{m+1}^d). \tag{1}$$

For recent advances in dimension-free estimates for M on $\mathbb{Z}^d$, we refer the readers to [23] and the references therein.

In the area of noncommutative analysis, remarkable progress has been made in terms of maximal inequalities, for applications in noncommtative ergodic theory and noncommutative harmonic analysis. Notable examples include the noncommutative versions of maximal ergodic inequalities [17, 11, 6], Hardy-Littlewood maximal inequalities [20], Stein's spherical maximal inequalities [9, 5, 19] etc. Despite the significant advances, the investigation of dimension-free bounds for noncommutative maximal inequalities remains largely unexplored, with only limited results in the literature.

The purpose of this paper is to establish the dimension-free L_p-bound (1) to the operator-valued setting, which then implies the corresponding noncommutative maximal inequalities. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. The noncommutative L_p-spaces associated to the pair $(\mathcal{M},\tau)$ is denoted by $L_p(\mathcal{M})$. For each integer k, define the convolution operators

$$T_kf(s)=\sigma_k * f(s)=\frac{1}{\binom{d}{k}m^k}\sum_{u:|u|=k}f(s-u)\ , \tag{2}$$

for operator valued functions $f:\mathbb{Z}_{m+1}^d\to\mathcal{M}$. Let $\mathcal{N}=L_\infty(\mathbb{Z}_{m+1}^d,\mathcal{M})\cong L_\infty(\mathbb{Z}_{m+1}^d)\bar{\otimes}\mathcal{M}$ be the von Neumann algebra tensor product equipped with the tensor trace $\varphi=\mu\otimes\tau$, where μ is the Haar measure on $\mathbb{Z}_{m+1}^d$. Note that for $0<p<\infty$, $L_p(\mathcal{N})$ can be identified as the Bochner L_p-space $L_p(\mathbb{Z}_{m+1}^d;L_p(\mathcal{M}))$. The following is our first main result.

Theorem 1.1. *Let $(T_k)_{1\leq k\leq d}$ be the convolution operators defined as (2). Then for all $p>1$ and for all operator-valued functions $f\in L_p(\mathcal{N})\cong L_p(\mathbb{Z}_{m+1}^d;L_p(\mathcal{M}))$*

$$\|(T_kf)_{1\leq k\leq d}\|_{L_p(\mathcal{N};\ell_\infty)}\leq C_{p,m}\|f\|_p,$$

where constant $C_{p,m}$ depends only on p and m (independent of d).

Remark 1.2. As noted in [14, Remark 1.6], the region $p>1$ is optimal because the optimal constant in the weak $(1,1)$ bound grows at least as fast as $\sqrt{d}$:

$$\sup_{f\neq 0}\frac{\|Mf\|_{L_{1,\infty}(\mathbb{Z}_{m+1}^d)}}{\|f\|_{L_1(\mathbb{Z}_{m+1}^d)}}\geq c\sqrt{d}.$$

An immediate difficulty in proving Theorem 1.1 is the fact that the classical maximal function $Mf=\sup_{1\leq k\leq d}|T_kf|$ does not make sense in general when (T_kf) is a sequence of operators (see [17]). Indeed, the right formulation of noncommutative L_p-maximal inequalities was achieved via the introduction of vector-valued noncommutative L_p-spaces $L_p(\mathcal{M};\ell_\infty)$, which play the roles of the L_p-norm of maximal function Mf in the noncommutative setting [16]. We refer to the next section for precise definitions.

As an application, we establish the noncommutative spherical maximal inequality for automorphism actions of $\mathbb{Z}_{m+1}^d$ on von Neumann algebras. Recall that for a von Neumann algebra $\mathcal{M}$, we say that $\alpha:\mathbb{Z}_{m+1}^d\to\mathrm{Aut}(\mathcal{M})$ is an action if for each u, $\alpha_u:\mathcal{M}\to\mathcal{M}$ is a $*$-preserving automorphism; and for all $s,u\in\mathbb{Z}_{m+1}^d$, $\alpha_s\circ\alpha_u=\alpha_{s+u}$. If in additional, $\tau\circ\alpha_u=\tau$ for all $u\in\mathbb{Z}_{m+1}^d$, we say α is a τ-preserving action or an action of τ-preserving automorphisms, denoted as $\alpha\curvearrowright(\mathcal{M},\tau)$. For each k, define

$$M_kx=\frac{1}{\binom{d}{k}m^k}\sum_{u:|u|=k}\alpha_ux,\ \ x\in\mathcal{M}. \tag{3}$$

Theorem 1.1, together with the noncommutative Calderón transference principle developed in [11], implies the following result. Readers may refer to Section 2 for the definition of ℓ_∞^c-valued $L_p(\mathcal{M})$ space $L_p(\mathcal{M};\ell_\infty^c)$.

Theorem 1.3. *Let α be an τ-preserving action of $\mathbb{Z}_{m+1}^d$ on a semi-finite von Neumann algebra $(\mathcal{M},\tau)$. Let $(M_k)_{1\leq k\leq d}$ be the defined as (3). Then for $p>1$,*

$$\|(M_kx)_{1\leq k\leq d}\|_{L_p(\mathcal{M};\ell_\infty)}\leq C_{p,m}\|x\|_{L_p(\mathcal{M})},\ \ \forall x\in L_p(\mathcal{M});$$

and for $p>2$,

$$\|(M_kx)_{1\leq k\leq d}\|_{L_p(\mathcal{M};\ell_\infty^c)}\leq\sqrt{C_{p,m}}\|x\|_{L_p(\mathcal{M})},$$

where the constant $C_{p,m}$ depends only on p and m.

Theorem 1.3 can be rephrased as a maximal inequality for multiple ergodic averages. Let $(L_i)_{1\leq i\leq d}$ be a commuting family (i.e. $L_iL_j=L_jL_i$) of τ-preserving automorphisms on $\mathcal{M}$ such that $L_i^{m+1}=id$. For each positive integer k, define

$$N_kx=\frac{1}{\binom{d}{k}m^k}\sum_{n:|n|=k}L^nx,\ \ x\in\mathcal{M}. \tag{4}$$

Here $n=(n_1,\cdots,n_d)$ and $L^n=L_1^{n_1}L_2^{n_2}\cdots L_d^{n_d}$. Then for $p>1$,

$$\|(N_kx)_{1\leq k\leq d}\|_{L_p(\mathcal{M};\ell_\infty)}\leq C_{p,m}\|x\|_{L_p(\mathcal{M})},\ \ \forall x\in L_p(\mathcal{M});$$

and $p > 2$,

$$\|(N_k x)_{1\le k\le d}\|_{L_p(\mathcal{M};\ell_\infty^c)} \le \sqrt{C_{p,m}}\|x\|_{L_p(\mathcal{M})},\ \forall x \in L_p(\mathcal{M}).$$

We conclude this introduction by outlining the structure of the paper. Section 2 reviews some basic background on noncommutative L_p-spaces. The proof of Theorem 1.1 are presented in Sections 3 and 4. We prove Theorem 1.3 in Section 5.

2. Preliminaries

Throughout this paper, we denote by C a generic positive constant whose value may change between occurrences. The notation $X \lesssim Y$ indicates $X \le CY$ for some constant $C > 0$ independent of essential parameters, while $X \approx Y$ signifies that both $X \lesssim Y$ and $Y \lesssim X$ hold simultaneously.

2.1. **Noncommutative L_p-spaces.** Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semi-finite faithful trace $\tau : \mathcal{M}_+ \to [0,\infty]$. Let $\mathcal{S}_{\mathcal{M}+}$ be the set of all positive elements $x \in \mathcal{M}$ such that $\tau(\mathrm{supp}(x)) < \infty$, where $\mathrm{supp}(x)$ denotes the support projection of x and $\mathcal{S}_{\mathcal{M}}$ be the linear span of $\mathcal{S}_{\mathcal{M}+}$. Then $\mathcal{S}_{\mathcal{M}}$ forms a w^*-dense $*$-subalgebra of $\mathcal{M}$. For $1 \le p < \infty$, the noncommutative L_p-space $L_p(\mathcal{M},\tau)$ (in short, $L_p(\mathcal{M})$) is the completion of $\mathcal{S}_{\mathcal{M}}$ under the norm,

$$\|x\|_p = [\tau(|x|^p)]^{1/p} \quad , x \in \mathcal{S}_{\mathcal{M}},$$

where $|x| = (x^*x)^{\frac{1}{2}}$. By convention, $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. We write $L_p^+(\mathcal{M})$ the positive cone of $L_p(\mathcal{M})$. We refer to [7, 24] for further information on noncommutative L_p-spaces.

2.2. **Vector-valued noncommutative L_p-spaces.** We refer to [16, 17] for the references on vector-valued noncommutative L_p-spaces. Let I be an index set. Given $1 \le p \le \infty$, the space $L_p(\mathcal{M};\ell_\infty(I))$ consists of all families $(x_n)_{n\in I}$ in $L_p(\mathcal{M})$ that admit a factorization $x_n = ay_nb$ with $a, b \in L_{2p}(\mathcal{M})$ and $(y_n)_{n\in I} \subset \mathcal{M}$, equipped with the norm

$$\|(x_n)_{n\in I}\|_{L_p(\mathcal{M};\ell_\infty(I))} = \inf\left\{\big\|a\big\|_{2p}\sup_{n\in I}\big\|y_n\big\|_\infty\big\|b\big\|_{2p}\right\},$$

where the infimum is taken over all possible factorizations $x_n = ay_nb, n \in I$. Following standard convention, this norm is often denoted concisely as $\|{\sup_{n\in I}}^+ x_n\|_p$.

Remark 2.1. For a sequence of positive operators $(x_n)_{n\in I} \subset L_p^+(\mathcal{M})$, $x = (x_n)_{n\in I}$ in $L_p(\mathcal{M};\ell_\infty(I))$ iff there exists $a \in L_p^+(\mathcal{M})$ such that $x_n \le a$ for all $n \in I$. Moreover,

$$\|x\|_{L_p(\mathcal{M};\ell_\infty(I))} = \inf\Big\{\|a\|_p : a \in L_p^+(\mathcal{M}) \text{ such that } x_n \le a,\ \forall n \in I\Big\}.$$

In the sequel, we omit the index set I when it will not cause confusions.

As established in [16], for every $p > 1$, the predual space of $L_p(\mathcal{M};\ell_\infty)$ is $L_{p'}(\mathcal{M};\ell_1)$, where $p' = \frac{p}{p-1}$ is the conjugate index of p. Given $1 \le p \le \infty$, the space $L_p(\mathcal{M};\ell_1)$ consists of all sequences $x = (x_n)_n$ in $L_p(\mathcal{M})$ admitting a factorization of the form

$$x_n = \sum_k u_{kn}^* v_{kn}, \quad \forall n$$

for two families $(u_{kn})_{k,n}$ and $(v_{kn})_{k,n}$ in $L_{2p}(\mathcal{M})$ such that

$$\sum_{k,n} u_{kn}^* u_{kn} \in L_p(\mathcal{M}) \text{ and } \sum_{k,n} v_{kn}^* v_{kn} \in L_p(\mathcal{M}).$$

The norm of x in $L_p(\mathcal{M};\ell_1)$ is defined by

$$\|x\|_{L_p(\mathcal{M};\ell_1)} = \inf \|\sum_{k,n} u_{kn}^* u_{kn}\|_p^{1/2} \|\sum_{k,n} v_{kn}^* v_{kn}\|_p^{1/2},$$

where the infimum runs over all possible decompositions $x_n = \sum_k u_{kn}^* v_{kn}$. The duality between $L_p(\mathcal{M};\ell_\infty)$ and $L_{p'}(\mathcal{M};\ell_1)$ is given by

$$\langle x,\ y\rangle = \sum_n \tau(x_n y_n), \quad x = (x_n)_n \in L_p(\mathcal{M};\ell_\infty),\ y = (y_n)_n \in L_{p'}(\mathcal{M};\ell_1).$$

Remark 2.2. A positive sequence $x = (x_n)_n$ belongs to $L_p(\mathcal{M};\ell_1)$ iff $\sum_n x_n \in L_p(\mathcal{M})$; and in this case,

$$\|x\|_{L_p(\mathcal{M};\ell_1)} = \big\| \sum_n x_n \big\|_p.$$

We collect some important properties of these two vector-valued noncommutative L_p-spaces. See [17] for the detailed proofs.

Proposition 2.3. *The following statements hold.*

(i) *Each element in the unit ball of $L_p(\mathcal{M};\ell_\infty)$ (resp. $L_p(\mathcal{M};\ell_1)$) can be written as a sum of sixteen (resp. eight) positive elements in the same ball.*

(ii) *Let $x = (x_n)$ be a positive sequence in $L_p(\mathcal{M};\ell_\infty)$. Then for any $1 \le p \le \infty$,*

$$\|{\sup_n}^+ x_n\|_p = \sup\Big\{ \sum_n \tau(x_n y_n) : y_n \in L_{p'}(\mathcal{M}) \text{ and } \big\|\sum_n y_n\big\|_{p'} \le 1 \Big\}.$$

In particular, for two positive sequences $(x_n) \le (y_n)$, $\|{\sup_n}^+ x_n\|_p \le \|{\sup_n}^+ y_n\|_p$.

(iii) *Let $x = (x_n)$ be a sequence in $L_p(\mathcal{M};\ell_\infty)$ and $(z_{n,k})_{n,k} \in \mathbb{C}$. Then for any $1 \le p \le \infty$,*

$$\big\|{\sup_n}^+ \sum_k z_{n,k} x_k\big\|_p \le \sup_n (\sum_k |z_{n,k}|) \|{\sup_n}^+ x_n\|_p.$$

(iv) *Let $1 \le p_0 < p_1 \le \infty$ and $0 < \theta < 1$. We have the following complex interpolation relation,*

$$L_p(\mathcal{M};\ell_1) = \big(L_{p_0}(\mathcal{M};\ell_1), L_{p_1}(\mathcal{M};\ell_1)\big)_\theta\ , \text{ isometrically}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

We then turn to the definition of $L_p(\mathcal{M};\ell_\infty^c)$. Let $2 \le p \le \infty$, the space $L_p(\mathcal{M};\ell_\infty^c)$ is defined as the family of all $(x_n)_n$ in $L_p(\mathcal{M})$ for which there are an $a \in L_p(\mathcal{M})$ and $(z_n)_n \subseteq L_\infty(\mathcal{M})$ such that

$$x_n = z_n a \text{ and } \sup_n \|z_n\|_\infty < \infty.$$

The norm $\|(x_n)_n\|_{L_p(\mathcal{M};\ell_\infty^c)}$ is then defined as to be the infimum $\inf\{\sup_n \|z_n\|_\infty \|a\|_p\}$ over all factorizations of $(x_n)_n = (z_n)a$ as above. Note that $(x_n)_n \in L_p(\mathcal{M};\ell_\infty^c)$ iff $(x_n^* x_n)_n \in L_{p/2}(\mathcal{M};\ell_\infty)$.

Finally, we introduce the ℓ_∞-valued noncommutative weak L_p-spaces $\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)$ (see e.g. [10]). Given a sequence $(x_n)_n$ in $L_p(\mathcal{M})$ with $1 \le p < \infty$, we define

$$\|(x_n)_n\|_{\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)} = \sup_{\lambda>0} \lambda \inf\Big\{\big(\tau(e^\perp)\big)^{\frac{1}{p}} : e \in \mathcal{P}(\mathcal{M}) \text{ s.t. } \|ex_ke\|_\infty \le \lambda \text{ for all } k\Big\},$$

where $\mathcal{P}(\mathcal{M})$ is the set of all projections in $\mathcal{M}$. The space $\Lambda_{p,\infty}(\mathcal{M};\ell_\infty)$ consists of all sequences $x = (x_n)_n$ for which this quasi-norm is finite.

2.3. **Noise operators.** Recall that the dual group of $\mathbb{Z}_{m+1}^d$ is itself $\mathbb{Z}_{m+1}^d$. In the following, we will denote points in the group by lowercase letters, and frequencies in the dual group by capital letters. For each $S, u \in \mathbb{Z}_{m+1}^d$, we define the L_2-normalized character

$$\chi_S(u) = \frac{1}{(m+1)^{d/2}}\xi_m^{S\cdot u} = \frac{1}{(m+1)^{d/2}}\prod_{i=1}^d \xi_m^{S(i)u(i)},$$

where $\xi_m = e^{2\pi i/(m+1)}$ is a primitive $(m+1)$-th root of unity and $S\cdot u = \sum_{i=1}^d S(i)u(i)$. A operator-valued function $f : \mathbb{Z}_{m+1}^d \to \mathcal{M}$ admits a Fourier expansion of the form:

$$f(u) = \sum_{S\in\mathbb{Z}_{m+1}^d} \widehat{f}(S)\chi_S(u),$$

where each Fourier coefficient is given by

$$\widehat{f}(S) = \sum_{u\in\mathbb{Z}_{m+1}^d} f(u)\overline{\chi_S(u)}.$$

For each $t > 0$, the noise operator N_t is defined by via its Fourier expansion

$$N_t f(u) = \sum_{S\in\mathbb{Z}_{m+1}^d} e^{-t|S|}\widehat{f}(S)\chi_S(u), \tag{5}$$

and thus the family of operators $(N_t)_{t\ge0}$ form a semigroup. For each $T > 0$, define

$$H_T f = \frac{1}{T}\int_0^T N_t f dt. \tag{6}$$

The boundedness theory for the family $(H_T)_{T\ge0}$ was first established by Lance [15] for the case $p = \infty$, and later by Yeadon [27] for $p = 1$. However, the case $1 < p < \infty$ remained open for thirty years until it was finally resolved by Junge and Xu [17] through the development of noncommutative Marcinkiewicz interpolation. Junge-Xu's result has since been widely applied in noncommutative ergodic theory and noncommutative harmonic analysis. We state their results as the following theorem.

Theorem 2.4 ([15, 27, 17])**.** *Let $\mathcal{N} = L_\infty(\mathbb{Z}_{m+1}^d)\bar{\otimes}\mathcal{M}$. The following assertions hold with a positive constant $C_{p,m}$ depending only on p and m:*

(i) *for $p = 1$ and any operator-valued function $f \in L_1(\mathcal{N})$,*

$$\|(H_T f)_{T>0}\|_{\Lambda_{1,\infty}(\mathcal{N};\ell_\infty)} \le C_{1,m}\|f\|_{L_1(\mathcal{N})};$$

(ii) *for $1 < p \le \infty$ and any operator-valued function $f \in L_p(\mathcal{N})$,*

$$\|(H_T f)_{T>0}\|_{L_p(\mathcal{N};\ell_\infty)} \le C_{p,m}\|f\|_{L_p(\mathcal{N})} \quad .$$

3. The smooth spherical maximal operators

In this section, we study the boundedness theory of two smooth spherical maximal operators. By triangle's inequality in $L_p(\mathcal{N};\ell_\infty)$,

$$\begin{aligned}\|(T_kf)_{1\le k\le d}\|_{L_p(\mathcal{N};\ell_\infty)} &\le \|(T_kf)_{1\le k\le \frac{md}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)} + \|(T_kf)_{\frac{md}{m+1}\le k\le d}\|_{L_p(\mathcal{N};\ell_\infty)}\\ &= \|(T_kf)_{1\le k\le \frac{md}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)} + \|(T_{d-k}f)_{1\le k\le \frac{d}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)},\end{aligned}$$

where the convolution operators $T_k, 1\le k\le d$ are defined in (2). Therefore, to establish Theorem 1.1, it suffices to estimate the aforementioned local term $(T_kf)_{1\le k\le \frac{md}{m+1}}$ and distant terms $(T_{d-k}f)_{1\le k\le \frac{d}{m+1}}$. For this purpose, we introduce the corresponding regularized operators for both local and distant terms. For $K\ge 0$, define

$$T_K^M := \frac{1}{K+1}\sum_{k\le K} T_k\ ,\ T_K^M f = \frac{1}{K+1}\sum_{k\le K}\sigma_k * f,$$

$$T_K^D := \frac{1}{K+1}\sum_{k\le K} T_{d-k}\ ,\ T_K^D f = \frac{1}{K+1}\sum_{k\le K}\sigma_{d-k} * f.$$

Proposition 3.1. *Let $1\le p\le\infty$. There exists a constant $C_{p,m}$ depending only on p and m such that*

(i) *for $p=1$ and any $f\in L_1(\mathcal{N}) = L_1(\mathbb{Z}_{m+1}^d; L_1(\mathcal{M}))$,*

$$\|(T_K^M f)_{0\le K\le \frac{md}{m+1}}\|_{\Lambda_{1,\infty}(\mathcal{N};\ell_\infty)} \le C_{1,m}\|f\|_{L_1(\mathcal{N})}\ ,$$

$$\|(T_K^D f)_{0\le K\le \frac{d}{m+1}}\|_{\Lambda_{1,\infty}(\mathcal{N};\ell_\infty)} \le C_{1,m}\|f\|_{L_1(\mathcal{N})}\ ;$$

(ii) *for $1<p\le\infty$ and any $f\in L_p(\mathcal{N}) = L_p(\mathbb{Z}_{m+1}^d; L_p(\mathcal{M}))$,*

$$\|(T_K^M f)_{0\le K\le \frac{md}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)} \le C_{p,m}\|f\|_{L_p(\mathcal{N})}\ ,$$

$$\|(T_K^D f)_{0\le K\le \frac{d}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)} \le C_{p,m}\|f\|_{L_p(\mathcal{N})}\ .$$

The proof of Proposition 3.1 relys on Theorem 2.4. To see this, given any fixed parameter $\eta\in(0,1)$, we define a probability measure μ_η on $\mathbb{Z}_{m+1}$ by $\mu_\eta(w) = 1-\eta$ if $w=0$ and $\mu_\eta(w) = \frac{\eta}{m}$ otherwise. This induces the product measure on $\mathbb{Z}_{m+1}^d$: for $u\in\mathbb{Z}_{m+1}^d$,

$$\mu_\eta^d(u) = \left(\frac{\eta}{m}\right)^{|u|}(1-\eta)^{d-|u|}.$$

It was proved from [14, Lemma 4.6] that

$$\hat{\mu}_\eta(S) = \left(1-\frac{(m+1)\eta}{m}\right)^{|S|}\ ,\ f*\mu_\eta^d(u) = \sum_{S\in\mathbb{Z}_{m+1}^d}\left(1-\frac{(m+1)\eta}{m}\right)^{|S|}\widehat{f}(S)\chi_S(u). \tag{7}$$

where $\hat{\mu}_\eta^d$ denotes the Fourier transform of μ_η^d. For $0<P\le\frac{m}{m+1}$, denote

$$\widetilde{N}_\eta f = f*\mu_\eta^d\ ,\ J_Pf := \frac{1}{P}\int_0^P \widetilde{N}_\eta f\,d\eta. \tag{8}$$

Note that $\widetilde{N}_\eta = N_{(1-\frac{(m+1)\eta}{m})}$ is a reparametrization of the noise operator.

Lemma 3.2. *There exists a constant $C_{p,m}$ depending only on p and m such that*

(i) *for $p = 1$,*

$$\|(J_P f)_{0<P\leq \frac{m}{m+1}}\|_{\Lambda_{1,\infty}(\mathcal{N};\ell_\infty)} \leq C_{1,m}\|f\|_1 \quad \forall f \in L_1(\mathcal{N});$$

(ii) *for $1 < p \leq \infty$,*

$$\|(J_P f)_{0<P\leq \frac{m}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)} \leq C_{p,m}\|f\|_p \quad \forall f \in L_p(\mathcal{N}).$$

Proof. In the proof, we may assume that f is positive. Indeed, by decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with positive f_j and $\|f_j\|_p \leq \|f\|_p$ for $j \in \{1,2,3,4\}$ and all $p \geq 1$, we can use (quasi-)triangle's inequality to restrict the proof to the case of positive operators.

(i) Following [14, Proposition 4.7], we use the short notation $\rho_m := \frac{m}{m+1}$ and define the measure

$$\nu_P = \Big\{\frac{\rho_m}{P} T e^{-T} \chi_{(0,-\ln[1-P/\rho_m])}\Big\} dT + \Big\{\Big(\frac{\rho_m}{P} - 1\Big)\Big(- \ln\Big[1 - \frac{P}{\rho_m}\Big]\Big)\Big\}\delta_{-\ln[1-P/\rho_m]},$$

for $0 < P \leq \rho_m$, where δ denotes the dirac measure. One can verify that ν_P has total mass 1. Moreover, it was shown in [14, Proposition 4.7] that

$$J_P f = \int_0^\infty \Big(\frac{1}{T}\int_0^T N_t f\, dt\Big) d\nu_P(T) = \int_0^\infty H_T f\, d\nu_P(T). \tag{9}$$

We now appeal to the weak type $(1,1)$ boundedness of $(H_T)_T$ stated in Theorem 2.4 (i) that for any $\lambda > 0$, there exists a projection $e \in \mathcal{N}$ such that $\mu \otimes \tau(1 - e) \leq C_{1,m}\frac{\|f\|_1}{\lambda}$ and

$$\sup_{0<P\leq \frac{m}{m+1}} \|eJ_P f e\|_\infty \leq \int_0^\infty \sup_{T>0} \|eH_T f e\|_\infty d\nu_P(T) \leq \lambda.$$

This proves (i).

(ii) The strong type (p,p) estimates follows analogously. Indeed, by Theorem 2.4 (ii) and Remark 2.1, there exist $F \in L_p^+(\mathcal{N})$ and a constant $C_{p,m}$ such that for all $T > 0$,

$$H_T f \leq F \quad \text{and} \quad \|F\|_p \leq C_{p,m}\|f\|_p.$$

It then follows from (9) that for all $0 < P \leq \rho_m$,

$$J_P f \leq \int_0^\infty F d\nu_P(T) = F.$$

Therefore, by using Remark 2.1 once more, we obtain

$$\|{\sup_{0<P\leq \frac{m}{m+1}}}^+ J_P f\|_p \leq \|F\|_p \leq C_{p,m}\|f\|_p,$$

which finishes the proof. □

Now we are ready to prove Proposition 3.1 .

Proof of Proposition 3.1. Without loss of generality, we can assume that f is positive. By the kernel estimate presented in [14, Proposition 4.8], for each $K \leq d$, one can choose $P(K) \in (0, \frac{m}{m+1}]$ and an absolute constant $C > 0$ such that

$$\frac{1}{K+1}\sum_{l\leq K}\sigma_l \leq C\frac{1}{P(K)}\int_0^{P(K)}\mu_\eta^d d\eta.$$

As a consequence,

$$T_K^M f = \frac{1}{K+1}\sum_{l\leq K}\sigma_l * f \leq C\frac{1}{P(K)}\int_0^{P(K)}\widetilde{N}_\eta f d\eta = CJ_{P(K)}f.$$

The desired bound for $T_K^M f$ then follows from the corresponding estimates in Lemma 3.2.

For T_K^D, by similar kernel estimate in [14, Proposition 4.8], we have that for any $0 \leq L \leq \frac{md}{m+1}$, there exists a constant C_m such that

$$\frac{1}{\lfloor\frac{L}{m}\rfloor + 1}\sum_{l\leq L/m}\sigma_{d-l} \leq C_m\frac{1}{L+1}\sum_{l\leq L}\sigma_l * \sigma_d,$$

where $\lfloor a \rfloor$ denotes the integer part of a. Hence, for any $K \leq \frac{d}{m+1}$

$$T_K^D f \leq C_m\frac{1}{L+1}\sum_{l\leq L}\sigma_l * (\sigma_d * f). \tag{10}$$

By the weak $(1,1)$-estimate of T_K^M, for any $\lambda > 0$ there exists a projection $e \in \mathcal{N}$ such that

$$\sup_{0\leq K\leq\frac{d}{m+1}}\|e(T_K^D f)e\|_\infty \leq C_m \sup_{L\leq\frac{md}{m+1}}\|e(T_L^M(\sigma_d * f))e\|_\infty \leq C_m\lambda$$

and

$$\mu\otimes\tau(1-e) \leq \frac{\|\sigma_d * f\|_1}{\lambda} \leq C_m\frac{\|f\|_1}{\lambda},$$

where we used the Young inequality (σ_d is a probability measure)

$$\|\sigma_d * f\|_p \leq \|f\|_p, \quad \forall\ 1 \leq p \leq \infty. \tag{11}$$

Then we combine (10), the strong type (p,p) estimate for T_K^M and (11) to obtain

$$\begin{aligned}\Big\|\sup_{0\leq K\leq\frac{d}{m+1}}{}^{+} T_K^D f\Big\|_p &\leq C_m\Big\|\sup_{0\leq L\leq\frac{md}{m+1}}{}^{+} T_L^M(\sigma_d * f)\Big\|_p \\ &\leq C_{p,m}\|\sigma_d * f\|_p \leq C_{p,m}\|f\|_p.\end{aligned}$$

This proves the strong type (p,p) estimates for T_K^D. $\square$

4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Let us begin by recalling the decomposition from the previous section:

$$\|(T_kf)_{1\le k\le d}\|_{L_p(\mathcal{N};\ell_\infty)} \le \|(T_kf)_{1\le k\le \frac{md}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)} + \|(T_{d-k}f)_{1\le k\le \frac{d}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)}.$$

Then it suffices to establish the following two estimates:

$$\|(T_kf)_{1\le k\le \frac{md}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)} \le C_{p,m}\|f\|_p; \tag{12}$$

$$\|(T_{d-k}f)_{1\le k\le \frac{d}{m+1}}\|_{L_p(\mathcal{N};\ell_\infty)} \le C_{p,m}\|f\|_p. \tag{13}$$

The proofs of (12) and (13) are inspired by the complex Cesaro sums method developed by Nevo and Stein [22]. We introduce some necessary notations. Given a complex number α and a nonnegative integer n, set

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!},\quad A_0^\alpha := 1,\ A_{-1}^\alpha := 0.$$

The following elementary properties on A_n^α are well-known, see [22, Lemma 3].

Lemma 4.1. *Let $\alpha=\beta+i\gamma$ be a complex number with $\beta,\gamma\in\mathbb{R}$. There exists $c_\beta>0$ such that*

(i) *for $\beta>-1$,*
$$c_\beta^{-1}(n+1)^\beta \le A_n^\beta \le c_\beta(n+1)^\beta;$$
(ii) *for $\beta>-1$,*
$$|A_n^\beta| \le |A_n^\alpha| \le c_\beta e^{2\gamma^2}|A_n^\beta|;$$
(iii) *for $\beta\in\mathbb{N}$,*
$$|(1+n)^\beta A_n^{-\beta+i\gamma}| \le c_\beta e^{3\gamma^2}.$$

For $\alpha\in\mathbb{C}$ and $n\in\mathbb{N}$, we define

$$S_n^\alpha f = \sum_{k=0}^n A_{n-k}^\alpha T_kf\ ,\ M_n^\alpha f = \frac{1}{(n+1)^{\alpha+1}}S_n^\alpha f;$$
$$U_n^\alpha f = \sum_{k=0}^n A_{n-k}^\alpha T_{d-k}f\ ,\ N_n^\alpha f = \frac{1}{(n+1)^{\alpha+1}}U_n^\alpha f.$$

We will establish the following more general estimate.

Theorem 4.2. *Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra. For any $\alpha\in\mathbb{C}$ and $p>1$, there exists a constant $C_{m,p,\alpha}$ depends only on m,p,α such that $f\in L_p(\mathcal{N}) = L_p(\mathbb{Z}_{m+1}^d; L_p(\mathcal{M}))$*

$$\|\sup_{0\le n\le\frac{md}{m+1}}{}^{+} M_n^\alpha(f)\|_p \le C_{m,p,\alpha}\|f\|_p;$$
$$\|\sup_{0\le n\le\frac{d}{m+1}}{}^{+} N_n^\alpha(f)\|_p \le C_{m,p,\alpha}\|f\|_p.$$

Indeed, Theorem 4.2 directly yields estimates (12) and (13) through the identifications $M_n^{-1}f = T_n f$ and $N_n^{-1}f = T_{d-n}f$. Our proof strategy for Theorem 4.2 follows the framework established by Nevo and Stein in [22, Section 3], which we adapt to the noncommutative setting. The proof of Theorem 4.2 will be divided into the following four lemmas.

Lemma 4.3. *Let $\alpha = \beta + i\gamma$ with $\beta \geq 0$. Then for all $p > 1$ and $f \in L_p(\mathcal{N})$,*

$$\|\sup_{0\leq n\leq \frac{md}{m+1}}{}^{+} M_n^{\alpha}(f)\|_p \leq C_{m,p,\beta} e^{2\gamma^2}\|f\|_p;$$

$$\|\sup_{0\leq n\leq \frac{d}{m+1}}{}^{+} N_n^{\alpha}(f)\|_p \leq C_{m,p,\beta} e^{2\gamma^2}\|f\|_p.$$

Proof. Without loss of generality, we assume that f is positive. Let $(g_n) \subset L_{p'}^{+}(\mathcal{N})$ with $\|\sum_n g_n\|_{L_{p'}(\mathcal{N})} \leq 1$. By applying Lemma 4.1, we have

$$\begin{aligned}
|\tau(M_n^{\alpha}(f)g_n)| &\leq (n+1)^{-\beta-1}\sum_{k=0}^{n}|A_{n-k}^{\alpha}|\tau(T_k f g_n)\\
&\leq c_\beta e^{2\gamma^2}(n+1)^{-\beta-1}\sum_{k=0}^{n}(n-k+1)^{\beta}\tau(T_k f g_n)\\
&\leq c_\beta e^{2\gamma^2}(n+1)^{-1}\sum_{k=0}^{n}\tau(T_k f g_n).
\end{aligned}$$

Therefore, recalling that $T_n^M f = (n+1)^{-1}\sum_{k\leq n} T_k f$, we obtain

$$\sum_{0\leq n\leq \frac{md}{m+1}} |\tau(M_n^{\alpha}(f)g_n)| \leq c_\beta e^{2\gamma^2}\sum_{0\leq n\leq \frac{md}{m+1}}\tau(T_n^M f g_n).$$

Taking the supremum over all $(g_n) \subset L_{p'}^{+}(\mathcal{N})$ with $\|\sum_n g_n\|_{L_{p'}(\mathcal{N})} \leq 1$, using Proposition 2.3 (i)(ii) and Proposition 3.1 (ii), we deduce the first assertion. The second statement can be proved similarly by using the definition of $T_n^D f$ and Proposition 3.1 (ii). ☐

The following L_2-estimates are identical to the scalar-valued cases in [14, Proposition 5.7], so we omit the proof. For an operator x, recall that $|x| = (x^*x)^{\frac{1}{2}}$.

Lemma 4.4. *Let $t \in \mathbb{N}$. Then there exists constant $C_{t,m} > 0$ depending only on t and m such that for all $f \in L_2^{+}(\mathcal{N})$,*

$$\|R_{t,m}^1 f\|_2 \leq C_{t,m}\|f\|_2 \quad \text{and} \quad \|R_{t,m}^2 f\|_2 \leq C_{t,m}\|f\|_2,$$

where $R_{t,m}^1 f$ and $R_{t,m}^2 f$ are respectively defined by

$$R_{t,m}^1 f = \Big(\sum_{0\leq k\leq \frac{md}{m+1}}(k+1)^{2t-1}|S_k^{-t-1}f|^2\Big)^{\frac{1}{2}};$$

$$R_{t,m}^2 f = \Big(\sum_{0\leq k\leq \frac{d}{m+1}}(k+1)^{2t-1}|U_k^{-t-1}f|^2\Big)^{\frac{1}{2}}.$$

With Lemma 4.4 in hand, the next step is to prove the maximal inequalities in $L_2(\mathcal{N})$ for M_n^{-t} and N_n^{-t} with $t\in\mathbb{N}$. For positive integers t and k, set

$$B_k^t = k(k-1)\cdots(k-t+1) \text{ for } t\le k \quad \text{and} \quad B_k^t = 0 \text{ for } t>k.$$

Lemma 4.5. *Let $t\in\mathbb{N}$. Then there exists $C_{t,m}$ such that for all $f\in L_2(\mathcal{N})$*

$$\|\sup_{0\le n\le \frac{md}{m+1}}{}^{+} M_n^{-t}(f)\|_2 \le C_{t,m}\|f\|_2;$$

$$\|\sup_{0\le n\le \frac{d}{m+1}}{}^{+} N_n^{-t}(f)\|_2 \le C_{t,m}\|f\|_2.$$

Proof. We prove the first inequality, as the second follows analogously. Again, we can assume that the operator-valued function f is positive. Observe that for any sequence (a_k),

$$\begin{aligned}\sum_{k=t}^{n} B_k^{t+1}\,(a_k-a_{k-1}) &= B_n^{t+1}\,a_n - \sum_{k=t}^{n-1}(B_{k+1}^{t+1}-B_k^{t+1})\,a_k - B_t^{t+1}a_{t-1}\\ &= B_n^{t+1}\,a_n-(t+1)\sum_{k=t}^{n-1}B_k^t\,a_k.\end{aligned}$$

Applying above formula to $a_k=S_k^{-t}$, we deduce

$$B_n^{t+1}\,S_n^{-t} = \sum_{k=t}^{n}B_k^{t+1}\Delta S_k^{-t} + (t+1)\sum_{k=t}^{n-1}B_k^tS_k^{-t}, \tag{14}$$

where $\Delta S_k^{-t} = S_k^{-t}-S_{k-1}^{-t}$. We claim that the desired maximal inequality is equivalent to

$$\Big\|\sup_{0\le n\le\frac{md}{m+1}}{}^{+}\frac{1}{(1+n)^2}B_n^{t+1}S_n^{-t}f\Big\|_2 \le C_t\|f\|_2. \tag{15}$$

Indeed, for each positive integer n, one has

$$\frac{1}{(1+n)^2}B_n^{t+1} \le \frac{(1+n)^{t+1}}{(1+n)^2} = (1+n)^{t-1}.$$

On the other hand, observe that

$$\frac{(1+n)^{t+1}}{B_n^{t+1}} = \frac{1+n}{n}\cdots\frac{1+n}{n-t} \le \Big(\frac{1+n}{n-t}\Big)^{t+1} = \Big(1+\frac{1+t}{n-t}\Big)^{t+1}.$$

If $0<t<\frac{n}{2}$, then

$$\frac{1+t}{n-t} < \frac{2t+2}{n} < \frac{2t+2}{2t} \le 2.$$

If $\frac{n}{2}\le t\le n-1$, then $\frac{1+t}{n-t}\le t+1$. Therefore, there is some constant C_t depending only on t such that $\frac{(1+n)^{t+1}}{B_n^{t+1}}\le C_t$. This proves the claim.

To prove (15), we use (14) to get

$$\frac{B_n^{t+1}}{(1+n)^2}\,S_n^{-t}f = \frac{1}{(1+n)^2}\sum_{k=t}^{n}B_k^{t+1}\Delta S_k^{-t}f + \frac{t+1}{(1+n)^2}\sum_{k=t}^{n-1}B_k^tS_k^{-t}f$$

$$=: G^1_{n,t}f + G^2_{n,t}f.$$

By the convexity of the operator-valued function $x \mapsto |x|^2$ (see e.g. [20, (1.13)]),

$$|G^1_{n,t}f| \le \frac{1}{(1+n)^2}\big(\sum_{k=t}^{n}(B^{t+1}_k)^2(k+1)^{-2t+1}\big)^{\frac12}\Big(\sum_{k=t}^{n}(k+1)^{2t-1}|\Delta S^{-t}_k f|^2\Big)^{\frac12}.$$

Observe that for $k \ge t+1$,

$$B^{t+1}_k = k(k-1)\cdots(k-t) \le k^{t+1},$$

which gives that

$$\frac{1}{(1+n)^2}\big(\sum_{k=t}^{n}(B^{t+1}_k)^2(k+1)^{-2t+1}\big)^{\frac12} \le \frac{1}{(1+n)^2}\big(\sum_{k=t}^{n}k^3\big)^{\frac12} \le 1.$$

On the other hand, it follows from [22, Lemma 2] that $\Delta S^{-t}_k = S^{-t-1}_k$. Therefore, we find

$$|G^1_{n,t}f| \le \Big(\sum_{k=n}^{2n}(k+1)^{2t-1}|S^{-t-1}_k f|^2\Big)^{\frac12} \le R^1_{t,m}f.$$

We now turn to the second term $G^2_{n,t}f$. Using the convexity of the operator-valued function $x \mapsto |x|^2$ again, we have

$$\begin{aligned}|G^2_{n,t}f| &\le \frac{t+1}{(1+n)^2}\big(\sum_{k=t}^{n-1}k^3\big)^{\frac12}\big(\sum_{k=t}^{n-1}k^{-3}(B^t_k)^2|S^{-t}_k f|^2\big)^{\frac12}\\ &\le C_t\big(\sum_{k=t}^{n-1}k^{2t-3}|S^{-t}_k f|^2\big)^{\frac12} \le C_t R^1_{t,m}f.\end{aligned}$$

Now let $(g_n) \subset L^+_2(\mathcal{N})$ with $\|\sum_n g_n\|_2 \le 1$. Then by applying the estimates of $G^1_{n,t}f$ and $G^2_{n,t}f$ obtained above, we get

$$\begin{aligned}\Big|\tau\Big(\frac{1}{(1+n)^2}B^{t+1}_n S^{-t}_n f g_n\Big)\Big| &\le \tau(|G^1_{n,t}f|g_n) + \tau(|G^2_{n,t}f|g_n)\\ &\le C_t\tau(R^1_{t,m}f g_n).\end{aligned}$$

Hence, by Lemma 4.5,

$$\Big|\sum_n \tau\Big(\frac{1}{(1+n)^2}B^{t+1}_n S^{-t}_n f g_n\Big)\Big| \le C_t\|R^1_{t,m}f\|_2\|\sum_n g_n\|_2 \le C_t\|f\|_2.$$

This implies (15). Thus the proof is finished. □

Lemma 4.6. *Let $t \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Then there exists a constant C_t such that for any $f \in L_2(\mathcal{N})$,*

$$\big\|\sup^+_{0\le n\le \frac{md}{m+1}} M^{-t+i\beta}_n f\big\|_2 \le C_t e^{3\beta^2}\|f\|_2,$$

$$\big\|\sup^+_{0\le n\le \frac{d}{m+1}} N^{-t+i\beta}_n f\big\|_2 \le C_t e^{3\beta^2}\|f\|_2.$$

Proof. We prove the first statement, as the other one can be done similarly. Assume that f is positive and $n \le 10t$ first. By the convolution formula stated in [22, Lemma 2], one has

$$S_n^{-t+i\beta} f = \sum_{k=0}^{n} A_{n-k}^{i\beta} S_k^{-t-1} f.$$

Note that $|A_{n-k}^{i\beta}| \le c_0 e^{2\beta^2}$ for all $k \le n$ by Lemma 4.1 (ii). Then it follows from the triangle inequality and Proposition 2.3 (iii) that

$$\begin{aligned}\| \sup_{0\le n\le \frac{md}{m+1}}{}^{+} M_n^{-t+i\beta} f\|_2 &\le \sup_{0\le k\le n} |A_{n-k}^{i\beta}| \sum_{k=0}^{n} \frac{(1+n)^{t-1}}{(1+k)^t} \| \sup_{0\le k\le \frac{md}{m+1}}{}^{+} M_k^{-t-1} f\|_2 \\ &\le c_t e^{2\beta^2} \|f\|_2,\end{aligned}$$

where the second inequality holds since $n \le 10t$ and Lemma 4.5.

Let us now analyze the case when $n > 10t$. Set $n_2 = \lfloor \frac{n}{2} \rfloor$ and consider the decomposition

$$S_n^{-t+i\beta} f = \sum_{k=0}^{n_2+t} A_{n-k}^{i\beta} S_k^{-t-1} f + \sum_{k=n_2+t+1}^{n} A_{n-k}^{i\beta} S_k^{-t-1} f =: F_{n,t}^1 f + F_{n,t}^2 f.$$

One can deduce from the triangle inequality and Proposition 2.3 that

$$\| \sup_{0\le n\le \frac{md}{m+1}}{}^{+} M_n^{-t+i\beta} f\|_2 \le \Big\| \sup_{0\le n\le \frac{md}{m+1}}{}^{+} \frac{F_{n,t}^1 f}{(1+n)^{-t+1}} \Big\|_2 + \Big\| \sup_{0\le n\le \frac{md}{m+1}}{}^{+} \frac{F_{n,t}^2 f}{(1+n)^{-t+1}} \Big\|_2.$$

We now estimate each term separately. For the second term, we use Proposition 2.3 (iii) to obtain

$$\Big\| \sup_{0\le n\le \frac{md}{m+1}}{}^{+} \frac{F_{n,t}^2 f}{(1+n)^{-t+1}} \Big\|_2 \le \sup_{0\le k\le n} |A_{n-k}^{i\beta}| \sum_{k=n_2+t+1}^{n} \frac{(1+n)^{t-1}}{(1+k)^t} \| \sup_{0\le n\le \frac{md}{m+1}}{}^{+} M_k^{-t-1} f\|_2.$$

The summation can be bounded as follows

$$\sum_{k=n_2+t+1}^{n} \frac{(1+n)^{t-1}}{(1+k)^t} \le \frac{(n-n_2-t)(1+n)^{t-1}}{(n_2+t+2)^t} \le \Big(\frac{n+1}{n_2+2}\Big)^{t-1} \le 2^t.$$

Combining this with Lemma 4.1 (ii) and Lemma 4.5 yields that

$$\Big\| \sup_{0\le n\le \frac{md}{m+1}}{}^{+} \frac{F_{n,t}^2 f}{(1+n)^{-t+1}} \Big\|_2 \le c_t e^{2\beta^2} \|f\|_2.$$

It remains to estimate the first term. Observe that for every nonnegative integer ℓ (see [18, Lemma 2.4]),

$$\sum_{k=0}^{m} A_{n-k}^{-\ell+i\beta} S_k^{-t-1+\ell} f = A_{n-m}^{-\ell+i\beta} S_m^{-t-1+\ell} f + \sum_{k=0}^{m-1} A_{n-k}^{-\ell-1+i\beta} S_k^{-t+\ell} f. \tag{16}$$

Then by repeated applications of identity (16), we obtain

$$\begin{aligned} F^1_{n,t}f &= A^{i\beta}_{n-(n_2+t)}S^{-t-1}_{n_2+t}f + \sum_{k=0}^{n_2+t-1} A^{-1+i\beta}_{n-k}S^{-t}_k f \\ &= A^{i\beta}_{n-(n_2+t)}S^{-t-1}_{n_2+t}f + \sum_{\ell=1}^{t-1} A^{-\ell+i\beta}_{n-(n_2+t-\ell)}S^{-t-1+\ell}_{n_2+t-\ell}f + \sum_{k=0}^{n_2} A^{-t+i\beta}_{n-k}S^{-1}_k f. \end{aligned}$$

Therefore, by Minkowski's inequality and Proposition 2.3 (iii), we get

$$\Big\| {\sup_{0\le n\le \frac{md}{m+1}}}^{+} \frac{F^1_{n,t}f}{(1+n)^{-t+1}} \Big\|_2 \le I \times II,$$

where

$$I = \sum_{\ell=0}^{t-1} \| {\sup_{0\le n\le \frac{md}{m+1}}}^{+} M^{-\ell}_n f\|_2$$

and

$$II = (1+n)^{t-1} \max\Big\{ \sum_{\ell=0}^{t-1} \frac{|A^{-\ell+i\beta}_{n-(n_2+t-\ell)}|}{(n_2+t-\ell+1)^{t-\ell}}, \sum_{k=0}^{n_2} |A^{-t+i\beta}_{n-k}| \Big\}.$$

By Lemma 4.5, $I \le C_t\|f\|_2$. On the other hand, for each $0 \le \ell \le t-1$, we use Lemma 4.1 to get

$$\begin{aligned} \frac{(1+n)^{t-1}|A^{-\ell+i\beta}_{n-(n_2+t-\ell)}|}{(n_2+t-\ell+1)^{t-\ell}} &\le \frac{(1+n)^{t-1}a_0e^{3\beta^2}(n-(n_2+t-\ell)+1)^{-\ell}}{(n_2+t-\ell+1)^{t-\ell}} \\ &\le c_{t,\ell}e^{3\beta^2}; \end{aligned}$$

and the sum

$$\begin{aligned} (1+n)^{t-1}|\sum_{k=0}^{n_2} A^{-t+i\beta}_{n-k}| &\le (1+n)^{t-1}(n_2+1)\cdot \max_{0\le k\le n_2}(n-k+1)^{-t}\cdot B_t e^{3\beta^2} \\ &\le c_t e^{3\beta^2}. \end{aligned}$$

Combining these two estimates, we have $II \le c_t e^{3\beta^2}$, which finishes the proof. □

Now we are in a position to prove Theorem 4.2. Our approach is based on Stein's complex interpolation. For the sake of self-completeness, we present a detailed proof.

Proof of Theorem 4.2. Write $\alpha = \beta + i\gamma$ with $\beta,\gamma \in \mathbb{R}$. Choose $\theta \in (0,1)$, $q \in (1,\infty)$, $a \in \mathbb{Z}$ and $b > 0$ such that

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q} \quad \text{and} \quad \beta = (1-\theta)a + \theta\, b.$$

Let $f \in L_p(\mathcal{N})$ and $g = (g_n)$ be a finite sequence in $L_{p'}(\mathcal{N})$ such that $\|f\|_p < 1$ and $\|g\|_{L_{p'}(\mathcal{N};\ell_1)} < 1$. Define

$$h(z) = u\,|f|^{\frac{p(1-z)}{2}+\frac{pz}{q}}, \quad z \in \mathbb{C},$$

where $f = u|f|$ is the polar decomposition of f. By construction, for any $t \in \mathbb{R}$, $\|h(it)\|_2 = \|f\|_p^{\frac{p}{2}} < 1$ and $\|h(1+it)\|_q = \|f\|_p^{\frac{p}{q}} < 1$. By Proposition 2.3 (iv), there is a function $w = (w_n)_n$ continuous on the strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$ and analytic in the interior such that $w(\theta) = g$ and

$$\sup_{t\in\mathbb{R}} \max\Big\{\|w(i\,t)\|_{L_2(\mathcal{N};\ell_1)}\ ,\ \|w(1+i\,t)\|_{L_{q'}(\mathcal{N};\ell_1)}\Big\} < 1.$$

Now define

$$F(z) = \exp\big(\delta(z^2-\theta^2)\big)\sum_n \varphi\big[M_n^{(1-z)a+zb+i\,\gamma}(h(z))\,w_n(z)\big],$$

where $\delta > 0$ is a constant to be specified later. Clearly, F is a function analytic in the open strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ (note that F is a finite sum). Applying Lemma 4.3 when $a \geq 0$ and Lemma 4.6 when $a \leq -1$, we have

$$\begin{aligned}
|F(it)| &\leq \exp\big(\delta(-t^2-\theta^2)\big)\,\big\|\big(M_n^{a+i(-ta+tb+\gamma)}(h(it))\big)_n\big\|_{L_2(\mathcal{N};\ell_\infty)}\,\big\|w(i\,t)\big\|_{L_2(\mathcal{N};\ell_1)}\\
&\leq C_\alpha \exp\big((-\delta+c_{m,\beta,b,\gamma})t^2-\delta\theta^2\big)\,\|h(it)\|_2\\
&\leq C_\alpha \exp\big((-\delta+c_{m,\beta,b,\gamma})t^2-\delta\theta^2\big)\ .
\end{aligned}$$

Similarly, by Lemma 4.3,

$$|F(1+it)| \leq C_{\alpha,q}\,\exp\big((-\delta+c'_{m,\beta,b,\gamma})t^2+\delta(1-\theta^2)\big)\ .$$

Choosing $\delta > \max(c_{m,\beta,b,\gamma},\ c'_{m,\beta,b,\gamma})$, we get

$$\sup_{t\in\mathbb{R}} \max\Big\{|F(i\,t)|\ ,\ |F(1+i\,t)|\Big\} \leq C_{m,\alpha,p}$$

for some constant $C_{m,\alpha,p} > 0$. Then by the maximum principle, $|F(\theta)| \leq C_{m,p,\beta,b,\gamma}$, which by duality implies

$$\big|\sum_n \varphi\big[M_n^\alpha(f)\,w_n\big]\big| \leq C_{m,\alpha,p}\ ,$$

as $g = (g_n)$ is an abitrary finite sequence such that $\|g\|_{L_{p'}(\mathcal{N};\ell_1)} < 1$. This proves the first inequality. The second conclusion follows similarly using the estimates involving $N_n^\alpha(f)$ from the previous lemmas. The details are left to readers. □

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra and $\alpha : \mathbb{Z}_{m+1}^d \to \mathrm{Aut}(\mathcal{M},\tau)$ be a τ-preserving action. Then for every $s \in \mathbb{Z}_{m+1}^d$, the map α_s extends to a positive isometry on $L_p(\mathcal{M})$ for all $p \geq 1$. It follows from [12, Proposition 5.3] that $(\alpha_s \otimes id_{\ell_\infty})_{s\in\mathbb{Z}_{m+1}^d}$ extends to a family of isometry on $L_p(\mathcal{M};\ell_\infty)$ for $1 \leq p \leq \infty$, still denoted by $(\alpha_s)_s$. Hence, for each $s \in \mathbb{Z}_{m+1}^d$,

$$\big\|(M_k x)_{1\leq k\leq d}\big\|_{L_p(\mathcal{M};\ell_\infty)} = \big\|(\alpha_{-s}M_k x)_{1\leq k\leq d}\big\|_{L_p(\mathcal{M};\ell_\infty)},$$

which implies that

$$\big\|(M_kx)_{1\le k\le d}\big\|^p_{L_p(\mathcal{M};\ell_\infty)}=\frac{1}{|\mathbb{Z}^d_{m+1}|}\sum_{s\in\mathbb{Z}^d_{m+1}}\big\|(\alpha_{-s}M_kx)_{1\le k\le d}\big\|^p_{L_p(\mathcal{M};\ell_\infty)}. \tag{17}$$

Given x, define an operator-valued function $f\in L_p(\mathbb{Z}^d_{m+1},L_p(\mathcal{M}))$ as

$$f(s)=\alpha_{-s}x,\quad s\in\mathbb{Z}^d_{m+1}.$$

Then clearly for each $s\in\mathbb{Z}^d_{m+1}$,

$$\alpha_{-s}M_kx=\frac{1}{\binom{d}{k}m^k}\sum_{u:|u|=k}\alpha_{u-s}x=\frac{1}{\binom{d}{k}m^k}\sum_{u:|u|=k}f(s-u)=T_kf(s). \tag{18}$$

By the definition of $\big\|(T_kf)_{1\le k\le d}\big\|_{L_p(L_\infty(\mathbb{Z}^d_{m+1})\bar{\otimes}\mathcal{M};\ell_\infty)}$, there exists a factorization $T_kf=ay_kb$ with $a,b\in L_{2p}(L_\infty(\mathbb{Z}^d_{m+1})\bar{\otimes}\mathcal{M})$ and $y_k\in L_\infty(\mathbb{Z}^d_{m+1})\bar{\otimes}\mathcal{M}$ such that

$$\|a\|_{2p}\sup_{1\le k\le d}\|y_k\|_\infty\|b\|_{2p}\le\big\|(T_kf)_{1\le k\le d}\big\|_{L_p(L_\infty(\mathbb{Z}^d_{m+1})\bar{\otimes}\mathcal{M};\ell_\infty)}+\delta.$$

By Hölder's inequality, one has

$$\begin{aligned}
&\sum_{s\in\mathbb{Z}^d_{m+1}}\big\|(T_kf(s))_{1\le k\le d}\big\|^p_{L_p(\mathcal{M};\ell_\infty)}\\
\le&\sum_{s\in\mathbb{Z}^d_{m+1}}\big\|a(s)\big\|^p_{2p}\sup_{1\le k\le d}\big\|y_k(s)\big\|^p_\infty\big\|b(s)\big\|^p_{2p}\\
\le&\Big(\sum_{s\in\mathbb{Z}^d_{m+1}}\big\|a(s)\big\|^{2p}_{2p}\Big)^{\frac12}\sup_{1\le k\le d}\sup_{s\in\mathbb{Z}^d_{m+1}}\big\|y_k(s)\big\|^p_\infty\Big(\sum_{s\in\mathbb{Z}^d_{m+1}}\big\|b(s)\big\|^{2p}_{2p}\Big)^{\frac12}\\
\le&\|a\|^p_{2p}\sup_{1\le k\le d}\|y_k\|^p_\infty\|b\|^p_{2p}\\
\le&\big(\|(T_kf)_{1\le k\le d}\|_{L_p(L_\infty(\mathbb{Z}^d_{m+1})\bar{\otimes}\mathcal{M};\ell_\infty)}+\delta\big)^p.
\end{aligned}$$

By the arbitrariness of δ, we obtain

$$\sum_{s\in\mathbb{Z}^d_{m+1}}\|(T_kf(s))_{1\le k\le d}\|^p_{L_p(\mathcal{M};\ell_\infty)}\le\|(T_kf)_{1\le k\le d}\|^p_{L_p(L_\infty(\mathbb{Z}^d_{m+1})\bar{\otimes}\mathcal{M};\ell_\infty)}. \tag{19}$$

Now using (17), (18) and (19), we deduce that

$$\begin{aligned}
\big\|(M_kx)_{1\le k\le d}\big\|^p_{L_p(\mathcal{M};\ell_\infty)}&=\frac{1}{|\mathbb{Z}^d_{m+1}|}\sum_{s\in\mathbb{Z}^d_{m+1}}\big\|(T_kf(s))_{1\le k\le d}\big\|^p_{L_p(\mathcal{M};\ell_\infty)}\\
&\le\frac{1}{|\mathbb{Z}^d_{m+1}|}\|(T_kf)_{1\le k\le d}\|^p_{L_p(L_\infty(\mathbb{Z}^d_{m+1})\bar{\otimes}\mathcal{M};\ell_\infty)}\\
&\le C_{m,p}\frac{1}{|\mathbb{Z}^d_{m+1}|}\|f\|^p_{L_p(\mathbb{Z}^d_{m+1};L_p(\mathcal{M}))}\\
&=C_{m,p}\frac{1}{|\mathbb{Z}^d_{m+1}|}\sum_{s\in\mathbb{Z}^d_{m+1}}\|\alpha_sx\|^p_{L_p(\mathcal{M})}.
\end{aligned}$$

$$= C_{m,p}\|x\|^p_{L_p(\mathcal{M})}.$$

where the second inequality above uses the strong type (p,p) estimate of $(T_k)_{1\le k\le d}$ by Theorem 1.1, and the last equality uses the fact that α_s is isometric on $L_p(\mathcal{M})$ for each $s\in\mathbb{Z}^d_{m+1}$. This proves the first estimate.

For the second assertions, let $x\in L_p(\mathcal{M})$. Note that for each k, M_k is unital and completely positive. Then by Kadison-Schwarz inequality for unital 2-positive map, we have $M_k(x^*x)\le M_k(x)^*M_k(x)$. Now applying the first assertion to x^*x, we deduce from Remark 2.1 that there exists positive $b\in L_{p/2}(\mathcal{M})$ such that

$$\|b\|_{p/2}\le C_{p,m}\|x^*x\|_{p/2}=C_{p,m}\|x\|_p^2 \text{ and } M_k(x^*x)\le b,\ \forall k.$$

Hence, $M_k(x)^*M_k(x)\le M_k(x^*x)\le b$, which implies that for each k, there is a $y_k\in\mathcal{M}$ such that $M_k(x)=y_kb^{1/2}$ and $\|y_k\|_\infty$. This gives the desired factorization of $(M_k(x))_k$ as an element in $L_p(\mathcal{M};\ell^c_\infty)$ and the proof is finished. □

Remark 5.1. The above transference method actually shows that Theorem 1.3 for the fully noncommutative setting and Theorem 1.1 for the operator-valued setting are equivalent with identical constants.

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