Finding Kissing Numbers with Game-theoretic Reinforcement Learning

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Since Isaac Newton first studied the Kissing Number Problem in 1694, determining the maximal number of non-overlapping spheres around a central sphere has remained a fundamental challenge. This problem represents the local analogue of Hilbert's 18th problem on sphere packing, bridging geometry, number theory, and information theory. Although significant progress has been made through lattices and codes, the irregularities of high-dimensional geometry and exponentially growing combinatorial complexity beyond 8 dimensions, which exceeds the complexity of Go game, limit the scalability of existing methods. Here we model this problem as a two-player matrix completion game and train the game-theoretic reinforcement learning system, *PackingStar*, to efficiently explore high-dimensional spaces. The matrix entries represent pairwise cosines of sphere center vectors; one player fills entries while another corrects suboptimal ones, jointly maximizing the matrix size, corresponding to the kissing number. This cooperative dynamics substantially improves sample quality, making the extremely large spaces tractable. *PackingStar* reproduces previous configurations and surpasses all human-known records from dimensions 25 to 31, with the configuration in 25 dimensions geometrically corresponding to the Leech lattice and suggesting possible optimality. It achieves the first breakthrough beyond rational structures from 1971 in 13 dimensions and discovers over 6000 new structures in 14 and other dimensions. These results demonstrate AI's power to explore high-dimensional spaces beyond human intuition and open new pathways for the Kissing Number Problem and broader geometry problems.

Introduction

The Kissing Number Problem was first posed in 1694 by Isaac Newton and David Gregory, asking for the maximal number of non-overlapping spheres that can touch a central sphere in three-dimensional space. It was later extended to n-dimensional Euclidean space, where $K(n)$ denotes the kissing number in n dimensions, sparking centuries of research in high-dimensional geometry. In 1900, David Hilbert included the sphere packing problem in n-dimensional space as the 18th problem of his famous 23 problems [1]. The Kissing Number Problem can be viewed as a local analogue of Hilbert's 18th problem [2]. Beyond geometry, it connects to number theory, group theory, and information theory [3, 4, 5]. Therefore, advancing this problem could foster progress across multiple fields.

Table 1: **Kissing number $K(n)$ in dimension n for** $1 \leq n \leq 24$ [6]. The bold items indicate the kissing numbers that are known to be optimal.

n	$K(n)$	n	$K(n)$	n	$K(n)$	n	$K(n)$	n	$K(n)$	n	$K(n)$
1	**2**	5	40	9	306	13	1154	17	5730	21	29768
2	**6**	6	72	10	510	14	1932	18	7654	22	49896
3	**12**	7	126	11	593	15	2564	19	11692	23	93150
4	**24**	8	**240**	12	840	16	4320	20	19448	24	**196560**

Unfortunately, determining kissing numbers in higher dimensions remains exceptionally difficult. Table 1 lists the kissing numbers in different dimensions. After centuries of effort, the problem has been solved exactly only in a few dimensions 1, 2, 3, 4, 8, and 24 [7, 8, 9, 10, 11, 12, 13]. As the dimension increases, human understanding becomes more limited due to the growing geometric complexity of high-dimensional spaces and combinatorial explosion of configurations. Over the past half-century, mathematicians have continued to push the boundaries using lattice constructions [14], code techniques [15], group representation theory [16], and carefully designed substructures derived from known configurations [17]. In dimensions 10, 11, 13, 14, and dimensions from 17 to 21, such approaches established new lower bounds [2, 16]. In dimensions 25–31, studies conducted in 2011 and 2016 further improved the bounds by exploiting subsets of Leech lattice vectors combined with heuristic optimization [18, 19]. While these results mark significant milestones, they are limited by reliance on prior knowledge, and painstaking human design. As dimensionality increases, the exponentially growing configuration space and the breakdown of full symmetry assumptions make existing approaches insufficient. These challenges highlight the need for novel paradigms to explore complex structures beyond human intuition.

In this paper, we introduce *PackingStar*, a multi-agent reinforcement learning system that reformulates the Kissing Number Problem as a two-player matrix completion game. This approach scales efficiently across dimensions, mitigates the combinatorial explosion of high-dimensional search spaces, and overcomes the challenge of extreme data scarcity due to the limited human knowledge of high-dimensional constructions. In *PackingStar*, two cooperative agents learn to play the matrix completion game to efficiently explore irregular structures in high-dimensional spaces without human prior knowledge. The first agent focuses on incrementally filling matrix entries to construct larger and denser configurations. However, in the early stages of exploration, the candidate space for potential entries is exponentially large, and the agent only observes local matrix information, making erroneous placements inevitable. To address this problem, the second agent is introduced to learn how to identify and remove invalid or inconsistent entries. By leveraging a more global understanding of the matrix geometry and high-dimensional structure, this agent dynamically corrects the mistakes made by the first agent, enabling the joint system to converge toward larger configurations—corresponding to higher kissing numbers. The key insight to *PackingStar*'s success lies in two complementary design principles. First, the framework reformulates the kissing number construction as an equivalent matrix completion problem in the Gram matrix domain. This reformulation eliminates numerical instability and inefficiency inherent in high-dimensional coordinate representations, enabling all computations to operate directly on the Gram matrix in a fully parallelizable manner. The formulation seamlessly couples problem modeling with its learning methods, allowing efficient GPU acceleration and large-scale sampling for AI-driven discovery of kissing numbers. Second, *PackingStar* features the finely designed game-theoretic interaction between the two agents. By strategically performing delayed and selective corrections, the second agent imposes geometric constraints on the first agent's expansive search, effectively regularizing exploration in an otherwise intractable space. This game dynamics not only improves the quality of sampled configurations and the efficiency of value estimation but also reduces the search com-

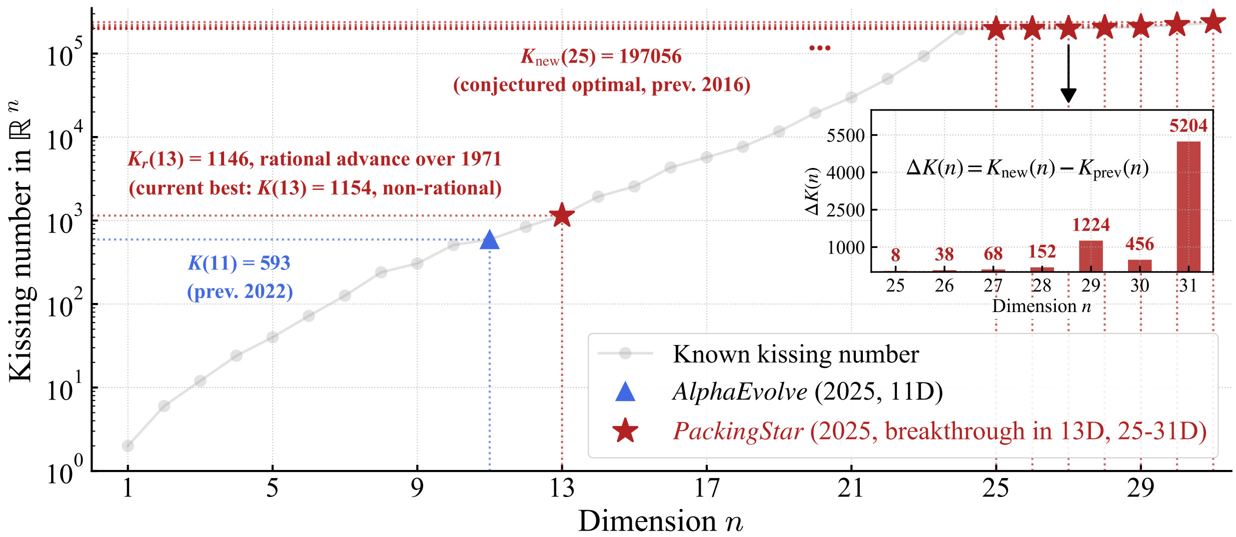

Figure 1: **AI breakthroughs in the Kissing Number Problem.** Grey markers show the best previously known lower bounds. The blue triangle marks the 11D improvement achieved by *AlphaEvolve* (2025) [20]. In 13D, *PackingStar* (red star) uncovers a new rational configuration with kissing number $K_r(13) = 1146$, distinct from the current best non-rational lower bound $K(13) = 1154$. For $25 \leq n \leq 31$, the red stars indicate new lower bounds $K_{\text{new}}(n)$ discovered by *PackingStar*, with the inset reporting the corresponding increments $\Delta K(n) = K_{\text{new}}(n) - K_{\text{prev}}(n)$ compared to the previous best $K_{\text{prev}}(n)$.

plexity in exponentially large spaces, enabling scalable discovery of complex high-dimensional configurations that were previously inaccessible to existing methods.

PackingStar demonstrates remarkable scalability, successfully reproducing all known kissing number configurations in dimensions up to 31. Figure 1 illustrates some of *PackingStar*'s breakthrough discoveries on this problem. From dimensions 25 to 31, *PackingStar* surpasses the lower bounds reported in 2016 and also exceeds the meta-construction form introduced in 2011 [18, 19], achieving what appears to be the optimal construction form along this path; the resulting construction form are summarized in Table 2. Notably, the breakthrough configuration discovered in 25 dimensions exhibits a clear geometric pattern that corresponds closely to the substructure of Leech lattice. Across multiple metrics, this configuration shows strong convergence, suggesting that it may represent the optimal 25-dimensional arrangement, although a rigorous mathematical proof is still lacking. In 13 dimensions, *PackingStar* surpasses the previously known rational structure with 1130 spheres from 1971 [15], reaching a kissing number of 1146. Rational structures are those in which all pairwise cosines between sphere centers are rational. The non-rational structure with 1154 spheres is obtained by slightly adjusting the coordinates of some spheres in the 1130-sphere configuration, later reached 1154 in 1999 [17]. Although the new 13-dimensional rational configuration does not exceed the non-rational record, it is of considerable mathematical value. Its fully rational inter-sphere cosines allow for exact analytic study without relying on numerical approximations. This offers insights into the effects of different constraints on kissing numbers and may inspire higher-density packings. Rational structures also facilitate more efficient encoding and decoding in information and communication applications, highlighting their practical significance alongside their theoretical and geometric value. Beyond these specific dimensions, *PackingStar* uncovers over 6000 new configurations in 14 dimensions and identifies additional configurations corresponding to current lower bounds in dimensions 12, 13, 15, and others. The sheer volume of these newly discovered arrangements provides rich opportunities for further investigation and may ultimately lead to configurations with even higher kissing numbers. Collectively, these results highlight AI's ability to understand and reason about high-dimensional geometric structures, opening a new paradigm for studying the Kissing Number Problem and demonstrating the potential of AI to explore a broader class of challenging mathematical problems.

Key difference to *AlphaEvolve*

The key difference between *PackingStar* and *AlphaEvolve* lies in the search space. *AlphaEvolve* optimizes explicit coordinates in $\mathbb{R}^n$, which is numerically unstable and scales exponentially with dimension; increasing computational resources cannot overcome this fundamental limitation. In contrast, *PackingStar* operates in the Gram-matrix domain with strict geometric constraints, enabling efficient large-scale sampling. This fundamental distinction explains why *PackingStar* scales to higher dimensions while *AlphaEvolve* cannot [20].

Although *AlphaEvolve* improved the kissing number in 11 dimensions from 592 to 593, likely by adjusting the coordinates of the known 592-sphere configuration, the result lacks clear geometric structure and does not generalize to other dimensions. In contrast, *PackingStar* not only sets new bounds across higher dimensions but also discovers geometrically meaningful, analyzable configurations, providing concrete objects for further mathematical study [20].

Results

We evaluate the *PackingStar* system across multiple high-dimensional regimes and report new state-of-the-art kissing configurations. Our results fall into three categories: (i) new lower bounds in dimensions 25–31, (ii) improved rational constructions in 13 dimensions, and (iii) diverse families of solutions in different dimensions. These findings demonstrate that *PackingStar* can both recover classical lattice-based structures and uncover entirely new high-dimensional arrangements, substantially expanding the landscape of known kissing configurations. The main kissing configurations are available at the data repository https://github.com/CDM1619/PackingStar.

New records from dimensions 25 to 31

We first evaluate *PackingStar* in dimensions $25 \le n \le 31$, where the best previously known lower bounds were all derived from a common structural template that the Leech lattice packing together with a family of disjoint subsets S_i of its minimal vectors [18, 19].

Table 2: **New kissing numbers $K_{\text{new}}(n)$ by *PackingStar* vs. previous $K_{\text{prev}}(n)$ for $25 \le n \le 31$.** The asterisk ★ indicates that under our new construction form, the theoretical lower bound in 31 dimensions should be 238350. The present version reports 238078 only due to computational and time limitations, rather than any intrinsic limitation of our method. The Future versions are expected to reach the theoretical bound. In the new construction form, bold items indicate the improved configuration forms discovered by *PackingStar* compared with previous forms.

n	***New construction form***	$\boldsymbol{K_{\text{new}}(n)}$	$K_{\text{prev}}(n)$ [18]
25	$K(24) + \|S_1\|$	**197056**	197048
26	$\boldsymbol{K(2)} + K(24) + 2\|S_1\| + 2\|S_2\|$	**198550**	198512
27	$\boldsymbol{K(3)} + K(24) + 2\|S_1\| + 2\|S_2\| + \sum_{i=3}^{5} \|S_i\|$	**200044**	199976
28	$\boldsymbol{K(4)} + K(24) + 2\sum_{i=1}^{8} \|S_i\|$	**204520**	204368
29	$\boldsymbol{K(5)} + K(24) + \boldsymbol{2\sum_{i=1}^{12} \|S_i\| + \sum_{i=13}^{14} \|S_i\|}$	**209496**	208272
30	$\boldsymbol{K(6)} + K(24) + 2\sum_{i=1}^{24} \|S_i\|$	**220440**	219984
31	$\boldsymbol{K(7)} + K(24) + \boldsymbol{2\sum_{i=1}^{42} \|S_i\|}$	**238078★**	232874

In this setting, the first objective is to identify the largest possible subsets S_i. Each S_i corresponds to a subset of shortest Leech lattice vectors whose pairwise cosine values are bounded by $1/4$; equivalently, in the *PackingStar* framework this corresponds to cosine sets $C_1 = C_2 = \{-1,\ 0,\ \pm 1/4\}$ together with the structural constraint that C_* matches the Leech minimal-vector set. Previous work identified subsets $|S_i| = 488$ with no apparent geometric structure [18]. *PackingStar* discovers subsets $|S_i| = 496$ exhibiting a clear internal geometric organization. Specifically, this 496-element subset is composed of 28 symmetric-frame structures X_8 in 8-dimensional subspaces, each containing 16 spheres, together with a symmetric-frame structure ("Conway-Curtis Cross" [14]) X_{24} in 24-dimensional space containing 48 spheres, which corresponds to an important substructure of the Leech lattice. Here the symmetric-frame structure is a Cross-polytope. Based on this clear geometric interpretation, we conjecture that this subset S_i may be optimal, and consequently, the 25-dimensional kissing configuration derived from this construction could also attain the maximum kissing number.

The second objective is to identify improved meta construction forms that assemble these subsets into full configurations. Both objectives are jointly addressed within the unified *PackingStar* framework, which simultaneously increases the achievable size of S_i and uncovers superior meta construction forms. Our findings show that *PackingStar* not only discovers larger subsets but also identifies a strictly better construction template than the meta-construction introduced in 2011; moreover, evidence from convergence and geometric consistency strongly suggests that the new template is optimal within this construction pathway. These advances yield especially pronounced improvements in dimensions 29 and 31. Table 2 reports additional new construction forms that deviate from the classical form [18, 19]. In dimensions 26–30, the system consistently augments the Leech-lattice component by adding lower-dimensional parts $K(k)$ where $k = n - 24$ (e.g., adding $K(2)$ in dimension 26). In 31 dimensions, *PackingStar* uncovers a novel assembly pattern by partitioning a 7-dimensional kissing configuration into 42 (optimal) disjoint unit-radius equilateral triangles, resulting in an 84-fold weighted S_i, surpassing the previous 75-fold weighted S_i. Similarly, in 29 dimensions, *PackingStar* embeds 12 (optimal) disjoint unit-radius equilateral triangles into the 5-dimensional kissing configuration to generate a 26-fold weighted S_i, exceeding the prior 24-fold weighted S_i. These advances yield strictly larger kissing configurations in all dimensions from 25 to 31. The resulting new lower bounds $K_{\mathrm{new}}(n)$ are summarized in Table 2. Compared with the previous best values $K_{\text{prev}}(n)$, our constructions increase the kissing number by 8, 38, 68, 152, 1224, 456, and 5204 spheres in dimensions 25 through 31, respectively. These improvements are also highlighted by the red markers in Fig. 1. Table 3 summarizes the cosine value among sphere pairs in the new configurations.

Improved rational configurations in 13 dimensions

We apply *PackingStar* to explore rational kissing configurations in dimension 13. In this setting, guided by the simulations in Step 1 of our method, we identify two feasible cosine sets for 13-dimensional rational arrangements: $C_1 = C_2 = \{-1, 0, \pm 1/4, \pm 1/2\}$ or $\{-1, -3/4, 0, \pm 1/4, \pm 1/2\}$. Table 3 summarizes the cosine value among sphere pairs in $K_r(13)$ configurations. Based on these cosine features, *PackingStar* discovers several new rational configuration with kissing number $K_r(13) = 1146$, surpassing the previously known rational structure from 1971, which achieved 1130 [15]. The new configuration exhibits a highly regular geometric pattern and a fully rational Gram matrix, allowing exact verification of all pairwise cosines without reliance on numerical approximations. This demonstrates the capability of *PackingStar* to systematically explore the space of rational configurations and discover previously unknown structures.

Remarkably, among the 1146-sphere configurations, we observe a variant that can be partitioned into two components: one containing 1008 spheres and the other 138 spheres. The 1008-sphere component exhibits two fundamentally distinct forms: one fully antipodal and the other completely non-antipodal. In the non-antipodal form, none of the spheres has an antipodal counterpart, corresponding to a cosine set of $\{-3/4, 0, \pm 1/4, \pm 1/2\}$, excluding -1. At the same time, it exhibits high uniformity, with each sphere tightly contacting 98 neighboring spheres. Such a structure is extremely rare and has no known analogues among previously discovered kissing configurations. This reveals that even in 13 and higher dimensions, kissing configurations may possess special non-symmetric arrangements that were previously unexplored.

Diverse constructions in different dimensions

Beyond the specific breakthroughs in 13 and in dimensions 25 through 31, *PackingStar* uncovers a rich diversity of kissing configurations whose kissing number reaches current lower bound across

Table 3: **Cosine sets of different kissing configurations.** For each configuration, the table lists the set of cosine values between distinct unit vectors. All maximal cosines are ≤ 0.5, ensuring that the geometric constraints of the kissing number problem are satisfied, i.e., no spheres overlap.

Configuration	Cosine set (excluding −1 and 0)
$K_{\mathrm{r}}(13)$	$\{\pm\frac{1}{4},\ \pm\frac{1}{2}\}$; variants: $\{-\frac{3}{4},\ \pm\frac{1}{4},\ \pm\frac{1}{2}\}$
$K_{\mathrm{new}}(25)$	$\{\pm\frac{1}{6},\ \pm\frac{\sqrt{6}}{12},\ \pm\frac{1}{4},\ \pm\frac{1}{3},\ \pm\frac{\sqrt{6}}{6},\ \pm\frac{1}{2}\}$
$K_{\mathrm{new}}(26)$	$\{-\frac{5}{6},\ -\frac{2}{3},\ \pm\frac{1}{6},\ \pm\frac{\sqrt{6}}{12},\ \pm\frac{1}{4},\ \pm\frac{1}{3},\ \pm\frac{\sqrt{6}}{6},\ \pm\frac{1}{2}\}$
$K_{\mathrm{new}}(27)$	$\{-\frac{5}{6},\ -\frac{2}{3},\ \pm\frac{2\sqrt{3}-\sqrt{6}}{12},\ \pm\frac{1}{6},\ \pm\frac{\sqrt{6}}{12},\ \pm\frac{1}{4},\ \pm\frac{1}{3},\ \pm\frac{\sqrt{6}}{6},\ \pm\frac{2\sqrt{3}+\sqrt{6}}{12},\ \pm\frac{1}{2}\}$
$K_{\mathrm{new}}(28)$	$\{-\frac{5}{6},\ -\frac{2}{3},\ \pm\frac{1}{6},\ \pm\frac{\sqrt{6}}{12},\ \pm\frac{1}{4},\ \pm\frac{1}{3},\ \pm\frac{\sqrt{6}}{6},\ \pm\frac{1}{2}\}$
$K_{\mathrm{new}}(29)$	$\{-\frac{5}{6},\ -\frac{2}{3},\ \pm\frac{2\sqrt{3}-\sqrt{6}}{12},\ \pm\frac{1}{6},\ \pm\frac{\sqrt{6}}{12},\ \pm\frac{1}{4},\ \pm\frac{\sqrt{3}}{6},\ \pm\frac{1}{3},\ \pm\frac{\sqrt{6}}{6},\ \pm\frac{2\sqrt{3}+\sqrt{6}}{12},\ \pm\frac{1}{2}\}$
$K_{\mathrm{new}}(30)$	$\{-\frac{5}{6},\ -\frac{2}{3},\ \pm\frac{1}{6},\ \pm\frac{\sqrt{6}}{12},\ \pm\frac{1}{4},\ \pm\frac{1}{3},\ \pm\frac{\sqrt{6}}{6},\ \pm\frac{1}{2}\}$
$K_{\mathrm{new}}(31)$	The full cosine set is available at the data repository.

multiple other dimensions. Among the thousands of discovered variants, we consistently observe the presence of asymmetric structures with cosine set $C_1 = C_2 = \{-1,\ -3/4,\ 0,\ \pm 1/4,\ \pm 1/2\}$, similar to those identified in the 13-dimensional rational configuration.

Methods

In this section, we present a learning-based system *PackingStar* to find high-dimensional kissing constructions. First, we extract dominant geometric features of candidate kissing configurations via simulation and Monte Carlo Tree Search. Second, we formulate the search for maximal kissing numbers as a cooperative two-player matrix completion game, where one agent iteratively fills the Gram matrix and the other corrects suboptimal entries, jointly maximizing the final matrix size.

Identifying geometric features of the kissing configuration via simulation

In this step, we will identify the basic geometric features of possible kissing structures through extensive simulations. This process corresponds to Step 1 in Figure 2. Given m existing spheres in $\mathbb{R}^n$ with $m \geq n-1$, we iteratively generate new sphere centers by considering every $(n-1)$-combination of existing spheres. Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1} \in \mathbb{R}^n$ be the corresponding coordinates, and construct $\mathbf{A} = [\mathbf{x}_1^T; \ldots; \mathbf{x}_{n-1}^T] \in \mathbb{R}^{(n-1)\times n}$, which must have full row rank. We solve for a new sphere center $\mathbf{x} \in \mathbb{R}^n$ on the unit sphere satisfying

$$\begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \|\mathbf{x}\|_2 = 1 \end{cases} \qquad \mathbf{b} = \tfrac{1}{2}\mathbf{1}_{n-1}. \tag{1}$$

The feasible solutions are

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} \pm \sqrt{1 - \|\mathbf{A}^{+}\mathbf{b}\|_2^2}\,\mathbf{z}, \quad \mathbf{z} \in \ker(\mathbf{A}),\ \|\mathbf{z}\|_2 = 1. \tag{2}$$

All feasible candidates $\mathbf{x}$ are added to the search space, and one is selected as a new center using Monte Carlo Tree Search (MCTS). Each node in the search tree corresponds to a partial configuration $s_t = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{m_t}\} \subset \mathbb{R}^{n\times m_t}$ of m_t existing sphere centers at step t. The MCTS algorithm selects the next candidate $\mathbf{x}_t$ according to the upper confidence bound:

$$\mathbf{x}_t = \arg\max_{\mathbf{x}} \left[Q(s_t, \mathbf{x}) + c\sqrt{\frac{\ln N(s_t)}{N(s_t, \mathbf{x})}} \right], \tag{3}$$

where $Q(s_t, \mathbf{x})$ is the estimated value, $N(s_t)$ and $N(s_t, \mathbf{x})$ are visit counts, and $c > 0$ balances exploration and exploitation. The reward, defined as the total number of spheres is backpropagated to guide the MCTS algorithm toward configurations that maximize the kissing number. This process repeats until no feasible $\mathbf{x}$ remain, yielding the cosine set $C_1 = \{\cos\theta_1, \ldots, \cos\theta_k\}$ of allowed cosine values for possible kissing configuration. We note that a similar concept of the cosine set appeared in a previous note [21], but in our method it is used fundamentally differently, with the construction relying solely on cosine relationships.

Finding Kissing Numbers in two-player matrix completion game

Let $\mathbf{G}^{(m_0)} \in \mathbb{R}^{m_0\times m_0}$ be the initial partial Gram matrix of the sphere center vectors. We formulate the matrix completion process as a two-player sequential game, corresponding to Step 2 and Step 3 in Figure 2. Each player is modeled as a learning agent, and the two agents are jointly optimized through game-theoretic reinforcement learning to cooperatively search for larger kissing numbers.

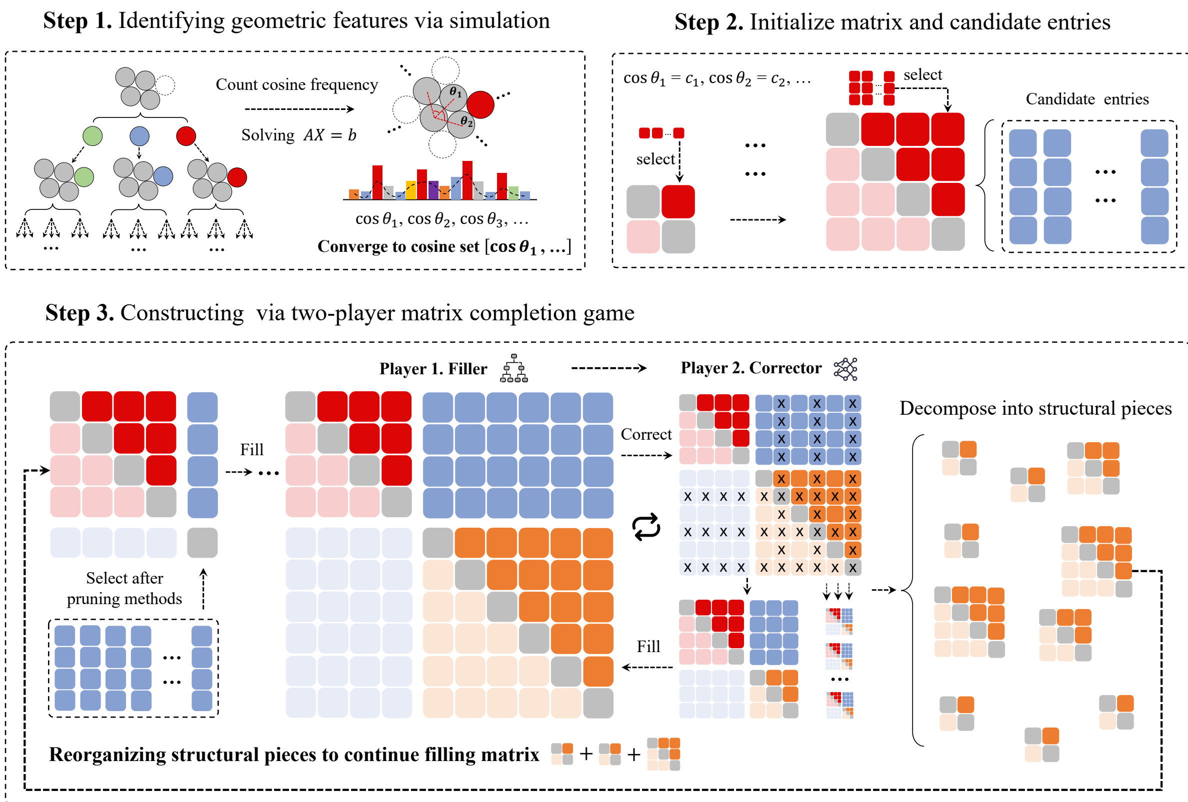

Figure 2: **Overview of *PackingStar* system.** The workflow consists of three steps for n dimensions: **Step 1.** New spheres tangent to $n-1$ existing spheres are iteratively solved from equations and added to the search space for simulation. Cosine frequencies are recorded until they converge to a discrete set to capture the geometric features. **Step 2.** The cosine set from Step 1 populates an initial matrix (red region) for the matrix completion game. Initialization can leverage geometric priors or start from scratch. **Step 3.** The first agent, Filler, fills the matrix with candidates, while the second agent, Corrector, removes previously added entries that are not good enough. After each round, the matrix is decomposed and reassembled for the next iteration based on geometric feature analysis.

This alternating filling-correcting process over N rounds can be formulated as a Markov game, where the matrix $\mathbf{G}^{(m)}$ represents the state, Player 1's action is the sampled entry $\mathbf{g}$ that extends the matrix, Player 2's action is the selected index set I for correcting entries, and the shared team reward is the number of rows of the final matrix.

Player 1 (Matrix Filler). At each step $m \geq m_0$, Player 1 samples an action from policy π_1, $\mathbf{g} \sim \pi_1(\cdot \mid \mathbf{G}^{(m)})$, where $\mathbf{g} \in \mathcal{A}^{(m)}$ to extend the current matrix $\mathbf{G}^{(m)} \in \mathbb{R}^{m \times m}$:

$$\mathbf{G}^{(m+1)}(\mathbf{g}) = \begin{bmatrix} \mathbf{G}^{(m)} & \mathbf{g} \\ \mathbf{g}^\top & 1 \end{bmatrix}. \tag{4}$$

The constraints for candidate action set $\mathcal{A}^{(m)}$ depends on m, For $m < n$

$$\mathcal{A}^{(m)} = \left\{ \mathbf{g} \in C_1^m \;\middle|\; \mathbf{G}^{(m+1)}(\mathbf{G}) \succeq 0,\ \operatorname{rank}(\mathbf{G}^{(m+1)}(\mathbf{g})) = m + 1 \right\}. \tag{5}$$

For $m \geq n$, split $\mathbf{g}$ into two parts:

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}^{(1)} \\ \mathbf{g}^{(2)} \end{bmatrix}, \quad \mathbf{g}^{(1)} \in \mathbb{R}^n,\ \mathbf{g}^{(2)} \in \mathbb{R}^{m-n}. \tag{6}$$

Define the top-left block $\mathbf{G}^{(m)}_{1:n,\,1:n} \in \mathbb{R}^{n \times n}$ and its Cholesky factorization $\mathbf{G}^{(m)}_{1:n,\,1:n} = \mathbf{M}\mathbf{M}^\top$, with pseudo-inverse $\mathbf{M}^+$. Let

$$\mathbf{G}' = \mathbf{G}^{(m)}_{n+1:m,\,1:n} \in \mathbb{R}^{(m-n) \times n}. \tag{7}$$

Compute the remaining part:

$$\mathbf{g}^{(2)} = \mathbf{G}'(\mathbf{G}^{(m)}_{1:n,\,1:n})^+ \mathbf{g}^{(1)}\,. \tag{8}$$

Then the candidate action set is

$$\mathcal{A}^{(m)} = \left\{ \mathbf{g} = [\mathbf{g}^{(1)}; \mathbf{g}^{(2)}] \;\middle|\; \mathbf{g}^{(1)} \in C_1^n,\ \|\mathbf{M}^+ \mathbf{g}^{(1)}\|_2 = 1,\ \mathbf{g}^{(2)} \in C_2^{m-n} \right\}. \tag{9}$$

C_2 is a predefined subset of allowed cosine values. C_2 may coincide with C_1, differ from it, or represent a continuous constraint set in different settings, e.g.,

$$C_2 = \left\{ \mathbf{g}^{(2)} \in \mathbb{R}^{m-n} \;\middle|\; g_i^{(2)} \leq 0.5,\ i = 1, \ldots, m-n \right\}. \tag{10}$$

In specific cases, an additional constraint set C_* can be directly applied to the candidate action set,

$$\mathcal{A}^{(m)} \leftarrow \mathcal{A}^{(m)} \cap C_* \tag{11}$$

Player 2 (Matrix Corrector). Once Player 1 completes M steps (or no feasible $\mathbf{g}$ remains), Player 2 samples an index set

$$I \sim \pi_2(\cdot \mid \mathbf{G}^{(M)}) \subseteq \{1, 2, \ldots, M\}, \tag{12}$$

and constructs the refined matrix

$$\mathbf{G}^* = \mathbf{G}^{(M)}_{I,I}. \tag{13}$$

Game Dynamics and Cooperative Learning Process. The policy π_2 is learned to identify and remove suboptimal entries filled by Player 1, so that in subsequent filling steps, Player 1 can expand the matrix further. Then the Player 1 resumes filling starting from $\mathbf{G}^*$, producing a new feasible set $\mathcal{A}^{(M-|I|)}$, and this alternation continues:

$$\mathbf{G}^{(M)} \xrightarrow{\pi_2} \mathbf{G}^* \xrightarrow{\pi_1} \mathbf{G}^{(M')} \xrightarrow{\pi_2} \cdots \tag{14}$$

The alternating filling-correcting process continues until no further feasible entries exist for Player 1, and Player 2 cannot correct suboptimal entries. Let $\mathbf{G}^{(M)}$ denote the final completed matrix. The cooperative reinforcement learning objective of the *PackingStar* system is to maximize the shared team reward $R_{\text{team}} = \left\|\operatorname{diag}\left(\mathbf{G}^{(M)}\right)\right\|_F^2$, where $\|\cdot\|_F$ denotes the Frobenius norm. The higher reward corresponds to a larger kissing number. Formally, the optimal joint policies (π_1^*, π_2^*) are obtained by solving

$$(\pi_1^*, \pi_2^*) = \arg\max_{\pi_1,\pi_2} \mathbb{E}_{\mathbf{g}\sim\pi_1, I\sim\pi_2}\left[\left\|\operatorname{diag}\left(\mathbf{G}^{(M)}(\{\mathbf{g}\}, I)\right)\right\|_F^2\right], \tag{15}$$

where the expectation accounts for the stochasticity of both policies, and $\mathbf{G}^{(M)}(\{\mathbf{g}\}, I)$ denotes the matrix obtained after all filling and correcting steps under policies π_1 and π_2. In short, the policy of Player 1 is implemented as a tree-search learner, where node values are updated based on the team reward obtained after each episode. The policy of Player 2 is parameterized by a neural network and optimized via policy gradient. After each round, both the tree search statistics of Player 1 and the policy parameters of Player 2 are jointly updated under the same team reward, enabling the two players to mutually improve and construct larger kissing configurations. More implementation details will be provided in the supplementary materials of a future version.

Conclusion

Trained from scratch without any human-designed priors, *PackingStar* automatically discovers high-dimensional kissing configurations that outperform all known constructions and exhibit clear mathematical structure. The key idea is to reformulate the Kissing Number Problem as a two-player matrix completion game over the Gram-matrix domain, enabling efficient exploration of an extremely large and sparse space without explicit coordinate representations. Notably, *PackingStar* improves all known lower bounds from dimensions 25 to 31. In 13 dimensions, it uncovers a rational configuration with a larger kissing number; and in 14 and other dimensions, it identifies thousands of new configurations that provide a rich experimental basis for structural classification and better geometric analysis. A current limitation is that *PackingStar* cannot provide optimality proofs for these new configurations. We anticipate that the interaction between reinforcement learning and rigorous proof techniques will not only open new pathways for the Kissing Number Problem, but also extend to broader geometry problems.

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Author contributions

C.M. conceived and led the project. Z.T. established geometric metrics and analyzed the properties of the structures. C.M. developed the game-theoretic reinforcement learning method. M.L., Z.T., Z.M. and C.M. explored geometric relationships between structures of different dimensions. H.C. explored the training of geometric generative models for high-dimensional structures. C.M., H.C., and M.L. modeled the process of cosine feature simulation and designed the simulation algorithm. M.L., C.M. and Z.T. discovered the cosine feature of constructions via numerical simulation. C.M., H.C., Z.T., M.L., P.L., Z.M. and Y.Y. completed the formulation of matrix completion and designed the algorithm. Z.M. provided practical insights and metrics on the exploration of lattices and error-correcting codes, particularly in the construction of 31D, which provides core insights. C.M., P.L., M.L., Z.T. and H.C. developed the project code. C.M., Z.T., P.L. and H.C. conducted the experiments. C.M. and Y.Y. wrote the manuscript. Y.Q., Y.Y. and Y.C. provided computational resources and contributed to the discussions.

Competing interests

The authors declare no competing interests.