# TETRAGONAL MODULAR QUOTIENTS OF $X_0(N)$

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Abstract. Let $N$ be a positive integer. For every $d \mid N$ such that $(d, N/d) = 1$ there exists an Atkin-Lehner involution $w_d$ of the modular curve $X_0(N)$. Let $B(N)$ be the group of all such involutions. In this paper we determine all $\mathbb{C}$ and $\mathbb{Q}$-tetragonal quotient curves $X_0(N)/W_N$, where $W_N \subseteq B(N)$ such that $4 \leq |W_N| \leq 2^{\omega(N)-1}$, thus completing the classification of all $\mathbb{C}$-tetragonal quotients of $X_0(N)$ by Atkin-Lehner involutions.

## 1. Introduction

Let $k$ be a field and $C$ a smooth projective curve over $k$. The *gonality* $\operatorname{gon}_k C$ of $C$ over $k$ is defined as the least degree of a non-constant morphism $f : C \to \mathbb{P}^1$, or equivalently the least degree of a non-constant function $f \in k(C)$. There exists an upper bound for $\operatorname{gon}_k C$, linear in terms of the genus $g$.

**Proposition 1.1** ([21, Proposition A.1.]). *Let $X$ be a curve of genus $g$ over a field $k$.*

*(i) If $L$ is a field extension of $k$, then* $\operatorname{gon}_L(X) \leq \operatorname{gon}_k(X)$.
*(ii) If $k$ is algebraically closed and $L$ is a field extension of $k$, then* $\operatorname{gon}_L(X) = \operatorname{gon}_k(X)$.
*(iii) If $g \geq 2$, then* $\operatorname{gon}_k(X) \leq 2g - 2$.
*(iv) If $g \geq 2$ and $X(k) \neq \emptyset$, then* $\operatorname{gon}_k(X) \leq g$.
*(v) If $k$ is algebraically closed, then* $\operatorname{gon}_k(X) \leq \frac{g+3}{2}$.
*(vi) If $\pi : X \to Y$ is a dominant $k$-rational map, then* $\operatorname{gon}_k(X) \leq \deg \pi \cdot \operatorname{gon}_k(Y)$.
*(vii) If $\pi : X \to Y$ is a dominant $k$-rational map, then* $\operatorname{gon}_k(X) \geq \operatorname{gon}_k(Y)$.

When $C$ is a modular curve, there also exists a linear lower bound for the $\mathbb{C}$-gonality. This was first proved by Zograf [26]. The constant was later improved by Abramovich [1] and by Kim and Sarnak in Appendix 2 to [14].

Modular curves, especially $X_0(N)$ and $X_1(N)$, are very important objects in arithmetic geometry and their gonality has been the subject of extensive research. Ogg [17] determined all hyperelliptic curves $X_0(N)$, Bars [2] determined all bielliptic curves $X_0(N)$, Hasegawa and Shimura determined all trigonal curves $X_0(N)$ over $\mathbb{C}$ and $\mathbb{Q}$, Jeon and Park determined all tetragonal curves $X_0(N)$ over $\mathbb{C}$, and Najman and Orlić [15] determined all curves $X_0(N)$ with $\mathbb{Q}$-gonality equal to 4, 5, or 6, and also determined the $\mathbb{Q}$ and $\mathbb{C}$-gonality for many other curves $X_0(N)$.

Modular quotients $X_0(N)/W_N$ are also important because points on them have a deep connection with $\mathbb{Q}$-curves, see [16, Section 2]. They also serve as a tool for studying the modular curve $X_0(N)$. Furumoto and Hasegawa [6] determined all hyperelliptic quotients $X_0(N)/W_N$, and Hasegawa and Shimura [9, 10, 11] determined all trigonal quotients $X_0(N)/W_N$ over $\mathbb{C}$. Jeon [12] determined all bielliptic curves $X_0^+(N)$, Bars, Gonzalez, and Kamel [3] determined

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all bielliptic quotients of $X_0(N)$ for squarefree levels $N$, Bars and Gonzalez determined all bielliptic curves $X_0^*(N)$, and Bars, Kamel, and Schweizer [4] determined all bielliptic quotients of $X_0(N)$ for non-squarefree levels $N$, completing the classification of bielliptic quotients.

The next logical step is to determine all tetragonal quotients of $X_0(N)$. All tetragonal quotients $X_0^{+d}(N) = X_0(N)/\langle w_d \rangle$ over $\mathbb{C}$ and $\mathbb{Q}$ were determined in [19, 18]. All $\mathbb{C}$-tetragonal quotients $X_0^*(N)$ were determined in [20], as well as all $\mathbb{Q}$-tetragonal quotients with the exception of level $N = 378$. The curve $X_0^*(378)$ is of genus 5 and it is not yet determined whether its $\mathbb{Q}$-gonality is 4 or 5.

Here, we will do the same for the remaining curves $X_0(N)/W_N$, that is, the curves with $4 \leq |W_N| \leq 2^{\omega(N)-1}$. We also determine all curves $X_0^*(N)$ of genus 4 that are trigonal over $\mathbb{Q}$, thus completing the classification of all $\mathbb{Q}$-trigonal quotients of $X_0(N)$ ([11] does not study the $\mathbb{Q}$-gonality of genus 4 quotients of $X_0(N)$).

Our main results are the following theorems.

**Theorem 1.2.** *Let $W_N \subseteq B(N)$ such that $4 \leq |W_N| \leq 2^{\omega(N)-1}$. The curve $X_0(N)/W_N$ is of genus 4 and has $\mathbb{Q}$-gonality equal to 3 if and only if*

$$\begin{aligned}(N, W_N) \in \{&(130, \langle w_5, w_{13}\rangle), (132, \langle w_4, w_{33}\rangle), (154, \langle w_2, w_{77}\rangle), (170, \langle w_5, w_{34}\rangle),\\ &(182, \langle w_2, w_{91}\rangle), (186, \langle w_3, w_{62}\rangle), (210, \langle w_2, w_{15}, w_{21}\rangle),\\ &(255, \langle w_5, w_{51}\rangle), (285, \langle w_3, w_{95}\rangle), (286, \langle w_2, w_{143}\rangle)\}.\end{aligned}$$

**Theorem 1.3.** *Let $W_N \subseteq B(N)$ such that $4 \leq |W_N| \leq 2^{\omega(N)-1}$. The curve $X_0(N)/W_N$ has $\mathbb{Q}$-gonality equal to 4 if and only if the pair $(N, W_N)$ is in Table 1 at the end of the paper. Furthermore, there are no curves $X_0(N)/W_N$ with $4 \leq |W_N| \leq 2^{\omega(N)-1}$ that are $\mathbb{C}$-tetragonal, but have $\mathbb{Q}$-gonality greater than 4.*

We use a variety of methods to prove these results. Each section roughly corresponds to one method. The paper is organized as follows:

- In Section 2 we eliminate all but finitely many levels $N$ for which the curve $X_0(N)/W_N$ is not $\mathbb{Q}$-tetragonal.
- In Section 3 we give lower bounds on the $\mathbb{Q}$-gonality via $\mathbb{F}_p$-gonality, either by counting points over $\mathbb{F}_{p^n}$ or computing the Riemann-Roch dimensions of degree 4 effective $\mathbb{F}_p$-rational divisors. We study the $\mathbb{F}_p$-gonality only for curves of genus $g \geq 10$ since for them we also get a bound on the $\mathbb{C}$-gonality, see Corollary 2.2.
- In Section 4 we determine the $\mathbb{C}$-gonality by computing Betti numbers $\beta_{i,j}$ for curves of genus $g \leq 9$. We use them both to prove and disprove the existence of degree 4 morphisms over $\mathbb{C}$ (and $\mathbb{Q}$) to $\mathbb{P}^1$.
- In Section 5 we use the Castelnuovo-Severi inequality to give a lower bound on the $\mathbb{C}$-gonality.
- In Section 6 we construct rational morphisms to $\mathbb{P}^1$. The two main methods are finding a degree 2 quotient map to another quotient or using `Magma` to find a morphism to $\mathbb{P}^1$ of the desired degree.
- In Section 7 we present isomorphisms between some quotients of $X_0(N)$ which enables us to determine their gonality more easily.
- In Section 8 we combine the results from previous sections to obtain proofs of Theorems 1.2 and 1.3. For the reader's convenience, at the end of the paper after the

proofs of these theorems we put Table 1 which contains all tetragonal quotient curves $X_0(N)/W_N$ with $4 \leq |W_N| \leq 2^{\omega(N)-1}$.

A significant number of results in this paper rely on `Magma` computations. The version of `Magma` used in the computations is V2.28-3. The codes that verify all computations in this paper can be found on

`https://github.com/orlic1/gonality_X0_star`.

All computations were performed on the Euler server at the Department of Mathematics, University of Zagreb with an Intel Xeon W-2133 CPU running at 3.60GHz and with 64 GB of RAM.


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## 2. Preliminaries

Since all quotient curves $X_0(N)/W_N$ have at least one rational cusp, Proposition 1.1 (iv) implies that their $\mathbb{Q}$-gonality is bounded from above by their genus. Moreover, due to the natural projection

$$X_0(N)/W_N \to X_0^*(N)$$

and Proposition 1.1 (vii), when searching for tetragonal curves $X_0(N)/W_N$, we may restrict ourselves to the levels $N$ for which the curve $X_0^*(N)$ has $\mathbb{C}$-gonality at most 4. All such levels $N$ are listed in [8, 10, 20] and for all of them we have that $\omega(N) \leq 4$. We may eliminate all levels $N$ with $\omega(N) \leq 2$ since for these levels the only quotient curves by Atkin-Lehner involutions are $X_0^{+d}(N)$ for $d \mid\mid N$ and $X_0^*(N)$.

This leaves us with levels $N$ such that $\omega(N) = 3, 4$. If $\omega(N) = 3$, there are 7 quotient curves $X_0(N)/W_N$ with $|W_N| = 4$. They are (for simplicity we suppose that $N = pqr$, $p \neq q \neq r \neq p$ are prime powers)

$$W_N = \langle w_p, w_q \rangle, \langle w_p, w_r \rangle, \langle w_q, w_r \rangle, \langle w_p, w_{qr} \rangle, \langle w_q, w_{pr} \rangle, \langle w_r, w_{pq} \rangle, \langle w_{pq}, w_{pr} \rangle.$$

If $\omega(N)=4$, there are 15 quotient curves $X_0(N)/W_N$ with $|W_N|=8$. They are (for simplicity we suppose that $N=p_1p_2p_3p_4$, $p$-s are pairwise different prime powers)

$$\begin{aligned} W_N = &\left\langle w_{p_i}, w_{p_j}, w_{p_k}\right\rangle \text{ (4 curves)}, \quad \left\langle w_{p_i}, w_{p_j}, w_{p_kp_l}\right\rangle \text{ (6 curves)},\\ &\left\langle w_{p_i}, w_{p_jp_k}\right\rangle \text{ (4 curves)}, \qquad \left\langle w_{p_1p_2}, w_{p_1p_3}, w_{p_1p_4}\right\rangle \text{ (1 curve)}. \end{aligned}$$

There are also 35 quotient curves $X_0(N)/W_N$ with $|W_N|=4$. They are

$$\begin{aligned} W_N = &\left\langle w_{p_i}, w_{p_j}\right\rangle \text{ (6 curves)}, \qquad \left\langle w_{p_i}, w_{p_jp_k}\right\rangle \text{ (12 curves)},\\ &\left\langle w_{p_ip_j}, w_{p_ip_k}\right\rangle \text{ (4 curves)}, \quad \left\langle w_{p_ip_j}, w_{p_kp_l}\right\rangle \text{ (3 curves)},\\ &\left\langle w_{p_i}, w_{p_ip_kp_l}\right\rangle \text{ (6 curves)}, \quad \left\langle w_{p_i}, w_{p_jp_kp_l}\right\rangle \text{ (4 curves)}. \end{aligned}$$

For many curves $X_0(N)/W_N$ we will be able to use $\mathbb{Q}$-gonality to give a lower bound on $\mathbb{C}$-gonality using the Tower theorem.

**Theorem 2.1** (The Tower Theorem, [21, Proposition 2.4])**.** *Let $C$ be a curve defined over a perfect field $k$ such that $C(k)\neq 0$ and let $f: C\to\mathbb{P}^1$ be a non-constant morphism over $\overline{k}$ of degree $d$. Then there exists a curve $C'$ defined over $k$ and a non-constant morphism $C\to C'$ defined over $k$ of degree $d'$ dividing $d$ such that the genus of $C'$ is $\leq (\frac{d}{d'}-1)^2$.*

**Corollary 2.2.** [15, Corollary 4.6. (ii)] *Let $C$ be a curve defined over $\mathbb{Q}$ with $\operatorname{gon}_\mathbb{C}(C)=4$ and $g(C)\geq 10$ and such that $C(\mathbb{Q})\neq\emptyset$. Then $\operatorname{gon}_\mathbb{Q}(C)=4$.*

Corollary 2.2 is applicable for all quotients of $X_0(N)$ since they all have at least 1 rational cusp. The next result is an application of Corollary 2.2. It deals with levels $N$ for which the curve $X_0^*(N)$ is tetragonal over $\mathbb{C}$, but not over $\mathbb{Q}$.

**Proposition 2.3.** *There are no $\mathbb{C}$-tetragonal curves $X_0(N)/W_N$ with $4\leq |W_N|\leq 2^{\omega(N)-1}$ for*

$$N\in\{506, 561, 590, 609, 615, 690, 858\}.$$

*Proof.* For these levels $N$ the curve $X_0^*(N)$ is of genus 6, $\mathbb{C}$-gonality equal to 4, and $\mathbb{Q}$-gonality greater than 4 by [20]. Therefore, the $\mathbb{Q}$-gonality of all curves $X_0(N)/W_N$ is also greater than 4 by Proposition 1.1 (vii).

For $N\in\{506, 561, 590, 609, 615\}$ the genera of all 7 quotients with $|W_N|=4$ are at least 10. Thus, Corollary 2.2 implies that their $\mathbb{C}$-gonality is also greater than 4.

For $N\in\{690, 858\}$ the genera of all 15 quotients with $|W_N|=8$ are at least 10. Thus, Corollary 2.2 implies that their $\mathbb{C}$-gonality is also greater than 4 and Proposition 1.1 (vii) implies the same for all 35 quotients with $|W_N|=4$ (because for every such curve there is a degree 2 rational map to a quotient by a group of order 8). ◻

The remaining levels $N$ are those with $\omega(N)=3,4$ for which the $\mathbb{Q}$-gonality of $X_0^*(N)$ is $\leq 4$ and $N=378$ (its $\mathbb{Q}$-gonality could be 4 or 5). Before moving on, we may eliminate levels

$$N\in\{42, 60, 66, 70, 78, 90, 105\}$$

because $g(X_0(N)/W_N)\leq 3$ for all groups $W_N$ of order 4.

## 3. $\mathbb{F}_p$-gonality

In this section we use the results on the $\mathbb{F}_p$-gonality to get a lower bound on the $\mathbb{Q}$-gonality of quotient curves $X_0(N)/W_N$. $\mathbb{F}_p$-gonality is often much easier to compute than $\mathbb{Q}$-gonality and, since we have an inequality

$$\operatorname{gon}_{\mathbb{F}_p}(C) \leq \operatorname{gon}_{\mathbb{Q}}(C)$$

for primes $p$ of good reduction for curves $C/\mathbb{Q}$, it is very useful to give a lower bound on the $\mathbb{Q}$-gonality.

**Proposition 3.1.** *The $\mathbb{F}_p$-gonality of the quotient curve $X_0(N)/W_N$ is at least* 5 *for the following pairs $(N, W_N)$:*

| $N$ | $W_N$ | $p$ | $N$ | $W_N$ | $p$ | $N$ | $W_N$ | $p$ |
|---|---|---|---|---|---|---|---|---|
| 210 | $\langle w_2, w_{15}\rangle$ | 11 | 210 | $\langle w_2, w_{11}\rangle$ | 11 | 210 | $\langle w_{15}, w_{42}\rangle$ | 11 |
| 210 | $\langle w_{21}, w_{30}\rangle$ | 11 | 260 | $\langle w_5, w_{13}\rangle$ | 3 | 260 | $\langle w_{20}, w_{52}\rangle$ | 3 |
| 264 | $\langle w_3, w_{11}\rangle$ | 5 | 264 | $\langle w_8, w_{11}\rangle$ | 5 | 264 | $\langle w_{24}, w_{33}\rangle$ | 5 |
| 280 | $\langle w_5, w_7\rangle$ | 3 | 280 | $\langle w_5, w_8\rangle$ | 3 | 280 | $\langle w_8, w_{35}\rangle$ | 3 |
| 300 | $\langle w_3, w_{25}\rangle$ | 7 | 300 | $\langle w_{12}, w_{25}\rangle$ | 7 | 300 | $\langle w_{12}, w_{75}\rangle$ | 7 |
| 300 | $\langle w_3, w_{100}\rangle$ | 7 | 306 | $\langle w_2, w_9\rangle$ | 7 | 306 | $\langle w_2, w_{17}\rangle$ | 5 |
| 306 | $\langle w_9, w_{17}\rangle$ | 5 | 308 | $\langle w_4, w_{11}\rangle$ | 3 | 308 | $\langle w_7, w_{11}\rangle$ | 3 |
| 308 | $\langle w_{28}, w_{44}\rangle$ | 3 | 322 | $\langle w_2, w_7\rangle$ | 5 | 322 | $\langle w_7, w_{46}\rangle$ | 3 |
| 322 | $\langle w_{14}, w_{23}\rangle$ | 3 | 330 | $\langle w_2, w_{11}\rangle$ | 7 | 330 | $\langle w_5, w_{11}\rangle$ | 7 |
| 330 | $\langle w_2, w_{33}\rangle$ | 7 | 330 | $\langle w_2, w_{55}\rangle$ | 7 | 330 | $\langle w_3, w_{22}\rangle$ | 7 |
| 330 | $\langle w_5, w_{22}\rangle$ | 7 | 330 | $\langle w_6, w_{11}\rangle$ | 7 | 330 | $\langle w_{10}, w_{11}\rangle$ | 7 |
| 330 | $\langle w_{22}, w_{30}\rangle$ | 7 | 330 | $\langle w_{30}, w_{33}\rangle$ | 7 | 330 | $\langle w_2, w_{165}\rangle$ | 7 |
| 330 | $\langle w_3, w_{110}\rangle$ | 7 | 336 | $\langle w_3, w_{16}\rangle$ | 5 | 336 | $\langle w_3, w_7\rangle$ | 5 |
| 336 | $\langle w_7, w_{16}\rangle$ | 5 | 336 | $\langle w_{21}, w_{48}\rangle$ | 5 | 336 | $\langle w_7, w_{48}\rangle$ | 5 |
| 336 | $\langle w_3, w_{112}\rangle$ | 5 | 340 | $\langle w_5, w_{17}\rangle$ | 3 | 340 | $\langle w_{17}, w_{20}\rangle$ | 3 |
| 340 | $\langle w_{20}, w_{68}\rangle$ | 3 | 340 | $\langle w_5, w_{68}\rangle$ | 3 | 342 | $\langle w_9, w_{19}\rangle$ | 5 |
| 342 | $\langle w_9, w_{38}\rangle$ | 5 | 345 | $\langle w_3, w_5\rangle$ | 7 | 345 | $\langle w_5, w_{23}\rangle$ | 2 |
| 345 | $\langle w_3, w_{23}\rangle$ | 2 | 345 | $\langle w_{15}, w_{23}\rangle$ | 2 | 345 | $\langle w_3, w_{115}\rangle$ | 2 |
| 350 | $\langle w_7, w_{50}\rangle$ | 3 | 350 | $\langle w_{14}, w_{25}\rangle$ | 3 | 354 | $\langle w_2, w_{59}\rangle$ | 5 |
| 354 | $\langle w_3, w_{59}\rangle$ | 5 | 360 | $\langle w_8, w_9\rangle$ | 7 | 360 | $\langle w_5, w_8\rangle$ | 7 |
| 360 | $\langle w_5, w_9\rangle$ | 7 | 360 | $\langle w_8, w_{45}\rangle$ | 7 | 364 | $\langle w_7, w_{13}\rangle$ | 3 |
| 364 | $\langle w_{13}, w_{28}\rangle$ | 3 | 364 | $\langle w_{28}, w_{152}\rangle$ | 3 | 364 | $\langle w_7, w_{52}\rangle$ | 3 |
| 366 | $\langle w_3, w_{61}\rangle$ | 5 | 366 | $\langle w_3, w_{122}\rangle$ | 5 | 366 | $\langle w_2, w_{183}\rangle$ | 5 |
| 370 | $\langle w_2, w_{37}\rangle$ | 3 | 370 | $\langle w_5, w_{74}\rangle$ | 3 | 370 | $\langle w_2, w_{185}\rangle$ | 3 |
| 372 | $\langle w_3, w_{31}\rangle$ | 5 | 372 | $\langle w_{12}, w_{31}\rangle$ | 5 | 372 | $\langle w_{12}, w_{93}\rangle$ | 5 |
| 372 | $\langle w_3, w_{124}\rangle$ | 5 | 374 | $\langle w_{11}, w_{17}\rangle$ | 3 | 374 | $\langle w_{11}, w_{34}\rangle$ | 3 |
| 378 | $\langle w_2, w_{27}\rangle$ | 5 | 378 | $\langle w_7, w_{27}\rangle$ | 5 | 378 | $\langle w_{14}, w_{27}\rangle$ | 5 |
| 378 | $\langle w_{14}, w_{54}\rangle$ | 5 | 378 | $\langle w_7, w_{54}\rangle$ | 5 | 378 | $\langle w_2, w_{189}\rangle$ | 5 |
| 385 | $\langle w_5, w_7\rangle$ | 3 | 385 | $\langle w_5, w_{11}\rangle$ | 2 | 385 | $\langle w_5, w_{77}\rangle$ | 2 |
| 390 | $\langle w_3, w_{65}\rangle$ | 11 | 390 | $\langle w_5, w_{39}\rangle$ | 7 | 390 | $\langle w_6, w_{26}\rangle$ | 7 |
| 390 | $\langle w_{26}, w_{30}\rangle$ | 7 | 390 | $\langle w_2, w_{195}\rangle$ | 7 | 396 | $\langle w_4, w_{11}\rangle$ | 5 |
| 396 | $\langle w_9, w_{11}\rangle$ | 5 | 396 | $\langle w_9, w_{44}\rangle$ | 5 | 396 | $\langle w_{36}, w_{44}\rangle$ | 5 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 402 | $\langle w_2, w_{67}\rangle$ | 7 | 402 | $\langle w_6, w_{134}\rangle$ | 5 | 402 | $\langle w_3, w_{134}\rangle$ | 5 |
| 402 | $\langle w_2, w_{201}\rangle$ | 5 | 406 | $\langle w_7, w_{29}\rangle$ | 3 | 406 | $\langle w_{14}, w_{29}\rangle$ | 3 |
| 406 | $\langle w_{14}, w_{58}\rangle$ | 3 | 406 | $\langle w_7, w_{58}\rangle$ | 3 | 406 | $\langle w_2, w_{203}\rangle$ | 3 |
| 410 | $\langle w_2, w_{41}\rangle$ | 3 | 410 | $\langle w_5, w_{41}\rangle$ | 3 | 410 | $\langle w_{10}, w_{41}\rangle$ | 3 |
| 410 | $\langle w_5, w_{82}\rangle$ | 3 | 410 | $\langle w_2, w_{205}\rangle$ | 3 | 414 | $\langle w_9, w_{23}\rangle$ | 5 |
| 414 | $\langle w_{18}, w_{23}\rangle$ | 5 | 418 | $\langle w_{11}, w_{19}\rangle$ | 3 | 418 | $\langle w_{22}, w_{38}\rangle$ | 3 |
| 418 | $\langle w_2, w_{209}\rangle$ | 3 | 435 | $\langle w_3, w_{29}\rangle$ | 2 | 435 | $\langle w_5, w_{87}\rangle$ | 2 |
| 435 | $\langle w_3, w_{145}\rangle$ | 2 | 438 | $\langle w_3, w_{146}\rangle$ | 5 | 438 | $\langle w_6, w_{146}\rangle$ | 5 |
| 442 | $\langle w_2, w_{13}\rangle$ | 3 | 442 | $\langle w_{13}, w_{17}\rangle$ | 3 | 442 | $\langle w_{17}, w_{26}\rangle$ | 3 |
| 442 | $\langle w_{26}, w_{34}\rangle$ | 3 | 442 | $\langle w_2, w_{221}\rangle$ | 3 | 444 | $\langle w_3, w_{37}\rangle$ | 5 |
| 444 | $\langle w_{12}, w_{111}\rangle$ | 5 | 456 | $\langle w_8, w_{19}\rangle$ | 5 | 456 | $\langle w_{19}, w_{24}\rangle$ | 5 |
| 456 | $\langle w_{24}, w_{57}\rangle$ | 5 | 456 | $\langle w_8, w_{57}\rangle$ | 5 | 456 | $\langle w_3, w_{152}\rangle$ | 5 |
| 470 | $\langle w_{10}, w_{47}\rangle$ | 3 | 494 | $\langle w_{13}, w_{38}\rangle$ | 3 | 504 | $\langle w_9, w_{56}\rangle$ | 5 |
| 520 | $\langle w_8, w_{13}\rangle$ | 7 | 558 | $\langle w_9, w_{31}\rangle$ | 5 | 558 | $\langle w_{18}, w_{62}\rangle$ | 5 |
| 630 | $\langle w_5, w_7, w_9\rangle$ | 11 | 630 | $\langle w_2, w_5, w_{63}\rangle$ | 11 | 630 | $\langle w_2, w_9, w_35\rangle$ | 11 |
| 630 | $\langle w_2, w_7, w_{45}\rangle$ | 11 | 630 | $\langle w_5, w_9, w_{14}\rangle$ | 11 | 630 | $\langle w_2, w_{35}, w_{45}\rangle$ | 11 |
| 630 | $\langle w_9, w_{10}, w_{14}\rangle$ | 11 | 660 | $\langle w_3, w_5, w_{11}\rangle$ | 7 | 660 | $\langle w_3, w_5, w_{44}\rangle$ | 7 |
| 660 | $\langle w_3, w_{11}, w_{20}\rangle$ | 7 | 660 | $\langle w_5, w_{11}, w_{12}\rangle$ | 7 | 660 | $\langle w_3, w_{20}, w_{44}\rangle$ | 7 |
| 660 | $\langle w_5, w_{12}, w_{33}\rangle$ | 7 | 660 | $\langle w_{11}, w_{12}, w_{15}\rangle$ | 7 | 660 | $\langle w_{12}, w_{15}, w_{33}\rangle$ | 7 |
| 714 | $\langle w_2, w_7, w_{17}\rangle$ | 5 | 714 | $\langle w_3, w_7, w_{17}\rangle$ | 5 | 714 | $\langle w_2, w_3, w_{119}\rangle$ | 5 |
| 714 | $\langle w_6, w_7, w_{17}\rangle$ | 5 | 714 | $\langle w_2, w_{21}, w_{51}\rangle$ | 5 | 714 | $\langle w_3, w_{14}, w_{34}\rangle$ | 5 |
| 714 | $\langle w_6, w_{14}, w_{34}\rangle$ | 7 | 770 | $\langle w_5, w_7, w_{22}\rangle$ | 3 | 780 | $\langle w_3, w_{13}, w_{20}\rangle$ | 7 |
| 780 | $\langle w_5, w_{12}, w_{39}\rangle$ | 7 | 798 | $\langle w_2, w_3, w_{133}\rangle$ | 5 | 798 | $\langle w_2, w_{19}, w_{21}\rangle$ | 5 |
| 798 | $\langle w_3, w_{14}, w_{38}\rangle$ | 5 | 798 | $\langle w_6, w_7, w_{38}\rangle$ | 5 | 798 | $\langle w_6, w_{14}, w_{19}\rangle$ | 5 |
| 990 | $\langle w_5, w_9, w_{11}\rangle$ | 7 | 990 | $\langle w_2, w_5, w_{99}\rangle$ | 7 | 990 | $\langle w_2, w_9, w_{55}\rangle$ | 7 |
| 990 | $\langle w_2, w_{11}, w_{45}\rangle$ | 7 | 990 | $\langle w_9, w_{10}, w_{11}\rangle$ | 7 | | | |

*Proof.* Using `Magma`, we compute that there are no morphisms $X_0(N)/W_N \to \mathbb{P}^1$ of degree $\leq 4$ defined over $\mathbb{F}_p$. We are able to reduce the number of degree 4 divisors that need to be checked by noting the following: If there exists a function $f : C \to \mathbb{P}^1$ over $\mathbb{F}_p$ of a certain degree and if $\#C(\mathbb{F}_p) > d(p+1)$, then there exists $c \in \mathbb{F}_p$ such that the function $g(x) := \frac{1}{f(x)-c}$ has the same degree and its polar divisor contains at least $d$ $\mathbb{F}_p$-rational points, see [15, Lemma 3.1].

Therefore, if $\#(X_0(N)/W_N)(\mathbb{F}_p) > p+1$, it is enough to check divisors of the form $1+1+1+1$ and $1+1+2$, and if $\#(X_0(N)/W_N)(\mathbb{F}_p) > 2(p+1)$, it is enough to just check divisors of the form $1+1+1+1$. $\square$

The computations for some curves took several days to finish. These curves either have a high genus ($g \geq 17$) or the prime number $p$ is large ($p \geq 7$). As the genus increases, the number of divisors that need to be checked grows significantly. The same is true when $p$ grows because Hasse's theorem tells us that $\#C(\mathbb{F}_{p^k}) \sim p^k$ for large $p$. See [15, Section 8.1] for more details. Here are some examples of curves with higher running time.

| $N$ | $W_N$ | $p$ | $g$ | time |
|---|---|---|---|---|
| 630 | $\langle w_5, w_9, w_{14}\rangle$ | 11 | 13 | 3 hours |
| 990 | $\langle w_2, w_9, w_{55}\rangle$ | 7 | 19 | 6 hours |
| 520 | $\langle w_8, w_{13}\rangle$ | 7 | 17 | 1 day |

We can also get a lower bound on the $\mathbb{Q}$-gonality by counting $\mathbb{F}_{p^k}$ points on the curve without actually computing the $\mathbb{F}_p$-gonality.

**Lemma 3.2.** [15, Lemma 3.5] *Let $C/\mathbb{Q}$ be a curve, $p$ a prime of good reduction for $C$, and $q$ a power of $p$. Suppose $\#C(\mathbb{F}_q) > d(q+1)$. Then $\mathrm{gon}_{\mathbb{Q}}(C) > d$.*

**Proposition 3.3.** *The $\mathbb{F}_p$-gonality of the quotient curve $X_0(N)/W_N$ is at least 5 for the following pairs of $(N, W_N)$:*

| $N$ | $W_N$ | $q$ | $\#(X_0(N)/W_N)(\mathbb{F}_q)$ | $N$ | $W_N$ | $q$ | $\#(X_0(N)/W_N)(\mathbb{F}_q)$ |
|---|---|---|---|---|---|---|---|
| 340 | $\langle w_5, w_{17}\rangle$ | 3 | 18 | 350 | $\langle w_7, w_{25}\rangle$ | 9 | 44 |
| 410 | $\langle w_5, w_{41}\rangle$ | 9 | 42 | 418 | $\langle w_2, w_{19}\rangle$ | 9 | 41 |
| 418 | $\langle w_{19}, w_{22}\rangle$ | 9 | 43 | 435 | $\langle w_5, w_{29}\rangle$ | 4 | 21 |
| 435 | $\langle w_{15}, w_{29}\rangle$ | 4 | 21 | 440 | $\langle w_5, w_{11}\rangle$ | 9 | 48 |
| 440 | $\langle w_{11}, w_{40}\rangle$ | 9 | 42 | 440 | $\langle w_{40}, w_{55}\rangle$ | 9 | 50 |
| 440 | $\langle w_8, w_{55}\rangle$ | 9 | 48 | 460 | $\langle w_5, w_{23}\rangle$ | 9 | 48 |
| 460 | $\langle w_{20}, w_{33}\rangle$ | 9 | 46 | 460 | $\langle w_{20}, w_{92}\rangle$ | 9 | 46 |
| 460 | $\langle w_5, w_{92}\rangle$ | 9 | 48 | 465 | $\langle w_5, w_{31}\rangle$ | 4 | 21 |
| 465 | $\langle w_{15}, w_{31}\rangle$ | 4 | 22 | 465 | $\langle w_{15}, w_{93}\rangle$ | 4 | 22 |
| 465 | $\langle w_3, w_{155}\rangle$ | 4 | 23 | 470 | $\langle w_{10}, w_{94}\rangle$ | 9 | 47 |
| 470 | $\langle w_5, w_{94}\rangle$ | 9 | 48 | 470 | $\langle w_2, w_{235}\rangle$ | 9 | 42 |
| 494 | $\langle w_{26}, w_{38}\rangle$ | 9 | 47 | 494 | $\langle w_2, w_{247}\rangle$ | 9 | 47 |
| 495 | $\langle w_9, w_{11}\rangle$ | 4 | 24 | 495 | $\langle w_9, w_{55}\rangle$ | 4 | 22 |
| 495 | $\langle w_{11}, w_{45}\rangle$ | 4 | 23 | 504 | $\langle w_7, w_9\rangle$ | 25 | 112 |
| 504 | $\langle w_{54}, w_{63}\rangle$ | 25 | 116 | 520 | $\langle w_5, w_8\rangle$ | 9 | 56 |
| 520 | $\langle w_{13}, w_{40}\rangle$ | 3 | 18 | 520 | $\langle w_{40}, w_{65}\rangle$ | 9 | 48 |
| 520 | $\langle w_8, w_{65}\rangle$ | 9 | 56 | 520 | $\langle w_5, w_{104}\rangle$ | 9 | 48 |
| 528 | $\langle w_{11}, w_{16}\rangle$ | 25 | 110 | 528 | $\langle w_{16}, w_{33}\rangle$ | 25 | 114 |
| 528 | $\langle w_{11}, w_{48}\rangle$ | 25 | 114 | 528 | $\langle w_{33}, w_{48}\rangle$ | 25 | 110 |
| 528 | $\langle w_3, w_{176}\rangle$ | 25 | 122 | 532 | $\langle w_7, w_{19}\rangle$ | 9 | 46 |
| 532 | $\langle w_7, w_{76}\rangle$ | 9 | 48 | 532 | $\langle w_{19}, w_{28}\rangle$ | 9 | 52 |
| 532 | $\langle w_{28}, w_{76}\rangle$ | 3 | 18 | 555 | $\langle w_{15}, w_{111}\rangle$ | 4 | 25 |
| 555 | $\langle w_5, w_{111}\rangle$ | 4 | 22 | 555 | $\langle w_3, w_{185}\rangle$ | 4 | 26 |
| 560 | $\langle w_5, w_{16}\rangle$ | 9 | 58 | 560 | $\langle w_5, w_{112}\rangle$ | 9 | 54 |
| 560 | $\langle w_{35}, w_{112}\rangle$ | 9 | 54 | 560 | $\langle w_{16}, w_{35}\rangle$ | 9 | 54 |
| 560 | $\langle w_7, w_{80}\rangle$ | 9 | 54 | 564 | $\langle w_{12}, w_{47}\rangle$ | 25 | 126 |
| 572 | $\langle w_{11}, w_{13}\rangle$ | 9 | 45 | 572 | $\langle w_{44}, w_{52}\rangle$ | 9 | 47 |
| 574 | $\langle w_7, w_{41}\rangle$ | 9 | 46 | 574 | $\langle w_{14}, w_{82}\rangle$ | 9 | 52 |
| 574 | $\langle w_2, w_{287}\rangle$ | 9 | 50 | 585 | $\langle w_5, w_9\rangle$ | 4 | 26 |
| 585 | $\langle w_5, w_{13}\rangle$ | 4 | 27 | 585 | $\langle w_9, w_{13}\rangle$ | 4 | 26 |
| 585 | $\langle w_9, w_{65}\rangle$ | 4 | 30 | 585 | $\langle w_{45}, w_{65}\rangle$ | 4 | 27 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 595 | $\langle w_7, w_{17}\rangle$ | 4 | 24 | 595 | $\langle w_{35}, w_{85}\rangle$ | 4 | 21 |
| 595 | $\langle w_5, w_{119}\rangle$ | 4 | 25 | 598 | $\langle w_{13}, w_{23}\rangle$ | 9 | 44 |
| 598 | $\langle w_{26}, w_{46}\rangle$ | 9 | 48 | 598 | $\langle w_2, w_{299}\rangle$ | 9 | 54 |
| 627 | $\langle w_{11}, w_{19}\rangle$ | 4 | 27 | 627 | $\langle w_{33}, w_{57}\rangle$ | 4 | 29 |
| 627 | $\langle w_3, w_{209}\rangle$ | 4 | 28 | 645 | $\langle w_{15}, w_{129}\rangle$ | 2 | 27 |
| 645 | $\langle w_3, w_{215}\rangle$ | 4 | 30 | 658 | $\langle w_7, w_{47}\rangle$ | 9 | 52 |
| 663 | $\langle w_3, w_{221}\rangle$ | 4 | 27 | 670 | $\langle w_{10}, w_{134}\rangle$ | 9 | 54 |
| 670 | $\langle w_2, w_{335}\rangle$ | 9 | 57 | 770 | $\langle w_2, w_7, w_{55}\rangle$ | 9 | 43 |
| 770 | $\langle w_5, w_{11}, w_{55}\rangle$ | 9 | 44 | 770 | $\langle w_2, w_{35}, w_{55}\rangle$ | 9 | 44 |
| 770 | $\langle w_{10}, w_{14}, w_{22}\rangle$ | 9 | 43 | 910 | $\langle w_5, w_7, w_{13}\rangle$ | 9 | 49 |
| 910 | $\langle w_2, w_5, w_{91}\rangle$ | 9 | 52 | 910 | $\langle w_2, w_{13}, w_{35}\rangle$ | 9 | 48 |
| 910 | $\langle w_5, w_{14}, w_{26}\rangle$ | 9 | 47 | 910 | $\langle w_{10}, w_{13}, w_{14}\rangle$ | 9 | 45 |

*Proof.* Using `Magma`, we calculate the number of $\mathbb{F}_q$ points on $X_0(N)/W_N$. It is now easy to check that $\#(X_0(N)/W_N)(\mathbb{F}_q) > 4(q+1)$ in all these cases.

For some curves we did this without the model of the curve using the function Fpnpoints-forQuotientcurveX0NWN() from

`https://github.com/FrancescBars/Magma-functions-on-Quotient-Modular-Curves/blob/main/funcions.m`.

It computes the newforms $f$ such that the corresponding abelian variety $A_f$ is (up to isogeny) in the Jacobian decomposition of the curve. After that, it finds the characteristic polynomial

$$P(x) = \prod_f (x^2 - a_p(f)x + p) = \prod_1^{2g} (x - \alpha_i).$$

The number of $\mathbb{F}_{p^k}$-points on the curve is then equal to $p^k + 1 - \sum_1^{2g} \alpha_i^k$.

This function can fail to give the correct decomposition and hence return a higher number of points than the correct one. In these cases we found the number of $\mathbb{F}_q$-points via the model of the curve. More details regarding the code are on Github.

The computations without the model were very fast, up to 1 minute. Some computations that needed the model took longer. For example, the curve $X_0(560)/\langle w_5, w_{16}\rangle$ is of genus 20 and it took 2 hours to count the $\mathbb{F}_9$-points on it. □

## 4. Betti numbers

Graded Betti numbers $\beta_{i,j}$ are helpful when determining the $\mathbb{C}$-gonality of a curve. We will use the indexation of Betti numbers as in [13, Section 1.]. The results we mention can be found there and in [15, Section 3.1.].

**Definition 4.1.** For a curve $X$ and divisor $D$ of degree $d$, $g_d^r$ is a subspace $V$ of the Riemann-Roch space $L(D)$ such that $\dim V = r+1$.

We see that the existence of $g_4^1$ is related to the existence of degree 4 maps to $\mathbb{P}^1$. Green's conjecture relates values of graded Betti numbers $\beta_{i,j}$ with the existence of $g_d^r$.

**Conjecture 4.2** (Green, [7]). *Let $X$ be a curve of genus $g$. Then $\beta_{p,2} \neq 0$ if and only if there exists a divisor $D$ on $X$ of degree $d$ such that a subspace $g^r_d$ of $L(D)$ satisfies $d \leq g-1$, $r = \ell(D) - 1 \geq 1$, and $d - 2r \leq p$.*

The "if" part of this conjecture has been proven in the same paper.

**Theorem 4.3** (Green and Lazarsfeld, Appendix to [7]). *Let $X$ be a curve of genus $g$. If $\beta_{p,2} = 0$, then there does not exist a divisor $D$ on $X$ of degree $d$ such that a subspace $g^r_d$ of $L(D)$ satisfies $d \leq g-1$, $r \geq 1$, and $d - 2r \leq p$.*

**Corollary 4.4** ([19, Corollary 3.20]). *Let $X$ be a curve of genus $g \geq 5$ with $\beta_{2,2} = 0$. Suppose that $X$ is neither hyperelliptic nor trigonal. Then $\mathrm{gon}_{\mathbb{C}}(X) \geq 5$.*

Even though the "only if" part is still an open problem, we can say something more about $g^1_4$ from the value of $\beta_{2,2}$.

**Theorem 4.5** ([24, Theorem 4.1, 4.4]). *Let $C \subseteq \mathbb{P}^{g-1}$ be a reduced irreducible canonical curve over $\mathbb{C}$ of genus $g \geq 7$. Then $\beta_{2,2}$ has one of the values in the following table.*

| $\beta_{2,2}$ | $(g-4)(g-2)$ | $\binom{g-2}{2} - 1$ | $g-4$ | $0$ |
|---|---|---|---|---|
| linear series | $\exists g^1_3$ | $\exists g^2_6$ or $g^3_8$ | $\exists$ a single $g^1_4$ | no $g^1_4$ |

Notice that the indexation of Betti numbers is different here and in [24], where they are indexed as $\beta_{2,4}$.

The existence of a single $g^1_4$ is useful as the following result tells us that in that case we have a rational morphism of degree $\leq 4$ to $\mathbb{P}^1$.

**Theorem 4.6** ([22, Theorem 5]). *Let $C/k$ be a smooth projective curve. Suppose that for a fixed $r$ and $d$ there is only one $g^r_d$, giving a morphism $f : C_{k^{\mathrm{sep}}} \to \mathbb{P}^r_{k^{\mathrm{sep}}}$. Then there exists a Brauer-Severi variety $\mathcal{P}$ defined over $k$ together with a $k$-morphism $g : C \to \mathcal{P}$ such that $g \oplus_k k^{\mathrm{sep}} : C_{k^{\mathrm{sep}}} \to \mathcal{P}_{k^{\mathrm{sep}}} \cong \mathbb{P}^r_{k^{\mathrm{sep}}}$ is equal to $f$.*

**Corollary 4.7** ([20, Corollary 5.7]). *Let $C/\mathbb{Q}$ be smooth projective curve such that there is only one $g^1_d$. Then there exists a conic $\mathcal{P}$ defined over $\mathbb{Q}$ and a degree $d$ rational map $g : C \to \mathcal{P}$. If additionally $C$ has at least one rational point, then $\mathcal{P}$ is $\mathbb{Q}$-isomorphic to $\mathbb{P}^1$.*

**Proposition 4.8.** *The $\mathbb{C}$-gonality of the quotient curve $X_0(N)/W_N$ is at least 5 for the following pairs $(N, W_N)$. Here $g$ is the genus of $X_0(N)/W_N$.*

| $N$ | $W_N$ | $g$ | $N$ | $W_N$ | $g$ | $N$ | $W_N$ | $g$ | $N$ | $W_N$ | $g$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 210 | $\langle w_2, w_7\rangle$ | 9 | 210 | $\langle w_5, w_7\rangle$ | 7 | 210 | $\langle w_2, w_{35}\rangle$ | 8 | 210 | $\langle w_3, w_{14}\rangle$ | 9 |
| 210 | $\langle w_3, w_{35}\rangle$ | 7 | 210 | $\langle w_5, w_6\rangle$ | 9 | 210 | $\langle w_5, w_{14}\rangle$ | 8 | 210 | $\langle w_5, w_{21}\rangle$ | 8 |
| 210 | $\langle w_6, w_{70}\rangle$ | 9 | 210 | $\langle w_{14}, w_{30}\rangle$ | 8 | 210 | $\langle w_2, w_{105}\rangle$ | 9 | 210 | $\langle w_5, w_{42}\rangle$ | 9 |
| 228 | $\langle w_3, w_{19}\rangle$ | 9 | 228 | $\langle w_{12}, w_{19}\rangle$ | 8 | 228 | $\langle w_{12}, w_{57}\rangle$ | 8 | 234 | $\langle w_2, w_9\rangle$ | 9 |
| 234 | $\langle w_2, w_{13}\rangle$ | 8 | 234 | $\langle w_9, w_{13}\rangle$ | 8 | 240 | $\langle w_3, w_5\rangle$ | 8 | 240 | $\langle w_5, w_4 8\rangle$ | 9 |
| 246 | $\langle w_2, w_{41}\rangle$ | 8 | 246 | $\langle w_3, w_{41}\rangle$ | 7 | 246 | $\langle w_3, w_{82}\rangle$ | 9 | 246 | $\langle w_6, w_{82}\rangle$ | 9 |
| 246 | $\langle w_6, w_{41}\rangle$ | 7 | 246 | $\langle w_2, w_{123}\rangle$ | 7 | 258 | $\langle w_3, w_{43}\rangle$ | 9 | 258 | $\langle w_3, w_{86}\rangle$ | 7 |
| 258 | $\langle w_2, w_{129}\rangle$ | 8 | 260 | $\langle w_{13}, w_{20}\rangle$ | 8 | 260 | $\langle w_5, w_{52}\rangle$ | 8 | 264 | $\langle w_3, w_8\rangle$ | 9 |
| 264 | $\langle w_3, w_{88}\rangle$ | 9 | 270 | $\langle w_5, w_{27}\rangle$ | 8 | 270 | $\langle w_5, w_{54}\rangle$ | 8 | 270 | $\langle w_2, w_{135}\rangle$ | 7 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 273 | $\langle w_7, w_{13}\rangle$ | 9 | 273 | $\langle w_7, w_{39}\rangle$ | 8 | 273 | $\langle w_{21}, w_{39}\rangle$ | 9 | 273 | $\langle w_{13}, w_{21}\rangle$ | 8 |
| 280 | $\langle w_7, w_8\rangle$ | 9 | 280 | $\langle w_7, w_{40}\rangle$ | 9 | 290 | $\langle w_2, w_{29}\rangle$ | 9 | 290 | $\langle w_5, w_{29}\rangle$ | 8 |
| 290 | $\langle w_{10}, w_{29}\rangle$ | 7 | 290 | $\langle w_5, w_{58}\rangle$ | 8 | 290 | $\langle w_2, w_{145}\rangle$ | 7 | 294 | $\langle w_2, w_{49}\rangle$ | 9 |
| 294 | $\langle w_3, w_{49}\rangle$ | 9 | 294 | $\langle w_6, w_{49}\rangle$ | 9 | 294 | $\langle w_3, w_{98}\rangle$ | 7 | 294 | $\langle w_2, w_{147}\rangle$ | 8 |
| 306 | $\langle w_{17}, w_{18}\rangle$ | 9 | 308 | $\langle w_4, w_7\rangle$ | 9 | 308 | $\langle w_7, w_{44}\rangle$ | 8 | 308 | $\langle w_4, w_{77}\rangle$ | 9 |
| 308 | $\langle w_{11}, w_{28}\rangle$ | 9 | 310 | $\langle w_2, w_{31}\rangle$ | 9 | 310 | $\langle w_{10}, w_{31}\rangle$ | 8 | 310 | $\langle w_{10}, w_{62}\rangle$ | 9 |
| 310 | $\langle w_2, w_{155}\rangle$ | 8 | 312 | $\langle w_3, w_{13}\rangle$ | 9 | 312 | $\langle w_8, w_{39}\rangle$ | 8 | 312 | $\langle w_3, w_{104}\rangle$ | 9 |
| 318 | $\langle w_3, w_{53}\rangle$ | 7 | 318 | $\langle w_6, w_{106}\rangle$ | 8 | 318 | $\langle w_2, w_{159}\rangle$ | 7 | 322 | $\langle w_7, w_{23}\rangle$ | 9 |
| 322 | $\langle w_{14}, w_{46}\rangle$ | 9 | 322 | $\langle w_2, w_{161}\rangle$ | 9 | 342 | $\langle w_{18}, w_{38}\rangle$ | 9 | 345 | $\langle w_{15}, w_{69}\rangle$ | 9 |
| 345 | $\langle w_5, w_{69}\rangle$ | 8 | 348 | $\langle w_{12}, w_{87}\rangle$ | 7 | 350 | $\langle w_{14}, w_{50}\rangle$ | 9 | 350 | $\langle w_2, w_{175}\rangle$ | 8 |
| 354 | $\langle w_6, w_{59}\rangle$ | 8 | 366 | $\langle w_6, w_{122}\rangle$ | 9 | 370 | $\langle w_{10}, w_{74}\rangle$ | 9 | 374 | $\langle w_{17}, w_{22}\rangle$ | 9 |
| 374 | $\langle w_2, w_{187}\rangle$ | 8 | 385 | $\langle w_{11}, w_{35}\rangle$ | 9 | 385 | $\langle w_{35}, w_{55}\rangle$ | 8 | 385 | $\langle w_7, w_{55}\rangle$ | 9 |

*Proof.* For all these curves we compute that $\beta_{2,2} = 0$. We know from [6, 11] that these curves are neither hyperelliptic nor trigonal. Thus by Corollary 4.4 we conclude that they are not tetragonal over $\mathbb{C}$.

The computations for $g = 7$ curves took only several seconds, for $g = 8$ up to several minutes, and for $g = 9$ they could take several hours. The computation for the genus 9 curve $X_0(234)/\langle w_2, w_9\rangle$ took around 4 hours.

The Betti numbers were computed with the `Magma` function BettiNumber(_,2,4). The indexation of $\beta_{i,j}$ in that function is different from that in this paper. It is the same indexation as in [23] and [24], and the reader can consult [23, Table 1] to check this result. ☐

**Proposition 4.9.** *The quotient curve $X_0(N)/W_N$ has $\mathbb{Q}$ and $\mathbb{C}$-gonality equal to 4 for the following pairs $(N, W_N)$:*

| $N$ | $W_N$ | $g$ | $\beta_{2,2}$ | $N$ | $W_N$ | $g$ | $\beta_{2,2}$ | $N$ | $W_N$ | $g$ | $\beta_{2,2}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 228 | $\langle w_3, w_{76}\rangle$ | 8 | 4 | 240 | $\langle w_5, w_{16}\rangle$ | 8 | 4 | 240 | $\langle w_{15}, w_{16}\rangle$ | 7 | 3 |
| 258 | $\langle w_2, w_{43}\rangle$ | 8 | 4 | 258 | $\langle w_6, w_{86}\rangle$ | 7 | 3 | 264 | $\langle w_8, w_{33}\rangle$ | 8 | 4 |
| 264 | $\langle w_{11}, w_{24}\rangle$ | 8 | 4 | 273 | $\langle w_3, w_7\rangle$ | 8 | 4 | 273 | $\langle w_3, w_{13}\rangle$ | 8 | 4 |
| 280 | $\langle w_{35}, w_{56}\rangle$ | 8 | 4 | 280 | $\langle w_5, w_{56}\rangle$ | 8 | 4 | | | | |

*Proof.* For these curves we computed that $\beta_{2,2} = g - 4$. Thus there exists a unique $g^1_4$ by Theorem 4.5. Since all quotients of $X_0(N)$ have a rational cusp, Corollary 4.7 now tells us that we have a rational map to $\mathbb{P}^1$ of degree 4. Therefore, all these curves have $\mathbb{Q}$ and $\mathbb{C}$-gonality 4 as they are not hyperelliptic nor trigonal by [6, 11]. ☐

## 5. Castelnuovo-Severi inequality

This is a very useful tool for producing a lower bound on the $\mathbb{C}$-gonality (see [25, Theorem 3.11.3] for a proof).

**Proposition 5.1** (Castelnuovo-Severi inequality)**.** *Let $k$ be a perfect field, and let $X$, $Y$, $Z$ be curves over $k$ with genera $g(X), g(Y), g(Z)$. Let non-constant morphisms $\pi_Y : X \to Y$ and $\pi_Z : X \to Z$ over $k$ be given, and let their degrees be $m$ and $n$, respectively. Assume that there is no morphism $X \to X'$ of degree $> 1$ through which both $\pi_Y$ and $\pi_Z$ factor. Then the following inequality holds:*

$$g(X) \leq m \cdot g(Y) + n \cdot g(Z) + (m-1)(n-1). \tag{1}$$

As we can see, Castelnuovo-Severi inequality is useful when we have a low degree morphism from a curve $X$ of high genus to a curve $Y$ of low genus. We will use this in the following results.

**Proposition 5.2.** *The $\mathbb{C}$-gonality of the quotient curve $X_0(N)/W_N$ is at least 5 for the following pairs $(N, W_N)$. Here $g$ denotes the genus of $X_0(N)/W_N$ and $g^*$ denotes the genus of the curve $X_0^*(N)$.*

| $N$ | $W_N$ | $g$ | $g^*$ | $N$ | $W_N$ | $g$ | $g^*$ | $N$ | $W_N$ | $g$ | $g^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 240 | $\langle w_3, w_{16}\rangle$ | 10 | 3 | 246 | $\langle w_2, w_3\rangle$ | 10 | 3 | 258 | $\langle w_2, w_3\rangle$ | 10 | 3 |
| 258 | $\langle w_6, w_{43}\rangle$ | 10 | 3 | 270 | $\langle w_2, w_5\rangle$ | 11 | 3 | 270 | $\langle w_2, w_{27}\rangle$ | 10 | 3 |
| 270 | $\langle w_{10}, w_{27}\rangle$ | 10 | 3 | 282 | $\langle w_2, w_3\rangle$ | 12 | 3 | 282 | $\langle w_6, w_{94}\rangle$ | 11 | 3 |
| 282 | $\langle w_3, w_{94}\rangle$ | 11 | 3 | 282 | $\langle w_2, w_{141}\rangle$ | 10 | 3 | 290 | $\langle w_2, w_5\rangle$ | 10 | 3 |
| 290 | $\langle w_{10}, w_{58}\rangle$ | 10 | 3 | 294 | $\langle w_2, w_3\rangle$ | 10 | 3 | 310 | $\langle w_2, w_5\rangle$ | 12 | 3 |
| 310 | $\langle w_5, w_{62}\rangle$ | 11 | 3 | 312 | $\langle w_3, w_8\rangle$ | 13 | 3 | 312 | $\langle w_8, w_{13}\rangle$ | 10 | 3 |
| 312 | $\langle w_{13}, w_{23}\rangle$ | 12 | 3 | 318 | $\langle w_2, w_3\rangle$ | 13 | 3 | 318 | $\langle w_2, w_{53}\rangle$ | 12 | 3 |
| 318 | $\langle w_3, w_{106}\rangle$ | 12 | 3 | 318 | $\langle w_6, w_{53}\rangle$ | 10 | 3 | 322 | $\langle w_2, w_{23}\rangle$ | 12 | 4 |
| 342 | $\langle w_2, w_9\rangle$ | 13 | 4 | 342 | $\langle w_2, w_{19}\rangle$ | 12 | 4 | 348 | $\langle w_3, w_{29}\rangle$ | 10 | 3 |
| 348 | $\langle w_{12}, w_{29}\rangle$ | 13 | 3 | 348 | $\langle w_3, w_{116}\rangle$ | 10 | 3 | 350 | $\langle w_2, w_7\rangle$ | 12 | 4 |
| 350 | $\langle w_2, w_{25}\rangle$ | 13 | 4 | 354 | $\langle w_2, w_3\rangle$ | 14 | 3 | 354 | $\langle w_6, w_{118}\rangle$ | 14 | 3 |
| 354 | $\langle w_3, w_{118}\rangle$ | 13 | 3 | 354 | $\langle w_2, w_{177}\rangle$ | 12 | 3 | 366 | $\langle w_2, w_3\rangle$ | 15 | 4 |
| 366 | $\langle w_2, w_{61}\rangle$ | 13 | 4 | 366 | $\langle w_6, w_{61}\rangle$ | 14 | 4 | 370 | $\langle w_2, w_5\rangle$ | 13 | 4 |
| 370 | $\langle w_5, w_{37}\rangle$ | 12 | 4 | 370 | $\langle w_{10}, w_{37}\rangle$ | 12 | 4 | 374 | $\langle w_2, w_{11}\rangle$ | 13 | 4 |
| 374 | $\langle w_2, w_{17}\rangle$ | 12 | 4 | 374 | $\langle w_{22}, w_{34}\rangle$ | 12 | 4 | 378 | $\langle w_2, w_7\rangle$ | 15 | 5 |
| 385 | $\langle w_7, w_{11}\rangle$ | 12 | 4 | 396 | $\langle w_4, w_9\rangle$ | 15 | 5 | 402 | $\langle w_2, w_3\rangle$ | 16 | 5 |
| 402 | $\langle w_3, w_{67}\rangle$ | 15 | 5 | 402 | $\langle w_6, w_{67}\rangle$ | 15 | 5 | 406 | $\langle w_2, w_7\rangle$ | 14 | 5 |
| 406 | $\langle w_2, w_{29}\rangle$ | 15 | 5 | 410 | $\langle w_2, w_5\rangle$ | 14 | 5 | 410 | $\langle w_{10}, w_{82}\rangle$ | 14 | 5 |
| 414 | $\langle w_2, w_9\rangle$ | 17 | 5 | 414 | $\langle w_2, w_{23}\rangle$ | 14 | 5 | 414 | $\langle w_9, w_{46}\rangle$ | 15 | 5 |
| 418 | $\langle w_2, w_{11}\rangle$ | 14 | 5 | 418 | $\langle w_{11}, w_{38}\rangle$ | 14 | 5 | 435 | $\langle w_3, w_5\rangle$ | 14 | 5 |
| 435 | $\langle w_{15}, w_{87}\rangle$ | 15 | 5 | 438 | $\langle w_2, w_3\rangle$ | 17 | 5 | 438 | $\langle w_2, w_{73}\rangle$ | 14 | 5 |
| 438 | $\langle w_3, w_{73}\rangle$ | 15 | 5 | 438 | $\langle w_6, w_{73}\rangle$ | 17 | 5 | 438 | $\langle w_2, w_{219}\rangle$ | 14 | 5 |
| 440 | $\langle w_5, w_8\rangle$ | 16 | 5 | 440 | $\langle w_8, w_{11}\rangle$ | 17 | 5 | 440 | $\langle w_5, w_{88}\rangle$ | 14 | 5 |
| 442 | $\langle w_2, w_{17}\rangle$ | 14 | 5 | 442 | $\langle w_{13}, w_{34}\rangle$ | 14 | 5 | 444 | $\langle w_{12}, w_{37}\rangle$ | 16 | 5 |
| 444 | $\langle w_3, w_{148}\rangle$ | 16 | 5 | 456 | $\langle w_3, w_8\rangle$ | 18 | 7 | 456 | $\langle w_3, w_{19}\rangle$ | 19 | 7 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 465 | $\langle w_3, w_5\rangle$ | 15 | 5 | 465 | $\langle w_3, w_{31}\rangle$ | 16 | 5 | 465 | $\langle w_5, w_{93}\rangle$ | 14 | 5 |
| 470 | $\langle w_2, w_5\rangle$ | 17 | 6 | 470 | $\langle w_2, w_{47}\rangle$ | 16 | 6 | 470 | $\langle w_5, w_{47}\rangle$ | 16 | 6 |
| 494 | $\langle w_2, w_{13}\rangle$ | 17 | 5 | 494 | $\langle w_2, w_{19}\rangle$ | 16 | 5 | 494 | $\langle w_{13}, w_{19}\rangle$ | 14 | 5 |
| 494 | $\langle w_{19}, w_{26}\rangle$ | 14 | 5 | 495 | $\langle w_5, w_9\rangle$ | 17 | 5 | 495 | $\langle w_5, w_{11}\rangle$ | 15 | 5 |
| 504 | $\langle w_7, w_8\rangle$ | 19 | 7 | 504 | $\langle w_8, w_9\rangle$ | 21 | 7 | 520 | $\langle w_5, w_{13}\rangle$ | 20 | 8 |
| 528 | $\langle w_3, w_{11}\rangle$ | 22 | 9 | 528 | $\langle w_3, w_{16}\rangle$ | 22 | 9 | 555 | $\langle w_3, w_5\rangle$ | 19 | 5 |
| 555 | $\langle w_5, w_{37}\rangle$ | 15 | 5 | 555 | $\langle w_3, w_{37}\rangle$ | 15 | 5 | 555 | $\langle w_{15}, w_{37}\rangle$ | 17 | 5 |
| 558 | $\langle w_2, w_9\rangle$ | 23 | 7 | 558 | $\langle w_2, w_{13}\rangle$ | 21 | 7 | 558 | $\langle w_9, w_{62}\rangle$ | 19 | 7 |
| 560 | $\langle w_7, w_{16}\rangle$ | 22 | 9 | 560 | $\langle w_5, w_7\rangle$ | 22 | 9 | 564 | $\langle w_3, w_{47}\rangle$ | 16 | 6 |
| 564 | $\langle w_{12}, w_{141}\rangle$ | 16 | 6 | 564 | $\langle w_3, w_{188}\rangle$ | 19 | 6 | 572 | $\langle w_{11}, w_{52}\rangle$ | 17 | 5 |
| 572 | $\langle w_{13}, w_{44}\rangle$ | 18 | 5 | 574 | $\langle w_2, w_7\rangle$ | 21 | 5 | 574 | $\langle w_2, w_{41}\rangle$ | 18 | 5 |
| 574 | $\langle w_{14}, w_{41}\rangle$ | 17 | 5 | 574 | $\langle w_7, w_{82}\rangle$ | 18 | 5 | 595 | $\langle w_5, w_7\rangle$ | 16 | 5 |
| 595 | $\langle w_5, w_{17}\rangle$ | 18 | 5 | 595 | $\langle w_{17}, w_{35}\rangle$ | 14 | 5 | 595 | $\langle w_7, w_{85}\rangle$ | 16 | 5 |
| 598 | $\langle w_2, w_{13}\rangle$ | 21 | 6 | 598 | $\langle w_2, w_{23}\rangle$ | 18 | 6 | 598 | $\langle w_{23}, w_{26}\rangle$ | 17 | 6 |
| 598 | $\langle w_{13}, w_{46}\rangle$ | 20 | 6 | 620 | $\langle w_5, w_{31}\rangle$ | 16 | 6 | 620 | $\langle w_{20}, w_{31}\rangle$ | 16 | 6 |
| 620 | $\langle w_{20}, w_{124}\rangle$ | 19 | 6 | 620 | $\langle w_5, w_{124}\rangle$ | 19 | 6 | 627 | $\langle w_3, w_{11}\rangle$ | 19 | 6 |
| 627 | $\langle w_3, w_{19}\rangle$ | 19 | 6 | 627 | $\langle w_{19}, w_{33}\rangle$ | 17 | 6 | 627 | $\langle w_{11}, w_{57}\rangle$ | 17 | 6 |
| 630 | $\langle w_2, w_5, w_9\rangle$ | 15 | 5 | 630 | $\langle w_2, w_5, w_7\rangle$ | 14 | 5 | 630 | $\langle w_2, w_7, w_9\rangle$ | 15 | 5 |
| 630 | $\langle w_7, w_9, w_{10}\rangle$ | 15 | 5 | 645 | $\langle w_3, w_5\rangle$ | 21 | 5 | 645 | $\langle w_3, w_{43}\rangle$ | 19 | 5 |
| 645 | $\langle w_5, w_{43}\rangle$ | 14 | 5 | 645 | $\langle w_{15}, w_{43}\rangle$ | 20 | 5 | 645 | $\langle w_5, w_{129}\rangle$ | 16 | 5 |
| 658 | $\langle w_2, w_7\rangle$ | 24 | 7 | 658 | $\langle w_{14}, w_{47}\rangle$ | 18 | 7 | 658 | $\langle w_{14}, w_{94}\rangle$ | 19 | 7 |
| 658 | $\langle w_7, w_{94}\rangle$ | 21 | 7 | 658 | $\langle w_2, w_{329}\rangle$ | 20 | 7 | 663 | $\langle w_3, w_{13}\rangle$ | 21 | 5 |
| 663 | $\langle w_3, w_{17}\rangle$ | 17 | 5 | 663 | $\langle w_{13}, w_{17}\rangle$ | 15 | 5 | 663 | $\langle w_{13}, w_{51}\rangle$ | 15 | 5 |
| 663 | $\langle w_{39}, w_{51}\rangle$ | 15 | 5 | 663 | $\langle w_{17}, w_{39}\rangle$ | 15 | 5 | 670 | $\langle w_2, w_5\rangle$ | 25 | 6 |
| 670 | $\langle w_2, w_{67}\rangle$ | 22 | 6 | 670 | $\langle w_5, w_{67}\rangle$ | 16 | 6 | 670 | $\langle w_{10}, w_{67}\rangle$ | 24 | 6 |
| 670 | $\langle w_5, w_{134}\rangle$ | 20 | 6 | 714 | $\langle w_2, w_3, w_7\rangle$ | 17 | 5 | 714 | $\langle w_2, w_3, w_{17}\rangle$ | 17 | 5 |
| 714 | $\langle w_2, w_7, w_{51}\rangle$ | 16 | 5 | 714 | $\langle w_2, w_{17}, w_{21}\rangle$ | 14 | 5 | 714 | $\langle w_3, w_7, w_{34}\rangle$ | 16 | 5 |
| 714 | $\langle w_3, w_{14}, w_{17}\rangle$ | 15 | 5 | 714 | $\langle w_6, w_7, w_{34}\rangle$ | 17 | 5 | 714 | $\langle w_6, w_{14}, w_{17}\rangle$ | 16 | 5 |
| 770 | $\langle w_2, w_5, w_7\rangle$ | 16 | 5 | 770 | $\langle w_2, w_5, w_{11}\rangle$ | 14 | 5 | 770 | $\langle w_2, w_7, w_{11}\rangle$ | 17 | 5 |
| 770 | $\langle w_5, w_7, w_{11}\rangle$ | 14 | 5 | 770 | $\langle w_2, w_5, w_{77}\rangle$ | 14 | 5 | 770 | $\langle w_2, w_{11}, w_{35}\rangle$ | 14 | 5 |
| 770 | $\langle w_7, w_{10}, w_{11}\rangle$ | 14 | 5 | 770 | $\langle w_5, w_{14}, w_{22}\rangle$ | 16 | 5 | 770 | $\langle w_7, w_{10}, w_{22}\rangle$ | 14 | 5 |
| 780 | $\langle w_3, w_5, w_{13}\rangle$ | 16 | 6 | 780 | $\langle w_3, w_5, w_{52}\rangle$ | 16 | 6 | 780 | $\langle w_5, w_{12}, w_{13}\rangle$ | 18 | 6 |
| 780 | $\langle w_3, w_{20}, w_{52}\rangle$ | 18 | 6 | 780 | $\langle w_{12}, w_{13}, w_{15}\rangle$ | 16 | 6 | 780 | $\langle w_{12}, w_{15}, w_{39}\rangle$ | 16 | 6 |
| 798 | $\langle w_2, w_3, w_7\rangle$ | 18 | 5 | 798 | $\langle w_2, w_3, w_{19}\rangle$ | 18 | 5 | 798 | $\langle w_2, w_7, w_{19}\rangle$ | 15 | 5 |
| 798 | $\langle w_3, w_7, w_{19}\rangle$ | 15 | 5 | 798 | $\langle w_2, w_7, w_{57}\rangle$ | 14 | 5 | 798 | $\langle w_3, w_7, w_{38}\rangle$ | 14 | 5 |
| 798 | $\langle w_3, w_{14}, w_{19}\rangle$ | 15 | 5 | 798 | $\langle w_6, w_7, w_{19}\rangle$ | 15 | 5 | 798 | $\langle w_2, w_{21}, w_{57}\rangle$ | 17 | 5 |
| 798 | $\langle w_6, w_{14}, w_{38}\rangle$ | 16 | 5 | 910 | $\langle w_2, w_5, w_7\rangle$ | 18 | 5 | 910 | $\langle w_2, w_5, w_{13}\rangle$ | 19 | 5 |
| 910 | $\langle w_2, w_7, w_{13}\rangle$ | 17 | 5 | 910 | $\langle w_2, w_7, w_65\rangle$ | 14 | 5 | 910 | $\langle w_5, w_7, w_{26}\rangle$ | 17 | 5 |
| 910 | $\langle w_5, w_{13}, w_{14}\rangle$ | 19 | 5 | 910 | $\langle w_7, w_{10}, w_{13}\rangle$ | 18 | 5 | 910 | $\langle w_2, w_{35}, w_{65}\rangle$ | 18 | 5 |
| 910 | $\langle w_7, w_{10}, w_{26}\rangle$ | 14 | 5 | 910 | $\langle w_{10}, w_{14}, w_{26}\rangle$ | 14 | 5 | 990 | $\langle w_2, w_5, w_9\rangle$ | 25 | 8 |
| 990 | $\langle w_2, w_5, w_{11}\rangle$ | 23 | 8 | 990 | $\langle w_2, w_9, w_{11}\rangle$ | 23 | 8 | 990 | $\langle w_5, w_9, w_{22}\rangle$ | 23 | 8 |
| 990 | $\langle w_5, w_{11}, w_{18}\rangle$ | 22 | 8 | 990 | $\langle w_9, w_{10}, w_{22}\rangle$ | 21 | 8 | | | | |

*Proof.* These curves $X_\Delta(N)$ are neither hyperelliptic nor trigonal by [6, 11]. Therefore, their $\mathbb{C}$-gonality is at least 4.

Suppose that there is a degree 4 morphism from $X_0(N)/W_N$ to $\mathbb{P}^1$ for some pair $(N, W_N)$ from this table. We apply the Castelnuovo-Severi inequality with the degree 2 quotient map $X_0(N)/W_N \to X_0^*(N)$ and this hypothetical degree 4 morphism. We can easily check that for all these curves we have

$$g > 2 \cdot g^* + 2 \cdot 0 + 1 \cdot 3 = 2 \cdot g^* + 3.$$

Therefore, this degree 4 morphism to $\mathbb{P}^1$ would have to factor through $X_0^*(N)$, implying that $X_0^*(N)$ would have to be hyperelliptic. However, [8] tells us that none of these curves $X_0^*(N)$ are hyperelliptic and we get a contradiction. $\square$

**Remark 5.3.** Proposition 5.2 fixes a small typo in [11]. In that paper it is stated that the curve $X_0(270)/\langle w_{10}, w_{27}\rangle$ is trigonal of genus 7. However, this curve is of genus 10 and is not trigonal as we have just seen (apply the Castelnuovo-Severi inequality with the degree 2 map to $X_0^*(270)$ and the hypothetical degree 3 map to $\mathbb{P}^1$).

Instead, the curve $X_0(270)/\langle w_{10}, w_{54}\rangle$ is trigonal of genus 7. We can prove its trigonality by computing the Betti number $\beta_{2,2} = 20 = (g-4)(g-2)$ which proves the existence of $g^1_3$ by Theorem 4.5.

**Proposition 5.4.** *The $\mathbb{C}$-gonality of the quotient curve $X_0(N)/W_N$ is at least* 5 *for the following pairs $(N, W_N)$. Here $g$ denotes the genus of $X_0(N)/W_N$ and $g'$ denotes the genus of $X_0(N)/W_n'$.*

| $N$ | $W_N$ | $g$ | $W_N'$ | $g'$ | $N$ | $W_N$ | $g$ | $W_N'$ | $g'$ |
|---|---|---|---|---|---|---|---|---|---|
| 210 | $\langle w_2, w_3\rangle$ | 10 | $\langle w_2, w_3, w_{35}\rangle$ | 3 | 210 | $\langle w_2, w_5\rangle$ | 10 | $\langle w_2, w_5, w_7\rangle$ | 3 |
| | $\langle w_3, w_5\rangle$ | 10 | $\langle w_3, w_5, w_7\rangle$ | 3 | | $\langle w_3, w_7\rangle$ | 10 | $\langle w_3, w_5, w_7\rangle$ | 3 |
| | $\langle w_3, w_1\rangle$ | 11 | $\langle w_3, w_{10}, w_{14}\rangle$ | 3 | | $\langle w_6, w_7\rangle$ | 10 | $\langle w_5, w_6, w_7\rangle$ | 3 |
| | $\langle w_7, w_{10}\rangle$ | 11 | $\langle w_2, w_5, w_7\rangle$ | 3 | | $\langle w_7, w_{15}\rangle$ | 10 | $\langle w_3, w_5, w_7\rangle$ | 3 |
| | $\langle w_{10}, w_{42}\rangle$ | 10 | $\langle w_3, w_{10}, w_{14}\rangle$ | 3 | | $\langle w_3, w_{70}\rangle$ | 10 | $\langle w_2, w_3, w_{35}\rangle$ | 3 |
| | $\langle w_7, w_{30}\rangle$ | 10 | $\langle w_5, w_6, w_7\rangle$ | 3 | | | | | |
| 330 | $\langle w_2, w_3\rangle$ | 16 | $\langle w_2, w_3, w_{11}\rangle$ | 6 | 330 | $\langle w_2, w_5\rangle$ | 17 | $\langle w_2, w_5, w_{11}\rangle$ | 6 |
| | $\langle w_3, w_5\rangle$ | 17 | $\langle w_3, w_5, w_{22}\rangle$ | 6 | | $\langle w_3, w_{11}\rangle$ | 14 | $\langle w_3, w_{10}, w_{11}\rangle$ | 5 |
| | $\langle w_2, w_{15}\rangle$ | 16 | $\langle w_2, w_{11}, w_{15}\rangle$ | 6 | | $\langle w_3, w_{10}\rangle$ | 16 | $\langle w_3, w_{10}, w_{11}\rangle$ | 6 |
| | $\langle w_3, w_{55}\rangle$ | 16 | $\langle w_2, w_3, w_{55}\rangle$ | 6 | | $\langle w_5, w_6\rangle$ | 15 | $\langle w_5, w_6, w_{11}\rangle$ | 5 |
| | $\langle w_5, w_{33}\rangle$ | 16 | $\langle w_5, w_6, w_{22}\rangle$ | 6 | | $\langle w_{11}, w_{15}\rangle$ | 13 | $\langle w_6, w_{10}, w_{11}\rangle$ | 4 |
| | $\langle w_6, w_{10}\rangle$ | 16 | $\langle w_6, w_{10}, w_{11}\rangle$ | 4 | | $\langle w_6, w_{22}\rangle$ | 16 | $\langle w_5, w_6, w_{22}\rangle$ | 6 |
| | $\langle w_{10}, w_{22}\rangle$ | 17 | $\langle w_2, w_5, w_{11}\rangle$ | 6 | | $\langle w_{15}, w_{33}\rangle$ | 17 | $\langle w_2, w_{15}, w_{33}\rangle$ | 6 |
| | $\langle w_6, w_{55}\rangle$ | 15 | $\langle w_5, w_6, w_{11}\rangle$ | 5 | | $\langle w_{10}, w_{33}\rangle$ | 16 | $\langle w_3, w_{10}, w_{11}\rangle$ | 5 |
| | $\langle w_{15}, w_{22}\rangle$ | 16 | $\langle w_2, w_{11}, w_{15}\rangle$ | 6 | | $\langle w_6, w_{110}\rangle$ | 12 | $\langle w_6, w_{10}, w_{11}\rangle$ | 4 |
| | $\langle w_{10}, w_{66}\rangle$ | 14 | $\langle w_6, w_{10}, w_{11}\rangle$ | 4 | | $\langle w_{15}, w_{66}\rangle$ | 12 | $\langle w_6, w_{10}, w_{11}\rangle$ | 4 |
| | $\langle w_{30}, w_{55}\rangle$ | 14 | $\langle w_6, w_{10}, w_{11}\rangle$ | 4 | | $\langle w_5, w_{66}\rangle$ | 14 | $\langle w_6, w_{10}, w_{11}\rangle$ | 4 |
| | $\langle w_{11}, w_{30}\rangle$ | 12 | $\langle w_6, w_{10}, w_{11}\rangle$ | 4 | | | | | |
| 390 | $\langle w_2, w_3\rangle$ | 20 | $\langle w_2, w_3, w_{13}\rangle$ | 7 | 390 | $\langle w_2, w_5\rangle$ | 20 | $\langle w_2, w_5, w_{13}\rangle$ | 8 |
| | $\langle w_2, w_{13}\rangle$ | 17 | $\langle w_2, w_{13}, w_{15}\rangle$ | 6 | | $\langle w_3, w_5\rangle$ | 20 | $\langle w_3, w_5, w_{13}\rangle$ | 6 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\langle w_3, w_{13}\rangle$ | 16 | $\langle w_3, w_5, w_{13}\rangle$ | 6 | | $\langle w_5, w_{13}\rangle$ | 18 | $\langle w_3, w_5, w_{13}\rangle$ | 6 |
| | $\langle w_2, w_{15}\rangle$ | 19 | $\langle w_2, w_{13}, w_{15}\rangle$ | 6 | | $\langle w_2, w_{39}\rangle$ | 16 | $\langle w_2, w_5, w_{39}\rangle$ | 6 |
| | $\langle w_2, w_{65}\rangle$ | 18 | $\langle w_2, w_{15}, w_{39}\rangle$ | 7 | | $\langle w_3, w_{10}\rangle$ | 19 | $\langle w_3, w_{10}, w_{13}\rangle$ | 7 |
| | $\langle w_3, w_{26}\rangle$ | 17 | $\langle w_3, w_{10}, w_{26}\rangle$ | 6 | | $\langle w_5, w_6\rangle$ | 19 | $\langle w_5, w_6, w_{26}\rangle$ | 5 |
| | $\langle w_5, w_{26}\rangle$ | 17 | $\langle w_5, w_6, w_{26}\rangle$ | 5 | | $\langle w_6, w_{13}\rangle$ | 20 | $\langle w_2, w_3, w_{13}\rangle$ | 7 |
| | $\langle w_{10}, w_{13}\rangle$ | 20 | $\langle w_2, w_5, w_{13}\rangle$ | 8 | | $\langle w_{13}, w_{15}\rangle$ | 17 | $\langle w_3, w_5, w_{13}\rangle$ | 6 |
| | $\langle w_6, w_{10}\rangle$ | 20 | $\langle w_6, w_{10}, w_{26}\rangle$ | 5 | | $\langle w_{10}, w_{26}\rangle$ | 15 | $\langle w_6, w_{10}, w_{26}\rangle$ | 5 |
| | $\langle w_{15}, w_{39}\rangle$ | 14 | $\langle w_6, w_{10}, w_{26}\rangle$ | 5 | | $\langle w_6, w_{65}\rangle$ | 15 | $\langle w_6, w_{10}, w_{26}\rangle$ | 5 |
| | $\langle w_{10}, w_{39}\rangle$ | 14 | $\langle w_6, w_{10}, w_{26}\rangle$ | 5 | | $\langle w_{15}, w_{26}\rangle$ | 15 | $\langle w_6, w_{10}, w_{26}\rangle$ | 5 |
| | $\langle w_6, w_{130}\rangle$ | 17 | $\langle w_5, w_6, w_{26}\rangle$ | 5 | | $\langle w_{10}, w_{78}\rangle$ | 17 | $\langle w_2, w_5, w_{39}\rangle$ | 6 |
| | $\langle w_{15}, w_{78}\rangle$ | 20 | $\langle w_3, w_5, w_{26}\rangle$ | 8 | | $\langle w_{30}, w_{39}\rangle$ | 15 | $\langle w_5, w_6, w_{26}\rangle$ | 5 |
| | $\langle w_{30}, w_{65}\rangle$ | 17 | $\langle w_3, w_{10}, w_{26}\rangle$ | 6 | | $\langle w_3, w_{130}\rangle$ | 18 | $\langle w_3, w_{10}, w_{13}\rangle$ | 7 |
| | $\langle w_5, w_{78}\rangle$ | 18 | $\langle w_2, w_5, w_{39}\rangle$ | 6 | | $\langle w_{13}, w_{30}\rangle$ | 17 | $\langle w_2, w_{13}, w_{15}\rangle$ | 6 |

*Proof.* These curves $X_\Delta(N)$ are neither hyperelliptic nor trigonal by [6, 11]. Therefore, their $\mathbb{C}$-gonality is at least 4.

Suppose that there is a degree 4 morphism from $X_0(N)/W_N$ to $\mathbb{P}^1$ for some pair $(N, W_N)$ from this table. We apply the Castelnuovo-Severi inequality with the degree 2 quotient map $X_0(N)/W_N \to X_0(N)/W'_N$ and this hypothetical degree 4 morphism. We can easily check that for all these curves we have

$$g > 2 \cdot g' + 2 \cdot 0 + 1 \cdot 3 = 2 \cdot g' + 3.$$

Therefore, this degree 4 morphism to $\mathbb{P}^1$ would have to factor through $X_0(N)/W'_N$ and the curve $X_0(N)/W'_N$ would have to be hyperelliptic. However, [6, 11] tell us that none of these curves $X_0(N)/W'_N$ are hyperelliptic and we get a contradiction. $\square$

## 6. Rational morphisms to $\mathbb{P}^1$

In previous sections we were giving lower bounds on $\mathbb{C}$ and $\mathbb{Q}$-gonality. Now we give upper bounds by giving rational morphisms from $X_0(N)/W_N$ to $\mathbb{P}^1$.

**Proposition 6.1.** *The following genus* 4 *curves* $X_0(N)/W_N$ *are trigonal over* $\mathbb{Q}$*:*

$$\begin{aligned}(N, W_N) \in \{&(130, \langle w_5, w_{13}\rangle), (132, \langle w_4, w_{33}\rangle), (154, \langle w_2, w_{77}\rangle), (170, \langle w_5, w_{34}\rangle),\\ &(182, \langle w_2, w_{91}\rangle), (186, \langle w_3, w_{62}\rangle), (210, \langle w_2, w_{15}, w_{21}\rangle),\\ &(255, \langle w_5, w_{51}\rangle), (285, \langle w_3, w_{95}\rangle), (286, \langle w_2, w_{143}\rangle)\}.\end{aligned}$$

*Proof.* For all these curves we used the `Magma` function Genus4GonalMap() to obtain a degree 3 morphism to $\mathbb{P}^1$ (it exists due to Proposition 1.1 (v)). This morphism is defined over $\mathbb{Q}$, proving that these curves are $\mathbb{Q}$-trigonal. $\square$

**Proposition 6.2.** *The following genus* 4 *curves* $X_0(N)/W_N$ *are not trigonal over* $\mathbb{Q}$*:*

| $N$ | $W_N$ |
|---|---|
| 102 | $\langle w_2, w_3\rangle,\ \langle w_3, w_{34}\rangle$ |
| 114 | $\langle w_2, w_3\rangle,\ \langle w_3, w_{19}\rangle,\ \langle w_3, w_{57}\rangle,\ \langle w_6, w_{19}\rangle$ |
| 120 | $\langle w_3, w_8\rangle,\ \langle w_3, w_{40}\rangle$ |
| 126 | $\langle w_2, w_7\rangle,\ \langle w_7, w_{18}\rangle$ |
| 130 | $\langle w_2, w_5\rangle,\ \langle w_{10}, w_{13}\rangle$ |
| 132 | $\langle w_3, w_{11}\rangle,\ \langle w_{12}, w_{33}\rangle$ |
| 138 | $\langle w_2, w_{69}\rangle$ |
| 140 | $\langle w_5, w_7\rangle,\ \langle w_5, w_{28}\rangle$ |
| 150 | $\langle w_2, w_{25}\rangle,\ \langle w_3, w_{25}\rangle,\ \langle w_6, w_{25}\rangle$ |
| 154 | $\langle w_7, w_{11}\rangle,\ \langle w_7, w_{22}\rangle$ |
| 165 | $\langle w_3, w_{11}\rangle,\ \langle w_5, w_{11}\rangle$ |
| 168 | $\langle w_3, w_{56}\rangle$ |
| 170 | $\langle w_2, w_{85}\rangle$ |
| 174 | $\langle w_6, w_{29}\rangle,\ \langle w_6, w_{58}\rangle$ |
| 182 | $\langle w_7, w_{26}\rangle,\ \langle w_{13}, w_{14}\rangle$ |
| 210 | $\langle w_2, w_3, w_7\rangle,\ \langle w_2, w_5, w_{21}\rangle,\ \langle w_2, w_7, w_{15}\rangle,\ \langle w_3, w_5, w_{14}\rangle$ |
| 220 | $\langle w_5, w_{11}\rangle,\ \langle w_{20}, w_{44}\rangle,\ \langle w_4, w_{55}\rangle$ |
| 222 | $\langle w_2, w_{111}\rangle$ |
| 231 | $\langle w_7, w_{33}\rangle,\ \langle w_{21}, w_{33}\rangle,\ \langle w_{11}, w_{21}\rangle$ |
| 330 | $\langle w_6, w_{10}, w_{11}\rangle$ |

*Proof.* We again used the `Magma` function Genus4GonalMap() to obtain a degree 3 morphism to $\mathbb{P}^1$. If the curve is $\mathbb{Q}$-trigonal, the given morphism will be defined over $\mathbb{Q}$. Otherwise, the function gives a morphism over a quadratic or a biquadratic field [5] (even though the trigonal map can always be defined over a number field of degree $\leq 2$).

For all these curves the degree 3 morphism is defined over a quadratic field, implying that they are not $\mathbb{Q}$-trigonal. □

**Proposition 6.3.** *The quotient curve $X_0(N)/W_N$ has $\mathbb{Q}$-gonality equal to 4 for*

$$(N, W_N) \in \{(234, \langle w_9, w_{26}\rangle), (240, \langle w_{15}, w_{48}\rangle), (282, \langle w_3, w_{47}\rangle),\\ (282, \langle w_6, w_{47}\rangle), (310, \langle w_5, w_{31}\rangle), (312, \langle w_{24}, w_{39}\rangle)\}.$$

*Proof.* These curves are of genus 6. We use the `Magma` function Genus6GonalMap() to obtain a degree 4 morphism to $\mathbb{P}^1$ (it exists due to Proposition 1.1 (v)). This morphism is defined over $\mathbb{Q}$ for all these curves, proving that these curves are $\mathbb{Q}$-tetragonal. □

**Proposition 6.4.** *The quotient curve $X_0(N)/W_N$ has $\mathbb{Q}$-gonality at most 4 for*

$$N \in \{120, 126, 132, 138, 140, 150, 154, 156, 165, 168, 170, 174, 180, 182, 186, 190, 195, 198, 204,\\ 210, 220, 222, 230, 231, 238, 240, 252, 255, 266, 276, 285, 286, 315, 330, 357, 380, 390\}.$$

*and all groups $W_N \subset B(N)$ of order $2^{\omega(N)-1}$.*

*Proof.* For these levels $N$ the quotient curve $X_0^*(N)$ has $\mathbb{Q}$-gonality equal to 2. For $N \neq 252, 315$ the genus of $X_0^*(N)$ is 1 or 2, and for $N = 252, 315$ the genus is equal to 3, but $X_0^*(N)$ is hyperelliptic due to [8].

Since we have a degree 2 quotient map $X_0(N)/W_N \to X_0^*(N)$, the composition map

$$X_0(N)/W_N \to X_0^*(N) \to \mathbb{P}^1$$

is a degree 4 rational morphism to $\mathbb{P}^1$. □

**Proposition 6.5.** *The quotient curve $X_0(210)/W$ has $\mathbb{Q}$-gonality equal to 4 for*

$$W \in \{\langle w_6, w_{10}\rangle, \langle w_6, w_{14}\rangle, \langle w_{10}, w_{14}\rangle, \langle w_{15}, w_{21}\rangle, \langle w_6, w_{35}\rangle, \langle w_{10}, w_{21}\rangle, \langle w_{14}, w_{15}\rangle\}.$$

*Proof.* The curve $X_0(210)/\langle w_6, w_{10}, w_{14}\rangle$ is a genus 2 hyperelliptic curve. We have a degree 2 quotient map $X_0(210)/W \to X_0(210)/\langle w_6, w_{10}, w_{14}\rangle$. Thus the composition map

$$X_0(210)/W \to X_0(210)/\langle w_6, w_{10}, w_{14}\rangle \to \mathbb{P}^1$$

is a degree 4 rational morphism to $\mathbb{P}^1$. □

**Proposition 6.6.** *The quotient curve $X_0(N)/W_N$ has $\mathbb{Q}$-gonality equal to 4 for the following pairs $(N, W_N)$. Here $g$ denotes the genus of $X_0(N)/W_N$.*

| $N$ | $W_N$ | $g$ | $N$ | $W_N$ | $g$ | $N$ | $W_N$ | $g$ |
|---|---|---|---|---|---|---|---|---|
| 210 | $\langle w_{30}, w_{35}\rangle$ | 8 | 234 | $\langle w_{18}, w_{26}\rangle$ | 7 | 240 | $\langle w_3, w_{80}\rangle$ | 7 |
| 273 | $\langle w_3, w_{91}\rangle$ | 7 | 282 | $\langle w_2, w_{47}\rangle$ | 7 | 294 | $\langle w_6, w_{98}\rangle$ | 7 |
| 306 | $\langle w_9, w_{34}\rangle$ | 10 | 336 | $\langle w_{16}, w_{21}\rangle$ | 12 | 360 | $\langle w_9, w_{40}\rangle$ | 13 |

*Proof.* For all these curves we used `Magma` we found a rational effective divisor that is a sum of 4 rational points and has Riemann-Roch dimension equal to 2.

When $g \leq 8$, we searched for rational points using the `Magma` function PointSearch(). These computations were very fast. They took up to 10 minutes.

For $g \geq 10$ the genus was too large and we were unable to use this function. Hence, we wrote a code that searched for rational points up to a certain height.

We then computed Riemann-Roch dimensions of divisors constructed from these rational points and found a divisor of dimension 2. The computation time for $N = 336$ was around 10 minutes, for $N = 306$ around 12 hours, and for $N = 360$ around 3 days. □

## 7. Isomorphic curves

In this section we present some isomorphisms between quotient curves $X_0(N)/W_N$ and use them to determine the gonality of these curves.

**Proposition 7.1** ([6, Proposition 4])**.** *Let $N = 4M$ with $M$ being odd. Let $W'$ be a subgroup of $B(N)$ generated by $w_4, w_{M_1}, \ldots, W_{M_s}$ $(M_i \mid\mid M)$. Then we have the following isomorphism*

$$X_0(N)/W' \cong X_0(2M)/\langle w_{M_i}\rangle.$$

*This isomorphism is defined over $\mathbb{Q}$, see* [8, Proposition 7].

**Proposition 7.2.** *The quotient curve $X_0(N)/W_N$ has $\mathbb{Q}$-gonality equal to 4 for the following pairs $(N, W_N)$.*

| $N$ | $W_N$ | $Y$ | $N$ | $W_N$ | $Y$ |
|---|---|---|---|---|---|
| 228 | $\langle w_3, w_4\rangle$ | $X_0(114)/w_3$ | 228 | $\langle w_4, w_{19}\rangle$ | $X_0(114)/w_{19}$ |
| 260 | $\langle w_4, w_{13}\rangle$ | $X_0(130)/w_{13}$ | 260 | $\langle w_4, w_{65}\rangle$ | $X_0(130)/w_{65}$ |
| 300 | $\langle w_4, w_{75}\rangle$ | $X_0(150)/w_{75}$ | | | |

*Proof.* We have the isomorphism $X_0(N)/W_N \cong Y$ defined over $\mathbb{Q}$ by Proposition 7.1 and we know from [18, Proposition 3.2] that these curves $Y$ have $\mathbb{Q}$-gonality equal to 4. $\square$

**Proposition 7.3.** *The quotient curve $X_0(N)/W_N$ has $\mathbb{C}$-gonality at least 5 for the following pairs $(N, W_N)$.*

| $N$ | $W_N$ | $Y$ | $N$ | $W_N$ | $Y$ |
|---|---|---|---|---|---|
| 228 | $\langle w_4, w_{57}\rangle$ | $X_0(114)/w_{57}$ | 260 | $\langle w_4, w_5\rangle$ | $X_0(130)/w_5$ |
| 300 | $\langle w_3, w_4\rangle$ | $X_0(150)/w_3$ | 300 | $\langle w_4, w_{25}\rangle$ | $X_0(150)/w_{25}$ |
| 340 | $\langle w_4, w_5\rangle$ | $X_0(170)/w_5$ | 340 | $\langle w_4, w_{17}\rangle$ | $X_0(170)/w_{17}$ |
| 340 | $\langle w_4, w_{85}\rangle$ | $X_0(170)/w_{85}$ | 348 | $\langle w_3, w_4\rangle$ | $X_0(174)/w_3$ |
| 348 | $\langle w_4, w_{29}\rangle$ | $X_0(174)/w_{29}$ | 348 | $\langle w_4, w_{87}\rangle$ | $X_0(174)/w_{87}$ |
| 364 | $\langle w_4, w_7\rangle$ | $X_0(182)/w_7$ | 364 | $\langle w_4, w_{13}\rangle$ | $X_0(182)/w_{13}$ |
| 364 | $\langle w_4, w_{91}\rangle$ | $X_0(182)/w_{91}$ | 372 | $\langle w_3, w_4\rangle$ | $X_0(186)/w_3$ |
| 372 | $\langle w_4, w_{31}\rangle$ | $X_0(186)/w_{31}$ | 372 | $\langle w_4, w_{93}\rangle$ | $X_0(186)/w_{93}$ |
| 444 | $\langle w_3, w_4\rangle$ | $X_0(222)/w_3$ | 444 | $\langle w_4, w_{37}\rangle$ | $X_0(222)/w_{37}$ |
| 444 | $\langle w_4, w_{111}\rangle$ | $X_0(222)/w_{111}$ | 460 | $\langle w_4, w_5\rangle$ | $X_0(230)/w_5$ |
| 460 | $\langle w_4, w_{23}\rangle$ | $X_0(230)/w_{23}$ | 460 | $\langle w_4, w_{115}\rangle$ | $X_0(230)/w_{115}$ |
| 532 | $\langle w_4, w_7\rangle$ | $X_0(266)/w_7$ | 532 | $\langle w_4, w_{19}\rangle$ | $X_0(266)/w_{19}$ |
| 532 | $\langle w_4, w_{133}\rangle$ | $X_0(266)/w_{133}$ | 564 | $\langle w_3, w_4\rangle$ | $X_0(282)/w_3$ |
| 564 | $\langle w_4, w_{47}\rangle$ | $X_0(282)/w_{47}$ | 564 | $\langle w_4, w_{141}\rangle$ | $X_0(282)/w_{141}$ |
| 572 | $\langle w_4, w_{11}\rangle$ | $X_0(286)/w_{11}$ | 572 | $\langle w_4, w_{13}\rangle$ | $X_0(286)/w_{13}$ |
| 572 | $\langle w_4, w_{143}\rangle$ | $X_0(286)/w_{143}$ | 620 | $\langle w_4, w_5\rangle$ | $X_0(310)/w_5$ |
| 620 | $\langle w_4, w_{31}\rangle$ | $X_0(310)/w_{31}$ | 620 | $\langle w_4, w_{155}\rangle$ | $X_0(310)/w_{155}$ |
| 660 | $\langle w_3, w_4, w_5\rangle$ | $X_0(330)/\langle w_3, w_5\rangle$ | 660 | $\langle w_3, w_4, w_{11}\rangle$ | $X_0(330)/\langle w_3, w_{11}\rangle$ |
| 660 | $\langle w_4, w_5, w_{11}\rangle$ | $X_0(330)/\langle w_5, w_{11}\rangle$ | 660 | $\langle w_3, w_4, w_{55}\rangle$ | $X_0(330)/\langle w_3, w_{55}\rangle$ |
| 660 | $\langle w_4, w_5, w_{33}\rangle$ | $X_0(330)/\langle w_5, w_{33}\rangle$ | 660 | $\langle w_4, w_{11}, w_{15}\rangle$ | $X_0(330)/\langle w_{11}, w_{15}\rangle$ |
| 660 | $\langle w_4, w_{15}, w_{33}\rangle$ | $X_0(330)/\langle w_{15}, w_{33}\rangle$ | 780 | $\langle w_3, w_4, w_5\rangle$ | $X_0(390)/\langle w_3, w_5\rangle$ |
| 780 | $\langle w_3, w_4, w_{13}\rangle$ | $X_0(390)/\langle w_3, w_{13}\rangle$ | 780 | $\langle w_4, w_5, w_{13}\rangle$ | $X_0(390)/\langle w_5, w_{13}\rangle$ |
| 780 | $\langle w_3, w_4, w_{65}\rangle$ | $X_0(390)/\langle w_3, w_{65}\rangle$ | 780 | $\langle w_4, w_5, w_{39}\rangle$ | $X_0(390)/\langle w_5, w_{39}\rangle$ |
| 780 | $\langle w_4, w_{13}, w_{15}\rangle$ | $X_0(390)/\langle w_{13}, w_{15}\rangle$ | 780 | $\langle w_4, w_{15}, w_{39}\rangle$ | $X_0(390)/\langle w_{15}, w_{39}\rangle$ |

*Proof.* We have the isomorphism $X_0(N)/W_N \cong Y$ defined over $\mathbb{Q}$ by Proposition 7.1.

For curves $Y = X_0^{+d}(2M)$ we know from [18] that they have $\mathbb{C}$-gonality greater than 4. For curves $Y = X_0(2M)/\langle w_{d_1}, w_{d_2}\rangle$ we gave a lower bound on $\mathbb{C}$-gonality in Propositions 3.1, 5.4. □

**Proposition 7.4** ([6, Proposition 5])**.** *Assume that* $9 \mid\mid N$*. Let* $W'$ *be a subgroup of* $B(N)$ *generated by* $w_{N_1}, \dots, w_{N_s}$ $(N_i \mid\mid N)$*, and let* $W'' = \left\langle w_9^{\epsilon(N_i)} w_{N_i} \right\rangle$*, where*

$$\epsilon(M) = \begin{cases} 0 & \text{if } M \equiv 1 \pmod 3 \text{ or if } 9 \mid\mid M \text{ and } M/9 \equiv 1 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

*Then we have the following isomorphism*

$$X_0(N)/W' \cong X_0(N)/W''.$$

*It is defined over* $\mathbb{Q}(\sqrt{-3})$*, see* [6, Lemma 1] *and* [4, Proposition 4.19].

Notice that Proposition 7.4 can give a non-trivial isomorphism only when $w_9 \notin W'$ (if $w_9 \in W'$, then $W' = W''$ by definition).

**Proposition 7.5.** *The quotient curve* $X_0(N)/W_N$ *has* $\mathbb{C}$*-gonality at least* 5 *for the following pairs* $(N, W_N)$*.*

| $N$ | $W_N$ | $W''$ | $N$ | $W_N$ | $W''$ |
|---|---|---|---|---|---|
| 306 | $\langle w_{18}, w_{34}\rangle$ | $\langle w_2, w_{17}\rangle$ | 306 | $\langle w_2, w_{153}\rangle$ | $\langle w_{17}, w_{18}\rangle$ |
| 342 | $\langle w_{18}, w_{19}\rangle$ | $\langle w_2, w_{19}\rangle$ | 342 | $\langle w_2, w_{171}\rangle$ | $\langle w_{18}, w_{38}\rangle$ |
| 360 | $\langle w_{40}, w_{45}\rangle$ | $\langle w_5, w_8\rangle$ | 360 | $\langle w_5, w_{72}\rangle$ | $\langle w_8, w_{45}\rangle$ |
| 396 | $\langle w_{11}, w_{36}\rangle$ | $\langle w_{36}, w_{44}\rangle$ | 396 | $\langle w_4, w_{99}\rangle$ | $\langle w_4, w_{11}\rangle$ |
| 414 | $\langle w_{18}, w_{46}\rangle$ | $\langle w_2, w_{23}\rangle$ | 414 | $\langle w_2, w_{207}\rangle$ | $\langle w_{18}, w_{23}\rangle$ |
| 495 | $\langle w_{45}, w_{55}\rangle$ | $\langle w_5, w_{11}\rangle$ | 495 | $\langle w_5, w_{99}\rangle$ | $\langle w_{11}, w_{45}\rangle$ |
| 504 | $\langle w_8, w_{63}\rangle$ | $\langle w_{56}, w_{63}\rangle$ | 504 | $\langle w_7, w_{72}\rangle$ | $\langle w_7, w_8\rangle$ |
| 558 | $\langle w_{18}, w_{31}\rangle$ | $\langle w_2, w_{31}\rangle$ | 558 | $\langle w_2, w_{279}\rangle$ | $\langle w_{18}, w_{62}\rangle$ |
| 585 | $\langle w_{13}, w_{45}\rangle$ | $\langle w_5, w_{13}\rangle$ | 585 | $\langle w_5, w_{117}\rangle$ | $\langle w_{45}, w_{65}\rangle$ |
| 630 | $\langle w_5, w_7, w_{18}\rangle$ | $\langle w_2, w_7, w_{45}\rangle$ | 630 | $\langle w_5, w_{14}, w_{18}\rangle$ | $\langle w_2, w_{35}, w_{45}\rangle$ |
| 630 | $\langle w_7, w_{10}, w_{18}\rangle$ | $\langle w_2, w_5, w_7\rangle$ | 630 | $\langle w_{10}, w_{14}, w_{18}\rangle$ | $\langle w_2, w_5, w_{63}\rangle$ |
| 990 | $\langle w_2, w_{45}, w_{55}\rangle$ | $\langle w_5, w_{11}, w_{18}\rangle$ | 990 | $\langle w_5, w_{18}, w_{22}\rangle$ | $\langle w_2, w_{11}, w_{45}\rangle$ |
| 990 | $\langle w_{10}, w_{11}, w_{18}\rangle$ | $\langle w_2, w_5, w_{99}\rangle$ | 990 | $\langle w_{10}, w_{18}, w_{22}\rangle$ | $\langle w_2, w_5, w_{11}\rangle$ |

*Proof.* We have the isomorphism $X_0(N)/W_N \cong X_0(N)/W''$ defined over $\mathbb{C}$ by Proposition 7.1. For curves $X_0(N)/W''$ of genus $\leq 9$ we know from Proposition 4.8 that they are not $\mathbb{C}$-tetragonal. For curves of genus $\geq 10$ we know from Propositions 3.1, 3.3, 5.2 that they are not $\mathbb{Q}$-tetragonal. Hence they are also not $\mathbb{C}$-tetragonal by Corollary 2.2. □

## 8. Proofs of the main theorems

The proof of Theorem 1.2 is given in Propositions 6.1 and 6.2 where we determine the fields of definition of trigonal maps. We now give the proof of Theorem 1.3.

*Proof of Theorem 1.3.* If $\omega(N) \leq 2$, then there are no subgroups $W_N \subseteq B(N)$ such that $4 \leq |W_N| \leq 2^{\omega(N)-1}$. Thus we may suppose that $\omega(N) \geq 3$. We may eliminate all levels $N$ such that $\operatorname{gon}_{\mathbb{C}}(X_0^*(N)) \geq 5$ by Proposition 1.1 (vii). Furthermore, in Proposition 2.3 we deal with the levels $N$ such that $\operatorname{gon}_{\mathbb{C}}(X_0^*(N)) = 4 < \operatorname{gon}_{\mathbb{Q}}(X_0^*(N))$. This leaves us with levels $N$ for which $\operatorname{gon}_{\mathbb{Q}}(X_0^*(N)) \leq 4$ (and $N = 378$). All such levels are listed in [8], [10], [20].

For all these levels $N$ we have $\omega(N) \leq 4$. For curves $X_0(N)/W_N$ listed in the theorem we use Propositions 6.3, 4.9, 6.4, 6.6, 7.2 to prove the existence of a rational degree 4 morphism to $\mathbb{P}^1$.

For other curves we use Propositions 3.1, 3.2, 4.8, 5.2, 5.4, 7.3, 7.5 to disprove the existence of a degree 4 morphism to $\mathbb{P}^1$. Note that Propositions 3.1, 3.2 only give a lower bound on the $\mathbb{Q}$-gonality, but all curves there are of genus $g \geq 10$ and we can use Corollary 2.2 to obtain a bound on the $\mathbb{C}$-gonality.

For

$$N \in \{630, 660, 714, 770, 780, 798, 910, 990\}$$

we only considered curves $X_0(N)/W_N$ with $|W_N| = 8$. This is because we proved that all such curves have $\mathbb{C}$-gonality at least 5. Proposition 1.1 (vii) then tells us that all curves $X_0(N)/W_N$ with $|W_N| = 4$ also have $\mathbb{C}$-gonality at least 5 (as every group $W_N$ of order 4 is a subgroup of some group of order 8). $\square$

For the end of the paper we summarize the result of Theorem 1.3. On the following page we present the table containing all $\mathbb{Q}$-tetragonal (and $\mathbb{C}$-tetragonal) curves $X_0(N)/W_N$ with $4 \leq |W_N| \leq 2^{\omega(N)-1}$.

Table 1: All $\mathbb{Q}$-tetragonal curves $X_0(N)/W_N$ with $4 \leq |W_N| \leq 2^{\omega(N)-1}$.

| $N$ | $W_N$ |
|---|---|
| 102 | $\langle w_2, w_3\rangle$, $\langle w_3, w_{34}\rangle$ |
| 114 | $\langle w_2, w_3\rangle$, $\langle w_3, w_{19}\rangle$, $\langle w_3, w_{57}\rangle$, $\langle w_6, w_{19}\rangle$ |
| 120 | $\langle w_3, w_8\rangle$, $\langle w_3, w_{40}\rangle$, $\langle w_5, w_8\rangle$ |
| 126 | $\langle w_2, w_7\rangle$, $\langle w_2, w_9\rangle$, $\langle w_7, w_{18}\rangle$ |
| 130 | $\langle w_2, w_5\rangle$, $\langle w_{10}, w_{13}\rangle$ |
| 132 | $\langle w_3, w_4\rangle$, $\langle w_3, w_{11}\rangle$, $\langle w_{12}, w_{33}\rangle$ |
| 138 | $\langle w_2, w_3\rangle$, $\langle w_6, w_{46}\rangle$, $\langle w_3, w_{46}\rangle$, $\langle w_2, w_{69}\rangle$ |
| 140 | $\langle w_4, w_5\rangle$, $\langle w_4, w_7\rangle$, $\langle w_5, w_7\rangle$, $\langle w_5, w_{28}\rangle$ |
| 150 | $\langle w_2, w_3\rangle$, $\langle w_2, w_{25}\rangle$, $\langle w_3, w_{25}\rangle$, $\langle w_6, w_{25}\rangle$ |
| 154 | all of size 4 except $\langle w_2, w_{77}\rangle$, $\langle w_{11}, w_{14}\rangle$ |
| 156 | $\langle w_3, w_4\rangle$, $\langle w_4, w_{13}\rangle$, $\langle w_{12}, w_{13}\rangle$, $\langle w_3, w_{52}\rangle$ |
| 165 | all of size 4 except $\langle w_5, w_{11}\rangle$ |
| 168 | all of size 4 except $\langle w_{24}, w_{56}\rangle$ |
| 170 | all of size 4 except $\langle w_{10}, w_{17}\rangle$, $\langle w_5, w_{34}\rangle$ |
| 174 | all of size 4 except $\langle w_3, w_{29}\rangle$, $\langle w_2, w_{87}\rangle$ |
| 180 | all of size 4 |
| 182 | all of size 4 except $\langle w_2, w_{91}\rangle$, $\langle w_{14}, w_{26}\rangle$ |
| 186 | all of size 4 except $\langle w_3, w_{62}\rangle$ |
| 190 | $\langle w_3, w_5\rangle$, $\langle w_3, w_{13}\rangle$, $\langle w_5, w_{13}\rangle$, $\langle w_{13}, w_{15}\rangle$ |
| 198 | all of size 4 |
| 204 | all of size 4 except $\langle w_3, w_{68}\rangle$ |
| 210 | $\langle w_6, w_{10}\rangle$, $\langle w_6, w_{14}\rangle$, $\langle w_{10}, w_{14}\rangle$, $\langle w_{15}, w_{21}\rangle$, $\langle w_6, w_{35}\rangle$, $\langle w_{10}, w_{21}\rangle$, $\langle w_{14}, w_{15}\rangle$, $\langle w_{30}, w_{35}\rangle$ <br> $\langle w_2, w_3, w_5\rangle$, $\langle w_2, w_3, w_7\rangle$, $\langle w_2, w_5, w_{21}\rangle$, $\langle w_2, w_7, w_{15}\rangle$, $\langle w_3, w_7, w_{10}\rangle$, $\langle w_3, w_5, w_{14}\rangle$ |
| 220 | all of size 4 |
| 222 | all of size 4 except $\langle w_6, w_{74}\rangle$ |
| 228 | $\langle w_3, w_4\rangle$, $\langle w_4, w_{19}\rangle$, $\langle w_3, w_{76}\rangle$ |
| 230 | all of size 4 |
| 231 | all of size 4 except $\langle w_3, w_{77}\rangle$ |
| 234 | $\langle w_{18}, w_{26}\rangle$, $\langle w_9, w_{26}\rangle$, $\langle w_2, w_{117}\rangle$ |
| 238 | $\langle w_2, w_7\rangle$, $\langle w_2, w_{17}\rangle$, $\langle w_7, w_{34}\rangle$, $\langle w_{14}, w_{17}\rangle$ |
| 240 | $\langle w_5, w_{16}\rangle$, $\langle w_{15}, w_{16}\rangle$, $\langle w_{15}, w_{48}\rangle$, $\langle w_3, w_{80}\rangle$ |
| 252 | all of size 4 |
| 255 | all of size 4 except $\langle w_5, w_{51}\rangle$ |
| 258 | $\langle w_2, w_{43}\rangle$, $\langle w_6, w_{86}\rangle$ |
| 260 | $\langle w_4, w_{13}\rangle$, $\langle w_4, w_{65}\rangle$ |
| 264 | $\langle w_8, w_{33}\rangle$, $\langle w_{11}, w_{24}\rangle$ |
| 266 | all of size 4 |
| 273 | $\langle w_3, w_7\rangle$, $\langle w_3, w_{13}\rangle$, $\langle w_3, w_{91}\rangle$ |
| 276 | all of size 4 |
| 282 | $\langle w_2, w_{47}\rangle$, $\langle w_3, w_{47}\rangle$, $\langle w_6, w_{47}\rangle$ |

| | |
|---|---|
| 285 | all of size 4 except $\langle w_3, w_{95}\rangle$ |
| 286 | all of size 4 except $\langle w_2, w_{143}\rangle$ |
| 294 | $\langle w_6, w_{98}\rangle$ |
| 300 | $\langle w_4, w_{75}\rangle$ |
| 310 | $\langle w_5, w_{31}\rangle$ |
| 312 | $\langle w_{24}, w_{39}\rangle$ |
| 315 | all of size 4 |
| 330 | all of size 8 except $\langle w_3, w_{10}, w_{11}\rangle$ |
| 336 | $\langle w_{16}, w_{21}\rangle$ |
| 357 | all of size 4 |
| 360 | $\langle w_9, w_{40}\rangle$ |
| 380 | all of size 4 |
| 390 | all of size 8 |

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