Quantization and Algebraic Index

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Dedicated to Prof. S.-T. Yau on the occasion of his 75th Birthday

Abstract

This article reviews the program on connecting Batalin-Vilkovisky (BV) quantization with index theories of algebraic type. We explain how the classical algebraic index theorem can be proved in terms of BV quantization of topological quantum mechanics. This is generalized to 2d chiral CFT in which we present an elliptic chiral analog of the algebraic index theory. As an application, we show how the generating function of all genus Gromov-Witten invariants on elliptic curves is mirror equivalent to an elliptic chiral index in the mirror BCOV theory.

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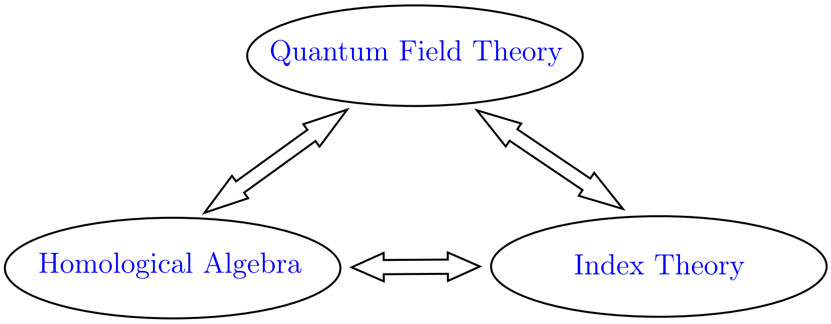
1 Introduction



It is well known that the Atiyah-Singer index theorem is closely related to supersymmetric/topological quantum mechanics [2, 26, 49, 50]. Though not rigorous, this physics interpretation provides a clear and deep insight into the origin of index theorem via the geometry of the loop space. As a natural generalization, one can replace the loop by the two-dimensional torus. This leads to Witten's proposal [51, 52] for the index of Dirac operators on loop spaces.

In [22, 42], Fedosov and Nest-Tsygan established the algebraic index theorem for deformation quantized algebras as the algebraic analogue of Atiyah-Singer index theorem. It was further shown [43] that the original Atiyah-Singer index theorem can be deduced from this algebraic one. In [29, 31], we established an exact connection between the algebraic index theorem and topological quantum mechanics via a trace map constructed in the Batalin-Vilkovisky(BV) formalism [4]. This sets up a mathematical understanding of the physics approach to index theorem in terms of an exact low-energy effective quantum field theory [31]. Such connection between quantization and algberaic index can be naturally extended to quantum field theory on other geometric objects, such as the torus. In [37, 30], we developed the effective BV quantization theory of two-dimensional chiral theory and established a chiral analogue of the algebraic index theory on the torus.

This paper reviews the program on BV quantization and index theories of algebraic type developed in [29, 31, 37, 30]. Here we summarize the main structures.

Let us denote by $\mathbf{k}$ the field of Laurent series $\mathbb{C}((\hbar))$. Roughly speaking, BV quantization in quantum field theory on X leads to the following data (we will give more details in the body of the text)

1. A factorization algebra of local observables (we follow the set-up in [16]).

   $$\mathrm{Obs}: \quad \text{a } \mathbf{k}\text{-module equipped with certain algebra structure.}$$

   It carries an algebraic structure called factorization product (or operator product expansion in physics terminology).

2. A (factorization) chain complex

   $$C_\bullet(\mathrm{Obs}): \quad \text{a } \mathbf{k}\text{-chain complex}, \qquad d: \quad \text{the differential.}$$

   It captures the algebraic structure and global information from local observables.

3. A BV algebra $(O_{\mathrm{BV}}, \Delta)$

   $$O_{\mathrm{BV}}: \quad \text{a BV algebra over } \mathbb{C}, \qquad \Delta: \quad \text{the BV operator}$$

   together with a BV integration map

   $$\int_{\mathrm{BV}}: O_{\mathrm{BV}} \to \mathbb{C}, \qquad \text{such that} \qquad \int_{\mathrm{BV}} \Delta(-) = 0.$$

In physics, O_{BV} are functions on the space of zero modes at low energy. $\int_{\mathrm{BV}}$ is a choice (related to the gauge fixing) of the integration map on zero modes. It will be $\hbar$-linearly extended when the quantum parameter $\hbar$ is involved.

4. A $\mathbf{k}$-linear map (encoding the path integral in physics)

$$\mathbf{Tr} : C_\bullet(\mathrm{Obs}) \to O_{\mathrm{BV},\mathbf{k}} = O_{\mathrm{BV}} \otimes_{\mathbb{C}} \mathbb{C}((\hbar))$$

satisfying the **quantum master equation** (QME)

$$(d + \hbar\Delta)\mathbf{Tr} = 0.$$

In other words, QME says $\mathbf{Tr}$ is a chain map intertwining d and $-\hbar\Delta$. In physics, it describes the quantum gauge consistency condition in terms of BV formalism. Index is obtained as the partition function of the model, which can be formulated as

$$\mathrm{Index} = \int_{\mathrm{BV}} \mathbf{Tr}(1).$$

In Section 2, we review the theory of effective BV quantization. In Section 3, we explain the 1d example of topological quantum mechanics on the circle and show how the above structures lead to the algebraic index theorem. In this case

- The factorization algebra is the Weyl algebra: $\mathrm{Obs} = \mathcal{W}_{2n}$.
- The factorization complex is the Hochschild chain complex $(C_\bullet(\mathcal{W}_{2n}), b)$.
- BV algebra on zero modes: $(\mathcal{A}, \Delta) = \left(\Omega^\bullet(\mathbb{R}^{2n}), \mathcal{L}_{\omega^{-1}}\right)$.
- Free correlation map

$$\langle - \rangle : C_\bullet(\mathcal{W}_{2n}) \to \Omega^\bullet(\mathbb{R}^{2n})((\hbar)), \quad b \mapsto \hbar\mathcal{L}_{\omega^{-1}}.$$

- $\mathrm{Index} = \int_{BV} \langle 1 \rangle = \left[e^{\omega_\hbar/\hbar}\widehat{A}\right]$.

In Section 4, we explain the 2d chiral example and the elliptic chiral analogue of algebra index via $\beta\gamma - bc$ system. In this case, the factorization complex is the chiral chain complex of the corresponding vertex operator algebra. The trace map arising from BV quantization on elliptic curves will be called the *elliptic trace map*.

The above two examples in Section 2 and 4 share a special property: they are both UV finite theories. A conjectured structure for BV quantization of general UV finite theory is presented in Section 2.4.

1d TQM	2d Chiral QFT
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology
BV QME: $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$	BV QME: $(\hbar\Delta + d_{ch})\langle - \rangle_{2d} = 0$
$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d}$ =integrals on the compactified configuration spaces of S^1	$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d}$ =regularized integrals of singular forms on Σ^n
Algebraic Index	Elliptic Chiral Algebraic Index

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2 Effective Theory of BV Quantization

In this section, we review Costello's homotopic theory of effective BV quantization [13]. This is the basic framework that we will use to establish the connection between BV quantization and index theories of algebraic type. We follow the presentation in [37].

2.1 (-1)-shifted Symplectic Structure

We first explain that classical field theories and their quantizations have a universal description in terms of (-1)-shifted symplectic structure. This is particularly convenient to quantize gauge theories in the BV framework.

We start with the finite dimensional toy model. Let (V, Q, ω) be a finite dimensional dg (differential graded) symplectic space. Here

- V is a finite dimensional graded vector space.
- $Q : V \longrightarrow V$ differential, $\deg Q = 1$ and $Q^2 = 0$.
- $\omega : \wedge^2 V \longrightarrow \mathbb{R}$ non-degenerate pairing of deg$=-1$, that is,

$$\omega(a, b) = 0, \text{ unless } |a| + |b| = 1.$$

- ω is Q-compatible $Q(\omega) = 0$, i.e.,

$$\omega(Q(a), b) + (-1)^{|a|}\omega(a, Q(b)) = 0.$$

The non-degeneracy of ω leads to linear isomorphisms

$$\begin{aligned} \omega : V^\vee &\xrightarrow{\sim} V[1] \\ \Longrightarrow \quad \wedge^2(V^\vee) &\xrightarrow{\sim} \wedge^2(V[1]) \simeq Sym^2(V)[2] \\ \omega &\longleftrightarrow K[2] \end{aligned}$$

Here $K = \omega^{-1} \in \mathrm{Sym}^2(V)$ is the Poisson Kernel and

$$\deg(K) = 1, \quad Q(K) = 0.$$

We obtain a triple (A, Q, Δ) as follows

- $A = \mathcal{O}(V) := \widehat{\mathrm{Sym}}(V^\vee)$ (formal power series on V)
- $Q : A \longrightarrow A$ derivation induced dually from $Q : V \longrightarrow V$
- BV operator

$$\Delta = \Delta_K : A \longrightarrow A$$

by contracting with the Poisson Kernel K

$$\Delta_K : \mathrm{Sym}^m(V) \longrightarrow \mathrm{Sym}^{m-2}(V).$$

Explicitly, for $\alpha_i \in V^\vee$

$$\Delta_K(\alpha_1 \otimes \cdots \otimes \alpha_m) = \sum_{i<j} \pm\langle K, \alpha_i \otimes \alpha_j\rangle \alpha_1 \otimes \cdots \widehat{\alpha_i} \otimes \cdots \widehat{\alpha_i} \otimes \cdots \otimes \alpha_m.$$

- Δ_K induces a BV bracket on A by

$$\{a, b\} := \Delta_K(ab) - (\Delta_K a)b - (-1)^{|a|} a\Delta_k b$$

Here $|a|$ is the degree of a.

- Since K is $Q-$closed, we have

$$[Q, \Delta_K] := Q\Delta_K + \Delta_K Q = 0$$

The triple (A, Q, Δ) is exactly the data of a DGBV. Given such a DGBV, we can talk about

- Classical master equation:

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0 \qquad \text{for} \quad I_0 \in A,\ \deg(I_0) = 0.$$

  Then the classical BRST operator $\delta = Q + \{I_0, -\}$ satisfies $\delta^2 = 0$.

- Quantum master equation:

$$QI + \hbar I + \frac{1}{2}\{I, I\} = 0 \Longleftrightarrow (Q + \hbar\Delta)e^{I/\hbar} = 0, \quad \text{for} \quad I = I_0 + \hbar I_1 + ... \in A[[\hbar]].$$

  Then the quantum BRST operator $\delta^\hbar = Q + \hbar\Delta + \{I, -\}$ satisfies $(\delta^\hbar)^2 = 0$.

Classical Field Theory

Now we discuss the QFT situation. For our purpose, we focus on theories where fields are sections of vector bundles. A classical field theory can be organized into $\infty-$dimensional (-1)-shifted dg symplectic space

$$(\mathcal{E}, Q, \omega)$$

- $\mathcal{E} = \Gamma(X, E^\bullet)$ the space of fields. Here $E^\bullet$ is a graded vector bundle on X.
- $(\mathcal{E}, Q)$ elliptic complex

$$... \longrightarrow \mathcal{E}^{-1} \xrightarrow{Q} \mathcal{E}^0 \xrightarrow{Q} \mathcal{E}^1 \xrightarrow{Q} ...$$

  For example, $Q = \bar\partial$ or d.

- ω: local (-1)-symplectic pairing

$$\omega(\alpha, \beta) = \int_X \langle\alpha, \beta\rangle, \ \ \forall\alpha, \beta \in \mathcal{E}$$

  and compatible with Q.

Example 2.1 (Chern-Simons Theory)**.** *Let X be a* $\dim = 3$ *manifold, and $\mathfrak{g}$ be a Lie algebra with trace pairing $Tr : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$. The space of fields is*

$$\mathcal{E} = \Omega^\bullet(X, \mathfrak{g})[1].$$

The degree shifting [1] *gives the following intepretation.*

	$\Omega^0(X, \mathfrak{g})$	$\Omega^1(X, \mathfrak{g})$	$\Omega^2(X, \mathfrak{g})$	$\Omega^3(X, \mathfrak{g})$
deg	*-1*	*0*	*1*	*2*
	c	A	$A^\vee$	$c^\vee$
	ghost	*field*	*anti-field*	*anti-ghost*

$Q = d$ *is the de Rham differential. The* (-1)*-symplectic pairing is*

$$\omega(\alpha, \beta) = \int_X Tr(\alpha \wedge \beta), \quad \alpha, \beta \in \mathcal{E}$$

which pairs 0*-forms with* 3*-forms and pairs* 1*-forms with* 2*-forms.*

Example 2.2 (Scalar Field Theory in BV formalism)**.** *The field complex $\mathcal{E}$ is*

$$C^\infty(M) \xrightarrow{Q=\Delta+m^2} C^\infty(M)$$

$$\deg = 0 \qquad\qquad \deg = 1$$

$$\phi \qquad\qquad \phi^\vee$$

The (-1)-symplectic pairing is

$$\omega(\phi, \phi^\vee) = \int_M \phi\phi^\vee.$$

UV Problem

Let us now perform the same construction of DGBV algebra following the toy model. We first need the notion of "functions" $\mathcal{O}(V) = \widehat{\mathrm{Sym}}(V^\vee)$ on V.

- linear function: we have to take a continuous linear dual and so

  $$\mathcal{E}^\vee = \mathrm{Hom}_X(\mathcal{E}, \mathbb{R})$$

  is given by distributions.
- $(\mathcal{E}^\vee)^{\otimes n} = \mathrm{Hom}_{X\times\ldots\times X}(\mathcal{E}^{\otimes n}, \mathbb{R})$ are distributions on X^n. Here

  $$\mathcal{E}^{\otimes n} = \Gamma(X^n, E^{\boxtimes n})$$

  is the completed tensor product. Thus

  $$\mathrm{Sym}^m(\mathcal{E}^\vee) := (\mathcal{E}^\vee)^{\otimes m}/S_m$$

  is well-defined by distributions on X^m. As a result, we can form

  $$\mathcal{O}(\mathcal{E}) = \prod_{m\geq 0} \mathrm{Sym}^m(\mathcal{E}^\vee)$$

  representing (formal) functions on $\mathcal{E}$.
- $Q : \mathcal{E} \longrightarrow \mathcal{E}$ induces duality $Q : \mathcal{E}^\vee \longrightarrow \mathcal{E}^\vee$ on distributions, and gives rise to

  $$Q : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E}).$$

- BV operator: Let $K = \omega^{-1}$ be the Poisson kernel as above. Since

  $$\omega = \int \langle -, - \rangle$$

  is an integral, its inverse K is a δ-function distribution supported on the diagonal of $X \times X$. Thus K is NOT a smooth element in $\mathrm{Sym}^2(\mathcal{E})$, but a distributional section. As a result, the naive BV operator

  $$\Delta_K : \mathrm{Sym}^m(\mathcal{E}^\vee) \to \mathrm{Sym}^{m-2}(\mathcal{E}^\vee)$$

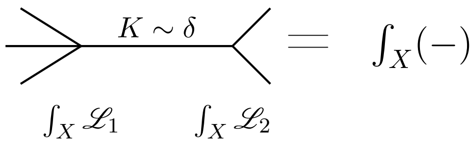
  is ill-defined since we can not pair two distributions. This is essentially the Ultra-Violet problem. Renormalization is needed in the quantum theory!

Before we move on to discuss the issue of renormalization, let us point out that the classical theory is actually well-behaved. Let $\mathcal{O}_{loc}(\mathcal{E}) \subset \mathcal{O}(\mathcal{E})$ denote the subspace of local functionals, i.e., those by integrals of lagrangian densities

$$\mathcal{O}_{loc}(\mathcal{E}) = \left\{ \int_X \mathscr{L}(...) \right\}$$

Although the BV operator Δ_K is ill-defined, the associated BV bracket $\{-,-\}$ is actually well-defined on local functionals since δ-function can be integrated.

$$\{-,-\} : \mathcal{O}_{loc}(\mathcal{E}) \otimes \mathcal{O}_{loc}(\mathcal{E}) \longrightarrow \mathcal{O}_{loc}(\mathcal{E})$$



In other words,

- CME makes sense for local functionals
- QME needs renormalization

We refer to [13] for detailed discussions on this issue.

Example 2.3 (Chern-Simons theory)**.** $\mathcal{E} = \Omega^\bullet(X, \mathfrak{g})[1]$

	$\Omega^0(X,\mathfrak{g})$	$\Omega^1(X,\mathfrak{g})$	$\Omega^2(X,\mathfrak{g})$	$\Omega^3(X,\mathfrak{g})$
deg	*-1*	*0*	*1*	*2*
	c	A	$A^\vee$	$c^\vee$
	ghost	*field*	*anti-field*	*anti-ghost*

Let $\mathcal{A} = C + A + A^\vee + C^\vee \in \mathcal{E}$ denote the master field collecting all components. Then the BV Chern-Simons action is

$$CS[\mathcal{A}] = \int_X Tr(\frac{1}{2}\mathcal{A} \wedge d\mathcal{A} + \frac{1}{6}\mathcal{A} \wedge [\mathcal{A}, \mathcal{A}]).$$

This takes the same form as ordinary Chern-Simons except that we have expanded $\mathcal{A}$ to get terms containing different components. The first quadratic term is denoted by S_{free}, the free part. The second cubic term is denoted by I, the interaction part.

CS satisfies the following classical master equation

$$\{CS, CS\} = 0.$$

This follows from the general argument that classical gauge theory is organized into a solution of classical master equation. Let us separate the free part and interaction

$$CS = S_{free} + I.$$

It is easy to see that

$$\{S_{free}, -\} = d \ (= Q)$$

which corresponds to the de Rham differential. Thus

$$\begin{aligned} & \{CS, CS\} = 0 \\ \Leftrightarrow \quad & \frac{1}{2}\{S_{free}, S_{free}\} + \{S_{free}, I\} + \frac{1}{2}\{I, I\} = 0 \\ \Leftrightarrow \quad & QI + \frac{1}{2}\{I, I\} = 0 \end{aligned}$$

This is precisely the form of classical master equation in our DGBV.

2.2 Effective Renormalization

Assume we have a classical field theory $(\mathcal{E} = \Gamma(X, E^\bullet), Q, \omega)$ with classical local functional I_0 (interaction) satisfying CME

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0.$$

As we explained before, quantization asks for

$$I_0 \longrightarrow I = I_0 + \hbar I_1 + \hbar I_2 + ... \in \mathcal{O}(\mathcal{E})[[\hbar]]$$

satisfying QME

$$\text{“}QI_0 + \frac{1}{2}\{I_0, I_0\} + \hbar\Delta I = 0\text{”}.$$

Problem: ΔI is NOT well-defined. In the following, we explain Costello's homotopic renormalization theory to solve this problem.

Toy Model

To motivate the construction, let us look back again at the toy model where (V, Q, ω) is finite dimensional (-1)-shifted dg symplectic space. The Poisson kernel

$$K_0 \in \mathrm{Sym}^2(V)$$

has $\deg(K_0) = 1$ and satisfies $Q(K_0) = 0$. This allows us to construction the BV operator Δ_0 by contracting with K_0 and obtain the DGBV triple (A, Q, Δ_0).

Let us now consider the change of K_0 by chain homotopy. Let

$$P \in \mathrm{Sym}^2(V), \quad \deg(P) = 0.$$

Define

$$K_P = K_0 + Q(P) = K_0 + (Q \otimes 1 + 1 \otimes Q)P.$$

We again have

- $K_P \in \mathrm{Sym}^2(V), \deg(K_P) = 1$
- $Q(K_P) = 0$

Thus we can construct a new BV operator

$$\Delta_P = \text{contraction with } K_P$$

such that $(\mathcal{O}(V), Q, \Delta_P)$ forms a new DGBV.

To see the relation with the original DGBV, denote

$$\partial_P : \mathrm{Sym}^m(V^\vee) \longrightarrow \mathrm{Sym}^{m-2}(V^\vee)$$

where ∂_P is a 2nd order operator of contracting with $P \in \mathrm{Sym}^2(V)$

Proposition 2.4. *The following diagram commutes*

$$\begin{tikzcd}
\mathcal{O}(V)[[\hbar]] \arrow[r, "e^{\hbar\partial_P}"] \arrow[d, "Q+\hbar\Delta_0"] & \mathcal{O}(V)[[\hbar]] \arrow[d, "Q+\hbar\Delta_P"] \\
\mathcal{O}(V)[[\hbar]] \arrow[r, "e^{\hbar\partial_P}"] & \mathcal{O}(V)[[\hbar]]
\end{tikzcd}$$

i.e.

$$(Q + \hbar\Delta_P)e^{\hbar\partial_P} = e^{\hbar\partial_P}(Q + \hbar\Delta_0).$$

Proof: This follows from the chain homotopy relation $K_P = K_0 + Q(P)$. $\square$

Corollary 2.5. *Assume $I \in \mathcal{O}(V)[[\hbar]]$ satisfies QME*

$$(Q + \hbar\Delta_0)e^{I/\hbar} = 0$$

in the DGBV $(\mathcal{O}(V), Q, \Delta_0)$. Then $\tilde{I} \in \mathcal{O}(V)[[\hbar]]$ satisfies QME

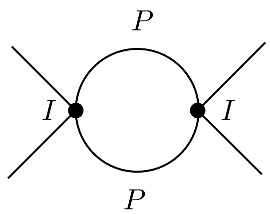
$$(Q + \hbar\Delta_P)e^{\tilde{I}/\hbar} = 0$$

in the DGBV $(\mathcal{O}(V), Q, \Delta_P)$. Here $\tilde{I}$ is related to I by

$$e^{\tilde{I}/\hbar} = e^{\hbar\partial_P} e^{I/\hbar}$$

The operator $e^{\hbar\partial_P}$ plays the role of integration with respect to the Gaussian measure. The relation $e^{\tilde{I}/\hbar} = e^{\hbar\partial_P} e^{I/\hbar}$ can be read via Wick's Theorem as

$$\tilde{I} = \sum_{\text{connected graphs}} \Bigg(\quad \Bigg) .$$



Here I serves as vertices and P for the propagator. Thus, Feynman diagrams give the required chain homotopy between different DGBV's.

Back to QFT

Now consider the QFT set-up $(\mathcal{E} = \Gamma(X, E^\bullet), Q, \omega)$. The problem is that the Poisson kernel $K_0 = \omega^{-1}$ is a δ-function distribution which leads to a singular BV operator. Nevertheless we know K_0 is Q-closed

$$Q(K_0) = 0.$$

Costello's approach: Using elliptic regularity

$$H^\bullet(\text{Distribution}, Q) = H^\bullet(\text{Smooth}, Q).$$

Here "Distribution" or "Smooth" means distributional or smooth sections of relevant tensor bundles of $E^\bullet$. Thus we can replace K_0 by a smooth object in its Q-cohomology class by

$$K_0 = K_r + Q(P_r).$$

Here K_r is smooth while P_r (called parametrix) is singular. Define

$$\Delta_r : \text{BV operator associated with } K_r$$

Since $K_r \in \mathrm{Sym}^2(\mathbb{E})$ is now smooth, the operator

$$\Delta_r : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E}) \text{ is well-defined.}$$

Definition 2.6. The DGBV $(\mathcal{O}(\mathcal{E}), Q, \Delta_r)$ will be called the **effective DGBV** with respect to the regularization r.

Let r' be another regularization with parametrix $P_{r'}$

$$K_0 = K_{r'} + Q(P_{r'}).$$

Then the two regularized Poisson kernels differ by a chain homotopy

$$K_{r'} - K_r = Q(P_r^{r'})$$

where $P_r^{r'} \in \mathrm{Sym}^2(\mathcal{E})$ is smooth. Let

$$\partial_{P_r^{r'}} : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$$

be the 2nd order operator of contracting with the smooth kernel $P_r^{r'}$.

The same argument as in the toy model gives the chain homotopy

$$\big(\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta_r\big) \xrightarrow{\exp\left(\hbar\partial_{P_r^{r'}}\right)} \big(\mathcal{O}(\mathcal{E})[[\hbar]], Q + \hbar\Delta'_r\big)$$

$\uparrow$

Homotopy RG flow (HRG)

Definition 2.7 (Costello[13]). An effective perturbative quantization of I_0 (which satisfies CME) is a family

$$I[r] \in \mathcal{O}(\mathcal{E})[[\hbar]]$$

(whichi is at least cubic modulo $\hbar$) for each choice of regularization r satisfying

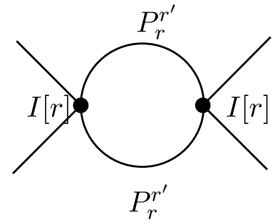
- Effective QME

$$(Q + \hbar\Delta_r)e^{I[r]/\hbar} = 0.$$

- Homotopy RG flow

$$e^{I[r']/\hbar} = e^{\hbar\partial_{P_r^{r'}}} e^{I[r]/\hbar}$$

  which is equivalent to the Feynman diagram expansion

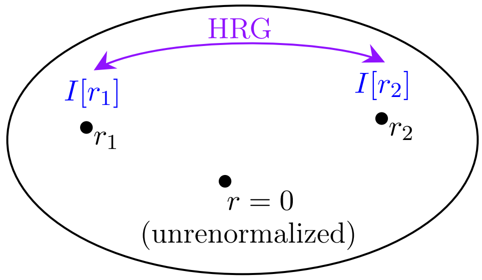
$$I[r'] = \sum_{\text{connected graphs}} \Bigg($$


$$\Bigg).$$

- $I[r]$ is asymptotic local when $r \longrightarrow 0$ and has the classical limit

$$\lim_{r\to 0} I_0[r] = I_0$$

Here is a pictue to illustrate what is going on. The situation is very similar to how residue is defined in algebraic geometry: we need to perturb the singularity and define residue at the deformed configuration, and show that all local deformations give the same answers. Here we use all "nearby" regularizations to define the unrenormalized point.



In practice, here are steps for constructing perturbative quantization.

① Construct counter-term $I^\varepsilon \in \hbar\mathcal{O}_{loc}(\mathcal{E})[[\hbar]]$ such that

$$e^{I[r]^{Naive}/\hbar} := \lim_{\varepsilon\to 0}(e^{\hbar P_\varepsilon^r} e^{(I_0+I^\varepsilon)/\hbar}) \quad \text{exists}$$

Then this naive family $\left\{I[r]^{Naive}\right\}_r$ satisfies HRG by construction.

② The choice of counter-terms is not unique. We need to further correct I^ε such that $e^{I[r]/\hbar}$ satisfies QME.

① is always possible by the method of counter-term. ② is NOT always possible: obstruction may exist which is called "**gauge anomaly**" in physics terminology. There is a deformation-obstruction theory, which shows that the gauge anomaly lies in

$$H^1(\mathcal{O}_{loc}(\mathcal{E}), Q + \{I_0, -\}).$$

2.3 Heat Kernel Regularization

There are many ways of regularizations. One method that connects to geometry is the heat kernel regularization. Typically, fixing a choice of metric, we have

- the adjoint of the elliptic operator $Q : \mathcal{E} \to \mathcal{E}$, denoted as $Q^\dagger : \mathcal{E} \to \mathcal{E}$,
- assume $\left[Q, Q^\dagger\right] = QQ^\dagger + Q^\dagger Q$ is a generalized Laplacian [1].

Thus we can define a heat operator $e^{-L[Q,Q^\dagger]}$ for $L > 0$. Let $K_L \in \mathrm{Sym}^2(\mathcal{E})$ be the kernel of the heat operator by

$$\left(e^{-L[Q,Q^\dagger]}\alpha\right)(x) = \int dy \,\langle K_L(x,y), \alpha(y)\rangle \quad \text{for } \alpha \in \mathcal{E}.$$

Here $\langle -, -\rangle$ is the pairing from ω. Note that

- $K_0 = \lim\limits_{L\to 0} K_L$ is the δ-function distribution ω^{-1},
- $K_L \in \mathrm{Sym}^2(\mathcal{E})$ is smooth for $L > 0$.

Let P_L be the kernel of the operator $\int_0^L dt\; Q^\dagger e^{-t[Q,Q^\dagger]}$. Explicitly, we have

$$P_L = \int_0^L dt \left(Q^\dagger \otimes 1\right) K_t.$$

The operator equation

$$\left[Q, \int_0^L dt\; Q^\dagger e^{-t[Q,Q^\dagger]}\right] = \int_0^L dt \left[Q, Q^\dagger\right] e^{-t[Q,Q^\dagger]} = 1 - e^{-L[Q,Q^\dagger]}$$

[1] We use $[-,-]$ for graded commutator in this paper.

can be translated into the kernel equation:

$$K_0 - K_L = (Q \otimes 1 + 1 \otimes Q)P_L$$

or simply written as

$$K_0 - K_L = Q\left(P_L\right).$$

We can use K_L to define the effective QME.

Similarly, for $0 < \varepsilon < L$, the operator equation is

$$\left[Q, \int_{\varepsilon}^{L} dt\ Q^{\dagger} e^{-t[Q,Q^{\dagger}]}\right] = e^{-\varepsilon[Q,Q^{\dagger}]} - e^{-L[Q,Q^{\dagger}]}$$

or

$$K_{\varepsilon} - K_L = (Q \otimes 1 + 1 \otimes Q)P_{\varepsilon}^{L},$$

where $P_{\varepsilon}^{L} = \int_{\varepsilon}^{L} dt \left(Q^{\dagger} \otimes 1\right) K_t$ is called the *regularized propagator*. Now we can use P_{ε}^{L} to connect the effective QME at ε with the effective QME at L via the HRG.

Remark 2.8. $P_0^{\infty} = \int_0^{\infty} dt \left(Q^{\dagger} \otimes 1\right) K_t$ is the *full propagator*. At $t = 0$, one will encounter ultraviolet (UV) divergence since there exists a singularity for the full propagator. On a non-compact manifold, one will encounter infrared (IR) divergence at $t = \infty$.

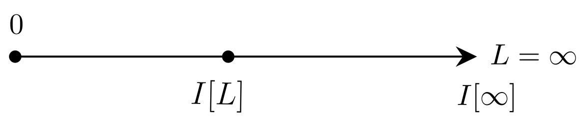
Consider the case when X is compact. Let

$$\mathbf{H} = \left\{\varphi \in \mathcal{E} \mid \left[Q, Q^{\dagger}\right]\varphi = 0\right\} = \left\{\varphi \in \mathcal{E} \mid Q\varphi = Q^{\dagger}\varphi = 0\right\} \simeq H^{\bullet}\left(\mathcal{E}, Q\right).$$

$\mathbf{H}$ is called the space of harmonics (or the zero modes), which is a finite-dimensional space (by Hodge theory). Then we have

$$\begin{array}{c} \infty - \text{dimensional } (-1) - \text{symplectic geometry } (\mathcal{E}, Q, \omega) \\ \Big\downarrow{\scriptstyle L \to \infty} \\ \text{finite-dimensional } (-1) - \text{symplectic geometry } (\mathbf{H}, \omega_{\mathbf{H}} = \omega|_{\mathbf{H}}) \end{array}$$

The BV operator $\Delta_{\mathbf{H}}$ associated with $\omega_{\mathbf{H}}^{-1}$ is $\Delta_{\mathbf{H}} = \Delta_{\infty}$. The essential story of effective BV quantization is depicted in the following diagram,



and $I[\infty]$ solves the QME for $(\mathscr{O}(\mathbf{H}), \Delta_{\mathbf{H}})$ at $L = \infty$. The limit $L \to \infty$ is an interesting point where we will find some finite-dimensional geometric data.

2.4 UV Finite Theory

In the BV formalism, the classical master equation

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0$$

is quantized to the quantum master equation

$$\text{“}QI + \hbar\Delta I + \frac{1}{2}\{I, I\} = 0\text{”}.$$

As we explained as above, this naive quantum master equation is ill-defined for local $I \in \mathcal{O}_{loc}(\mathcal{E})$, and we have to use regularization to formulate the renormalized quantum master equation

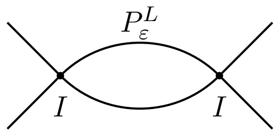
$$QI[r] + \hbar\Delta_r I[r] + \frac{1}{2}\{I, I\}_r = 0.$$

If the effective action at regularization r can be found as

$$e^{I[r]/\hbar} = \lim_{\varepsilon \to 0} e^{\hbar\partial_{P_\varepsilon^r}} e^{I/\hbar}$$

for $I \in \mathcal{O}_{loc}(\varepsilon)[[\hbar]]$, i.e., the ε-dependent counter-term is **<u>NOT</u>** needed, we say the theory is UV finite. That is, for all regularized Feynman diagrams

$\lim_{\varepsilon \to 0}$



exist.

In this way, we can consider the limit

$$I[r] \to I, \qquad r \to 0,$$

and the $r \to 0$ limit of the renormalized quantum master equation

$$QI + \frac{1}{2}\{I, I\} + \cdots = 0$$

will have a local expression that deforms the CME.

Conjecture. *For UV finite theory, we expect to describe effective QME at $r \to 0$ limit by*

$$l_1^\hbar I + \frac{1}{2} l_2^\hbar(I, I) + \frac{1}{3!} l_3^\hbar(I, I, I) + \cdots = 0$$

where $\{l_1^\hbar, l_2^\hbar, \cdots\}$ defines a family of L_∞-algebra parametrized by $\hbar$. They can be viewed as traded from Δ in terms of the renormalization procedure.

There are two main classes of UV finite theories.

① Topological theory (Chern-Simons type) where $\mathcal{E}$ is of the form of de Rham complex. The UV finite property was established by Kontsevich [34] and Axelrod-Singer [3] using the compactified configuration space.

② Holomorphic theory where $\mathcal{E}$ is of the form of Dolbeault complex. In this case, the Feynman graph integral can not be extended to the compactified configuration space. Fortunately, the UV finite property still holds in general.

- $\dim_{\mathbb{C}} = 1$: the UV property for chiral deformations is known to physicists via the method of point-splitting regularization (see for example Douglas [20] and Dijkgraaf [19]). This method is essentially Cauchy principal value, and a homological theory for such regularization was systematically developed in Li-Zhou [39]. In the framework of effective BV quantization, the UV finite property was established in Li [36, 37].

– $\dim_{\mathbb{C}} > 1$: the method in [37] has been generalized for one loop graphs in Costello-Li [15] and Williams [48]. At higher loops, Budzik-Gaiotto-Kulp-Wu-Yu [10] presented a strategy to prove UV finiteness for Laman graphs. In [46], Wang proved the UV finite property for all graphs on all $\mathbb{C}^n$ using a compactified Schwinger space. This is further generalized to Kähler manifolds in Wang-Yan [47].

It is an extremely interesting question to figure out $\{l_1^\hbar, l_2^\hbar, \cdots\}$ in these examples. In Section 3 and Section 4, we explain the simplest example in each of these two categories (Conjecture holds there) to illustrate the underlying rich structures.

3 Topological Quantum Mechanics

In this section we consider the example of topological quantum mechanics and illustrate its connection with deformation quantization and algebraic index theorem.

3.1 Deformation Quantization

The method of deformation quantization was developed in the series of papers by Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer [5]. The space of the real-valued (or complex-valued) functions on a phase space admits two algebraic structures: a structure of *associative algebra* given by the usual product of functions and a structure of Lie algebra given by the *Poisson bracket*. The study of the properties of the deformations (in a suitable sense) of these two structures gives a new invariant approach for quantum mechanics.

$$\begin{tikzcd} \text{Poisson algebra} \arrow[r, bend left, "\text{quantization}"] & \text{Associative algebra} \arrow[l, bend left, "\hbar\to 0"] \end{tikzcd}$$

This is essentially the quantization method in quantum mechanics, in which a function f on the classical phase space is quantized to an operator $\widehat{f}$.

Definition 3.1. A **Poisson manifold** is a pair (X, P), where X is a smooth manifold, and $P \in \Gamma(X, \bigwedge^2 TX)$ satisfying $\{P, P\}_{\mathrm{SN}} = 0$.

Here $\{-,-\}_{\mathrm{SN}}$ is the *Schouten-Nijenhuis bracket*. P is called the Poisson tensor/bivector. In local coordinates, we can write

$$P = \sum_{i,j} P^{ij}(x)\partial_i \wedge \partial_j.$$

It defines a *Poisson bracket* $\{-,-\}_P$ on $C^\infty(X)$:

$$\{f, g\}_P := \sum_{i,j} P^{ij}\partial_i f \partial_j g, \quad \forall f, g \in C^\infty(X).$$

The Poisson condition $\{P, P\}_{\mathrm{SN}} = 0$ implies $\{-,-\}_P$ satisfies Jacobi identity. Hence $\{-,-\}_P$ naturally defines the Poisson algebra $(C^\infty(X), \{-,-\}_P)$.

Example 3.2. *Let (X, ω) be a symplectic manifold, where $\omega = \frac{1}{2}\sum_{i,j}\omega_{ij}dx^i \wedge dx^j$ is the symplectic 2-form. Let*

$$P = \omega^{-1} = \frac{1}{2}\sum_{i,j}\omega^{ij}\partial_i \wedge \partial_j,$$

where (ω^{ij}) *is the inverse of* (ω_{ij})*. Then*

$$d\omega = 0 \Leftrightarrow \{P, P\}_{SN} = 0.$$

Hence (X, ω^{-1}) *defines a Poisson manifold.*

Definition 3.3. A **star-product** on a Poisson manifold (X, P) is a $\mathbb{R}[[\hbar]]$-bilinear map

$$\begin{aligned} C^\infty(X)[[\hbar]] \times C^\infty(X)[[\hbar]] &\to C^\infty(x)[[\hbar]] \\ f \times g \quad &\mapsto f \star g = \sum_{k \geq 0} \hbar^k c_k(f, g) \end{aligned}$$

such that

(1) $\star$ is associative: $(f \star g) \star h = f \star (g \star h)$,

(2) $f \star g = fg + \mathcal{O}(\hbar), \quad \forall f, g \in C^\infty(X)$,

(3) $\frac{1}{2}(f \star g - g \star f) = \hbar\{f, g\} + \mathcal{O}(\hbar^2), \quad \forall f, g \in C^\infty(X)$,

(4) $c_k : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ is a bidifferential operator.

Then $(C^\infty(X)[[\hbar]], \star)$ is called a **deformation quantization** of (X, P).

The definition of deformation quantization is purely algebraic. The *existence* of deformation quantization is highly nontrivial. DeWilde-Lecomte first [17] obtained the general existence of deformation quantization on symplectic manifolds via cohomological method. Fedosov [21] presented another beautiful approach on symplectic manifolds via differential geometric method. In the general case, Kontsevich [35] gave the complete solution for arbitrary Poisson manifold. The parameter $\hbar$ is formal in the above definition of deformation quantization (only formal power series of $\hbar$ is concerned). There is also a notion of strict deformation quantization introduced by Rieffel [45] in terms of C^*-algebras where $\hbar$ is not formal.

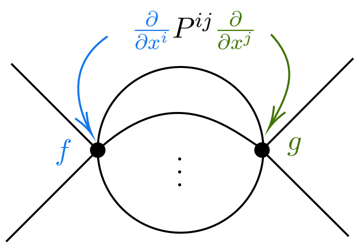
Example 3.4. *Let* $X = \mathbb{R}^{2n}$*, with Poisson tensor*

$$P = \frac{1}{2}\sum_{i,j} P^{ij}\partial_i \wedge \partial_j$$

Here P^{ij} *are constants. Given* $f(x), g(x)$*, define the (formal) Moyal product* $\star$ *by*

$$(f \star g)(x) = \exp\left(\frac{\hbar}{2}\sum_{i,j} P^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)\Bigg|_{y=z=x} f(y)g(z)$$

or pictorially,



Then $\star$ *defines a deformation quantization.*

Remark 3.5. If $P^{ij} \neq$ constant, then the above formula does not work. How to correct this to a star product is the celebrated Formality Theorem in [35].

Definition 3.6. Let (V, ω) be a linear symplectic space, where $V \simeq \mathbb{R}^{2n}$ and $\omega : \bigwedge^2 V \to \mathbb{R}$ is a symplectic pairing. $P = \omega^{-1}$ is the Poison tensor. Write

$$\widehat{\mathscr{O}}(V) := \widehat{\mathrm{Sym}}(V^\vee) = \prod_{k \geq 0} \mathrm{Sym}^k(V^\vee),$$

where $V^\vee = \mathrm{Hom}(V, \mathbb{R})$ is the linear dual of V. Then the Moyal product defines an associative algebra $\left(\widehat{\mathscr{O}}(V)[[\hbar]], \star\right)$, called the **(formal) Weyl algebra**.

3.2 Fedosov's Geometric Method

We will focus on *symplectic manifolds* in the rest of this section. Fedosov [21] gave a simple and geometric construction of deformation quantization as follows. On a symplectic manifold (X, ω), the tangent plane T_pX at each point $p \in X$ is a linear symplectic space. Quantum fluctuations deform the algebra of functions on T_pX to the associated *Weyl algebra*. These pointwise Weyl algebras form a vector bundle — the *Weyl bundle* $\mathcal{W}(X)$ on X.

Definition 3.7. Let (X, ω) be a symplectic manifold. We define the **Weyl bundle**

$$\mathcal{W}(X) := \prod_{k \geq 0} \mathrm{Sym}^k \left(T^*X\right)[[\hbar]].$$

So at each point $p \in X$, its fiber is

$$\mathcal{W}(X)|_p = \widehat{\mathscr{O}}\left(T_pX\right)[[\hbar]].$$

Here $\widehat{\mathscr{O}}$ refers to formal power series functions.

A local section of $\mathcal{W}(X)$ is

$$\sigma(x, y) = \sum_{k,l \geq 0} \hbar^k a_{k,i_1 \cdots i_l}(x) y^{i_1} \cdots y^{i_l},$$

where $\left\{x^i\right\}$ are the base coordinates and $\left\{y^i\right\}$ are the fiber coordinates; $a_{k,i_1\cdots i_l}(x)$ are smooth functions. Since $\left(T_pX, \omega|_{T_pX}\right)$ is linear symplectic, we have a fiberwise Moyal product, still denoted by $\star$. Thus $(\mathcal{W}(X), \star)$ defines the ∞-dimensional bundle of algebras.

Let ∇ be a connection on TX which is torsion-free and compatible with ω (i.e. $\nabla\omega = 0$). Such connection is called a **symplectic connection** (which always exists and is not unique). ∇ induces a connection on all tensors. In particular, it defines a connection on $\mathcal{W}(X)$, still denoted by ∇. Its curvature is

$$\nabla^2 \sigma = \frac{1}{\hbar}\left[R_\nabla, \sigma\right]_\star, \quad \forall \sigma \in \Gamma(X, \mathcal{W}(X))$$

where

$$R_\nabla = \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l \in \Omega^2(X, \mathcal{W}(X))$$

is a 2-form valued in the Weyl bundle $\mathcal{W}(X)$; $R_{ijkl} = \omega_{im} R^m{}_{jkl}$ is a curvature form contracted with the symplectic pairing.

Given a sequence of closed 2-forms $\{\omega_k\}_{k \geq 1}$ on X, Fedosov showed that there exists a unique (up to gauge) connection on $\mathcal{W}(X)$ of the form

$$\nabla + \frac{1}{\hbar}[\gamma, -]_\star$$

where $\gamma \in \Omega^1(X, \mathcal{W}(X))$ is a $\mathcal{W}(X)$-valued 1-form, satisfying certain initial conditions and the equation

$$\nabla\gamma + \frac{1}{2\hbar}[\gamma,\gamma]_\star + R_\nabla = \omega_\hbar \qquad \text{(Fedosov equation)}$$

where $\omega_\hbar = -\omega + \sum_{k\geq 1} \hbar^k \omega_k$. Let

$$D = \nabla + \frac{1}{\hbar}[\gamma, -]_\star$$

be the Fedosov connection. Then the Fedosov equation implies

$$D^2 = \frac{1}{\hbar}[\omega_\hbar, -]_\star = 0$$

since $\omega_\hbar$ is constant along each fiber, thus a central term. So we obtain a flat connection D on $\mathcal{W}(X)$. Fedosov equation has the geometric interpretation of BV quantum master equation [29, 31].

Let $\mathcal{W}_D(X) := \{\sigma \in \Gamma(X, \mathcal{W}(X)) \mid D\sigma = 0\}$ be the space of flat sections. Then $(\mathcal{W}_D(X), \star)$ is an associative algebra. Let

$$\sigma : \mathcal{W}_D(X) \to C^\infty(X)[[\hbar]]$$

be the symbol map by sending $y \mapsto 0$. Then σ is an isomorphism, and

$$f \star g \mapsto \sigma\left(\sigma^{-1}(f) \star \sigma^{-1}(g)\right)$$

defines a deformation quantization. $\omega_\hbar$ is the corresponding characteristic class (or moduli).

3.3 Algebraic Index Theorem

Given a deformation quantization $(C^\infty(X)[[\hbar]], \star)$ on a symplectic manifold with characteristic class $\omega_\hbar$, there exists a unique **trace map**

$$\mathrm{Tr} : C^\infty(X)[[\hbar]] \to \mathbb{R}((\hbar))$$

satisfying a normalization condition and the trace property:

$$\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f).$$

Then the index is obtained as the partition function of the theory, which can be formulated as

$$\text{Index} = \mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar} \widehat{A}(X),$$

where $\widehat{A}(X)$ is the *(formal) $\widehat{A}$-genus* of X. This is the simplest version of **algebraic index theorem** formulated by Fedosov [22] and Nest-Tsygan [42] as the algebraic analogue of Atiyah-Singer index theorem.

In the case of a vector bundle E on X, one can similarly construct a *deformation quantization* for $C^\infty(X, \mathrm{End}(E))[[\hbar]]$ and construct the trace map. In this case

$$\mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar}\, \mathrm{Ch}(E)\widehat{A}(X),$$

where $\mathrm{Ch}(E)$ is the Chern character of the vector bundle E over X.

Relation with QFT

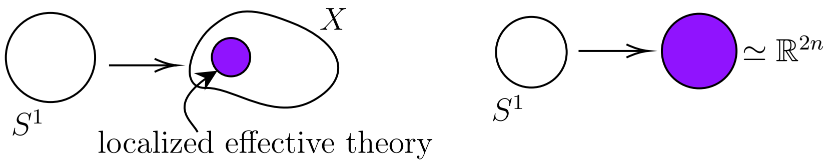
In supersymmetric (SUSY) QFT, *localization* often appears, in which the path integral on $\mathcal{E}$ is often localized effectively to an equivalent integral on a finite-dimensional space $M \subset \mathcal{E}$ describing some interesting moduli space:

$$\int_{\mathcal{E}} e^{iS/\hbar} = \int_M (-)\,.$$

In topological QM, we find

$$\int_{\mathrm{Map}(S^1,X)} e^{-S/\hbar} \overset{\hbar\to 0}{=} \int_X (-)\,,$$

where $\mathrm{Map}(S^1, X)$ is a loop space, and $\int_X$ indicates the localization to a *constant map*. The path integral will be captured exactly by an effective theory in the formal neighborhood of constant maps inside the full mapping space. Exact semi-classical approximation in $\hbar \to 0$ allows us to reduce the path integral into a meaningful integral on the moduli space of constant maps, i.e., X. The left-hand side usually gives a physics presentation of the analytic index of certain elliptic operator; the right-hand side will end up with integrals of various curvature forms representing the topological index.



Geometrically, a loop space is mapped to a localized neighborhood of a point in X (specified by the constant map), where a localized effective theory exists. Locally, by the choice of Darboux coordinates, X can be thought of as a standard phase space, $\mathbb{R}^{2n}$. The loop spaces are then glued together on X as a family of effective field theory. This can be done rigorously within the framework of effective BV quantization [31]: we find the following dictionary

- Effective action $\rightsquigarrow \gamma$,
- Quantum master equation $\rightsquigarrow$ Fedosov equation,
- BV integral $\rightsquigarrow$ trace map,
- Partition function $\rightsquigarrow$ algebraic index.

3.4 Local Theory

In this section we study topological quantum mechanics in terms of the effective renormalization method, and explain how to use it to prove the algebraic index theorem. We follow the presentation in [29, 31].

Local model

Let us consider the standard phase space (V, ω), where $V \simeq \mathbb{R}^{2n}$ with coordinates

$$(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{2n}) = (q^1, \cdots, q^n, p_1, \cdots, p_n)$$

and

$$\omega = \sum_{i=1}^n dp_i \wedge dq^i.$$

Let S^1_{dR} be the (locally ringed) space with underlying topology of the circle S^1 and the structure sheaf $\mathscr{O}(S^1_{dR}) = \Omega^\bullet_{S^1}$, which is a differential graded ring of differential forms with the de Rham differential operator d. Consider the *local model* describing the space of maps

$$\varphi : S^1_{dR} \to V \simeq \mathbb{R}^{2n}.$$

Such a φ can be identified with an element in $\Omega^\bullet_{S^1} \otimes V$. Explicitly, let θ be the coordinates on S^1 (with the identification $\theta \sim \theta + 1$). The space of maps can then be written as

$$\{\varphi\} = \left\{\mathbb{P}_i(\theta), \mathbb{Q}^i(\theta)\right\}_{i=1,\cdots,n}, \quad \mathbb{P}_i, \mathbb{Q}^i \in \Omega^\bullet_{S^1}.$$

Writing in form component,

$$\mathbb{P}_i(\theta) = p_i(\theta) + \eta_i(\theta)d\theta, \quad \mathbb{Q}^i(\theta) = q^i(\theta) + \xi^i(\theta)d\theta.$$

So the space of fields is

$$\mathcal{E} = \Omega^\bullet_{S^1} \otimes V.$$

The triple $\left(\Omega^\bullet_{S^1} \otimes V, d, \int_{S^1} \langle -, - \rangle_\omega\right)$ is an ∞-dimensional (-1)-dg symplectic space. The topological action is the free one:

$$\begin{aligned} S[\varphi] &:= \int_{S^1} \langle \varphi, d\varphi \rangle_\omega \\ &= \sum_i \int_{S^1} \mathbb{P}_i d\mathbb{Q}^i = \sum_i \int_{S^1} p_i(\theta) dq^i(\theta). \end{aligned}$$

Remark 3.8. This is the first-order formalism of topological quantum mechanics along the line of the *AKSZ* construction [1].

Propagator

Let us choose the standard flat metric on S^1. Let d^* be the adjoint of d. The Laplacian is

$$dd^* + d^*d = -\left(\frac{d}{d\theta}\right)^2.$$

Let

$$\Pi = \omega^{-1} = \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} = \frac{1}{2} \sum_i \left(\frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q^i} - \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_i}\right) \in \bigwedge^2 V$$

be the Poisson bivector (or Poisson kernel). Let

$$h_t(\theta_1, \theta_2) = \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} e^{-\frac{(\theta_1 - \theta_2 + n)^2}{4t}}$$

be the standard heat kernel on S^1. Then the regularized propagator is

$$P^L_\varepsilon = \int_\varepsilon^L \partial_{\theta_1} h_t(\theta_1, \theta_2)\, dt \otimes \Pi \;\in \mathcal{E} \otimes \mathcal{E},$$

where $\int_\varepsilon^L \partial_{\theta_1} h_t(\theta_1, \theta_2)\, dt \in C^\infty(S^1 \times S^1)$ and $\Pi \in V \otimes V$. Let us denote

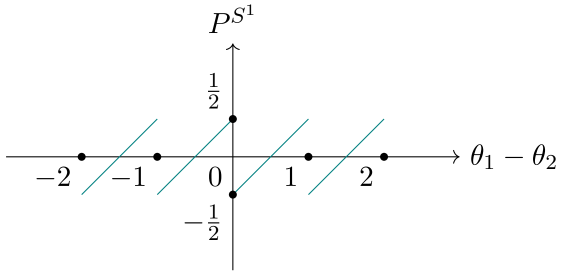
$$P^{S^1}(\theta_1, \theta_2) = \int_0^\infty \partial_{\theta_1} h_t(\theta_1, \theta_2)\, dt.$$

Then the full propagator is given by

$$P^\infty_0 = P^{S^1} \otimes \Pi.$$

Proposition 3.9. *$P^{S^1}(\theta_1, \theta_2)$ is the following periodic function of $\theta_1 - \theta_2 \in \mathbb{R}/\mathbb{Z}$*

$$P^{S^1}(\theta_1, \theta_2) = \theta_1 - \theta_2 - \frac{1}{2} \quad \text{if} \quad 0 < \theta_1 - \theta_2 < 1.$$



P^{S^1} is NOT a smooth function on $S^1 \times S^1$ (as expected), but it is bounded.

Correlation map

Let us denote the formal Weyl algebra

$$\mathcal{W}_{2n} = \left(\mathbb{R}[[p_i, q^i]]((\hbar)), \star\right) ,$$

and the formal Weyl subalgebra

$$\mathcal{W}^+_{2n} = \left(\mathbb{R}[[p_i, q^i]][[\hbar]], \star\right) ,$$

where $\star$ is the Moyal product. We can identify the formal Weyl subalgebra as (formal) functions on V (via deformation quantization):

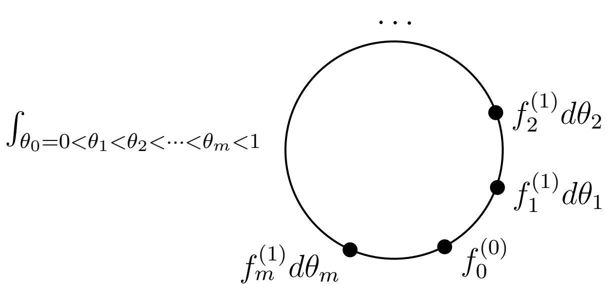
$$\mathcal{W}^+_{2n} \simeq \left(\widehat{\mathscr{O}}(V)[[\hbar]], \star\right).$$

Given $f_0, f_1, \cdots, f_m \in \mathcal{W}_{2n}$, we define $\mathcal{O}_{f_0, f_1, \cdots, f_m} \in \mathscr{O}(\mathcal{E})((\hbar))$ by

$$\mathcal{O}_{f_0, f_1, \cdots, f_m}[\varphi] := \int_{0<\theta_1<\theta_2<\cdots<\theta_m<1} d\theta_1 d\theta_2 \cdots d\theta_m f_0^{(0)}(\varphi(\theta_0))\, f_1^{(1)}(\varphi(\theta_1)) \cdots f_m^{(1)}(\varphi(\theta_m))\,.$$

Here $\varphi \in \Omega^\bullet_{S^1} \otimes V$, $f(\varphi(\theta)) = f(\mathbb{P}_i(\theta), \mathbb{Q}^i(\theta)) \in \Omega^\bullet_{S^1}$, and we decompose it as

$$f(\varphi(\theta)) = f^{(0)}(\varphi(\theta)) + f^{(1)}(\varphi(\theta)) d\theta.$$



Remark 3.10. $f^{(1)}(\varphi)$ is the topological descent of $f^{(0)}(\varphi)$ in topological field theory.

Now let us apply the HRG flow

$$\exp\left(\hbar P_0^\infty\right)\left(\mathcal{O}_{f_0,f_1,\cdots,f_m}\right).$$

Since P_0^∞ is bounded, it is convergent and well-defined! This is the *UV finite property*. As we have discussed, at $L = \infty$, we can view it as defining a function on zero modes

$$\mathbb{H} = H^\bullet\left(\Omega^\bullet_{S^1} \otimes V, d\right) = H^\bullet(S^1) \otimes V = V \oplus V d\theta.$$

On zero modes we have $\widehat{\mathscr{O}}(\mathbb{H}) = \widehat{\Omega}^{-\bullet}_{2n}$ forms on V.

Definition 3.11. We define the following correlation map:

$$\langle \cdots \rangle_{free} : \mathcal{W}_{2n} \otimes \cdots \otimes \mathcal{W}_{2n} \to \widehat{\Omega}^{-\bullet}_{2n}((\hbar))$$

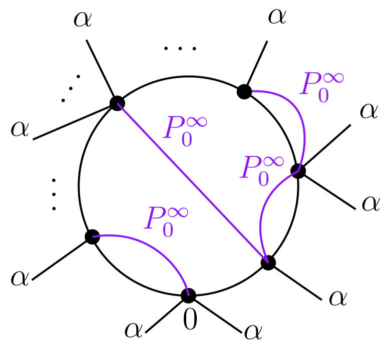
by

$$\langle f_0 \otimes f_1 \otimes \cdots \otimes f_m \rangle_{free} := \exp\left(\hbar P_0^\infty\right)\left(\mathcal{O}_{f_0,f_1,\cdots,f_m}\right)|_{\mathbb{H}}.$$

In the path integral perspective, this is

$$\langle f_0 \otimes f_1 \otimes \cdots \otimes f_m \rangle_{free}(\alpha) = \int_{\mathrm{Im}\, d^* \subset \mathcal{E}} [D\varphi]\, e^{-S[\varphi+\alpha]/\hbar} \mathcal{O}_{f_0,f_1,\cdots,f_m}[\varphi+\alpha], \quad \alpha \in \mathbb{H} = H^\bullet(S^1) \otimes V.$$

Here the zero mode α is viewed as the background field. It can also be represented as a Feynman diagram as follows.



Remark 3.12. See [38] for a probabilistic approach where the topological correlations above are constructed in terms of a large variance limit.

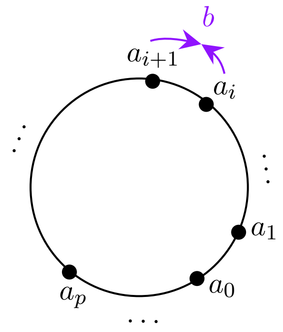
(Cyclic) Hochschild complex reviewed

Let A be a unital associative algebra and $\overline{A} := A/(\mathbb{C} \cdot 1)$. Let $C_{-p}(A) := A \otimes \overline{A}^{\otimes p}$ be the cyclic p-chains. It carries a natural Hochschild differential

$$b : C_{-p}(A) \to C_{-p+1}(A), \quad p \geq 1$$

by

$$b(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p.$$



Then the associativity implies $b \circ b = 0$. Thus, $(C_{-\bullet}(A), b)$ defines the Hochschild chain complex. We can also define the Connes operator:

$$B : C_{-p}(A) \to C_{-p-1}(A)$$

by

$$B(a_0 \otimes \cdots \otimes a_p) = 1 \otimes a_0 \otimes \cdots \otimes a_p + \sum_{i=1}^{p} (-1)^{pi} 1 \otimes a_i \otimes \cdots \otimes a_p \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

We have the following relations:

$$b^2 = 0, \quad B^2 = 0, \quad [b, B] = bB + Bb = 0.$$

Let u be a formal variable of deg $= 2$. Then $(b + uB)^2 = 0$. This defines a complex

$$CC_{-\bullet}^{per}(A) = \left(C_{-\bullet}(A)[u, u^{-1}], b + uB\right),$$

called the periodic cyclic complex. For a systematic reference, see Loday [41].

Back to Correlation map

It is not hard to see that

$$\langle \cdots \rangle_{free} : C_{-p}(\mathcal{W}_{2n}) \to \widehat{\Omega}_{2n}^{-p}((\hbar)),$$

i.e. $\langle f_0 \otimes f_1 \otimes \cdots \otimes f_p \rangle_{free}$ is a p-form. Recall that $\widehat{\Omega}_{2n}^{-\bullet}$ is equipped with a BV operator $\Delta = \mathcal{L}_{\omega^{-1}} = \mathcal{L}_{\Pi}$, the Lie derivative with respect to the Poisson bi-vector.

Proposition 3.13 ([31])**.**

$$\langle b(-) \rangle_{free} = \hbar \Delta \langle \cdots \rangle_{free},$$
$$\langle B(-) \rangle_{free} = d_{2n} \langle \cdots \rangle_{free}.$$

Here $d_{2n} : \widehat{\Omega}_{2n}^{-\bullet} \to \widehat{\Omega}_{2n}^{-(\bullet+1)}$ *is the de Rham differential.*

In other words, the correlation map:

$$\langle \cdots \rangle_{free} : C_{-\bullet}(\mathcal{W}_{2n}) \to \widehat{\Omega}_{2n}^{-\bullet}((\hbar))$$

intertwines b with $\hbar\Delta$ and B with d_{2n}. We can combine the above two to get

$$\langle \cdots \rangle_{free} : CC_{-\bullet}^{per}(\mathcal{W}_{2n}) \to \widehat{\Omega}_{2n}^{-\bullet}((\hbar))[u, u^{-1}]$$

which intertwines $b + uB$ with $\hbar\Delta + ud_{2n}$.

BV integral on zero modes

We can define a *BV integration* map on the BV algebra $\left(\widehat{\Omega}_{2n}^{-\bullet}, \Delta\right)$ which is only non-zero on top forms $\widehat{\Omega}_{2n}^{-2n}$ and sends

$$\beta \in \widehat{\Omega}_{2n}^{-2n} \mapsto \left.\frac{\hbar^n}{n!}\iota_\Pi^n \beta\right|_{p=q=0}.$$

This is the Berezin integral [7] over the purely fermionic super Lagrangian. We can extend this BV integration to an S^1-equivariant version by

$$\int_{BV} : \widehat{\Omega}_{2n}^{-\bullet}[u, u^{-1}] \to \mathbb{R}((\hbar))[u, u^{-1}], \quad \beta \mapsto \left.\left(u^n e^{\hbar\iota_\Pi/u}\beta\right)\right|_{p=q=0}.$$

Then it has the following property

$$\int_{BV} (\hbar\Delta + ud_{2n})(-) = 0.$$

Remark 3.14. For $\beta \in \widehat{\Omega}_{2n}^{-\bullet}$, the non-equivariant limit

$$\lim_{u\to 0}\int_{BV} \beta = \left.\frac{\hbar^n}{n!}\iota_\Pi^n \beta\right|_{p=q=0}$$

gives back the Berezin integral.

Combining the above maps, we define

$$\mathrm{Tr} := \int_{BV} \circ \langle\cdots\rangle_{free} : CC_{-\bullet}^{per}(\mathcal{W}_{2n}) \to \mathbb{R}((\hbar))[u, u^{-1}]$$

which satisfies the following equation:

$$\mathrm{Tr}\,((b + uB)(-)) = 0.$$

Therefore Tr descends to periodic cyclic homology. This essentially leads to the trace formula in Feigin-Felder-Shoikhet [23].

Quantum Master Equation

We can generalize slightly by considering a graded vector space V with a deg $= 0$ symplectic pairing ω. We still have the canonical quantization $\left(\widehat{\mathscr{O}}(V)[[\hbar]], \star\right)$ by Moayl product and similarly can define the BV algebra of forms

$$\left(\widehat{\Omega}_V^{-\bullet}, \Delta = \mathcal{L}_{\omega^{-1}}\right).$$

The same trace map gives

$$\langle\cdots\rangle_{free} : C_{-\bullet}\left(\widehat{\mathscr{O}}(V)[[\hbar]]\right) \to \widehat{\Omega}_V^{-\bullet}((\hbar)), \quad b \mapsto \hbar\Delta.$$

Given $\gamma \in \widehat{\mathscr{O}}(V)[[\hbar]]$, $\deg(\gamma) = 1$, it defines an action functional:

$$I_\gamma = \int_{S^1} \gamma(\varphi) \quad \forall \varphi \in \Omega^\bullet(S^1) \otimes V.$$

Let us treat I_γ as an interaction and consider

$$\underbrace{\frac{1}{2}\int_{S^1}\langle\varphi, d\varphi\rangle}_{\text{free part}} + \underbrace{\int_{S^1}\gamma(\varphi)}_{I_\gamma}.$$

Then we run the HRG flow to get

$$e^{\frac{1}{\hbar}I_\gamma[\infty]} := e^{\hbar\partial_{P_0^\infty}} e^{\frac{1}{\hbar}I_\gamma}$$

which is well-defined since P_0^∞ is bounded.

Let us now analyze the QME. By construction,

$$e^{\frac{1}{\hbar}I_\gamma[\infty]} = \left\langle 1\otimes e^{\gamma/\hbar}\right\rangle_{free}.$$

Assume $\gamma\star\gamma = \frac{1}{2}[\gamma,\gamma]_\star = 0$. Then

$$\hbar\Delta e^{\frac{1}{\hbar}I_\gamma[\infty]} = \left\langle b\left(1\otimes e^{\gamma/\hbar}\right)\right\rangle_{free} = 0.$$

Proposition 3.15 ([29]). *If $[\gamma,\gamma]_\star = 0$, then the local interaction $I_\gamma = \int_{S^1}\gamma(\varphi)$ defines a family of solutions of effective QME $I_\gamma[L]$ at scale $L > 0$ by*

$$e^{\frac{1}{\hbar}I_\gamma[L]} := \lim_{\varepsilon\to\infty} e^{\hbar\partial_{P_\varepsilon^L}} e^{\frac{1}{\hbar}I_\gamma}.$$

3.5 Global Theory

Recall in Section 3.4, we have discussed the first-order formalism of TQM such that in a local model with maps $\varphi : \Omega^\bullet_{S^1} \to V \simeq \mathbb{R}^{2n}$, the correlation map

$$\langle\cdots\rangle_{free} : C_{-\bullet}(\mathcal{W}_{2n}) \to \widehat{\Omega}^{-\bullet}_{2n}((\hbar))$$

intertwines b with $\hbar\Delta$ and B with d_{2n}.

In this section, we are going to glue this construction to a symplectic manifold and establish the algebraic index to universal Lie algebra cohomology computations. The basic idea is to *glue* the local model $\Sigma \to T^{Model} \subset X$. In the following discussion, we borrow the presentation in [31], where extensive references are given for related material.

Gluing via Gelfand-Kazhdan formal geometry

Definition 3.16. A **Harish-Chandra pair** is a pair $(\mathfrak{g}, K)$, where $\mathfrak{g}$ is a Lie algebra, K is a Lie group, with

- an action of K on $\mathfrak{g}$: $K \xrightarrow{\rho} \mathrm{Aut}(\mathfrak{g})$,
- a natural embedding: $\mathrm{Lie}(K) \overset{i}{\hookrightarrow} \mathfrak{g}$, where $\mathrm{Lie}(K)$ is the Lie algebra associated with K,

such that they are compatible:

$$\begin{tikzcd} \mathrm{Lie}(K) \arrow[r, hook, "i"] \arrow[dr, "d\rho"'] & \mathfrak{g} \arrow[d, "adjoint"] \\ & \mathrm{Der}(\mathfrak{g}) \end{tikzcd}$$

Definition 3.17. A **$(\mathfrak{g}, K)$-module** is a vector space V with

- an action of K on V: $K \xrightarrow{\varphi} \mathrm{GL}(V)$,
- a Lie algebra morphism: $\mathfrak{g} \to \mathrm{End}(V)$,

such that they are compatible:

$$\begin{tikzcd}
\mathrm{Lie}(K) \arrow[r, hook, "i"] \arrow[dr, "d\varphi"'] & \mathfrak{g} \arrow[d] \\
& \mathrm{End}(V)
\end{tikzcd}$$

Definition 3.18. A **flat $(\mathfrak{g}, K)$-bundle** over X is

- a principal K-bundle $P \xrightarrow{\pi} X$,
- a K-equivariant $\mathfrak{g}$-valued 1-form $\gamma \in \Omega^1(P, \mathfrak{g})$ on P,

satisfying the following conditions:

(1) $\forall a \in \mathrm{Lie}(K)$, let $\xi_a \in \mathrm{Vect}(P)$ generated by a. Then we have the contraction $\gamma(\xi_a) = a$ such that

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Lie}(K) \arrow[r] \arrow[dr, hook, "i"'] & \mathrm{Vect}(P) \arrow[d, "\gamma"] \\
& & \mathfrak{g}
\end{tikzcd}$$

(2) γ satisfies the Maurer-Cartan equation

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0,$$

where d is the de Rham differential on P, and $[-,-]$ is the Lie bracket in $\mathfrak{g}$.

Given a flat $(\mathfrak{g}, K)$-bundle $P \to X$ and $(\mathfrak{g}, K)$-module V, let

$$\Omega^\bullet(P, V) := \Omega^\bullet(P) \otimes V$$

denote differential forms on P valued in V. It carries a connection

$$\nabla^\gamma = d + \gamma : \Omega^\bullet(P, V) \to \Omega^{\bullet+1}(P, V)$$

which is *flat* by the Maurer-Cartan equation. The group K acts on $\Omega^\bullet(P)$ and V, and hence inducing a natural action on $\Omega^\bullet(P, V)$. Let

$$V_P := P \times_K V$$

be the vector bundle on X associated to the K-representation V. Let

$$\Omega^\bullet(X; V_P)$$

be differential forms on X valued in the bundle $V_P \to X$. Similar to the usual principal bundle case, ∇^γ induces a flat connection on $V_P \to X$. This defines a (de Rham) chain complex $(\Omega^\bullet(X; V_P), \nabla^\gamma)$, and $H^\bullet(X; V_P)$ denotes the corresponding de Rham cohomology.

We can descend Lie algebra cohomologies to geometric objects on X.

Definition 3.19. Let V be a $(\mathfrak{g}, K)$-module. Define the $(\mathfrak{g}, K)$ **relative Lie algebra cochain complex** $(C^{\bullet}_{\mathrm{Lie}}(\mathfrak{g}, K; V), \partial_{Lie})$ by

$$C^p_{\mathrm{Lie}}(\mathfrak{g}, K; V) = \mathrm{Hom}_K \left(\textstyle\bigwedge^p \left(\mathfrak{g}/\,\mathrm{Lie}(K)\right), V\right).$$

Here Hom_K means K-equivariant linear maps. ∂_{Lie} is the Chevalley-Eilenberg differential if we view $C^p_{\mathrm{Lie}}(\mathfrak{g}, K; V)$ as a subspace of the Lie algebra cochain $C^p_{\mathrm{Lie}}(\mathfrak{g}; V)$. Explicitly, for $\alpha \in C^p_{\mathrm{Lie}}(\mathfrak{g}, K; V)$,

$$\begin{aligned}(\partial_{\mathrm{Lie}}\alpha)(a_1 \wedge \cdots \wedge a_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i-1} a_i \cdot \alpha\left(a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_{p+1}\right) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha\left([a_i, a_j] \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_{p+1}\right).\end{aligned}$$

The corresponding cohomology is $H^{\bullet}_{\mathrm{Lie}}(\mathfrak{g}, K; V)$.

Given a $(\mathfrak{g}, K)$-module V and flat $(\mathfrak{g}, K)$-bundle $P \to X$ with the flat connection $\gamma \in \Omega^1(P, \mathfrak{g})$. We can define the **descent map** from the $(\mathfrak{g}, K)$ relative Lie algebra cochain complex to V-valued de Rham complex on P by

$$\mathrm{desc} : (C^{\bullet}_{Lie}(\mathfrak{g}, K; V), \partial_{Lie}) \to (\Omega^{\bullet}(X; V_P), \nabla^{\gamma}), \quad \alpha \mapsto \alpha(\gamma, \cdots, \gamma)$$

inducing the cohomology descent map

$$\mathrm{desc} : H^{\bullet}_{Lie}(\mathfrak{g}, K; V) \to H^{\bullet}(X; V_P).$$

Fedosov connection revisited

Recall the (formal) Weyl algebras

$$\mathcal{W}_{2n} = \mathbb{R}[[p_i, q^i]]((\hbar)), \quad \mathcal{W}^+_{2n} = \mathbb{R}[[p_i, q^i]][[\hbar]]$$

with the induced Lie algebra structure such that the Lie bracket is defined by

$$[f, g] \coloneqq \frac{1}{\hbar}[f, g]_{\star} = \frac{1}{\hbar}(f \star g - g \star f).$$

Let Sp_{2n} be the symplectic group of linear transformations preserving the Poisson bivector Π. It acts on Weyl algebras by inner automorphisms. We can identify the Lie algebra $\mathfrak{sp}_{2n}$ of Sp_{2n} with quadratic polynomials in $\mathbb{R}[p_i, q^i]$, and $\mathfrak{sp}_{2n}$ is a Lie subalgebra of $\mathcal{W}^+_{2n}$. The action $\mathrm{Sp}_{2n} \curvearrowright \mathbb{R}^{2n}$ induces the action $\mathrm{Sp}_{2n} \curvearrowright \mathcal{W}^+_{2n}$. Hence, $\left(\mathcal{W}^+_{2n}, \mathrm{Sp}_{2n}\right)$ and $(\mathcal{W}_{2n}, \mathrm{Sp}_{2n})$ are Harish-Chandra pairs.

Let (X, ω) be a symplectic manifold, and $F_{\mathrm{Sp}}(X)$ be the symplectic frame bundle. We have the Weyl bundles

$$\mathcal{W}^+_X = F_{\mathrm{Sp}}(X) \times_{\mathrm{Sp}_{2n}} \mathcal{W}^+_{2n}, \quad \mathcal{W}_X = F_{\mathrm{Sp}}(X) \times_{\mathrm{Sp}_{2n}} \mathcal{W}_{2n}.$$

Consider the Harish-Chandra pair

$$(\bar{\mathfrak{g}}, K) = (\mathfrak{g}/Z(\mathfrak{g}), \mathrm{Sp}_{2n}),$$

where $\mathfrak{g} = \mathcal{W}^+_{2n}$, and $Z(\mathfrak{g}) = \mathbb{R}[[\hbar]]$ is the center of $\mathfrak{g}$, $Z(\mathfrak{g}) \cap \mathfrak{sp}_{2n} = 0$. Fedosov constructed a flat $(\bar{\mathfrak{g}}, K)$-bundle $F_{\mathrm{Sp}}(X) \to X$ and $H^0(X; \mathcal{W}^+_X)$ gives a *deformation quantization*. Choose the trivial $(\bar{\mathfrak{g}}, K)$-module $\mathbb{R}((\hbar))$. Then

$$\mathrm{desc} : C^{\bullet}_{\mathrm{Lie}}\left(\overline{\mathcal{W}^+_{2n}}, \mathfrak{sp}_{2n}; \mathbb{R}((\hbar))\right) \to \Omega^{\bullet}_X((\hbar)).$$

This is the **Gelfand-Fuks map**. Here

$$C^{\bullet}_{\mathrm{Lie}}\left(\overline{\mathcal{W}^+_{2n}}, \mathfrak{sp}_{2n}; \mathbb{R}((\hbar))\right) \simeq C^{\bullet}_{Lie}\left(\mathcal{W}^+_{2n}, \mathfrak{sp}_{2n} \oplus Z(\mathcal{W}^+_{2n}); \mathbb{R}((\hbar))\right).$$

Characteristic classes

Let us review the Chern-Weil construction of characteristic classes in Lie algebra cohomology. They will descend to the usual characteristic forms via the Gelfand-Fuks map.

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ be its Lie subalgebra. Let the projection map

$$\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$$

be the $\mathfrak{h}$-equivariant splitting of the embedding $\mathfrak{h} \subset \mathfrak{g}$. In general pr is not a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$. The *failure* of pr being a Lie algebra homomorphism gives $R \in \mathrm{Hom}\left(\bigwedge^2 \mathfrak{g}, \mathfrak{h}\right)$ by

$$R(\alpha, \beta) = [\mathrm{pr}(\alpha), \mathrm{pr}(\beta)]_{\mathfrak{h}} - \mathrm{pr}\,[\alpha, \beta]_{\mathfrak{g}}\,, \quad \alpha, \beta \in \mathfrak{g}.$$

The $\mathfrak{h}$-equivariance of pr implies that $R \in \mathrm{Hom}_{\mathfrak{h}}\left(\bigwedge^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}\right)$. R is called the **curvature form**. Let $\mathrm{Sym}^m(\mathfrak{h}^\vee)^{\mathfrak{h}}$ be $\mathfrak{h}$-invariant polynomials on $\mathfrak{h}$ of homogeneous degree deg $= m$. Given $P \in \mathrm{Sym}^m(\mathfrak{h}^\vee)^{\mathfrak{h}}$, we can associate a cochain

$$P(R) \in C^{2m}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$$

by the composition

$$P(R) : \bigwedge^{2m} \mathfrak{g} \xrightarrow{\bigwedge^m R} \mathrm{Sym}^m(\mathfrak{h}) \xrightarrow{P} \mathbb{R}.$$

It can be checked that $\partial_{\mathrm{Lie}} P(R) = 0$, defining a cohomology class

$$[P(R)] \in H^{2m}(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$$

which does not depend on the choice of pr. Therefore we have the analogue of Chern-Weil characteristic map

$$\chi : \mathrm{Sym}^\bullet(\mathfrak{h}^\vee)^{\mathfrak{h}} \to H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}), \quad P \mapsto \chi(P) := [P(R)]\,.$$

Now we apply the above construction to the case where

$$\mathfrak{g} = \mathcal{W}^+_{2n}, \quad \mathfrak{h} = \mathfrak{sp}_{2n} \oplus Z(\mathfrak{g}).$$

Any element f in $\mathfrak{g} = \mathcal{W}^+_{2n}$ can be uniquely written as a polynomial $f = f(y^i, \hbar)$, with coordinates $(y^1, \cdots, y^n, y^{n+1}, \cdots, y^{2n}) = (p_1, \cdots, p_n, q^1, \cdots, q^n)$. Define the $\mathfrak{h}$-equivariant projections

$$\begin{aligned} \mathrm{pr}_1(f) &= \frac{1}{2} \sum_{i,j} \partial_i \partial_j f \Big|_{y=\hbar=0} y^i y^j \in \mathfrak{sp}_{2n}, \\ \mathrm{pr}_3(f) &= f|_{y=0} \in Z(\mathfrak{g}). \end{aligned}$$

We obtain the corresponding curvature

$$\begin{aligned} R_1 &:= [\mathrm{pr}_1(-), \mathrm{pr}_1(-)] - \mathrm{pr}_1\,[-,-] \ \in \mathrm{Hom}(\textstyle\bigwedge^2 \mathfrak{g}, \mathfrak{sp}_{2n}), \\ R_3 &:= -\,\mathrm{pr}_3\,[-,-] \ \in \mathrm{Hom}(\textstyle\bigwedge^2 \mathfrak{g}, \mathbb{R}[[\hbar]]). \end{aligned}$$

Remark 3.20. A more general case can be considered when we incorporate vector bundles, where $\mathfrak{g} = \mathcal{W}^+_{2n} + \hbar\left(\mathfrak{gl}\left(\mathcal{W}^+_{2n}\right)\right)$, $\mathfrak{h} = \mathfrak{sp}_{2n} \oplus \hbar\mathfrak{gl} \oplus Z(\mathfrak{g})$. There the extra projection pr_2 and its corresponding curvature R_2 are defined as elements in $\hbar\mathfrak{gl}$ and $\mathrm{Hom}\left(\bigwedge^2, \mathfrak{gl}\right)$, respectively. It is worthwhile to point out that all the Hom's here are only $\mathbb{R}$-linear map, but not $\mathbb{R}[[\hbar]]$-linear, although $\mathfrak{g}$ is a $\mathbb{R}[[\hbar]]$-module.

We now define the $\widehat{A}$**-genus**

$$\widehat{A}(\mathfrak{sp}_{2n}) := \left[\det\left(\frac{R_1/2}{\sinh(R_1/2)}\right)^{\frac{1}{2}}\right] \in H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}).$$

Under the descent map desc : $H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}((\hbar))) \to H^\bullet(X)((\hbar))$ via the Fedosov connection, it can be shown that

$$\operatorname{desc}\left(\widehat{A}(\mathfrak{sp}_{2n})\right) = \widehat{A}(X),$$
$$\operatorname{desc}(R_3) = \omega_\hbar - \hbar\omega.$$

Universal trace map

Recall that using $\Omega^\bullet_{S^1} \to \mathbb{R}^{2n}$, we have obtained

$$\operatorname{Tr} = \int_{BV} \circ \langle - \rangle_{free} : CC^{per}_{-\bullet}(\mathcal{W}_{2n}) \to \mathbb{K} := \mathbb{R}((\hbar))[u, u^{-1}].$$

Let us write

$$\operatorname{Tr} \in \operatorname{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right).$$

This is a $\left(\mathcal{W}^+_{2n}, \mathrm{Sp}_{2n}\right)$-module. Via the flat $\left(\mathcal{W}^+_{2n}, \mathrm{Sp}_{2n}\right)$-bundle $F_{\mathrm{Sp}}(X) \to X$, we obtain the associated bundle

$$E^{per} := F_{\mathrm{Sp}}(X) \times_{\mathrm{Sp}_{2n}} \operatorname{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right)$$

with induced flat connection ∇^γ.

Recall the Weyl bundle $\mathcal{W}(X) = F_{\mathrm{Sp}}(X) \times_{\mathrm{Sp}_{2n}} \mathcal{W}_{2n}$ with flat connection ∇^γ. We would like to glue Tr on X. Let us denote δ for the differential on $\operatorname{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right)$ induced from $b + uB$. So

$$\delta \operatorname{Tr} = \operatorname{Tr}((b + uB)(-)) = 0.$$

We can view Tr as defining an element in

$$C^0_{Lie}\left(\mathfrak{g}, \mathfrak{h}; \operatorname{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right)\right),$$

where we take

$$\mathfrak{g} = \mathcal{W}^+_{2n}/Z(\mathcal{W}^+_{2n}), \quad \mathfrak{h} = \mathfrak{sp}_{2n}.$$

However, Tr is NOT $\mathfrak{g}$-invariant, i.e. $\partial_{\mathrm{Lie}} \operatorname{Tr} \neq 0$. In other words, Tr is NOT a map of $(\mathfrak{g}, \mathrm{Sp}_{2n})$-module. So Tr can not be glued directly.

It is observed that $\partial_{\mathrm{Lie}} \operatorname{Tr} = \delta(-)$. It turns out that we have a canonical way to lift Tr to

$$\widehat{\operatorname{Tr}} \in C^\bullet_{Lie}\left(\mathfrak{g}, \mathfrak{h}; \operatorname{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right)\right)$$

such that

$$\widehat{\operatorname{Tr}} = \operatorname{Tr} + \text{terms in } C^{>0}_{Lie}\left(\mathfrak{g}, \mathfrak{h}; \operatorname{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right)\right)$$

and satisfying the coupled cocycle condition

$$(\partial_{Lie} + \delta)\widehat{\operatorname{Tr}} = 0.$$

$\widehat{\operatorname{Tr}}$ is called the **universal trace map**. Let us insert $1 \in \mathcal{W}_{2n}$, then $\widehat{\operatorname{Tr}}(1)$ is ∂_{Lie}-closed, which defines the **universal index**, $\left[\widehat{\operatorname{Tr}}(1)\right] \in H^\bullet_{Lie}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$.

Theorem 3.21 (Universal algebraic index theorem)**.**

$$\left[\widehat{\mathrm{Tr}}(1)\right] = u^n e^{-R_3/(u\hbar)}\widehat{A}(\mathfrak{sp}_{2n})_u,$$

where for $A = \sum_{p\ even} A_p$*,* $A_P \in H^p(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$*,*

$$A_u = \sum_p u^{-p/2} A_p.$$

This theorem is developed in the works of Feigin-Tsygan [24], Feigin-Felder-Shoikhet [23], Bressler-Nest-Tsygan [9], and many others. This can be naturally generalized to the bundle case [31] (as well as an explicit formula as a byproduct).

Now we apply the Gelfand-Fuks (descent) map on $\widehat{\mathrm{Tr}}$, such that

$$\begin{tikzcd} C^\bullet_{Lie}\left(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right)\right) \arrow[d, "\mathrm{desc}"] \\ \Omega^\bullet\left(X, \mathrm{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}(X)), \mathbb{K}\right)\right) \end{tikzcd}$$

Let $\mathcal{W}_D(X)$ be the space of flat sections of $\mathcal{W}(X)$ that gives a *deformation quantization.* Then

$$\mathrm{desc}(\widehat{\mathrm{Tr}}) : CC^{per}_{-\bullet}(\mathcal{W}_D(X)) \to \Omega^\bullet(X)((\hbar))[u, u^{-1}], \quad b + uB \mapsto d_X.$$

In particular, it defines a *trace map* in deformation quantization by

$$f \in \mathcal{W}_D(X) \mapsto \int_X \mathrm{desc}(\widehat{\mathrm{Tr}})(f) \in \mathbb{R}((\hbar)).$$

We can show that $\int_X \mathrm{desc}(\widehat{\mathrm{Tr}})(f)$ does not depend on u. By the *universal algebraic index theorem*, we have

$$\int_X \mathrm{desc}(\widehat{\mathrm{Tr}})(1) = \int_X e^{-\omega_\hbar/\hbar}\widehat{A}(X).$$

This gives the algebraic index theorem.

Construction of universal trace map $\widehat{\mathrm{Tr}}$

We have the following relations [31].

$$\begin{tikzcd} \boxed{\text{background symmetry}} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \boxed{\text{connection form}} \arrow[dl, leftrightarrow] \\ & \boxed{\text{interaction}} & \end{tikzcd}$$

Let $\Theta : \mathfrak{g} \to \mathcal{W}^+_{2n}/Z(\mathcal{W}^+_{2n}) = \mathfrak{g}$ be the canonical identity map. For each $f \in \mathcal{W}^+_{2n}/Z(\mathcal{W}^+_{2n})$, we have defined the local functional on $\mathcal{E} = \Omega^\bullet(S^1) \otimes \mathbb{R}^{2n}$ by

$$I_f(\varphi) = \int_{S^1} f(\varphi), \quad \varphi \in \mathcal{E}.$$

Then Θ gives a map

$$I_\Theta : \mathfrak{g} \to \mathcal{O}_{loc}(\mathcal{E}), \quad f \mapsto I_{\Theta(f)}.$$

We can view this map as

$$I_\Theta \in C^1(\mathfrak{g}, \mathcal{O}_{loc}(\mathcal{E})) = \mathfrak{g}^\vee \otimes \mathcal{O}_{loc}(\mathcal{E}).$$

This allows constructing $\widehat{\mathrm{Tr}} \in C^\bullet_{Lie}\left(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{K}}\left(CC^{per}_{-\bullet}(\mathcal{W}_{2n}), \mathbb{K}\right)\right)$ explicitly [31] by

$$\begin{aligned}\widehat{\mathrm{Tr}}\left(f_0 \otimes f_1 \otimes \cdots \otimes f_m\right) &:= \int_{BV} \exp\left(\hbar P_0^\infty\right)\left(\mathcal{O}_{f_0,f_1,\cdots,f_m} e^{\frac{1}{\hbar}I_\Theta}\right) \in C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K}), \quad f_i \in \mathcal{W}_{2n} \\ \text{``} &= \int_{BV}\int_{\mathrm{Im}\, d^* \subset \mathcal{E}} e^{-\frac{1}{2\hbar}\int_{S^1}\langle\varphi, d\varphi\rangle + \frac{1}{\hbar}I_\Theta}\mathcal{O}_{f_0,f_1,\cdots,f_m}\text{''}.\end{aligned}$$

Computation of index

The Weyl algebra $\mathcal{W}_{2n}$ can be viewed as a family of associative algebras parameterized by $\hbar$. This leads to the Gauss-Manin-Getzler connection [27] $\nabla_{\hbar\partial_\hbar}$ on $CC^{per}_{-\bullet}(\mathcal{W}_{2n})$. The calculation of index consists of the following steps [31]:

(1) *Feynman diagram computation* implies

$$\widehat{\mathrm{Tr}}(1) = u^n e^{-R_3/(u\hbar)}\left(\underbrace{\widehat{A}(\mathfrak{sp}_{2n})_u}_{1-\text{loop computation}} + \mathcal{O}(\hbar)\right).$$

(2) Computation of *Gauss-Manin-Getzler connection* shows

$$\nabla_{\hbar\partial_\hbar}\left(e^{R_3/(u\hbar)}\widehat{\mathrm{Tr}}(1)\right) \text{ is } \partial_{Lie}\text{-exact}.$$

(3) Combining (1) and (2), we find

$$\left[\widehat{\mathrm{Tr}}(1)\right] = \left[u^n e^{-R_3/(u\hbar)}\widehat{A}(\mathfrak{sp}_{2n})_u\right] \in H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K}).$$

4 Two-dimensional Chiral Theory

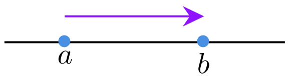
We have discussed the first-order formalism of topological QM, where the fields are differential forms $\Omega^\bullet(S^1, V)$ on S^1 valued in the vector bundle V with the de Rham differential d. Here d being part of the BRST operator implies that "translation is homologically trivial." This defines a topological theory.

We will now consider 2d chiral models where the fields are differential forms $\Omega^{0,\bullet}(\Sigma, h)$ with the Dolbeault differential $\bar\partial$. The Dolbeault differential being part of the BRST operator implies that "anti-holomorphic translation is homologically trivial," which in turn defines a chiral (or holomorphic) theory.

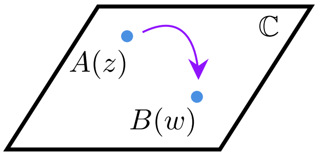
In topological QM, the theory is *UV finite*. The general consideration in Section 2.4 applies and we find that the renormalized QME is traded to Moyal commutator and Fedosov equation. We will see that 2d chiral theory is also *UV finite* and we have a similar geometric result for QME [37].

4.1 Vertex Algebra

As illustrated by the picture below, in 1d topological theory we have an associative algebra defined by the fusion of two operators $a \cdot b$; in 2d chiral theory we have (chiral) vertex algebra defined by $A_{(n)}B$. The algebras are found when one operator approaches another either on a line (for 1d) or on a plane (for 2d).



$a \cdot b$



$A(z)B(w) \sim \sum_n \frac{(A_{(n)}B)(w)}{(z-w)^{n+1}}$

On a plane, the "product" (binary operation) depends on the location holomorphically, leading to infinitely many binary operations.

Definition 4.1. A **vertex algebra** is a collection of data:

- (space of states) a $\mathbb{Z}$-graded superspace $\mathcal{V} = \mathcal{V}_{even} \oplus \mathcal{V}_{odd}$,
- (vacuum) a vector $|0\rangle \in \mathcal{V}_{even}$,
- (translation operator) an even linear map $T : \mathcal{V} \to \mathcal{V}$,
- (state-field correspondence) an even linear operation (vertex operation)

$$Y(-,z) : \mathcal{V} \to \mathrm{End}\, \mathcal{V}[[z,z^{-1}]], \quad A \mapsto Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$$

  such that $Y(A,z)B \in \mathcal{V}((z))$ for any $A, B \in \mathcal{V}$.

The data are required to satisfy the following axioms:

- (vacuum axiom) $Y(|0\rangle, z) = 1_{\mathcal{V}}$, i.e. for any $A \in \mathcal{V}$,

$$Y(A,z)|0\rangle \in \mathcal{V}[[z]] \quad \text{and} \quad \lim_{z\to 0} Y(A,z)|0\rangle = A,$$

- (translation axiom) $T|0\rangle = 0$, i.e. for any $A \in \mathcal{V}$,

$$[T, Y(A,z)] = \partial_z Y(A,z),$$

- (locality axiom) all $\{Y(A,z)\}_{a\in\mathcal{V}}$ are mutually local.

Roughly speaking, mutual locality implies for any $A, B \in \mathcal{V}$, we can expand as

$$Y(A,z)Y(B,w) = \sum_{n\in\mathbb{Z}} \frac{Y(A_{(n)} \cdot B, w)}{(z-w)^{n+1}}.$$

This is called the **operator product expansion (OPE)**. $\{A_{(n)} \cdot B\}$ from the expansion coefficient can be viewed as defining an infinite tower of products. For simplicity, we will write

$$A(z) \equiv Y(A,z) \quad \text{for } A \in \mathcal{V}.$$

Then the OPE can be written as

$$A(z)B(w) = \sum_{n\in\mathbb{Z}} \frac{A_{(n)} \cdot B(w)}{(z-w)^{n+1}}.$$

We also write, whenever only the *singular* parts matter,

$$A(z)B(w) \sim \sum_{n\geq 0} \frac{A_{(n)} \cdot B(w)}{(z-w)^{n+1}}.$$

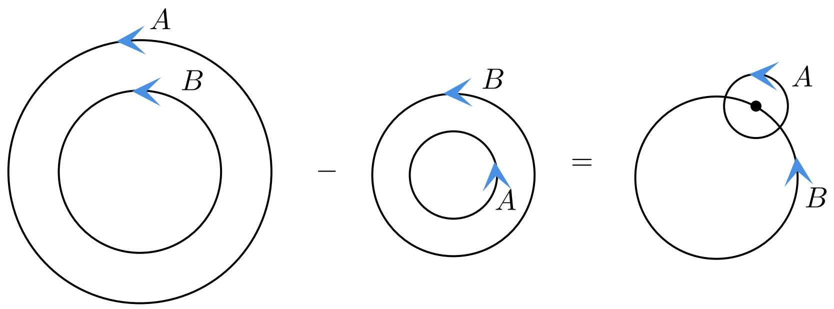
Given a vertex algebra, we can define its **modes Lie algebra**

$$\oint \mathcal{V} := \mathrm{Span}_{\mathbb{C}} \left\{ \oint dz\ z^k A(z) = A_{(k)} \right\}_{A \in \mathcal{V},\ k \in \mathbb{Z}}.$$

The Lie bracket of contour integrals is determined by the OPE,

$$\left[\oint dz\ z^m A(z),\ \oint dw\ w^n B(w) \right] = \oint dw\ w^n \oint_w dz\ z^m \sum_{j \in \mathbb{Z}} \frac{A_{(j)} \cdot B(w)}{(z-w)^{j+1}},$$

where only the singular part matters in the integration. The Lie bracket is represented diagrammatically as follows.



We refer to [32, 25] for a systematic discussion on vertex algebras.

Example 4.2 ($\beta\gamma$-system)**.** *The $\beta\gamma$-system is generated by two* bosonic *fields $\beta(z), \gamma(z)$ with OPE*

$$\beta(z)\gamma(w) \sim \frac{\hbar}{z-w} \sim -\gamma(z)\beta(w).$$

The vertex algebra $\mathcal{V}$ is identified with the differential ring

$$\mathcal{V} = :\mathbb{C}[[\partial^i \beta, \partial^i \gamma]]:\ [[\hbar]],$$

where $:\ :$ is the normal ordering operator*. The general OPE is obtained via **Wick contractions**. For example,*

$$:\beta(z)\gamma(z):\ :\beta(w)\gamma(w):\ = \underbrace{\frac{\hbar}{z-w}\ :\gamma(z)\beta(w):\ - \frac{\hbar}{z-w}\ :\beta(z)\gamma(w):}_{1\ contraction} - \underbrace{\left(\frac{\hbar}{z-w}\right)^2}_{2\ contractions}$$
$$= \sum_{k \geq 0} \frac{\hbar}{z-w} \frac{(z-w)^k}{k!}\ :\partial^k \gamma(w)\beta(w) - \partial^k \beta(w)\gamma(w):\ - \frac{\hbar^2}{(z-w)^2}.$$

Example 4.3 (bc-system)**.** *The bc-system is generated by two* fermionic *fields $b(z), c(z)$ with OPE*

$$b(z)c(w) \sim \frac{\hbar}{z-w} \sim c(z)b(w).$$

The vertex algebra $\mathcal{V}$ is identified with the differential ring

$$\mathcal{V} = :\mathbb{C}[[\partial^i b, \partial^i c]]:\ [[\hbar]].$$

The general OPE is generated in the similar way as the $\beta\gamma$-system (but we need to take care of the signs).

More generally, we can define a general $\beta\gamma{-}bc$ system by considering a $\mathbb{Z}_2$-graded space

$$h = h_0 \oplus h_1$$

with an even symplectic pairing

$$\langle -,- \rangle : \bigwedge^2 h \to \mathbb{C}.$$

Let $\{a_i\}$ be a basis of h, then we can define a vertex algebra $\mathcal{V}_h$ by

$$\mathcal{V}_h = :\mathbb{C}[[\partial^k a_i]]: \ [[\hbar]].$$

The OPE is generated by

$$a_i(z)a_j(w) \sim \frac{\hbar}{z-w} \langle a_i, a_j \rangle .$$

In particular, h_0 represents the copies of $\beta\gamma$-system; h_1 represents the copies of bc-system.

4.2 Chiral Deformation of $\beta\gamma - bc$ Systems

We consider the following data:

- an elliptic curve E (topologically a torus T^2) with linear coordinate z such that $z \sim z+1 \sim z+\tau$,
- a graded symplectic space $h = h_0 \oplus h_1$ with an even symplectic pairing $\langle -,- \rangle$.

This defines a field theory in BV formalism where the space of fields is

$$\mathcal{E} = \Omega^{0,\bullet}(E) \otimes h$$

with (-1)-symplectic pair by

$$\omega(\varphi_1, \varphi_2) = \int_E dz \, \langle \varphi_1, \varphi_2 \rangle , \quad \varphi_i \in \mathcal{E}.$$

Note that ω has deg $= -1$ since we need exactly one $\overline{dz}$ from φ_1, φ_2 to be integrated.

The free theory is given by

$$\frac{1}{2} \int_E dz \, \langle \varphi, \bar{\partial} \varphi \rangle , \quad \varphi \in \mathcal{E}.$$

The local quantum observables form exactly $\beta\gamma - bc$ system. The propagator is given by the Szegö kernel

$$\bar{\partial}^{-1} \sim \frac{1}{z-w} + \text{regular}.$$

We would like to consider a general interacting theory by turning on **chiral deformations** of the form

$$\int \mathcal{L} \left(\varphi, \partial_z \varphi, \partial_z^2 \varphi, \cdots \right)$$

which involves only *holomorphic* derivatives. This is related to the vertex algebra

$$\mathcal{V}_{h^\vee} = \mathbb{C}[[\partial^i h^\vee]][[\hbar]]$$

as follows. Define a map

$$I : \mathcal{V}_{h^\vee} \to \mathscr{O}_{loc}(\mathcal{E}), \quad \gamma \mapsto I_\gamma.$$

Explicitly, if $\gamma = \sum \partial^{k_1} a_1 \cdots \partial^{k_m} a_m$, then

$$I_\gamma(\varphi) = i \int_E dz \sum \pm \partial_z^{k_1} a_1(\varphi) \cdots \partial_z^{k_m} a_m(\varphi).$$

Here $a_i \in h^\vee$ and $a_i(\varphi) \in \omega^{0,\bullet}(E)$.

Theorem 4.4 ([37])**.** *For any $\gamma \in \mathcal{V}_{h^\vee}$, the chiral deformed theory*

$$\frac{1}{2} \int_E dz \left\langle \varphi, \bar{\partial}\varphi \right\rangle + I_\gamma(\varphi)$$

*is **UV finite** in the sense that the limit*

$$e^{\frac{1}{\hbar} I_\gamma[L]} = \lim_{\varepsilon \to 0} e^{\hbar \partial_{P_\varepsilon^L}} e^{\frac{1}{\hbar} I_\gamma} \quad \textit{exists.}$$

Remark 4.5. The proof of the UV finiteness theorem is a bit technical. The reason is different from topological QM, where we saw that the propagator is bounded (although not continuous). Here the graph integral is NOT absolutely convergent. See Section 4.3 for another geometric interpretation [39] of this fact.

Once we have a well-defined $I_\gamma[L]$ described above, we can formulate the effective QME

$$\bar{\partial} I_\gamma[L] + \hbar \Delta_L I_\gamma[L] + \frac{1}{2} \{I_\gamma[L], I_\gamma[L]\}_L = 0$$

and ask for the condition of γ to satisfy the equation. It turns out that the answer is very simple.

Theorem 4.6 ([37])**.** *Consider $\gamma \in \mathcal{V}_{h^\vee}$ and the effective functional $I_\gamma[L]$ defined above via the UV finiteness. Then $I_\gamma[L]$ satisfies the effective QME*

$$\bar{\partial} I_\gamma[L] + \hbar \Delta_L I_\gamma[L] + \frac{1}{2} \{I_\gamma[L], I_\gamma[L]\}_L = 0$$

if and only if

$$\left[\oint \gamma, \oint \gamma \right] = 0 \ \in \oint \mathcal{V}.$$

Remark 4.7. The local quantum observable of the chiral deformed theory is the vertex algebra $H^\bullet(\mathcal{V}_{h^\vee}, [\oint \gamma, -])$. So $[\oint \gamma, -]$ plays the role of BRST reduction. Reversing this reasoning, vertex algebras coming from the BRST reduction of free field realizations can be realized via the model of chiral deformations above.

The above theorem can be glued for a *chiral σ-model*

$$\varphi : E \to X$$

which produces a bundle $\mathcal{V}(X) \to X$ of chiral vertex algebras on X. Then the solution of effective QME asks for a flat connection on $\mathcal{V}(X)$ of the form

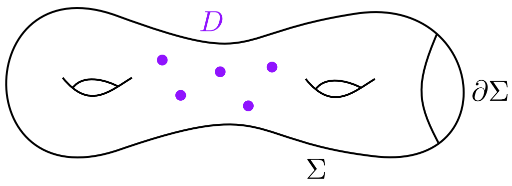
$$D = d + \frac{1}{\hbar} \left[\oint \gamma, - \right], \text{ such that } D^2 = 0.$$

Here $\gamma \in \Omega^1(X, \mathcal{V}(X))$ and $\oint \gamma$ is fiberwise chiral mode operator. This can be viewed as the *chiral analogue of Fedosov connection.*

4.3 Regularized Integral and UV Finiteness

The propagator $\bar{\partial}^{-1}$ is given by the Szegö kernel which exhibits holomorphic poles $\frac{1}{z-w}$ along the diagonal. In general, the Feynman diagram involves $\int_{\Sigma^n} \Omega$, where Ω exhibits holomorphic poles of arbitrary order when $z_i \to z_j$. It turns out that such looking divergent integral has an intrinsic *regularization* via its conformal structure.

For simplicity, we start by considering such an integral $\int_\Sigma \omega$. Here Σ is a Riemann surface, possibly with boundary $\partial\Sigma$, ω is a 2-from on Σ with meromorphic poles of arbitrary order along a finite set $D \subset \Sigma$ and $D \cap \partial\Sigma = \varnothing$.



Let $p \in D$ and z be a local coordinate centered at p. Then locally ω can be written as

$$\omega = \frac{\eta}{z^n}$$

where η is smooth, and $n \in \mathbb{Z}$. Since the pole order can be arbitrarily large, the naive integral $\int_\Sigma \omega$ is divergent in general. One homological way out of this divergence problem [39] is as follows. We can decompose ω into

$$\omega = \alpha + \partial\beta,$$

where α is a 2-form with at most *logarithmic pole* along D, β is a $(0,1)$-form with *arbitrary order of poles* along D, and $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham differential. Such a decomposition *exists* and is *not unique*.

Definition 4.8 ([39])**.** Define the **regularized integral**

$$\fint_\Sigma \omega := \int_\Sigma \alpha + \int_{\partial\Sigma} \beta$$

as a recipe to integrate the singular form ω on Σ. It has the following properties

- it does not depend on the choice of α, β, and is equivalent to the Cauchy principal value,
- $\fint_\Sigma$ is invariant under conformal transformations,
- $\fint_\Sigma \partial(-) = \int_{\partial\Sigma}(-)$ on $(0,1)$-form with meromorphic poles,
- $\fint_\Sigma \bar{\partial}(-) = \mathrm{Res}(-)$ on $(1,0)$-form with meromorphic poles.

The regularized integral extends the usual integral for smooth forms, i.e., the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}^2(\Sigma) \arrow[rr, hook] \arrow[dr, "\int_\Sigma"'] & & \mathcal{A}^2(\Sigma, \star D) \arrow[dl, "\fint_\Sigma"] \\ & \mathbb{C} & \end{tikzcd}$$

Here $\mathcal{A}^2(\Sigma)$ means smooth 2-forms on Σ, and $\mathcal{A}^2(\Sigma, \star D)$ means smooth 2-forms on $\Sigma - D$ with meromorphic poles of arbitrary order around D.

We can use this to define integrals on configuration space of Σ

$$\mathrm{Conf}_n(\Sigma) = \Sigma^n - \Delta = \{ (p_1, \cdots, p_n) \in \Sigma^n |\, p_i \neq p_j, \forall i \neq j\}$$

and define

$$\fint_{\Sigma^n} : \mathcal{A}^{2n}(\Sigma^n, \star\Delta) \to \mathbb{C}$$

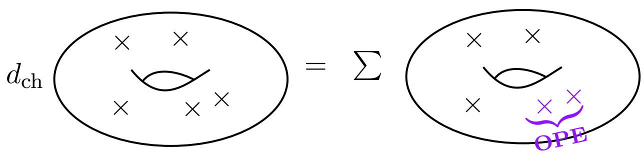
by iterating

$$\fint_{\Sigma^n}(-) = \fint_{\Sigma}\fint_{\Sigma}\cdots\fint_{\Sigma}(-).$$

It does NOT depend on the choice of the ordering of the factors in Σ^n: Fubini-type theorem holds. This gives an intrinsically regularized meaning for $\fint_{\Sigma^n}\Omega$, where Ω is the Feynman diagram integrand. This explains why the theory is UV finite.

4.4 Chiral Homology and Elliptic Trace

Intuitively, chiral chain complex can be viewed as a 2d chiral analogue of Hochschild chain complex.



- In [54], Zhu studied the space of genus 1 conformal blocks (i.e. the 0th elliptic chiral homology).
- In [6], Beilinson and Drinfeld developed the chiral homology theory on general algebraic curves.

The construction of Beilinson-Drinfeld. We briefly review the construction of Beilinson-Drinfeld and refer to [30] for further details related to the purpose of the current discussion. Let S denote the category of finite non-empty sets whose morphisms are surjections. Given the following data:

- a category of right $\mathcal{D}$-modules $\mathcal{M}(X)$ on $X = \Sigma$,
- a category of right $\mathcal{D}$-modules $\mathcal{M}(X^S)$ on X^S, such that each element $M \in \mathcal{M}(X^S)$ is a collection that assigns every finite index set $I \in S$ a right $\mathcal{D}$-module $\mathcal{M}_{X^I}$ on the product X^I satisfying certain compatibility conditions,
- there is an exact fully faithful embedding

  $$\Delta_\star^{(S)} : \mathcal{M}(X) \hookrightarrow \mathcal{M}(X^S)$$

  via the diagonal map $\Delta^{(I)} : X \hookrightarrow X^I$,
- $\mathcal{M}(X^S)$ carries a (chiral) tensor structure $\otimes^{\mathrm{ch}}$,

Then a chiral algebra $\mathcal{A}$ is a **Lie algebraic object** via $\Delta_\star^{(S)}$.

Remark 4.9. The chiral algebra $\mathcal{A}$ collects all “normal ordered operators” in physics terminology.

We consider the Chevalley-Eilenberg (CE) complex

$$(C(\mathcal{A}), d_{\mathrm{CE}}) = \left(\bigoplus_{\bullet>0} \mathrm{Sym}^{\bullet}_{\otimes^{\mathrm{ch}}} \left(\Delta_{\star}^{(S)} \mathcal{A}[1] \right), d_{\mathrm{CE}} \right).$$

The chiral homology for this complex is

$$C^{\mathrm{ch}}(X, \mathcal{A}) = R\Gamma_{DR}(X^S, C(\mathcal{A})).$$

We will focus on $\beta\gamma-bc$ system, where the vertex operator algebra (VOA) $\mathcal{V}^{\beta\gamma-bc}$ gives rise to a chiral algebra $\mathcal{A}^{\beta\gamma-bc}$. The following theorem gives the corresponding elliptic trace map in terms of renormalization group flow.

Theorem 4.10 ([30])**.** *Let E be an elliptic curve. Then the HRG flow gives a map*

$$\langle - \rangle_{2d} : C^{\mathrm{ch}}(E, \mathcal{A}^{\beta\gamma-bc}) \to \mathcal{O}_{\mathrm{BV}}((\hbar))$$

satisfying the QME

$$(d_{\mathrm{ch}} + \hbar\Delta) \langle - \rangle_{2d} = 0.$$

- *Here $\mathcal{O}_{\mathrm{BV}}$ is the space of functions on zero modes of the $\beta\gamma-bc$ system, which carries a structure of BV algebra. Δ is the corresponding BV operator.*
- *$\langle - \rangle_{2d}$ is defined by*

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{E^n} \left(\prod_{i=1}^{n} \frac{d^2 z_i}{\mathrm{Im}\,\tau} \right) \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$$

  where $\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is the correlation function computed via Feynman diagrammatics, and $\fint$ is the regularized integral.
- *The QME says that $\langle - \rangle_{2d}$ intertwines the chiral differential of the elliptic chiral chain complex with the BV operator $-\hbar\Delta$ of the zero-mode algebra $\mathcal{O}_{\mathrm{BV}}((\hbar)))$. Moreover, $\langle - \rangle_{2d}$ is shown to be a quasi-isomorphism.*
- *The BV trace with universal background leads to* ***Witten genus****.*

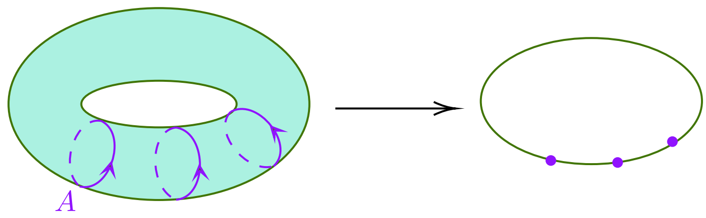
Theorem 4.10 establishes the construction of BV quantization and trace map outlined in the introduction. The Witten genus can be viewed as describing an elliptic chiral analogue of the algebraic index. The computation of Witten genus in BV quantization follows essentially from similar arguments in Costello [12] and Gorbounov-Gwilliam-Williams [28].

4.5 2d → 1d Reduction

We summarize our discussion as follows.

1d TQM	**2d chiral QFT**
Associative algebra	Vertex/chiral algebra
Hochschild homology	Chiral homology
QME $(\hbar\Delta + b) \langle - \rangle_{1d} = 0$	QME $(\hbar\Delta + d_{\mathrm{ch}}) \langle - \rangle_{2d} = 0$
$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d} = \int_{\overline{\mathrm{Conf}_n(S^1)}}$	$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} = \fint_{\Sigma^n}$

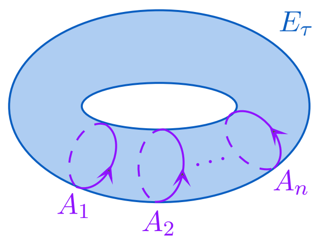
In physics, the partition functions/correlation functions on elliptic curves are described by reducing to a quantum mechanical system on S^1.



$$\langle - \rangle_{2d} \longrightarrow \mathrm{Tr}_{\mathcal{H}} \ \langle \cdots \rangle$$

Now we can define 2d chiral correlation function using *regularized integral* $\fint_E$. In 1d, operators are described by A-cycle $\oint_A$. These two integrals are not exactly the same, but related to each other by *holomorphic limit.*

Theorem 4.11 ([39]). *Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbb{H}$ which is holomorphic away from diagonals. Let $A_1, \cdots, A_n$ be n disjoint A-cycles on the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$.*



Then the regularized integral

$$\fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{Im}\tau} \right) \Phi(z_1, \cdots, z_n; \tau)$$

lies in $\mathscr{O}_{\mathbb{H}}\left[\frac{1}{\operatorname{Im}\tau}\right]$. Moreover, we have

$$\lim_{\bar\tau \to \infty} \fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{Im}\tau} \right) \Phi = \frac{1}{n!} \sum_{\sigma \in S_n} \oint_{A_{\sigma(1)}} dz_1 \cdots \oint_{A_{\sigma(n)}} dz_n \ \Phi,$$

where S_n is the n-th permutation group and

$$\lim_{\bar\tau \to \infty} : \mathscr{O}_{\mathbb{H}} \left[\frac{1}{\operatorname{Im}\tau} \right] \to \mathscr{O}_{\mathbb{H}} \quad \textit{is the map sending} \quad \frac{1}{\operatorname{Im}\tau} \to 0.$$

This theorem gives a precise relation on reduction of torus to circle

$$\fint_{E^n} \xrightarrow{\lim_{\bar\tau\to\infty}} \text{averaged} \oint_A .$$

The anti-holomorphic dependence of $\fint_{E^n}$ on the moduli τ is actually fully described by the holomorphic anomaly equation [40].

Furthermore, if $\Phi(z_1, \cdots, z_n; \tau)$ is modular of weight m, then its regularized integral $\fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{Im}\tau} \right) \Phi(z_1, \cdots, z_n; \tau)$ is modular of weight m and thus an almost holomorphic modular forms [33]. The holomorphic limit by averaged $\oint_A$ is a quasi-modular form of weight m.

We apply the above theorem to 2d chiral correlations on elliptic curves. This leads to the following relation between the elliptic trace map in Theorem 4.10 and Weyl-ordered operators by A-cycle integrals

$$
\begin{aligned}
\lim_{\bar\tau\to\infty} \langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} &= \lim_{\bar\tau\to\infty} \fint_{E_\tau^n} \left(\prod_{i=1}^{n} \frac{d^2 z_i}{\operatorname{Im}\tau} \right) \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \oint_{A_{\sigma(1)}} dz_1 \cdots \oint_{A_{\sigma(n)}} dz_n \ \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle .
\end{aligned}
$$

This can be viewed as a reduction formula from 2d to 1d. This formula illustrates an interesting relationship between regularization and modularity/quasi-modularity.

4.6 Application to Mirror Symmetry

Mirror symmetry is about a duality between

$$
\boxed{\text{symplectic geometry}} \ (\text{A-model}) \iff \boxed{\text{complex geometry}} \ (\text{B-model})
$$

Here is a cartoon to illustrate how such mirror relation arises from physics.

$$
\begin{tikzcd}
\int_{\mathrm{Map}(\Sigma_g, X)} (\text{A-model}) \arrow[r, equal, "\text{Fourier transform}"] \arrow[d, "\text{SUSY localize}"] & \int_{\mathrm{Map}(\Sigma_g, X')} (\text{B-model}) \arrow[d, "\text{SUSY localize}"'] \\
\int_{\text{Holomorphic maps}(\Sigma_g, X)} \arrow[r, leftrightarrow, dashed] \arrow[d, Rightarrow] & \int_{\text{Constant maps}(\Sigma_g, X')} \arrow[d, Rightarrow] \\
\text{Gromov-Witten Theory} & \text{Hodge theory/Kodaira-Spencer gravity}
\end{tikzcd}
$$

Consider the example of elliptic curves, whose mirrors are elliptic curves as well. The full quantum B-model (quantum BCOV theory as developed in [14]) on elliptic curves (including all gravitational descendents) is completely solved in [37]. The so-called stationary sector is described by the chiral deformation of chiral boson

$$
S = \int \partial\phi \wedge \bar\partial\phi + \sum_{k \geq 0} \int \eta_k \frac{W^{(k+2)}(\partial_z \phi)}{k+2}
$$

where

$$
W^{(k)}(\partial_z \phi) = (\partial_z \phi)^k + O(\hbar)
$$

are the bosonic realization of the $W_{1+\infty}$-algebra. The holomorphic limit $\bar\tau \to \infty$ (explained in Section 4.5) of the generating function of S on the elliptic curve coincides with the Gromov-Witten invariants on the mirror computed by Dijkgraaf[18] and Okounkov-Pandharipande[44]. In this case, we find [37]

Quantum Mirror Symmetry=Boson-Fermion Correspondence.

References

[1] Alexandrov, M., Schwarz, A., Zaboronsky, O., Kontsevich, M. *The geometry of the master equation and topological quantum field theory.* International Journal of Modern Physics A, 12.07.(1997), 1405-1429.

[2] Atiyah, M., *Circular symmetry and stationary-phase approximation.* Astérisque 131 (1985): 43-59.

[3] Axelrod, S. and Singer, I.M., *Chern-Simons Perturbation Theory II.* J. Differ. Geom. 39 (hep–th/9304087), 173–213 (1993)

[4] Batalin, I. and Vilkovisky, G., *Gauge algebra and quantization*, Physics Letters B **102**(1), 27–31 (1981).

[5] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D., *Deformation theory and quantization. I. Deformations of symplectic structures.* Annals of Physics 111, no. 1 (1978): 61-110. *Deformation theory and quantization. II. Physical applications.* Annals of Physics 111, no. 1 (1978): 111-151.

[6] Beilinson, A., and Drinfeld, V., *Chiral algebras*, volume 51, American Mathematical Soc., 2004.

[7] Berezin, F.. *Feynman path integrals in a phase space.* Soviet Physics Uspekhi 23.11 (1980): 763.

[8] Bordemann, M., *Deformation quantization: a survey.* Journal of Physics: Conference Series. Vol. 103. No. 1. IOP Publishing, 2008.

[9] Bressler, P., Nest, R., and Tsygan, B., *Riemann–Roch Theorems via Deformation Quantization, I*, Advances in Mathematics **167**(1), 1–25 (2002).

[10] Budzik, K., Gaiotto, D., Kulp, J., Wu, J., and Yu, M. *Feynman diagrams in four-dimensional holomorphic theories and the Operatope.* Journal of High Energy Physics, 2023(7), 1-40.

[11] Cattaneo, A.S., and Felder, G.. *Poisson sigma models and deformation quantization.* Modern Physics Letters A 16.04n06 (2001): 179-189.

[12] Costello, K. *A geometric construction of the Witten genus, I.* In Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures (pp. 942-959).

[13] Costello, K., *Renormalization and effective field theory*, volume 170 of *Mathematical Surveys and Monographs*, American Mathematical Soc., 2011.

[14] Costello, K. and Li, S., *Quantum BCOV theory on Calabi-Yau manifolds and the higher genus B-model* , arXiv:1201.4501[math.QA] .

[15] Costello, K. and Li, S., *Quantization of open-closed BCOV theory, I*, arXiv:1505.06703[hep-th]

[16] Costello, K. and Gwilliam, O., *Factorization algebras in quantum field theory*, Vol 1, 2. Cambridge University Press, 2021.

[17] De Wilde, M. and Lecomte, P., *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds.* Letters in Mathematical Physics 7 (1983): 487-496.

[18] Dijkgraaf, R., *Mirror symmetry and elliptic curves.* The moduli space of curves. Boston, MA: Birkhäuser Boston, 1995. 149-163.

[19] Dijkgraaf, R. *Chiral deformations of conformal field theories.* Nuclear physics B, 493(3), 588-612, (1997)

[20] Douglas, M. R. *Conformal field theory techniques in large N Yang-Mills theory.* In Quantum field theory and string theory (pp. 119-135). (1995). Boston, MA: Springer US.

[21] Fedosov, B., *A simple geometrical construction of deformation quantization.* Journal of differential geometry 40.2 (1994): 213-238.

[22] Fedosov, B., *Deformation quantization and index theory.* Mathematical topics 9 (1996).

[23] Feigin, B., Felder, G. and Shoikhet, B., *Hochschild cohomology of the Weyl algebra and traces in deformation quantization.* Duke Mathematical Journal 127 (3) 487 - 517, (2005)

[24] Feigin, B. and Tsygan, B., *Riemann-Roch theorem and Lie algebra cohomology*, in *Proceedings of the Winter School "Geometry and Physics"*, pages 15–52, Circolo Matematico di Palermo, 1989.

[25] Frenkel, E., and Ben-Zvi, D. *Vertex algebras and algebraic curves.* No. 88. American Mathematical Soc..(2004).

[26] Getzler, E., *Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem*, Communications in mathematical physics **92** (1983), no. 2, 163–178.

[27] Getzler, E. *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology.* In Israel Math. Conf. Proc, Vol. 7, pp. 65-78. (1993).

[28] Gorbounov, V., Gwilliam, O., and Williams, B. R. *Chiral differential operators via quantization of the holomorphic σ-model.* Astérisque. (2020)

[29] Grady, R., Li, Q. and Li, S.. *Batalin–Vilkovisky quantization and the algebraic index.* Advances in Mathematics 317 (2017): 575-639.

[30] Gui, Z. and Li, S., *Elliptic Trace Map on Chiral Algebras*, (12 2021), arXiv: 2112.14572.

[31] Gui, Z., Li. S., and Xu, K., *Geometry of localized effective theories, exact semi-classical approximation and the algebraic index.* Communications in Mathematical Physics 382.1 (2021): 441-483.

[32] Kac, V. G. *Vertex algebras for beginners.* Vol. 10. American Mathematical Soc. (1998).

[33] Kaneko, M., and Zagier, D. *A generalized Jacobi theta function and quasimodular forms.* In The moduli space of curves, pp. 165-172. Boston, MA: Birkhäuser Boston. (1995).

[34] Kontsevich, M., *Feynman diagrams and low-dimensional topology.* First European Congress of Mathematics Paris, July 6–10, 1992: Vol. II: Invited Lectures (Part 2). Basel: Birkhäuser Basel, 1994.

[35] Kontsevich, M, *Deformation quantization of Poisson manifolds.* Letters in Mathematical Physics 66 (2003): 157-216.

[36] Li, S., *Feynman graph integrals and almost modular forms*, Comm. in Number Theory and Physics, 6, 2012.

[37] Li, S., *Vertex algebras and quantum master equation.* J.Diff.Geom. 123 (2023), no 3, 461 - 521.

[38] Li, S., Wang, Z., Yang, P., *Stochastic Calculus and Hochschild Homology*, arXiv:2501.12360 [math.PR]

[39] Li, S., Zhou, J., *Regularized Integrals on Riemann Surfaces and Modular Forms* . Commun. Math. Phys. 388, 1403–1474 (2021).

[40] Li, S., Zhou, J., *Regularized Integrals on Elliptic Curves and Holomorphic Anomaly Equations*. Commun. Math. Phys. 401, pp 613–645 (2023).

[41] Loday, J. L. *Cyclic homology* (Vol. 301). Springer Science & Business Media. (2013).

[42] Nest, R., and Tsygan, B., *Algebraic index theorem.* Communications in Mathematical Physics 172, 223-262 (1995)

[43] Nest, R., and Tsygan, B., *Formal versus analytic index theorems*, International Mathematics Research Notices **1996** (1996), no. 11, 557–564.

[44] Okounkov, A., and Pandharipande, R., *Virasoro constraints for target curves.* Inventiones mathematicae 163.1 (2006): 47-108.

[45] Rieffel, M. A. *Deformation quantization of Heisenberg manifolds.* Communications in mathematical physics, 122(4), 531-562. (1989)

[46] Wang, M., *Feynman Graph Integrals on* $\mathbb{C}^d$. Commun. Math. Phys. 406.5 (2025): 116.

[47] Wang, M, Yan, J., *Feynman Graph Integrals on Kähler Manifolds*, arXiv:2507.09170 [math-ph]

[48] Williams, B. R. *Renormalization for holomorphic field theories.* Communications in Mathematical Physics, 374(3), 1693-1742.

[49] Windey, P., *Supersymmetric quantum mechanics and the Atiyah-Singer index theorem*, Acta Phys. Pol. B **15** (1983), no. CERN-TH-3758, 435–452.

[50] Witten, E., *Constraints on supersymmetry breaking*, Nuclear Physics B **202** (1982), no. 2, 253–316.

[51] Witten, E., *Elliptic genera and quantum field theory*, Communications in Mathematical Physics **109** (1987), no. 4, 525–536.

[52] Witten, E., *The index of the Dirac operator in loop space*, Elliptic curves and modular forms in algebraic topology, Springer, 1988, pp. 161–181.

[53] Witten, E., *Analytic continuation of Chern-Simons theory.* AMS/IP Stud. Adv. Math 50 (2011): 347.

[54] Zhu, Y., *Global vertex operators on Riemann surfaces*, Commun. Math. Phys. **165**(3), 485–531 (1994).

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