NULL FLUID GRAVITATIONAL FIELDS ON KERR MANIFOLDS AND OPTICAL LIFTS OF SASAKI STRUCTURES

MASOUD GANJI, CRISTINA GIANNOTTI, AND ANDREA SPIRO

Abstract. Building on the characterisation in [C. D. Hill, J. Lewandowski and P. Nurowski, Indiana Univ. Math. J. **57** (2008), 3131–3176] of 4-dimensional Lorentzian metrics adapted to an optical structure and satisfying the null fluid Einstein equations, we give an explicit parameterisation of this class under the assumption that the optical structure is of Kerr type. As immediate consequences, we obtain: (1) a new method for constructing solutions to the Einstein equations with a null fluid energy momentum tensor, yielding a large family of explicit metrics that naturally includes the classical Kerr black hole metrics and all Ricci flat examples described in [M. Ganji, C. Giannotti, G. Schmalz and A. Spiro, Ann. Physics **75** (2025), Paper No. 169908, 28]; (2) a solution to a conjecture in Hill, Lewandowski and Nurowski's paper on the local existence of smooth optical lifts in the case of Sasaki CR structures.

1. Introduction

We recall that an *optical structure* on an n-dimensional manifold M is a pair $\mathcal{Q} = (\mathcal{K}, \{g\})$ given by

- a 1-dimensional distribution $\mathcal{K} \subset TM$,
- the unique family $\{g\}$ of all Lorentzian metrics g, such that: (1) the curves tangent to $\mathcal{K}$ determine a geodesic shearfree null congruence on M; (2) the g-orthogonal distribution $\mathcal{K}^{\perp_g}$ is a fixed distribution $\mathcal{W} \subset TM$, independent of $g \in \{g\}$ (so $\mathcal{W} = \mathcal{K}^{\perp_g}$ for all g).

The metrics in the class $\{g\}$ are said to be *compatible* with the optical structure.

The notion of optical structure was introduced by Robinson and Trautman to encode the geometric properties of Lorentzian manifolds admitting electromagnetic plane waves (or appropriate higher dimensional generalisations) propagating along a prescribed foliation by null geodesics (see e.g. [16, 18, 17, 8, 2, 3, 20, 15] and references therein). A standard way to construct explicit examples of optical structures is as follows.

Consider a $(2k-1)$-dimensional manifold $\mathcal{S}$, equipped with a $(2k-2)$-dimensional contact distribution $\mathcal{D}$ and with a field J of complex structures $J_x : \mathcal{D}_x \to \mathcal{D}_x$ on the spaces of the distribution $\mathcal{D}$ satisfying, for any vector fields $X, Y \in \mathcal{D}$,

$$[JX, JY] - [X, Y] \in \mathcal{D} , \qquad [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) = 0 .$$

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A triple $(\mathcal{S}, \mathcal{D}, J)$ of this kind is called *Levi non-degenerate CR manifold.* If there exists a contact form θ for $(\mathcal{S}, \mathcal{D})$ (i.e., a 1-form with $\ker\theta_x = \mathcal{D}_x$, $x \in M$) such that $d\theta_x(J_x\cdot, \cdot)$ is a positive definite scalar product on each $\mathcal{D}_x \subset T_xM$, then $(\mathcal{S}, \mathcal{D}, J)$ is called *strongly pseudoconvex.*

Consider the trivial principal $\mathbb{R}$-bundle $\pi^{\mathcal{S}} : M = \mathcal{S}\times\mathbb{R} \to \mathcal{S}$ over a strongly pseudoconvex CR manifold $(\mathcal{S}, \mathcal{D} = \ker\theta, J)$. Denote by ϑ and p_o the pulled-back 1-form $\vartheta = \pi^{\mathcal{S}*}\theta$ on M and the vertical vector field $\mathrm{p}_o = \frac{\partial}{\partial u}$ generating the natural right $\mathbb{R}$-action on $M = \mathcal{S} \times \mathbb{R}$, respectively. Then let $\mathcal{Q}^{(\mathcal{S})} = (\mathcal{K}, \{g\})$ be the pair given by the 1-dimensional distribution $\mathcal{K} = \langle \mathrm{p}_o\rangle \subset TM$ and the class $\{g\}$ of Lorentzian metrics on M having the form

$$g = \sigma\pi^{\mathcal{S}*}(d\theta_x(J_x\cdot, \cdot)) + \vartheta \vee \left(\widetilde{\alpha}\mathrm{p}_o^* + \mathfrak{d} + \frac{\widetilde{\beta}}{2}\vartheta\right) , \qquad \text{where}$$

(i) p_o^* is an $\mathbb{R}$-invariant 1-form with $\mathrm{p}_o^*(\mathrm{p}_o) = 1$ and $\mathcal{H} := \ker\mathrm{p}_o^* \cap \ker\vartheta$ is the unique $\mathbb{R}$-invariant distribution in TM that projects onto $\mathcal{D} \subset T\mathcal{S}$,
(ii) $\sigma, \widetilde{\alpha}, \widetilde{\beta}$ are freely specifiable smooth functions with $\sigma, \widetilde{\alpha}$ nowhere vanishing,
(iii) $\mathfrak{d}$ is a freely specifiable 1-form with $\ker\mathfrak{d}$ containing a distribution $\mathcal{V} \subset TM$ which is complementary to $\langle \mathrm{p}_o\rangle + \mathcal{D}$.

One checks directly that any such $\mathcal{Q}^{(\mathcal{S})} = (\mathcal{K}, \{g\})$ is an optical structure ([17, 12, 3, 4, 8]). These optical structures provide an important link between the geometry of the strongly pseudoconvex CR manifolds $(\mathcal{S}, \mathcal{D}, J)$ and Lorentzian metrics g compatible with the corresponding optical structures $\mathcal{Q}^{(\mathcal{S})}$. Indeed, such a relation led to very interesting results in both CR and Lorentzian geometry; see for instance [12] for a number of remarkable results and applications in both areas.

In this paper we deal with the optical structures built as above from a 3-dimensional strongly pseudoconvex CR manifold $(\mathcal{S}, \mathcal{D} = \ker\theta, J)$ under the assumption that: (a) $\mathcal{S}$ is a principal $\mathbb{R}$-bundle $\pi : \mathcal{S} \to N$ over a 2-dimensional Kähler manifold (N, J, g_o); (b) the distribution $\mathcal{D}$ is $\mathbb{R}$-invariant and complementary to the vertical distribution of $\mathcal{S}$; (c) the contact 1-form θ is a connection 1-form $\theta : T\mathcal{S} \to \mathbb{R}$ satisfying $d\theta = \pi^*\omega_o$, where $\omega_o := g_o(J\cdot, \cdot)$ is the Kähler form of (N, J, g_o). We mention that the CR structures satisfying (a) – (c) belong to the class of the so called *Sasaki structures.* We call the optical structures built upon these Sasaki manifolds *of Kerr type* and we call any 4-manifold $M \simeq \mathcal{S} \times \mathbb{R}$, equipped with such an optical structure, a *Kerr manifold.* These names are motivated by the fact that the classical Kerr metrics of rotating black holes are compatible metrics of Kerr type optical structures (see [9] and references therein for details).

Our first main result can be summarised as follows. In [12] Hill, Lewandowski and Nurowski determined a local characterisation of the 4-dimensional Lorentzian metrics g that are compatible with an optical structure determined as above by an *embeddable* strongly pseudoconvex CR manifold (i.e. equivalent to a strongly pseudoconvex real hypersurface in $\mathbb{C}^2$) and satisfy an equation of the form

$$\mathrm{Ric} = \Lambda g + \mathsf{f}\vartheta \vee \vartheta \qquad \text{for some function } \mathsf{f} . \tag{1.1}$$

Note that, for any such a compatible metric g, the tensor field $\vartheta \vee \vartheta$ is null and (1.1) is the Einstein equation for the gravitational field of a null fluid in a background metric with cosmological constant Λ. In addition, by the generalisation in [11, Thm. 5.10] of the Goldberg-Sachs Theorem, it is known that the curvature of any such compatible metric is algebraically special at all points. Following [10, 19], we call metrics of this kind *quasi-Einstein*.

The results in [12] on quasi-Einstein metrics are presented using complex functions in a special system of *complex* coordinates (z, w) on $\mathbb{C}^2$ and satisfying a particular set of differential constraints. Subsequently, in [10] the authors studied the quasi-Einstein metrics compatible with Kerr type optical structures, a class to which Hill, Lewandowski, and Nurowski's results immediately apply with substantial simplifications. In this paper, we further develop [12, 10] to provide a fully explicit and coordinate-free local parametrisation of quasi-Einstein compatible metrics on a 4-dimensional Kerr manifold. We present them in terms of eight *real* functions on the Kähler surface, onto which the Kerr manifold projects, constrained by a system of five partial differential equations, the first four readily solvable.

This result has at least two remarkable consequences. First, it simplifies the presentation of the quasi-Einstein compatible metrics on a Kerr manifold so greatly that we can resolve, in a very simple way, the following conjecture by Hill, Lewandowski and Nurowski in the setting of Kerr manifolds. In [12] it was proved that *for any choice of the cosmological constant* Λ *and for any real analytic and embeddable strongly pseudoconvex* 3*-manifold* $(\mathcal{S}, \mathcal{D}, J)$*, there locally exists at least one quasi-Einstein Lorentzian metric with curvature tensors of Petrov type II or D, which is compatible with the optical structure on* $M = \mathcal{S} \times \mathbb{R}$ *described above.* We call these compatible metrics *(local) optical lifts* of the CR structure $(\mathcal{D}, J)$. In [12] the authors conjectured that the real analyticity is not essential for the above result and that it holds under weaker regularity hypotheses. Using results on logistic elliptic equations, this was partially confirmed in [10] for Kerr type optical structures, under the additional assumptions that the cosmological constant Λ is positive and that the underlying Kähler surface N is biholomorphic to $\mathbb{C}$. Exploiting our new parametrisation of quasi-Einstein metrics on Kerr manifolds, we now show that, for Kerr type optical structures, the conjecture holds with no restriction on the sign of Λ and on the Kähler surface N. In fact, this follows directly from a classical general result on non-linear elliptic equations.

The second consequence of our first result is that, by setting two of the eight parametrising functions equal to zero and assuming two of the remaining functions proportional to each other, we obtain a family of quasi-Einstein metrics effectively parametrised by: (a) three freely specifiable constants, (b) one freely specifiable harmonic function, and (c) a real function of two variables subject to a single non-linear elliptic equation. This family contains the classical Kerr metrics and the large class of explicit Ricci flat Lorentzian metrics recently constructed in [9]; we therefore refer to it as the *enlarged Kerr family.* Using arguments from [9], we construct a large subclass of quasi-Einstein metrics within the enlarged Kerr family. These furnish an infinite family of pairwise non isometric metrics representing gravitational fields sourced by non-vanishing null fluid energy momentum tensors, which, to the best of our knowledge, have not previously appeared in the literature. Note also that, as a consequence of the cited curvature properties of quasi-Einstein metrics and the fact that the Weyl scalar Ψ_2 with respect to the tetrads considered in [12, 14], is generically non-zero, the enlarged Kerr family provides a broad family of metrics whose curvature tensors are of Petrov type II or D.

The paper is organised as follows. Preliminaries are presented in §2 and §3. In §4 we establish our first main result, i.e., a new parametrisation of quasi-Einstein metrics compatible with Kerr type optical structures. In §5 we introduce and analyse the above mentioned enlarged Kerr family of quasi-Einstein metrics. In §6 we prove the announced resolution of the conjecture in [12] on the local existence of $\mathscr{C}^\infty$ quasi-Einstein optical lifts in the context of Kerr type optical structures.

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2. Preliminaries (I): Optical geometries and optical lifts

2.1. **Geodesic shearfree null congruences and optical geometries.** Let (M, g) be a Lorentzian n-manifold. A *null congruence on* M is a foliation by null curves. Let $\{\mathrm{p}\}$ be the class of null vector fields tangent to the curves of such a null congruence, $\mathcal{W}$ the g-orthogonal distribution to all vector fields in $\{\mathrm{p}\}$, and $h = g|_{\mathcal{W}\times\mathcal{W}}$ the degenerate metric induced on $\mathcal{W}$. The null congruence is called *shearfree* if there exists a vector field p of the class $\{\mathrm{p}\}$, whose (local) flow preserves the distribution $\mathcal{W}$ and the conformal class $[h]$ of the degenerate metric h on $\mathcal{W}$. It is known that this condition is equivalent to (see e.g. [3, Lemma 2.1])

$$\mathcal{L}_{\mathrm{p}} g = f g + \mathrm{p}^\flat \vee \eta \qquad \text{for some function } f \text{ and a 1-form } \eta \tag{2.1}$$

and that, if this holds, all curves of the null congruence are geodesics ([3, Prop. 2.4]). Due to this, in Physics, a foliation of this kind is usually called *geodesic shearfree null congruence.*

An *optical geometry* on a manifold M is a quadruple $\mathcal{Q} = (\mathcal{W}, [h]_\pm, \mathcal{K}, \{g\})$, given by

(i) a codimension one distribution $\mathcal{W} \subset TM$,
(ii) the union $[h]_\pm = [h] \cup [-h]$ of two conformal classes $[h]$, $[-h]$, of degenerate metrics on $\mathcal{W}$ with one-dimensional kernels, the first consisting of semipositive metrics, the second made of their opposites,
(iii) the 1-dimensional distribution $\mathcal{K}$ defined by $\mathcal{K}|_x = \ker h_x \subset \mathcal{W}_x$, $x \in M$, $h \in [h]_\pm$,
(iv) the class $\{g\}$ of all Lorentzian metrics (some with mostly plus, others with mostly minus signature) that induce on $\mathcal{W}$ at least one of the degenerate metrics in $[h]_\pm$,

such that the following holds: *the class* $\{g\}$ *is not empty and there exists a vector field* p *in* $\mathcal{K}$ *satisfying* (2.1). The metrics of the class $\{g\}$ of the optical structure $\mathcal{Q}$ are said to be *compatible with* $\mathcal{Q}$.

We recall that for any optical geometry $\mathcal{Q} = (\mathcal{W}, [h]_\pm, \mathcal{K}, \{g\})$, either one of the pairs $(\mathcal{W}, [h]_\pm)$ or $(\mathcal{K}_h = \ker h, \{g\})$ is sufficient to recover the other terms of the quadruple.

The pairs $(\mathcal{W}, [h]_\pm)$ that appear in optical geometries are called *shearfree structures* in [3], while the corresponding pairs $(\mathcal{K}, \{g\})$ are precisely the *optical geometries* as they were defined by Robinson and Trautman in [16] (provided that both definitions are relaxed to allow mostly plus and mostly minus signatures, of course).

Given an optical geometry $\mathcal{Q}$, all metrics that are compatible with $\mathcal{Q}$ determine the same shearfree null congruence, i.e., the foliation by the integral curves of $\mathcal{K}$. For this reason, an optical geometry can be regarded as the geometry underlying a prescribed shearfree null congruence.

2.2. **Robinson and Trautman optical geometries and CR structures.** Consider now an optical geometry $\mathcal{Q} = (\mathcal{W}, [h]_\pm, \mathcal{K}, \{g\})$ on a manifold M, where the 1-dimensional distribution $\mathcal{K}$ is generated by a vector field p, which is not only preserves the codimension one distribution $\mathcal{W}$, but is also complete and generates a group $A = \{e^{t\mathrm{p}}, t \in \mathbb{R}\}$ of diffeomorphisms (isomorphic to $\mathbb{R}$ or S^1) acting freely and properly. This is equivalent to saying that $\mathcal{Q}$ is an optical geometry on the total space of the principal A-bundle $\pi : M \to \mathcal{S} = M/A$, with the distribution $\mathcal{K}$ equal to the vertical distribution $\mathcal{K} \subset T^{\mathrm{vert}}M$ of the bundle. Following the terminology introduced in [3], we call the optical geometries of this kind *Robinson and Trautman geometries* (or just *RT geometries*, for short).

Remark 2.1. Notice that *any optical geometry is* <u>*locally*</u> *identifiable with an RT geometry.* Indeed, if $\tilde{\mathrm{p}}$ is a vector field on a manifold M, that is tangent to a shearfree congruence and whose local flow preserves the distribution $\mathcal{W}$ and the conformal class $[h]$ of an optical geometry $\mathcal{Q}$, then around any $x \in M$, we may choose coordinates $(x^0, x^1, \ldots x^{n-1})$, in which the flow of $\tilde{\mathrm{p}}$ takes the translational form $\Phi^{\tilde{\mathrm{p}}}_t(x^0, x^1, \ldots, x^{n-1}) = (x^0 + t, x^1, \ldots, x^n)$. In this way, we may determine a local diffeomorphism between a neighbourhood of $x \in M$ and an open set of a trivial bundle of the form $\pi : M' = \mathcal{S} \times A \to \mathcal{S}$ with $A = S^1$ or $A = \mathbb{R}$, mapping $\tilde{\mathrm{p}}$ into the vector field $\mathrm{p} = \frac{\partial}{\partial r}$ whose flow generates the standard translations of the 1-dimensional group A. One can directly check that under such a local diffeomorphism, the optical geometry $\mathcal{Q}$ is locally mapped into an RT geometry $\mathcal{Q}'$ on $M' = \mathcal{S} \times A$. Due to this observation, *if one is interested only in* local *properties, there is essentially no loss of generality in restricting to the class of RT geometries.*

Each RT geometry on the total space of a bundle $\pi : M \to \mathcal{S} = M/A$ is known to be determined by a uniquely associated *sub-conformal structure* on the base space $\mathcal{S}$, i.e., by a codimension one distribution $\mathcal{D}$ equipped with an equivalence class $[g^{\mathcal{D}}]_\pm$ of Riemannian or negative definite metrics on $\mathcal{D}$, each determined from the others by multiplication with a nowhere-vanishing conformal factor. In fact,

Proposition 2.2 ([3, Prop. 3.6])**.** *Given an A-bundle $\pi : M \to \mathcal{S}$, with $A = \mathbb{R}$ or S^1, there is a natural one-to-one correspondence between the RT geometries $\mathcal{Q} = (\mathcal{W}, [h]_\pm, \mathcal{K}, \{g\})$ on M with $\mathcal{K} = T^{\mathrm{vert}}M$ and the sub-conformal structures $(\mathcal{D}, [g^{\mathcal{D}}]_\pm)$ on $\mathcal{S}$.*

A distinguished class of RT geometries is given by the even-dimensional total spaces of A-bundles M, associated with sub-conformal structures $(\mathcal{D}, [g^{\mathcal{D}}])$ on their base spaces, where $\mathcal{D} \subset T(M/A)$ is a contact distribution. These are called *(maximally) twisting* RT geometries. They are particularly interesting because there exists a natural correspondence between them and an appropriate class of almost CR structures. Let us briefly review this correspondence.

Let $\mathcal{S}$ be an odd-dimensional manifold with $\dim \mathcal{S} = 2n-1$. A *(codimension one) almost CR structure on* $\mathcal{S}$ is a pair $(\mathcal{D}, J)$, consisting of:

- a codimension-one distribution $\mathcal{D} \subset T\mathcal{S}$, and
- a smooth family J of endomorphisms of the subspaces $\mathcal{D}_x \subset T_x\mathcal{S}$, $x \in \mathcal{S}$, such that $J_x^2 = -\operatorname{Id}_{\mathcal{D}_x}$ for all x.

An almost CR structure on $\mathcal{S}$ is called *integrable* if the following two conditions hold:

(1) For any pair of vector fields X, Y in $\mathcal{D}$, the vector field $[X, JY] + [JX, Y]$ also lies in $\mathcal{D}$;

(2) The Nijenhuis tensor field $N_J \in \mathcal{D}^* \otimes \mathcal{D}^* \otimes \mathcal{D}$ of J, defined by
$$N_J(X, Y) = [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) \ , \quad X, Y \in \mathcal{D},$$
vanishes identically. .

A *CR structure* is an integrable almost CR structure. If an almost CR structure satisfies only condition (1) (but not necessarily (2)), it is called *partially integrable*.

Note that if $\dim \mathcal{S} = 3$, then any almost CR structure on $\mathcal{S}$ is integrable.

Let $(\mathcal{D}, J)$ be a partially integrable CR structure on $\mathcal{S}$ and let θ be a *defining* 1*-form for* $\mathcal{D}$, i.e., a 1-form such that $\ker \theta_x = \mathcal{D}_x$ for every $x \in \mathcal{S}$. One can directly check that, in this case, the type $(0, 2)$ tensor field on $\mathcal{D}$ defined by
$$h^\theta \in \mathcal{D}^* \times \mathcal{D}^* \ , \qquad h^\theta(X, Y) := d\theta(X, JY)$$
is symmetric. This is called the *Levi form* of the partially integrable CR structure, and it is independent of the choice of θ *up to a multiplication by a nowhere vanishing conformal factor*. Note that when $\mathcal{D}$ is contact, the Levi form $h^\theta = d\theta(\cdot, J\cdot)$ is non-degenerate. If it is positive definite, the partially integrable CR structure is said to be *strongly pseudoconvex*.

We are now ready to state the precise relation between twisting RT geometries and CR structures. The following is a direct application of [3, Cor. 4.5].

Theorem 2.3. *Let $\pi : M \to \mathcal{S} = M/A$ be a principal bundle with* 1*-dimensional structure group $A = \mathbb{R}$ or S^1, and with even-dimensional total space M. Then there are natural (with respect to A-bundle automorphisms) bijections between the following three classes:*

(1) The twisting RT geometries $\mathcal{Q} = (\mathcal{W}, [h]_\pm, \mathcal{K}, \{g\})$ on M with $\mathcal{K} = T^{\mathrm{vert}}M$;

(2) The sub-conformal structures $(\mathcal{D}, [g^{\mathcal{D}}]_\pm)$ on $\mathcal{S}$ where $\mathcal{D}$ is a contact distribution;

(3) The pairs $((\mathcal{D}, J), [B]_\pm)$, consisting of

- *a strongly pseudoconvex partially integrable CR structure $(\mathcal{D}, J)$;*
- *a class $[B]_\pm$ of smooth fields of endomorphisms $B_x : \mathcal{D}_x \to \mathcal{D}_x$, $x \in M$, each defined up to multiplication by a nowhere vanishing factor, and either positive definite or negative definite with respect to the positive Levi form of $(\mathcal{D}, J)$.*

Explicit expressions for the bijections between the classes (1) – (3) in Theorem 2.3, given in full detail, can be found in [3].

2.3. **Optical lifts of Sasaki CR structures.** An immediate and important consequence of Theorem 2.3 is that the twisting RT geometries on a fixed $2n$-dimensional principal A-bundle $\pi : M \to \mathcal{S}$, $A = \mathbb{R}$ or S^1, are naturally grouped according to the corresponding partially integrable CR structures on $\mathcal{S}$. More precisely, two RT geometries

$$\mathcal{Q} = (\mathcal{W}, [h]_\pm, \mathcal{K}, \{g\}) , \qquad \mathcal{Q}' = (\mathcal{W}', [h']_\pm, \mathcal{K}' = \mathcal{K}, \{g'\})$$

on the A-bundle $\pi : M \to \mathcal{S}$ are said to *have the same underlying CR structure* if the corresponding pairs $((\mathcal{D}, J), [B]_\pm)$ and $((\mathcal{D}', J'), [B']_\pm)$ in Theorem 2.3 (3) satisfy

$$(\mathcal{D}, J) = (\mathcal{D}', J') .$$

Due to the naturality of the bijection given in Theorem 2.3, if $(\mathcal{D}, J)$ is a strongly pseudoconvex partially integrable CR structure on $\mathcal{S}$, and $\{\{g\}\}$ denotes the class of Lorentzian metrics on M compatible with at least one RT geometry having $(\mathcal{D}, J)$ as underlying CR structure, then any bundle automorphism of $\pi : M \to \mathcal{S}$, that projects onto an automorphism of the CR structure $(\mathcal{D}, J)$ maps the class $\{\{g\}\}$ into itself. Conversely, any automorphism of the CR structure $(\mathcal{D}, J)$ that lifts to a bundle automorphism of $\pi : M \to \mathcal{S}$, preserves the class $\{\{g\}\}$.

Following the terminology adopted in [12, 10], throughout this paper we refer to the metrics in such invariant class $\{\{g\}\}$ as the *optical (Lorentzian) lifts* of the almost CR structure $(\mathcal{D}, J)$. By the above discussion, any invariant of the optical lifts *considered determined up to the action of a bundle diffeomorphism preserving the entire class* $\{\{g\}\}$, canonically corresponds to an invariant of the underlying almost CR structure $(\mathcal{D}, J)$.

Let $\mathcal{S}$ be a $(2n-1)$-dimensional manifold equipped with a strongly pseudoconvex *integrable* CR structure $(\mathcal{D}, J)$. The triple $(\mathcal{S}, \mathcal{D}, J)$ is called a *regular Sasaki manifold* if it is the total space of a principal A'-bundle $\pi : \mathcal{S} \to N$, with $A' = \mathbb{R}$ or S^1, such that the right action of A' leaves the CR structure invariant.

If $(\mathcal{S}, \mathcal{D}, J)$ is a regular Sasaki manifold, then:

(a) There exists an A'-invariant defining 1-form θ for $\mathcal{D}$, which also serves as a connection 1-form for the A'-bundle;
(b) The complex structure J and the Levi form h^θ of $(\mathcal{D}, J)$ project onto an integrable complex structure J and a Riemannian metric g_o on N, making (N, g_o, J) a Kähler manifold;
(c) The Kähler form $\omega_o := g_o(J\cdot, \cdot)$ of g_o satisfies $d\theta = \pi^*\omega_o$.

Conversely, a Kähler manifold (N, J, g_o) with Kähler form ω_o is called *quantisable* if there exists a principal A'-bundle $\pi : \mathcal{S} \to N$, $A' = S^1$ or $\mathbb{R}$, equipped with a connection 1-form $\theta : T\mathcal{S} \to \mathbb{R}$ satisfying the condition $d\theta = \pi^*\omega_o$. In this case, the horizontal distribution $\mathcal{D} = \ker\theta$ on $\mathcal{S}$ is contact, and the lift of the complex structure J on N to $\mathcal{D}$ makes $(\mathcal{S}, \mathcal{D}, J)$ a regular Sasaki manifold.

We stress the fact that *the correspondence between regular Sasaki manifolds and quantisable Kähler manifolds described above is a bijection* (see [1]).

Consider now a regular Sasaki $(2n-1)$-manifold $(\mathcal{S}, \mathcal{D} = \ker\theta, J)$ and denote by (N, J, g_o) its associated quantisable Kähler $2n-2$-manifold. Let also $\pi^{\mathcal{S}} : M = \mathcal{S}\times\mathbb{R} \to \mathcal{S}$ be the trivial $\mathbb{R}$-bundle over $\mathcal{S}$, and let ϑ and p_o denote, respectively, the pulled-back 1-form $\vartheta = \pi^{\mathcal{S}*}\theta$ and the vertical vector field $\mathrm{p}_o = \frac{\partial}{\partial r}$ generating the $\mathbb{R}$-action on $M = \mathcal{S}\times\mathbb{R}$. According to Theorem 2.3 there is a one-to-one correspondence between the twisting RT geometries on $M = \mathcal{S}\times\mathbb{R}$, with 1-dimensional distribution $\mathcal{K}_o = \langle \mathrm{p}_o\rangle = T^{\mathrm{vert}}M \subset TM$ and the pairs $((\mathcal{D}, J), [B]_\pm)$ as in item (3) of that theorem, where $(\mathcal{D}, J)$ is the CR structure on $\mathcal{D}$. We recall that, following the proof of [3, Cor. 4.5], all such RT geometries can be explicitly determined.

Among them, the simplest RT geometry is the one corresponding to the class of endomorphisms $[B = \mathrm{Id}_{\mathcal{D}}]_\pm$. Such RT geometry turns out to be the quadruple $\mathcal{Q}_o = (\mathcal{W}_o, [h_o]_\pm, \mathcal{K}_o = \langle \mathrm{p}_o\rangle, \{g\})$, defined as follows (for further details, see e.g., [3, 4, 9]):

- $\{g\}$ is the set of the Lorentzian metrics on M having the form
$$g = \sigma\breve{\pi}^* g_o + \vartheta \vee \left(\sigma\alpha \mathrm{p}_o^* + \sigma\mathfrak{d} + \frac{\sigma\beta}{2}\vartheta\right), \tag{2.2}$$
where we use the following notation:
  (i) $\breve{\pi} := \pi\circ\pi^{\mathcal{S}}$;
  (ii) p_o^* is the unique $\mathbb{R}$-invariant 1-form, satisfying the condition $\mathrm{p}_o^*(\mathrm{p}_o) = 1$, with $\mathcal{H} := \ker \mathrm{p}_o^* \cap \ker\vartheta$ equal to the unique $\mathbb{R}$-invariant horizontal distribution projecting onto $\mathcal{D}\subset T\mathcal{S}$,
  (iii) σ, α, β are freely specifiable $\mathscr{C}^\infty$ functions with $\sigma, \alpha \neq 0$ at all points, and $\mathfrak{d}$ is a freely specifiable 1-form with the property that $\ker\mathfrak{d}$ contains the vertical distribution of $\breve{\pi} : M \to N$,
- $\mathcal{W}$ is the g-orthogonal distribution $\mathcal{W} = \mathcal{K}^{\perp_g}$ for one (and hence, for any) metric $g \in \{g\}$;
- $[h_o]_\pm$ is the pair of conformal classes of the degenerate metrics $h = \pm g|_{\mathcal{W}\times\mathcal{W}}$ with $g\in\{g\}$.

We call this the *standard RT geometry of the $A\times\mathbb{R}$-bundle* $\breve{\pi} = \pi^{\mathcal{S}}\circ\pi : M = \mathcal{S}\times\mathbb{R}\to N$. Note that each compatible metric g of this optical geometry is completely determined by the quadruple $(\sigma, \alpha, \beta, \mathfrak{d})$, consisting of three real functions σ, α, β, with σ, α nowhere vanishing, and the tensor field $\mathfrak{d}\in\mathcal{H}^*$. We call this quadruple the *(canonical) datum* of g.

3. Preliminaries (II): Null fluid gravitational fields on Kerr manifolds

3.1. **An important class of optical geometries: the Kerr structures.** Following the same line of reasoning as in Remark 2.1, we point out that the total space of a principal bundle $\pi^{\mathcal{S}} : M \to \mathcal{S}$ with a 1-dimensional structure group $A = \mathbb{R}$ or S^1, is always locally diffeomorphic to a trivial $\mathbb{R}$-bundle $\pi^{\mathcal{S}} : \mathcal{S}'\times\mathbb{R}\to\mathcal{S}'$ for some open subset $\mathcal{S}'\subset\mathcal{S}$. Moreover, if $\mathcal{S}$ is the total space of an A'-bundle $\pi : \mathcal{S}\to N$, with a 1-dimensional structure group $A' = \mathbb{R}$ or S^1 as well, the trivial bundle $\mathcal{S}'\times\mathbb{R}$ is in turn locally diffeomorphic to a

Cartesian product $\pi \circ \pi^{\mathcal{S}} : \mathcal{V} \times \mathbb{R} \times \mathbb{R}$ over an open subset $\mathcal{V} \subset N$. Because of this, from a purely local point of view, there is essentially no loss of generality in restricting to the RT geometries on bundles over Sasaki manifolds for which the structure groups A and A' are both isomorphic to $\mathbb{R}$. This observation, together with the celebrated examples provided by the 4-dimensional Kerr metrics, is the main motivations for focusing on the class of Lorentzian manifolds equipped with the following structure.

Definition 3.1. A *Kerr structure of dimension* $n = 2(k+1)$ is a quadruple

$$\mathcal{M} = (M = \mathcal{S} \times \mathbb{R}, \pi : \mathcal{S} \to N, (J, g_o), \mathcal{H}) ,$$

given by:

(a) An n-dimensional manifold M, identifiable with the Cartesian product $M = \mathcal{S} \times \mathbb{R}$, where $\mathcal{S}$ is a regular Sasaki manifold that is an $\mathbb{R}$-bundle $\pi : \mathcal{S} \to N$ over a quantisable Kähler manifold N as described in (b);

(b) A quantisable Kähler manifold (N, J, g_o) of dimension $2k$ with Kähler form $\omega_o := g_o(J\cdot, \cdot)$;

(c) A connection $\mathcal{H} \subset T\mathcal{S}$ on the $\mathbb{R}$-bundle $\pi^{\mathcal{S}} : \mathcal{S} \to N$, which is the kernel distribution of a connection 1-form $\theta^{\mathcal{S}}$ satisfying $d\theta^{\mathcal{S}} = \pi^*\omega_o$.

A manifold M as in (a), equipped with the above-defined Kerr structure $\mathcal{M}$, is called a *Kerr manifold*. By a slight abuse of language, we also refer to any connected open subset $M' = \mathcal{S} \times I$, with $I \subset \mathbb{R}$, of a Kerr manifold $M = \mathcal{S} \times \mathbb{R}$ as a "Kerr manifold".

Being a (trivial) bundle over a regular Sasaki manifold, any Kerr manifold $M = \mathcal{S} \times \mathbb{R}$ is naturally endowed with its standard RT geometry $\mathcal{Q}_o$ as defined in §2.3. For brevity, the compatible metrics of $\mathcal{Q}_o$ are referred to as *compatible metrics of the Kerr manifold*.

3.2. **Kähler potentials and complex structures on Kerr manifolds.** Let $M = \mathcal{S} \times \mathbb{R}$ be a Kerr manifold with Kerr structure $\mathcal{M} = (M = \mathcal{S} \times \mathbb{R}, \pi : \mathcal{S} \to N, (J, g_o), \mathcal{H})$ and underlying quantisable Kähler manifold (N, J, g_o). Let also q_o and p_o be the vector fields of M, whose flows give the right actions of the structure groups of the bundles $\pi : \mathcal{S} \to N$ and $\pi^{\mathcal{S}} : M = \mathcal{S} \times \mathbb{R} \to \mathcal{S}$, respectively (here, we trivially identify the generator q_o of the right action on $\mathcal{S}$ with a vector field on the cartesian product $M = \mathcal{S} \times \mathbb{R}$). Then, for each $x = (U, r) \in M = \mathcal{S} \times \mathbb{R}$, we may consider the standard identification

$$T_{(U,r)}M \simeq T_U\mathcal{S} + T_r\mathbb{R} = T_U\mathcal{S} + \mathbb{R}\mathrm{p}_o|_r \ . \tag{3.1}$$

Such a Kerr manifold is naturally equipped with the family of (locally defined) complex structures, one per each Kähler potential for g_o, which we define as follows.

Consider a local trivialisation $\mathcal{S}|_{\mathcal{V}} \simeq \mathcal{V} \times \mathbb{R}$ for the $\mathbb{R}$-bundle $\pi : \mathcal{S} \to N$ over an open subset $\mathcal{V} \subset N$ and the corresponding trivialisation $\mathcal{S}|_{\mathcal{V}} \times \mathbb{R} \simeq \mathcal{V} \times \mathbb{R}^2$ of $M = \mathcal{S} \times \mathbb{R}$. Assume also that $\mathcal{V}$ is sufficiently small so that:

(a) there exists a Kähler potential $\varphi : \mathcal{V} \to \mathbb{R}$ for g_o and

(b) there is a system of complex coordinate $(z^1 = x^1 + \mathrm{i}\, y^1, \ldots, z^{n-1} = x^{n-1} + \mathrm{i}\, y^{n-1})$ on the open subset $\mathcal{V}$ of the complex manifold (N, J).

Let us also adopt the shorthand notation $\varphi_{x^i} := \frac{\partial \varphi}{\partial x^i}$ and $\varphi_{y^i} := \frac{\partial \varphi}{\partial y^i}$. One can directly verify (see e.g. [9, Lemma 2.7]) that the distribution $\mathcal{D}$ of the CR structure of $\mathcal{S}|_{\mathcal{V}}$ (which is also identifiable with a codimension two distribution of M through the identification (3.1)) is spanned by the vector fields:

$$\begin{aligned} \widehat{\frac{\partial}{\partial x^i}} &= \frac{\partial}{\partial x^i} - d^c\varphi\left(\frac{\partial}{\partial x^i}\right)\mathrm{q}_o = \frac{\partial}{\partial x^i} + \varphi_{y^i}\frac{\partial}{\partial u} \ , \\ \widehat{\frac{\partial}{\partial y^i}} &= \frac{\partial}{\partial y^i} - d^c\varphi\left(\frac{\partial}{\partial y^i}\right)\mathrm{q}_o = \frac{\partial}{\partial y^i} - \varphi_{x^i}\frac{\partial}{\partial u} \end{aligned} \tag{3.2}$$

and the complex structure J of the CR structure $(\mathcal{D}, J)$ is given by the family of involutive linear isomorphisms of the spaces $\mathcal{D}_{(U,r)}$ defined by

$$J\left(\widehat{\frac{\partial}{\partial x^i}}\right) = \widehat{\frac{\partial}{\partial y^i}} \ , \qquad J\left(\widehat{\frac{\partial}{\partial y^i}}\right) = -\left(\widehat{\frac{\partial}{\partial x^i}}\right) \ . \tag{3.3}$$

Consider now the diffeomorphism

$$\Phi^\varphi : \mathcal{V}\times\mathbb{R}^2 \longrightarrow \mathcal{V}\times\mathbb{R}^2 \ , \qquad \Phi^\varphi(x^i, y^j, u, r) = (x^i, y^j, u, r + \varphi(x^i, y^j)) \ ,$$

whose associated pushing-forward map Φ^φ_* is such that

$$\Phi^\varphi_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \varphi_{x^i}\frac{\partial}{\partial r} \ , \qquad \Phi^\varphi_*\left(\frac{\partial}{\partial y^j}\right) = \frac{\partial}{\partial y^j} + \varphi_{y^j}\frac{\partial}{\partial r} \ ,$$
$$\Phi^\varphi_*\left(\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u} \ , \qquad \Phi^\varphi_*\left(\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial r} \ .$$

Thus, Φ^φ_* sends the CR structure $(\mathcal{D}, J)$ into the CR structure $(\mathcal{D}', J'|_{\mathcal{D}'})$ of $\mathcal{V}\times\mathbb{R}^2$, given by the distribution $\mathcal{D}'$ spanned by the vector fields

$$\Phi^\varphi_*\left(\widehat{\frac{\partial}{\partial x^i}}\right) = \frac{\partial}{\partial x^i} + \varphi_{y^i}\frac{\partial}{\partial u} + \varphi_{x^i}\frac{\partial}{\partial r} \ , \qquad \Phi^\varphi_*\left(\widehat{\frac{\partial}{\partial x^i}}\right) = \frac{\partial}{\partial y^i} - \varphi_{x^i}\frac{\partial}{\partial u} + \varphi_{y^i}\frac{\partial}{\partial r} \ ,$$

and the restriction to $\mathcal{D}'$ of the (integrable) complex structure J' on $\mathcal{V}\times\mathbb{R}\times\mathbb{R}$, defined by

$$J'\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} \ , \qquad J'\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i} \ , \qquad J'\left(\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial r} \ , \qquad J'\left(\frac{\partial}{\partial r}\right) = -\frac{\partial}{\partial u} \ .$$

The pulled-back (integrable) complex structure $J^{(M,\varphi)} := F^{\varphi *}(J')$ on $\mathcal{S}|_{\mathcal{V}}\times\mathbb{R} \simeq \mathcal{V}\times\mathbb{R}^2$ is called the *complex structure canonically determined by the Kähler potential* φ.

By construction, the CR structures $(\mathcal{D}, J)$ of the hypersurfaces $\mathcal{S}|_{\mathcal{V}}\times\{r_o\} \subset \mathcal{S}|_{\mathcal{V}}\times\mathbb{R}$, $r_o \in \mathbb{R}$, coincide with the CR structures which are induced on these hypersurfaces by $J^{(M,\varphi)}$. Notice also that the $\mathbb{C}$-valued coordinates on $\mathcal{S}|_{\mathcal{V}}\times\mathbb{R}$ defined by

$$(z^i := x^i + \mathrm{i}\, y^i, w := u + \mathrm{i}(r + \varphi(x, y)) \tag{3.4}$$

are given by the expression of the map Φ^φ in terms of the (holomorphic) complex coordinates (z^i, w) for the complex manifold $\big(\mathcal{S}|_{\mathcal{V}} \times \mathbb{R}, J'\big)$. Thus (3.4) are (holomorphic) complex coordinates for the complex manifold $\big(\mathcal{S}|_{\mathcal{V}} \times \mathbb{R}, J^{(M,\varphi)}\big)$.

3.3. **Null fluid gravitational fields and quasi-Einstein metrics.** Let (M, g) be an n-dimensional Lorentzian manifold. From now on, *we adopt the notation* $\mathrm{sign}(g)$ *to indicate the integer* $+1$ *or* -1*, depending on whether the signature of* g *is mostly plus or mostly minus, respectively.*

Let g represent a gravitational field determined by a so-called *perfect null fluid*. From a mathematical point of view, this condition means that the metric g is a solution of an equation of the form

$$\mathrm{Ric} - \frac{1}{2}\,\mathrm{Scal}\, g = -\Lambda' g + \mathsf{k}\, T \tag{3.5}$$

where k is the physical constant $\mathsf{k} = \frac{8\pi G}{c^4}$ and where:

- Ric and Scal are the Ricci and scalar curvatures of g, respectively,
- $\Lambda' := \mathrm{sign}(g)\Lambda$, with Λ denoting the cosmological constant of a background metric on M,
- T is a symmetric tensor field of type $(0, 2)$, which represents the energy momentum tensor of a perfect null fluid.

We recall that the latter condition on T means that the tensor $\mathsf{k}T$ has the form

$$\mathsf{k}T = \mathsf{f}\,\vartheta \vee \vartheta$$

for some smooth real function $\mathsf{f} : M \to \mathbb{R}$ and a 1-form ϑ that is null with respect to g.

Note that:

– The scalar curvature Scal of a metric satisfying an equation of the form (3.5) is necessarily constant – actually, taking the g-trace on both sides of (3.5), one obtains that $\mathrm{Scal} = \frac{2n}{n-2}\Lambda'$, so that (3.5) is equivalent to

$$\mathrm{Ric} = \frac{2}{n-2}\,\Lambda' g + \mathsf{k}T\ . \tag{3.6}$$

– The solutions to (3.5) (or to (3.6)) are analogs of the "Lorentzian quasi-Einstein metrics" defined in [19], that is, solutions to equations of the form (3.5) for some *time-like* ϑ.

If M is a Kerr manifold with Kerr structure $\mathcal{M} = (M = \mathcal{S} \times \mathbb{R}, \pi : \mathcal{S} \to N, (J, g_o), \mathcal{H})$, and if $\vartheta = \pi^{\mathcal{S}*}\theta$ is the pulled-back 1-form on M corresponding to the contact form of $\mathcal{S}$, we know that ϑ is null with respect to *any* of the compatible metrics of M. Thus, a natural ansatz for finding solutions to (3.5) is to assume that the unknown g is a compatible metric of a Kerr manifold. This is the main motivation for considering the following

Definition 3.2. A compatible metric g of a Kerr manifold $M = \mathcal{S} \times \mathbb{R}$ is called *quasi-Einstein* if it satisfies an equation of the form

$$\mathrm{Ric} = \frac{2}{n-2}\,\Lambda' g + \mathsf{f}\,\vartheta \vee \vartheta\ , \qquad \text{with } \vartheta = \pi^{\mathcal{S}*}\theta\ , \tag{3.7}$$

for some smooth real function f and some real constant Λ'.

We conclude this section with the following useful remark: *if* $\mathsf{F} : M \to M$ *is an automorphism of* ϑ *(i.e., a diffeomorphism such that* $\mathsf{F}^*\vartheta = \vartheta$*), then a metric* g *is a solution to* (3.7) *if and only if* $g' := \mathsf{F}^* g$ *is a solution of that equation too.*

Very simple examples of automorphisms of the 1-form ϑ of a Kerr manifold $M = \mathcal{S} \times \mathbb{R}$ are those whose coordinate expressions have the form

$$\mathsf{F}(x, y, u, r) = \Big(x, y, u, r'(x, y, u, r)\Big) \tag{3.8}$$

where we denote by (x, y, u) a set of local coordinates for $\mathcal{S}$ and by r the standard coordinate for the fiber $\mathbb{R}$. The (local) diffeomorphisms of the form (3.8) will be called *bending diffeomorphisms*. It is clear that the quadruple

$$\mathcal{Q}_o^{\mathsf{F}} = \Big(\mathsf{F}_*(\mathcal{W}_o), [\mathsf{F}_*(h_o)]_{\pm}, \mathsf{F}_*(\mathcal{K}_o), \{\mathsf{F}_* g\}\Big) \tag{3.9}$$

is still an optical geometry on M and the compatible metrics of $\mathcal{Q}_o$ are isometric to the compatible metrics of $\mathcal{Q}_o^{\mathsf{F}}$. On the other hand, the shearfree congruence of the optical geometry $\mathcal{Q}_o$ is in general different from the one of $\mathcal{Q}_o^{\mathsf{F}}$. Indeed, the first is given by the integral curves of the generator $\mathrm{p}_o = \frac{\partial}{\partial r}$ of the null distribution $\mathcal{K}_o$, while the shearfree congruence of $\mathcal{Q}_o^{\mathsf{F}}$ is given by the integral curves of $\mathrm{p} := \mathsf{F}_*(\mathrm{p}_o)$. The corresponding optical geometry $\mathcal{Q}_o^{\mathsf{F}}$ (resp. metric compatible with $\mathcal{Q}_o^{\mathsf{F}}$) is said to be F-*bended*.

4. Quasi-Einstein metrics on 4-dimensional Kerr manifolds

In [12] the authors determined a complete parametrisation of the local coordinate expressions of the 4-dimensional Lorentzian metrics that satisfy the following conditions:

(a) they are compatible with a prescribed twisting optical geometry, projecting onto a strongly pseudoconvex 3-dimensional *embeddable* CR manifold (i.e. CR equivalent to a real hypersurface of $\mathbb{C}^2$);
(b) they satisfy null fluid equations of the form (3.6).

The parametrisation in [12] is given in terms of the solutions of a particular system of complex p.d.e.'s. Building on this result, we give here a detailed description of the quasi-Einstein Lorentzian metrics, which are compatible with the optical geometry of a Kerr 4-manifold. This is possible because any such optical geometry is twisting and projecting onto a Sasaki CR structure. Indeed, since the Sasaki CR structures are well known to be embeddable (for instance, see [5, 13, 7]), the optical geometries we consider in this paper nicely fit within the class that is considered in [12].

Throughout the following, $M = \mathcal{S} \times \mathbb{R}$ is a Kerr 4-manifold, equipped with a Kerr structure $\mathcal{M} = (M = \mathcal{S} \times \mathbb{R}, \pi : \mathcal{S} \to N, (J, g_o), \mathcal{H})$ and the corresponding standard RT geometry $\mathcal{Q}_o$. We use the notation $\pi^{\mathcal{S}} : M = \mathcal{S} \times \mathbb{R} \to \mathcal{S}$ and $\breve{\pi} = \pi \circ \pi^{\mathcal{S}} : M \to N$ to indicate the two projections of M onto $\mathcal{S}$ and N, respectively. We also recall that, given a

Lorentzian metric g, we define $\mathrm{sign}(g)$ to be equal to $+1$ (resp.-1) if the signature of g is $(+,+,+,-)$ (resp. $(-,-,-,+)$).

Theorem 4.1. *Let g be a $\mathcal{Q}_o$-compatible Lorentzian metric on $M = \mathcal{S}\times\mathbb{R}$, i.e. of the form*

$$g = \sigma\breve{\pi}^* g_o + \vartheta \vee \left(\sigma\alpha \mathrm{p}_o^* + \sigma\mathfrak{d} + \frac{\sigma\beta}{2}\vartheta\right),$$

where $\vartheta := \pi^*\theta$ is the pulled-back 1-form of the contact form θ of $\mathcal{S}$

$$\textit{(thus, with } d\vartheta = \breve{\pi}^*(\omega_o)\ ,\ \omega_o := g_o(J\cdot,\cdot)) \tag{4.1}$$

and assume that it has constant scalar curvature $\mathrm{Scal} = 4\Lambda'$ *for some $\Lambda' \in \mathbb{R}$ and has a datum $(\sigma,\alpha,\mathfrak{d},\beta)$ satisfying the conditions*

$$\mathrm{q}_o(\sigma) = \mathrm{q}_o(\alpha) = \mathrm{q}_o(\beta) = 0\ , \qquad \mathcal{L}_{\mathrm{q}_o}\mathfrak{d} = 0\ . \tag{4.2}$$

Let also $\mathcal{V}\subset N$ be an open subset of N, on which there exists a potential φ for the Kähler form $\omega_o = dd^c\varphi$ and for which there are coordinates $\xi = (x,y,u,r)$ on $\breve{\pi}^{-1}(\mathcal{V})$ such that

- *x and y are the real and imaginary parts of a holomorphic coordinate $z = x + \mathrm{i}\,y$ on $\mathcal{V}$,*
- *u is a fiber coordinate for the $\mathbb{R}$-bundle $\mathcal{S}|_{\mathcal{V}} \simeq \mathcal{V}\times\mathbb{R}$, so that $\mathrm{q}_o = \frac{\partial}{\partial u}$,*
- *r is the standard coordinate of the factor $\mathbb{R}$ of $M = \mathcal{S}\times\mathbb{R}$, so that $\mathrm{p}_o = \frac{\partial}{\partial r}$.*

This metric satisfies an equation of the form $\mathrm{Ric} = \Lambda' g + \mathsf{f}\,\vartheta\vee\vartheta$ *for some smooth function f on $\breve{\pi}^{-1}(\mathcal{V})$ if and only if, on such a set, it has the form*

$$\begin{aligned} g = {}& \frac{1}{4}\left(\kappa + \frac{r'^2}{\kappa}\right)\breve{\pi}^* g_o + \vartheta \vee \Bigg\{ dr' + \underbrace{\kappa_y dx - \kappa_x dy}_{=-d^c\kappa} + \\ & + \left(\kappa + \frac{r'^2}{\kappa}\right)(t_2 dx + t_1 dy) + 2\big(t_2\kappa - r' t_1\big)dx + 2\big(t_1\kappa + r' t_2\big)dy + \frac{1}{\Delta\varphi}\left(\kappa + \frac{r'^2}{\kappa}\right)\mathcal{H}\,\vartheta\Bigg\}\ , \end{aligned} \tag{4.3}$$

where:

(i) $t_i = t_i(x,y)$, $i = 1,2$, are the real and imaginary parts of a solution $t = t_1 + \mathrm{i}\,t_2$ to

$$\partial_z t - (\partial_z(\log\Delta\varphi) + t)t = 0\ ; \tag{4.4}$$

(ii) $\kappa = \kappa(x,y)$ is a function of the form $\kappa := \mathrm{sign}(g)\widetilde{\kappa}^2$ for a solution $\widetilde{\kappa} = \widetilde{\kappa}(x,y)$ of

$$\Delta\widetilde{\kappa} = \frac{2^8\,\mathrm{Re}(m)}{(\Delta\varphi)^2}\frac{1}{\widetilde{\kappa}^3} + \left(\frac{1}{4}\Delta\left(\log\Delta\varphi\right) + 3\left(\partial_z\bar{t} + \partial_{\bar{z}}t + |t|^2\right)\right)\widetilde{\kappa} + \left(\frac{\Lambda\Delta\varphi}{6}\right)\widetilde{\kappa}^3\ , \tag{4.5}$$

with $\Lambda = \mathrm{sign}(g)\Lambda'$ and $m(x,y) = m_1(x,y) + \mathrm{i}\,m_2(x,y)$ solution to

$$\partial_z m - 3\Big(\partial_z\left(\log\Delta\varphi\right) + t\Big)m = 0 \quad \textit{(here, } t = t_1 + \mathrm{i}\,t_2 \textit{ is the function in (i))}\ . \tag{4.6}$$

(iii) denoting $f := \log\left(\frac{\Delta\varphi}{4}\right)$, the function $\mathcal{H}$ depends only on x, y and is equal to the sum $\mathcal{H} := h_o + h_m + h_t$, in which we use the notation

$$h_o := \frac{1}{6(\kappa^2 + r'^2)}\Bigg(10\Lambda\kappa^3\Delta\varphi + 2\Lambda\kappa\Delta\varphi r'^2 + \\ + 3\kappa^2\Delta(\log(\Delta\varphi)) + 3\left(\kappa_x^2 + \kappa_y^2\right) - 3\kappa\Delta\kappa\Bigg)\ , \tag{4.7}$$

$$h_t := \frac{1}{\kappa^2 + r'^2}\Bigg(-t_{2x}r'\kappa - t_1\kappa_y r' + t_1 f_x\kappa^2 - t_1 f_x r'^2 - t_2 f_y\kappa^2 + t_2 f_y r'^2 - t_{1y}r'\kappa \\ + 2\kappa t_2\kappa_y - 2\kappa t_1\kappa_x - t_2\kappa_x r' - \frac{\kappa^{\frac{5}{2}}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} f_x t_1}{2} - \frac{\kappa^{\frac{3}{2}}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}}\kappa_x t_1}{2} - \\ - \frac{\sqrt{\kappa}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} r'^2 f_y t_2}{2} + \kappa^{\frac{3}{2}}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} r' f_y t_1 + \kappa^{\frac{3}{2}}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} r' f_x t_2 - \frac{\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} r'^2\kappa_y t_2}{2\sqrt{\kappa}} \\ + \frac{\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} r'^2\kappa_x t_1}{2\sqrt{\kappa}} + \sqrt{\kappa}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} r'\kappa_y t_1 + \sqrt{\kappa}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} r'\kappa_x t_2 + 11t_1^2\kappa^2 + t_1^2 r'^2 \\ + 5t_{1x}\kappa^2 - 5t_{2y}\kappa^2 + 11t_2^2\kappa^2 + t_2^2 r'^2 + \frac{\kappa^{\frac{5}{2}}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} f_y t_2}{2} + \frac{\kappa^{\frac{3}{2}}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}}\kappa_y t_2}{2} \\ - 2f_y t_1 r'\kappa - 2f_x t_2 r'\kappa + \frac{t_2\kappa_y r'^2}{\kappa} - \frac{t_1\kappa_x r'^2}{\kappa} + \frac{\sqrt{\kappa}\sqrt{2}\,\mathrm{e}^{\frac{f}{2}} r'^2 f_x t_1}{2}\Bigg)\ , \tag{4.8}$$

$$h_m := \frac{2^6\kappa\left(\kappa m_1 - m_2 r'\right)}{(\Delta\varphi)^2\left(\kappa^2 + r'^2\right)^2}\ , \tag{4.9}$$

where $t = t_1 + \mathrm{i}\,t_2$, κ and $m = m_1 + \mathrm{i}\,m_2$ are the functions defined in (i) and (ii);

(iv) $r' = r'(x, y, r)$ is a real function having the form

$$r'(x, y, r) = \kappa(x, y)\tan\left(\frac{\frac{\alpha}{2}r + \varphi(x, y) + s(x, y)}{2}\right)\ , \tag{4.10}$$

where $s{=}s(x, y)$ is a freely specifiable function of x and y, and κ is the function in (ii).

Remark 4.2. As noted in the Introduction, all these metrics have algebraically special curvature tensors at all points. Moreover, from the results in [12], for any such metric the Weyl scalar $\Psi_2 = \Psi_2(x, y, r)$ with respect to the adapted tetrads considered in that paper, is given in terms of the functions φ, κ, r' and $m = m_1 + \mathrm{i}\,m_2$ by

$$\Psi_2 = \frac{8\,\mathrm{sign}(g)}{\kappa^2 e^{3f}}\frac{m}{(\kappa - \mathrm{i}\,r')^3}\ . \tag{4.11}$$

It follows that, if $m = m_1 + \mathrm{i}\,m_2$ is a non-zero solution to (4.6) (resp. if $m \equiv 0$), then the corresponding metric has a Weyl scalar Ψ_2 that is not identically zero (resp. has the Weyl scalar Ψ_2 identically zero) and the Petrov type of its curvature is II or D on the set $\{\Psi_2 \neq 0\}$ (resp. III, N or 0 at all points).

Proof. Let (x, y, u, r) be coordinates on $\breve{\pi}^{-1}(\mathcal{V})$ as in the statement and denote by $\mathsf{F} : \mathcal{S}|_{\mathcal{V}} \times \mathbb{R} \to \mathcal{S}|_{\mathcal{V}} \times \mathbb{R}$ the bending diffeomorphism

$$\mathsf{F}(x, y, u, r) = \left(x, y, u, \breve{r} = \frac{\alpha(x, y, r)r}{2}\right) . \tag{4.12}$$

Under this diffeomorphism, $\mathcal{Q}_o$ is mapped to the optical structure (3.9) and, in particular, the $\mathcal{Q}_o$-compatible metric g is mapped to the $\mathcal{Q}^{\mathsf{F}}_o$-compatible metric $g^{\mathsf{F}} := \mathsf{F}_* g$. Since

$$\mathsf{F}_*(\breve{\pi}^* g_o) = \breve{\pi}^* g_o \ , \quad \mathsf{F}_*(\vartheta) = \vartheta \ , \quad \mathsf{F}_*(\mathrm{p}^*_o) = (\mathsf{F}^{-1})^* dr = \frac{2}{\alpha} d\breve{r} - \frac{2\breve{r}}{\alpha^2} d\alpha \ ,$$

the metric g^{F} has the form

$$g^{\mathsf{F}} = \sigma \breve{\pi}^* g_o + \vartheta \vee \left(2\sigma\, d\breve{r} + \sigma \mathfrak{d}' + \frac{\sigma\beta}{2}\vartheta\right) , \qquad \text{where } \mathfrak{d}' := \mathfrak{d} - \frac{2\breve{r}}{\alpha} d\alpha \ . \tag{4.13}$$

By the remarks in §3.3, g is quasi-Einstein if and only if g^{F} is. Moreover (1) g^{F} is $\mathcal{Q}^{\mathsf{F}}_o$-compatible and (2) $\mathcal{Q}^{\mathsf{F}}_o$ is the standard optical geometry for the Kerr structure on $\breve{\pi}^{-1}(\mathcal{V})$ which is given if one identifies $\breve{\pi}^{-1}(\mathcal{V})(= \mathcal{S}|_{\mathcal{V}} \times \mathbb{R})$ with the cartesian product $\mathcal{S}|_{\mathcal{V}} \times \mathbb{R}$ via the diffeomorphism $\mathsf{F} : \mathcal{S}|_{\mathcal{V}} \times \mathbb{R} \to \mathcal{S}|_{\mathcal{V}} \times \mathbb{R}$ (instead of just using the identity map!). Hence, by replacing the original Kerr structure with the one just described, expression (4.13) shows that there is no loss of generality if, from now on, we assume that g has a datum $(\sigma, \alpha, \mathfrak{d}, \beta)$ satisfying the following more restrictive conditions:

$$\alpha \equiv 2 \ , \qquad \mathcal{L}_{\mathrm{q}_o} \mathfrak{d} = 0 \ , \qquad \mathrm{q}_o(\sigma) = \mathrm{q}_o(\beta) = 0 \ . \tag{4.14}$$

In fact, in order to recover general results from those under the restricted assumption (4.14), it is sufficient to make the very simple replacements $r \dashrightarrow \frac{\alpha r}{2}$ and $\mathfrak{d} \dashrightarrow \mathfrak{d} - \frac{2r}{\alpha} d\alpha$.

Consider now the complex coordinates $(z := x + \mathrm{i}\, y, w := u + \mathrm{i}\, \widetilde{r})$ with $\widetilde{r} := r + \varphi(x, y)$, defined in (3.4). We observe that:

- (z, w) are holomorphic coordinates for the *integrable* complex structure $J^{(M,\varphi)}$;
- $J^{(M,\varphi)}$ induces the CR structure $(\mathcal{D}, J)$ on each hypersurface

$$\mathcal{S}^{(\mathsf{b})} := \mathcal{S}|_{\mathcal{V}} \times \{\mathsf{b}\} \subset \breve{\pi}^{-1}(\mathcal{V}) \simeq \mathcal{S}|_{\mathcal{V}} \times \mathbb{R} \ , \qquad \mathsf{b} \in \mathbb{R} \ ;$$

- Each $\mathcal{S}^{(\mathsf{b})}$ is equal to the set of solutions to the equation

$$u - F(z, \bar{z}) - \mathsf{b} = 0 \qquad \text{where we use the notation} \qquad F(z, \bar{z}) := \varphi\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\,\mathrm{i}}\right) \ ;$$

- The tensor fields $\breve{\pi}^* g_o$, $\vartheta = \pi^{\mathcal{S}*}\, \theta$ and $\mathrm{p}^*_o = dr$ have the form

$$\breve{\pi}^* g_o = \Delta\varphi(dx \vee dx + dy \vee dy) = 4F_{z\bar{z}} dz \vee d\bar{z} \ ,$$
$$\vartheta = du - \varphi_y dx + \varphi_x dy = du - iF_z dz + iF_{\bar{z}} d\bar{z} \ , \qquad \mathrm{p}^*_o = d\widetilde{r} - dF \ ;$$

- The conditions (4.14) imply that σ, β, $\mathfrak{d}$ are independent on $u = \mathrm{Re}(w)$;
- The signature of g is mostly plus (resp. mostly minus) if and only if $\sigma > 0$ (resp. $\sigma < 0$), meaning that $\mathrm{sign}(g) = \mathrm{sign}(\sigma)$.

We now introduce the real functions $\mathcal{P}$, $\mathcal{H}$ and the complex function $\mathcal{W}$ defined by

$$\mathcal{P} := \sqrt{2|\sigma|F_{z\bar z}}\ , \qquad \mathcal{H} := \frac{\beta F_{z\bar z}}{2}\ , \qquad \mathrm{Re}(\mathcal{W}dz) := \frac{\mathfrak{d}}{4} - \frac{dF}{2}\ . \tag{4.15}$$

Recalling that $dr = d\widetilde{r} - d\varphi = d\widetilde{r} - dF$, a direct check shows that, after a multiplication by $\mathrm{sign}(g)$, any metric (4.1) satisfying (4.14) (in particular, with $\alpha \equiv 2$) has the form

$$\mathrm{sign}(g)g = 2\mathcal{P}^2\left(dz \vee d\bar z + \frac{1}{2F_{z\bar z}}\vartheta \vee \left(d\widetilde{r} + 2\,\mathrm{Re}\left(\mathcal{W}dz\right) + \frac{\mathcal{H}}{2F_{z\bar z}}\vartheta\right)\right)\ , \tag{4.16}$$

namely the form of the metrics considered in [12, 10] ([1]). In other words, the mostly plus metrics $\mathrm{sign}(g)g$ satisfying (4.14) are nothing but the metrics considered by Hill, Lewandowski and Nurowski satisfying the additional assumption that the three functions (4.15) are independent on u. Combining this condition with the results in [12], in [10] it is proved that $\mathrm{sign}(g)g$ is a solution to an equation of the form (3.6) with cosmological constant $\Lambda = \mathrm{sign}(g)\Lambda'$ if and only if the functions (4.15) are as follows:

$$\mathcal{P}(x,y,r) = \frac{p(x,y)}{\cos(\frac{r+F(x,y)+s(x,y)}{2})}\ , \tag{4.17}$$

$$\mathcal{W}(x,y,r) = \mathrm{i}\left(-t - f_z + \frac{2p_z}{p}\right)e^{-\,\mathrm{i}(r+F(z,\bar z)+s(z,\bar z))} - \mathrm{i}\,t + \mathrm{i}\left(-t - f_z + \frac{2p_z}{p}\right) + s_z\ , \tag{4.18}$$

$$\begin{aligned}\mathcal{H}(x,y,r) = 2\,\mathrm{Re}\Bigg(\frac{m}{p^4}\,\mathrm{e}^{2\,\mathrm{i}(r+F(z,\bar z)+s)} + \Bigg(&\frac{3m+\bar m}{p^4} + \frac{2}{3}\Lambda p^2 + \frac{p_z\,p_{\bar z} - pp_{z\bar z}}{p^2} - 2tp_{\bar z} - \bar t\frac{p_z}{p}\\ &+ \frac{3}{2}t_{\bar z} + f_{\bar z}t + \frac{1}{2}f_z\bar t + \frac{5}{2}t\bar t + \bar t_z + f_{z\bar z}\Bigg)\mathrm{e}^{\mathrm{i}(r+F(z,\bar z)+s)} +\\ &+ \frac{3m+3\bar m}{p^4} + 2\Lambda p^2 + \frac{2p_z\,p_{\bar z} - 2pp_{z\bar z}}{p^2} - 3(t\frac{p_{\bar z}}{p} + \bar t\frac{p_z}{p}) +\\ &+ \frac{5}{2}(\bar t_z + t_{\bar z}) + 6t\bar t + \frac{3}{2}(f_z\bar t + f_{\bar z}t) + 2f_{z\bar z}\Bigg)\ ,\end{aligned} \tag{4.19}$$

where:

- $s = s(z,\bar z)$ is a freely specifiable smooth real valued functions of z and $\bar z$;
- $p = p(z,\bar z) \neq 0$ is a nowhere vanishing real function ([2]) and $t = t_1(z,\bar z) + \mathrm{i}\,t_2(z,\bar z)$ and $m = m_1(z,\bar z) + \mathrm{i}\,m_2(z,\bar z)$ are complex function, both depending just on z and $\bar z$,

[1]For a correct comparison, keep in mind that the λ of the formulas in [12, 10] is actually equal to $\lambda = \frac{1}{2F_{z\bar z}}\vartheta$.

[2]The constraint $p(z,\bar z) \neq 0$ at all points follows from the fact that, by (4.17), if $p(x_o,y_o) = 0$ at some (x_o,y_o), then $\mathcal{P}$ would be zero on a non empty set $\{(x_o,y_o)\times\mathbb{R}\times(a,b)\}$ and (4.16) would not be a metric.

constrained by the following system of p.d.e.'s:

$$\partial_z t - (\partial_z f + t)t = 0\ , \tag{4.20}$$

$$\partial_z m - 3(\partial_z f + t)m = 0\ , \tag{4.21}$$

$$\left(2\partial_z\partial_{\bar z} - f_{\bar z}\partial_z - f_z\partial_{\bar z} + \frac{1}{2}|f_z|^2 - \frac{3}{2}f_{z\bar z} - \frac{3}{2}\left(\partial_z\bar t + \partial_{\bar z}t + |t|^2\right)\right)p = \frac{m+\bar m}{p^3} + \frac{2}{3}\Lambda p^3, \tag{4.22}$$

where $f := \log(F_{z\bar z})$.

For a given $s = s(z,\bar z)$ and a triple (t,m,p) solving (4.20) – (4.21), we set

$$\kappa(z,\bar z) := \mathrm{sign}(g)\frac{8p^2(z,\bar z)}{\Delta F(z,\bar z)}\ ,\quad r'(z,\bar z,r) := \kappa(z,\bar z)\tan\left(\frac{r + F(z,\bar z) + s(z,\bar z)}{2}\right)\ . \tag{4.23}$$

After a (long and tedious) sequence of straightforward manipulations (for the benefit of the reader, a sketch of these manipulations is given in Appendix A), one gets that a necessary and sufficient condition for $\mathrm{sign}(g)g$ to have the form (4.16) with $\mathcal{P}$, $\mathcal{W}$ and $\mathcal{H}$ as in (4.17) – (4.19) (thus, a strictly plus metric satisfying a quasi-Einstein equation) is that g has the form (4.3) with r' as in (4.10) with $\alpha \equiv 2$. As we have pointed out before, the expression for a compatible metric with α arbitrary function independent on u is obtained starting from the one we just described via the overall replacement $r \dashrightarrow \frac{r\alpha}{2}$. This leads to (4.3) in full generality.

Since (4.20) and (4.21) are the conditions (4.4) and (4.6), in order to conclude, we just need to show that the constraint (4.22) on the triple (t,m,p) is equivalent to the constraint (4.5) on the triple (t,m,κ) where, being $\mathrm{sign}(g)\kappa > 0$, the function κ can be written as $\kappa = \mathrm{sign}(g)\widetilde\kappa^2$ for some smooth function $\widetilde\kappa$. In order to check this, let us write (4.22) as

$$2\frac{p_{z\bar z}}{p} - f_{\bar z}\frac{p_z}{p} - f_z\frac{p_{\bar z}}{p} + \frac{1}{2}|f_z|^2 - \frac{3}{2}f_{z\bar z} - \frac{3}{2}\left(\partial_z\bar t + \partial_{\bar z}t + |t|^2\right) = \frac{m+\bar m}{p^4} + \frac{2}{3}\Lambda p^2. \tag{4.24}$$

We now observe that, from $p^2 = \frac{\mathrm{sign}(g)\kappa F_{z\bar z}}{2} = \frac{\mathrm{sign}(g)\kappa e^f}{2}$,

$$\begin{aligned}2\frac{p_{z\bar z}}{p} &= \partial_{z\bar z}\left(\log p^2\right) + \frac{1}{2}\ \partial_z\left(\log p^2\right)\partial_{\bar z}\left(\log p^2\right) = \\ &= \partial_{z\bar z}\left(\log(\mathrm{sign}(g)\kappa)\right) + f_{z\bar z} + \frac{1}{2}\left(\frac{\kappa_z}{\kappa} + f_z\right)\left(\frac{\kappa_{\bar z}}{\kappa} + f_{\bar z}\right) = \\ &= \frac{\kappa_{z\bar z}}{\kappa} - \frac{1}{2}\frac{\kappa_z}{\kappa}\frac{\kappa_{\bar z}}{\kappa} + f_{z\bar z} + \frac{1}{2}\frac{\kappa_z}{\kappa}f_{\bar z} + \frac{1}{2}f_z\frac{\kappa_{\bar z}}{\kappa} + \frac{1}{2}|f_z|^2\ .\end{aligned}$$

Plugging this in (4.24) and using the relation $\Lambda = \mathrm{sign}(g)\Lambda'$, we get

$$\frac{\kappa_{z\bar z}}{\kappa} - \frac{1}{2}\frac{\kappa_z}{\kappa}\frac{\kappa_{\bar z}}{\kappa} - \frac{1}{2}f_{z\bar z} - \frac{3}{2}\left(\partial_z\bar t + \partial_{\bar z}t + |t|^2\right) = \frac{4(m+\bar m)e^{-2f}}{\kappa^2} + \frac{1}{3}\,\mathrm{sign}(g)\Lambda'\kappa e^f\ . \tag{4.25}$$

Since $\mathrm{sign}(g)\kappa = \widetilde\kappa^2$, $\kappa_z = 2\,\mathrm{sign}(g)\widetilde\kappa\widetilde\kappa_z$, $\kappa_{z\bar z} = \mathrm{sign}(g)(2\widetilde\kappa_{\bar z}\widetilde\kappa_z + 2\widetilde\kappa\widetilde\kappa_{z\bar z})$, it follows that (4.25) is equivalent to

$$2\frac{\widetilde\kappa_{z\bar z}}{\widetilde\kappa} - \frac{1}{2}f_{z\bar z} - \frac{3}{2}\left(\partial_z\bar t + \partial_{\bar z}t + |t|^2\right) = \frac{4(m+\bar m)e^{-2f}}{\widetilde\kappa^4} + \frac{1}{3}\Lambda\widetilde\kappa^2 e^f\ . \tag{4.26}$$

Recalling that $f = \log(F_{z\bar{z}}) = \log\left(\frac{\Delta\varphi}{4}\right)$, this is equivalent to (4.5), as claimed. □

Remark 4.3. The above arguments show that (4.5) is actually equivalent to (4.25), i.e. to the following elliptic equation on $\kappa = \mathrm{sign}(g)\widetilde{\kappa}^2$:

$$\frac{\Delta\kappa}{\kappa} - \frac{1}{2}\frac{\kappa_x^2 + \kappa_y^2}{\kappa^2} - \frac{1}{2}\Delta(\log\Delta\varphi) - 6\left(\partial_z\bar{t} + \partial_{\bar{z}}t + |t|^2\right) - \frac{2^9\,\mathrm{Re}(m)}{\kappa^2(\Delta\varphi)^2} - \frac{1}{3}\Lambda'\kappa\Delta\varphi = 0\ . \quad (4.27)$$

5. A new large class of examples of null fluid gravitational fields

By the results in §4, the $\mathcal{Q}_o$-compatible Lorentzian metrics on a Kerr manifold M corresponding to null fluid gravitational fields are locally of the form (4.3). In this section we focus on the following subclass of gravitational fields, which we will shortly see naturally include and generalise the well known Ricci flat Kerr metrics.

Definition 5.1. We say that a Lorentzian metric on a Kerr manifold, as in Theorem 4.1, *belongs to the enlarged Kerr family* if the corresponding functions κ, t and m satisfy the following conditions for some constant $\mathsf{B} \neq 0$:

$$\kappa = \mathsf{B}\varphi\ , \qquad t = t_1 + \mathrm{i}\,t_2 \equiv 0\ , \qquad m = m_1 + \mathrm{i}\,m_2 \text{ is nowhere vanishing}\ . \quad (5.1)$$

The third condition in (5.1) and (4.6) imply that, locally, m is a solution to the equation $\partial_z \log\left(\frac{m_1 + \mathrm{i}\,m_2}{(\Delta\varphi)^3}\right) = 0$ for some holomorphic logarithm. If $\mathcal{V} \subset N$ is a simply connected domain where such a holomorphic logarithm exists, and if $(x_o, y_o) \in \mathcal{V}$ is a fixed point, then all real functions $m_1 = \mathrm{Re}(m)$ and $m_2 = \mathrm{Im}(m)$ can be described as follows:

$$m_1 = (\Delta\varphi)^3(\widetilde{\mathsf{m}} + \mathsf{A})\ , \qquad m_2 = (\Delta\varphi)^3(\mathsf{m} + \mathsf{A}^*)\ , \quad (5.2)$$

where:
- m and $\widetilde{\mathsf{m}}$ are two freely specifiable real constants;
- $\mathsf{A}(x, y)$ is an arbitrary harmonic function on $\mathcal{V}$ which is 0 at (x_o, y_o);
- $\mathsf{A}^*(x, y)$ is the unique harmonic function, which is 0 at (x_o, y_o) and is such that the complex function $\mathsf{A}(x, y) + \mathrm{i}\,\mathsf{A}^*(x, y)$ is anti-holomorphic ([3]).

The next proposition and the following remarks motivate our interest for this class of metrics. In what follows $\mathcal{V} \subset N$ is a simply connected domain on which m_1 and m_2 are as in (5.2) and B is the constant which determines κ as in (5.1). We also assume that the local identification between the Kerr manifold $M|_{\mathcal{V}}$ and the cartesian product $\mathcal{S}|_{\mathcal{V}} \times \mathbb{R}$ is determined by an appropriate bending diffeomorphism (not necessarily the identity map), so that the function (4.10) and the 1-form dr' are just $r'(x, y, r) = r$ and $dr' = dr$.

Proposition 5.2. *A metric with scalar curvature* $\mathrm{Scal} = 4\Lambda'$ *is in the enlarged Kerr family on* $\breve{\pi}^{-1}(\mathcal{V}) \simeq \mathcal{S}|_{\mathcal{V}} \times \mathbb{R} \subset M$ *with associated constant* $\mathsf{B} \neq 0$ *if and only if there are*

(a) two real constants m, $\widetilde{\mathsf{m}}$,

(b) a harmonic function $\mathsf{A}(x, y)$ *vanishing at* (x_o, y_o) *and*

[3] If we denote by $\widetilde{\mathsf{A}}$ the unique harmonic conjugate of A vanishing at (x_o, y_o), then $\mathsf{A}^*(x, y) := \widetilde{\mathsf{A}}(x, -y)$.

(c) a solution $\varphi = \varphi(x, y)$ to the equation

$$\varphi\Delta\varphi - \frac{1}{2}(\varphi_x^2 + \varphi_y^2) - \frac{\varphi^2}{2}\Delta(\log\Delta\varphi) - \Delta\varphi\left(\frac{2^9(\widetilde{\mathsf{m}} + \mathsf{A})}{\mathsf{B}^2} + \frac{1}{3}\Lambda'\mathsf{B}\varphi^3\right) = 0\ , \tag{5.3}$$

such that $g = \mathsf{B}\breve{g}$ with

$$\breve{g} = \frac{1}{4}\left(\varphi^2 + \frac{r^2}{\mathsf{B}^2}\right)\frac{\Delta\varphi}{\varphi}(dx^2 + dy^2) + \vartheta \vee \left(d\left(\frac{r}{\mathsf{B}} + u\right) + \left(-\frac{1}{2} + \widetilde{\beta}_o\right)\vartheta\right)\ , \tag{5.4}$$

where we use the notation

$$\widetilde{\beta}_o := 2^6\left(\frac{-7(\frac{\widetilde{\mathsf{m}}}{\mathsf{B}^2} + \frac{\mathsf{A}}{\mathsf{B}^2})\varphi^2 - \varphi(\frac{\mathsf{m}}{\mathsf{B}^2} + \frac{\mathsf{A}^*}{\mathsf{B}^2})\frac{r}{\mathsf{B}} - 8(\frac{\widetilde{\mathsf{m}}}{\mathsf{B}^2} + \frac{\mathsf{A}}{\mathsf{B}^2})(\frac{r}{\mathsf{B}})^2}{\varphi(\varphi^2 + \left(\frac{r}{\mathsf{B}}\right)^2)}\right) + \\ + \Lambda'\mathsf{B}\left(\frac{(5\,\mathrm{sign}(g) - 4)}{3}\varphi^2 + \frac{\mathrm{sign}(g)}{3}\left(\frac{r}{\mathsf{B}}\right)^2\right)\ . \tag{5.5}$$

Proof. Assuming that g has the form (4.3) and that (5.1) is satisfied, from the identity $\vartheta = du + d^c\varphi$ and (5.2) we may write g as

$$g = \frac{1}{4}\left(\mathsf{B}\varphi + \frac{r^2}{\mathsf{B}\varphi}\right)\breve{\pi}^* g_o + \vartheta \vee \left\{dr - \mathsf{B}d^c\varphi + \frac{1}{\Delta\varphi}\left(\mathsf{B}\varphi + \frac{r^2}{\mathsf{B}\varphi}\right)(h_o + h_m)\,\vartheta\right\} = \\ = \frac{1}{4}\frac{\mathsf{B}^2\varphi^2 + r^2}{\mathsf{B}}\frac{\Delta\varphi}{\varphi}(dx^2 + dy^2) + \vartheta \vee \left\{d\,(r + \mathsf{B}u) + \mathsf{B}\left(-1 + \frac{\mathsf{B}^2\varphi^2 + r^2}{\mathsf{B}^2\varphi\Delta\varphi}(h_o + h_m)\right)\vartheta\right\}\ , \tag{5.6}$$

with

$$h_o := \frac{1}{\mathsf{B}^2\varphi^2 + r^2}\left(\frac{5}{3}\Lambda\mathsf{B}^3\varphi^3\Delta\varphi + \frac{1}{3}\Lambda\mathsf{B}\ \varphi\Delta\varphi r^2 + \right. \\ \left. + \frac{1}{2}\mathsf{B}^2\varphi^2\Delta(\log(\Delta\varphi)) + \frac{1}{2}\ \mathsf{B}^2\left(\varphi_x^2 + \varphi_y^2\right) - \frac{1}{2}\mathsf{B}^2\varphi\Delta\varphi\right)\ , \tag{5.7}$$

$$h_m := \frac{\mathsf{B}\varphi\Delta\varphi}{\mathsf{B}^2\varphi^2 + r^2}\frac{2^6\left(\mathsf{B}\varphi(\widetilde{\mathsf{m}} + \mathsf{A}) - (\mathsf{m} + \mathsf{A}^*)r\right)}{\mathsf{B}^2\varphi^2 + r^2}\ . \tag{5.8}$$

Since $t = 0$, by (4.27) the Kähler potential $\varphi = \frac{1}{\mathsf{B}}\kappa$ is constrained by

$$\mathsf{B}^2\varphi\Delta\varphi - \frac{1}{2}\mathsf{B}^2(\varphi_x^2 + \varphi_y^2) - 2^9\Delta\varphi(\widetilde{\mathsf{m}} + \mathsf{A}) - \frac{1}{3}\Lambda'\mathsf{B}^3\varphi^3\Delta\varphi = \frac{1}{2}\mathsf{B}^2\varphi^2\Delta(\log\Delta\varphi)\ . \tag{5.9}$$

This equality implies that h_o is also given by

$$h_o := \frac{\mathsf{B}\varphi\Delta\varphi}{\mathsf{B}^2\varphi^2 + r^2}\left(\frac{(5\,\mathrm{sign}(g) - 4)}{3}\Lambda'\mathsf{B}^2\varphi^2 + \frac{\mathrm{sign}(g)}{3}\Lambda' r^2 + \frac{1}{2}\mathsf{B} - 2^9\frac{\widetilde{\mathsf{m}} + \mathsf{A}}{\mathsf{B}\varphi}\right)\ , \tag{5.10}$$

so that

$$
\begin{aligned}
&-1+\frac{\mathsf{B}^2\varphi^2+r^2}{\mathsf{B}^2\varphi\Delta\varphi}(h_o+h_m)=\\
&=-1-2^9\frac{\widetilde{\mathsf{m}}+\mathsf{A}}{\mathsf{B}^2\varphi}+\frac{2^6\left(\mathsf{B}\varphi(\widetilde{\mathsf{m}}+\mathsf{A})-(\mathsf{m}+\mathsf{A}^*)r\right)}{\mathsf{B}(\mathsf{B}^2\varphi^2+r^2)}+\Lambda'\left(\frac{(5\operatorname{sign}(g)-4)}{3}\mathsf{B}\varphi^2+\frac{\operatorname{sign}(g)}{3\mathsf{B}}r^2\right)+\frac{1}{2}=\\
&=-\frac{1}{2}-2^9\frac{\widetilde{\mathsf{m}}+\mathsf{A}}{\mathsf{B}^2\varphi}+\frac{2^6\left(\mathsf{B}\varphi(\widetilde{\mathsf{m}}+\mathsf{A})-(\mathsf{m}+\mathsf{A}^*)r\right)}{\mathsf{B}(\mathsf{B}^2\varphi^2+r^2)}+\Lambda'\left(\frac{(5\operatorname{sign}(g)-4)}{3}\mathsf{B}\varphi^2+\frac{\operatorname{sign}(g)}{3\mathsf{B}}r^2\right)=\\
&\qquad=-\frac{1}{2}+2^6\left(\frac{-7(\widetilde{\mathsf{m}}+\mathsf{A})\mathsf{B}^2\varphi^2-\mathsf{B}\varphi(\mathsf{m}+\mathsf{A}^*)r-8(\widetilde{\mathsf{m}}+\mathsf{A})r^2}{\mathsf{B}^2\varphi(\mathsf{B}^2\varphi^2+r^2)}\right)+\\
&\qquad\qquad+\Lambda'\left(\frac{(5\operatorname{sign}(g)-4)}{3}\mathsf{B}\varphi^2+\frac{\operatorname{sign}(g)}{3\mathsf{B}}r^2\right)\ . \qquad (5.11)
\end{aligned}
$$

This implies the claim. □

Proposition 5.2 shows that the quasi-Einstein metrics with a scalar curvature $\mathrm{Scal} = 4\Lambda'$ belonging to the enlarged Kerr family are locally parametrised by:

(a) The freely specifiable constants $\mathsf{B} \neq 0$, m, and $\widetilde{\mathsf{m}}$,
(b) The freely specifiable harmonic function A vanishing at a fixed point,
(c) The solutions to (5.9).

Note also that the metric $\breve{g}$ defined in (5.4) is isometric to a metric $\widetilde{g}$ with $\mathrm{Scal} = 4\mathsf{B}\Lambda'$ and associated with the constants $\mathsf{B}' = 1$, $\mathsf{m}' = \frac{\mathsf{m}}{\mathsf{B}^2}$, $\widetilde{\mathsf{m}}' = \frac{\widetilde{\mathsf{m}}}{\mathsf{B}^2}$ and with the function $\mathsf{A}' = \frac{\mathsf{A}}{\mathsf{B}^2}$ (in order to pass from $\breve{g}$ to $\widetilde{g}$ it suffices to make the replacement $r \dashrightarrow \mathsf{B}r$). Hence *up to an homothety and a change of signature, for any quasi-Einstein metric of the enlarged Kerr family the corresponding constant* B *can be assumed to be* $\mathsf{B} = 1$.

We claim that the set of solutions of (c) (and, consequently, the corresponding set of metrics) is very large. In fact, in case $\widetilde{\mathsf{m}} = 0$ and $\mathsf{A} \equiv 0$ (so that $\mathsf{A}^* \equiv 0$ as well), (5.3) reduces to

$$\frac{\Delta\varphi}{\varphi}-\frac{1}{2\varphi^2}(\varphi_x^2+\varphi_y^2)-\frac{1}{2}\Delta(\log\Delta\varphi)+\frac{\varphi\Delta\varphi}{3}\Lambda'\mathsf{B}=0\ . \qquad (5.12)$$

If we limit ourselves to the case $\Lambda' = 0$ and we introduce the function $\psi = \frac{\Delta\varphi}{|\varphi|}$, the solutions φ to (5.12) appear to be in bijection with the pairs (φ, ψ) that satisfy

$$\Delta\varphi-\operatorname{sign}(\varphi)\psi\varphi=0\ ,\qquad \Delta(\log\psi)-\operatorname{sign}(\varphi)\ \psi=0\ .$$

As it is observed in the proof of [9, Thm. 4.1], we may always select coordinates (x, y) on $\mathcal{V}$, in which a solution ψ to the second equation takes the form $\frac{8}{(1-\operatorname{sign}(\varphi)(x^2+y^2))^2}$. This yields that in such coordinates φ is either a solution to

$$
\begin{aligned}
&\Delta\varphi+\frac{8\varphi}{(1+x^2+y^2)^2}=0 \qquad \text{in case} \qquad \varphi<0 \qquad \text{or to}\\
&\Delta\varphi-\frac{8\varphi}{(1-x^2-y^2)^2}=0 \qquad \text{in case} \qquad \varphi>0\ .
\end{aligned}
\qquad (5.13)
$$

Explicit expressions for a countable set of solutions to these conditions and corresponding explicit expressions for metrics with $\Lambda' = 0$, $\mathsf{B} = 1$, $\widetilde{\mathsf{m}} = 0$ and $\mathsf{A} \equiv 0$ have been given in [9, § 6]. There it is shown that the metrics of this kind with φ solution to (5.13) (which is equivalent to (5.12)) are Ricci flat and constitute a class that properly include all metrics of Kerr black holes. As we observed above, replacing $\mathsf{B} = 1$ by any other non-zero value $\mathsf{B}' \neq 1$ determines a homothetic metric with a possibly opposite signature, thus Ricci flat as well. From this and the existence of infinite non-isometric metrics in the class considered in [9], we may conclude that *the enlarged family of Kerr metrics surely contains a very large class of solutions to the quasi-Einstein equations, actually a very large class of* Ricci flat *metrics.*

Let us now focus on the (possibly larger) class of metrics with $\mathsf{B} = 1$, m, $\widetilde{\mathsf{m}}$ and Λ' arbitrary and $\mathsf{A} \equiv 0$. Following the same approach described above, the solutions to (5.3) are in bijection with the pairs of functions $(\phi, \psi = e^{\widetilde{\psi}})$ solving a system of the form

$$\begin{aligned} &\Delta\varphi - \mathrm{sign}(\varphi) e^{\widetilde{\psi}} \phi = 0 \ , \\ &\Delta\widetilde{\psi} - e^{\widetilde{\psi}} \left(\frac{c_1}{\varphi} + \mathrm{sign}(\varphi) + c_2 \varphi \right) = 0 \ , \end{aligned} \tag{5.14}$$

where the constants c_i, $i = 1, 2$ (which are determined by Λ', m and $\widetilde{\mathsf{m}}$), are possibly non-zero. If we limit our discussion to the solutions $(\varphi = \varphi(\mathfrak{r}), \psi = e^{\widetilde{\psi}(\mathfrak{r})})$ depending just on $\mathfrak{r} = \sqrt{x^2 + y^2}$, the system (5.14) simplifies into

$$\begin{aligned} &\frac{d^2\varphi}{d\mathfrak{r}^2} + \frac{1}{\mathfrak{r}} \ \frac{d\varphi}{d\mathfrak{r}} - \mathrm{sign}(\varphi) e^{\widetilde{\psi}} \phi = 0 \ , \\ &\frac{d^2\widetilde{\psi}}{d\mathfrak{r}^2} + \frac{1}{\mathfrak{r}} \ \frac{d\widetilde{\psi}}{d\mathfrak{r}} - e^{\widetilde{\psi}} \left(\frac{c_1}{\varphi} + \mathrm{sign}(\varphi) + c_2 \varphi \right) = 0 \ . \end{aligned} \tag{5.15}$$

By standard results on systems of ordinary differential equations, any such system admits smooth solutions on sufficiently small neighbourhoods $(\mathfrak{r}_o - \varepsilon, \mathfrak{r}_o + \varepsilon)$ of any point $\mathfrak{r}_o \neq 0$. This yields a large class of (local) $\mathscr{C}^\infty$ solutions to (5.3) for any choice of the cosmological Λ' and hence also a corresponding class of $\mathscr{C}^\infty$ quasi-Einstein metrics in the enlarged Kerr family with non-zero cosmological constants. We did not succeed in finding expressions for such solutions in some closed form, but it is manifest that explicit solutions can be easily determined numerically.

We conclude by noting that questions on the existence of solutions to the equation (5.3) in certain Banach spaces (as, for instance, in suitable Sobolev spaces of functions on a fixed smooth bounded domain in $\mathbb{R}^2$), under the assumption that A, Λ', m and $\widetilde{\mathsf{m}}$ are sufficiently close to 0, can be answered affirmatively. Starting from the case $A \equiv 0$, $\Lambda' = \mathsf{m} = \widetilde{\mathsf{m}} = 0$, one can apply the Implicit Function Theorem for Banach spaces to deformations of elliptic systems of p.d.e.'s. A detailed discussion of these issues are left to future work.

6. Existence of quasi-Einstein lifts for Sasaki CR 3-manifolds of class $\mathscr{C}^\infty$

Let $(\mathcal{D}, J)$ be a strongly pseudoconvex CR structure on a 3-dimensional manifold $\mathcal{S}$. We focus on its optical lifts over $M = \mathcal{S} \times \mathbb{R}$ that satisfy the quasi-Einstein condition, namely solutions of (3.7) for a fixed cosmological constant $\Lambda = \mathrm{sign}(g)\Lambda'$. Proving (or disproving) the local existence of quasi-Einstein optical lifts of this kind has two main consequences: on the one hand, it enables the determination of CR invariants via the isometric invariants of the associated Lorentzian lifted metrics; on the other hand, explicit or numerical methods for solving equations of the form (3.7) for prescribed strongly pseudoconvex CR structures provide valuable tools for studying gravitational fields generated by null fluids.

In [12], Hill, Lewandowski and Nurowski obtained quite remarkable results on strongly pseudoconvex 3-dimensional CR manifolds with quasi-Einstein optical lifts. In particular they proved the following

Theorem 6.1 ([12, Thm. 2.5])**.** *For any real analytic strongly pseudoconvex CR 3-dimensional manifold and any choice of a constant Λ, there locally exists a quasi-Einstein optical lift (with mostly plus signature) of Petrov type II or D with the prescribed cosmological constant Λ.*

The proof of this result relies crucially on the following properties of CR 3-manifolds:

(a) Any strongly pseudoconvex, real-analytic CR 3-manifold is locally embeddable and can be identified with a real-analytic submanifold of $\mathbb{C}^2$.
(b) In holomorphic coordinates on $\mathbb{C}^2$, the system of equations characterising the quasi-Einstein optical lifts of a CR 3-manifold as in (a) is real-analytic and, by standard results on integrable real-analytic systems of p.d.e.'s, locally solvable.

However, in [12, Rem. 3.25] the authors conjectured that the above existence result for quasi-Einstein optical lifts of type II or D should hold under weaker regularity assumptions, provided the CR 3-manifold is embeddable. In support of this expectation, in [12, Cor. 3.26] it is shown that, under a much weaker hypothesis than real analyticity, there do exist optical lifts with curvatures of type III, N or 0 for embeddable CR 3-manifolds. Furthermore, the conjecture was partially confirmed in [10], where, using results on logistic elliptic equations, the existence of a globally defined quasi-Einstein optical lift of the desired Petrov types was established under the following hypotheses:

(1) The underlying CR 3-manifold is $\mathscr{C}^\infty$ and regular Sasaki (hence, also locally embeddable) and projects onto a Kähler surface (N, g_o, J) diffeomorphic to $\mathbb{R}^2$;
(2) The Kähler metric g_o of N is such that both limits $\lim_{|x|\to\infty} g_o$ and $\lim_{|x|\to\infty} -\log(vol_{g_o})$ exist and are strictly positive;
(3) The prescribed cosmological constant Λ for the optical lift is positive.

In this concluding section, we combine Theorem 4.1 with a classical result on non-linear elliptic equations to improve the results of [10] and confirm Hill, Lewandowski and Nurowski's conjecture for $\mathscr{C}^\infty$ 3-dimensional Sasaki CR manifolds. Specifically, we establish the following.

Corollary 6.2. *Let Λ be an arbitrary real constant, and let $(\mathcal{S}, \mathcal{D} = \ker\theta, J)$ be a regular Sasaki 3-manifold, fibering over a quantisable Kähler 2-dimensional manifold (N, g_o, J). For any sufficiently small open subset $\mathcal{V} \subset N$, there exists a quasi-Einstein optical lift g on $\pi^{-1}(\mathcal{V}) \times \mathbb{R} \subset \mathcal{S} \times \mathbb{R}$ with scalar curvature $\mathrm{Scal} = 4\Lambda'$, where $\Lambda' = \mathrm{sign}(g)\Lambda$, and whose curvature tensors are of Petrov type II or D at all points.*

Proof. By Theorem 4.1 the class of the quasi-Einstein optical lifts of the form (4.1) for the CR structure $(\mathcal{D}, J)$ on a set of the form $\mathcal{S}|_{\mathcal{V}}$, $\mathcal{V} \subset N$, is non-empty if and only if the family of triples of $\mathscr{C}^\infty$ functions $(t, m, \widetilde{\kappa})$, solving the corresponding system on $\mathcal{V}$

$$\begin{gathered}
\partial_z t - (\partial_z(\log\Delta\varphi) + t)t = 0 \ , \qquad \partial_z m - 3\bigg(\partial_z\left(\log\Delta\varphi\right) + t\bigg)m = 0 \ , \\
\Delta\widetilde{\kappa} = \left(\frac{256\,\mathrm{Re}(m)}{(\Delta\varphi)^2}\right)\frac{1}{\widetilde{\kappa}^3} + \left(\frac{1}{4}\Delta\left(\log\Delta\varphi\right) + 3\left(\partial_z\bar{t} + \partial_{\bar{z}}t + |t|^2\right)\right)\widetilde{\kappa} + \left(\frac{\Lambda\Delta\varphi}{6}\right)\widetilde{\kappa}^3
\end{gathered} \tag{6.1}$$

is non-empty. Since the first two equations are independent on $\widetilde{\kappa}$ and admit local solutions on any open subset of $\mathbb{R}^2$ (the first admits the trivial solution $t \equiv 0$, and the second is linear in m), the existence of triples $(t, m, \widetilde{\kappa})$ solving (6.1) reduces to the following question: given a $\mathscr{C}^\infty$ Kähler potential φ for the Kähler manifold $(\mathcal{V}, g_o, J)$, a cosmological constant Λ and two $\mathscr{C}^\infty$ solutions t, m to the first two equations, does there exist a local $\mathscr{C}^\infty$ solution $\widetilde{\kappa}$ to the third equation in (6.1)? Without loss of generality we may assume $\mathrm{Re}(m) \equiv 0$ and $\mathrm{Im}(m)$ nowhere vanishing, in which case the third equation takes the form $\Delta\widetilde{\kappa} = \mathbf{h}(x, y, \widetilde{\kappa})$ for a well-defined $\mathscr{C}^\infty$ function $\mathbf{h}$ on $\mathcal{V} \times \mathbb{R}$. By a classical result on non-linear elliptic equations of this kind (see e.g. [6, p. 373]) and standard properties of elliptic equations, a $\mathscr{C}^\infty$ solution to the last equation in (6.1) exists on any sufficiently small open subset $\mathcal{V}' \subset \mathcal{V}$. The conclusion is then a consequence of Remark 4.2 and the hypothesis that m is nowhere vanishing. □

Appendix A. The proof that (4.16) is equivalent to (4.3)

From $r' := \kappa\tan\left(\frac{r+F+s}{2}\right)$, we may express the coordinate $\widetilde{r} = r + \varphi = r + F$ in terms of r'as $\widetilde{r} = 2\arctan\left(\frac{r'}{\kappa}\right) - s$. Since $F_{z\bar{z}} = \frac{1}{4}\Delta\varphi$, the following equalities are immediate:

$$g_o = \Delta\varphi(dx^2 + dy^2) = 4F_{z\bar{z}}dz \vee d\bar{z} \ , \tag{A.1}$$

$$d\widetilde{r} = dr + dF = \frac{2}{1 + \frac{r'^2}{\kappa^2}}\left(\frac{1}{\kappa}dr' - \frac{r'}{\kappa^2}\left(\kappa_x dx + \kappa_y dy\right)\right) - s_x dx - s_y dy \ , \tag{A.2}$$

$$\frac{\mathcal{P}^2}{F_{z\bar{z}}} = \frac{4\mathcal{P}^2}{\Delta F} = \frac{1}{2}\frac{8p^2}{\Delta F\cos^2\left(\frac{r+F+s}{2}\right)} = \frac{\mathrm{sign}(g)\kappa}{2}\frac{1}{\cos^2\left(\frac{r+F+s}{2}\right)} = \frac{\mathrm{sign}(g)}{2}\kappa\left(\frac{r'^2}{\kappa^2} + 1\right) \ , \tag{A.3}$$

$$\frac{\mathcal{P}^2}{F_{z\bar{z}}}d\widetilde{r} = \mathrm{sign}(g)\left(dr' - \left(\frac{\kappa_x r'}{\kappa} + \frac{s_x r'^2}{2\kappa} + \frac{\kappa s_x}{2}\right)dx - \left(\frac{\kappa_y r'}{\kappa} + \frac{s_y r'^2}{2\kappa} + \frac{\kappa s_y}{2}\right)dy\right) \ . \tag{A.4}$$

Using (A.1) – (A.4), (4.17) and (4.18), the product between (4.16) and sign(g) becomes

$$
\begin{aligned}
g = \frac{1}{4}\left(\kappa + \frac{r'^2}{\kappa}\right) g_o + \vartheta \vee \Bigg\{ dr' - \left(\frac{\kappa_x r'}{\kappa} + \frac{s_x r'^2}{2\kappa} + \frac{\kappa s_x}{2}\right) dx - \\
- \left(\frac{\kappa_y r'}{\kappa} + \frac{s_y r'^2}{2\kappa} + \frac{\kappa s_y}{2}\right) dy + \\
+ \frac{1}{2}\left(\kappa + \frac{r'^2}{\kappa}\right) 2\,\mathrm{Re}\left(\left(\mathrm{i}\left(-t + c + \frac{2p_z}{p}\right)\left(e^{-\mathrm{i}(\widetilde{r}+s)} + 1\right) - \mathrm{i}\,t + s_z\right) dz\right) + \\
+ \frac{1}{2}\left(\kappa + \frac{r'^2}{\kappa}\right) \mathcal{H}\vartheta \Bigg\}\ , \qquad \text{(A.5)}
\end{aligned}
$$

In this formula, the term $\mathrm{Re}\left(\mathrm{i}\left(-t + c + \frac{2p_z}{p}\right)\left(e^{-\mathrm{i}(\widetilde{r}+s)} + 1\right) - \mathrm{i}\,t + s_z\right)$ can be written as

$$
\begin{aligned}
\mathrm{Re}\Bigg(\mathrm{i}\left(-t_1 - \mathrm{i}\,t_2 - \frac{f_x}{2} + \mathrm{i}\,\frac{f_y}{2} + \frac{p_x}{p} - \mathrm{i}\,\frac{p_y}{p}\right)(\cos(\widetilde{r}+s) - \mathrm{i}\sin(\widetilde{r}+s) + 1) \\
- \mathrm{i}\,(t_1 + \mathrm{i}\,t_2) + \frac{s_x}{2} - \mathrm{i}\,\frac{s_y}{2}\Big)\,(dx + \mathrm{i}\,dy)\Bigg) = \\
= t_2\cos(\widetilde{r}+s)dx - \frac{f_y}{2}\cos(\widetilde{r}+s)dx + \frac{p_y}{p}\cos(\widetilde{r}+s)dx + \\
- \sin(\widetilde{r}+s))t_1 dx - \sin(\widetilde{r}+s)\frac{f_x}{2}dx + \sin(\widetilde{r}+s))\frac{p_x}{p}dx + t_2 dx - \frac{f_y}{2}dx + \frac{p_y}{p}dx + t_2 dx + \frac{s_x}{2}dx + \\
+ t_1\cos(\widetilde{r}+s)dy + \frac{f_x}{2}\cos(\widetilde{r}+s)dy - \frac{p_x}{p}\cos(\widetilde{r}+s)dy + \\
+ \sin(\widetilde{r}+s)t_2 dy - \sin(\widetilde{r}+s)\frac{f_y}{2}dy + \sin(\widetilde{r}+s)\frac{p_y}{p}dy + t_1 dy + \frac{f_x}{2}dy - \frac{p_x}{p}dy + t_1 dy + \frac{s_y}{2}dy\ .
\end{aligned}
$$

By classical trigonometric relations, $\cos(\widetilde{r}+s) = \frac{\kappa^2 - r'^2}{\kappa^2 + r'^2}$, $\sin(\widetilde{r}+s) = \frac{2r'\kappa}{\kappa^2 + r'^2}$, so that the previous expression becomes

$$
\begin{aligned}
& t_2\frac{\kappa^2 - r'^2}{\kappa^2 + r'^2}dx - \frac{f_y}{2}\frac{\kappa^2 - r'^2}{\kappa^2 + r'^2}dx + \frac{p_y}{p}\frac{\kappa^2 - r'^2}{\kappa^2 + r'^2}dx + \\
& - \frac{2r'\kappa}{\kappa^2 + r'^2}t_1 dx - \frac{2r'\kappa}{\kappa^2 + r'^2}\frac{f_x}{2}dx + \frac{2r'\kappa}{\kappa^2 + r'^2}\frac{p_x}{p}dx + t_2 dx - \frac{f_y}{2}dx + \frac{p_y}{p}dx + t_2 dx + \frac{s_x}{2}dx + \\
& + t_1\frac{\kappa^2 - r'^2}{\kappa^2 + r'^2}dy + \frac{f_x}{2}\frac{\kappa^2 - r'^2}{\kappa^2 + r'^2}dy - \frac{p_x}{p}\frac{\kappa^2 - r'^2}{\kappa^2 + r'^2}dy + \\
& + \frac{2r'\kappa}{\kappa^2 + r'^2}t_2 dy - \frac{2r'\kappa}{\kappa^2 + r'^2}\frac{f_y}{2}dy + \frac{2r'\kappa}{\kappa^2 + r'^2}\frac{p_y}{p}dy + t_1 dy + \frac{f_x}{2}dy - \frac{p_x}{p}dy + t_1 dy + \frac{s_y}{2}dy.
\end{aligned}
$$

Multiplying the above expression with $\frac{\kappa^2+r'^2}{\kappa}$, through some tedious but straightforward manipulation, the above expression simplifies into

$$\left(t_2\frac{\kappa^2-r'^2}{\kappa}-f_y\frac{\kappa^2-r'^2}{2\kappa}+p_y\frac{\kappa^2-r'^2}{p\kappa}-2r't_1-f_xr'+p_x\frac{2r'}{p}\right.$$
$$\left.+\frac{\kappa^2+r'^2}{\kappa}t_2-\frac{\kappa^2+r'^2}{\kappa}\frac{f_y}{2}+\frac{\kappa^2+r'^2}{\kappa}\frac{p_y}{p}+\frac{\kappa^2+r'^2}{\kappa}t_2+\frac{\kappa^2+r'^2}{\kappa}\frac{s_x}{2}\right)dx+$$
$$+\left(t_1\frac{\kappa^2-r'^2}{\kappa}+f_x\frac{\kappa^2-r'^2}{2\kappa}-p_x\frac{\kappa^2-r'^2}{p\kappa}+2r't_2-f_yr'+p_y\frac{2r'}{p}+\frac{\kappa^2+r'^2}{\kappa}t_1\right.$$
$$\left.+\frac{\kappa^2+r'^2}{\kappa}\frac{f_x}{2}-\frac{\kappa^2+r'^2}{\kappa}\frac{p_x}{p}+\frac{\kappa^2+r'^2}{\kappa}t_1+\frac{\kappa^2+r'^2}{\kappa}\frac{s_y}{2}\right)dy.$$

Due to this, the following sum (appearing in (A.5))

$$-\left(\frac{\kappa_xr'}{\kappa}+\frac{s_xr'^2}{2\kappa}+\frac{\kappa s_x}{2}\right)dx-\left(\frac{\kappa_yr'}{\kappa}+\frac{s_yr'^2}{2\kappa}+\frac{\kappa s_y}{2}\right)dy+$$
$$+\frac{1}{2}\left(\kappa+\frac{r'^2}{\kappa}\right)2\operatorname{Re}\left(\left(\mathrm{i}\left(-t+c+\frac{2p_z}{p}\right)\left(e^{-\mathrm{i}(\tilde{r}+s)}+1\right)-\mathrm{i}\,t+s_z\right)dz\right)$$

simplifies into

$$\left(3t_2\kappa-f_y\kappa+\frac{2p_y\kappa}{p}-2r't_1-f_xr'+\frac{2p_xr'}{p}+\frac{r'^2}{\kappa}t_2-\frac{\kappa_xr'}{\kappa}\right)dx+$$
$$+\left(3t_1\kappa+f_x\kappa-\frac{2p_x\kappa}{p}+2r't_2-f_yr'+\frac{2p_yr'}{p}+\frac{r'^2}{\kappa}t_1-\frac{\kappa_yr'}{\kappa}\right)dy. \quad \text{(A.6)}$$

Since $\kappa=2\operatorname{sign}(g)p^2e^{-f}$, $\frac{2p_x\kappa}{p}=\kappa_x+\kappa f_x$ and $\frac{2p_y\kappa}{p}=\kappa_y+\kappa f_y$, (A.6) is equal to

$$\kappa_ydx-\kappa_xdy+\left(3t_2\kappa-2r't_1+\frac{r'^2}{\kappa}t_2\right)dx+\left(3t_1\kappa+2r't_2+\frac{r'^2}{\kappa}t_1\right)dy\ . \quad \text{(A.7)}$$

From (A.5) and (A.7), we get that g is as in (4.3). It only remains to show that $\mathcal{H}$ is equal to the sum of the three functions h_o, h_m and h_t. But this is a straightforward consequence of (4.19) and the definition (4.23) of κ and r'.

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Masoud Ganji
School of Science and Technology
University of New England,
Armidale NSW 2351
Australia

E-mail: mganjia2@une.edu.au

Cristina Giannotti and Andrea Spiro
Scuola di Scienze e TecnolSogie
Università di Camerino
I-62032 Camerino (Macerata)
Italy

E-mail: cristina.giannotti@unicam.it
E-mail: andrea.spiro@unicam.it