On Tangential and Projectively Adjacent Approach Regions

Fausto Di Biase
Dipartimento di Economia
Università "G. D'Annunzio" di Chieti-Pescara,
Viale Pindaro, 42, I-65127, Pescara, Italy
email: fausto.dibiase@unich.it

Haguma Gratien
Department of Mathematics
University of Rwanda
KN 67 Nyarugenge, Kigali, 3900 Kigali, Rwanda
email: hagugrat@gmail.com

Olof Svensson
Department of Science and Technology
Linköping University
Campus Norrköping
SE-60174 Norrköping, Sweden
email: olof.svensson@liu.se

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Abstract

In 1906 Fatou proved that bounded holomorphic functions on the unit disc converge a.e. on the boundary along nontangential approach regions. In 1927 Littlewood proved a "negative" result, i.e., that a.e. convergence fails for certain approach regions: More precisely, it fails for the rotationally invariant families of tangential approach regions that end *curvilinearly* at the boundary. The fact that tangential approach regions which end *sequentially* at the boundary may instead be very well conducive to a.e. convergence was understood more recently by W. Rudin (in 1979) and A. Nagel and E.M. Stein (in 1984), in contributions that provided additional and much needed insight, and that prompted the question of giving an *a priori* description of those families of tangential approach regions which end *sequentially* at the boundary and for which a.e. convergence *fails*.

Our main result is the first one of this kind. Indeed, we prove the failure of a.e. convergence for a class of approach regions, introduced in this work, which we call *projectively adjacent*, that contains curvilinear approach regions as well as sequential ones.

Our result recaptures the aforementioned theorem of Littlewood as well as some of the other theorems of that type, but its novelty lies in the fact that, while all previous "negative" results for tangential approach (Littlewood, 1927; Lohwater and Piranian, 1957; Aikawa, 1990-1991, and others) dealt with approach regions that either are *curves* or share with curves a certain topological property that *excludes* the possibility that they could end sequentially at the boundary, our "negative" result deals with a class of approach regions that contains both the curvilinear and the sequential ones. Hence we present a significant extension of the class of those families of approach regions for which a.e. convergence fails.

Keywords. Fatou type theorems, Littlewood type theorems, bounded holomorphic functions, bounded harmonic functions unit disc, approach regions, almost everywhere convergence, tangential approach regions, nontangential approach regions, very oscillatory approach regions, Littlewood sets, the Nagel–Stein phenomenon, projectively adjacent approach regions. MSC 30H05, 30J99, 31A20.

1 Introduction

The goal of this work is to advance our knowledge of the *qualitative* boundary behavior of functions in $\mathsf{H}^\infty(\mathbb{D})$, i.e., *in the sense of almost everywhere convergence along certain approach regions*, where

$$\mathbb{D} \overset{\text{def}}{=\!=} \{z \in \mathbb{C} : |z| < 1\}$$

is the open unit disc and $\mathsf{H}^\infty(\mathbb{D})$ is the Banach algebra of holomorphic functions defined on $\mathbb{D}$. For several decades, our knowledge in this area has been shaped by two results, due to P. Fatou in 1906 [19] and to J.E. Littlewod in 1927 [32]. More recently, the discoveries made by W. Rudin in 1979 [39] and by A. Nagel and E.M. Stein in 1984 [35] have provided additional and much needed insight.

Most, but (as we shall see) not all of the results on the qualitative boundary behavior of functions in $\mathsf{H}^\infty(\mathbb{D})$ fall in two main categories: Those of a "positive" type, which describe approach regions along which all functions in $\mathsf{H}^\infty(\mathbb{D})$ converge a.e., and those of a "negative" type, which describe approach regions along which, on the contrary, the almost everywhere convergence described above *fails.*

The fact that the nontangential approach regions are conducive to a.e. convergence and thus yield results of a "positive" nature was discovered quite early in the literature (Fatou, 1906) [19], and left the impression that results of a "positive" type along *tangential* approach regions did not exist.

The first result of a "negative" type, obtained by Littlewood in 1927 [32], involved rotationally invariant families of *tangential* approach regions, and was proved under the additional assumption that they were *curvilinear.*

The fact that tangential approach regions that end *sequentially* at the boundary may very well be conducive to a.e. convergence was understood more recently by W. Rudin in 1979 [39] and A. Nagel and E.M. Stein in 1984 [35]:

This discovery prompted the question of giving an *a priori* description of those families of tangential approach regions that end *sequentially* at the boundary and for which a result of a "negative" type holds.

Our main result is the first one of this kind. While all previous "negative" results for tangential approach (Littlewood in 1927 [32]; Lohwater and Piranian in 1957 [33]; Aikawa in 1990 and 1991 [2], [3], and [14]) dealt with approach regions that either are *curves* or share with curves a certain topological property that *excludes* the possibility of being countably infinite, our main result deals with approach regions that satisfy a certain condition, called **projective adjacency**, introduced for the first time in this work, which does allow them to be countably infinite, *without* excluding the possibility that they could be curvilinear. In other words, ours is the first result of a "negative" type which applies also to approach regions that are countable and infinite. In fact, our condition of projective adjacency concerns the approach behavior of the region, and not its continuous or discrete nature.

1.1 Four notable classes of approach regions

We first present the general notation used in our work.

1.1.1 Notation

The **power set** of a set X, denoted by $\mathcal{P}(X)$, is the collection of all subsets of X. If $z \in \mathbb{C}$, $|z|$ is its absolute value and $B(y, r)$ is **the (Euclidean) open ball** in $\mathbb{C}$ of center $y \in \mathbb{C}$ and radius $r > 0$, i.e.,

$$\mathrm{B}(y, \rho) \stackrel{\text{def}}{=\!=} \{x \in \mathbb{C} : |x - y| < r\}$$

If $\mathsf{A} \in \mathcal{P}(\mathbb{C})$ and $r > 0$, we set

$$\mathsf{A}(y, r) \stackrel{\text{def}}{=\!=} \mathsf{A} \cap \mathrm{B}(y, r). \tag{1.1}$$

If $\mathsf{A}, \mathsf{V} \in \mathcal{P}(\mathbb{C})$, we write

$$\boxed{\mathsf{A} \sim_{\mathtt{w}} \mathsf{V}} \tag{1.2}$$

and say that A **and** V **have the same germ at** $\mathtt{w}$ if there exists $r > 0$ such that $\mathsf{A}(\mathtt{w}, r) = \mathsf{V}(\mathtt{w}, r)$; cf. [7, I §6.10, p. 68]. The sets in (1.1), for $r > 0$, are called the **tails of** A **at** $\mathtt{w}$; cf. [13].

The set $\{z \in \mathbb{C} : |z| = 1\}$ has two lives: If seen as a **metric measure space**, endowed with the metric inherited by the Euclidean metric of $\mathbb{C}$ and Lebesgue one-dimensional measure (based on arc-length) it is denoted by $\partial\mathbb{D}$; if seen as a (compact) **topological group** under the familiar identification with the quotient group $\mathbb{R}/\mathbb{Z}$, it is denoted by $\mathbb{T}$. The fact that the same set $\{z \in \mathbb{C} : |z| = 1\}$ is the underlying set of two different structures can be best expressed in the language of category theory, but here we shall not dwell on these matters and we simply employ different symbols. We denote by $|\mathtt{S}|$ the **Lebesgue measure** (based on **arc-length**) of a measurable subset $\mathtt{S}$ of $\partial\mathbb{D}$.

An **arc in** $\partial\,\mathbb{D}$ is a *nonempty* subset of $\partial\,\mathbb{D}$ equal to the intersection of $\partial\,\mathbb{D}$ with an open ball of $\mathbb{C}$: It is either equal to $\partial\,\mathbb{D}$ or of the form

$$\mathtt{arc}_{+}[\mathtt{w};\theta] \stackrel{\text{def}}{=\!=} \{\mathtt{w}\,e^{i\alpha} : 0<\alpha<\theta\} \text{ (where } 0<\theta\le 2\pi \text{ and } \mathtt{w}\in\partial\,\mathbb{D}) \tag{1.3}$$

The points $\mathtt{w}$ and $\mathtt{w}\,e^{i\theta}$ are called the **endpoints** of the arc (1.3), where $\mathtt{w}$ is the **left-endpoint** of (1.3) and $\mathtt{w}\,e^{i\theta}$ is the **right-endpoint**. The right-endpoint is equal to the left-endpoint if and only if $\theta = 2\pi$. We set

$$\mathtt{Arcs}(\mathtt{w}) \stackrel{\text{def}}{=\!=} \{\mathtt{J} : \mathtt{J} \text{ is an arc in } \partial\,\mathbb{D} \text{ with } \mathtt{w} \text{ as an endpoint}\} \tag{1.4}$$

and

$$\mathtt{Arcs}(\mathtt{w}\!\leftarrow) \stackrel{\text{def}}{=\!=} \{\mathtt{J} : \mathtt{J}\in\mathtt{Arcs}(\mathtt{w}) \text{ and } \mathtt{w} \text{ is a left-endpoint of } \mathtt{J}\} \tag{1.5}$$

The **arc in** $\partial\,\mathbb{D}$ **centered at** $\mathtt{w}$ **of arc-length** θ is defined as follows:

$$\mathtt{arc}[\mathtt{w};\theta] \stackrel{\text{def}}{=\!=} \{\mathtt{w}\,e^{i\epsilon} : |\epsilon|<\theta/2\} \tag{1.6}$$

(where $\theta\le 2\pi$). If $\mathtt{w} = e^{i\omega}$, with $\omega\in\mathbb{R}$, then $\mathtt{arc}[\mathtt{w};\theta] \equiv \{e^{i(\omega+\epsilon)} : |\epsilon|<\theta/2\}$. Recall that $|\mathtt{S}|$ is the Lebesgue measure (based on arc-length) of a measurable subset $\mathtt{S}$ of $\partial\,\mathbb{D}$. In particular, if $\mathtt{w}\in\partial\,\mathbb{D}$ and $\theta\le 2\pi$ then

$$|\mathtt{arc}[\mathtt{w};\theta]| = \theta \tag{1.7}$$

A **null set** in $\partial\,\mathbb{D}$ is a measurable subset $\mathtt{U}\subset\partial\,\mathbb{D}$ with $|\mathtt{U}|=0$, and we set

$$\boldsymbol{\mathcal{N}} \stackrel{\text{def}}{=\!=} \{\mathtt{U}\in\mathcal{P}(\partial\,\mathbb{D}) : \mathtt{U} \text{ is a null set in } \partial\,\mathbb{D}\}.$$

If $\mathtt{S}$ and $\mathtt{Q}$ are subsets of $\partial\,\mathbb{D}$ we say that $\mathtt{S}$ is **a.e. contained in** $\mathtt{Q}$, and write

$$\mathtt{S} \overset{\text{a.e.}}{\subset} \mathtt{Q}$$

if $\mathtt{S}\setminus\mathtt{Q}\in\boldsymbol{\mathcal{N}}$: This means that almost all of $\mathtt{S}$ is a subset of $\mathtt{Q}$. Observe that the sets $\mathtt{S}$ and $\mathtt{Q}$ need not be measurable (see, e.g., Theorem 2.18) and that

$$\mathtt{A}\setminus\mathtt{B} \overset{\text{a.e.}}{\subset} \mathtt{C} \iff \mathtt{A}\setminus\mathtt{C} \overset{\text{a.e.}}{\subset} \mathtt{B} \tag{1.8}$$

since $(\mathtt{A}\setminus\mathtt{B})\setminus\mathtt{C} = (\mathtt{A}\setminus\mathtt{C})\setminus\mathtt{B}$, as both are equal to $\mathtt{A}\setminus(\mathtt{B}\cup\mathtt{C})$. We say that $\mathtt{S}$ and $\mathtt{Q}$ are **almost everywhere equal**, and write

$$\mathtt{S} \overset{\text{a.e.}}{=} \mathtt{Q}$$

if $\mathtt{S} \overset{\text{a.e.}}{\subset} \mathtt{Q}$ and $\mathtt{Q} \overset{\text{a.e.}}{\subset} \mathtt{S}$, i.e., if $(\mathtt{S}\setminus\mathtt{Q})\cup(\mathtt{Q}\setminus\mathtt{S})\in\boldsymbol{\mathcal{N}}$. A set $\mathtt{S}\subset\partial\,\mathbb{D}$ has **full measure** if $\mathtt{S} \overset{\text{a.e.}}{=} \partial\,\mathbb{D}$, i.e., if $\partial\,\mathbb{D}\setminus\mathtt{S}\in\boldsymbol{\mathcal{N}}$. A property is said to hold **a.e.** if the set of points in $\partial\,\mathbb{D}$ for which it holds has full measure. A set $\mathtt{S}\subset\mathtt{Q}$ has **full measure in** $\mathtt{Q}$ if $\mathtt{S} \overset{\text{a.e.}}{=} \mathtt{Q}$.

The following notation from category theory is borrowed (and adapted to our needs) from [41]: If B and L are sets, the **set of all functions from B to L** is denoted by

$$[\![B \to L]\!]$$

In particular, the elements of $[\![\partial\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]$ are functions $\mathbf{A} : \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})$ that map $\mathtt{w} \in \partial\mathbb{D}$ to a subset $\mathbf{A}(\mathtt{w})$ of $\mathbb{D}$. If $\mathbf{A} \in [\![\partial\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]$, $\mathtt{w} \in \partial\mathbb{D}$, and $r > 0$, then we write, along the lines of (1.1)

$$\mathbf{A}(\mathtt{w}, r) \overset{\text{def}}{=\!=} \mathbf{A}(\mathtt{w}) \cap \mathrm{B}(\mathtt{w}, r) \tag{1.9}$$

If B, L are objects in a concrete category, a subscript in parentheses is used to indicate the category where the hom-set is considered. For example, if **top** is the category of topological spaces, then $[\![B \to L]\!]_{(\mathbf{top})}$ is the collection of continuous functions from B to L.

The closure of $S \in \mathcal{P}(\mathbb{C})$, where S is seen as a subset of the **Riemann sphere** $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and its closure where S is seen as a subset of $\mathbb{C}$, are equal only if S is bounded. We will denote the former by $\mathsf{cl}_\infty(S)$, the latter by $\mathsf{cl}(S)$. In general, $\mathsf{cl}(S) \subset \mathbb{C}$, $\mathsf{cl}_\infty(S) \subset \widehat{\mathbb{C}}$. We denote by ∂S the **boundary** of $S \in \mathcal{P}(\mathbb{C})$, where S is seen as a subset of $\mathbb{C}$.

Now we introduce the notions needed in the statement of our main result.

1.1.2 Approach regions and limiting values

Let $\mathtt{w} \in \partial\mathbb{D}$. If $\mathsf{f} \in [\![\mathbb{D} \to \mathbb{C}]\!]$, $\ell \in \mathbb{C}$, and $\mathsf{A} \in \mathcal{P}(\mathbb{D})$, we write

$$\lim_{\mathsf{A} \to \mathtt{w}} \mathsf{f} = \ell \tag{1.10}$$

and say that ℓ is **the limiting value of f along A at $\mathtt{w}$** if

$$\bigcap_{r>0} \mathsf{cl}_\infty(\{\mathsf{f}(\mathsf{z}) : \mathsf{z} \in \mathsf{A}(\mathtt{w}, r)\}) = \{\ell\} \tag{1.11}$$

Cf. [10]. Observe that (1.11) is equivalent to the following conditions

$$\begin{cases} \mathtt{w} \in \mathsf{cl}(\mathsf{A}) \\ \forall \epsilon > 0\ \exists \delta > 0\ \ \mathsf{A}(\mathtt{w}, \delta) \subset \{\mathsf{z} \in \mathbb{D} : \mathsf{f}(\mathsf{z}) \in \mathrm{B}(\ell, \epsilon)\} \end{cases} \tag{1.12}$$

and (1.10) is usually written with a dummy variable, in the form $\lim\limits_{\substack{\mathsf{z} \to \mathtt{w} \\ \mathsf{z} \in \mathsf{A}}} \mathsf{f}(\mathsf{z}) = \ell$.

An **approach region (in $\mathbb{D}$) ending at $\mathtt{w}$** is a set $\mathsf{A} \in \mathcal{P}(\mathbb{D})$ such that $\mathtt{w} \in \mathsf{cl}(\mathsf{A})$. We set

$$\mathbb{D}[\mathtt{w}] \overset{\text{def}}{=\!=} \{\mathsf{A} \in \mathcal{P}(\mathbb{D}) : \mathsf{A} \text{ is an approach region in } \mathbb{D} \text{ ending at } \mathtt{w}\}$$

The presence of $\mathtt{w}$ in the notation (1.10) is necessary since it may happen the $\mathsf{A} \in \mathbb{D}[\mathtt{w}] \cap \mathbb{D}[\mathtt{w}']$ with $\mathtt{w}' \neq \mathtt{w}$, as for example in the case $\mathsf{A} = \mathbb{D}$.

The following two notable subsets of $\mathbb{D}[\mathtt{w}]$ play a special role in the subject, as showed in 1927 by J.E. Littlewood [32] (Section 2.16). Observe that in the following definitions we assume $\mathsf{A} \in \mathbb{D}[\mathtt{w}]$; in dealing with a general $\mathsf{A} \in \mathcal{P}(\mathbb{D})$ it is useful to recall that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

($\mathbf{t}[\mathtt{w}]$) We say that $\mathsf{A} \in \mathbb{D}[\mathtt{w}]$ **ends tangentially at** $\mathtt{w} \in \partial\,\mathbb{D}$ if

$$0 = \inf_{r>0} \sup_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \frac{1 - |\,\mathsf{z}\,|}{|\,\mathsf{z} - \mathtt{w}\,|} \tag{1.13}$$

and we set $\mathbf{t}[\mathtt{w}] \overset{\text{def}}{=\!=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A} \text{ ends tangentially at } \mathtt{w}\}$.

($\mathbf{c}[\mathtt{w}]$) We say that $\mathsf{A} \in \mathcal{P}(\mathbb{D})$ is **curvilinear at** $\mathtt{w}$ if

$$\exists \varphi \in [\![[0,1) \to \mathbb{D}]\!]_{(\mathsf{top})},\ \lim_{s \uparrow 1} \varphi(s) = \mathtt{w} \text{ and } \mathsf{A} \sim_{\mathtt{w}} \{\varphi(s) : 0 \le s < 1\} \tag{1.14}$$

and we set $\mathbf{c}[\mathtt{w}] \overset{\text{def}}{=\!=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A} \text{ is curvilinear at } \mathtt{w}\}$.

The following subset of $\mathbb{D}[\mathtt{w}]$ is a natural one but its role in the subject was revealed only recently, in 1979 [39] and in 1984 [35], as we will see momentarily.

($\mathbf{s}[\mathtt{w}]$) We say that $\mathsf{A} \in \mathcal{P}(\mathbb{D})$ **ends sequentially at** $\mathtt{w}$ if

$$\exists \varphi \in [\![\mathbb{N} \to \mathbb{D}]\!] \text{ with } \lim_{j \to +\infty} \varphi_j = \mathtt{w} \text{ and } \mathsf{A} \sim_{\mathtt{w}} \{\varphi_j : j \in \mathbb{N}\} \tag{1.15}$$

and we set $\mathbf{s}[\mathtt{w}] \overset{\text{def}}{=\!=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A} \text{ ends sequentially at } \mathtt{w}\}$.

The notions described above have already been established and used in the literature [2], [3], [10], [32], [33], [35], [39], [50]. The following notion is much more recent and was introduced in 2006 [14].

($\mathbf{a}[\mathtt{w}]$) We say that $\mathsf{A} \in \mathcal{P}(\mathbb{D})$ is **attached at** $\mathtt{w}$ if

$$\exists\, \mathrm{S} \in \mathcal{P}(\mathbb{D}),\ \{\mathtt{w}\} \cup \mathrm{S} \text{ is connected, and } \mathsf{A} \sim_{\mathtt{w}} \mathrm{S} \tag{1.16}$$

and we set $\mathbf{a}[\mathtt{w}] \overset{\text{def}}{=\!=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A} \text{ is attached at } \mathtt{w}\}$.

Observe that since, in this setting, $\mathtt{w} \in \mathsf{cl}(\mathrm{S})$, then the hypothesis that $\{\mathtt{w}\} \cup \mathrm{S}$ is connected is strictly weaker that the hypothesis that S is connected. We say that S is **connected to** $\mathtt{w}$ if the set $\{\mathtt{w}\} \cup \mathrm{S}$ is connected.

The following result says that the collection $\mathbf{a}[\mathtt{w}]$ strictly contains $\mathbf{c}[\mathtt{w}]$ but it is still disjoint from $\mathbf{s}[\mathtt{w}]$, precisely as $\mathbf{c}[\mathtt{w}]$ is.

Lemma 1.1 $\mathbf{c}[\mathtt{w}] \subsetneq \mathbf{a}[\mathtt{w}]$ *and* $\mathbf{a}[\mathtt{w}] \cap \mathbf{s}[\mathtt{w}] = \emptyset$.

Proof. The proof of Lemma 1.1 will be given in Section 3.1. $\square$

1.2 Projectively adjacent approach regions

The main contribution of the present work, Theorem 1.3, is based on the following new notion, introduced in the present work.

($\mathbf{p}[\mathtt{w}]$) We say that $\mathsf{A} \in \mathcal{P}(\mathbb{D})$ is **projectively adjacent to** $\mathtt{w}$, if

$$\exists b \in \mathbb{N} : \forall r > 0 \, \exists \, \mathtt{J} \in \mathtt{Arcs}(\mathtt{w}) : \forall \, \mathtt{x} \in \mathtt{J} \, \exists \, \mathtt{z} \in \mathsf{A}(\mathtt{w}, r) : |\, \mathtt{x} - \mathtt{z} \,| < (1 + b)(1 - |\, \mathtt{z} \,|) \quad (1.17)$$

and we set $\mathbf{p}[\mathtt{w}] \overset{\text{def}}{=\!=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A} \text{ is projectively adjacent at } \mathtt{w}\}$.

We will present momentarily a more geometric rendition of the notion described in rather dry terms in (1.17). Observe that $\mathbf{p}[\mathtt{w}] \subset \mathbb{D}[\mathtt{w}]$. For the time being, it is important to keep in mind the following facts. We have seen in Lemma 1.1 that $\mathbf{a}[\mathtt{w}]$ is strictly larger than $\mathbf{c}[\mathtt{w}]$ and it has *empty* intersection with $\mathbf{s}[\mathtt{w}]$ (just as $\mathbf{c}[\mathtt{w}]$ has). Proposition 1.2 says that $\mathbf{p}[\mathtt{w}]$ is strictly larger than $\mathbf{a}[\mathtt{w}]$ but it has *nonempty* intersection with $\mathbf{s}[\mathtt{w}]$. Hence a result that holds assuming that certain approach regions belong to $\mathbf{p}[\mathtt{w}]$ is significantly stronger than one that holds assuming it belongs to $\mathbf{a}[\mathtt{w}]$, as we will see in detail in Section 1.3.

Proposition 1.2 *If* $\mathtt{w} \in \partial\,\mathbb{D}$ *then*
(a) $\mathbf{a}[\mathtt{w}] \subsetneq \mathbf{p}[\mathtt{w}]$, **(b)** $\mathbf{s}[\mathtt{w}] \cap \mathbf{t}[\mathtt{w}] \cap \mathbf{p}[\mathtt{w}] \neq \emptyset$, **(c)** $(\mathbf{s}[\mathtt{w}] \cap \mathbf{t}[\mathtt{w}]) \setminus \mathbf{p}[\mathtt{w}] \neq \emptyset$.

Proof. The proof will be presented in Section 3. □

Further background: [2], [3], [4], [9], [10], [11], [14], [15], [20], [22], [23], [25], [27], [28], [29], [34], [33], [36], [37], [40], [42], [43], [44], [46], [47], [48], [51].

1.3 Our main result, its significance, with a look at open problems

The reason that underlies the importance of the class of projectively adjacent approach regions, introduced in the present work, and makes our main contribution, Theorem 1.3, stated below, a significant advance in the understanding of the boundary behaviour of functions in $\mathsf{H}^{\infty}(\mathbb{D})$, may be better understood if we recall **Fatou's Theorem (1906)** [19] (cf. Section 2.15), where Pierre Fatou proved that for each $\mathsf{f} \in \mathsf{H}^{\infty}(\mathbb{D})$ there exists $\mathtt{E}_{\mathsf{f}} \in \boldsymbol{\mathcal{N}}$, such that for each $\mathtt{w} \in \partial\,\mathbb{D} \setminus \mathtt{E}_{\mathsf{f}}$ and for each $\mathbf{A} \in [\![\, \partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]$, if

(nt) $\quad 0 < \sup_{r>0} \inf_{\mathtt{z} \in \mathbf{A}(\mathtt{w},r)} (1 - |\, \mathtt{z} \,|)/|\, \mathtt{z} - \mathtt{w} \,| < +\infty$

then the limiting value $\lim\limits_{\mathbf{A}(\mathtt{w}) \nrightarrow \mathtt{w}} \mathsf{f}$ exists [19]. The condition in **(nt)** says that $\mathbf{A}(\mathtt{w})$ is an approach region ending *nontangentially* at $\mathtt{w}$ (cf. Section 2). Observe that since $\inf \emptyset = +\infty$, the condition $\sup_{r>0} \inf_{\mathtt{z} \in \mathbf{A}(\mathtt{w},r)} (1 - |\, \mathtt{z} \,|)/|\, \mathtt{z} - \mathtt{w} \,| < +\infty$ ensures that $\mathbf{A}(\mathtt{w}, r) \neq \emptyset \;\forall r > 0$, i.e., that $\mathbf{A}(\mathtt{w}) \in \mathbb{D}[\mathtt{w}]$.

Our main contribution is a result of a "negative" nature where *projectively adjacent* approach regions, defined in (1.17), play a central role.

Theorem 1.3 *Assume that* $\mathbf{A} \in [\![\, \partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]$ *and that*

(t) $\mathbf{A}(\mathtt{w}) \in \mathbf{t}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ ends tangentially at $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\,\mathbb{D}$.

Moreover, assume that $\mathbf{A}$ *has the following properties.*

[p] $\mathbf{A}(\mathtt{w}) \in \mathbf{p}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ is projectively adjacent to $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\,\mathbb{D}$.

[reg] $U \subset \mathbb{D}$ *is open* $\implies \{\mathtt{w} \in \partial\,\mathbb{D} : U \cap \mathbf{A}(\mathtt{w}) \neq \emptyset\}$ *is measurable.*

Then there exists $\mathsf{h} \in \mathsf{H}^\infty(\mathbb{D})$ *such that for a.e.* $\mathtt{w} \in \partial\,\mathbb{D}$ $\displaystyle\lim_{\mathbf{A}(\mathtt{w}) \ni \to \mathtt{w}} \mathsf{h}$ **does not exist**.

1.3.1 The significance of our main result

In order to appreciate how Theorem 1.3 improves our understanding of the subject, recall that while in the previously known "negative" results of this kind the various approach regions could *not* possibly end *sequentially* at the boundary, our Theorem 1.3 also allows for this possibility. Indeed, Proposition 1.2 shows that approach regions that are projectively adjacent to $\mathtt{w}$ may very well end sequentially *and* tangentially at $\mathtt{w}$, while in the previously known "negative" results of this kind, either the approach regions are assumed to end *curvilinearly* at the various boundary points [2], [3], [10], [32], [33], [50], or they are assumed to be *attached* at them [14], and Lemma 1.1 shows that these assumptions *exclude* the possibility that they could end *sequentially* at the boundary.

It is useful to illustrate these observations in some more detail. Recall that the prototype result of a "negative" nature is Littlewood's Theorem (1927) [32] (see Section 2.16). Indeed, all the subsequent "negative" results of this kind are called "Theorems of Littlewood Type".

Littlewood's Theorem (1927) ([32])**.** Assume that $\mathbf{A} \in [\![\, \partial\,\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]$ and that

(t) $\mathbf{A}(\mathtt{w}) \in \mathbf{t}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ ends tangentially at $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\,\mathbb{D}$.

Moreover, assume that $\mathbf{A}$ has the following properties.

[c] $\mathbf{A}(\mathtt{w}) \in \mathbf{c}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ is curvilinear at $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\,\mathbb{D}$.

[$\mathbb{T}$ - invariant] $\mathbf{A}$ is rotationally invariant: $\mathsf{z} \in \mathbf{A}(\mathtt{w}) \iff \mathsf{z}\, e^{i\theta} \in \mathbf{A}(\mathtt{w}e^{i\theta})$.

Then there exists $\mathsf{f} \in \mathsf{H}^\infty(\mathbb{D})$ such that for a.e. $\mathtt{w} \in \partial\,\mathbb{D}$ $\displaystyle\lim_{\mathbf{A}(\mathtt{w}) \ni \to \mathtt{w}} \mathsf{f}$ **does not exist**.

Observe Theorem 1.3 implies Littlewood's Theorem, since **[c]** is stronger than **[p]** and [$\mathbb{T}$ - **invariant**] is stronger than **[reg]**. However, **[c]** *excludes* that the approach regions could end *sequentially* at the boundary (cf. Lemma 1.1).

After 1927, theorems of Littlewood type were proved in 1949 [50], 1957 [33], 1990-1991 [2]&[3], 2006 [14], and, in other contexts, in 1983 [21] and 2005 [22].

For reasons that will be clarified momentarily, it is now convenient to describe a theorem of Littlewood type published in 2006 in [14] (cf. Section 2.16).

A Theorem of Littlewood Type (2006) ([14])**.** If $\mathbf{A} \in [\![\, \partial\,\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]$ and

(t) $\mathbf{A}(\mathtt{w}) \in \mathbf{t}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ ends tangentially at $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\,\mathbb{D}$,

and, moreover, if $\mathbf{A}$ has the following properties:

[a] $\mathbf{A}(\mathtt{w}) \in \mathbf{a}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ is attached at $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\,\mathbb{D}$;

[reg] $\quad U \subset \mathbb{D}$ is open $\implies \{\mathtt{w} \in \partial\,\mathbb{D} : U \cap \mathbf{A}(\mathtt{w}) \neq \emptyset\}$ is measurable;

then there exists $\mathsf{f} \in \mathsf{H}^\infty(\mathbb{D})$ such that for a.e. $\mathtt{w} \in \partial\,\mathbb{D}$ $\lim\limits_{\mathbf{A}(\mathtt{w}) \ni \mathtt{w}} \mathsf{f}$ **does not exist**.

The statement of this theorem differs from the statement of Littlewood's Theorem since instead of **[c]** the following condition appears:

[a] $\mathbf{A}(\mathtt{w}) \in \mathbf{a}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ is attached at $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\,\mathbb{D}$.

and instead of **[$\mathbb{T}$ - invariant]** the condition **[reg]** appears. The condition **[reg]** was introduced in this context for the first time in [14].

Observe that the theorem of Littlewood type stated above, proved in 2006 in [14], implies Littlewood's Theorem, since [$\mathbb{T}$ - **invariant**] is stronger than [**reg**] and [**c**] is stronger than [**a**]. However, [**a**] *excludes* that the approach regions could end *sequentially* at the boundary (cf. Lemma 1.1).

We are now able to put Theorem 1.3 in context. Since **[a]** is stronger than **[p]**, it follows that Theorem 1.3 implies Theorem LT (2006). However, [**p**] *allows* the approach regions to end *sequentially* at $\mathtt{w}$ (cf. Proposition 1.2), while this outcome is excluded by [**c**] and [**a**]. The significance of this fact can be better appreciated if we recall the results obtained by W. Rudin in 1979 [39] and by A. Nagel and E.M. Stein in 1984 [35], presented in Section 2, where the necessary background of the present work is given. Indeed, these authors showed that approach regions that end *sequentially* and *tangentially* at the boundary may very well be conducive to almost everywhere convergence for functions in $\mathsf{H}^\infty(\mathbb{D})$. Our result is the first "negative" one where the approach regions are allowed to be countably infinite rather than curvilinear.

1.3.2 Comment on our proof

The original proof of Littlewood's Theorem (1927) was based on a nontrivial measure-theoretic result in Diophantine approximation proved by A. Ya. Khinchin. In 1949, A. Zygmund gave two other proofs of Littlewood's Theorem: The first one was based on complex analysis, while the other was entirely within the realm of real analysis. The general outline of the proof of our main result is similar to the real-variable proof given by A. Zygmund of Littlewood's Theorem, but the application of that outline to our case is by no means immediate or straightforward, as we will see, since the notion of *projective adjacency*, on which our proof hinges, does not depend on the continuous or discrete nature of the approach regions. Indeed, while Littlewood's Theorem (1927) says that

> no rotationally invariant family of *curvilinear* and tangential approach regions can be of Fatou type

our main result, Theorem 1.3, says that

> no regular family of *projectively adjacent* and tangential approach regions can be of Fatou type

and the class of *projectively adjacent* approach regions contains countably infinite approach regions as well as curvilinear ones. Moreover, the class of projectively adjacent approach regions also contains the nontangential approach regions.

1.3.3 Extension to harmonic functions in higher dimensions

An open problem that lies ahead is first of all the extension of the notion of projective adjacency to higher dimensions, first of all in the study of the boundary behavior of harmonic function in the upper-half spaces $\mathbb{R}^n \times (0, +\infty)$, and, correspondingly, the extension of our main result, Theorem 1.3, to those higher-dimensional settings, and to others as well. Of course we hope to be able to return to these matters in the near future. For the time being, recall that in 1991 Hiroaki Aikawa [3] (cf. [2]) proved the following result.

Aikawa's Theorem of Littlewood Type (1991) ([3]). If $\mathbf{A} \in [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]$ and

(t) $\mathbf{A}(\mathtt{w}) \in \mathbf{t}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ ends tangentially at $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\mathbb{D}$,

and, moreover, we assume that $\mathbf{A}$ has the following properties:

[c] $\mathbf{A}(\mathtt{w}) \in \mathbf{c}[\mathtt{w}]$ (i.e., $\mathbf{A}(\mathtt{w})$ is curvilinear at $\mathtt{w}$) $\forall\, \mathtt{w} \in \partial\mathbb{D}$;

[unif] The curves $\{\mathbf{A}(\mathtt{w})\}_{\mathtt{w}\in\partial\mathbb{D}}$ are uniformly bi-Lipschitz equivalent;

then there exists $\mathsf{f} \in \mathsf{H}^\infty(\mathbb{D})$ such that for a.e. $\mathtt{w} \in \partial\mathbb{D}$ $\lim\limits_{\mathbf{A}(\mathtt{w}) \ni \to \mathtt{w}} \mathsf{f}$ **does not exist**.

Here we shall not give a more precise definition of the property described in **[unif]** (see [3]), but limit ourselves to observe that Aikawa has used a condition of this kind to prove a Littlewood type theorem in the context of the study of the boundary behavior of harmonic function in Euclidean half-spaces. However, we would rather be interested in using **[reg]** rather than **[unif]**, in a hypothetical higher-dimensional extension of Theorem 1.3.

The fact that the extension of our results to higher dimensions is not as straightforward as it may appear may be seen by observing that the aforementioned extension to the study of the boundary behavior of harmonic functions in Euclidean half-spaces could possibly be done along two directions, which have a different level of generality. To fix ideas, we examine the case $n = 2$, i.e., $\mathbb{R}^2 \times (0, +\infty)$. In this case, one may look at approach regions which amount to (i) hypersurfaces, or amount to (ii) curves.

In the first case, it could perhaps be possible to extend our results to (i) without too many technical difficulties, provided of course the appropriate higher-dimensional version of the notion of projective adjacency can be found.

In the second case, the difficulties of dealing with (ii), which arise from the fact that now the approach regions are "smaller", can perhaps be solved by replacing the regularity hypothesis in Theorem 1.3 with a "quantitative" hypothesis such as the one in **[unif]**, as H. Aikawa did in [3], where one assumes, roughly speaking, that the approach regions at different boundary points have *comparable order of tangency.* However, if we seek a theorem having the same outline as Theorem 1.3, where one only assumes the regularity hypothesis, which is of a *qualitative* nature, and does not assume that, in some sense, the approach regions at different boundary points have the same order of tangency (a hypothesis of a *quantitative nature*), then the extension to higher-dimensional settings of the results introduced in the present work, in the sense of (ii), is more difficult and requires new tools. We hope to be able to address these issues in future work.

1.3.4 Further analysis of $\mathbb{D}[\mathtt{w}]$ and other open problems

In order to summarise the relations between the various subsets of $\mathbb{D}[\mathtt{w}]$ that we met so far, the following notation will be useful. If b and d are subsets of $\mathbb{D}[\mathtt{w}]$, the expression $b \perp d$ means that $b \cap d = \emptyset$; the expression $b \between d$ means that the sets $b \setminus d$, $d \setminus b$, and $b \cap d$ are all nonempty; the expression $b \dot{\cup} d$ denotes the union of b and d under the assumption that these sets are disjoint. Here is a summary of the relations between $\boldsymbol{\gamma}[\mathtt{w}], \mathbf{o}[\mathtt{w}], \mathbf{t}[\mathtt{w}], \mathbf{c}[\mathtt{w}], \mathbf{a}[\mathtt{w}], \mathbf{p}[\mathtt{w}], \mathbf{s}[\mathtt{w}]$.

- $\mathbb{D}[\mathtt{w}] = \boldsymbol{\gamma}[\mathtt{w}] \,\dot{\cup}\, \mathbf{o}[\mathtt{w}] \,\dot{\cup}\, \mathbf{t}[\mathtt{w}]$
- $\mathbf{c}[\mathtt{w}] \subsetneq \mathbf{a}[\mathtt{w}] \subsetneq \mathbf{p}[\mathtt{w}], \quad \mathbf{a}[\mathtt{w}] \perp \mathbf{s}[\mathtt{w}], \quad \text{but} \quad \mathbf{p}[\mathtt{w}] \between \mathbf{s}[\mathtt{w}]$
- $\boldsymbol{\gamma}[\mathtt{w}] \subsetneq \mathbf{p}[\mathtt{w}]$
- $(\mathbf{s}[\mathtt{w}] \cap \mathbf{t}[\mathtt{w}]) \between \mathbf{p}[\mathtt{w}]$
- $\boldsymbol{\gamma}[\mathtt{w}] \between \mathbf{s}[\mathtt{w}], \quad \mathbf{o}[\mathtt{w}] \between \mathbf{s}[\mathtt{w}], \quad \mathbf{t}[\mathtt{w}] \between \mathbf{s}[\mathtt{w}],$

This brief outline makes it apparent that it would be interesting to shed light on a part of $\mathbb{D}[\mathtt{w}]$ that seems totally unexplored, as far as we know, i.e., the complement of $\mathbf{s}[\mathtt{w}] \cup \mathbf{p}[\mathtt{w}]$ in $\mathbb{D}[\mathtt{w}]$, that will be denoted by $\mathbf{r}[\mathtt{w}]$ (where the letter r stands for *residual*). In particular, it would be interesting to understand how the families of approach regions whose values lie in $\mathbf{r}[\mathtt{w}]$ for each $\mathtt{w} \in \partial\,\mathbb{D}$ behave in theorems of Littlewood type. We hope to be able to return to this issue on a future occasion.

The goal of this work is to advance our understanding of the *qualitative* boundary behavior of functions in $\mathsf{H}^\infty(\mathbb{D})$, i.e., in the sense of the almost everywhere convergence *along certain approach regions.* The more general viewpoint, where one considers the boundary behavior along *filters*, has not be studied here. Other points of view, such as the purely *pointwise* boundary behavior or the *quantitative* boundary behavior, will not be addressed here; see [13] for background and references.

1.4 Organization of the paper

Section 2 contains further results and additional terminology that form the necessary background of our contribution. Section 3 contains the proof of Lemma 1.1 and Proposition 1.2. Section 4 contains the proof of our main result, Theorem 1.3.

2 Background

The meaning of the new contribution presented in this work may be appreciated in full if we first give a brief outline of the results that form its background, starting from the fundamental theorem of Pierre Fatou (1906). In order to do so, we need some preliminary definitions and notation.

2.1 The function $\boldsymbol{\tau}$, nontangential approach regions, and Fatou points

Observe that the description, given in (1.13), of the class $\mathbf{t}[\mathtt{w}]$ of approach regions which end tangentially at the given boundary point is based on the quantity $(1-|\mathsf{z}|)/(|\mathtt{w}-\mathsf{z}|)$, which, given $\mathsf{z}\in\mathbb{D}$ and $\mathtt{w}\in\partial\mathbb{D}$, measures how close is z to the boundary *compared* to its distance from $\mathtt{w}$: It is the *normalized* distance to the boundary. Since this same quantity will appear again in another important definition, it is useful to describe it formally as a function $\tau:\partial\mathbb{D}\times\mathbb{D}\to(0,1]$

$$\tau(\mathtt{w},\mathsf{z}) \overset{\text{def}}{=\!=} (1-|\mathsf{z}|)/(|\mathtt{w}-\mathsf{z}|) \quad (\mathtt{w}\in\partial\mathbb{D},\mathsf{z}\in\mathbb{D}).$$

The following approach regions appeared in the work of Otto Stolz [45] in 1875.

($\boldsymbol{\gamma}[\mathtt{w}]$) We say that $\mathsf{A}\in\mathbb{D}[\mathtt{w}]$ ends **nontangentially** at $\mathtt{w}\in\partial\mathbb{D}$ if

$$0<\sup_{r>0}\inf_{\mathsf{z}\in\mathsf{A}(\mathtt{w},r)}\tau(\mathtt{w},\mathsf{z}) \tag{2.1}$$

and we set $\boldsymbol{\gamma}[\mathtt{w}] \overset{\text{def}}{=\!=} \{\mathsf{A}\in\mathbb{D}[\mathtt{w}] : \mathsf{A} \text{ ends } \textit{nontangentially} \text{ at } \mathtt{w}\}$.

Example 2.1. If $\mathtt{w}\in\partial\mathbb{D}$ and $b\geq 1$ we set

$$\Gamma_b(\mathtt{w}) \overset{\text{def}}{=\!=} \{\mathsf{z}\in\mathbb{D} : \tau(\mathtt{w},\mathsf{z})>1/(1+b)\} \tag{2.2}$$

and $\Gamma_0(\mathtt{w}) \overset{\text{def}}{=\!=} \{r\,\mathtt{w} : 0\leq r<1\}$. Then $\Gamma_b(\mathtt{w})\in\boldsymbol{\gamma}[\mathtt{w}]$.

Remark 2.2. Observe that if $\mathsf{A}\in\boldsymbol{\gamma}[\mathtt{w}]$ then there exists $\mathsf{A}'\in\mathbb{D}[\mathtt{w}]$ such that $\mathsf{A}\sim_{\mathtt{w}}\mathsf{A}'$ and $n\in\mathbb{N}$ such that $\mathsf{A}'\subset\Gamma_n(\mathtt{w})$.

If $\mathsf{f} \in [\![\mathbb{D} \nrightarrow \mathbb{C}]\!]$ then $\mathtt{w} \in \partial\mathbb{D}$ is called a **Fatou point** of f if $\lim\limits_{\mathsf{A} \nrightarrow \mathtt{w}} \mathsf{f}$ exists and is finite for each $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$, and we set

$$\mathsf{f}_\flat(\mathtt{w}) \stackrel{\text{def}}{=\!=} \lim_{\mathsf{A} \nrightarrow \mathtt{w}} \mathsf{f} \tag{2.3}$$

The following result says that the definition in (2.3) is well-posed.

Lemma 2.3 *If* $\mathtt{w} \in \partial\mathbb{D}$, $\mathsf{f} \in [\![\mathbb{D} \nrightarrow \mathbb{C}]\!]$, *and* $\lim\limits_{\mathsf{A} \nrightarrow \mathtt{w}} \mathsf{f}$ *exists and is finite for each* $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$, *then its value does not depend on the choice of* $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$.

Proof. It suffices to observe that if $\mathsf{A}_i \in \boldsymbol{\gamma}[\mathtt{w}], i = 1, 2$, then $\mathsf{A}_1 \cup \mathsf{A}_2 \in \boldsymbol{\gamma}[\mathtt{w}]$. □

2.2 Very oscillatory approach regions

The following approach regions appeared in 1979 in a seminal work of W. Rudin [39], as we will see momentarily.

($\mathbf{o}[\mathtt{w}]$) We say that $\mathsf{A} \in \mathcal{P}(\mathbb{D})$ is **very oscillatory** at $\mathtt{w} \in \partial\mathbb{D}$ if

$$0 = \sup_{r>0} \inf_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w}, \mathsf{z}) < \inf_{r>0} \sup_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w}, \mathsf{z}) \tag{2.4}$$

and we set $\mathbf{o}[\mathtt{w}] \stackrel{\text{def}}{=\!=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A} \text{ is very oscillatory at } \mathtt{w}\}$.

2.3 A partition of $\mathbb{D}[\mathtt{w}]$ related to tangency conditions

The classification of approach regions (ending at a given boundary point $\mathtt{w}$) in tangential, nontangential, and very oscillatory, can be clarified by the following notions. If $\mathtt{w} \in \partial\mathbb{D}$, $\mathsf{A} \in \mathbb{D}[\mathtt{w}]$, and $r > 0$, consider the functions

$$\boldsymbol{\tau}_* : \mathbb{D}[\mathtt{w}] \to [\![(0,1) \nrightarrow [0,1]]\!] \text{ and } \boldsymbol{\tau}^* : \mathbb{D}[\mathtt{w}] \to [\![(0,1) \nrightarrow [0,1]]\!] \tag{2.5}$$

that map $\mathsf{A} \in \mathbb{D}[\mathtt{w}]$ to $\boldsymbol{\tau}_*\mathsf{A}, \boldsymbol{\tau}^*\mathsf{A} \in [\![(0,1) \nrightarrow [0,1]]\!]$ where, for $r > 0$,

$$\boldsymbol{\tau}_*\mathsf{A}(r) \stackrel{\text{def}}{=\!=} \inf_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w}, \mathsf{z}) \text{ and } \boldsymbol{\tau}^*\mathsf{A}(r) \stackrel{\text{def}}{=\!=} \sup_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w}, \mathsf{z}). \tag{2.6}$$

Observe that

$$0 \le \boldsymbol{\tau}_*\mathsf{A}(r) \le \boldsymbol{\tau}^*\mathsf{A}(r) \le 1, \ \forall r > 0 \tag{2.7}$$

and, as functions of $r > 0$

$$\text{the function } \boldsymbol{\tau}_*\mathsf{A} \text{ is nonincreasing;} \quad \text{the function } \boldsymbol{\tau}^*\mathsf{A} \text{ is nondecreasing} \tag{2.8}$$

It follows that the values

$$\mathsf{A}_\natural \stackrel{\text{def}}{=\!=} \lim_{r \downarrow 0} \boldsymbol{\tau}_*\mathsf{A}(r) = \sup_{r>0} \boldsymbol{\tau}_*\mathsf{A}(r) \text{ and } \mathsf{A}^\natural \stackrel{\text{def}}{=\!=} \lim_{r \downarrow 0} \boldsymbol{\tau}^*\mathsf{A}(r) = \inf_{r>0} \boldsymbol{\tau}^*\mathsf{A}(r) \tag{2.9}$$

are well-defined, and $0 \le \mathsf{A}_\natural \le \mathsf{A}^\natural \le 1$.

Lemma 2.4 *If* $\mathtt{w} \in \partial\mathbb{D}$ *then*

$$\mathbb{D}[\mathtt{w}] = \boldsymbol{\gamma}[\mathtt{w}] \ \dot\cup\ \mathbf{o}[\mathtt{w}] \ \dot\cup\ \mathbf{t}[\mathtt{w}] \tag{2.10}$$

(where the symbol $\dot\cup$ *means that the sets are disjoint).*

Proof. If $\mathsf{A}_\natural > 0$ then $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$. If $\mathsf{A}_\natural = 0$ (i.e., if $\boldsymbol{\tau}_*\mathsf{A}(r) \equiv 0\ \forall r > 0$), then either $\mathsf{A}^\natural = 0$ or $\mathsf{A}^\natural > 0$. In the first case, $\mathsf{A} \in \mathbf{t}[\mathtt{w}]$, in the second case, $\mathsf{A} \in \mathbf{o}[\mathtt{w}]$. $\square$

The approach regions in $\mathbf{o}[\mathtt{w}]$ play a central role in Rudin's results (1979), to be presented momentarily, that played an important role in advancing our understanding of the subject. As observed in [39], and as conveyed, in some sense, in the proof of Lemma 2.4, the *tangential* approach are only a "small" portion of those that are *not nontangential*. This observation is made precise in the following result.

Proposition 2.5 *The set* $\mathbb{D}[\mathtt{w}]$ *is endowed with a topology for which the set* $\mathbf{o}[\mathtt{w}]$ *is open and the sets* $\boldsymbol{\gamma}[\mathtt{w}]$ *and* $\mathbf{t}[\mathtt{w}]$ *are closed with empty interior.*

Proof. The proof will be given in Section 2.3.2. $\square$

Firstly, we have to describe the topology on $\mathbb{D}[\mathtt{w}]$.

2.3.1 The rough topology on $\mathbb{D}[\mathtt{w}]$

We now define a topology on $\mathbb{D}[\mathtt{w}]$ which is, informally speaking, rather *rough*, and will thus be called the *rough topology* on $\mathbb{D}[\mathtt{w}]$. It is defined by selecting an appropriate subbase $\mathcal{C}$; cf. [38, pp. 176-177]. If $\mathsf{A} \in \mathbb{D}[\mathtt{w}]$ and $r > 0$ we set $\mathsf{C}(\mathsf{A}, r) \overset{\text{def}}{=} \{\mathsf{V} : \mathsf{V} \in \mathbb{D}[\mathtt{w}] \text{ and } \mathsf{V} \supset \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r)\}$ and we define the collection $\mathcal{C} \overset{\text{def}}{=} \{\mathsf{C}(\mathsf{A}, r) : \mathsf{A} \in \mathbb{D}[\mathtt{w}], r > 0\}$. The **rough topology** on $\mathbb{D}[\mathtt{w}]$ is defined as the weakest topology on $\mathbb{D}[\mathtt{w}]$ that contains $\mathcal{C}$.

Lemma 2.6 *The function* $\mathbb{D}[\mathtt{w}] \to [0,1], \mathsf{A} \mapsto \mathsf{A}^\natural$ *is lower semicontinuous; the function* $\mathbb{D}[\mathtt{w}] \to [0,1], \mathsf{A} \mapsto \mathsf{A}_\natural$ *is upper semicontinuous.*

Proof. Let $c \in [0,1)$. We claim that the set $O \overset{\text{def}}{=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A}^\natural > c\}$ is open. Let $\mathsf{A} \in O$. Then $\mathsf{A} \supset \mathsf{A} \cap \mathrm{B}(\mathtt{w}, 1/2)$, hence $\mathsf{A} \in \mathsf{C}(\mathsf{A}, 1/2)$. We claim that $\mathsf{C}(\mathsf{A}, 1/2) \subset O$. Let $\mathsf{V} \in \mathsf{C}(\mathsf{A}, 1/2)$. Then $\mathsf{V} \supset \mathsf{A} \cap \mathrm{B}(\mathtt{w}, 1/2)$. It follows that if $0 < r < 1/2$ then $\mathsf{V} \cap \mathrm{B}(\mathtt{w}, r) \supset \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r)$, hence

$$\sup_{\mathsf{z} \in \mathsf{V} \cap \mathrm{B}(\mathtt{w}, r)} \tau(\mathtt{w}, \mathsf{z}) \geq \sup_{\mathsf{z} \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r)} \tau(\mathtt{w}, \mathsf{z})$$

i.e., $\boldsymbol{\tau}^*\mathsf{V}(r) \geq \boldsymbol{\tau}^*\mathsf{A}(r)$ if $0 < r < 1/2$, hence $\mathsf{V}^\natural \geq \mathsf{A}^\natural > c$, thus $\mathsf{V} \in O$. We have proved that $\mathsf{A} \in \mathsf{C}(\mathsf{A}, 1/2) \subset O$, and since $\mathsf{C}(\mathsf{A}, 1/2)$ is open, we have proved that for each $\mathsf{A} \in O$ there exists an open set U in $\mathbb{D}[\mathtt{w}]$ such that $\mathsf{A} \in U \subset O$. Hence O is open. The proof that $\mathsf{A} \mapsto \mathsf{A}_\natural$ is upper semicontinuous is symmetrical and will be omitted. $\square$

Lemma 2.7 *Each of the sets* $\{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : A_\natural = 0\}$ *and* $\{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : A^\natural = 1\}$ *is open.*

Proof. Let $O \overset{\text{def}}{=\!=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : A_\natural = 0\}$ and $\mathsf{A} \in O$. Then $\boldsymbol{\tau}_*\mathsf{A}(r) \equiv 0$ for all $r > 0$. We claim that $\mathsf{C}(\mathtt{w}, 1/2) \subset O$. Indeed, if $\mathsf{V} \in \mathsf{C}(\mathsf{A}, 1/2)$ and $0 < r < 1/2$ then $\mathsf{V} \cap \mathrm{B}(\mathtt{w}, r) \supset \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r)$, since $\mathsf{V} \supset \mathsf{A} \cap \mathrm{B}(\mathtt{w}, 1/2)$. Hence if $0 < r < 1/2$

$$\inf_{\mathsf{z} \in \mathsf{V} \cap \mathrm{B}(\mathtt{w}, r)} \tau(\mathtt{w}, \mathsf{z}) \le \inf_{\mathsf{z} \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r)} \tau(\mathtt{w}, \mathsf{z})$$

i.e., $\boldsymbol{\tau}_*\mathsf{V}(r) \le \boldsymbol{\tau}_*\mathsf{A}(r)$ and since $\boldsymbol{\tau}_*\mathsf{A}(r) \equiv 0$ for each $r > 0$, it follows that $\boldsymbol{\tau}_*\mathsf{V}(r) \equiv 0$, i.e., $\mathsf{V}_\natural = 0$, hence $\mathsf{V} \in O$. Since $\mathsf{A} \in \mathsf{C}(\mathsf{A}, 1/2) \subset O$ and $\mathsf{C}(\mathsf{A}, 1/2)$ is open, it follows that O is open. The proof that $\{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : A^\natural = 1\}$ is open is symmetrical and will be omitted. □

2.3.2 Proof of Proposition 2.5

Proof that the set $\mathbf{o}[\mathtt{w}]$ is open in $\mathbb{D}[\mathtt{w}]$. Since

$$\mathbf{o}[\mathtt{w}] = \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A}_\natural = 0\} \cap \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A}^\natural > 0\}$$

the conclusion follows from Lemma 2.6 and Lemma 2.7. Q.E.D.

Proof that the set $\boldsymbol{\gamma}[\mathtt{w}]$ is closed in $\mathbb{D}[\mathtt{w}]$. The conclusion follows from Lemma 2.7, since $\mathbb{D}[\mathtt{w}] \setminus \boldsymbol{\gamma}[\mathtt{w}] = \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A}_\natural = 0\}$. Q.E.D.

Proof that the set $\mathbf{t}[\mathtt{w}]$ is closed in $\mathbb{D}[\mathtt{w}]$. The conclusion follows from Lemma 2.6, since $\mathbb{D}[\mathtt{w}] \setminus \mathbf{t}[\mathtt{w}] = \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A}^\natural > 0\}$.
Q.E.D.

Proof that the set $\boldsymbol{\gamma}[\mathtt{w}]$ has empty interior. We may assume, without loss of generality, that $\mathtt{w} = 1$. Let $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$ and assume that there exists an open set $U \subset \mathbb{D}[\mathtt{w}]$ such that $\mathsf{A} \in U \subset \boldsymbol{\gamma}[\mathtt{w}]$. Since $\mathcal{C}$ is a subbase of the topology of $\mathbb{D}[\mathtt{w}]$, it follows that U is the intersection of finitely many sets in the collection $\mathcal{C}$. To fix ideas, let us assume that U is the intersection of precisely two sets in the collection $\mathcal{C}$, i.e., $U = \mathsf{C}(\mathsf{V}_1, r_1) \cap \mathsf{C}(\mathsf{V}_2, r_2)$, where $\mathsf{V}_k \in \mathbb{D}[\mathtt{w}]$ and $r_k > 0$, $k = 1, 2$. Hence

[$\sharp$] $\quad \mathsf{A} \in \mathsf{C}(\mathsf{V}_1, r_1) \cap \mathsf{C}(\mathsf{V}_2, r_2) \subset \boldsymbol{\gamma}[\mathtt{w}]$

Let $r \overset{\text{def}}{=\!=} \min\{r_1, r_2\}$, let $\mathsf{W} \overset{\text{def}}{=\!=} \{(1 - n^{-2})e^{i/n} : n \in \mathbb{N}, n \ge 2\}$, and define

$$\mathsf{V}_3 \overset{\text{def}}{=\!=} [\mathsf{V}_1 \cap \mathrm{B}(\mathtt{w}, r_1)] \cup [\mathsf{V}_2 \cap \mathrm{B}(\mathtt{w}, r_2)] \cup [\mathsf{W} \cap \mathrm{B}(\mathtt{w}, r)]$$

We claim that

[$\sharp\sharp$] $\quad \mathsf{V}_3 \in \mathsf{C}(\mathsf{V}_1, r_1) \cap \mathsf{C}(\mathsf{V}_2, r_2) \setminus \boldsymbol{\gamma}[\mathtt{w}]$

in contradiction with **[↯]**. In order to prove **[↯↯]**, observe that

$$\mathsf{V}_3 \supset \mathsf{V}_1 \cap \mathrm{B}(\mathtt{w}, r_1) \text{ and } \mathsf{V}_3 \supset \mathsf{V}_2 \cap \mathrm{B}(\mathtt{w}, r_2)$$

and hence $\mathsf{V}_3 \in \mathsf{C}(\mathsf{V}_1, r_1) \cap \mathsf{C}(\mathsf{V}_2, r_2)$. Since $\mathsf{V}_3 \supset \mathsf{W} \cap \mathrm{B}(\mathtt{w}, r)$ and $\mathsf{W} \in \mathbf{t}[\mathtt{w}]$, it follows that $\mathsf{V}_3 \notin \boldsymbol{\gamma}[\mathtt{w}]$. Hence **[↯↯]** is proved, in contradiction with **[↯]**. The case where U is the intersection of more than two sets of the collection $\mathcal{C}$ is similar and will be omitted. Q.E.D.

The set $\mathbf{t}[\mathtt{w}]$ has empty interior. The proof is similar to the proof of the fact that the set $\boldsymbol{\gamma}[\mathtt{w}]$ has empty interior, and will be omitted. Q.E.D.

2.4 Families of approach regions defined by pointwise conditions

Our understanding of these matters is greatly enhanced if the theoretical results are formulated in terms of the collection $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]$ and some of its subsets. The main idea is that we are first of all interested in those elements $\mathbf{A} \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]$ such that $\mathbf{A}(\mathtt{w}) \in \mathbb{D}[\mathtt{w}]$ for each $\mathtt{w} \in \partial\mathbb{D}$. We set

$$[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} \stackrel{\text{def}}{=\!=} \{\, \mathbf{A} \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!] : \mathbf{A}(\mathtt{w}) \in \mathbb{D}[\mathtt{w}],\ \forall\, \mathtt{w} \in \partial\mathbb{D} \,\} \tag{2.11}$$

The elements of $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ are called **families of approach regions in** $\mathbb{D}$.

The symbol ℓ in $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ is used to remind us that a family of approach regions is used to evaluate limiting values, whenever they exist. The following notable subset of $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ is defined by a "pointwise" condition.

$$[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\gamma}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} : \mathbf{A}(\mathtt{w}) \in \boldsymbol{\gamma}[\mathtt{w}] \quad \forall\, \mathtt{w} \in \partial\mathbb{D}\}; \tag{2.12}$$

Elements of $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\gamma}}$ are families of nontangential approach regions.

Example 2.8. $\Gamma_b \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\gamma}}$ (where $\Gamma_b(\mathtt{w})$ is seen as a function of $\mathtt{w}$).

Some of the other subsets of $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ that are of interest in the subject are also defined in terms of **pointwise** conditions and will also be denoted by adding one or more subscripts.

2.4.1 Subscript notation for pointwise conditions

The following **subscript notation** is a generalization of the notation employed in (2.12) and will be systematically adopted. If

$$\{\mathbf{x}, \mathbf{z}\} \subset \{\gamma, \mathbf{t}, \mathbf{o}, \mathbf{c}, \mathbf{a}, \mathbf{p}, \mathbf{s}\} \tag{2.13}$$

and $\mathrm{Y} \subset [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ then

$$\mathrm{Y}_{\mathbf{x}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in \mathrm{Y} : \mathbf{A}(\mathtt{w}) \in \mathbf{x}(\mathtt{w}) \quad \forall\, \mathtt{w} \in \partial\mathbb{D}\} \tag{2.14}$$

and

$$\mathrm{Y}_{\mathbf{xz}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in \mathrm{Y} : \mathbf{A}(\mathtt{w}) \in \mathbf{x}(\mathtt{w}) \cap \mathbf{z}(\mathtt{w}) \quad \forall\, \mathtt{w} \in \partial\,\mathbb{D}\} \tag{2.15}$$

If $\mathrm{Y} = [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ **the subscript $\boldsymbol{\ell}$ is dropped**, since it is redundant. Hence if $\mathbf{x} = \boldsymbol{\gamma}$ then (2.14) yields (2.12), and if $\{\mathbf{x}, \mathbf{z}\} = \{\mathbf{t}, \mathbf{c}\}$ then (2.15) yields

$$[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathbf{tc}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} : \mathbf{A}(\mathtt{w}) \in \mathbf{t}[\mathtt{w}] \cap \mathbf{c}[\mathtt{w}] \quad \forall\, \mathtt{w} \in \partial\,\mathbb{D}\}$$

The class $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathbf{a}}$ appeared in 2006 in [14] and enabled us a deeper understanding of Littlewood's result [32], as we shall see.

2.4.2 A note on pointwise conditions

The constructions in Section 2.4.1 can be subsumed under the following general procedure, with which our notation is consistent. A subset $\mathbf{b}$ of $\partial\,\mathbb{D} \times \mathcal{P}(\mathbb{D})$ is called a **pointwise condition**. If $X \subset [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]$ and $\mathbf{b}$ is a pointwise condition, the set

$$X_{\mathbf{b}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in X : (\mathtt{w}, \mathbf{A}(\mathtt{w})) \in \mathbf{b} \ \forall\, \mathtt{w} \in \partial\,\mathbb{D}\} \tag{2.16}$$

is called **the subset of X defined pointwise by $\mathbf{b}$**. This construction may be readily adapted when more than one pointwise condition is involved. We illustrate this fact in the case of two pointwise conditions. Hence if $X \subset [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]$ and $\mathbf{b}$ and $\mathbf{d}$ are two pointwise conditions, then

$$X_{\mathbf{bd}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in X : (\mathtt{w}, \mathbf{A}(\mathtt{w})) \in \mathbf{b} \text{ and } (\mathtt{w}, \mathbf{A}(\mathtt{w})) \in \mathbf{d},\ \forall\, \mathtt{w} \in \partial\,\mathbb{D}\}$$

is **the subset of X defined pointwise by $\mathbf{b}$ and $\mathbf{d}$**. Observe that

$$\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D})) \ \text{ and } [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}}$$

(introduced in Section 2.13.1 and in Section 2.14) cannot be obtained by the method of imposing pointwise conditions.

2.5 The shadow along a family of approach regions

The following construction occurs implicitly in the notion of regularity that appears in our main result, Theorem 1.3; cf. [14].

If $\mathbf{A} \in [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$, the **A-shadow** (or **A-projection**) of $\mathsf{U} \in \mathcal{P}(\mathbb{D})$ is the set

$$\mathbf{A}^*(\mathsf{U}) \stackrel{\text{def}}{=\!=} \{\mathtt{w} \in \partial\,\mathbb{D} : \mathbf{A}(\mathtt{w}) \cap \mathsf{U} \neq \emptyset\}. \tag{2.17}$$

Observe that the map $\mathbf{A}^* : \mathcal{P}(\mathbb{D}) \to \mathcal{P}(\partial\,\mathbb{D})$ is monotone increasing, i.e.,

$$\text{if } \mathsf{U} \subset \mathsf{U}' \text{ then } \ \mathbf{A}^*(\mathsf{U}) \subset \mathbf{A}^*(\mathsf{U}'). \tag{2.18}$$

Lemma 2.9 *If $b \in \mathbb{N}$ and $b \geq 1$, then for each $\mathsf{z} \in \mathbb{D}$ the set $\Gamma_b^*(\mathsf{z})$ is either equal to $\partial\,\mathbb{D}$ or it is an arc in $\partial\,\mathbb{D}$.*

Proof. Observe that $\mathtt{x} \in \Gamma_b^*(\mathsf{z})$ if and only if $\mathtt{x} \in \partial\,\mathbb{D}$ and $\mathsf{z} \in \Gamma_b(\mathtt{x})$. The second condition means that $\frac{1}{1+b} < \tau(\mathtt{x}, \mathsf{z})$, i.e., $|\,\mathsf{z} - \mathtt{x}\,| < (1+b)(1 - |\,\mathsf{z}\,|)$, hence

(s) $\Gamma_b^*(\mathsf{z}) = \partial\,\mathbb{D} \cap \mathrm{B}(\mathsf{z}, (1+b)(1 - |\,\mathsf{z}\,|))$

Observe that (s) shows that $\Gamma_b^*(\mathsf{z})$ is the intersection of $\partial\,\mathbb{D}$ with an open ball in $\mathbb{C}$, the conclusion will follow once we show that $\Gamma_b^*(\mathsf{z}) \neq \emptyset$. If $\mathsf{z} = 0$ then $\Gamma_b^*(\mathsf{z}) = \partial\,\mathbb{D}$, and if $\mathsf{z} \neq 0$ then $\frac{\mathsf{z}}{|\,\mathsf{z}\,|} \in \Gamma_b^*(\mathsf{z})$. □

2.6 Sets adjacent to a point

Let $\mathtt{w} \in \partial\,\mathbb{D}$. Recall from (1.4) that $\mathtt{Arcs}(\mathtt{w})$ is the collection of all arcs with $\mathtt{w}$ as an endpoint. If $\mathtt{S} \in \mathcal{P}(\partial\,\mathbb{D})$, we say that $\mathtt{S}$ **is adjacent to** $\mathtt{w}$, and write

$$\boxed{\mathtt{S} \rightsquigarrow \mathtt{w}}$$

if there exists an arc $\mathtt{J} \in \mathtt{Arcs}(\mathtt{w})$ such that $\mathtt{J} \subset \mathtt{S}$. Then $\mathtt{S}$ is *adjacent to the right (left) of* $\mathtt{w}$ if $\mathtt{w}$ is a left-endpoint (right-endpoint, resp.) of $\mathtt{J}$. Observe that

$$\text{If } \mathtt{S}' \rightsquigarrow \mathtt{w} \text{ and } \mathtt{S}' \subset \mathtt{S} \text{ then } \mathtt{S} \rightsquigarrow \mathtt{w} \tag{2.19}$$

Lemma 2.10 *If* $\mathtt{S}$ *contains an arc* $\mathtt{U}$ *and* $\mathtt{w} \in \mathtt{U}$ *then* $\mathtt{S} \rightsquigarrow \mathtt{w}$.

Proof. Since $\mathtt{U}$ contains contains the arc $\{e^{i\theta} : -\alpha < \theta < \alpha\}$ for some $\alpha \in (0, \pi/2)$, it contains the arc $\mathtt{S}' \overset{\text{def}}{=\!=} \{e^{i\theta} : 0 < \theta < \alpha\}$, of which $\mathtt{w}$ is a left-endpoint. Now apply (2.19). □

2.7 A new look at projectively adjacent approach regions

We are now ready to express in a more geometrical flavor the notion described in (1.17).

Lemma 2.11 *If* $\mathsf{A} \subset \mathbb{D}$ *and* $\mathtt{w} \in \partial\,\mathbb{D}$*, the following conditions are equivalent.*

(1) A *is projectively adjacent to* $\mathtt{w}$.

(2) *There exists* $b \in \mathbb{N}$ *such that, for each* $r > 0$,

$$\Gamma_b^*(\mathsf{A}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w} \tag{2.20}$$

(3) $\exists\, b \in \mathbb{N}$ *and* $\exists\, r_0 > 0$ *such that, for each* $r \in (0, r_0]$, (2.20) *holds.*

Proof. Consider the following statements: (i) $\mathsf{z} \in \Gamma_b(\mathtt{x})$; (ii) $\tau(\mathtt{x}, \mathsf{z}) > 1/(1+b)$; (iii) $|\,\mathtt{x} - \mathsf{z}\,| < (1+b)(1 - |\,\mathsf{z}\,|)$. Since these statements are equivalent, to say that $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$ means that $\exists\, b \in \mathbb{N}$ such that for each $r > 0$ there exists $\mathtt{J} \in \mathtt{Arcs}(\mathtt{w})$ such that $\forall\, \mathtt{x} \in \mathtt{J}$, $\exists\, \mathsf{z} \in \mathsf{A}(\mathtt{w}, r) \cap \Gamma_b(\mathtt{x})$, i.e., $\mathsf{A}(\mathtt{w}, r) \cap \Gamma_b(\mathtt{x}) \neq \emptyset$, i.e., $u \in \Gamma_b^*(\mathsf{A}(\mathtt{w}, r))$, hence $\mathtt{J} \subset \Gamma_b^*(\mathsf{A}(\mathtt{w}, r))$, thus $\Gamma_b^*(\mathsf{A}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$. Hence the equivalence between **(1)** and **(2)** has been proved.

Assume that **(3)** holds. If $r \geq r_0$ then $\mathsf{A}(\mathtt{w}, r) \supset \mathsf{A}(\mathtt{w}, r_0)$ and hence from (2.18) it follows that $\Gamma_b^*(\mathsf{A}(\mathtt{w}, r)) \supset \Gamma_b^*(\mathsf{A}(\mathtt{w}, r_0))$, and since $\Gamma_b^*(\mathsf{A}(\mathtt{w}, r_0)) \rightsquigarrow \mathtt{w}$, (2.19) implies that (2.20) holds, hence (2.20) holds for each $r > 0$, and **(2)** holds.

If **(2)** holds then **(3)** holds for (say) $r_0 = 1/2$. □

If $b \in \mathbb{N}$, $\mathsf{A} \in \mathcal{P}(\mathbb{D})$, and $\mathtt{w} \in \partial\,\mathbb{D}$, we say that A is ***b*-projectively adjacent to** $\mathtt{w}$ if

$$\forall r > 0,\ \Gamma_b^*(\mathsf{A}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$$

and we set

$$\mathbf{p}_b[\mathtt{w}] \stackrel{\text{def}}{=\!=} \{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \mathsf{A}\,\text{is } b\text{-projectively adjacent to } \mathtt{w}\}.$$

Observe that $\mathbf{p}[\mathtt{w}] = \bigcup\limits_{b \in \mathbb{N}} \mathbf{p}_b[\mathtt{w}]$ and $\mathbf{p}_b[\mathtt{w}] \subset \mathbf{p}_{b+1}[\mathtt{w}]$.

Lemma 2.12 *If* $\mathtt{w} \in \partial\,\mathbb{D}$ *then* $\boldsymbol{\gamma}[\mathtt{w}] \subset \mathbf{p}[\mathtt{w}]$.

Proof. Let $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$. Then $\mathsf{A}_\natural > 0$, i.e., $0 < \sup\limits_{r>0} \boldsymbol{\tau}_*\mathsf{A}(r)$. Hence $\exists\ r_0 > 0$ such that $0 < \boldsymbol{\tau}_*\mathsf{A}(r_0)$. Let $b_0 \in \mathbb{N}$ such that $\frac{1}{1+b_0} < \boldsymbol{\tau}_*\mathsf{A}(r_0)$. Then for each $r \in (0, r_0)$, $\frac{1}{1+b_0} < \boldsymbol{\tau}_*\mathsf{A}(r)$, hence if $0 < r < r_0$ and $\mathtt{z} \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r)$ then $\frac{1}{1+b_0} < \tau(\mathtt{w}, \mathtt{z})$, *i.e.*, $\mathtt{z} \in \Gamma_{b_0}(\mathtt{w})$, thus

$$\text{if } 0 < r < r_0 \text{ and } \mathtt{z} \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r) \text{ then } \mathtt{w} \in \Gamma_{b_0}^*(\mathtt{z})$$

Recall from Lemma 2.9 and Lemma 2.10 that $\Gamma_{b_0}^*(\mathtt{z})$ is an arc in $\partial\,\mathbb{D}$ that contains $\mathtt{w}$, and hence $\Gamma_{b_0}^*(\mathtt{z}) \rightsquigarrow \mathtt{w}$. Since $\Gamma_{b_0}^*(\mathtt{z}) \subset \Gamma_{b_0}^*(\mathsf{A} \cap \mathrm{B}(\mathtt{w}, r))$, it follows that $\Gamma_{b_0}^*(\mathsf{A} \cap \mathrm{B}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$. Lemma (2.11) implies that $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$. □

2.8 A variant of the polar coordinates

Recall that $z \in \mathbb{C}$ has **polar coordinates** (r, θ) if $z = r\exp(i\theta)$, where $r \geq 0$, $\theta \in \mathbb{R}$. Points in a neighborhood of $\partial\,\mathbb{D}$ may be conveniently parameterized by a natural variant of the polar coordinates: We say that z has **boundary coordinates** (θ, δ) if $z = (1-\delta)e^{i\theta}$. Hence δ is the *signed distance to the boundary*, since $\delta < 0$ if z lies outside of $\mathbb{D}$. For example, if O is the neighborhood of $\mathtt{w} = 1$ given by

$$O \stackrel{\text{def}}{=\!=} \{(1-\delta)e^{i\theta} : |\delta| < \frac{1}{4}, |\theta| < \frac{1}{4}\} \tag{2.21}$$

and we denote by S the square $(-\frac{1}{4}, \frac{1}{4})^2 \stackrel{\text{def}}{=\!=} \{(\theta, \delta) : |\theta| < \frac{1}{4}, |\delta| < \frac{1}{4}\}$, then for the function $p \in [\![S \to O]\!]$ defined by $p(\theta, \delta) \stackrel{\text{def}}{=\!=} (1-\delta)e^{i\theta}$ we have the following approximation, which is rough but useful.

Lemma 2.13 *If* $(\theta_k, \delta_k) \in S$, $k = 1, 2$, *then*

$$\frac{3}{64}\left[(\theta_1-\theta_2)^2 + (\delta_1-\delta_2)^2\right] \leq |p(\theta_1, \delta_1) - p(\theta_2, \delta_2)|^2 \\ \leq \frac{125}{64}\left[(\theta_1-\theta_2)^2 + (\delta_1-\delta_2)^2\right] \tag{2.22}$$

Proof. A laborious but essentially straightforward calculation based on Taylor's second order approximation of cos yields

$$\left|(1-\delta_1)e^{i\theta_1} - (1-\delta_2)e^{i\theta_2}\right|^2 = (\delta_1-\delta_2)^2 + (\theta_1-\theta_2)^2(1+\beta)$$

where (for some number ξ in the arc from 0 to $\theta_1 - \theta_2$)

$$\beta = -\delta_1 - \delta_2 + \delta_1\delta_2 - \frac{1}{3}(1-\delta_1)(1-\delta_2)(\theta_1-\theta_2)\sin(\xi)$$

and since $|\beta| < \frac{61}{64}$ for $(\theta_k, \delta_k) \in S$, (2.22) follows. $\square$

2.9 Partial functions.

Partial functions arise naturally in our subject, as we will see momentarily. A **partial function** f from a set B to a set L is a function whose values lie in L, defined on a (possibly empty) subset of B, called the **effective domain** of the partial function and denoted by $\mathrm{E_d}(f)$. Hence $\mathrm{E_d}(f) \subset B$. In order to avoid confusion, instead of the familiar arrow notation, the *sparrow tail notation*

$$B \rightarrowtail L$$

is used to denote a partial function, and the set of all *partial functions* from B to L is denoted by $\{\!\{B \nrightarrow L\}\!\}$. In particular, the function

$$\mathrm{E_d} : \{\!\{\,\partial\mathbb{D} \nrightarrow \mathbb{C}\,\}\!\} \to \mathcal{P}(\partial\mathbb{D})$$

maps each partial function $\partial\mathbb{D} \rightarrowtail \mathbb{C}$ to its effective domain.

2.9.1 Limiting values as partial functions.

Implicit in (1.10) is the definition of the partial function

$$\lim : \partial\mathbb{D} \times [\![\,\mathbb{D} \nrightarrow \mathbb{C}\,]\!] \times \mathcal{P}(\mathbb{D}) \rightarrowtail \mathbb{C} \tag{2.23}$$

whose effective domain is the set

$$\mathrm{E_d}(\lim) = \left\{ (\mathtt{w}, \mathsf{f}, \mathsf{A}) \in \partial\mathbb{D} \times [\![\,\mathbb{D} \nrightarrow \mathbb{C}\,]\!] \times \mathcal{P}(\mathbb{D}) : \lim_{\mathsf{A} \nrightarrow \mathtt{w}} \mathsf{f} \text{ is defined} \right\}$$

where $\lim(\mathtt{w}, \mathsf{f}, \mathsf{A}) \overset{\text{def}}{=\!=} \lim_{\mathsf{A} \nrightarrow \mathtt{w}} \mathsf{f}$ whenever this limiting value is defined. In particular, currying the partial function (2.23), for $\mathtt{w} \in \partial\mathbb{D}$ and $\mathsf{f} \in [\![\,\mathbb{D} \nrightarrow \mathbb{C}\,]\!]$, we obtain the partial function

$$\lim\nolimits_{\mathtt{w}} \mathsf{f} : \mathbb{D}[\mathtt{w}] \rightarrowtail \mathbb{C} \tag{2.24}$$

whose effective domain is the set $\{\mathsf{A} \in \mathbb{D}[\mathtt{w}] : \lim_{\mathsf{A} \nrightarrow \mathtt{w}} \mathsf{f}$ exists and is finite$\}$ and whose value at $\mathsf{A} \in \mathbb{D}[\mathtt{w}]$ is equal to the limiting value in (1.10), if it exists.

2.9.2 The convergence set

Recall from (2.11) that $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell}$ is the collection of all families of approach regions. If $\mathbf{A} \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell}$ and $\mathsf{f} \in [\![\, \mathbb{D} \nrightarrow \mathbb{C}]\!]$, the effective domain of the partial function

$$\mathtt{lim}_{\mathbf{A}}\, \mathsf{f} : \partial\mathbb{D} \;\rightarrowtail\; \mathbb{C} \tag{2.25}$$

is called the **convergence set of f along A** and is the set

$$\mathtt{conv}[\mathsf{f}, \mathbf{A}] \overset{\text{def}}{=\!=} \{\mathtt{w} \in \partial\mathbb{D} : (2.26) \text{ exists and is finite}\},$$

$$(\mathtt{lim}_{\mathbf{A}}\, \mathsf{f})(\mathtt{w}) \overset{\text{def}}{=\!=} \lim_{\mathbf{A}(\mathtt{w}) \nrightarrow \mathtt{w}} \mathsf{f} \tag{2.26}$$

As we will see, we seek conditions on $\mathbf{A} \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell}$ that insure, or exclude, that for certain functions $\mathsf{f} \in [\![\, \mathbb{D} \nrightarrow \mathbb{C}]\!]$ the set $\mathtt{conv}[\mathsf{f}, \mathbf{A}]$ has full measure. Observe that we have implicitly defined the function

$$\boxed{\mathtt{lim} : [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell} \times [\![\, \mathbb{D} \nrightarrow \mathbb{C}]\!] \to \{\!\!\{\, \partial\mathbb{D} \nrightarrow \mathbb{C} \}\!\!\}}$$

which maps $(\mathbf{A}, \mathsf{f}) \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell} \times [\![\, \mathbb{D} \nrightarrow \mathbb{C}]\!]$ to $\mathtt{lim}_{\mathbf{A}}\, \mathsf{f}$, defined in (2.25).

2.9.3 The divergence set

The following notion is also useful. If $(\mathbf{A}, \mathsf{f}) \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell} \times [\![\, \mathbb{D} \nrightarrow \mathbb{C}]\!]$, the set

$$\mathtt{DIV}[\mathsf{f}, \mathbf{A}] \overset{\text{def}}{=\!=} \{\mathtt{w} \in \partial\mathbb{D} : (2.26) \text{ does not exist}\} \tag{2.27}$$

is called the **divergence set of f along A**.

2.10 A natural partial order relation in $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell}$

The "pointwise" application of the inclusion relation between subsets of the unit disc yields a natural partial order relation in $[\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell}$, as follows. If $\mathbf{A}, \mathbf{W} \in [\![\, \partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\ell}$ and $\mathbf{W}(\mathtt{w}) \subset \mathbf{A}(\mathtt{w})$, $\forall \mathtt{w} \in \partial\mathbb{D}$, we write

$$\mathbf{W} \subset \mathbf{A}$$

(with a slight abuse of language) in much the same way as we write $f \leq g$, for real-valued functions f and g with the same domain D, if $f(x) \leq g(x)\ \forall x \in D$. Observe that families of "smaller" approach regions are more likely to allow convergence, since $\mathtt{DIV}_{(\cdot)}[\mathsf{f}]$ is increasing and $\mathtt{conv}_{(\cdot)}[\mathsf{f}]$ is decreasing:

$$\text{if } \mathbf{W} \subset \mathbf{A} \text{ then } \mathtt{conv}[\mathsf{f}, \mathbf{A}] \subset \mathtt{conv}[\mathsf{f}, \mathbf{W}] \text{ and } \mathtt{DIV}[\mathsf{f}, \mathbf{W}] \subset \mathtt{DIV}[\mathsf{f}, \mathbf{A}].$$

2.11 Germs of approach regions and the notion of stability

The relation defined in (1.2) is an equivalence relation that plays a natural role in the study of the limiting value in (1.10).

Lemma 2.14 *The relation $\sim_{\mathtt{w}}$ is an equivalence relation in $\mathbb{D}[\mathtt{w}]$.*

Proof. Reflexivity: If $\mathsf{A} \in \mathbb{D}[\mathtt{w}]$ then $\mathsf{A}(\mathtt{w}, 2) = \mathsf{A}(\mathtt{w}, 2)$, hence $\mathsf{A} \sim_{\mathtt{w}} \mathsf{A}$. *Symmetry:* If $\mathsf{A} \sim_{\mathtt{w}} \mathsf{V}$ then $\mathsf{A}(\mathtt{w}, r) = \mathsf{V}(\mathtt{w}, r)$ for some $r > 0$, hence $\mathsf{V}(\mathtt{w}, r) = \mathsf{A}(\mathtt{w}, r)$, thus $\mathsf{V} \sim_{\mathtt{w}} \mathsf{A}$. *Transitivity:* If $\mathsf{A} \sim_{\mathtt{w}} \mathsf{V}$ then $\exists\ r_1 > 0$ such that $\mathsf{A}(\mathtt{w}, r_1) = \mathsf{V}(\mathtt{w}, r_1)$, and hence (i) $\mathsf{A}(\mathtt{w}, r) = \mathsf{V}(\mathtt{w}, r)$ for each $r \in (0, r_1)$. If, moreover, $\mathsf{V} \sim_{\mathtt{w}} \mathsf{W}$, then $\exists\ r_2 > 0$ such that $\mathsf{V}(\mathtt{w}, r_2) = \mathsf{W}(\mathtt{w}, r_2)$, and, for the same reason, (ii) $\mathsf{V}(\mathtt{w}, r) = \mathsf{W}(\mathtt{w}, r)$ for each $r \in (0, r_2)$. Let $r_{\min} \stackrel{\text{def}}{=\!=} \min\{r_1, r_2\}$. Then (i) and (ii) together imply that $\mathsf{A}(\mathtt{w}, r_{\min}) = \mathsf{V}(\mathtt{w}, r_{\min}) = \mathsf{W}(\mathtt{w}, r_{\min})$ and hence $\mathsf{A}(\mathtt{w}, r_{\min}) = \mathsf{W}(\mathtt{w}, r_{\min})$, i.e., $\mathsf{A} \sim_{\mathtt{w}} \mathsf{W}$. $\square$

Remark 2.15. If $\mathsf{f} \in [\![\, \mathbb{D} \nrightarrow \mathbb{C}\,]\!]$ then the partial function (2.23) preserves this equivalence relation, in the sense that if $\mathsf{A} \sim_{\mathtt{w}} \mathsf{V}$ and (1.10) exists then $\lim\limits_{(\mathtt{w};\mathsf{V})} \mathsf{f}$ also exists and has the same value.

We now employ a notion from [5], which applies to any equivalence relation, and say that a set $\mathbf{v} \subset \mathbb{D}[\mathtt{w}]$ is **stable** — *saturée* in [5, E.R.24, §5.6]] — if

$$\boxed{\mathsf{A} \in \mathbf{v}, \mathsf{V} \in \mathbb{D}[\mathtt{w}] \text{ and } \mathsf{V} \sim_{\mathtt{w}} \mathsf{A} \implies \mathsf{V} \in \mathbf{v}}$$

Lemma 2.16 *Each of the collections* $\mathbf{t}[\mathtt{w}]$, $\mathbf{c}[\mathtt{w}]$, $\mathbf{a}[\mathtt{w}]$, $\mathbf{s}[\mathtt{w}]$, $\mathbf{p}[\mathtt{w}]$, $\mathbf{o}[\mathtt{w}]$, *and* $\boldsymbol{\gamma}[\mathtt{w}]$ *is a stable subset of* $\mathbb{D}[\mathtt{w}]$.

Proof. $\mathbf{c}[\mathtt{w}]$ **is stable.** If $\mathsf{A} \in \mathbf{c}[\mathtt{w}]$ then $\exists\ \varphi \in [\![[0,1) \nrightarrow \mathbb{D}\,]\!]$ continuous such that $\lim\limits_{s\uparrow 1} \varphi(s) = \mathtt{w}$ and $\mathsf{A} \sim_{\mathtt{w}} \{\varphi(s) : 0 \le s < 1\}$. If $\mathsf{V} \in \mathbb{D}[\mathtt{w}]$ and $\mathsf{V} \sim_{\mathtt{w}} \mathsf{A}$ then $\mathsf{V} \sim_{\mathtt{w}} \{\varphi(s) : 0 \le s < 1\}$, by the transitive property of the equivalence relation $\sim_{\mathtt{w}}$, hence $\mathsf{V} \in \mathbf{c}[\mathtt{w}]$. Q.E.D.

$\mathbf{a}[\mathtt{w}]$ **is stable.** If $\mathsf{A} \in \mathbf{a}[\mathtt{w}]$, $\mathsf{V} \in \mathbb{D}[\mathtt{w}]$, and $\mathsf{V} \sim_{\mathtt{w}} \mathsf{A}$, then $\exists\ \mathrm{S} \in \mathcal{P}(\mathbb{D})$ which is connected to $\mathtt{w}$ and such that $\mathsf{A} \sim_{\mathtt{w}} \mathrm{S}$, and since $\sim_{\mathtt{w}}$ is an equivalence relation, it follows that $\mathsf{V} \sim_{\mathtt{w}} \mathrm{S}$, i.e., $\mathsf{V} \in \mathbf{a}[\mathtt{w}]$. Q.E.D.

$\mathbf{s}[\mathtt{w}]$ **is stable.** If $\mathsf{A} \in \mathbf{s}[\mathtt{w}]$, $\mathsf{V} \in \mathbb{D}[\mathtt{w}]$, and $\mathsf{V} \sim_{\mathtt{w}} \mathsf{A}$, then there exists $\varphi \in [\![\, \mathbb{N} \nrightarrow \mathbb{D}\,]\!]$ with $\lim\limits_{j\to+\infty} \varphi_j = \mathtt{w}$ and $\mathsf{A} \sim_{\mathtt{w}} \{\varphi_j : j \in \mathbb{N}\}$, and since $\sim_{\mathtt{w}}$ is an equivalence relation, it follows that $\mathsf{V} \sim_{\mathtt{w}} \{\varphi_j : j \in \mathbb{N}\}$, i.e., $\mathsf{V} \in \mathbf{s}[\mathtt{w}]$. Q.E.D.

$\mathbf{p}[\mathtt{w}]$ **is stable.** If $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$ then $\exists\ b \in \mathbb{N}$ with $\mathsf{A} \in \mathbf{p}_b[\mathtt{w}]$. If $\mathsf{V} \in \mathbb{D}[\mathtt{w}]$, and $\mathsf{V} \sim_{\mathtt{w}} \mathsf{A}$ then $\exists\ \rho > 0$ with $\mathsf{A}(\mathtt{w}, \rho) = \mathsf{V}(\mathtt{w}, \rho)$. Now let $r > 0$. If $r > \rho$ then $\Gamma^*_b(\mathsf{V}(\mathtt{w}, r)) \supset \Gamma^*_b(\mathsf{V}(\mathtt{w}, \rho)) = \Gamma^*_b(\mathsf{A}(\mathtt{w}, \rho))$, and since $\Gamma^*_b(\mathsf{A}(\mathtt{w}, \rho)) \rightsquigarrow \mathtt{w}$, it follows that $\Gamma^*_b(\mathsf{V}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$. If $r = \rho$ then $\Gamma^*_b(\mathsf{V}(\mathtt{w}, r)) = \Gamma^*_b(\mathsf{V}(\mathtt{w}, \rho)) = \Gamma^*_b(\mathsf{A}(\mathtt{w}, \rho))$, and since $\Gamma^*_b(\mathsf{A}(\mathtt{w}, \rho)) \rightsquigarrow \mathtt{w}$, it follows that $\Gamma^*_b(\mathsf{V}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$. If $r < \rho$ then $\mathsf{A}(\mathtt{w}, r) = \mathsf{V}(\mathtt{w}, r)$, hence $\Gamma^*_b(\mathsf{V}(\mathtt{w}, r)) = \Gamma^*_b(\mathsf{A}(\mathtt{w}, r))$, and since $\Gamma^*_b(\mathsf{A}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$, it follows that $\Gamma^*_b(\mathsf{V}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$. We have proved that $\forall r > 0\ \ \Gamma^*_b(\mathsf{V}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$, hence $\mathsf{V} \in \mathbf{p}_b[\mathtt{w}]$, thus $\mathsf{V} \in \mathbf{p}[\mathtt{w}]$.

Each of the collections $\mathsf{o}[\mathtt{w}]$, $\gamma[\mathtt{w}]$, and $\mathbf{t}[\mathtt{w}]$ is a stable subset of $\mathbb{D}[\mathtt{w}]$. Observe that if $\mathsf{A} \sim_{\mathtt{w}} \mathsf{V}$ then there exists $r_0 > 0$ such that $\mathsf{A}(\mathtt{w}, r_0) = \mathsf{V}(\mathtt{w}, r_0)$. Since $\mathsf{A}(\mathtt{w}, r) = \mathsf{V}(\mathtt{w}, r)$ for $0 < r < r_0$, then, in the notation of Section 2.3

$$\text{if } 0 < r < r_0 \text{ then } \boldsymbol{\tau}_* \mathsf{A}(r) = \boldsymbol{\tau}_* \mathsf{V}(r) \text{ and } \boldsymbol{\tau}^* \mathsf{A}(r) = \boldsymbol{\tau}^* \mathsf{V}(r) \tag{$*$}$$

Since the condition that an approach region A belongs to one of the collections $\mathsf{o}[\mathtt{w}]$, $\gamma[\mathtt{w}]$, and $\mathbf{t}[\mathtt{w}]$ can be described in terms of the behavior of the functions $\boldsymbol{\tau}_* \mathsf{A}(r)$ and $\boldsymbol{\tau}^* \mathsf{A}(r)$ for $r \downarrow 0$, $(*)$ implies that if A belongs to one of these collections then V also belongs to the same collection. $\square$

We now extend the notion of stability to subsets of $[\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$.

2.12 Stable subsets of $[\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$

A nonempty subset $Q \subset [\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ is **stable** if the following condition holds:

$$\mathbf{A} \in Q,\ \mathbf{W} \in [\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}, \text{ and } \mathbf{A}(\mathtt{w}) \sim_{\mathtt{w}} \mathbf{W}(\mathtt{w})\ \forall\, \mathtt{w} \in \partial \mathbb{D} \implies \mathbf{W} \in Q.$$

Lemma 2.17 *If $\{\mathbf{x}, \mathbf{z}\}$ is as in* (2.13) *then $[\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathbf{x}}$ and $[\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathbf{xz}}$ are stable subsets of $[\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$.*

Proof. The result follows at once from Lemma 2.16. $\square$

2.13 Strongly stable subsets of $[\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$

If $R \subset [\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ and $Q \subset R$, $Q \neq \emptyset$, we say that Q is **strongly stable in** R if the following condition holds:

(strong stability in $\boldsymbol{R} \subset [\![\, \boldsymbol{\partial} \mathbb{D} \nrightarrow \boldsymbol{\mathcal{P}}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$)

$$\mathbf{A} \in Q,\ \ \mathbf{W} \in R, \text{ and } \mathbf{A}(\mathtt{w}) \sim_{\mathtt{w}} \mathbf{W}(\mathtt{w}) \text{ for a.e. } \mathtt{w} \in \partial \mathbb{D} \implies \mathbf{W} \in Q.$$

If $R = [\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ we simply say that Q **is strongly stable**.

The subset of $[\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ introduced in Section 2.13.1 is strongly stable. It appeared in 2006 in [14], and enabled us to reach a deeper understanding of Littlewood's 1927 result [32], as we shall see.

2.13.1 Regular families of approach regions

We say that $\mathbf{A} \in [\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ is **regular** if

$$O \subset \mathbb{D} \text{ is open } \implies \mathbf{A}^*(O) \subset \partial \mathbb{D} \text{ is measurable} \tag{2.28}$$

where $\mathbf{A}^*(O)$ is defined in (2.17). We set

$$\mathrm{R}(\partial \mathbb{D}, \mathcal{P}(\mathbb{D})) \overset{\text{def}}{=\!=} \{\mathbf{A} \in [\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} : \ \mathbf{A} \text{ is regular } \} \tag{2.29}$$

Since an a.e. change in a family of approach regions only alters its shadow on a set of measure zero, $\mathrm{R}(\partial \mathbb{D}, \mathcal{P}(\mathbb{D}))$ is a strongly stable subset of $[\![\, \partial \mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$.

2.13.2 The subscript notation applied to $\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))$

The subscript notation in Section 2.4.1, where $\mathrm{Y} = \mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))$, yields

$$\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathbf{t}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in \mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D})) : \mathbf{A}(\mathtt{w}) \in \mathbf{t}[\mathtt{w}] \quad \forall\, \mathtt{w} \in \partial\,\mathbb{D}\}$$

and $\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathbf{tc}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in \mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D})) : \mathbf{A}(\mathtt{w}) \in \mathbf{t}[\mathtt{w}] \cap \mathbf{c}[\mathtt{w}] \quad \forall\, \mathtt{w} \in \partial\,\mathbb{D}\}.$

2.14 Rotational invariance

The main results in the subject concern some subsets Q of $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$, in the sense that they are valid for all $\mathbf{A} \in Q$, provided Q is suitably defined. One of the first example of this sort is the class of all *rotationally invariant* families of approach regions, that appeared in 1927 in Littlewood's work [32]. Recall that the set $\{z \in \mathbb{C} : |z| = 1\}$, seen as a *topological group*, under the familiar identification with the quotient group $\mathbb{R}\,/\,\mathbb{Z}$, is denoted by $\mathbb{T}$.

2.14.1 Notation for invariant elements

If $\mathbb{T}$ acts on a set X, then

$$X^{\mathbb{T}} \stackrel{\text{def}}{=\!=} \{x \in X : ux = x \;\forall u \in \mathbb{T}\}$$

is the set of elements *of* X that are $\mathbb{T}$-**invariant**, and if $Q \subset X$, then

$$Q^{\mathbb{T}} \stackrel{\text{def}}{=\!=} Q \cap X^{\mathbb{T}} \tag{2.30}$$

is the set of elements *of* Q that are $\mathbb{T}$-**invariant**.

2.14.2 Rotationally invariant families of approach regions

We will apply this general notational device to the case where

$$X = [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$$

Indeed, the natural action of $\mathbb{T}$ on $\mathbb{D}$ and on $\partial\,\mathbb{D}$ yields an action of $\mathbb{T}$ on $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$. In particular,

$$[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}} = \{\mathbf{A} \in [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} : u\,\mathbf{A} = \mathbf{A} \;\;\forall u \in \mathbb{T}\} \tag{2.31}$$

If $\mathbf{A} \in [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}}$ we say that $\mathbf{A}$ is **rotationally invariant**: This means that for $\mathtt{w} \in \partial\,\mathbb{D}$, $s \in \mathbb{R}$, and $\mathtt{z} \in \mathbb{D}$ $\mathtt{z} \in \mathbf{A}(\mathtt{w}) \iff \mathtt{z}\,e^{is} \in \mathbf{A}(\mathtt{w}e^{is})$. Hence $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}} \subsetneq [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$.

2.14.3 A remark on notation

Observe that the notation for invariant elements introduced in Section 2.14.1 and the subscript notation of Section 2.4.1 are robust, in the sense that they commute and do not lead to ambiguities. For example, the expression

$$[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{tc}}^{\mathbb{T}} \tag{2.32}$$

could be interpreted in two ways, namely as $([\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{tc}})^{\mathbb{T}}$, arising from

$$[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} \mapsto [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{tc}} \mapsto ([\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{tc}})^{\mathbb{T}}$$

or as $([\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}})_{\mathsf{tc}}$, arising from

$$[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} \mapsto [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}} \mapsto ([\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}})_{\mathsf{tc}}$$

and the two interpretations have the same meaning, since both of them denote the set $\{\mathbf{A} \in [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} : \mathbf{A}(\mathtt{w}) \in \mathbf{t}[\mathtt{w}] \cap \mathbf{c}[\mathtt{w}]\,\forall\,\mathtt{w} \in \partial\,\mathbb{D}\ \&\ ux = x\,\forall u \in \mathbb{T}\}$.

2.14.4 Tight subsets of $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$

A nonempty subset $Q \subset [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ is **tight** if it has the following property: whenever $\mathbf{A}, \mathbf{V} \in [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ differ only at finitely many points (in the sense that the set $\{\mathtt{w} \in \partial\,\mathbb{D} : \mathbf{A}(\mathtt{w}) \neq \mathbf{V}(\mathtt{w})\}$ is nonempty and finite) then $\mathbf{A}$ and $\mathbf{V}$ cannot both belong to Q. A tight subset $Q \subset [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ is similar to the set of continuous functions from $\partial\,\mathbb{D}$ to $\mathbb{C}$, seen as a subset of $[\![\,\partial\,\mathbb{D} \nrightarrow \mathbb{C}\,]\!]$. If (2.13) holds then the collections $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}}$ and $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{xz}}^{\mathbb{T}}$ are tight.

2.14.5 Examples

The collection $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}}$ is tight. A tight subset of $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ cannot be strongly stable. Observe that

$$[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}} \subsetneq \mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D})),$$

where $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}^{\mathbb{T}}$ is tight and $\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))$ is strongly stable. On the other hand, $\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{tc}}$ is stable but not strongly stable. However, $\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{tc}}$ is strongly stable in $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{tc}}$ and, for similar reasons, $\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{tp}}$ is strongly stable in $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{tp}}$.

2.15 Fatou's Theorem (1906) and the nontangential boundary function

The nontangential approach regions appeared in 1875 in the work of Otto Stolz [45], who employed them in theorems of a *pointwise* nature; cf. [13], [42], [44], and they appeared again in 1906, in the work of Pierre Fatou [19], in a theorem that is not *pointwise* in nature but *qualitative*, i.e., where an *almost everywhere* existence of boundary values is proved; see [13] for background.

It is useful to state Fatou's Theorem in terms of the following notion (see Section 2.1). The **Fatou map**

$$\mathtt{Fatou} : [\![\mathbb{D} \to \mathbb{C}]\!] \to \mathcal{P}(\partial\mathbb{D})$$

is defined as follows: If $\mathsf{f} \in [\![\mathbb{D} \to \mathbb{C}]\!]$ then the **Fatou set** of f is the set

$$\mathtt{Fatou}(\mathsf{f}) \stackrel{\text{def}}{=\!=} \{\mathtt{w} \in \partial\mathbb{D} : \mathtt{w} \text{ is a Fatou point of } \mathsf{f}\} \tag{2.33}$$

For a generic $\mathsf{f} \in [\![\mathbb{D} \to \mathbb{C}]\!]$, the set $\mathtt{Fatou}(\mathsf{f})$ may be empty. The **nontangential boundary function** of f is the following partial function (recall Section 2.1)

$$\mathsf{f}_\flat : \partial\mathbb{D} \rightarrowtail \mathbb{C} \tag{2.34}$$

whose effective domain is the Fatou set of f and whose value $(\mathsf{f}_\flat)(\mathtt{w})$ is defined by (2.3), for $\mathtt{w} \in \mathtt{Fatou}(\mathsf{f})$.

Fatou's Theorem (1906) ([19])**.** If $\mathsf{f} \in \mathsf{H}^\infty(\mathbb{D})$, $\mathtt{Fatou}(\mathsf{f})$ has full measure.

If $\mathsf{f} \in \mathsf{H}^\infty(\mathbb{D})$ then by Fatou's Theorem the effective domain of the partial function (2.34) has full measure.

2.15.1 A bootstrap result

The approach regions (2.2) have the following bootstrap property [11, pp.33-36], [13, p.25, p.106].

Theorem 2.18 ([11]) *If* $\mathsf{f} \in [\![\mathbb{D} \to \mathbb{R}]\!]$, $\mathtt{J} \subset \partial\mathbb{D}$ *has positive outer measure, and for each* $\mathtt{w} \in \mathtt{J}$ *the limiting value* $\lim\limits_{\Gamma_1(\mathtt{w}) \to \mathtt{w}} \mathsf{f}$ *exists and is finite, then* $\mathtt{J} \overset{\text{a.e.}}{\subset} \mathtt{Fatou}(\mathsf{f})$ *and, in particular,* $\mathsf{f}_\flat(\mathtt{w})$ *exists for a.e.* $\mathtt{w} \in \mathtt{J}$.

Observe that in the statement of Theorem 2.18 neither the set $\mathtt{J}$ nor the function f need be measurable.

2.15.2 The relative Fatou's set

If $\mathbf{A} \in [\![\partial\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]_\ell$ the **relative Fatou set of** $\mathsf{f} \in [\![\mathbb{D} \to \mathbb{C}]\!]$ **along** $\mathbf{A}$ is the set $\mathtt{Fatou}_{\mathbf{A}}(\mathsf{f}) \subset \mathtt{Fatou}(\mathsf{f})$ defined by

$$\mathtt{Fatou}_{\mathbf{A}}(\mathsf{f}) \stackrel{\text{def}}{=\!=} \{\mathtt{w} \in \mathtt{Fatou}(\mathsf{f}) : \lim_{\mathbf{A}(\mathtt{w}) \to \mathtt{w}} \mathsf{f} \text{ exists } \textit{and} \text{ is equal to } \mathsf{f}_\flat(\mathtt{w})\}$$

Observe that in general

$$\mathtt{Fatou}_{\mathbf{A}}(\mathsf{f}) \subsetneq \{\mathtt{w} \in \partial\mathbb{D} : \lim_{\mathbf{A}(\mathtt{w}) \to \mathtt{w}} \mathsf{f} \text{ exists}\}$$

Lemma 2.19 *If* $\mathbf{A} \in [\![\partial\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]_\gamma$ *then* $\mathtt{Fatou}_{\mathbf{A}}(\mathsf{f}) = \mathtt{Fatou}(\mathsf{f})$.

Proof. If $\mathtt{Fatou}(\mathsf{f}) = \emptyset$ then the conclusion holds since $\mathtt{Fatou}_{\mathbf{A}}(\mathsf{f}) \subset \mathtt{Fatou}(\mathsf{f})$. Assume that $\mathtt{Fatou}(\mathsf{f}) \neq \emptyset$. If $\mathtt{w} \in \mathtt{Fatou}(\mathsf{f})$ then, by definition, $\lim\limits_{\mathsf{A} \to \mathtt{w}} \mathsf{f}$ exists and is finite for each $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$, and its value, denoted by $\mathsf{f}_\flat(\mathtt{w})$, does not depend on A. Since $\lambda(\mathtt{w}) \in \boldsymbol{\gamma}[\mathtt{w}]$, it follows that $\lim\limits_{\lambda(\mathtt{w}) \to \mathtt{w}} \mathsf{f}$ exists and it is equal to $\mathsf{f}_\flat(\mathtt{w})$. $\square$

2.15.3 A corollary of Fatou's theorem

Fatou's Theorem has the following corollary.

Corollary 2.20 *If* $\mathsf{f} \in \mathsf{H}^\infty(\mathbb{D})$ *and* $\mathbf{A} \in [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_\gamma$ *then the effective domain of the function* $\lim\limits_{\mathbf{A}} \mathsf{f}$ *defined in* (2.25) *has full measure.*

Proof. If $\mathbf{A} \in [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_\gamma$ then $\lambda(\mathtt{w}) \in \boldsymbol{\gamma}[\mathtt{w}]$ implies that

$$\mathtt{Fatou}(\mathsf{f}) \subset \{\mathtt{w} \in \partial\mathbb{D} : \lim_{\mathbf{A}(\mathtt{w}) \nrightarrow \mathtt{w}} \mathsf{f} \text{ exists}\} \equiv \mathrm{E_d}(\lim_{\mathbf{A}} \mathsf{f}) \tag{2.35}$$

and Fatou's Theorem yields the conclusion. $\square$

Observe that inclusion in (2.35) may be strict. For example, f may have a radial limiting value at $\mathtt{w}$ even though $\mathtt{w} \notin \mathtt{Fatou}(\mathsf{f})$.

2.15.4 Theorems of Fatou type

We say that $\mathbf{A} \in [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ is of **Fatou type for** $\mathsf{H}^\infty(\mathbb{D})$ if for each $\mathsf{f} \in \mathsf{H}^\infty(\mathbb{D})$ the set $\mathtt{Fatou}_{\mathbf{A}}(\mathsf{f})$ has full measure, and we set

$$[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}} \stackrel{\text{def}}{=\!=} \{\mathbf{A} \in [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}} : \mathbf{A} \text{ is of } \textit{Fatou type} \text{ for } \mathsf{H}^\infty(\mathbb{D})\}$$

A theorem that gives sufficient conditions on $\mathrm{Q} \subset [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ implying Q is contained in $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$ is called a **theorem of Fatou type**. In particular,

$$\boxed{[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_\gamma \subset [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}} \subset [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}}$$

2.16 Littlewood's Theorem (1927) and related results

Since Littlewood was interested in determining the possible existence of families of approach regions belonging to $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$ beyond those of $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_\gamma$, it was natural for him to look at $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathbf{tc}}$.

Littlewood's Theorem (1927) ([32])**.** If $\mathbf{A}$ is any family of tangential and curvilinear approach regions that is rotationally invariant, i.e., if

$$\mathbf{A} \in [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathbf{tc}}$$

then there exists $\mathsf{f} = \mathsf{f}_{\mathbf{A}} \in \mathsf{H}^\infty(\mathbb{D})$ such that $\mathtt{DIV}[\mathsf{f}, \mathbf{A}]$ has full measure.

In particular, Littlewood proved that no family of tangential and curvilinear approach regions that is rotationally invariant is of Fatou type.

LT (1927). $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathbf{tc}}$ is **d i s j o i n t** from $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$.

Observe that the statement

$$«[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{t}} \text{ is } \mathbf{d\,i\,s\,j\,o\,i\,n\,t} \text{ f r o m } [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}» \tag{2.36}$$

does *not* follow from Littlewood's Theorem. Indeed, Littlewood proved that *a subset of* $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{t}}$, namely $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{tc}}$, is disjoint from $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$, *not* $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{t}}$ itself. By the same token, the statement

$$«[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{tc}} \text{ is } \mathbf{d\,i\,s\,j\,o\,i\,n\,t} \text{ from } [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}» \tag{2.37}$$

does not follow from Littlewood's Theorem. The issue of the truth-value of (2.37) appeared in [32] and was posed again in [39] .

2.16.1 Rudin's Theorem (1979)

Fatou's Theorem and Littlewood's Theorem have led to the belief that (2.36) is true. How deeply entrenched was this credence may be inferred by the fact that *in the unit disc* $\mathbb{D} \subset \mathbb{C}$ it was *disproved* only in 1979, by Walter Rudin; cf. [26]. In [39], W. Rudin proved the following result.

Rudin's Theorem (1979) ([39])**.** The sets $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{ts}}$ and $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{oc}}$ are not disjoint from $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$, i.e.,

$$\begin{aligned} [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{ts}} \cap [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}} \neq \emptyset \\ [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{oc}} \cap [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}} \neq \emptyset \end{aligned} \tag{2.38}$$

Rudin's Theorem implies that the set $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{t}}$ is *not* disjoint from $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$, and thus (2.36) is disproved. Hence the property of being tangential, for $\mathbf{A} \in [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$, is not *by itself* an impediment for $\mathbf{A}$ to be of Fatou type.

In his work ([39], [40]) W. Rudin also asked whether (2.37) could possibly be proved. The issue about the truth-value of (2.37) also appeared in [32]. The answer was found in 2006 in [14], and it exhibited an additional level of difficulty; cf. [49].

2.16.2 An independence result

An Independence Result (2006) ([14])**.** The statement (2.37) can be neither proved nor disproved: It is independent of ZFC.

The proof of this theorem relies upon an insight about a precise connection between modern logic and harmonic analysis, that makes it possible to combine methods of both areas in a fruitful manner. The method of modern logic used therein have been developed after 1929; see [12, 17, 24, 31] for background.

2.16.3 The Nagel–Stein Phenomenon (1984)

We shall return momentarily to Rudin's Theorem (1979). For the time being, we point out that an indication of the fact that in 1979 our understanding of these matters was still rudimentary may be found in a conjecture that Rudin made in the same paper, that amounts, in effect, to the following statement.

Rudin's Conjecture

$$\text{«}[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{ts}} \cup [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{oc}} \text{ is } \textbf{disjoint} \text{ from } [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}\text{»} \tag{2.39}$$

In 1984 by A. Nagel and E.M. Stein have disproved (2.39), showing once more how insidiously slippery the subject is.

Nagel and Stein's Theorem (1984).

$$[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{ts}} \cap [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}} \neq \emptyset \text{ and } [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{oc}} \cap [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}} \neq \emptyset$$

We shall return on Nagel and Stein's Theorem (1984) momentarily.

2.16.4 Theorems of Littlewood Type

The general picture can be conveniently clarified if we adopt the following terminology: A **Littlewood set** is a subset of $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ that is d i s j o i n t from $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$. For example, Littlewood's Theorem (1927) implies that $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{tc}}$ is a Littlewood set. By the same token, a **theorem of Littlewood type** is a theorem that provides conditions on $\mathrm{Q} \subset [\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$ that imply that Q is a Littlewood set. The following result, published in 2006, is a theorem of Littlewood type.

Theorem LT (2006) ([14])**.** For each family of approach regions

$$\mathbf{A} \in \mathrm{R}(\partial\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{ta}}$$

there exists $\mathsf{f} = \mathsf{f}_{\mathbf{A}} \in \mathsf{H}^{\infty}(\mathbb{D})$ such that $\mathtt{DIV}[\mathsf{f}, \mathbf{A}]$ has full measure.

The following statement is a corollary of Theorem LT (2006).

TLT (2006). $\mathrm{R}(\partial\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{ta}}$ is **d i s j o i n t** from $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$.

Observe that the larger the set Q is, the more "difficult" it will be for it to be disjoint from $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$, hence larger sets Q (in theorems of Littlewood type) yield sharper results. Since $[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{tc}} \subsetneq \mathrm{R}(\partial\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{ta}}$, it follows that Theorem LT (2006) is sharper than Littlewood's Theorem, since it yields a larger Littlewood set.

In view of the Independence Result (2006) [14], the condition in (2.28) should be interpreted as a *regularity condition* in Theorem LT (2006) [14], i.e., as a part of the hypotheses that is not formally necessary to give meaning to its conclusion. Some theorems in mathematical analysis do fail if we omit a regularity condition contained therein (see e.g. [6, p. 198]), while others, typically those involving null sets, remain valid without regularity hypothesis (see e.g. [42, p. 251]). It is not clear how to determine *a priori* if a theorem belongs to the first or to the second group with respect to a given regularity condition.

Summing up, of the sets that appear in the chain in (2.40)

$$[\![\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{tc}} \subsetneq \mathrm{R}(\partial\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{ta}} \subsetneq \mathrm{R}(\partial\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{tp}} \tag{2.40}$$

(i) The first was shown to be disjoint from $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$ by Littlewood in 1927 [32];

(ii) The second was shown to be disjoint from $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$ in 2006 [14];

(iii) Theorem 1.3 says that the third one is disjoint from $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$.

Observe that, on the one hand, Theorem 1.3 is a sharper theorem of Littlewood type, not only because $\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{ta}} \subsetneq \mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{tp}}$, but also because $\mathrm{R}(\partial\,\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{tp}}$ is a Littlewood set that is not disjoint from $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]^{\mathbb{T}}_{\mathsf{ts}}$, a set which, in turn, does indeed contain elements of $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$, as shown by A. Nagel and E.M. Stein in 1984.

2.16.5 More on Rudin's Theorem (1979)

It is useful to provide more details on W. Rudin's 1979 contribution, where he proved the following result.

A Very Oscillatory Inner Function (1979) ([39]). There exists $\varphi \in \mathsf{H}^{\infty}(\mathbb{D})$ such that, for a.e. $\mathtt{w} \in \partial\,\mathbb{D}$, $|\varphi_{\flat}(\mathtt{w})| = 1$ and

$$0 = \lim_{r\downarrow 0}\ \inf_{0<s<r} \frac{1 - |\varphi(s\,\mathtt{w})|}{|\varphi(s\,\mathtt{w}) - \varphi_{\flat}(\mathtt{w})|} < \lim_{r\downarrow 0}\ \sup_{0<s<r} \frac{1 - |\varphi(s\,\mathtt{w})|}{|\varphi(s\,\mathtt{w}) - \varphi_{\flat}(\mathtt{w})|}$$

Thanks to standard fact on conformal mapping [36], it follows that

$$\begin{aligned} &[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{ts}} \cap [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}} \neq \emptyset \\ &[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{oc}} \cap [\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}} \neq \emptyset \end{aligned} \tag{2.41}$$

and, moreover, that (2.38) holds.

Observe that (2.38) does not yield a whole class of elements in each of the sets appearing therein, but only one particular element, linked to the inner function constructed above. A more in-depth understanding of (2.38) was given by A. Nagel and E.M. Stein in 1984, as we will now show in more detail.

2.16.6 The Nagel–Stein Phenomenon (1984)

The Nagel–Stein Phenomenon is best illustrated by recalling another of W. Rudin's works, in collaboration with A. Nagel and J.H. Shapiro, where certain families of tangential approach regions, in the context of the *Dirichlet space*, also made their appearance [34]. We will not describe this result in detail, and limit ourselves to observe that this second contribution is important for two reasons. Firstly, not just one particular element of $[\![\,\partial\,\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{t}}$ which is *of Fatou type for the Dirichlet space* is constructed, but an infinite subset. Secondly, the authors have employed the powerful technique of maximal function estimates.

Rudin's Theorem (1979), as well as his work in collaboration with A. Nagel and J.H. Shapiro [34], have attracted the attention of Elias Menachem Stein, who, in collaboration with Alexander Nagel, reached a deeper understanding of these matters, disproved (2.39), and showed that the sets $\llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{t}}$ and $\llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{o}}$ contain infinitely many elements of $\llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket_{\mathsf{Fatou}}$.

The Nagel–Stein Phenomenon (1984) ([35])**.** The sets

$$\llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{ts}} \text{ and } \llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{oc}}$$

are not Littlewood sets. There exists a map

$$\begin{array}{c} \llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{tc}} \\ \big\downarrow S \\ \llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{ts}} \cap \llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket_{\mathsf{Fatou}} \end{array} \tag{2.42}$$

such that $S(\mathbf{A}) \subset \mathbf{A}$ for all $\mathbf{A} \in \llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{tc}}$.

In particular, the sets $\llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{t}}$ and $\llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\mathsf{o}}$ are not Littlewood sets, hence Rudin's Conjecture 2.39 is disproved.

Observe that the Nagel–Stein Phenomenon is centered on $\llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\boldsymbol{\ell}}$, i.e, rotationally invariant families of approach regions: If $\mathbf{A} \in \llbracket \partial\mathbb{D} \to \mathcal{P}(\mathbb{D})\rrbracket^{\mathbb{T}}_{\boldsymbol{\ell}}$, A. Nagel and E.M. Stein understood the relevance of the so-called *cross-section condition* (henceforth denoted CS condition), which may be more easily illustrated in the upper-half plane $\mathbb{U} \overset{\text{def}}{=\!=} \{z \in \mathbb{C} : z = x + iy, x \in \mathbb{R}, y > 0\}$, and *which only is relevant for group-invariant families of approach regions.* Apart from an additional technicality, the CS-condition requires that the measure of the radial projection to the boundary of the set of points in the approach regions which lie at distance r from the boundary must be of the order of r. A. Nagel and E.M. Stein proved that the CS-condition is equivalent to the $w(1,1)$ boundedness of the associated maximal operator. Observe that the CS-condition holds if $\mathbf{A} = \Gamma_n$ for a fixed n (see the approach region S in Figure 1), and it does not hold for a tangential approach regions (see the approach region T in Figure 1). However, if the approach region has a *lacunary* nature, i.e., if its cross-section has many "holes", then the *CS-condition may hold even if the approach region contains a tangential sequence* (see the approach region O in Figure 1). See [30] for a new proof in the group-invariant case, and [11] for a detailed presentation of this topic as well as a treatment of the general case, where there is no group acting on the space, and the CS-condition alone is *not* sufficient to guarantee that the associated maximal operator is $w(1,1)$.

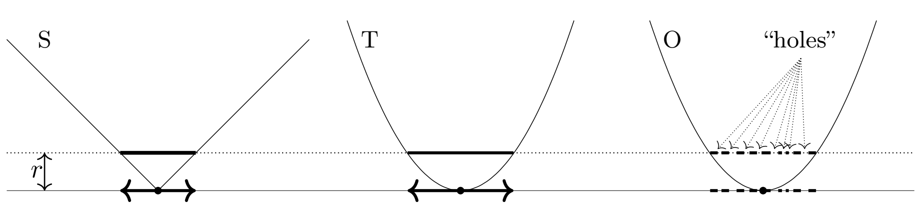

Figure 1: The cross-section condition fails in (T) but holds in (S) and (O).

2.16.7 A brief summary

Some of the results obtained so far are summarized in Figure 2 and in Figure 3, were the sets contained in the rectangles are *not* Littlewood sets, the sets contained in the ellipse *is* a Littlewood set, and the "starburst" shape contains a set whose property of being a Littlewood set is independent of ZFC.

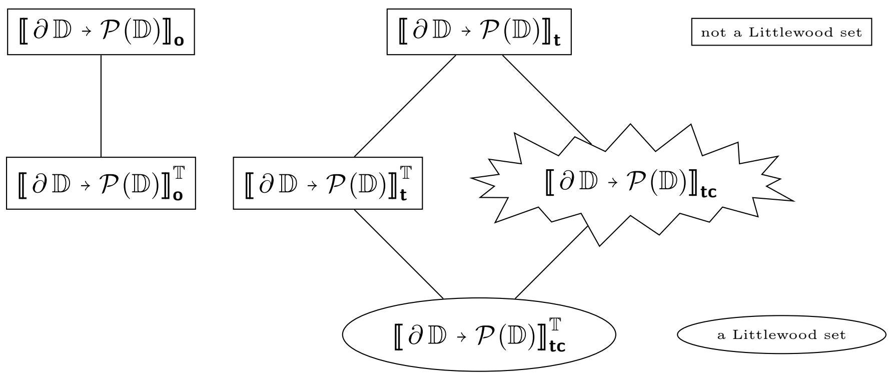

Figure 2: A diagram with Littlewood, or not Littlewood, subsets of $[\![\partial\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]_{\boldsymbol{\ell}}$.

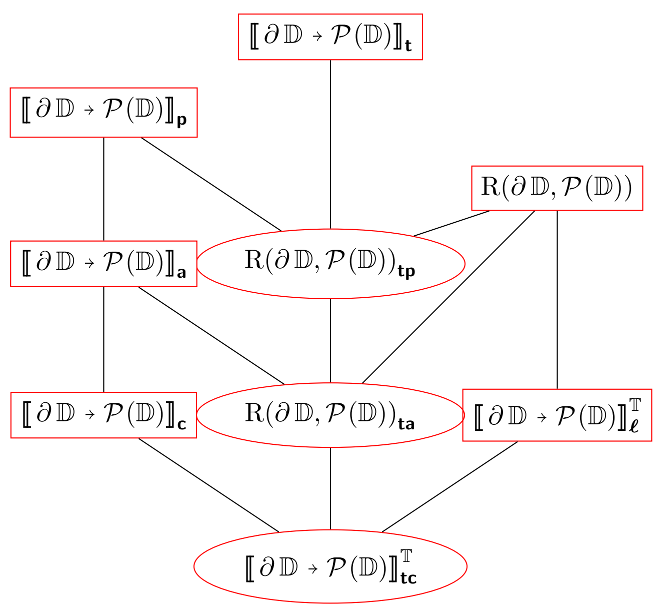

Figure 3: A larger Littlewood set.

The definition of the class $[\![\partial\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]_{\mathbf{p}}$, that plays a key role in Theorem 1.3, is one of the contribution of this work. Since $[\![\partial\mathbb{D} \to \mathcal{P}(\mathbb{D})]\!]_{\mathbf{a}}$ is disjoint from $[\![\,\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{s}}$, the Littlewood set determined in [2], [3], [14], [32], [33] are all disjoint from $[\![\,\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{s}}$. The main contribution of this paper is the introduction of a new Littlewood set, described in Theorem 1.3, namely, $\mathrm{R}(\partial\mathbb{D}, \mathcal{P}(\mathbb{D}))_{\mathsf{tp}}$, that is *not* disjoint from $[\![\,\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{s}}$. The point is that $[\![\,\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{s}}$ is not disjoint from $[\![\,\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{Fatou}}$, as discovered by W. Rudin and by A. Nagel and E.M. Stein. As far as we know, this is the first Littlewood set that is not disjoint from $[\![\,\partial\mathbb{D} \nrightarrow \mathcal{P}(\mathbb{D})]\!]_{\mathsf{s}}$.

3 Proofs of Lemma 1.1 and Proposition 1.2

In dealing with approach regions which belong to $\mathbf{a}[\mathtt{w}]$ the Jordan curve theorem [36, p. 2] is a useful tool. It will be applied in the following form: If $\mathsf{c} : [a,b] \to \mathbb{C}$ is a closed Jordan curve and $V \subset \mathbb{C}$ is a connected set which meets both the "inside" of c and its "outside", then V meets the curve itself.

3.1 Proof of Lemma 1.1

Firstly, we prove that $\mathbf{c}[\mathtt{w}] \subset \mathbf{a}[\mathtt{w}]$. We may assume, without loss of generality that $\boxed{\mathtt{w} = 1}$.

Proof. Let $\mathsf{A} \in \mathbf{c}[\mathtt{w}]$. Then $\exists\ \varphi \in [\![[0,1) \nrightarrow \mathbb{D}\,]\!]_{(\mathsf{top})}$ with $\mathsf{A} \sim_{\mathtt{w}} \{\varphi(s) : 0 \leq s < 1\}$ and $\lim\limits_{s\uparrow 1} \varphi(s) = \mathtt{w}$. Hence it suffices to show that the set $\{\mathtt{w}\} \cup \{\varphi(s) : 0 \leq s < 1\}$ is connected. Since

$$\{\varphi(s) : 0 \leq s < 1\} \subset \{\mathtt{w}\} \cup \{\varphi(s) : 0 \leq s < 1\} \subset \mathsf{cl}(\{\varphi(s) : 0 \leq s < 1\}).$$

and the set $\{\varphi(s) : 0 \leq s < 1\}$ is connected, by [7, Prop. 4, TG I.82], the conclusion follows from [7, Prop. 1, TG I.81]. □

Secondly, we show that $\mathbf{a}[\mathtt{w}] \setminus \mathbf{c}[\mathtt{w}] \neq \emptyset$.

Proof. Since $\mathtt{w} = 1$, the set $\mathsf{A} \overset{\text{def}}{=\!=} (0,1) \cup \bigcup_{n=2}^{\infty}\{(1-n^{-1})\exp(i\theta) : 0 \leq \theta < n^{-1}\}$ belongs to $\mathbf{a}[\mathtt{w}]$, as may be seen by [7, Corollaire, TG I.81]. Now observe that

$$\text{for each } r > 0 \text{ the set } [\mathsf{A} \cup \{\mathtt{w}\}] \cap \mathsf{cl}(\mathrm{B}(\mathtt{w}, r)) \text{ is not compact} \tag{$*1$}$$

If $\mathsf{A} \in \mathbf{c}[\mathtt{w}]$ then $\exists\ \varphi \in [\![[0,1) \nrightarrow \mathbb{D}\,]\!]_{(\mathsf{top})}$ such that $\lim\limits_{s\uparrow 1} \varphi(s) = \mathtt{w}$ and $\exists\ r_1 > 0$ such that $\mathsf{A} \cap \mathrm{B}(\mathtt{w}, r_1) = \{\varphi(s) : 0 \leq s < 1\} \cap \mathrm{B}(\mathtt{w}, r_1)$. It follows that

$$[\mathsf{A} \cup \{\mathtt{w}\}] \cap \mathrm{B}(\mathtt{w}, r_1) = [\{\varphi(s) : 0 \leq s < 1\} \cup \{\mathtt{w}\}] \cap \mathrm{B}(\mathtt{w}, r_1).$$

Let $r_2 \overset{\text{def}}{=\!=} r_1/2$. Since

$$\mathsf{cl}(\mathrm{B}(\mathtt{w}, r_2)) \subset \mathrm{B}(\mathtt{w}, r_1)$$

it follows that

$$[\mathsf{A} \cup \{\mathtt{w}\}] \cap \mathsf{cl}(\mathrm{B}(\mathtt{w}, r_2)) = [\{\varphi(s) : 0 \leq s < 1\} \cup \{\mathtt{w}\}] \cap \mathsf{cl}(\mathrm{B}(\mathtt{w}, r_2)) \tag{$\lightning$}$$

Observe that the right-hand term of ($\lightning$) is compact, being the intersection of compact sets, hence the left-hand is also compact, in contradiction with ($*1$). $\square$

Finally, we show that $\mathsf{a}[\mathtt{w}] \cap \mathsf{s}[\mathtt{w}] = \emptyset$.

Proof. Recall that $\mathtt{w} = 1$. If $0 < r_1 < r_2 \leq +\infty$, we define

$$\mathrm{C}(\mathtt{w}; r_1, r_2) \stackrel{\text{def}}{=\!=} \{z \in \mathbb{C} : r_1 < |z - \mathtt{w}| < r_2\} \tag{3.1}$$

Let $\mathsf{A} \in \mathsf{a}[\mathtt{w}] \cap \mathsf{s}[\mathtt{w}]$. Since $\mathsf{A} \in \mathsf{a}[\mathtt{w}]$, there exists $S \in \mathcal{P}(\mathbb{D})$ such that S is connected to w and $A \sim_{\mathtt{w}} S$, hence there exists $\rho > 0$ such that

$$\mathsf{A} \cap \mathrm{B}(\mathtt{w}, \rho) = S \cap \mathrm{B}(\mathtt{w}, \rho) \tag{$*2$}$$

Since $\mathsf{A} \in \mathsf{s}[\mathtt{w}]$, $\exists\ \varphi \in [\![\mathbb{N} \to \mathbb{D}]\!]$ with $\lim_{j\to+\infty} \varphi_j = \mathtt{w}$ and $\mathsf{A} \sim_{\mathtt{w}} \{\varphi_j : j \in \mathbb{N}\}$, hence $\exists\ r_* > 0$ with

$$\mathsf{A} \cap \mathrm{B}(\mathtt{w}, r_*) = \mathsf{A} \cap \{\varphi_j : j \in \mathbb{N}\}. \tag{$*3$}$$

Hence $\{|\mathsf{z} - \mathtt{w}| : \mathsf{z} \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r_*)\} = \{|\mathsf{z} - \mathtt{w}| : \mathsf{z} \in \mathsf{A} \cap \{\varphi_j : j \in \mathbb{N}\}\}$ and thus

$$\{|\mathsf{z} - \mathtt{w}| : \mathsf{z} \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r_*)\} \text{ is a countably infinite subset of } (0, r_*). \tag{$*4$}$$

To fix ideas, assume that $\rho \leq r_*$. Since $\mathsf{A} \in \mathbb{D}[\mathtt{w}]$, $\exists\ y_0 \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, \rho)$. Then ($*4$) implies that $\exists\ s > 0$ such that $s < |y_0 - \mathtt{w}| < \rho$ and

$$\text{the set } \{\mathsf{z} \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r_*) : |\mathsf{z} - \mathtt{w}| = s\} \text{ is empty} \tag{$*5$}$$

Then ($*2$) and ($*5$) imply that

$$\{\mathtt{w}\} \cup S = [(\{\mathtt{w}\} \cup S) \cap \mathrm{B}(\mathtt{w}, s)] \cup [(\{\mathtt{w}\} \cup S) \cap \mathrm{C}(\mathtt{w}; s, +\infty)] \tag{$\lightning$}$$

and ($\lightning$) contradicts the fact that $\{\mathtt{w}\} \cup S$ is connected. Indeed, each of the sets appearing within the square brackets in ($\lightning$) is nonempty and open in the subspace topology of $\{\mathtt{w}\} \cup S$.

If $r_* < \rho$ then ($*3$) implies that $\exists\ k_0 \in \mathbb{N}$ such that $\varphi_{k_0} \in \mathsf{A} \cap \mathrm{B}(\mathtt{w}, r_*)$. Let $s > 0$ with $s < |\varphi_{k_0} - \mathtt{w}| < r_*$ and such that ($*5$) holds. Then ($\lightning$) follows and again we have a contradiction. $\square$

3.2 Proof of Proposition 1.2 (b)

As in the proof of Lemma 1.1, we may assume, without loss of generality that $\boxed{\mathtt{w} = 1}$. Define the sequence $\mathsf{z} \in [\![\mathbb{N} \to \mathbb{D}]\!]$ by $\mathsf{z}_n \stackrel{\text{def}}{=\!=} (1 - n^{-2})e^{i/n}$ and let $\mathsf{A} \stackrel{\text{def}}{=\!=} \{\mathsf{z}_n : n \in \mathbb{N}\}$. We claim that $\mathsf{A} \in \mathsf{s}[\mathtt{w}] \cap \mathsf{t}[\mathtt{w}] \cap \mathsf{p}[\mathtt{w}]$.

Proof. $\mathsf{A} \in \mathsf{s}[\mathtt{w}]$, since $\lim_{n\to+\infty} \mathsf{z}_n = 1$. The boundary coordinates of z_n are (θ_n, δ_n) with $\theta_n = n^{-1}$ and $\delta_n = n^{-2}$, hence $\tau(1, \mathsf{z}_n)^2 \leq \frac{64}{3} n^{-2}(1 + o(1))$, by (2.22), if n is large enough, hence $\mathsf{A} \in \mathsf{t}[\mathtt{w}]$.

We now prove that $\mathsf{A} \in \mathbf{p}_2[\mathtt{w}]$. Let $r > 0$. We may assume, without loss of generality, that $r < 1/4$. If $\alpha > 0$ let $I_\alpha \stackrel{\text{def}}{=\!=} \{e^{i\theta} : 0 < \theta < \alpha\}$. We now prove that $\exists\ \alpha_r \in (0, 1/4)$ such that $I_{\alpha_r} \subset \Gamma_2^*(\mathsf{A}(\mathtt{w}, r))$. We claim that $\alpha_r \stackrel{\text{def}}{=\!=} 2^2 5^{-2} r$ will do. Indeed, if $0 < \theta < \alpha_r$ there exists a unique $n \equiv n_\theta \in \mathbb{N}$ with $n_\theta^{-1} \le \theta < (n_\theta - 1)^{-1}$. It follows that $n_\theta \ge \theta^{-1} > (\alpha_r)^{-1} > 5^2$, hence $\mathsf{z}_{n_\theta} \in O$, with O as in (2.21). Moreover, we claim that $\mathsf{z}_{n_\theta} \in \mathrm{B}(1, r)$, i.e., that $|\mathsf{z}_{n_\theta} - 1|^2 < r^2$. Indeed, since $\mathsf{z}_{n_\theta} \in O$, (2.22) implies that

$$|\mathsf{z}_n - 1|^2 \le 5^3 \cdot 2^{-6} n^{-2} (1 + n^{-2}) \stackrel{(i)}{<} 5^3 \cdot 2^{-6} n^{-2} (1 + 1) \stackrel{(ii)}{<} r^2$$

where (i) follows from $n \ge 25$ and (ii) follows from $n^{-1} \le \theta < \alpha_r \equiv 2^2 5^{-2} r$. Finally, we prove that $\mathsf{z}_{n_\theta} \in \Gamma_2(e^{i\theta})$, i.e., that $\tau(e^{i\theta}, \mathsf{z}_{n_\theta}) > \frac{1}{3}$. Indeed,

$$\begin{aligned}
[\tau(e^{i\theta}, \mathsf{z}_{n_\theta})]^2 &= \frac{n^{-4}}{|e^{i\theta} - \mathsf{z}_{n_\theta}|^2} \stackrel{(i)}{\ge} \frac{64}{125} \frac{n^{-4}}{|\theta - n^{-1}|^2 + n^{-4}} \\
&\stackrel{(ii)}{\ge} \frac{64}{125} \frac{n^{-4}}{n^{-4}(1 - n^{-1})^2 + n^{-4}} = \frac{64}{125} \frac{1}{(1 - n^{-1})^2 + 1} \stackrel{(iii)}{>} \frac{64}{125} \frac{1}{1 + 1} > \frac{1}{9}
\end{aligned}$$

where (i) follows from (2.22), which may be applied since $e^{i\theta}$ and z_n belong to O; (ii) follows from $n^{-1} \le \theta < (n - 1)^{-1}$ and hence $|\theta - n^{-1}| \le n^{-1}(n - 1)^{-1}$; and (iii) follows from $(1 - n^{-1})^2 < 1$.

Summing up, we have proved that for each $\theta \in (0, \alpha_r)\ \exists\ n \in \mathbb{N}$ such that $\mathsf{z}_n \in \mathrm{B}(1, r) \cap O$ and $\mathsf{z}_n \in \Gamma_2(e^{i\theta})$, and since $\mathsf{A} \stackrel{\text{def}}{=\!=} \{\mathsf{z}_j : j \in \mathbb{N}\}$, this means that $e^{i\theta} \in \Gamma_2^*(\mathsf{A}(\mathtt{w}, r))$. We have proved that for each $r \in (0, 1/4)\ \exists\ \alpha_r > 0$ with $I_{\alpha_r} \subset \Gamma_2^*(\mathsf{A}(\mathtt{w}, r))$, and hence the same statement holds for each $r > 0$. $\square$

3.3 Proof of Proposition 1.2 (c)

Recall that $\boxed{\mathtt{w} = 1}$. If $n \in \mathbb{N}$ define $\mathsf{z}_n \stackrel{\text{def}}{=\!=} (1 - 4^{-2^n}) \exp(i 4^{-2^{n-1}})$. The boundary coordinates of z_n are $\theta_n = 4^{-d(n-1)}$ and $\delta_n = 4^{-d(n)}$, where $d(n) \stackrel{\text{def}}{=\!=} 2^n$.

Let $\mathsf{A} \stackrel{\text{def}}{=\!=} \{\mathsf{z}_n : n \in \mathbb{N}\}$. We claim that $\mathsf{A} \in \mathbf{s}[\mathtt{w}] \cap \mathbf{t}[\mathtt{w}] \setminus \mathbf{p}[\mathtt{w}]$.

Proof. $\mathsf{A} \in \mathbf{s}[\mathtt{w}]$, since $\lim_{n \to +\infty} \mathsf{z}_n = 1$. Observe that

$$\tau(1, \mathsf{z}_n)^2 = (1 - |\mathsf{z}_n|)^2 / |\mathsf{z}_n - 1|^2 = 4^{-2d(n)} / |\mathsf{z}_n - 1|^2$$

and if n is large enough then (2.22) implies that

$$|\mathsf{z}_n - 1|^2 \ge \frac{3}{64} \left[\left(4^{-d(n)}\right)^2 + \left(4^{-d(n-1)}\right)^2 \right] = \frac{3}{64} 4^{-d(n)} [1 + o(1)]$$

hence $\tau(1, \mathsf{z}_n)^2 \le \frac{64}{3} 4^{-d(n)} [1 + o(1)]$ thus $\mathsf{A} \in \mathbf{t}[\mathtt{w}]$.

We now show that $\mathsf{A} \notin \mathbf{p}[\mathtt{w}]$, i.e., that if $b \in \mathbb{N}$ then $\mathsf{A} \notin \mathbf{p}_b[\mathtt{w}]$. We may assume, without loss of generality, that $b \geq 10.$ We claim that $\exists\, r \equiv r_b$ such that the set $\Gamma_b^*(\mathsf{A}(\mathtt{w}, r_b))$ is not adjacent to $\mathtt{w}$. Let $n_b \in \mathbb{N}$ be such that $n_b \geq 10$ and

(**\$1**) if $n \geq n_b$ then $64 \cdot 3^{-1} \cdot 2 \cdot 4^{-d(n)} < \frac{1}{(1+b)^2}$

and define

(**\$2**) $r_b \overset{\text{def}}{=\!=} \min\{|1 - \mathtt{z}_n| : n < 10 + n_b\}$

We claim that the set $\Gamma_b^*(\mathsf{A}(\mathtt{w}, r_b))$ is not adjacent to $\mathtt{w}$, i.e., that

($*$) if $\mathtt{J}$ is an arc in $\partial\mathbb{D}$ with endpoint at 1, then $\mathtt{J} \not\subset \Gamma_b^*(\mathsf{A}(\mathtt{w}, r_b))$

Observe that if $\mathtt{J}$ is an arc in $\partial\mathbb{D}$ with endpoint at 1 then

($**$) either $\mathtt{J} \equiv \{e^{i\theta} : -\epsilon < \theta < 0\}$ or $\mathtt{J} \equiv \{e^{i\theta} : 0 < \theta < \epsilon\}$ for some $\epsilon > 0$

We may assume, without loss of generality, that $\epsilon < 1/10.$

Firstly, we assume that the first case displayed in ($**$) holds, i.e., that $\mathtt{w}$ is a right-endpoint of $\mathtt{J}$. Select $k \in \mathbb{N}$ with $k \geq 10$ and $k^{-1} < \epsilon$. We claim that

$$e^{-i/k} \in \mathtt{J} \setminus \Gamma_b^*(\mathsf{A}(\mathtt{w}, r_b))$$

On the one hand, $k^{-1} < \epsilon$ yields $e^{-i/k} \in \mathtt{J}$. On the other hand, we claim that

$$e^{-i/k} \notin \Gamma_b^*(\mathsf{A}(\mathtt{w}, r_b)), \text{ i.e., } \Gamma_b(e^{-i/k}) \cap \mathsf{A}(\mathtt{w}, r_b) = \emptyset, \text{ i.e., that}$$

($\heartsuit$) if $\mathtt{z}_n \in \mathrm{B}(\mathtt{w}, r_b)$ then $\mathtt{z}_n \notin \Gamma_b(e^{-i/k})$

If $\mathtt{z}_n \in \mathrm{B}(\mathtt{w}, r_b)$ then (\$2) implies that $n \geq 10 + n_b,$ and since $\mathtt{z}_n$ and $e^{-i/k}$ both belong to O, (2.22) implies that

$$\begin{aligned}
\left[\tau(e^{-i/k}, \mathtt{z}_n)\right]^2 &\leq 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + |4^{-d(n-1)} + k^{-1}|^2} \\
&\leq 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + |4^{-d(n-1)}|^2} = 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + 4^{-d(n)}} \\
&< 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-d(n)}} = 64 \cdot 3^{-1} 4^{-d(n)} \overset{(\$1)}{\leq} \frac{1}{(1+b)^2}
\end{aligned}$$

hence $\tau(e^{-i/k}, \mathtt{z}_n) < \frac{1}{1+b}$, i.e., $\mathtt{z}_n \notin \Gamma_b(e^{-i/k})$, and the proof of ($\heartsuit$) is complete.

Secondly, we assume that the second case displayed in ($**$) holds, i.e., that $\mathtt{w}$ is a left-endpoint of $\mathtt{J}$. Recall that $\epsilon < 1/10$, define $\boxed{c_j \overset{\text{def}}{=\!=} \frac{1}{2}(\theta_j + \theta_{j+1})}$ $\forall j \geq 1$, and choose $k \geq 10 + n_b$ such that $c_k < \epsilon/2.$ We claim that

$$e^{ic_k} \in \mathtt{J} \setminus \Gamma_b^*(\mathsf{A}(\mathtt{w}, r_b))$$

Indeed, $e^{ic_k} \in \mathtt{J}$ follows from $c_k < \epsilon$. We now show that

$$e^{ic_k} \notin \Gamma_b^*(\mathsf{A}(\mathtt{w}, r_b)), \text{ i.e., } \Gamma_b(e^{ic_k}) \cap \mathsf{A}(\mathtt{w}, r_b) = \emptyset, \text{ i.e., that}$$

($\heartsuit\heartsuit$) if $\mathtt{z}_n \in \mathrm{B}(\mathtt{w}, r_b)$ then $\mathtt{z}_n \notin \Gamma_b(e^{ic_k})$

If $\mathtt{z}_n \in \mathrm{B}(\mathtt{w}, r_b)$ then ($\$\$$) implies that $n \geq 10 + n_b$, and since $\mathtt{z}_n$ and e^{ic_k} both belong to O, (2.22) implies that

($\diamondsuit$) $\left[\tau(e^{ic_k}, \mathtt{z}_n)\right]^2 \leq 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{|4^{-d(n)}|^2 + |\theta_n - c_k|^2}$

Recall that $k \geq 10 + n_b$ and $n \geq 10 + n_b$. We now claim that

($\diamondsuit\diamondsuit$) $|\theta_n - c_k| \geq |\theta_n - c_n|$

Indeed, observe that

$$\theta_1 > c_1 > \theta_2 > c_2 > \ldots > \theta_{n-1} > c_{n-1} > \boldsymbol{\theta_n} > c_n > \theta_{n+1} > c_{n+1} > \ldots$$

hence ($\diamondsuit\diamondsuit$) will follow if we show that

($\triangleright$) $|\theta_n - c_{n-1}| \geq |\theta_n - c_n|$

Now, ($\triangleright$) amounts to show that $|\theta_n - \frac{1}{2}(\theta_{n-1} + \theta_n)| \geq |\theta_n - \frac{1}{2}(\theta_n + \theta_{n+1})|$, i.e.,

$$\frac{\theta_n - \theta_{n+1}}{\theta_{n-1} - \theta_n} = \frac{4^{-d(n-1)} - 4^{-d(n)}}{4^{-d(n-2)} - 4^{-d(n-1)}} = \frac{4^{-d(n-2)} - 4^{-3d(n-2)}}{1 - 4^{-d(n-2)}} < 1$$

and since

$$\frac{4^{-d(n-2)} - 4^{-3d(n-2)}}{1 - 4^{-d(n-2)}} = 4^{-d(n-2)} \frac{1 - 4^{-2d(n-2)}}{1 - 4^{-d(n-2)}} = 4^{-d(n-2)}(1 + 4^{-d(n-2)})$$

the conclusion follows from $n \geq 10 + n_b$. Now, ($\diamondsuit$) and ($\diamondsuit\diamondsuit$) imply that

$$\begin{aligned}
&\left[\tau(e^{ic_k}, \mathtt{z}_n)\right]^2 \leq 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + |\theta_n - c_n|^2} \\
&= 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + [2^{-1}(\theta_n - \theta_{n+1}]^2} = 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + 4^{-1}[4^{-d(n-1)} - 4^{-d(n)}]^2} \\
&= 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + 4^{-1}[4^{-d(n-1)}(1 - 4^{-d(n-1)})]^2} \\
&= 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + 4^{-1}4^{-d(n)}[1 - 4^{-d(n-1)}]^2} \\
&\leq 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + 4^{-1}4^{-d(n)}4^{-1}} = 64 \cdot 3^{-1} \frac{4^{-2d(n)}}{4^{-2d(n)} + 4^{-d(n)-2}} \\
&= 64 \cdot 3^{-1} \frac{4^{-d(n)+2}}{4^{-d(n)+2} + 1} \overset{(\triangleleft)}{\leq} 64 \cdot 3^{-1} \cdot 2 \cdot 4^{-d(n)} \overset{(\$)}{<} \frac{1}{(1+b)^2}
\end{aligned}$$

where ($\triangleleft$) follows by $0 \leq 4^2 + 2 \cdot 4^{d(n)}$. Hence $\tau(e^{ic_k}, \mathtt{z}_n) < \frac{1}{1+b}$, and the proof of ($\heartsuit\heartsuit$) is complete. $\square$

3.4 Proof of Proposition 1.2 (a)

Recall that $\mathtt{w} = 1.$ We now show that $\mathbf{a}[\mathtt{w}] \subset \mathbf{p}[\mathtt{w}]$. Since Lemma 1.1 and Proposition 1.2 (b) imply that $\mathbf{p}[\mathtt{w}] \setminus \mathbf{a}[\mathtt{w}] \neq \emptyset$, the proof that $\mathbf{a}[\mathtt{w}] \subsetneq \mathbf{p}[\mathtt{w}]$ will then be completed.

Proof. Let $\mathsf{A} \in \mathbf{a}[\mathtt{w}]$. We will show that $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$ in each of three possible cases.

(1) $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$, i.e., A ends nontangentially at $\mathtt{w}$.

(2) $\mathsf{A} \in \mathbf{o}[\mathtt{w}]$, i.e., A is very oscillatory at $\mathtt{w}$:

(3) $\mathsf{A} \in \mathbf{t}[\mathtt{w}]$, i.e., A ends tangentially at $\mathtt{w}$:

Case (1). We now show that if $\mathsf{A} \in \mathbf{a}[\mathtt{w}] \cap \boldsymbol{\gamma}[\mathtt{w}]$ then $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$. Recall that $\mathsf{A} \in \boldsymbol{\gamma}[\mathtt{w}]$ means that $0 < \mathsf{A}_\natural$, as in (2.5)-(2.9), i.e.,

$$\boxed{0 < \sup_{r>0} \inf_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z})}$$

Let $b \in \mathbb{N}$ such that $\dfrac{1}{1+b} < \dfrac{1}{2} \sup_{r>0} \inf_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z})$. Then there exists $r_1 > 0$ with $\dfrac{1}{1+b} < \dfrac{1}{2} \sup_{r>0} \inf_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z}) < \inf_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r_1)} \tau(\mathtt{w},\mathsf{z})$, hence

$$\frac{1}{1+b} < \tau(\mathtt{w},\mathsf{z}) \quad \forall\, \mathsf{z} \in \mathsf{A}(\mathtt{w}, r_1)$$

i.e., $\mathsf{A}(\mathtt{w}, r_1) \subset \Gamma_b(\mathtt{w})$. Let $r_0 \overset{\text{def}}{=\!=} \frac{r_1}{2}$. We claim that

$$\text{if } r \leq r_0 \text{ then (2.20) holds}$$

Indeed, if $r \leq r_0$, then $\mathsf{A}(\mathtt{w}, r) \subset \Gamma_b(\mathtt{w})$, and since $\mathsf{A}(\mathtt{w}, r) \neq \emptyset$, $\exists\, \mathsf{z}_r \in \mathsf{A}(\mathtt{w}, r)$. Hence $\mathsf{z}_r \in \Gamma_b(\mathtt{w})$, i.e., $\mathtt{w} \in \Gamma_b^*(\mathsf{z}_r)$, Since $\mathsf{z}_r \in \mathsf{A}(\mathtt{w}, r)$, (2.18) implies that

$$\Gamma_b^*(\mathsf{z}_r) \subset \Gamma_b^*(\mathsf{A}(\mathtt{w}, r))$$

Lemma 2.9 implies that $\Gamma_b^*(\mathsf{z}_r)$ is an arc, and we know that $\Gamma_b^*(\mathsf{z}_r)$ contains $\mathtt{w}$. It follows that $\Gamma_b^*(\mathsf{A}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$. Now, Lemma 2.11 (3) implies that $\mathsf{A} \in \mathbf{p}_b[\mathtt{w}]$. Q.E.D

Case (2). We now show that if $\mathsf{A} \in \mathbf{a}[\mathtt{w}] \cap \mathbf{o}[\mathtt{w}]$ then $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$. Recall that $\mathsf{A} \in \mathbf{o}[\mathtt{w}]$ means that $0 = \mathsf{A}_\natural < \mathsf{A}^\natural$, as in (2.5)-(2.9), i.e.,

$$\boxed{0 = \sup_{r>0} \inf_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z}) < \inf_{r>0} \sup_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z})}$$

Choose $b \in \mathbb{N}$ such that $\dfrac{1}{1+b} < \inf\limits_{r>0} \sup\limits_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z})$. Let $r > 0$. Then

$$\frac{1}{1+b} < \sup_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z})$$

thus $\exists\ \mathsf{z}_r \in \mathsf{A}(\mathtt{w},r)$ with $\frac{1}{1+b} < \tau(\mathtt{w},\mathsf{z}_r)$, i.e., $\mathsf{z}_r \in \Gamma_b(\mathtt{w}) \cap \mathsf{A}(\mathtt{w},r)$, and (2.18) implies that $\Gamma_b^*(\mathsf{z}_r) \subset \Gamma_b^*(\mathsf{A}(\mathtt{w},r))$ and since $\Gamma_b^*(\mathsf{z}_r)$ is an arc that contains $\mathtt{w}$, by Lemma 2.9, $\Gamma_b^*(\mathsf{A}(\mathtt{w},r)) \rightsquigarrow \mathtt{w}$, and hence $\mathsf{A} \in \mathbf{p}_b[\mathtt{w}]$. Q.E.D

Case (3). Let $\mathsf{A} \in \mathbf{a}[\mathtt{w}] \cap \mathbf{t}[\mathtt{w}]$. The following variant of (2.21) will be useful. If $0 < r < 1$ we define

$$O(r) \stackrel{\text{def}}{=\!=} \{(1-\delta)e^{i\theta} : |\delta| < r, |\theta| < r\} \tag{3.2}$$

Observe that $\mathrm{B}(1,r) \not\subset O(r)$, since $\left(\frac{1}{1+r^2}, \frac{r+\sqrt{r^2(1+r^2)^2-r^4}}{2(1+r^2)}\right) \in \mathrm{B}(1,r) \setminus O(r)$, but

$$O(r) \subset \mathrm{B}(1,2r) \text{ and } \mathrm{B}(1,r) \subset O(2r) \tag{3.3}$$

Moreover,

$$O(r) \cap \mathbb{D} = Q^-(r) \cup \{1-\delta : 0 < \delta < r\} \cup Q^+(r) \tag{3.4}$$

where

$$\begin{aligned} Q^-(r) &\stackrel{\text{def}}{=\!=} \{(1-\delta)e^{i\theta} : 0 < \delta < r, -r < \theta < 0\} \\ Q^+(r) &\stackrel{\text{def}}{=\!=} \{(1-\delta)e^{i\theta} : 0 < \delta < r, 0 < \theta < r\} \end{aligned} \tag{3.5}$$

Since $\mathsf{A} \in \mathbf{a}[\mathtt{w}] \cap \mathbf{t}[\mathtt{w}]$, $\exists\ \mathrm{S} \in \mathcal{P}(\mathbb{D})$ such that $\{\mathtt{w}\} \cup \mathrm{S}$ is connected and $\mathsf{A} \sim_{\mathtt{w}} \mathrm{S}$. Let $\rho_0 \stackrel{\text{def}}{=\!=} \frac{1}{2}\sup\{|\mathtt{w} - s| : s \in S\}$ and let $\rho_1 \in (0, \min\{\rho_0, 1/10\})$ with

(∗0) $\mathsf{A} \cap \mathrm{B}(\mathtt{w},\rho_1) = S \cap \mathrm{B}(\mathtt{w},\rho_1)$.

Since $\mathsf{A} \in \mathbf{t}[\mathtt{w}]$, $0 = \sup\limits_{r>0} \inf\limits_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z}) = \inf\limits_{r>0} \sup\limits_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r)} \tau(\mathtt{w},\mathsf{z})$, hence

(∗1) $\forall\, \epsilon > 0\ \exists\, r_\epsilon \in (0,\rho_1)$ such that $\sup_{\mathsf{z} \in \mathsf{A}(\mathtt{w},r_\epsilon)} \tau(\mathtt{w},\mathsf{z}) < \epsilon$.

In particular, (∗1) implies that there exists $\rho_2 \in (0, \dfrac{1}{2}\rho_1)$ such that

(∗2) $\sup_{\mathsf{z} \in \mathsf{A}(\mathtt{w},\rho_2)} \tau(\mathtt{w},\mathsf{z}) < 8/25$

and there exists a sequence $\{\mathsf{z}_n\}_{n\in\mathbb{N}}$ with $\mathsf{z}_n = (1-\delta_n)e^{i\theta_n}$ such that

(∗3) $\mathsf{z}_n \in \mathsf{A} \cap \mathrm{S} \cap \mathrm{B}(1,\rho_2)$ and $\tau(\mathtt{w},\mathsf{z}_n) < 1/(2+n)$ for each $n \in \mathbb{N}$;
$0 < \delta_{n+1} < \delta_n$ and $|\theta_{n+1}| < |\theta_n|$ $\forall n \in \mathbb{N}$; $\lim\limits_{n\to+\infty} \mathsf{z}_n = 1$.

Since $\mathtt{z}_n \in \mathrm{B}(1, 1/10) \cap \mathbb{D} \subset O(2/10) \cap \mathbb{D}$ and $\tau(\mathtt{w}, \mathtt{z}_n) < 1$ it follows from (3.4) that either $\mathtt{z}_n$ belongs to $Q^+(2/10)$ or to $Q^-(2/10)$. Indeed, if $\mathtt{z}_n$ belonged to the set $\{1 - \delta : 0 < \delta < r\}$ in (3.4), then $\tau(\mathtt{w}, \mathtt{z}_n)$ would be equal to 1, since $\mathtt{w} = 1$. Hence there are three possibilities:

(i) $\exists\, n_0 \in \mathbb{N}$ such that $\mathtt{z}_n \in Q^+(2/10)$ for all $n \geq n_0$;

(ii) $\exists\, n_0 \in \mathbb{N}$ such that $\mathtt{z}_n \in Q^-(2/10)$ for all $n \geq n_0$;

(iii) a subsequence of the sequence $\{\mathtt{z}_n\}_{n \in \mathbb{N}}$ belongs to $Q^+(2/10)$.

Remark 3.1. We claim that it suffices to prove that $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$ under the assumption that (i) holds. Indeed, as it will be clear momentarily, the case (ii) is symmetrical to (i), and the case (iii) is implicit in (i).

Proof of $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$ assuming that (i) holds. Let $\rho_3 \overset{\text{def}}{=\!=} \dfrac{1}{4}\rho_2$ and let

$$\mathtt{V} \overset{\text{def}}{=\!=} \{r \in (0, \rho_3] : \exists\, \delta \in (0, \rho_3) \text{ with } (1-\delta)e^{ir} \in \mathsf{A}\}$$

and

$$\mathtt{W} \overset{\text{def}}{=\!=} \{r \in (0, \rho_3] : (0, r) \subset \mathtt{V}\}.$$

Claim 1.

(⊙) $\mathtt{W} \neq \emptyset$

Proof Claim 1. In order to prove (⊙), we observe that either (i) $\rho_3 \in \mathtt{W}$ or (ii) $\rho_3 \notin \mathtt{W}$. If (i) holds then the claim is proved. If (ii) holds then $\exists$ $r_1 \in (0, \rho_3)$ such that $r_1 \notin \mathtt{V}.$ Observe that $(*3)$ implies that $\exists\, n_1 \in \mathbb{N}$ with

(∗4) $0 < \theta_{n_1} < r_1$ and $0 < \delta_{n_1} < \rho_3$

Let $\theta_* \overset{\text{def}}{=\!=} \frac{1}{2}(\theta_{1+n_1} + \theta_{n_1})$. Then either (a) $\theta_* \in \mathtt{W}$ or (b) $\theta_* \notin \mathtt{W}$. If (a) holds then the claim holds. Assume that (b) holds. Then $\exists$ $r_2 \in (0, \theta_*)$ such that $r_2 \notin \mathtt{V}.$ It follows that

[↯] $0 < r_2 < \theta_{n_1} < r_1 < \rho_3, \quad \delta_{n_1} < \rho_3, \quad r_1 \notin \mathtt{V}, \text{ and } r_2 \notin \mathtt{V}.$

We claim that (↯) yields a contradiction. Indeed, consider the open set

(∗5) $D \overset{\text{def}}{=\!=} \{(1-\delta)e^{ir} : r_2 < r < r_1,\ 0 < \delta < \rho_3\}$

and observe that (↯) implies that $\mathtt{z}_{n_1} \in D$, and hence $D \cap \mathrm{S} \neq \emptyset.$ Now consider the open set $D' \overset{\text{def}}{=\!=} \mathbb{C} \setminus \mathsf{cl}(D)$ and observe that $D' \cap (\{\mathtt{w}\} \cup \mathrm{S}) \neq \emptyset$ since $\mathtt{w} \in D'$ and, moreover, since $\rho_3 < \rho_0$. Since $\{\mathtt{w}\} \cup \mathrm{S}$ is connected and

$$D \cap (\{\mathtt{w}\} \cup \mathrm{S}) \neq \emptyset \text{ and } D' \cap (\{\mathtt{w}\} \cup \mathrm{S}) \neq \emptyset$$

it follows that

$$(\{\mathtt{w}\} \cup \mathrm{S}) \cap \partial D \neq \emptyset$$

and hence $\exists$ $\mathsf{z}_0 \in (\{\mathtt{w}\} \cup \mathrm{S}) \cap \partial D.$ Since $\mathsf{z}_0 \in \partial D$ implies that $\mathtt{z} \neq \mathtt{w}$, it follows that $\mathsf{z}_0 \in \mathrm{S} \cap \partial D$. Since $\mathrm{S} \subset \mathbb{D}$, it follows that one of the following holds

[$\not\downarrow$1] $\mathsf{z}_0 \in \{(1-\delta)e^{ir_2} : 0 < \delta < \rho_3\}$

[$\not\downarrow$2] $\mathsf{z}_0 \in \{(1-\delta)e^{ir_1} : 0 < \delta < \rho_3\}$

[$\not\downarrow$3] $\mathsf{z}_0 \in \{(1-\rho_3)e^{ir} : r_2 < r < r_1\}$

Since

($) $\partial D \subset \mathsf{cl}(O(\rho_3)) \subset O(2\rho_3) = O(\frac{1}{2}\rho_2) \subset \mathrm{B}(\mathtt{w}, \rho_2) \subset \mathrm{B}(\mathtt{w}, \rho_1)$

then $(*0)$ and $(*2)$ imply that $\mathsf{z}_0 \in \mathsf{A}$ and $\tau(\mathtt{w}, \mathsf{z}_0) < 8/25.$ Observe that then ($\not\downarrow$1) yields a contradiction with the fact that $r_2 \notin \mathtt{V}$. Similarly, ($\not\downarrow$2) yields a contradiction with the fact that $r_1 \notin \mathtt{V}$. If ($\not\downarrow$3) holds, then

$$\mathsf{z}_0 = (1-\rho_3)e^{ir} \text{ with } 0 < r_2 < r < r_1 < \rho_3$$

and $\tau(\mathtt{w}, \mathsf{z}_0) < \frac{8}{25}$. Hence

$$(1 - |\,\mathsf{z}_0\,|)^2 < \frac{8^2}{25^2}|\,\mathsf{z}_0 - \mathtt{w}\,|^2 < \frac{8^2}{25^2}\frac{125}{64}({\rho_3}^2 + r^2)$$

thus ${\rho_3}^2 < \frac{1}{5}({\rho_3}^2 + r^2) = \frac{1}{5}{\rho_3}^2 + \frac{1}{5}r^2$ hence $\frac{4}{5}{\rho_3}^2 < \frac{1}{5}r^2$ and therefore

($\not\downarrow\not\downarrow$) $2\rho_3 < r$

in contradiction with $0 < r < r_1 < \rho_3$. The proof of ($\odot$) is complete. Q.E.D

We now show that $\mathtt{W} \neq \emptyset$ implies that $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$. Since $\mathtt{W} \neq \emptyset$, there exists $r_3 \in \mathtt{W}$. Let $r_0 \overset{\text{def}}{=\!=} \frac{1}{4}r_3$.

Claim 2.

($\heartsuit$) for each $r \in (0, r_0]$, $\quad \Gamma_0^*(\mathsf{A}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$

Proof of Claim 2. Let $r \in (0, r_0]$. We claim that

($\heartsuit\heartsuit$) $\{e^{i\theta} : 0 < \theta < r/2\} \subset \Gamma_0^*(\mathsf{A}(\mathtt{w}, r))$

Since the set $\{e^{i\theta} : 0 < \theta < r/2\}$ belongs to $\mathtt{Arcs}(\mathtt{w})$, ($\heartsuit$) follows from ($\heartsuit\heartsuit$).

If $0 < \theta < r/2$, then $\theta \in \mathtt{V}$, since $r_3 \in \mathtt{W}$, hence there exists $\delta \in (0, \rho_3)$ such that $(1-\delta)e^{i\theta} \in \mathsf{A}.$ Since $(1-\delta)e^{i\theta} \in O(\rho_3)$, $(1-\delta)e^{i\theta} \in \mathsf{A}(\mathtt{w}, \rho_2)$, by ($), and $(*2)$ implies that $\tau(\mathtt{w},(1-\delta)e^{i\theta}) < 8/25$, thus $\left(\tau(\mathtt{w},(1-\delta)e^{i\theta})\right)^2 < 8^2/25^2$, hence

$$(1-|(1-\delta)e^{i\theta}|)^2 \leq \frac{8^2}{25^2}|\,\mathtt{w}-(1-\delta)e^{i\theta}|^2$$

thus $\delta^2 \leq \frac{8^2}{25^2}\frac{125}{64}(\delta^2+\theta^2) = \frac{1}{5}\delta^2+\frac{1}{5}\theta^2$, hence $\frac{4}{5}\delta^2 \leq \frac{1}{5}\theta^2$, therefore $0 < \delta < \dfrac{1}{2}\theta.$

Summing up, we have proved the following statement:

(&1) if $0 < \theta < r/2$ then $\exists\,\delta \in (0,\theta/2)$ with $(1-\delta)e^{i\theta} \in \mathsf{A}$

Since $(1-\delta)e^{i\theta} \in O(r/2) \subset \mathrm{B}(\mathtt{w},r)$, by (3.3), and $(1-\delta)e^{i\theta} \in \mathsf{A}$, it follows that $(1-\delta)e^{i\theta} \in \mathsf{A}(\mathtt{w},r)$, hence (&1) says that

(&2) if $0 < \theta < r/2$ then $\exists\,\delta \in (0,\theta/2)$ with $(1-\delta)e^{i\theta} \in \mathsf{A}(\mathtt{w},r)$

and since $(1-\delta)e^{i\theta} \in \Gamma_0(e^{i\theta})$, it follows that $e^{i\theta} \in \Gamma_0^*((1-\delta)e^{i\theta})$. Hence

(&3) $\forall\theta \in (0,r/2)\ \exists\,\delta$ with $e^{i\theta} \in \Gamma_0^*((1-\delta)e^{i\theta})$ and $(1-\delta)e^{i\theta} \in \mathsf{A}(\mathtt{w},r)$

and this statement implies ($\heartsuit\heartsuit$). The proof of the claim is complete. Now observe that Lemma 2.11 implies that $\mathsf{A} \in \mathbf{p}_0[\mathtt{w}]$, hence $\mathsf{A} \in \mathbf{p}[\mathtt{w}]$.

Conclusion of the proof. As we observed in Remark 3.1, the case (ii) is symmetrical to (i) and proceeds along similar lines, while if (iii) holds then one can proceed as in (i), using the subsequence instead of the original sequence. Hence the proof is concluded. $\square$

4 Proof of Theorem 1.3

We assume that $\mathbf{A}$ satisfies the hypotheses of Theorem 1.3.

Denote by $\mathsf{h}^\infty(\mathbb{D})$ the Banach space of real-valued and harmonic functions defined on $\mathbb{D}$ and bounded therein. If $\mathsf{u} \in \mathsf{h}^\infty(\mathbb{D})$ and $\mathtt{w} \in \partial\,\mathbb{D}$, we define

$$\operatorname{osc}(\mathsf{u},w) \stackrel{\text{def}}{=\!=} \limsup_{\substack{\mathsf{z}\to\mathtt{w}\\ \mathsf{z}\in\mathbf{A}(\mathtt{w})}} \mathsf{u}(\mathsf{z}) - \liminf_{\substack{\mathsf{z}\to\mathtt{w}\\ \mathsf{z}\in\mathbf{A}(\mathtt{w})}} \mathsf{u}(\mathsf{z})$$

In order to prove Theorem 1.3, it suffices to prove the following statement

$$\text{there exists } \mathsf{u} \in \mathsf{h}^\infty(\mathbb{D}) \text{ with } \mathsf{u} > 0 \text{ and } \operatorname{osc}(\mathsf{u},\mathtt{w}) > 0 \text{ for a.e. } \mathtt{w} \in \partial\,\mathbb{D} \tag{4.1}$$

Indeed, if v is the harmonic conjugate to u then $\mathsf{h} \stackrel{\text{def}}{=\!=} e^{-\mathsf{u}-i\mathsf{v}}$ will have the required properties.

4.1 Carleson tents and the Zygmund map

The **Carleson tent above** an arc $\mathtt{arc}[\mathtt{y};\theta] \subset \partial\mathbb{D}$ is defined by

$$\Delta(\mathtt{arc}[\mathtt{y};\theta]) \stackrel{\text{def}}{=\!=} \mathbb{D} \cap \mathrm{B}(\mathtt{y}, |\mathtt{y} - \mathtt{y}e^{i\frac{\theta}{2}}|). \tag{4.2}$$

The **Zygmund map** is the map $\mathtt{Z} \in [\![\mathcal{P}(\partial\mathbb{D}) \to \mathcal{P}(\partial\mathbb{D})]\!]$ defined as follows: If $\mathtt{V} \subset \partial\mathbb{D}$ then $\mathtt{Z}(\mathtt{V}) \subset \partial\mathbb{D}$ is the set of points of $\partial\mathbb{D}$ with the following properties:

$$\text{(i) } \mathtt{w} \in \partial\mathbb{D} \setminus \mathtt{V}; \quad \text{(ii) } \forall \epsilon > 0 \, \exists \text{ arc } \mathtt{J}_\epsilon \subset \mathtt{V} \cap \mathrm{B}(\mathtt{w}, \epsilon) \text{ with } \mathbf{A}(\mathtt{w}) \cap \Delta(\mathtt{J}_\epsilon) \neq \emptyset \tag{4.3}$$

4.2 Poisson integrals and the basic Carleson tent estimate

If $\mathtt{f} \in [\![\partial\mathbb{D} \to \mathbb{R}]\!]$ is Lebesgue integrable then the **Poisson integral** of $\mathtt{f}$ is denoted by $\mathsf{P}[\mathtt{f}]$. Hence $\mathsf{P}[\mathtt{f}] \in [\![\mathbb{D} \to \mathbb{R}]\!]$.

If $\mathtt{S} \subset \partial\mathbb{D}$ then $\mathtt{1}_\mathtt{S} \in [\![\partial\mathbb{D} \to \mathbb{R}]\!]$ is defined by

$$\mathtt{1}_\mathtt{S}(\mathtt{w}) = 1 \text{ if } \mathtt{w} \in \mathtt{S}, \ \mathtt{1}_\mathtt{S}(\mathtt{w}) = 0 \text{ if } \mathtt{w} \notin \mathtt{S}.$$

The following result is well-known (cf. [8], [44]). The constant $c_0 > 0$ that appears therein is called the **Carleson tent constant**. It is not important to determine its precise value, and we will not do it.

Lemma 4.1 *There exists a constant $c_0 \in (0, +\infty)$ with the following property: For each arc $\mathtt{J} \subset \partial\mathbb{D}$ and each $\mathtt{z} \in \Delta(\mathtt{J})$, $\mathsf{P}[\mathtt{1}_\mathtt{J}](\mathtt{z}) \geq c_0$.*

Proof. See e.g. Lemma 3.6 in [14]. □

Lemma 4.2 *If $\mathtt{V} \subset \partial\mathbb{D}$ is open then, for all $\mathtt{w} \in \mathtt{Z}(\mathtt{V})$,*

$$\limsup_{\substack{\mathtt{z}\to\mathtt{w} \\ \mathtt{z}\in\mathbf{A}(\mathtt{w})}} \mathsf{P}[\mathtt{1}_\mathtt{V}](\mathtt{z}) \geq c_0$$

Proof. It follows from Lemma 4.1, since if $\mathtt{J} \subset \mathtt{V}$ then $\mathsf{P}[\mathtt{1}_\mathtt{J}] \leq \mathsf{P}[\mathtt{1}_\mathtt{V}]$. □

4.3 Basic constructions

The following constructions will be employed in the proof of Theorem 1.3.

Recall that since $\mathbf{A}(\mathtt{w}) \in \mathbf{p}[\mathtt{w}]$ for each $\mathtt{w} \in \partial\mathbb{D}$ and $\mathbf{p}[\mathtt{w}] = \bigcup_{b\in\mathbb{N}} \mathbf{p}_b[\mathtt{w}]$, then for each $\mathtt{w} \in \partial\mathbb{D}$ there exists $b = b(\mathtt{w}) \in \mathbb{N}$ such that $\mathbf{A}(\mathtt{w}) \in \mathbf{p}_b[\mathtt{w}]$. Since

$$\mathbf{p}_b[\mathtt{w}] \subset \mathbf{p}_{b+1}[\mathtt{w}] \tag{4.4}$$

we may assume, without loss of generality, that

$$b(\mathtt{w}) \geq 10 \quad \forall \, \mathtt{w} \in \partial\mathbb{D} \tag{4.5}$$

Consider the sequence of everywhere defined functions $f_j : \partial\mathbb{D} \to (0,\infty)$ gauging the order of tangency of $\mathbf{A}(\mathtt{w})$ at various points:

$$f_j(\mathtt{w}) \stackrel{\text{def}}{=\!=} \sup\{\tau(\mathtt{w},\mathtt{z}) : \mathtt{z} \in \mathsf{A}(\mathtt{w}, (11/10)\cdot 2\pi/j)\} \tag{4.6}$$

Consider the sequence $v \in [\![\mathbb{N} \to \mathbb{N}]\!]$ whose values are, in the natural order:

$$1+1, 1+2, 1+1, 1+2, 1+3, 1+1, 1+2, 1+3, 1+4, 1+1, 1+2, 1+3, \ldots$$

so that for each $b' \in \mathbb{N}\setminus\{1\}$ there are infinitely many values of $j \in \mathbb{N}$ such that $v_j = b'$. Since $\mathbf{A}(\mathtt{w})$ ends tangentially at $\mathtt{w}$, for each $\mathtt{w} \in \partial\mathbb{D}$ the sequence $\{f_n(\mathtt{w})\}_{n\in\mathbb{N}}$ decreases to 0. Moreover, since $\mathbf{A}$ is regular, for each $n \in \mathbb{N}$ the function $f_n : \partial\mathbb{D} \to (0,1]$ is measurable. Egorov's Theorem [38] implies that for each $j \in \mathbb{N}$ then there is a set $\mathtt{C}_j \subset \partial\mathbb{D}$ whose Lebesgue measure is greater than $2\pi - 2^{-j}$ and such that the sequence $\{f_n\}$ converges uniformly to 0 on $\mathtt{C}_j$. Observe that if a sequence $\{f_n\}$ converges uniformly to 0 on a set $\mathtt{A}$ and on a set $\mathtt{B}$, then it converges uniformly to 0 on $\mathtt{A}\cup\mathtt{B}$. Hence we may assume that

$$\mathtt{C}_j \subset \mathtt{C}_{j+1} \quad \forall\, j \in \mathbb{N} \tag{4.7}$$

It follows that there exists $\phi \in [\![\mathbb{N} \to \mathbb{N}]\!]$ such that, for each $j \in \mathbb{N}$

$$\text{(i)}\ \sup\left\{f_{\phi_j}(\mathtt{v}) : \mathtt{v} \in \mathtt{C}_j\right\} < c\frac{10}{22}\frac{2^{-j}}{v_j},\ \text{(ii)}\ \phi_j > j,\ \text{(iii)}\ \phi_{j+1} > \phi_j \tag{4.8}$$

where c is a constant that will be chosen momentarily. Define

$$\mathtt{C} \stackrel{\text{def}}{=\!=} \bigcup_{j\in\mathbb{N}} \mathtt{C}_j \tag{4.9}$$

Then $\mathtt{C} \stackrel{\text{a.e.}}{=} \partial\mathbb{D}$. Let $\mathtt{w} \in \partial\mathbb{D}$ and $|\theta| \le 2\pi$. Recall that $\mathtt{arc}[\mathtt{w};\theta]$, defined in (1.6), is the arc in $\partial\mathbb{D}$ centered at $\mathtt{w}$ of arc-length θ. If $\mathtt{Y} \subset \partial\mathbb{D}$ we define

$$\cup_{\mathtt{arcs}}[\mathtt{Y};\theta] \stackrel{\text{def}}{=\!=} \bigcup_{\mathtt{w}\in\mathtt{Y}} \mathtt{arc}[\mathtt{w};\theta]$$

If $n \in \mathbb{N}$ and $n \ge 1$, we define

$$\mathtt{U}_n \stackrel{\text{def}}{=\!=} \{e^{i2\pi p/n} : p = 0, 1, \ldots, n-1\}. \tag{4.10}$$

Then $|\cup_{\mathtt{arcs}}[\mathtt{U}_n; 2\pi/n]| = 2\pi$, since $\cup_{\mathtt{arcs}}[\mathtt{U}_n; 2\pi/n]$ is the union of n disjoint arcs whose union is equal to $\partial\mathbb{D}\setminus\{\mathtt{u}_n{}^p e^{i\pi/n} : p = 0, 1, \ldots n-1\}$. Indeed,

$$\cup_{\mathtt{arcs}}[\mathtt{U}_n; 2\pi/n] = \bigcup_{\mathtt{w}\in\mathtt{U}_n} \mathtt{arc}[\mathtt{w}; 2\pi/n] = \bigcup_{p=0}^{n-1} \mathtt{arc}[e^{i2\pi p/n}; 2\pi/n]$$

and the arcs appearing in the union are disjoint. Define

$$\mathtt{O}_n \stackrel{\text{def}}{=\!=} \cup_{\mathtt{arcs}}[\mathtt{U}_{\phi_n}; 2^{-n}2\pi/\phi_n] \tag{4.11}$$

Observe that $|\mathtt{O}_n| = 2^{-n}2\pi$ and hence $\lim_{j\to+\infty}\left|\bigcup_{k\geq j}\mathtt{O}_k\right| = 0$. Define

$$\mathtt{V}_j \stackrel{\text{def}}{=\!=} \bigcup_{k\geq j}\mathtt{O}_k \tag{4.12}$$

The set $\mathtt{V}_j$ is open and dense in $\partial\mathbb{D}$, and, if j is large, it has small measure.

4.4 Basic lemmata

Lemma 4.3 *If* $\mathbf{A}$ *satisfies the hypotheses of Theorem 1.3 then each* $n\in\mathbb{N}$, $\mathtt{C}\setminus\mathtt{V}_n \subset \mathtt{Z}(\mathtt{V}_n)$.

Proof. Before we delve into the proof, we would like to explain why it is more involved than one would perhaps think. It is indeed true that, given $\mathtt{w},\mathtt{v}\in\partial\mathbb{D}$, with $\mathtt{w}\neq\mathtt{v}$, and given $b\in\mathbb{N}$ and $r>0$ there exists $n=n(\mathtt{w},\mathtt{v},b,r)\in\mathbb{N}$ such that $\Gamma_b(\mathtt{v})\setminus\Gamma_n(\mathtt{w})\subset \mathrm{B}(\mathtt{v},r)$: However, in this statement, n depends upon $\mathtt{w}$, $\mathtt{v}$, b, and r, while, in the statement that we need to prove, the variables appear in a different order and hence the dependence of n on the other variables is not admissible.

Let $\mathtt{w}\in\mathtt{C}\setminus\mathtt{V}_n$. Then there exists $d_o\in\mathbb{N}$ with $\mathtt{w}\in\mathtt{C}_{d_o}$, and it follows that

$$j\geq d_o \implies \mathtt{w}\in\mathtt{C}_j \text{ and hence (4.8) implies that } f_{\phi_j}(\mathtt{w})\leq c\frac{10}{22}\frac{2^{-j}}{v_j} \tag{4.13}$$

Moreover,

$$\forall j\geq n,\ \mathtt{w}\notin\mathtt{O}_j \tag{4.14}$$

and there exists $b'\in\mathbb{N}$ such that

$$\mathbf{A}(\mathtt{w})\in\mathbf{p}_{b'}[\mathtt{w}],\ b'\geq 10 \tag{4.15}$$

Let $\epsilon>0$. Since $\mathbf{A}(\mathtt{w})\in\mathbf{t}[\mathtt{w}]$, there exists $r>0$ such that

$$\sup\{\tau(\mathtt{w},\mathtt{z}):\mathtt{z}\in\mathsf{A}(\mathtt{w},r)\} < \frac{1}{10}\frac{1}{(1+b')^2} \text{ and } r<\epsilon\cdot 2^{-10} \tag{4.16}$$

Since $\mathbf{A}(\mathtt{w})\in\mathbf{p}_{b'}[\mathtt{w}]$, it follows that

$$\Gamma^*_{b'}(\mathsf{A}(\mathtt{w},r)) \rightsquigarrow \mathtt{w} \tag{4.17}$$

We may assume, without loss of generality, that there exists $\mathtt{J}\in\mathtt{Arcs}(\mathtt{w}\leftarrow)$ such that $\mathtt{J}\subset\Gamma^*_{b'}(\mathsf{A}(\mathtt{w},r))$, i.e., there exists $\theta>0$ such that

$$\text{(i) } \mathtt{arc}_+[\mathtt{w};\theta]\equiv\{\mathtt{w}\,e^{i\beta}:0<\beta<\theta\}\subset\Gamma^*_{b'}(\mathsf{A}(\mathtt{w},r)),\ \text{(ii) } \theta<r\cdot 2^{-10}. \tag{4.18}$$

Select $j\in\mathbb{N}$ such that

$$\text{(i) } j>n,\quad \text{(ii) } 2\pi/\phi_j<\theta\cdot 2^{-10},\quad \text{(iii) } j>d_o,\quad \text{and (iv) } v_j=1+b' \tag{4.19}$$

It follows that there exists $p \in \mathbb{Z}$ such that if $\mathtt{v} \stackrel{\text{def}}{=\!=} e^{i2\pi p/\phi_j}$ then

$$\mathtt{w} \in \mathtt{arc}_+[\mathtt{v}; 2\pi/\phi_j] \equiv \{\mathtt{v}e^{i\alpha} : 0 < \alpha < 2\pi/\phi_j\} \tag{4.20}$$

Let $\mathtt{y} \stackrel{\text{def}}{=\!=} e^{i2\pi(p+1)/\phi_j} \equiv \mathtt{v}e^{i2\pi/\phi_n}$. Since $\mathtt{w} \notin \mathtt{O}_j$ it follows that

$$\mathtt{w} \notin \mathtt{arc}[\mathtt{v}; 2^{-j}\frac{2\pi}{\phi_j}] \text{ and } \mathtt{w} \notin \mathtt{arc}[\mathtt{y}; 2^{-j}\frac{2\pi}{\phi_j}] \tag{4.21}$$

We claim that $\mathtt{y} \in \mathtt{arc}_+[\mathtt{w}; \theta]$. Indeed, (4.20) and (4.19) (ii) imply that $\mathtt{w} = \mathtt{v}e^{i\alpha}$ with $0 < \alpha < 2\pi/\phi_j < \theta \cdot 2^{-10}$ and $\mathtt{y} = \mathtt{v}e^{i2\pi/\phi_j} = \mathtt{v}e^{i\alpha}e^{i(2\pi/\phi_j - \alpha)} = \mathtt{w}\,e^{i(2\pi/\phi_j-\alpha)}$ with $0 < 2\pi/\phi_j - \alpha < 2\pi/\phi_j < \theta \cdot 2^{-10} < \theta$. Hence

$$\mathtt{y} = \mathtt{w}\,e^{i\beta} \text{ with } 0 < \beta < 2\pi/\phi_j \tag{4.22}$$

where $\beta \stackrel{\text{def}}{=\!=} 2\pi/\phi_j - \alpha$. Since $0 < \beta < 2\pi/\phi_j < \theta/20 < \theta$, the claim is proved.

Since $\mathtt{y} \in \mathtt{arc}_+[\mathtt{w}; \theta]$, (4.18) implies that $\mathtt{y} \in \Gamma^*_{b'}(\mathsf{A}(\mathtt{w}, r))$, hence

$$\text{there exists } \tilde{\mathtt{z}} \in \mathsf{A}(\mathtt{w}, r) \text{ such that } \tilde{\mathtt{z}} \in \Gamma_{b'}(\mathtt{y}) \tag{4.23}$$

Since $\tilde{\mathtt{z}} \in \mathsf{A}(\mathtt{w}, r)$, (4.16) implies that $\tau(\mathtt{w}, \tilde{\mathtt{z}}) < \frac{1}{10}\frac{1}{(1+b')^2}$, i.e.

$$1 - |\tilde{\mathtt{z}}| < |\,\mathtt{w} - \tilde{\mathtt{z}}|\frac{1}{10}\frac{1}{(1+b')^2} \tag{4.24}$$

Since $\tilde{\mathtt{z}} \in \Gamma_{b'}(\mathtt{y})$ it follows that $\tau(\mathtt{y}, \tilde{\mathtt{z}}) > 1/(1+b')$, i.e.

$$|\mathtt{y} - \tilde{\mathtt{z}}| < (1 - |\tilde{\mathtt{z}}|)(1+b') \tag{4.25}$$

Then (4.24) and (4.25) imply that

$$|\mathtt{y} - \tilde{\mathtt{z}}| < |\,\mathtt{w} - \tilde{\mathtt{z}}|\frac{1}{10}\frac{1}{1+b'} \tag{4.26}$$

Then $|\,\mathtt{w} - \tilde{\mathtt{z}}| \le |\,\mathtt{w} - \mathtt{y}| + |\mathtt{y} - \tilde{\mathtt{z}}| < |\,\mathtt{w} - \mathtt{y}| + |\,\mathtt{w} - \tilde{\mathtt{z}}|\frac{1}{10}\frac{1}{1+b'}$, hence

$$|\,\mathtt{w} - \tilde{\mathtt{z}}|\left[1 - \frac{1}{10}\frac{1}{1+b'}\right] < |\,\mathtt{w} - \mathtt{y}| \tag{4.27}$$

Observe that (4.22) implies that

$$|\,\mathtt{w} - \mathtt{y}| < 2\pi/\phi_j \tag{4.28}$$

and then (4.27) implies that

$$|\,\mathtt{w} - \tilde{\mathtt{z}}| < \frac{1 + \frac{1}{b'}}{1 + \frac{9}{10b'}}\frac{2\pi}{\phi_j} < \frac{11}{10}\frac{2\pi}{\phi_j} \tag{4.29}$$

Hence

$$\tilde{\mathtt{z}} \in \mathsf{A}(\mathtt{w}, \frac{11}{10}\frac{2\pi}{\phi_j}) \tag{4.30}$$

and thus (4.6) implies that

$$\tau(\mathtt{w},\tilde{\mathtt{z}}) < f_{\phi_j}(\mathtt{w}) \tag{4.31}$$

Since $\mathtt{w} \in \mathtt{C}_j$, (4.8) and (4.19) imply that

$$f_{\phi_j}(\mathtt{w}) < c\frac{10}{22}\frac{2^{-j}}{v_j} = c\frac{10}{22}\frac{2^{-j}}{1+b'} \tag{4.32}$$

(4.31) and (4.32) imply that $\tau(\mathtt{w},\tilde{\mathtt{z}}) = \frac{1-|\tilde{\mathtt{z}}|}{|\mathtt{w}-\tilde{\mathtt{z}}|} < c\frac{10}{22}\frac{2^{-j}}{1+b'}$, i.e.,

$$1 - |\tilde{\mathtt{z}}| < c|\mathtt{w}-\tilde{\mathtt{z}}|\frac{10}{22}\frac{2^{-j}}{1+b'} \tag{4.33}$$

and now (4.25), (4.33), and (4.29) imply that, thanks to (2.22)

$$|\mathtt{y} - \tilde{\mathtt{z}}| < c\frac{1}{2}2^{-j}\frac{2\pi}{\phi_j} \leq c\sqrt{\frac{64}{3}}|\mathtt{y} - \mathtt{y}e^{i\frac{2^{-j}2\pi/\phi_j}{2}}| = |\mathtt{y} - \mathtt{y}e^{i\frac{2^{-j}2\pi/\phi_j}{2}}| \tag{4.34}$$

where $c = \sqrt{\frac{3}{64}}$ and hence

$$\tilde{\mathtt{z}} \in \Delta(\mathtt{arc}[\mathtt{y}; 2^{-j}\frac{2\pi}{\phi_j}]) \tag{4.35}$$

This fact concludes the proof. □

Lemma 4.4 *If* $\mathbf{A}$ *satisfies the hypotheses of Theorem 1.3,* $\mathtt{V} \subset \partial\mathbb{D}$ *is open, and* $|\partial\mathbb{D}\backslash\mathtt{V}| > 0$*, then*

$$\partial\mathbb{D}\backslash\mathtt{V} \overset{\text{a.e.}}{\subset} \left\{ \mathtt{w} \in \partial\mathbb{D} : \liminf_{\substack{\mathtt{z}\to\mathtt{w} \\ \mathtt{z}\in\mathbf{A}(\mathtt{w})}} (\mathsf{P}[\mathtt{1}_{\mathtt{V}}])(\mathtt{z}) = 0 \right\} \tag{4.36}$$

Proof. If we denote by $\mathtt{Q}$ the set in the right-hand side of (4.36), then we have to prove that $\partial\mathbb{D}\backslash\mathtt{V} \overset{\text{a.e.}}{\subset} \mathtt{Q}$, i.e., that $(\partial\mathbb{D}\backslash\mathtt{V}) \setminus \mathtt{Q} \in \boldsymbol{\mathcal{N}}$. The proof of Lemma 4.4 rests on the following claim.

Claim 4.5. For each $\delta > 0$ there exists $\mathtt{E} \in \mathcal{P}(\partial\mathbb{D})$ with $|\mathtt{E}| < \delta$ and

$$(\partial\mathbb{D}\backslash\mathtt{V}) \setminus \mathtt{E} \overset{\text{a.e.}}{\subset} \mathtt{Q} \tag{4.37}$$

Claim 4.5 and (1.8) imply that $(\partial\mathbb{D}\backslash\mathtt{V})\setminus\mathtt{Q} \overset{\text{a.e.}}{\subset} \mathtt{E}$, and hence $|(\partial\mathbb{D}\backslash\mathtt{V})\setminus\mathtt{Q}| < \delta$, and since $\delta > 0$ is arbitrary, it follows that $(\partial\mathbb{D}\backslash\mathtt{V}) \setminus \mathtt{Q} \in \boldsymbol{\mathcal{N}}$, thus completing the proof of Lemma 4.4.

Proof of Claim 4.5. Let $\delta > 0$. A well-known extension of Fatou's Theorem to Poisson integrals (see e.g. [44]) says that

$$\mathtt{Fatou}(\mathsf{P}[1_{\mathtt{V}}]) \overset{\text{a.e.}}{=} \partial\mathbb{D} \quad \text{and} \quad (\mathsf{P}[1_{\mathtt{V}}])_{\mathfrak{b}}(\mathtt{w}) = 1_{\mathtt{V}}(\mathtt{w}) \;\; \forall\, \mathtt{w} \in \mathtt{Fatou}(P(1_{\mathtt{V}})).$$

It follows that $(\mathsf{P}[1_{\mathtt{V}}])_{\mathfrak{b}}(\mathtt{w}) = 0 \quad$ for a.e. $\mathtt{w} \in \partial\mathbb{D}\setminus\mathtt{V}$. We denote by $\mathtt{X}$ the subset of $\partial\mathbb{D}\setminus\mathtt{V}$ of full measure in $\partial\mathbb{D}\setminus\mathtt{V}$ with the property that $(\mathsf{P}[1_{\mathtt{V}}])_{\mathfrak{b}}(\mathtt{w}) = 0$ for all $\mathtt{w} \in \mathtt{X}$, and define, for each $(j,b) \in \mathbb{N}\times\mathbb{N}$, the function

$$g_{(j,b)} : \mathtt{X} \to (0,1] \quad \text{by } g_{(j,b)}(\mathtt{w}) \overset{\text{def}}{=} \sup\{\mathsf{P}[1_{\mathtt{V}}](\mathtt{z}) : \mathtt{z} \in \Gamma_b(\mathtt{w}) \cap \mathrm{B}(\mathtt{w}, 1/j)\}.$$

Observe that $\forall b, j \in \mathbb{N},\; g_{(j,b)} \geq g_{(j+1,b)}$ and $g_{(j,b)} \leq g_{(j,b+1)}$ and that

$$\forall\, \mathtt{w} \in \mathtt{X},\; \lim_{j\to+\infty} g_{(j,b)}(\mathtt{w}) = 0$$

Egorov's theorem implies that for each $k \geq 1$ there exists a set $\mathtt{E}(k) \subset \mathtt{X}$ with

(a) $|\mathtt{E}(k)| < \delta \cdot 2^{-k}$;

(b) the sequence $\{g_{(j,k)}\}_{j\in\mathbb{N}}$ converges to zero uniformly on $\mathtt{X}\setminus\mathtt{E}(k)$;

(c) the set $\mathtt{X}\setminus\mathtt{E}(k)$ is perfect.

Let $\mathtt{E} \overset{\text{def}}{=} \cup_{k\geq 1}\mathtt{E}(k)$. Then $|\mathtt{E}| < \delta$, $\mathtt{X}\setminus\mathtt{E} = \cap_{k\geq 1}\mathtt{X}\setminus\mathtt{E}(k) \subset \mathtt{X}\setminus\mathtt{E}(b)$ for each $b \geq 1$. The proof of Claim 4.5 rests on the following result.

Claim 4.6.

$$\mathtt{X}\setminus\mathtt{E} \overset{\text{a.e.}}{\subset} \mathtt{Q} \tag{4.38}$$

Observe that, since $\mathtt{X} \subset \partial\mathbb{D}\setminus\mathtt{V}$ has full measure in $\partial\mathbb{D}\setminus\mathtt{V}$, (4.38) implies (4.37) and concludes the proof of Claim 4.5 and hence of Lemma 4.4.

Proof of Claim 4.6. Let $\mathtt{w} \in \mathtt{X}\setminus\mathtt{E}$ and let $b \in \mathbb{N}$ such that $\mathbf{A}(\mathtt{w}) \in \mathbf{p}_b[\mathtt{w}]$. Then $\mathtt{w} \in \mathtt{X}\setminus\mathtt{E}(b)$. Since $\mathtt{X}\setminus\mathtt{E}(b)$ is perfect, there is a subset of $\mathtt{X}\setminus\mathtt{E}(b)$, which is at most countable, such that, if $\mathtt{w} \in \mathtt{X}\setminus\mathtt{E}(b)$ does not belong to this subset, then there exists a sequence of points of $\mathtt{X}\setminus\mathtt{E}(b)$ that converges to $\mathtt{w}$ from the right and one that converges to $\mathtt{w}$ from the left.

Let $\epsilon > 0$. Since the sequence $\{g_{(j,b)}\}_{j\in\mathbb{N}}$ converges to zero uniformly on $\mathtt{X}\setminus\mathtt{E}(b)$, there exists $n_0 \in \mathbb{N}$ such that if $j \geq n_0$ then

$$\sup\{(\mathsf{P}[1_{\mathtt{V}}])(\mathtt{z}) : \mathtt{z} \in \Gamma_b(\mathtt{x}) \cap \mathrm{B}(\mathtt{x}, 1/j)\} = g_{(j,b)}(\mathtt{x}) < \epsilon \quad \forall\, \mathtt{x} \in \mathtt{X}\setminus\mathtt{E}(b) \tag{4.39}$$

Let $r < \frac{2^{-10}}{n_0}$. Since $\mathbf{A}(\mathtt{w}) \in \mathbf{p}_b[\mathtt{w}]$, $\Gamma_b^*(\mathsf{A}(\mathtt{w}, r)) \rightsquigarrow \mathtt{w}$. We may assume, without loss of generality, that $\Gamma_b^*(\mathsf{A}(\mathtt{w}, r))$ is adjacent to $\mathtt{w}$ from the right. Hence there exists $\theta > 0$ such that

$$\mathtt{arc}_+[\mathtt{w}; \theta] \subset \Gamma_b^*(\mathsf{A}(\mathtt{w}, r)) \tag{4.40}$$

We may assume, without loss of generality, that $\theta < 2^{-10}r$. Let $\theta_0 > 0$ with $\theta_0 < \theta$ and $\mathtt{w}\, e^{i\theta_0} \in \mathtt{X} \setminus \mathtt{E}(b)$. Then (4.40) implies that

$$\mathtt{w}\, e^{i\theta_0} \in \Gamma_b^*(\mathsf{z}) \quad \text{for some } \mathsf{z} \in \mathsf{A}(\mathtt{w}, r)$$

hence $\mathsf{z} \in \Gamma_b(\mathtt{w}\, e^{i\theta_0}) \cap \mathbf{A}(\mathtt{w}) \cap \mathrm{B}(\mathtt{w}, r)$. Observe that

$$|\,\mathsf{z} - \mathtt{w}\, e^{i\theta_0}| \le |\,\mathsf{z} - \mathtt{w}\,| + |\,\mathtt{w} - \mathtt{w}\, e^{i\theta_0}| < r + \theta_0 < \frac{2^{-10}}{n_0} + \frac{2^{-20}}{n_0} < \frac{1}{n_0}$$

Since $\mathsf{z} \in \mathrm{B}(\mathtt{w}\, e^{i\theta_0}, \frac{1}{n_0}) \cap \Gamma_b(\mathtt{w}\, e^{i\theta_0})$ and $\mathtt{w}\, e^{i\theta_0} \in \mathtt{X} \setminus \mathtt{E}(b)$ then (4.39) (with $\mathtt{x} = \mathtt{w}\, e^{i\theta_0}$) implies that

$$(\mathsf{P}[1_{\mathtt{V}}])(\mathsf{z}) \le g_{(n_0,b)}(u) < \epsilon$$

where $\mathsf{z} \in \mathbf{A}(\mathtt{w}) \cap \mathrm{B}(\mathtt{w}, r)$. Thus, we have proved that $\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}$ such that $\forall\, r \in (0, \frac{2^{-10}}{n_0})$ there exists $\mathsf{z} \in \mathbf{A}(\mathtt{w}) \cap \mathrm{B}(\mathtt{w}, r)$ with $(\mathsf{P}[1_{\mathtt{V}}])(\mathsf{z}) < \epsilon$, and this implies that

$$\liminf_{\substack{\mathsf{z}\to\mathtt{w} \\ \mathsf{z}\in\mathbf{A}(\mathtt{w})}} (\mathsf{P}[1_{\mathtt{V}}])(\mathsf{z}) = 0, \text{ i.e., } \mathtt{w} \in \mathtt{Q}. \tag{4.41}$$

Hence we have proved that $\mathtt{w} \in \mathtt{Q}$ for a.e. $\mathtt{w} \in \mathtt{X} \setminus \mathtt{E}$. $\square$

4.5 Proof of Theorem 1.3

Observe that, for each $j \in \mathbb{N}$, $\mathsf{P}[1_{\mathtt{V}_j}] \in \mathsf{h}^\infty(\mathbb{D})$. The basic lemmata in Section 4.4 and the Carleson tent estimate imply that for a.e. $\mathtt{w} \in \mathtt{C} \setminus \mathtt{V}_j$

$$\operatorname{osc}(\mathsf{P}[1_{\mathtt{V}_j}], \mathtt{w}) \ge c_0.$$

where c_0 is the Carleson tent constant. Let $s \stackrel{\text{def}}{=\!=} 1 + \frac{1+c_0}{c_0}$. Following [50],

$$\mathtt{f} = \sum_{j\ge1} s^{-j}\, 1_{\mathtt{V}_j}$$

It follows that

$$\mathsf{P}[\mathtt{f}] = \sum_{j\ge1} s^{-j}\, \mathsf{P}[1_{\mathtt{V}_j}]$$

Define $\mathtt{W} \stackrel{\text{def}}{=\!=} \cap_{j\ge1} \mathtt{V}_j$ and observe that $|\mathtt{W}| = 0$.

Claim 4.7. For a.e. $\mathtt{w} \in \mathtt{C} \setminus \mathtt{W}$, $\operatorname{osc}(\mathsf{P}[\mathtt{f}], \mathtt{w}) > 0$.

Since the set $\mathtt{C} \setminus \mathtt{W}$ has full measure in $\partial\,\mathbb{D}$, Claim 4.7 completes the proof of Theorem 1.3.

Proof of Claim 4.7. Let j be the smallest integer n such that $\mathtt{w} \notin \mathtt{V}_n$. Then $\mathtt{w}$ belongs to the open set

$$\bigcap_{k=1}^{j-1} \mathtt{V}_k. \tag{4.42}$$

For $k = 1, 2, \ldots, j-1$, the function $\mathtt{1}_{\mathtt{V}_k}$ is equal to 1 on the set (4.42); since this set is open, it follows that $\operatorname{osc}(\mathsf{P}[\mathtt{1}_{\mathtt{V}_k}], \mathtt{w}) = 0$ for each $k = 1, 2, \ldots, l-1$. On the other hand, $\operatorname{osc}(s^{-j}\,\mathsf{P}[\mathtt{1}_{\mathtt{V}_j}], \mathtt{w}) \geq s^{-j} c_0$ and

$$\operatorname{osc}(\sum_{k \geq j+1} s^{-k}\,\mathsf{P}[(\mathtt{1}_{\mathtt{V}_k})], \mathtt{w}) \leq \sum_{k \geq j+1} s^{-k} \leq s^{-j} \frac{1}{s-1}$$

It follows that

$$\operatorname{osc}(\mathsf{P}[\mathtt{f}], \mathtt{w}) \geq s^{-j} c_0 - s^{-j} \frac{1}{s-1} > 0$$

Since the set $(\mathtt{C} \setminus \mathtt{W}) \setminus \mathtt{N}$ has measure equal to 2π, the proof of Claim 4.7 is completed.

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