

# THRESHOLD GRAPHS ARE GLOBALLY SYNCHRONIZING

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ABSTRACT. The Kuramoto model describes phase oscillators on the unit circle whose interactions are encoded by a graph. Each edge acts like a spring that pulls the two adjacent oscillators toward each other whenever their phases differ. A central question is to determine which graphs are globally synchronizing, meaning that trajectories of the Kuramoto dynamics converge to the fully synchronized state from almost all initial conditions. This property is tightly linked to the benign nonconvexity of the model’s energy landscape. Existing guarantees for global synchronization rely on minimum-degree thresholds, which require the graph to be highly dense. In this work, we show that connected threshold graphs, whose density may vary from  $2/n$  to 1, are globally synchronizing. Our proof relies on a phasor-geometric analysis of the stationary points of the associated energy landscape.

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## 1. INTRODUCTION

The study of synchronization traces back to the 17th century, when Christiaan Huygens first observed the spontaneous synchronization of pendulum clocks. Since then, synchronization phenomena have been widely studied across both natural and technological systems. For example, the

flashing of fireflies [BB68, SS93], the rhythmic beating of heart cells [Win67, MS06], synchronization in power grids [DB12], coordinated motion in robotic networks [OS06], modern artificial intelligence models such as neural networks [WCP<sup>+</sup>24], and Transformer architecture [GLPR25, CRMB24].

A major obstacle in the study of synchronization was the absence of a model that simultaneously offered analytical tractability and sufficient dynamical richness. The Kuramoto model, introduced by Yoshiki Kuramoto in 1975 [Kur75], provided precisely such a balance, and has since become a central and widely accepted model for analyzing synchronization phenomena.

The Kuramoto model describes  $n$  oscillators on the unit circle, coupled according to a graph  $G = (V, E)$  with adjacency matrix  $A$ . Mathematically, it is a set of ordinary differential equations

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^n A_{ij} \sin(\theta_j - \theta_i), \quad i \in V, \quad (1)$$

where  $\theta_i : \mathbb{R} \rightarrow \mathbb{S}^1$  denotes the phase of oscillator  $i$ . Specifically, each oscillator is driven by its intrinsic frequency  $\omega_i$  and by the nonlinear interactions determined by the phase differences with its neighbors.

Early studies of the Kuramoto model primarily considered systems of oscillators with heterogeneous intrinsic frequencies  $\{\omega_i\}_{i=1}^n$ , coupled through a complete graph with uniform weights  $A_{ij} = K/n$ . Here,  $K$  denotes the strength of the coupling between oscillators. A fundamental question in this setting is to determine the critical coupling strength  $K_c$ , such that for  $K > K_c$ , the oscillators converge to a common frequency, a phenomenon known as *frequency synchronization*<sup>1</sup>, and for  $K < K_c$ , they remain desynchronized. This problem is of central interest across many scientific domains, ranging from biological synchronization, rhythmic phenomena [MMJ87], technological applications such as deep brain stimulation [Tas03, NM11] and power systems [CCC95, DB12]. For comprehensive surveys of this line of work and its applications, we refer to [Str00, ABPV<sup>+</sup>05, DB14].

Another line of research focuses on the homogeneous Kuramoto model where all intrinsic frequencies are identical, i.e.,  $\omega_i = \omega$  for all  $i \in \{1, \dots, n\}$ . In this case, frequency synchronization is always achieved. This becomes immediate after applying a co-rotating frame  $\theta_i(t) \leftarrow \theta_i(t) - \omega t$ , under which the dynamics reduce to the gradient system

$$\frac{d\theta_i}{dt} = \sum_{j=1}^n A_{ij} \sin(\theta_j - \theta_i), \quad i \in V. \quad (2)$$

By LaSalle’s invariance principle, every trajectory converges to the invariant set of equilibria satisfying  $\dot{\theta}_i = 0$  for all  $i$ . Thus, in the co-rotating frame, the asymptotic frequency of every oscillator is zero. Transforming back to the original coordinates, this implies that the oscillators always achieve frequency synchronization. In contrast, when intrinsic frequencies are non-identical, the dynamics may exhibit significantly more complex long-term behavior, including chaotic attractors [MPT05].

In the homogeneous setting, the remaining challenge is to determine whether *phase synchronization* occurs. This clearly depends on the initial conditions. If all oscillators start within a half circle, they converge to a common phase. However, beyond the half-circle region, the global behavior remained poorly understood for a long time.

Over the past two decades, a central question that has emerged is the following: under what graph structures does the homogeneous Kuramoto system achieve phase synchronization for all initial conditions, except for a set of measure zero? This problem is not only mathematically appealing but also closely connected to several recent developments across different research areas, including the Burer–Monteiro approach for semidefinite programming [BBV16, LXB19, Lin23] and the dynamics of Transformer architectures [GLPR25, GLPR23].

In this work, we follow this line of research, and beginning with a precise formulation of the problem.

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<sup>1</sup>That is, all derivatives  $\frac{d\theta_i(t)}{dt}$  become identical as  $t \rightarrow \infty$ .

### Problem Setting.

**Definition 1.1** (Globally synchronizing graph). We say that a graph  $G = (V, E)$  is *globally synchronizing* if, for all initial conditions  $\boldsymbol{\theta}(0) \in \mathbb{R}^n$  except for a set of measure zero, the solution to the Kuramoto model (2) on  $G$  satisfies

$$\lim_{t \rightarrow \infty} (\theta_i(t) - \theta_j(t)) = 0 \quad \text{for all } i, j \in V.$$

The key question is now stated as:

*Which graphs are globally synchronizing?*

To build some intuition for the model (2) and the definition (1.1), one may imagine placing  $n$  oscillators on the unit circle and connecting them by edges specified by the adjacency matrix  $A$ . If there is an edge between nodes  $i$  and  $j$ , then whenever their phases differ, the interaction tends to reduce this difference and pull them toward a common phase.

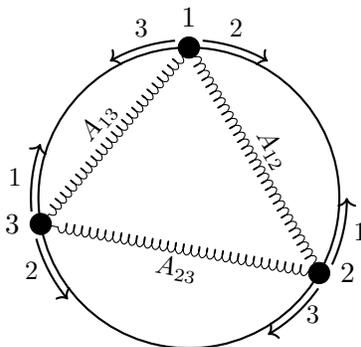


FIGURE 1. A spring analogy for a coupled oscillator network. The label on each curved arrow marks which node exerts the force.

From the above illustration of the complete graph  $K_3$ , one may ask whether the three oscillators will eventually converge to a common position when placed as shown in Figure 1. Each node is subject to two forces, for instance, node 1 is pulled toward node 2 on one side and toward node 3 on the other. So it is not immediately clear in which direction it will move, let alone whether all three nodes will ultimately coincide. Even in this simple example, the difficulty of the problem becomes apparent. Nevertheless, one may form an intuitive if somewhat heuristic expectation: since in a complete graph each oscillator tends to move closer to every other oscillator, the abundance of interactions may indeed drive the system to synchronize from almost all initial configurations.

Historically, Verwoerd and Mason [VM07] conjectured in 2007 that every complete graph  $K_n$  ( $n \geq 2$ ) is globally synchronizing. This conjecture was resolved by Richard Taylor in 2012 [Tay12], and global synchronization of complete graphs has since become a standard benchmark result in this line of research. For classical results on the Kuramoto model on complete graphs, we refer to [ABPV+05, Str00, PR15, BGLM20].

In the same work [Tay12], Taylor also initiated what is now known as the *extremal combinatorics* version of the problem [BKMR25], which sparked substantial interest and subsequent developments. A more detailed discussion of these advances will be given in *Prior Work*.

One standard approach to this problem is to study the corresponding energy landscape. Note that the Kuramoto model (2) can be formulated as a gradient flow of a certain energy function. Precisely, the system of ordinary differential equations (2) defines a *gradient dynamical system* (or

called *gradient flow*<sup>2</sup>) with energy function  $E_G(\boldsymbol{\theta})$ , satisfying

$$\frac{d\boldsymbol{\theta}}{dt} = -\nabla E_G(\boldsymbol{\theta}) \quad (3)$$

where  $E_G(\boldsymbol{\theta}) : (\mathbb{S}^1)^{|V|} \rightarrow \mathbb{R}$  with graph  $G = (V, E)$  is defined as

$$\begin{aligned} E_G(\boldsymbol{\theta}) &:= \frac{1}{2} \sum_{1 \leq i, j \leq n} \mathbf{A}_{ij} (1 - \cos(\theta_i - \theta_j)) \\ &= \frac{1}{2} \sum_{1 \leq i, j \leq n} \mathbf{A}_{ij} (1 - \langle v_i, v_j \rangle). \end{aligned} \quad (4)$$

Here,  $v_i$  is the unit vector  $(\cos \theta_i, \sin \theta_i) \in \mathbb{R}^2$ .

Clearly, this function is nonconvex. Moreover, understanding the geometry of (4) provides direct insight into the long-term dynamics of (2). In particular, the synchronous states correspond exactly to the global minima of this energy function, and those minima that are local but not global (called *spurious local minima*) correspond to non-synchronous stable equilibria. If such spurious local minima exist, the gradient dynamics may converge to them whenever the initialization lies within their basin of attraction, in which case the graph  $G$  is not globally synchronizing. In fact, if one can show that the energy function (4) admits no spurious second-order stationary point (in such case we say the function is *benign*), then the graph  $G$  is globally synchronizing. A rigorous proof of this statement can be established using the Łojasiewicz gradient inequality [Lag07] together with the center manifold theorem; see Lemma A.1 in [GLPR25] for an excellent exposition.

**Prior Work.** Existing guarantees for global synchronization on deterministic graphs focus on the *extremal combinatorics* version of the problem. The aim is to identify minimum-degree thresholds above which a graph is guaranteed to be globally synchronizing.

**Problem 1.** Let  $\delta(G)$  denote the minimum degree of  $G$ . Find the smallest constant  $\mu_c \in [0, 1]$  such that

$$\delta(G) \geq \mu_c(n-1) \implies G \text{ is globally synchronizing.}$$

The intuition behind this problem is that when every node is connected to a sufficiently large fraction of the network, the graph becomes dense enough for the collective attractive forces to guarantee convergence to synchrony. A remarkable property of these degree-based criteria is that, although they are only sufficient, they apply any graph meeting the threshold. This means global synchronization is guaranteed regardless of the graph's finer structural details.

Several advances on this problem have been made in [Tay12, LXB19, LS21, KST21, Can22, YTT21]. The currently best known bounds on the critical minimum-degree threshold  $\mu_c$  satisfy  $0.6875 \leq \mu_c \leq 0.75$ , where the upper bound is established in [KST21], and the strongest known lower bound follows from [Can22]. The conjecture asserts that the minimum-degree threshold for global synchronization is exactly  $\mu_c = 0.75$ , and that below this density, even highly connected graphs may fail to synchronize. A formal statement of this conjecture can be found as Conjecture 5 (Density threshold for global synchrony) in [BKMR25].

Global synchrony on random graphs constitutes another important line of inquiry. Ling, Xu, and Bandeira [LXB19] proved that for Erdős–Rényi graphs, global synchronization occurs with high probability as  $n \rightarrow \infty$  provided that the edge probability satisfies  $p \gg \log(n)/n^{1/3}$ . This bound was later improved by Kassabov, Strogatz, and Townsend to  $p \gg \log^2(n)/n$  [KST22]. Both works raised the conjecture that synchronization should already occur as soon as the graph is connected, that is, for any  $\epsilon > 0$  and  $p \geq (1 + \epsilon) \log(n)/n$ . This conjecture was recently resolved in [ABK<sup>+</sup>24].

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<sup>2</sup>We call a dynamical system a *gradient flow* on  $E(x)$  if it has the form  $\frac{dx(t)}{dt} = -\nabla E(x)$ . Its trajectories evolve along the steepest descent direction of  $E$ .

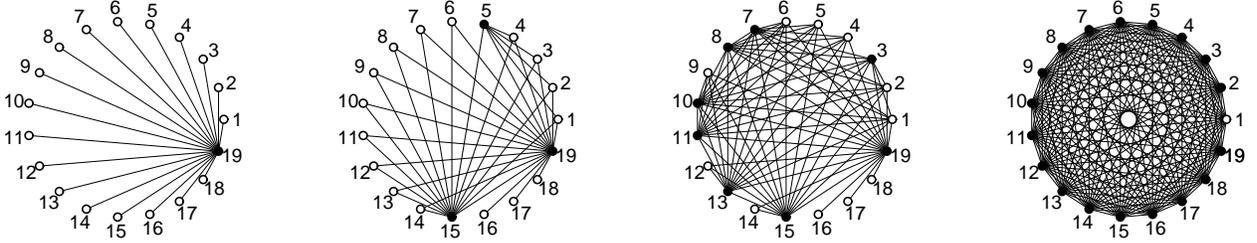


FIGURE 2. Four connected threshold graphs on 19 vertices, from the sparsest ( $S_{18}$ ) to the densest ( $K_{19}$ ). Node labels indicate the construction order: vertex 1 is the initial white vertex, and all subsequent labels follow the binary code of the construction. White and black circles denote isolated and dominating vertices (bit 0 and bit 1). Codes from left to right: 000000000000000001, 00010000000010001, 010001101101010001, 111111111111111111.

Further improvement has been made in [JMS25, McR25]. For results on global synchrony of random geometric graphs, see [ABI24].

Interestingly, recent work has pointed out that the idealized *transformer dynamics* admits a similar optimization form as the Kuramoto energy function [GLPR25]. This gives rise to the so-called *nonlinear* setting of the Kuramoto model, obtained by placing a nonlinear function on top of the inner product  $E_f(\boldsymbol{\theta}) = \sum_{1 \leq i, j \leq n} A_{ij} f(\cos(\theta_i - \theta_j))$ . The classical Kuramoto case corresponds to the linear function  $f(t) = t$ , while in the *transformer dynamics* one takes  $f(t) = e^{\beta t}$  where  $\beta > 0$ . In this setting, the central question is to determine the values of  $\beta$  for which global synchronization occurs, that is, the tokens collapse into a single cluster. Equivalently, this means that all token are making uniform predictions. A well-known conjecture predicted that global synchronization holds for all  $\beta \geq 0$ . The problem was originally formulated in [GLPR25], with an initial result for  $\beta \geq \Omega(1/n)$ . It was subsequently improved in [CRMB24] to cover the range  $\beta \in [0, 1]$ , and was very recently settled by Polyanskiy, Rigollet, and Yao [PRY25]. For more work connecting the Kuramoto model and Transformer (or attention) dynamics, see [GLPR23, KPR24, CLPR25, PRY25, Rig25, GKPR24] and the references therein.

**Our Contribution.** Existing sufficient conditions for global synchronization of deterministic graphs require the graph to be sufficiently dense. In this work, we present a class of graphs that does not satisfy this minimum-degree requirement and whose densities range from  $2/n$  to 1, yet globally synchronizes nevertheless. Through this result. Our main result is the following:

**Theorem 1.2.** *Any connected threshold graph globally synchronizes.*

In the following, we briefly introduce the definition of threshold graphs and the motivation for studying their global synchronization. We conclude this section with a discussion of the structural mechanism that underlies the global synchronization of this graph class.

(1) *What Are Threshold Graphs?* Threshold graphs can be constructed starting from a single vertex by iteratively adding each new vertex as either: an isolated vertex connecting to no existing vertex, or a dominating vertex connecting to all existing vertices. This generative process is uniquely encoded by a binary sequence of length  $n-1$ , where 0 denotes an isolated vertex and 1 a dominating one. Note that the binary sequence of any connected threshold graph must end with a 1; otherwise, the graph is not connected since there exists at least one isolated node. A full formal definition and additional characterizations are given in Section 2.

(2) *Why Threshold Graphs?* Many theorems begin with initial observations from which suitable conjectures are formulated, and our case is no exception. The investigation into global synchronization on threshold graphs began with several simple observations:

*Observation 1.* Star graphs and complete graphs are respectively the densest and sparsest connected threshold graphs, and both exhibit global synchronization.

$$\begin{array}{ccc} \underbrace{(0 \ 0 \ \dots \ 0, \ 1)}_{\text{all 0}} & & \underbrace{(1 \ 1 \ \dots \ 1)}_{\text{all 1}} \\ \text{Star graph } S_n & & \text{Complete graph } K_n \end{array}$$

Certainly, this is not sufficient to propose the conjecture that all threshold graphs globally synchronize. Indeed, although the two extreme cases synchronize, this does not by itself imply that the entire spectrum of threshold graphs does. However, two further observations strengthened our confidence in this conjecture.

*Observation 2.* One of the many equivalent definitions of threshold graphs requires that their neighborhood-inclusion preorder is total, i.e., the neighborhoods of any two nodes are nested: for any pair  $u, v$ , either  $N(u) \subseteq N[v]$  or  $N(v) \subseteq N(u)$ . This total preorder of vicinities [MP95] suggests that influence may propagate step by step from the most dominant node to the rest without interference, hinting at the possibility of global synchronization.

*Observation 3.* While far from exhaustive, our numerical experiments did not reveal any non-synchronous stable equilibria on connected threshold graphs. Our experiments are twofold: (a) To examine global synchronization precisely in small systems, we employed computational algebraic geometry following [MDDH15]. More explicitly, we reformulated the equilibrium equations of the Kuramoto model as a system of polynomial equations and used the Bertini software package [BHSW13] to compute all solutions. We then evaluated the Jacobian at each solution to determine its stability. (b) For larger size systems (up to 200), using `ode45` in MATLAB, we initialized a large number of random initial conditions and observed that all trajectories converged to the fully synchronized state.

(3) *Differences Between Our Approach and Existing Techniques.* Many existing proofs of global synchronization rely, either explicitly or implicitly, on the so-called *half-circle lemma*. This lemma states that if all oscillators are confined to an open half-circle, then their dynamics will contract toward synchrony. Based on this observation, the bootstrap strategy attempts to show that all second-order stationary points must lie within a half-circle. If successful, this rules out the existence of non-synchronous state that the dynamics may converge to and thereby implies global synchronization.

However, for threshold graphs, it remains unclear how to apply this strategy. It typically depends on sufficient connectivity in the network thus naturally suited to dense graphs, where the large number of edges ensures that nodes inside the half-circle collectively exert sufficient force to pull outside nodes back into alignment. In contrast, threshold graphs span the full spectrum from extremely sparse examples such as stars to dense cliques, and their highly inhomogeneous, layered structure complicates the application of this strategy. As a result, existing techniques offer limited insight into whether and why these graphs synchronize globally.

In this work, we bypass the half-circle approach by reducing the high-dimensional geometric question of whether the energy landscape is benign to a planar vector-geometric condition. This condition naturally leads us to study twins in the graph-theoretic sense, as well as a broader notion that we introduce, which we call *geometric twins*. The latter depends jointly on the graph structure and on the equilibrium configuration  $\theta$ . These notions clarify how the relative positions of phases behave at second-order stationary point of the energy landscape  $E(\theta)$ . Building on this understanding, we employ an inductive strategy: synchronization is first established locally on small substructures, and then shown to propagate across the whole graph along the neighborhood-inclusion hierarchy of threshold graphs.

(4) *Understanding the Mechanism Behind Global Synchrony in Threshold Graphs.* Previous global synchronization results focus on density-based conditions. For instance, any graph whose minimum degree exceeds  $0.7889(n-1)$  is globally synchronizing [LS20]. In other words, these results require the graph to be sufficiently dense. Threshold graphs, however, lie outside this regime. Depending on the positions of dominating vertices in the generating sequence, their density can vary from  $O(2/n)$  (stars) to 1 (cliques), and many intermediate densities occur. Nevertheless, we prove in this work that they also exhibit global synchronization. What deeper insight can we draw from this result?

Our analysis shows that the global synchronization of threshold graphs is not a consequence of density. Instead, it arises from a different mechanism, namely the presence of local symmetries. Some of these symmetries come from the graph structure itself: as we show in Lemma 5.2, closed twins must synchronize at any second-order stationary point. There are also symmetries that do not originate purely from combinatorial structure, that is, certain configurations exhibit a form of geometric symmetry. As we will show in Section 5.3, in the configurations we call synchronous pendants, pairs of vertices that are not structurally symmetric nonetheless become geometrically equivalent at second-order stationary points and hence synchronize. Such symmetric configurations arise inductively along the construction process of threshold graphs. Their synchronization propagates backward from later-added vertices to earlier ones, which shows that every second-order stationary point is a synchronous state. Consequently, every threshold graph is globally synchronizing.

**Notation.** Throughout this work, we consider only connected simple graphs, i.e., graphs without self-loops or weighted edges. Following the convention of dynamical systems, we refer to the variable  $\theta = (\theta_1, \dots, \theta_n)$  of the Kuramoto model (2) as a *state*, and call the solution  $\theta(t)$  with respect to the initial condition  $\theta(0)$  in the Kuramoto model (2) a *trajectory* starting from  $\theta(0)$ . Moreover,  $\mathbb{S}^1 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$  denotes the unit circle. And  $\mathbf{x} \uparrow \mathbf{y}$  means that  $\mathbf{x}$  and  $\mathbf{y}$  are nonzero and positively collinear:  $\mathbf{x} = \lambda \mathbf{y}$  for some  $\lambda > 0$ . We write  $u \sim v$  to indicate that vertices  $u$  and  $v$  are adjacent in a graph, and  $u \approx v$  otherwise. For a vertex  $i$  and a subset  $Q$  of vertices, we denote

$$N_Q(i) = \{q \in Q : q \sim i\},$$

the set of neighbors of  $i$  that lie in  $Q$ .

## 2. PRELIMINARIES OF THRESHOLD GRAPHS

Threshold graphs are a well-studied class of graphs that play a central role in graph theory and have found applications across various disciplines such as scheduling theory, psychology, computer science [MP95]. Threshold graphs have been defined by Chvátal and Hammer as follows [CH73, CH77].

**Definition 2.1.** A graph  $G$  is called a *threshold graph* if it can be constructed from the one-vertex graph by repeatedly applying one of the following two operations:

- (i) add an *isolated* vertex, adjacent to no existing vertex;
- (ii) add a *dominating* vertex, adjacent to all existing vertices.

The resulting graph on vertices  $\{1, \dots, n\}$  is uniquely encoded by the binary sequence  $(b_2, \dots, b_n)$ , where  $b_i = 0$  if vertex  $i$  is added as isolated and  $b_i = 1$  if it is added as dominating.

Although in this work we rely only on the construction process of threshold graphs to analyze their global synchronization, it is of independent interest to recall that they admit several equivalent characterizations [MP95].

**Theorem 2.2.** *Let  $G = (V, E)$  be a graph. The following statements are equivalent:*

- (i)  $G$  is a threshold graph.
- (ii)  $G$  is obtained by iteratively adding isolated or dominating vertices to the one-vertex graph.

(iii)  $G$  admits a weight function  $w : V \rightarrow \mathbb{R}$  and a threshold  $t \in \mathbb{R}$  such that

$$\{u, v\} \in E \iff w(u) + w(v) \geq t, \quad \forall u \neq v.$$

(iv)  $G$  is  $\{P_4, C_4, 2K_2\}$ -free.

(vi) For any distinct  $u, v \in V$ ,  $N(u) \subseteq N[v]$  or  $N(v) \subseteq N[u]$ .

**Remark 2.3** (Edge densities of threshold graphs). *The edge density of threshold graphs ranges from  $O(1/n)$  to 1. Indeed, if  $(b_2, \dots, b_n)$  is the generating sequence, then*

$$|E| = \sum_{i:b_i=1} (i-1), \quad \text{dens}(G) = \frac{|E|}{\binom{n}{2}}.$$

By choosing the positions at which  $b_i = 1$ , one obtains connected threshold graphs whose densities range from  $2/n$  (stars) to 1 (the complete graph), spanning many intermediate values. Moreover, threshold graphs are exactly the graphs whose degree sequences saturate the Erdős–Gallai inequalities in the characteristic Ferrers pattern corresponding to nested neighborhoods [MP95].

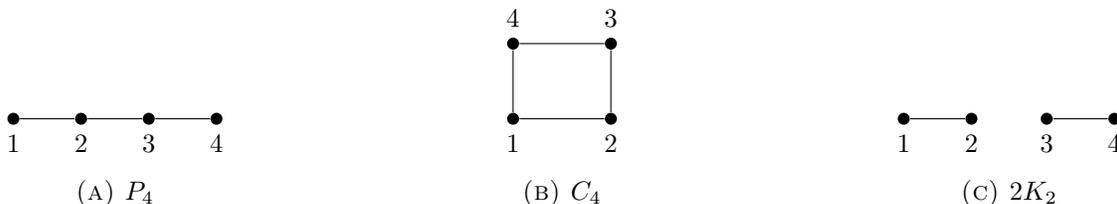


FIGURE 3. Forbidden induced subgraphs for threshold graphs.

**Remark 2.4.** *Interestingly, threshold graphs appeared in the synchronization literature long before their role in the Kuramoto model was considered. To the best of our knowledge, the earliest instance dates back to 1977, with the seminal work of Ausiello, Messina, and Protasi [AMP77], who provided a graph-theoretic characterization of the  $PV_c$  class of synchronizing primitives and introduced the notion of  $PV_c$ -definable graphs. This line of work emerged almost simultaneously with the independent introduction of threshold graphs by Chvátal and Hammer in their study of aggregation of inequalities in integer programming [CH77].*

### 3. PRELIMINARIES ON KURAMOTO MODEL

In this section, we introduce two families of concepts for the Kuramoto model: one arising from dynamical systems and the other from nonconvex optimization. Presenting these notions side by side is important, as many of the local synchronization lemmas developed later rely on interpreting stability to infer the relative positions of phasors at second-order stationary points of the energy landscape.

**3.1. Equilibrium and Local Stability.** In this section, we introduce the notion of *equilibrium* and *stable equilibrium* of the Kuramoto model (2).

By the definition, equilibrium points of a dynamical system are the points where the system remains at rest, meaning that the first-order derivative of the state vanishes [Kha02]. Setting the right-hand side of (2) to zero provides the characterization of an equilibrium point in the homogeneous Kuramoto model on the graph  $G$ .

**Definition 3.1** (Equilibrium of the Kuramoto model (2)). We call  $\theta \in \mathbb{R}^n$  an *equilibrium* of the Kuramoto model (2) if it satisfies

$$\sum_{j=1}^n \mathbf{A}_{ij} \sin(\theta_j - \theta_i) = 0, \quad \forall i \in [n]. \quad (5)$$

An equilibrium point is considered stable if small perturbations around it remain small over time [Kha02]. The linearization of a nonlinear system provides a necessary condition for local stability: an equilibrium point is stable if the *Jacobian* matrix is negative semi-definite.

**Proposition 3.2** (Characterization of stable equilibrium). *An equilibrium point  $\theta$  is stable if the Jacobian matrix  $\mathbf{J}$ , defined as:*

$$\mathbf{J}_{ij} = \begin{cases} -\sum_{j=1}^n \mathbf{A}_{ij} \cos(\theta_i - \theta_j), & \text{if } i = j, \\ \mathbf{A}_{ij} \cos(\theta_i - \theta_j), & \text{if } i \neq j, \end{cases}$$

*is negative semi-definite.*

### 3.2. Stationary Point and Local Optimality.

**Definition 3.3** (Stationary point of the energy function (4)). We call  $\theta \in \mathbb{R}^n$  a *stationary point* of the energy function (4) if it satisfies  $\nabla E(\theta) = 0$ , namely

$$\sum_{j=1}^n \mathbf{A}_{ij} \sin(\theta_j - \theta_i) = 0, \quad \forall i \in [n]. \quad (6)$$

Obviously, from Proposition 3.1 and Definition 3.3, a configuration  $\theta \in \mathbb{R}^n$  is a stationary point of the energy function (4) if and only if it is an equilibrium of the Kuramoto model (2).

**Definition 3.4** (Local minimum of the energy function). A configuration  $\theta \in \mathbb{R}^n$  is called a *local minimum* of the energy function  $E$  if there exists  $\varepsilon > 0$  such that

$$E(\theta) \leq E(\theta'), \quad \forall \theta' \in \mathbb{R}^n \text{ with } \|\theta' - \theta\| < \varepsilon.$$

**Proposition 3.5** (First- and second-order necessary conditions). *If a configuration  $\theta$  is a local minimum of  $E$ , then*

- (1)  $\nabla E(\theta) = 0$  (first-order condition), and
- (2) the Hessian  $\nabla^2 E(\theta)$  is positive semidefinite.

A point satisfying the above two conditions is also called a second-order stationary point (SOSP) of (4).

**3.3. Correspondence between Stability and Optimality.** In this section, we briefly outline the relationship between the equilibrium stability of the dynamical system (2) and the local optimality of the energy function (4). Understanding this correspondence provides valuable intuition. In particular, in this work, we often exploits the fact that a small perturbation causes the system to leave that state (therefore this state is unstable), in order to show a certain state is not a second-order stationary point.

At the first-order level, the two notions coincide. Indeed, a configuration  $\theta$  is a first-order stationary point of the energy function (4) *if and only if* it is an equilibrium of the dynamical system (2) on the graph  $G$  with adjacency matrix  $A$ , since both are characterized by

$$\sum_{j=1}^n A_{ij} \sin(\theta_j - \theta_i) = 0 \quad \text{for all } i \in V.$$

One can further relate the stability of an equilibrium to the second-order necessary condition of the energy landscape. It is precisely because of the real analyticity of the energy function (4), the local stability of an equilibrium of (2) is equivalent to the local minimality of the energy function (4). The relation mentioned here is summarized in Figure 4. Such an equivalence was established in the celebrated work [AK06].

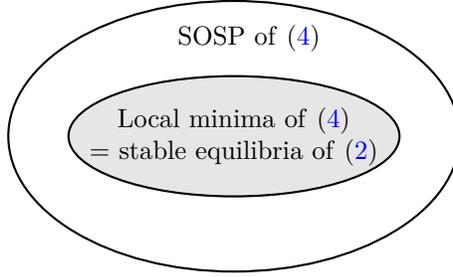


FIGURE 4. Relation between different concepts.

#### 4. VECTOR GEOMETRY OF EQUILIBRIA

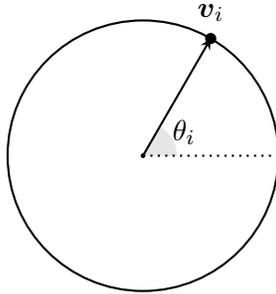
In this section, we recast the algebraic first- and second-order stationary conditions in Section 3 into geometric ones. This geometric viewpoint replaces the nonlinear equations (6) and the Hessian constraints by combinatorial relations among the vectors associated with the nodes. They form the starting point of our analysis. Additionally, stronger geometric constraints will be derived in later sections and combined with the results here to rule out all non-synchronous second-order stationary points of the energy function  $E_G(\boldsymbol{\theta})$  where  $G$  is any connected threshold graph.

**4.1. Vector-labeled Graph.** The basic object underlying this geometric perspective is the *vector-labeled graph* corresponding to angle configuration of  $\boldsymbol{\theta}$ . Defining such object allows us reasoning about whether a given state  $\boldsymbol{\theta}$  is an equilibrium, and whether an equilibrium is stable, i.e., can we move the angles so as to decrease the energy.

Given a state  $\boldsymbol{\theta} \in \mathbb{R}^n$  of Kuramoto model on graph  $G = (V, E)$ . Associate each node  $i \in V$  its corresponding unit vector

$$\mathbf{v}_i := (\cos \theta_i, \sin \theta_i) \in \mathbb{R}^2.$$

This is called a *phasor*, following the standard convention in the Kuramoto model. The object  $(G, \{\mathbf{v}_i\}_{i \in V})$ , called *vector-labeled graph* encoding  $\boldsymbol{\theta}$  combinatorially and geometrically.

FIGURE 5. Representation of  $\mathbf{v}_i$  with angle  $\theta_i$ .

**4.2. Alignment and Equilibrium.** Given a vector-labeled graph  $(G, \{\mathbf{v}_i\})$ , we now formulate the equilibrium condition of (2) in geometric terms.

**Lemma 4.1** (Equilibrium condition rephrased geometrically). *A state  $\boldsymbol{\theta}$  is an equilibrium of the Kuramoto model (2) on  $G = (V, E)$  with adjacency  $\mathbf{A}$  if and only if, for each  $i \in V$ ,*

$$\sum_{j \in N(i)} \mathbf{v}_j = \mu_i \mathbf{v}_i \quad \text{where} \quad \mu_i = \sum_{j \in N(i)} \cos(\theta_j - \theta_i). \quad (7)$$

The proof can be found in [MP05], we present here for completeness.

*Proof of Lemma 4.1.* Let us firstly derive an equivalent form of (7). Rewrite all vectors  $\mathbf{v}_i$  in (7) in terms of their Cartesian components  $(\cos \theta_i, \sin \theta_i)$ , yielding

$$\sum_{j \in N(i)} (\cos \theta_j, \sin \theta_j) = \mu_i (\cos \theta_i, \sin \theta_i).$$

Equivalently, this can be written as

$$\begin{cases} \sum_{j \in N(i)} \cos \theta_j = \mu_i \cos \theta_i, \\ \sum_{j \in N(i)} \sin \theta_j = \mu_i \sin \theta_i. \end{cases} \quad (8)$$

This is equivalent to verifying the equality between two complex numbers:

$$\sum_{j \in N(i)} \cos \theta_j + I \sum_{j \in N(i)} \sin \theta_j = \mu_i \cos \theta_i + I \mu_i \sin \theta_i,$$

which can be compactly rewritten as

$$\sum_{j \in N(i)} e^{I\theta_j} = \mu_i e^{I\theta_i}. \quad (9)$$

using the Euler formula  $e^{I\theta_i} = \cos \theta_i + I \sin \theta_i$ . Now that we have derived the equivalence between conditions (7) and (9), our goal now reduces to proving the equivalence between (9) and (4), namely

$$\sum_{j \in N(i)} e^{I\theta_j} = \mu_i e^{I\theta_i} \iff \sum_{j \in N(i)} \sin(\theta_j - \theta_i) = 0.$$

This indeed holds.

To prove the implication from the right-hand side to the left-hand side, we construct such a  $\mu_i$  that satisfies the left-hand side. Dividing both sides of (9) by  $e^{I\theta_i}$  (noting that  $e^{I\theta_i} \neq 0$ ), we obtain

$$\frac{\sum_{j \in N(i)} e^{I\theta_j}}{e^{I\theta_i}} = \mu_i.$$

Such a  $\mu_i$  exists and is in fact given by

$$\mu_i = \sum_{j \in N(i)} \cos(\theta_j - \theta_i),$$

at an equilibrium point  $\boldsymbol{\theta}$ , as we will show below. Expanding the fraction, we obtain

$$\begin{aligned} \frac{\sum_{j \in N(i)} e^{I\theta_j}}{e^{I\theta_i}} &= \sum_{j \in N(i)} e^{I(\theta_j - \theta_i)} \\ &= \sum_{j \in N(i)} (\cos(\theta_j - \theta_i) + I \sin(\theta_j - \theta_i)). \end{aligned} \quad (10)$$

Since  $\boldsymbol{\theta}$  is an equilibrium, the imaginary part must vanish, leading to

$$\frac{\sum_{j \in N(i)} e^{I\theta_j}}{e^{I\theta_i}} = \sum_{j \in N(i)} \cos(\theta_j - \theta_i) \in \mathbb{R}.$$

Thus, we conclude

$$\sum_{j \in N(i)} e^{I\theta_j} = \sum_{j \in N(i)} \cos(\theta_j - \theta_i) \cdot e^{I\theta_i}.$$

Conversely, to prove the left-hand side implies the right-hand side, we show that if there exists a constant  $\mu_i$  such that

$$\sum_{j \in N(i)} e^{I\theta_j} = \mu_i e^{I\theta_i},$$

then  $\theta$  is an equilibrium point. This is equivalent to showing that

$$I \sum_{j \in N(i)} \sin(\theta_j - \theta_i) + \sum_{j \in N(i)} \cos(\theta_j - \theta_i) \in \mathbb{R}.$$

Indeed, we compute

$$\begin{aligned} & I \sum_{j \in N(i)} \sin(\theta_j - \theta_i) + \sum_{j \in N(i)} \cos(\theta_j - \theta_i) \\ &= \sum_{j \in N(i)} e^{I(\theta_j - \theta_i)} \\ &= \frac{\sum_{j \in N(i)} e^{I\theta_j}}{e^{I\theta_i}} \\ &= \mu_i. \end{aligned} \tag{11}$$

Thus, the proof is complete.  $\square$

**Remark 4.2** (Normalization and local coherence). *One can rewrite (4.1) into*

$$\mathbf{v}_i = \frac{1}{\mu_i} \sum_{j \in N(i)} \mathbf{v}_j, \quad \text{where } \mu_i = \left\| \sum_{j \in N(i)} \mathbf{v}_j \right\| = \sum_{j \in N(i)} \cos(\theta_j - \theta_i).$$

*The expression describes a geometric condition of equilibrium: At equilibrium, each phasor aligns with the direction of the sum of its neighbors' vectors and is scaled to unit length. Although introduced as a normalization factor, the quantity  $\mu_i$  carries physical meanings. If the neighbors are highly coherent, the sum vector is large, requiring a larger  $\mu_i$  to normalize  $\mathbf{v}_i$  to unit length; if the directions are dispersed,  $\mu_i$  is small. In this sense,  $\mu_i$  measures how strongly the phasors of node  $i$ 's neighbors align directionally. Indeed, defining the local order parameter  $R_i := \frac{1}{\deg(i)} \left| \sum_{j \in N(i)} e^{i\theta_j} \right|$ , we have the relation  $\mu_i = \deg(i) \cdot R_i$ , showing that  $\mu_i$  is proportional to the standard notion of local phase coherence.*

**Remark 4.3** (Equilibrium and force). *This equation admits a natural physical interpretation using the concept of force. Write each phasor  $\mathbf{v}_i$  in complex form as  $e^{i\theta_i}$ . Then, intuitively, each neighboring node  $j \in N(i)$  pulls node  $i$  with a force given by  $e^{I(\theta_j - \theta_i)}$ . Under this interpretation, equation (4.1) states that: At equilibrium, the total force acting on node  $i$  sums to a real scalar. In other words, the net force is purely radial and contributes only to scaling. Due to the unit circle geometry  $|\mathbf{v}_i| = 1$  of the Kuramoto model, such a scaling force does not induce any motion while a tangential component can cause rotation.*

**4.3. Angle and Stability.** Given an equilibrium  $\theta$ , if there exists a node  $i \in V$  whose corresponding force satisfies  $\mu_i < 0$ , then  $\theta$  cannot be a second-order stationary point. Indeed, a perturbation that moves  $\mathbf{v}_i$  toward the direction of  $\sum_{j \in N(i)} \mathbf{v}_j$  strictly decreases the energy.

**Lemma 4.4.** *If  $\theta$  is a second-order stationary point of (4), then either  $\sum_{j \in N(i)} \mathbf{v}_j = \mathbf{0}$  or*

$$\angle \left( \sum_{j \in N(i)} \mathbf{v}_j, \mathbf{v}_i \right) = 0.$$

*Proof of Lemma 4.4.* The  $i$ -th diagonal entry of  $\text{Hess } E(\theta)$  can be written as

$$\text{Hess } E(\theta)_{ii} := \sum_{j=1}^n \mathbf{A}_{ij} \cos(\theta_i - \theta_j) = \sum_{j=1}^n \mathbf{A}_{ij} e^{I(\theta_j - \theta_i)} = \mu_i$$

where the second equality follows from the equilibrium condition (4.1), after rewriting the unit vectors in complex form. If some  $\mu_i < 0$ , then  $\text{Hess } E(\theta)_{ii} < 0$ , contradicting the positive semidefiniteness of  $\text{Hess } E(\theta)$  at any second-order stationary point [Bha07] [HJ85].  $\square$

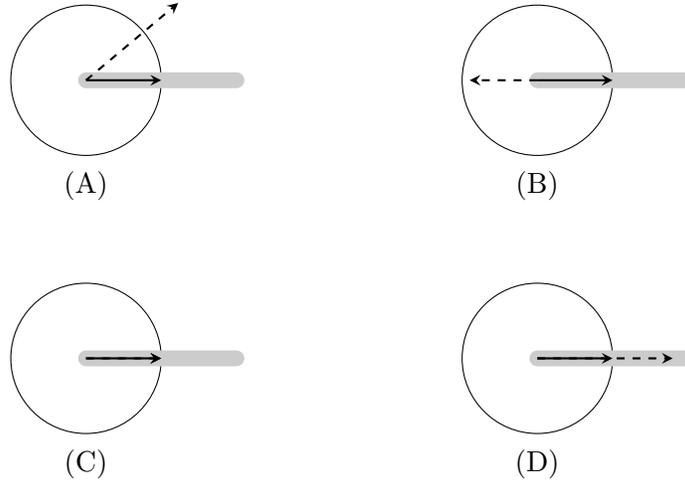


FIGURE 6. Geometric intuition for the first- and second-order stationary conditions. The solid arrow represents  $\mathbf{v}_i$ , and the dashed arrow represents  $\sum_{j \in N(i)} \mathbf{v}_j$ . For each node  $i$ , the four panels show all possible configurations satisfying the first-order condition. Among them, cases (A) and (B) violate the second-order condition because the aggregated phasor lies outside the feasible (gray) region corresponding to  $\mu_i \geq 0$ .

**Remark 4.5** (Mechanical insight into node at stable equilibrium). *The equilibrium condition in Lemma 4.4 for node  $i \in V$  has a mechanical interpretation. The sum of the phasors representing its neighbors can be visualized as a set of unit-length rods with directions. We place one phasor's starting point at the origin, then connect the remaining phasors head-to-tail. The head-to-tail connected rods can freely adjust their orientations, but the endpoint of the last phasor must remain on the right-end unrestricted sliding track, to ensure node  $i$  being stable there.*

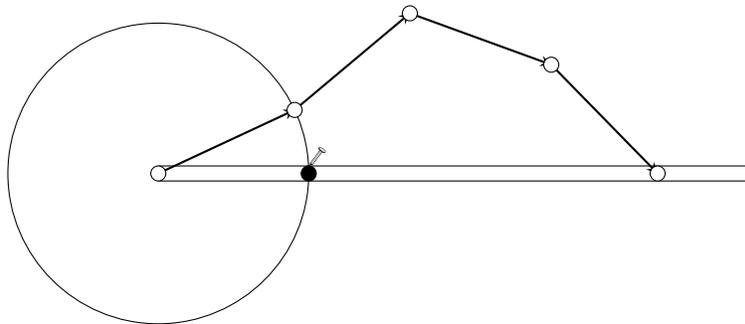


FIGURE 7. With  $\mathbf{v}_i$  fixed at angle 0, the endpoint of the aggregated phasor  $\sum_j A_{ij} \mathbf{v}_j$  is constrained to the right-end sliding track. The rods may freely adjust their orientations, but the endpoint of the last rod must lie in the feasible sliding interval. The solid vectors show the unit-length phasors of the neighbors of node  $i$ .

The geometric conditions of second-order stationary points presented here are only the most basic one. Further necessary geometric conditions will be established in subsequent sections, and these will be essential for showing that every connected threshold graph is globally synchronizing.

## 5. LOCAL SYNCHRONIZATION PRIMITIVES

**5.1. A Fundamental Geometric Fact.** We begin with a geometric lemma that characterizes all possible relative positions of two unit vectors satisfying a pair of linear relations. This fact will serve as the local synchronization primitive used in the induction argument in Section 6.

**Lemma 5.1.** *Let  $\mathbf{v}_a, \mathbf{v}_b \in \mathbb{S}^1 \subset \mathbb{R}^2$ . Suppose there exist a vector  $\mathbf{q} \in \mathbb{R}^2$  and scalars  $\mu_a, \mu_b \in \mathbb{R}$  such that*

$$\mathbf{v}_b + \mathbf{q} = \mu_a \mathbf{v}_a \quad \text{and} \quad \mathbf{v}_a + \mathbf{q} = \mu_b \mathbf{v}_b. \quad (12)$$

*Then the positions of  $\mathbf{v}_a$  and  $\mathbf{v}_b$  fall into one of the following three cases:*

- (1)  $\mathbf{v}_a = \mathbf{v}_b$  and  $\mu_a = \mu_b$ ;
- (2)  $\mathbf{v}_a = -\mathbf{v}_b$ ,  $\mu_a + \mu_b = -2$ , and  $(\mu_a, \mu_b) \neq (-1, -1)$ ;
- (3)  $\mathbf{v}_a, \mathbf{v}_b \in \mathbb{S}^1$ ,  $\mu_a = \mu_b = -1$ , and  $\mathbf{v}_a + \mathbf{v}_b + \mathbf{q} = 0$ .

*Proof.* According to (12), we have

$$\mathbf{q} = \mu_a \mathbf{v}_a - \mathbf{v}_b = \mu_b \mathbf{v}_b - \mathbf{v}_a$$

Thus

$$(\mu_a + 1)\mathbf{v}_a = (\mu_b + 1)\mathbf{v}_b. \quad (13)$$

*Case 1:*  $\mu_a = \mu_b = -1$ . In this case, we have

$$\mathbf{q} = \mathbf{v}_a + \mathbf{v}_b$$

*Case 2:*  $\mu_a \neq -1$  and  $\mu_b \neq -1$ . In this case, from (13), it must hold that

$$\mathbf{v}_a \parallel \mathbf{v}_b.$$

This leads to two subcases: (1) If  $\mu_a = \mu_b \neq -1$ , then  $\mathbf{v}_a = \mathbf{v}_b$ ; (2) If  $\mu_a + 1 = -(\mu_b + 1)$ , then equivalently, we have  $\mu_a + \mu_b = -2$ . This implies  $\mathbf{v}_a = -\mathbf{v}_b$ .  $\square$

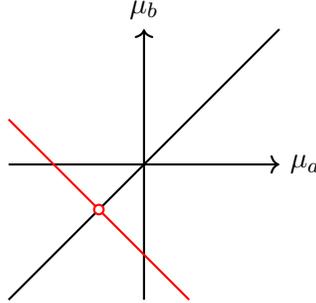


FIGURE 8. The feasible region for  $(\mu_a, \mu_b)$  consists of: (1) the black line  $\mu_a = \mu_b$ , excluding the intersection point, corresponding to  $\mathbf{v}_a = \mathbf{v}_b$ ; (2) the red line  $\mu_a + \mu_b = -2$ , excluding the intersection point, corresponding to the antipodal case  $\mathbf{v}_a = -\mathbf{v}_b$ ; and (3) their intersection point  $(-1, -1)$ , marked by a hollow dot, at which no additional constraint is imposed on the vectors  $\mathbf{v}_a$  and  $\mathbf{v}_b$ .

**Corollary 5.2.** *Under the assumptions of Lemma 5.1, if in addition  $\mu_a, \mu_b \geq 0$ , then  $\mathbf{v}_a = \mathbf{v}_b$ .*

*Proof.* With  $\mu_a, \mu_b \geq 0$ , Lemma 5.1 leaves only case (1), hence  $\mathbf{v}_a = \mathbf{v}_b$ .  $\square$

**Definition 5.3** (Geometric twins). At state  $\theta$  of Kuramoto model on  $G$ , nodes  $i$  and  $j$  are called *geometric twins* if their phasors  $\mathbf{v}_i$  and  $\mathbf{v}_j$  satisfy the compatibility relations of Lemma 5.1.

## 5.2. Phasor Geometry of Closed Twins.

**Definition 5.4.** Let  $G = (V, E)$  be a graph. Two nodes  $i, j \in V$  are called *closed twins* if

$$N[i] = N[j], \quad (14)$$

where  $N[i]$  is defined as  $N(i) \cup \{i\}$ .

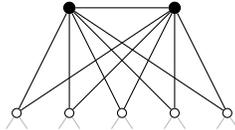


FIGURE 9. The two black nodes form a pair of closed twins.

Open twins, defined by  $N(i) = N(j)$ , will not be needed in this work.

**Corollary 5.5.** Let  $\theta$  be a second-order stationary point of the energy function  $E_G(\theta)$ . If nodes  $a$  and  $b$  form a pair of closed twins, then  $\theta_a = \theta_b$ .

*Proof.* From Lemma 4.4, since  $\theta$  is a second-order stationary point, there exist  $\mu_a, \mu_b \geq 0$  such that

$$\begin{cases} \sum_{j \in N(a)} \mathbf{v}_j = \mu_a \mathbf{v}_a, \\ \sum_{j \in N(b)} \mathbf{v}_j = \mu_b \mathbf{v}_b. \end{cases}$$

Because  $N[a] = N[b]$ , we have  $N(a) \setminus \{b\} = N(b) \setminus \{a\}$ , and we denote this common set by  $M$ . Thus,

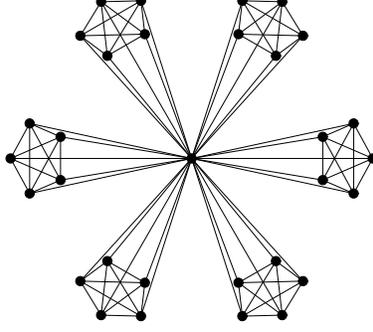
$$\begin{cases} \sum_{j \in M} \mathbf{v}_j + \mathbf{v}_b = \mu_a \mathbf{v}_a, \\ \sum_{j \in M} \mathbf{v}_j + \mathbf{v}_a = \mu_b \mathbf{v}_b. \end{cases}$$

By Corollary 5.2 and the fact that  $\mu_a, \mu_b \geq 0$ , we conclude that  $\mathbf{v}_a = \mathbf{v}_b$ , and therefore  $\theta_a = \theta_b$ .  $\square$

Clearly, this is a case where geometric twins defined in (5.3) formed. Moreover, Corollary 5.5 implies immediately that windmill graphs are globally synchronizing.

**Theorem 5.6.** Every windmill graph is globally synchronizing.

*Proof.* Let  $\theta$  be a second-order stationary point of  $E_G(\theta)$ , and let  $G$  be a windmill graph. Within each clique of  $G$ , all non-central nodes have identical closed neighborhoods and therefore form closed twins. By Corollary 5.5, all nodes in each clique must share the same phase. Contracting each clique into a single node yields a star graph, which is globally synchronizing. Hence all nodes of  $G$  have the same phase.  $\square$

FIGURE 10. Windmill graphs  $W_{5,6}$ .

**5.3. Synchronous Pendant Extension.** In this section, we present a scenario in which a group of nodes satisfy the phasor relations of Lemma 5.1 without being closed twins. More explicitly, the main result of this section is Lemma 5.8, which shows that even when a set of nodes are not twins, they may still be forced to synchronize at any second-order stationary point, provided that their distinct neighbors form a synchronous pendant whose combined phasor contribution collapses to a single, non-adversarial direction.

The next lemma, Lemma 5.7, serves as a preparatory step toward establishing Lemma 5.8.

**Lemma 5.7.** *Let  $G = (V, E)$  be a graph, and let  $W \subseteq V$  be partitioned into*

$$W = Q \uplus S \uplus P,$$

*subject to the following structural conditions:*

- (i) *Every node in  $Q$  has all its neighbors inside  $S$ , i.e.,  $N(i) \subseteq S$  for all  $i \in Q$ ;*
- (ii)  *$S$  induces a clique in  $G$ ;*
- (iii) *For each  $i \in S$ , its neighborhood is  $P \cup N_Q(i) \cup (S \setminus \{i\})$ , where  $N_Q(i) = \{q \in Q : q \sim i\}$ .*

*Assume further that the configuration  $(\mathbf{v}_i)_{i \in V}$  is a second-order stationary point of the Kuramoto energy (4), and that  $\mathbf{v}_i = \mathbf{v}$  for all  $i \in Q \uplus S$ . Then the phasor sum over  $P$  satisfies*

$$\sum_{i \in P} \mathbf{v}_i \uparrow \mathbf{v} \quad \text{or} \quad \sum_{i \in P} \mathbf{v}_i = \mathbf{0}.$$

*Proof.* For any node  $i \in S$ ,

$$N(i) = Q \uplus (S \setminus \{i\}) \uplus P \tag{15}$$

and

$$\sum_{j \in N(i)} \mathbf{v}_j = \mu_i \mathbf{v}_i. \tag{16}$$

Plug (15) into (16), we obtain

$$\sum_{j \in Q \uplus (S \setminus \{i\}) \uplus P} \mathbf{v}_j = \mu_i \mathbf{v}_i.$$

Since  $Q \uplus S$  forms a synchronized group, we have

$$(|Q| + |S| - 1) \mathbf{v}_i + \sum_{j \in P} \mathbf{v}_j = \mu_i \mathbf{v}_i, \quad \mu_i \geq 0.$$

Thus

$$\sum_{j \in P} \mathbf{v}_j = (\mu_i - (|Q| + |S| - 1)) \mathbf{v}_i, \quad \mu_i \geq 0.$$

We now show that  $\mu_i - (|Q| + |S| - 1) < 0$  is impossible, because in this case there would exist a direction along which the energy decreases in a neighborhood of  $\boldsymbol{\theta}$ .

To see this, let  $\mathbf{v}$  be the  $n$ -dimensional indicator vector of  $Q \uplus S$ , i.e.,  $v_i = 1$  for  $i \in Q \uplus S$  and  $v_i = 0$  otherwise. Then

$$\begin{aligned}
\mathbf{v}^T H \mathbf{v} &= \sum_{i \in Q \uplus S} \sum_{j \in P} \cos(\theta_i - \theta_j) \\
&= \sum_{i \in S} \sum_{j \in P} \cos(\theta_i - \theta_j) \\
&= \sum_{i \in S} \sum_{j \in P} \mathbf{v}_i \cdot \mathbf{v}_j \\
&= \sum_{i \in S} \mathbf{v}_i \cdot \sum_{j \in P} \mathbf{v}_j \\
&= |S| \mathbf{v}_i \cdot \sum_{j \in P} \mathbf{v}_j.
\end{aligned} \tag{17}$$

where the second equality uses the fact that there are no edges between  $Q$  and  $P$ . Clearly, if  $\mu_i - (|Q| + |S| - 1) < 0$ , then the above quantity is negative, which contradicts the second-order stationarity of  $\boldsymbol{\theta}$ .  $\square$

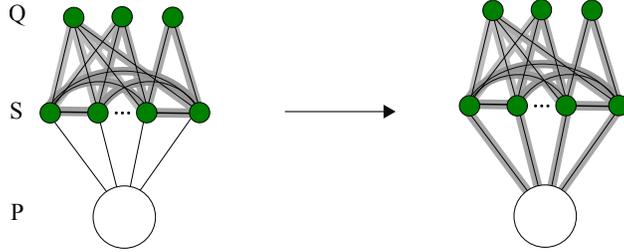


FIGURE 11. Illustration of Lemma 5.7. The green vertices correspond to nodes for which  $\mu_i \mathbf{v}_i = \sum_{j \in N(i)} \mathbf{v}_j$  holds with  $\mu_i \geq 0$ . The large circular node represents the set  $P$  of vertices. Gray edges indicate synchronization interactions between the corresponding nodes.

**Lemma 5.8.** *Let  $G = (V, E)$  be a graph, and suppose there exists a subset of nodes  $W \subseteq V$  partitioned into four disjoint subsets:*

$$W = Q \uplus S_1 \uplus S_2 \uplus P.$$

*under the following structural assumptions:*

- (i) *For all  $i \in Q$ , we have  $N(i) \subseteq S_1$ ;*
- (ii) *The subgraph  $G[S_1 \uplus S_2]$  is a complete graph;*
- (iii) *For any  $i \in S_1 \uplus S_2$ , its neighbourhood can be decomposed into three parts: the set  $P$ , its neighbours inside  $Q$ , and the rest of the clique  $S_1 \uplus S_2$ . Formally, if we denote  $Q_i = N(i) \cap Q$ , then*

$$N(i) = P \uplus Q_i \uplus ((S_1 \uplus S_2) \setminus \{i\}).$$

*Equivalently,*

$$N(i) \setminus (Q \uplus (S_1 \uplus S_2)) = P.$$

*Assume further that all nodes in  $Q \uplus S_1$  are synchronized, i.e.,*

$$\mathbf{v}_i = \mathbf{v}_j, \quad \forall i, j \in Q \uplus S_1.$$

*Then the set  $S_1 \uplus S_2$  synchronize.*

*Proof.* Observe that the set  $S_1$  induces a fully-connected subgraph, and  $Q$  is a synchronous pendant to  $S_1$ , then according to Lemma 5.8, we have

$$\forall i \in S_1, \quad \mathbf{v}_i \uparrow \sum_{k \in S_2 \uplus P} \mathbf{v}_k \quad \text{or} \quad \sum_{k \in S_2 \uplus P} \mathbf{v}_k = \mathbf{0}.$$

Combine this with the fact

$$\mathbf{v}_i = \mathbf{v}_j \quad \forall i \in S_1, j \in S_2,$$

we obtain

$$\forall i \in S_1, \quad \mathbf{v}_i \uparrow \sum_{k \in S_2 \uplus P} \mathbf{v}_j + \sum_{k \in S_1 \setminus \{i\}} \mathbf{v}_j \quad \text{or} \quad \sum_{k \in S_2 \uplus P} \mathbf{v}_k + \sum_{k \in S_1 \setminus \{i\}} \mathbf{v}_k = \mathbf{0}.$$

Equivalently, we can write

$$\forall i \in S_1, \quad \mathbf{v}_i \uparrow \sum_{k \in S_1 \uplus P \uplus S_2 \setminus \{i\}} \mathbf{v}_k \quad \text{or} \quad \sum_{k \in S_1 \uplus P \uplus S_2 \setminus \{i\}} \mathbf{v}_k = \mathbf{0}. \quad (18)$$

Another observation is on nodes in  $S_2$  that

$$\forall i \in S_2, \quad \mathbf{v}_i \uparrow \sum_{k \in S_2 \setminus \{i\}} \mathbf{v}_k + \sum_{j \in S_1 \uplus P} \mathbf{v}_k \quad \text{or} \quad \sum_{k \in S_2 \setminus \{i\}} \mathbf{v}_k + \sum_{j \in S_1 \uplus P} \mathbf{v}_k = \mathbf{0}$$

since every node in  $S_2$  attains stable equilibrium in  $\theta$ . Equivalently, we can write

$$\forall i \in S_2, \quad \mathbf{v}_i \uparrow \sum_{k \in S_1 \uplus P \uplus S_2 \setminus \{i\}} \mathbf{v}_k \quad \text{or} \quad \sum_{k \in S_1 \uplus P \uplus S_2 \setminus \{i\}} \mathbf{v}_k = \mathbf{0}. \quad (19)$$

Combined the facts (18) and (19), we get nodes in  $S_1 \uplus S_2$  are geometric stable closed twins.  $\square$

Clearly, this is another situation in which geometric twins, as defined in Definition 5.3, appear in addition to closed twins at the second-order stationary point.

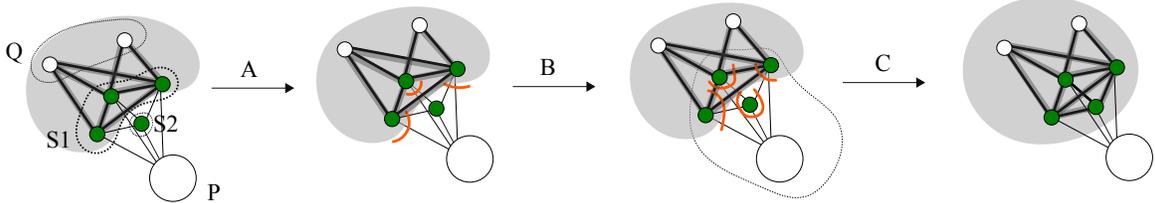


FIGURE 12. Illustration of Lemma 5.8, illustrating how the synchronized block  $Q \uplus S_1$  extends to include  $S_2$  through propagation enabled by the specific local structure. The green vertices correspond to nodes for which  $\mu_i \mathbf{v}_i = \sum_{j \in N(i)} \mathbf{v}_j$  holds with  $\mu_i \geq 0$ . The gray-shaded nodes are synchronized; an edge with a gray background indicates synchronization between the corresponding nodes. Each orange arc separates a small green node from a group of nodes it connects to, indicating that the vector of the green node is in the same direction as the sum of the vectors of the nodes on the opposite side of the arc.

**5.4. Twin Attachment.** If a set of nodes all have identical neighborhoods, they form a group of closed twins. When an additional node is connected exclusively to this group, the entire collection of nodes, must take the same phase at any second-order stationary point.

**Lemma 5.9.** *Let  $\theta$  be a second-order stationary point of  $E_G(\theta)$ . If  $S \subseteq V$  is a set of closed twins in  $G$  and  $A \in V$  satisfies  $N(A) = S$ , then  $\theta_i = \theta_j$  for all  $i, j \in S \cup \{A\}$ .*

*Proof.* Since  $\theta$  is a second-order stationary point of  $E_G(\theta)$  and  $S$  forms a set of closed twins, Lemma 5.5 yields

$$\theta_i = \theta_j \quad \text{for all } i, j \in S.$$

Because  $N(A) = S$ , Lemma 4.4 implies

$$\theta_A = \theta_i \quad \text{for all } i \in S.$$

Hence all nodes in  $S \cup \{A\}$  synchronize. □

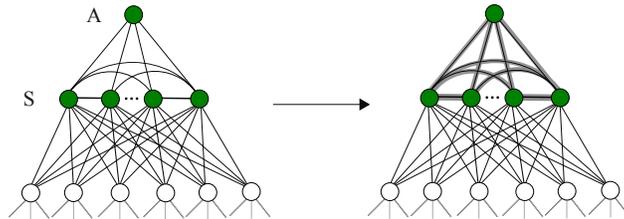


FIGURE 13. Illustration of Lemma 5.9. All green nodes attain stable equilibrium. The set  $S$  (middle row) forms a set of false twins, and node  $A$  (top) has neighborhood exactly  $S$ . Thin lines represent the edges in the original graph. The conclusion is that all green nodes must synchronize, indicated by the thick edges in the right diagram.

## 6. MAIN PROOF: THE INDUCTION ARGUMENT

**6.1. A Prototype Example.** Let us take a look at the threshold graph corresponding to the sequence 01010101. The first vertex is denoted by  $A$ , and we label the vertices  $A$  through  $I$  in the order in which they appear in the sequence, with each bit determining whether the added vertex is isolated (0) or dominating (1).

Let  $G$  be the threshold graph corresponding to the sequence 01010101. To prove that  $G$  is globally synchronizing, we show that every second-order stationary point  $\theta$  of  $E_G(\theta)$  is synchronous; that is,  $\theta_1 = \dots = \theta_n$ . The following figure illustrates how synchronization must propagate, step by step, in any second-order stationary configuration on this graph.

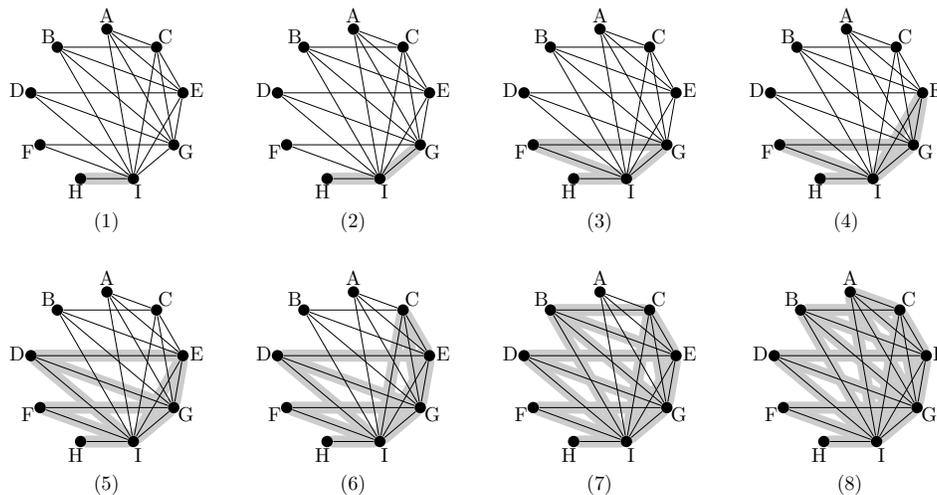


FIGURE 14. Synchronization propagates in eight steps, forcing every second-order stationary point to be synchronous.

*Step (1)* We begin with the edge between  $H$  and  $I$ . Applying Lemma 4.4 to node  $H$  shows that, since  $\theta$  is a second-order stationary point, we must have  $v_H \uparrow v_I$ . Equivalently, nodes  $H$  and  $I$  synchronize.

*Step (2)* Observe that  $N(G) = N(I) \cup H$  and  $\theta_H = \theta_I$ , applying Lemma by letting  $Q = \{H\}$ ,  $S_1 = \{I\}$ ,  $S_2 = \{G\}$ ,  $P = \{F, D, B, A, C, E\}$ , we obtain  $\theta_G = \theta_I$ . Intuitively, the only difference between the neighborhoods of  $G$  and  $H$  is that  $H$  has one additional neighbor, namely  $I$ . This extra neighbor does not obstruct the synchronization between  $G$  and  $I$ , because  $H$  is already synchronized with  $I$  and therefore behaves as a synchronous pendant at this stage.

*Step (3)* Note that node  $F$  connects only to  $G$  and  $I$ , which are synchronized as established in Step (2). Therefore, by Lemma 5.9, we have  $v_F = v_G = v_I$ .

*Step (4)* Observe that nodes  $F$  and  $H$  connect only to  $G$  and  $I$ , and the four-node set  $\{F, H, G, I\}$  behaves as a synchronizing group. Furthermore, ignoring the leaf-like nodes  $F$  and  $H$  and edges in the induced clique on  $\{E, G, I\}$ , the two nodes  $G$  and  $I$  share exactly the same neighbors:

$$N(G) \setminus (\{F\} \cup \{E, I\}) = N(I) \setminus (\{F, H\} \cup \{E, G\}) = N(E) \setminus \{G, I\} = \{D, B, A, C\}.$$

Therefore, by applying Lemma 5.8 with  $Q = \{F, H\}$ ,  $S_1 = \{G, I\}$ ,  $S_2 = \{E\}$ , and  $P = \{D, B, A, C\}$ , we conclude that the nodes  $E$ ,  $G$ , and  $I$  form a synchronizing group.

*Step (5)* Note that node  $D$  connects to  $E$ ,  $G$ , and  $I$ , which are synchronized as established in Step (4). Therefore, by Lemma 5.9, we have  $v_D = v_E = v_G = v_I$ .

*Step (6)* Observe that nodes  $D$ ,  $F$ , and  $H$  connect only to  $E$ ,  $G$ , and  $I$ , and therefore the six-node set  $\{D, F, H, E, G, I\}$  behaves as a synchronizing group. Furthermore, ignoring the leaf-like nodes  $D$ ,  $F$ , and  $H$  and edges in the induced clique on  $\{C, E, G, I\}$ , the four nodes  $C$ ,  $E$ ,  $G$ , and  $I$  share exactly the same neighbors, namely  $\{A, B\}$ . Therefore, by applying Lemma 5.8 with  $Q = \{D, F, H\}$ ,  $S_1 = \{E, G, I\}$ ,  $S_2 = \{C\}$ , and  $P = \{A, B\}$ , we conclude that the nodes  $C$ ,  $E$ ,  $G$ , and  $I$  form a synchronizing group.

*Step (7)* Note that node  $B$  connects to  $C$ ,  $E$ ,  $G$ , and  $I$ , which are synchronized as established in Step (6). Therefore, by Lemma 5.9, we have  $v_B = v_C = v_E = v_G = v_I$ .

*Step (8)* As in Step (7), Lemma 5.9 implies that  $A$  synchronizes with  $C$ ,  $E$ ,  $G$ , and  $I$ , since these nodes form a synchronizing group and constitute all neighbors of  $A$ .

In conclusion,  $\theta$  must assign the same angle to every node. Equivalently, every second-order stationary point is synchronous, and hence the threshold graph is globally synchronizing.

## 6.2. Proof of Theorem 1.2: General Case.

*Proof of Theorem 1.2.* Any connected threshold graph can be described by a bit sequence of 1's and 0's, as introduced in Definition 2.1. Since the graph is connected, the last bit must be a 1. We decompose the sequence into maximal contiguous blocks of 1's and 0's. Let  $U_1, U_2, \dots, U_k$  denote the blocks of 1's (indexed from bottom to top), and let  $I_i$  denote the 0-block that lies immediately above  $U_i$ , if such a block exists. In this notation, the sequence has one of the two forms

$$U_1, I_2, U_2, \dots, I_{k-1}, U_{k-1}, I_k, U_k,$$

or

$$I_1, U_1, I_2, U_2, \dots, I_{k-1}, U_{k-1}, I_k, U_k,$$

depending on whether or not it begins with a block of 0's. We focus on the first case; the second follows by an entirely analogous argument.

Given that  $\theta$  is a second-order stationary point of  $E(\theta)$ , we aim to show that  $\theta_1 = \dots = \theta_n$ , and we proceed by induction. We proceed by induction. For the base case, observe that all nodes within a block of 1's form a set of closed twins. Hence, at any second-order stationary point  $\theta$ , the nodes in the last block  $U_k$  must synchronize, as follows directly from Lemma 5.1.

Assume as the inductive hypothesis that the nodes in the last  $m$  blocks of 1's,  $U_{k-m+1}, \dots, U_k$ , are synchronized. Lemma 5.9 then implies that the adjacent 0-blocks  $I_{k-m+1}, \dots, I_k$  synchronize with them as well. We next show that the preceding 1-block  $U_{k-m}$  also synchronizes with  $U_{k-m+1}, \dots, U_k$ . Indeed, define

$$\begin{aligned} Q &:= I_{k-m+1} \uplus \dots \uplus I_k, \\ S_1 &:= U_{k-m+1} \uplus \dots \uplus U_k, \\ S_2 &:= U_{k-m}, \\ P &:= U_1 \uplus \dots \uplus U_{k-m-1} \uplus I_2 \uplus \dots \uplus I_{k-m}. \end{aligned}$$

We next verify that the sets  $Q, S_1, S_2$ , and  $P$  satisfy the assumptions of Lemma 5.8. First, for every node  $i$  in  $Q$ , its neighborhood is contained in  $S_1$ ; indeed, these nodes are isolated at this stage and connect only to the dominating nodes that are added later. Second, the sets  $S_1$  and  $S_2$  together induce a clique. Third, for any node in  $S_1 \cup S_2$ , its neighbors consist exactly of all other nodes in  $S_1 \cup S_2$ , all nodes in  $P$ , and possibly some nodes in  $Q$ . Hence all conditions of Lemma 5.8 are satisfied, and we conclude that  $S_1$  and  $S_2$  must synchronize. In particular, the block  $U_{k-m}$  synchronizes with  $U_{k-m+1} \uplus U_{k-m+2} \uplus \dots \uplus U_k$ .

By induction, all nodes synchronize at  $\theta$ . Therefore, every second-order stationary point is a synchronous state. It follows that every connected threshold graph is globally synchronizing.  $\square$

## 7. DISCUSSION AND OPEN PROBLEMS

In this work, we present a class of globally synchronizing graphs — threshold graphs — whose density vary from  $2/n$  to 1, revealing a new mechanism of global synchronization that has not been articulated before. Our proof crucially exploits the constructive nature of threshold graphs: they form the smallest graph class closed under the addition of isolated nodes and universal nodes. This closure property is essential to our inductive argument, as it allows us to show that global synchronization is preserved when starting from a threshold graph and adding either an isolated node or a universal node.

A natural question is whether this closure phenomenon extends beyond threshold graphs. Specifically, suppose  $G$  is an arbitrary globally synchronizing graph. If we add a universal node to  $G$ , is the resulting graph still globally synchronizing? We expect the answer to be yes, since attaching a node that is connected to all existing vertices does not appear to introduce new geometric directions for the oscillators beyond the synchronized configuration. However, we do not currently know how to prove this, and we leave this problem for future work.

Moreover, we expect that our results may admit further generalizations. Possible directions include extending the analysis to oscillators evolving on higher-dimensional spheres, incorporating nonlinear interaction terms as in [GLPR25], and considering broader classes of graphs. For instance, one may study systems in which oscillators are coupled through both positive and negative edges.

**Acknowledgements.** We would like to thank Afonso S. Bandeira for insightful discussions, and Kevin Lucca for pointing out relevant references on the known bounds of the density threshold.

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