

Black hole spacetimes with dark matter spikes: Energy-momentum tensor and backreaction effects

Wei Xiong and Peng-Cheng Li*

School of Physics and Optoelectronics, South China University of Technology, Guangzhou 510641, People's Republic of China

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We study the energy–momentum tensor of a dark matter (DM) spike formed during the adiabatic growth of a black hole embedded in a DM halo, and investigate its backreaction on the spacetime geometry. Within the Einstein cluster framework, we derive the complete tensor, explicitly incorporating the kinetic contribution to the energy density and the anisotropic pressure arising from noncircular particle orbits. Adopting the Hernquist profile as an illustrative model of DM halo and employing parameters appropriate to the Milky Way, we find that near the spike, the kinetic term enhances the total energy density by approximately 50% relative to the rest-mass component, while the nonzero radial pressure induces a mild anisotropy in the stress tensor. The derived tensor satisfies all standard energy conditions. By treating it as a fixed source in Einstein’s equations, we numerically obtain a static, spherically symmetric metric that deviates from the Schwarzschild solution by an amount more than twice that found when only the mass density is considered. These results demonstrate that including the full dynamical structure of the DM spike is essential for accurately modeling the backreaction of DM on black hole spacetimes.

I. INTRODUCTION

Dark matter (DM) constitutes a dominant but invisible component of the Universe, inferred from its gravitational influence on galactic rotation curves, gravitational lensing, and the dynamics of large-scale structures [1]. Despite its overwhelming abundance, the microscopic nature of DM remains elusive [2–4]. In the absence of direct detection, one promising avenue to probe DM properties is through its gravitational effects in strong-field regimes, particularly in the environments surrounding black holes (BHs) [5–9].

When a BH grows adiabatically inside a DM halo, the gradual deepening of its gravitational potential compresses the surrounding DM distribution into a steep density enhancement known as a DM spike [5, 6]. In the adiabatic approximation, the BH is assumed to grow on a timescale much longer than the orbital period of DM particles. Under such slow evolution, the system remains close to equilibrium, and the orbital actions—serving as adiabatic invariants are conserved. This conservation allows one to relate each particle’s initial orbit in the halo to its final orbit after the BH growth, thereby reconstructing the modified density profile. The resulting spike follows a power-law form whose index depends on the slope of the original DM halo. For instance, a Navarro-Frenk-White (NFW) halo [10] with $\gamma = 1$ yields a spike index $\gamma_{\text{sp}} \approx 3/2$, producing a density enhancement of several orders of magnitude near the BH. This framework has since been extended to general-relativistic settings, where the spike structure is computed in the Schwarzschild [11, 12] or Kerr spacetime [13]. Relativistic corrections become important close to the event horizon and can alter both the inner slope and the maximum density of the spike. The inclusion of strong-field effects is particularly relevant for modeling potential observational signatures, such as gamma-ray emission from DM annihilation

or gravitational waves [14–37].

In most treatments, however, the DM spike has been modeled under simplified assumptions about its energy-momentum content. Many analyses approximate the system as an ideal or anisotropic fluid characterized mainly by its rest-mass density, while neglecting kinetic and anisotropic pressure terms. In the simplest cases, the radial pressure is set to zero, corresponding to circular orbits, whereas more general approaches allow for a tangential component to represent the ensemble-averaged orbital motion of DM particles [38–48]. Although these approximations greatly simplify the Einstein equations and are sufficient for estimating the density profile, they omit part of the stress-energy associated with the distribution of bound geodesic orbits. Such missing contributions are responsible for the kinetic and anisotropic pressures that also gravitate and hence can influence the spacetime geometry. Including these effects allows one to assess the backreaction of the DM spike on the metric—a key step toward a more self-consistent description of BHs embedded in dense DM environments [49–62].

To incorporate these dynamical effects more consistently, we adopt the Einstein cluster model [63, 64], which describes a self-gravitating system of collisionless particles with an anisotropic velocity distribution. In our framework, the DM spike is assumed to form under adiabatic evolution. This allows the spike to be constructed self-consistently from an initial DM halo through conservation of orbital actions. For the host halo, we adopt a Hernquist profile [65], which provides a realistic finite-mass model with an inner slope similar to that of the NFW profile while offering analytic simplicity. The corresponding spike distribution function (DF) is then used to compute the full energy-momentum tensor of the DM spike from the particle distribution function, explicitly accounting for the kinetic contribution to the energy density and the non-vanishing pressure components arising from noncircular orbits. This approach bridges the gap between oversimplified fluid models and the fully self-consistent Einstein-Vlasov description, while remaining tractable for numerical implemen-

* pchli2021@scut.edu.cn

tation. Using the derived energy-momentum tensor as a fixed source, we numerically solve the Einstein field equations for a static, spherically symmetric spacetime containing a BH surrounded by a DM spike. By using the parameters for the Milky way fitted in [66], we find that the kinetic term enhances the total energy density near the spike by roughly 50% compared with the rest-mass contribution, while the radial pressure, though subdominant, introduces mild anisotropy in the stress components. We find the derived tensor satisfies all the standard energy conditions. Furthermore, the resulting spacetime exhibits a slight but non-negligible deviation from the Schwarzschild solution—roughly a factor of two larger than that found in previous treatments considering only the mass density. Although our model is semi-consistent, whose energy-momentum tensor and metric are solved iteratively rather than solved simultaneously, this approximation is physically well justified. In realistic systems, the DM spike around a supermassive BH is expected to be extremely dilute, leading to a weak gravitational backreaction. Hence, the iterative treatment adopted here captures the essential physics and represents a realistic step toward a fully self-consistent relativistic description of BHs embedded in DM halos.

The remainder of this paper is organized as follows. Sect. II reviews the theoretical framework used to describe DM spikes formed under adiabatic evolution and derives the full energy-momentum tensor of the spike from the DF constructed for a Hernquist-type halo. Sect. III derives the Einstein field equation for solving a static, spherically symmetric solution and imposes the boundary conditions. The results are presented in Sect. IV. Sec. V summarizes our conclusions and outlines possible extensions. The geometric units $G = c = 1$ are maintained throughout this paper except specifically mentioned.

II. THEORETICAL FRAMEWORK FOR DM SPIKES

A. Distribution function

Within the framework of classical mechanics, the phase-space DF $f^{(3)}$ for a self-gravitating system can be defined as

$$\int f^{(3)}(\vec{r}, \vec{p}) d^3r d^3p = 1, \quad (1)$$

where the integral extends over all phase space $(\vec{r}, \vec{p})$. In other words, $f^{(3)}(\vec{r}, \vec{p})$ describes the probability density for DM particles at specified phase-space coordinates. The superscript (3) of the DF represents that the particle orbit are evaluated in the nonrelativistic framework. We denote the DM particle mass, total particle number and total mass of this system as μ , N and M , respectively. These three parameters are related to each other by the formula

$$M = \mu N. \quad (2)$$

The mass density profile $\rho_M(r)$ of this system is then expressed as the integral over the momentum space

$$\rho_M(\vec{r}) \equiv \int [M f^{(3)}(\vec{r}, \vec{p})] d^3p, \quad (3)$$

since $M f^{(3)}(\vec{r}, \vec{p})$ represents the mass density in the phase space. The mean value of any quantity Q can be imposed by

$$\langle Q \rangle \equiv \frac{1}{n(\vec{r})} \int Q [N f^{(3)}(\vec{r}, \vec{p})] d^3p, \quad (4)$$

where $n(\vec{r}) \equiv \int [N f^{(3)}(\vec{r}, \vec{p})] d^3p$ denotes the number density in the coordinate space. For the relativistic cases, the necessary changes consist of modifying the volume element d^3p to $\sqrt{-g} d^4p$ and performing the integral in the mass shell ($p^\mu p_\mu = -\mu^2$ for the mass particle) in phase space.

Note that DFs in the literature often represent not the probability density in phase space, but rather the number density or mass density. These distinctions merely correspond to multiple the probability density f by N or M , respectively. In this paper, all DFs in the following of this work are referred to probability density at specified phase-space coordinates. While citing a DF from other works, we will explicitly indicate their physical meaning (probability density, particle number density, or mass density) to avoid confusion.

For a galactic system with a spherical gravitational potential $\Phi(r)$, the Hamiltonian of a DM particle per unit mass can be expressed as $H \equiv \frac{1}{2}v^2 + \Phi$, and we define the relative potential Ψ and the relative energy ϵ for a particle per unit mass by

$$\Psi \equiv -\Phi, \quad \epsilon \equiv -H = \Psi - \frac{1}{2}v^2, \quad (5)$$

where v is the velocity of the particle. The relative potential is fixed to vanish at infinity and nonnegative for any r , while the relative energy is hence equal to the binding energy of a particle. As we only consider bound particles for this system, ϵ ranges from 0 to $\Psi(r)$ for a given position. It is worth noting that the DF $f^{(3)}$ is ergodic for this spherical isotropic system (5), i.e., $f^{(3)}$ can be expressed solely as a function $f^{(3)}(\epsilon)$ of energy ϵ [67].

From (5) and the definition $\vec{p} \equiv \mu \vec{v}$, the equation (3) for a spherical and isotropic system can be written as

$$\begin{aligned} \rho_M &= \int [\mu^3 M f^{(3)}(\epsilon)] d^3v \\ &= 4\pi \int [\mu^3 M f^{(3)}(\epsilon)] v^2 dv \\ &= 4\sqrt{2}\pi \int_0^\Psi [\mu^3 M f^{(3)}(\epsilon)] \sqrt{\Psi - \epsilon} d\epsilon \end{aligned} \quad (6)$$

and the derivation gives

$$\frac{1}{\sqrt{8}\pi} \frac{d\rho_M}{d\Psi} = \int_0^\Psi d\epsilon \frac{\mu^3 M f^{(3)}(\epsilon)}{\sqrt{\Psi - \epsilon}}, \quad (7)$$

which is an Abel integral equation having a solution

$$\mu^3 M f^{(3)}(\epsilon) = \frac{1}{\sqrt{8\pi^2}} \frac{d}{d\epsilon} \int_0^\epsilon \frac{d\Psi}{\sqrt{\epsilon - \Psi}} \frac{d\rho_M}{d\Psi}. \quad (8)$$

The above equation (8) imposes the standard formula for transforming a observed (or numerically simulated) mass density to the DF for the spherical and isotropic system. This method is known as Eddington inversion [67].

In this paper, for the DM halo we employ the Hernquist profile [65]

$$\rho_M^{(H)}(r) = \frac{M}{2\pi} \frac{a}{r} \frac{1}{(r+a)^3}, \quad (9)$$

where a is a lengthscale denoting the radius containing a mass of $M/4$. The parameters of the Hernquist model for the Milky Way are fitted in [66], which gives $M = 1.55 \times 10^{12} M_\odot$ and $a = 16.9$ kpc. We adopt the mass of the BH at the center of the Milky Way as $M_{BH} = 4 \times 10^6 M_\odot$. In the geometric units, the ratio of the above quantities is

$$M = 387500 M_{BH}, \quad a = 227775 M. \quad (10)$$

The corresponding relative potential and the mass distribution of the Hernquist profile is given by

$$\Psi^{(H)}(r) = \frac{M}{r+a}, \quad M^{(H)}(r) = \frac{Mr^2}{(r+a)^2}, \quad (11)$$

with $M(a) = M/4$. For the Hernquist profile (11), the DF $f^{(H)}(\epsilon)$ is obtained as

$$\begin{aligned} \mu^3 M f^{(H)}(\epsilon) = & \frac{\sqrt{\epsilon}(2a\epsilon - M)(-8a^2\epsilon^2 + 8aM\epsilon + 3M^2)}{8\sqrt{2}\pi^3 a M^3 (a\epsilon - M)^2} \\ & + \frac{3M^4 \tan^{-1}\left(\sqrt{\frac{a\epsilon}{M-a\epsilon}}\right)}{8\sqrt{2}\pi^3 a M^3 (M - a\epsilon)^{5/2}}. \end{aligned} \quad (12)$$

Here we regain the result presented by Hernquist [65].

We adopt the Hernquist profile primarily because it possesses a known analytical DF (12). While other profiles could, in principle, be used, they pose no fundamental difficulty to our calculations. The pair of equations (6) and (8) provides a crucial bridge between the microscopic dynamics, described by the distribution function $f(\epsilon)$, and the macroscopic density profile $\rho(r)$ observed in simulations or astronomical observations.

B. Adiabatic growth of a BH

In the previous subsection, the motion of bound DM particles is completely determined by their self-gravity with no central BH present. The actions of the particles can be de-

fined

$$\begin{aligned} I_r^{(0)} &\equiv \oint v_r dr = \oint \sqrt{2\Psi - 2\epsilon - L^2/r^2}, \\ I_\theta^{(0)} &\equiv \oint v_\theta d\theta = 2\pi(L - L_z), \\ I_\varphi^{(0)} &\equiv \oint v_\varphi d\varphi = 2\pi L_z, \end{aligned} \quad (13)$$

where v_i ($i = r, \theta, \varphi$) represent the velocity's components. However, the presence of a supermassive BH at the center of most galaxies must be accounted for. A effective approach is adiabatically growing a BH within an initial halo that lacks one. This process, though it alters the central gravitational potential, is assumed to occur over timescales much longer than the typical orbital period of DM particles. Under this adiabatic approximation, the actions of the particles Eq. (13) remain constant as the potential changes. These conserved quantities are known as adiabatic invariants.

We now proceed to construct the adiabatic invariants for DM particles orbiting a central BH within the halo. However, this task is not straightforward. The standard formulation of adiabatic invariants is derived within Newtonian mechanics, which presents a conceptual challenge since a BH is an inherently relativistic object. Consequently, the orbits of bound particles in its vicinity must be re-examined. Following [11], we assume that the geodesics of DM particles are entirely governed by the Schwarzschild metric of the BH. The validity of this assumption will be assessed in the following subsection.

The line element of the Schwarzschild BH is given by

$$ds^2 = -\left(1 - \frac{2M_{BH}}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M_{BH}}{r}} + r^2 d\Omega^2, \quad (14)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. The complete integrability of timelike geodesics in the spacetime (14) ensures that they can be described completely by four orbital conservation quantities: the mass of the DM particles μ , the energy per unit mass E , the angular momentum per unit mass L and its z -components L_z

$$\mu^2 = -p^\nu p_\nu, \quad (15)$$

$$E = -u_t, \quad (16)$$

$$L_z = u_\varphi, \quad (17)$$

$$L^2 = u_\theta^2 + \frac{u_\varphi^2}{\sin^2 \theta}, \quad (18)$$

where $p_\nu \equiv \mu u_\nu$ represents the 4-momentum of the DM particles.

From (15-18), one can express the radial effective potential of the DM particles as

$$\begin{aligned} V_{eff} &\equiv -u_r^2 \left(1 - \frac{2M_{BH}}{r}\right)^2 \\ &= -E^2 + \left(\frac{r - 2M_{BH}}{r}\right) \left(1 + \frac{L^2}{r^2}\right). \end{aligned} \quad (19)$$

Intuitively, for a given energy and angular momentum, the ra-

dial motion of a DM particle is permitted only where the effective potential Eq. (19) is nonpositive. Such particles can either plunge into the horizon, escape to infinity, or be confined between an inner and an outer turning point. Equation (19) indicates that the energy of a bound particle must be less than unity. Furthermore, since bound particles do not cross the horizon, their energy also has a lower bound. The characteristic trajectory of a bound particle is elliptical orbits, oscillating periodically between its pericenter and apocenter. A circular orbit represents a special case of this, where the pericenter and apocenter coincide.

Similar to (13), the adiabatic invariants of the timelike geodesic in the Schwarzschild spacetime can also be constructed by the 4-velocity components

$$\begin{aligned} I_r^{(1)} &\equiv \oint u_r dr = \oint dr \left(1 - \frac{2M_{BH}}{r}\right)^{-1} \sqrt{-V_{eff}}, \\ I_\theta^{(1)} &\equiv \oint u_\theta d\theta = 2\pi(L - L_z), \\ I_\varphi^{(1)} &\equiv \oint u_\varphi d\varphi = 2\pi L_z, \end{aligned} \quad (20)$$

where the second equality in each row uses (19), (18) and (17), respectively. The adiabatic growth of a BH in the DM halo requires the equality between the initial and final adiabatic invariants throughout the slow evolution of the potential

$$I_r^{(0)} = I_r^{(1)}, \quad I_\theta^{(0)} = I_\theta^{(1)}, \quad I_\varphi^{(0)} = I_\varphi^{(1)}. \quad (21)$$

By comparing (13) and (20), a simple conclusion is that the angular momentum L and its z -component of a DM particle remains unchanged during the adiabatic growth, while its energy must vary. The initial energy ϵ and final energy E during the adiabatic growth can be related through

$$\begin{aligned} \oint dr \sqrt{2\Psi - 2\epsilon - \frac{L^2}{r^2}} &= \\ \oint dr \left(1 - \frac{2M_{BH}}{r}\right)^{-1} \sqrt{E^2 - \left(1 - \frac{2M_{BH}}{r}\right) \left(1 + \frac{L^2}{r^2}\right)}. \end{aligned} \quad (22)$$

This relation generates an implicit function $\epsilon = \epsilon(E, L^2)$ for the initial particle energy which can be evaluated numerically.

C. DM spike around a BH

Research [5, 6, 11, 12] indicates that a Schwarzschild BH leads to the formation of a high-amplitude DM density spike, which can be several orders of magnitude denser than the initial Hernquist profile. Additionally, a Kerr BH's spin is known to further increase the DM density near the horizon [13]. Building upon this foundation, we begin by reviewing the derivation of the DM density profile influenced by a Schwarzschild BH. The primary new contribution of this subsection is the first presentation of the energy-momentum tensor for this DM profile following adiabatic BH growth. This step is essential because the energy-momentum tensor serves as the source term that couples matter in the Einstein field equation.

Given our focus on the backreaction of the DM spike on the BH, a simplified treatment of the spike profile is warranted. We adopt the standard approach from [5, 6, 11, 12], assuming the geodesics of DM particles are governed solely by the Schwarzschild metric of the central BH. Our calculation is confined to the radial interval $r_H \leq r \leq 10^5$, where r_H is the event horizon radius and we set $M_{BH} = 1$. This restriction is justified by Fig. 1, which shows that within this range, the total potential Ψ is dominated by the BH and is indistinguishable from that of an isolated point mass. Furthermore, the enclosed mass of the Hernquist profile at $r = 10^5$ is negligible compared to the BH mass, validating the point-mass approximation for the BH's potential at these distances.

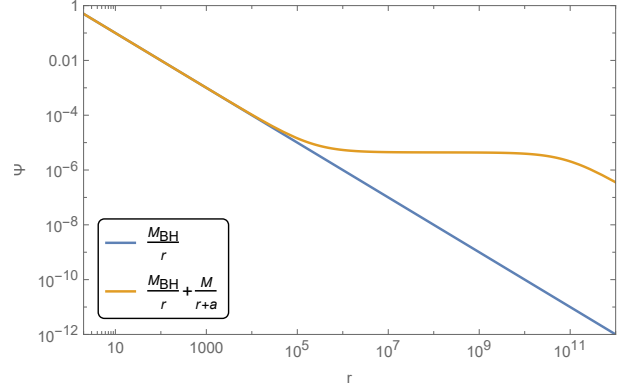


FIG. 1. The relative potential for a BH and an isolated point mass, respectively. This plot shows that these two potentials are similar before $r = 10^5$, where we set $M_{BH} = 1$.

For the relativistic galactic dynamics, the mass current density j^μ of identical particles for a DM distribution is defined by $j_\mu \equiv \rho_M(x) \langle u_\mu \rangle$. From (4), this can be expressed by

$$j_\mu(x) = \int u_\mu \left[M f^{(4)}(x, p) \right] \sqrt{-g^{(S)}} d^4 p, \quad (23)$$

where the integral is performed in the mass shell $p^\mu p_\mu = -\mu^2$. The function $M f^{(4)}(x, p)$ represents the mass density of DM particles at specified phase-space coordinates, and $\sqrt{-g^{(S)}} = r^2 \sin \theta$ for the Schwarzschild BH. The superscript (4) means that $f^{(4)}(x, p)$ is a relativistic DF different from its non-relativistic counterpart $f^{(3)}(\vec{r}, \vec{p})$.

The energy-momentum tensor $T^\mu{}_\nu$ of a galactic system has been statistically established by Einstein, known as the Einstein cluster [63]. Following this construction, we express the energy-momentum tensor as

$$T_\mu{}^\nu(x) \equiv \frac{n(x)}{\mu} \langle p_\mu p^\nu \rangle, \quad (24)$$

where $n(x)$ represents the particle number density. The marker $\langle p_\mu p^\nu \rangle$ denotes the mean value of $p_\mu p^\nu$ for DM particles at spacetime coordinate x , which can be expressed as $\frac{1}{n(x)} \int p_\mu p^\nu \left[N f^{(4)}(x, p) \right] \sqrt{-g^{(S)}} d^4 p$ via the DF. The

energy-momentum tensor is then given by

$$T_{\mu}^{\nu}(x) = \int u_{\mu} u^{\nu} \left[M f^{(4)}(x, p) \right] \sqrt{-g^{(S)}} d^4 p, \quad (25)$$

using (2). The expressions (23) and (25) align with the form of (16) and (17) in relativistic stellar dynamics [68], where the DF “f” represents the particle number density in phase space and corresponds to $N f^{(4)}(x, p)$ in this paper.

Following [68], the DF $f^{(4)}$ further takes the form

$$f^{(4)}(x, p) = f(E, L^2) \delta(\mu' - \mu), \quad (26)$$

where the Dirac delta function is generated from DM particles with identical mass, and the phase space distribution $f(E, L^2)$ has been proven to preserve the form for the adiabatic growth of a BH within the DM halo

$$f(E, L^2) = f^{(H)}[\epsilon(E, L^2)], \quad (27)$$

along with the transformation (22) [11, 12]. Given that $f(E, L^2)$ depends solely on the conserved quantities, it is convenient to transform the integral from the momentum space into the conserved quantity space. The volume element of the momentum space in integrations (23) and (25) can then be converted into

$$\begin{aligned} \sqrt{-g^S} d^4 p &= \sqrt{-g^S} \left| \frac{\partial(p^t, p^r, p^{\theta}, p^{\varphi})}{\partial(\mu', E, L^2, L_z)} \right| d\mu' dE dL^2 dL_z \\ &= \frac{\mu'^3}{2r^2 |u^r| |u_{\theta}| \sin \theta} d\mu' dE dL^2 dL_z, \end{aligned} \quad (28)$$

where

$$u^r = \pm \sqrt{-V_{eff}}, \quad u_{\theta} = \pm \sqrt{L^2 - L_z^2 \sin^{-2} \theta}, \quad (29)$$

read from (18) and (19).

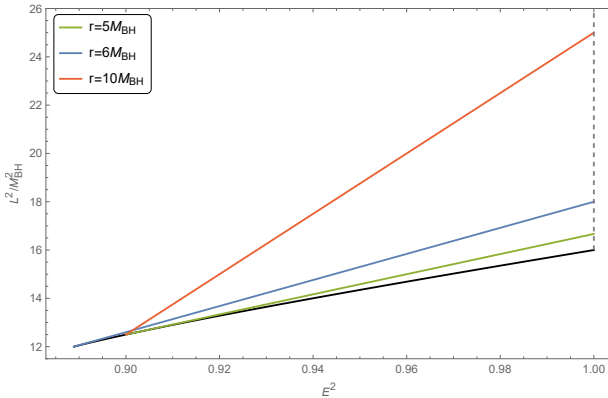


FIG. 2. This plot shows the integration region (E, L) of integrals (30) and (31) for each fixed r . The colored lines represent the maximal L^2 for a given r and E .

Combining (26), (27) and (28), the mass current density

(23) and the energy-momentum tensor (25) can be written as

$$j_{\mu} = \int u_{\mu} \left(\mu^3 M f^{(H)}[\epsilon(E, L^2)] \right) \times \frac{dE dL^2 dL_z}{2r^2 |u^r| |u_{\theta}| \sin \theta}, \quad (30)$$

$$T_{\mu}^{\nu} = \int u_{\mu} u^{\nu} \left(\mu^3 M f^{(H)}[\epsilon(E, L^2)] \right) \times \frac{dE dL^2 dL_z}{2r^2 |u^r| |u_{\theta}| \sin \theta}. \quad (31)$$

It is remarkable that, although we begin from the (x, p) phase space, the resulting expressions for the mass current density Eq. (30) and the energy-momentum tensor Eq. (31) are independent of the DM particle mass μ , except for the term $\mu^3 M f^{(3)}[\epsilon(E, L^2)]$. The latter, however, corresponds exactly to the DF Eq. (12) of the Hernquist profile, which depends only on the macroscopic parameters (M, a, ϵ) . Therefore, the calculation does not require any knowledge of the specific DM particle; it relies solely on macroscopic parameters—obtained from observations or numerical simulations—together with the orbital properties of a unit-mass timelike particle near the BH. This approach circumvents the need for a specific DM particle model and offers a universal description of the DM spike around a BH.

The integration ranges in Eqs. (30) and (31) are defined by the set of all bound timelike geodesics in the Schwarzschild background given by Eq. (14). As specified by Eqs. (17) and (18), the integral over L_z for fixed L and θ is taken over the symmetric interval $(-L \sin \theta, L \sin \theta)$. The critical (minimum) and maximum values of the total angular momentum L^2 are given, respectively, by

$$L_{\text{cri}}^2 = \frac{32M_{BH}^2}{-27E^4 + 36E^2 + E(9E^2 - 8)^{3/2} - 8}, \quad (32)$$

$$L_{\text{max}}^2 = \frac{E^2 r^3}{r - 2M_{BH}} - r^2. \quad (33)$$

The critical angular momentum L_{cri}^2 is represented by the black curve in the (E^2, L^2) plane of Fig. 2, while L_{max}^2 is indicated by colored curves for specific radii: green for $r = 5M_{BH}$, blue for $r = 6M_{BH}$, and red for $r = 10M_{BH}$. The gray dashed line in the same figure marks the bound condition $E^2 \leq 1$, indicating that the particle lacks sufficient energy to escape to infinity. Thus, for a given radius r , the integration domain in the (E^2, L^2) -plane is the region enclosed by the corresponding maximal L^2 curve (green, blue, or red), the critical L^2 curve (black), and the horizontal line $E^2 = 1$ (gray dashed). The intersection point of the critical L^2 curve and the maximal L^2 curve defines the minimum energy E within the integration domain for a given r , which can be expressed as

$$E_{\text{min}} = \begin{cases} \left(1 - \frac{2M_{BH}}{r}\right) \cdot \left(1 - \frac{3M_{BH}}{r}\right)^{-\frac{1}{2}}, & 4 \leq \frac{r}{M_{BH}} \leq 6, \\ \left(1 + \frac{2M_{BH}}{r}\right) \cdot \left(1 + \frac{6M_{BH}}{r}\right)^{-\frac{1}{2}}, & \frac{r}{M_{BH}} \geq 6. \end{cases} \quad (34)$$

Among the remaining variables in the volume element (28), the variable μ' is constrained (or fixed) by the delta function in (26) and is thus integrated out. We refer readers to reference [11, 12] for further details of the derivation.

We now examine each component of j_μ . As shown in Eq. (29), the r - and θ -components of the 4-velocity u_μ each carry a $\pm$ sign for a given set of (E, L^2, L_z) . Since the in-

tegration runs over both positive and negative values of u_r and u_θ , these contributions cancel, and thus j_r and j_θ in Eq. (30) vanish. For the φ -component, we note that $u_\varphi = L_z$, and the integration over L_z spans the symmetric interval $(-L \sin \theta, L \sin \theta)$. Since the integrand in j_φ is an odd function of L_z , integration over this symmetric range also yields zero. Therefore, the only nonvanishing component of Eq. (30) is

$$\begin{aligned} j_t &= 4 \int (-E) \times \frac{\mu^3 M f^{(H)} [\epsilon(E, L^2)]}{2r^2 \sqrt{E^2 - \left(\frac{r-2M_{BH}}{r}\right) \left(1 + \frac{L^2}{r^2}\right)} \sqrt{L^2 \sin^2 \theta - L_z^2}} dE dL^2 dL_z \\ &= -4\pi \int_{E_{min}}^1 E dE \int_{L_{cri}}^{L_{max}} L dL \frac{\mu^3 M f^{(H)} [\epsilon(E, L^2)]}{r^2 \sqrt{E^2 - \left(\frac{r-2M_{BH}}{r}\right) \left(1 + \frac{L^2}{r^2}\right)}}, \end{aligned} \quad (35)$$

where the extra factor of 4 in front of the integral in the first line is attributed to the inclusion of both positive and negative values of u_r and u_θ . This extra factor 4 is also included in the following derivation of the energy-momentum tensor. Here we have used (16) and (12).

The analysis of the energy-momentum tensor follows the reasoning previously applied to j_μ . Components of T_μ^ν containing a single subscript or superscript r or θ vanish upon in-

tegration, since the integrand takes both positive and negative values of u_r and u_θ . Furthermore, the integral for T_r^θ (or T_θ^r) reduces to an expression of the form $\int [2u_r u_\theta - 2u_r u_\theta] \dots$, which cancels identically. Similarly, the components T_t^φ and T_φ^t vanish for the same reason as j_φ : the integrand is an odd function of L_z , and the integration domain is symmetric about zero. As a result, the only nonvanishing components of T_μ^ν are the diagonal ones, which can be expanded as

$$T_t^t = -2\pi \int_{E_{min}}^1 E^2 dE \int_{L_{cri}}^{L_{max}} dL \frac{\mu^3 M f^{(H)} [\epsilon(E, L^2)]}{(r^2 - 2M_{BH} r) \sqrt{E^2 - \left(\frac{r-2M_{BH}}{r}\right) \left(1 + \frac{L^2}{r^2}\right)}}, \quad (36)$$

$$T_r^r = 2\pi \int_{E_{min}}^1 dE \int_{L_{cri}}^{L_{max}} dL \sqrt{E^2 - \left(\frac{r-2M_{BH}}{r}\right) \left(1 + \frac{L^2}{r^2}\right)} \frac{\mu^3 M f^{(H)} [\epsilon(E, L^2)]}{r^2 - 2M_{BH} r}, \quad (37)$$

$$T_\theta^\theta = T_\varphi^\varphi = \pi \int_{E_{min}}^1 dE \int_{L_{cri}}^{L_{max}} L^2 dL \frac{\mu^3 M f^{(H)} [\epsilon(E, L^2)]}{r^4 \sqrt{E^2 - \left(\frac{r-2M_{BH}}{r}\right) \left(1 + \frac{L^2}{r^2}\right)}}. \quad (38)$$

For the nonnegative DF (12), the component T_t^t is nonpositive for $r > r_H = 2M_{BH}$, while the remaining diagonal components are both nonnegative, which indicates that the system exhibit positive energy and positive pressure.

III. SPACETIME OF THE BH EMBEDDED IN THE DM SPIKE

In this section, we investigate the backreaction of a DM spike on the BH. We begin by reviewing the standard static, spherically symmetric ansatz and the corresponding Einstein and conservation equations. We then discuss the boundary conditions for solving these equations and briefly present the numerical method.

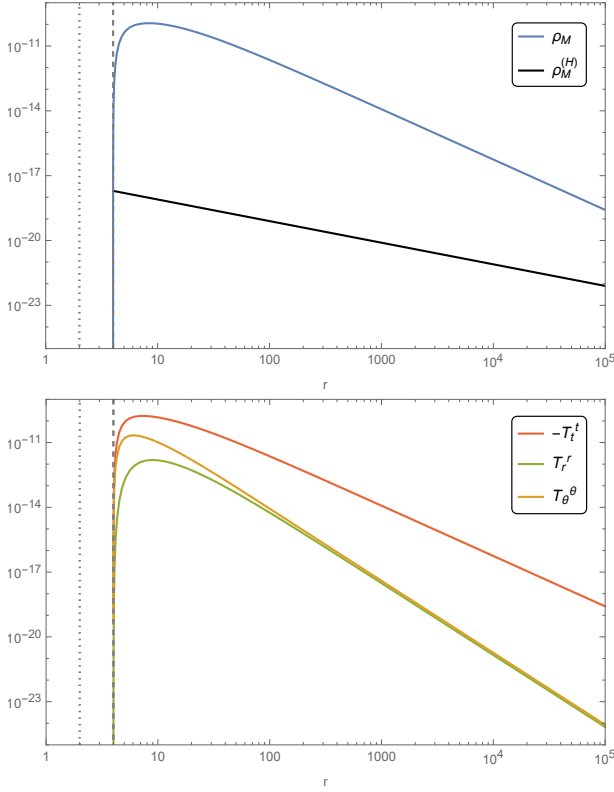


FIG. 3. The mass density (upper) and energy-momentum tensor (bottom) for the DM profile around a Schwarzschild BH. The black line in the upper panel shows the initial mass density for the Hernquist profile. After the adiabatic growth of a BH, ρ_M forms a “spike” near the BH. Components of T_μ^ν exhibit a similar spike, although their peak values differ by orders of magnitude.

A. Einstein equation

The general metric ansatz for a static, spherically symmetric BH is given by

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2, \quad (39)$$

with two unknown functions $g(r)$ and $m(r)$. Conventionally, the DM can be modeled as an anisotropic fluid with the energy-momentum tensor in the comoving frame given by

$$T_\mu^\nu = \text{Diag}\{-\rho_E(r), p_r(r), p_t(r), p_t(r)\}, \quad (40)$$

where $\rho_E(r)$ denotes the energy density of fluid, $p_r(r)$ is the radial pressure and $p_t(r)$ is the tangential pressure. The equality $T_\theta^\theta = T_\varphi^\varphi = p_t$ follows from the spherical symmetry of the spacetime, as can be verified from (38).

The Einstein equation and the conservation equation can

then be given by

$$\mathcal{E}_\mu^\nu \equiv G_\mu^\nu - 8\pi T_\mu^\nu = 0, \quad (41)$$

$$C_\nu \equiv \nabla^\mu T_{\mu\nu} = 0. \quad (42)$$

Employing the ansatz (39), the nonvanishing components of this system are given by

$$\mathcal{E}_t^t = 8\pi\rho_E(r) - \frac{2m'(r)}{r^2} \quad (43)$$

$$\mathcal{E}_r^r = \frac{(r - 2m(r))g'(r)}{r^2g(r)} - \frac{2m(r)}{r^3} - 8\pi p_r(r) \quad (44)$$

$$\begin{aligned} \mathcal{E}_\theta^\theta = & \frac{(r - 2m(r))g''(r)}{2rg(r)} - \frac{g'(r)(r(m'(r) - 1) + m(r))}{2r^2g(r)} \\ & - \frac{(r - 2m(r))g'(r)^2}{4rg(r)^2} + \frac{m(r) - rm'(r)}{r^3} - 8\pi p_t(r) \end{aligned} \quad (45)$$

$$C_r = \frac{1}{2}p_r(r) \left(\frac{g'(r)}{g(r)} + \frac{4}{r} \right) + \frac{\rho_E(r)g'(r)}{2g(r)} + p_r'(r) - \frac{2p_t(r)}{r}, \quad (46)$$

with $\mathcal{E}_\varphi^\varphi = \mathcal{E}_\theta^\theta$. There are four equations with five independent variables. However, we can show that (45) is equivalent to (46). From (43) and (44), one can find that $m'(r) \rightarrow 4\pi r^2 \rho_E(r)$ and $g(r) \rightarrow c_1 \exp \left(\int_{r_h}^r \frac{2[4\pi p_r(\hat{r})\hat{r}^3 + m(\hat{r})]}{\hat{r}[\hat{r} - 2m(\hat{r})]} d\hat{r} \right)$, here c_1 represents an integration constant. Employing such substitutions, (45) and (46) become

$$\begin{aligned} \frac{1}{4\pi r} \mathcal{E}_\theta^\theta = C_r = & \frac{p_r(r)(-3m(r) + 4\pi r^3 p_r(r) + 2r)}{r(r - 2m(r))} \\ & + \frac{\rho_E(r)(m(r) + 4\pi r^3 p_r(r))}{r^2 - 2rm(r)} + p_r'(r) - \frac{2p_t(r)}{r}. \end{aligned} \quad (47)$$

Therefore, the number of independent equations is actually three.

By substituting the expressions (36), (37) and (38) for the energy-momentum tensor and applying the Schwarzschild metric ($g \rightarrow 1 - 2M_{BH}/r$, $m \rightarrow 1$) to the conservation equation (46), we find that equation (46) is automatically satisfied, regardless of the specific form of the DF. However, under these conditions, the equations (43), (44) and (45) are not satisfied by the Schwarzschild solution, which is known to fulfill only the vacuum Einstein equation $G_\mu^\nu = 0$. Expanding the expression (25) of the energy-momentum tensor in a general metric shows that the conservation equation still holds. This indicates that the validity of the conservation equation does not rely on the Einstein field equations themselves, provided that the definition (25) of T_μ^ν employs the same metric functions (g, m) appearing in equation (46).

We compute the energy-momentum tensor of the DM spike in a fixed Schwarzschild BH spacetime and investigate its backreaction on the geometry. In this analysis, equations (43) and (44) are adopted as constraint equations to determine the metric functions (g, m). The choice is motivated by

the fact that the conservation equation (47) is inherently satisfied in the Schwarzschild background, where the DM energy-momentum tensor is constructed via a statistical DF that ensures covariant conservation. Once the backreaction of the DM spike is taken into account, however, the spacetime deviates from Schwarzschild, and this statistical construction no longer guarantees exact conservation, hence (47) cannot be simultaneously fulfilled in the perturbed metric.

In realistic astrophysical systems [66], the DM spike surrounding a supermassive BH is expected to be extremely dilute, resulting in only a weak gravitational backreaction. Therefore, the iterative treatment adopted here adequately captures the essential physics. To fully solve the system of equations (43), (44) and (47), one would need to compute the energy-momentum tensor in a fully numerical BH background. Such a computation lies beyond the scope of the present work. As will be demonstrated in the next section, the resulting BH solution remains close to the Schwarzschild metric but exhibits a slightly stronger deviation compared to the alternative approach presented in [47].

For comparison, an alternative methodology adopted in [39, 47] solves the full system of equations (43), (44) and (47) by taking ρ_E or ρ_m as an input quantity. This input is typically obtained either by multiplying the DM halo profile by a cutoff factor or by evaluating it in a fixed Schwarzschild BH spacetime within the Einstein cluster framework. In this formulation, the DM spike is modeled as an anisotropic fluid with vanishing radial pressure. Consequently, the three independent variables in the system can be consistently matched to the three independent field equations, enabling a self-consistent treatment of the coupled system.

B. Boundary conditions

We discuss the boundary conditions for solving equations (43) and (44). We begin with the simple equation (43), which possesses a straightforward solution $m(r) = \int_{r_H}^{\infty} 4\pi r'^2 \rho_E(r') dr'$. The function $m(r)$ is hence named the “mass function” for the metric, and represents the energy distribution on the spacetime. Recall that T_{μ}^{ν} vanishes at radius $r_{cri} = 4M_{BH}$, $m(r)$ is solely contributed by the BH mass M_{BH} at $r < r_{cri}$. The boundary condition of $m(r)$ at the event horizon is given by

$$m(r_H) = m(r_{cri}) = M_{BH}. \quad (48)$$

From $g_{rr} = (1 - \frac{2m(r)}{r})^{-1}$, the radius for the event horizon is simply $r_H = 2M_{BH}$. Substituting (48) and $p_r = 0$ into (44), this equation becomes

$$(r^2 - 2M_{BH} r)g'(r) = 2M_{BH} g(r), \quad r_H \leq r < r_{cri}, \quad (49)$$

which gives the solution $g(r) = 1 - \frac{2M_{BH}}{r}$ in this interval. One can find that the solutions of metric functions are exactly the Schwarzschild BH for $r < r_{cri}$, which is attributed to the truncation of the DM distribution at $r = r_{cri}$.

Since the contribution of the DM spike begins at radius r_{cri} ,

we numerically solve the first-order differential equations (43) and (44) in the interval $r_{cri} \leq r \leq 10^5$. Boundary conditions at r_{cri} can be imposed

$$g(r_{cri}) = 1 - \frac{2M_{BH}}{r_{cri}} = \frac{1}{2}, \quad m(r_{cri}) = M_{BH}, \quad (50)$$

based on the requirement of continuity.

The system of equations (43) and (44), subject to the boundary conditions (50), is solved using a spectral method. We expand the metric functions in a series of Chebyshev polynomials. The governing equations and boundary conditions are then discretized at Chebyshev collocation points, transforming the problem into a system of algebraic equations for the series coefficients. Starting from an initial guess (the Schwarzschild solution in this work), this nonlinear system is solved iteratively via the Newton-Raphson method.

IV. RESULTS

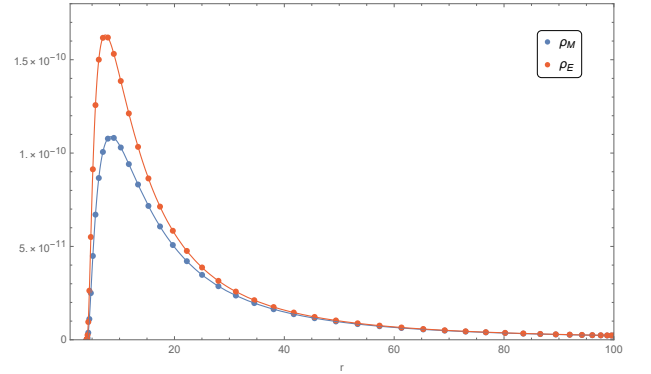


FIG. 4. The comparison between the mass density ρ_M (blue) and the energy density ρ_E (red) of the DM spike. At the peak, the maximum of ρ_E is approximately 1.50 times the maximum of ρ_M .

This section presents our results for both the energy-momentum tensor of the DM spike and the corresponding BH solution influenced by it. As in previous sections, the BH mass is fixed at $M_{BH} = 1$ for all calculations.

We begin with the radial distributions of the mass density (upper panel) and the components of T_{μ}^{ν} (lower panel) in Fig. 3. The dotted and dashed lines in both panels mark the radius of the BH horizon and the marginally bound orbit, respectively. The upper panel compares the initial Hernquist profile $\rho_M^{(H)}$ (black) with the mass density ρ_M (blue) after the adiabatic growth of the BH. The formation of a dense spike in ρ_M , orders of magnitude higher than the initial profile, successfully reproduces the findings of [12]. In the lower panel, the components of the energy-momentum tensor reveal a hierarchy in their peak magnitudes within the spike: the energy density ρ_E surpasses the radial pressure by two orders of magnitude. This finding offers some justification for the neglect of radial pressure in studies like [39, 47]. However, it is crucial

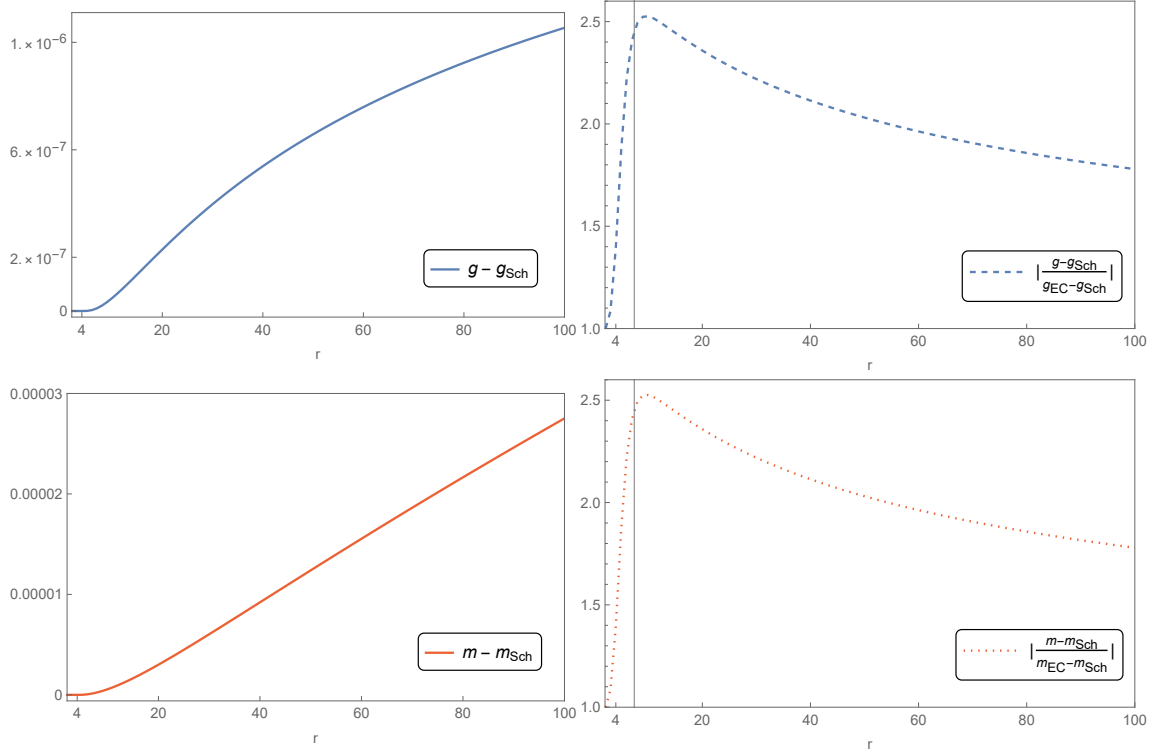


FIG. 5. The deviation of the numerical solutions $g(r)$ (upper left) and $m(r)$ (bottom left) from their Schwarzschild counterparts g_{Sch} and m_{Sch} , and the ratio of deviations of our resulting metric to those of g_{EC} and m_{EC} obtained by an alternative approach from [47] in the right panels. The subscript “EC” represents that the alternative approach with the Einstein cluster model. The gray thin lines in right panels show the radius where the energy density ρ_E becomes a maximum value.

to note that this result is derived under the specific assumption that all particles at a given point follow circular geodesics [64, 69]. We emphasize that the radial component T_r^r is non-vanishing. Beyond the spike region, the radial pressure p_r and tangential pressure p_t become comparable in magnitude, indicating that p_r cannot be neglected, at least in the far zone. In fact, T_r^r can vanish only if the square root term in Eq. (37) is zero. For the integral in Eq. (37) at a fixed r , generic elliptical geodesics with non-zero square roots (except at $r = 4$ or ∞) contribute, leading to a nonzero T_r^r .

To validate our solution, we check for potential violations of the energy conditions by the derived energy-momentum tensor. The conditions are defined as follows: the Null Energy Condition requires $\rho_E + p_i \geq 0$; the Weak Energy Condition requires $\rho_E \geq 0$ and $\rho_E + p_i \geq 0$; the Strong Energy Condition requires $\rho_E + p_i \geq 0$ and $\rho_E + \sum_i p_i \geq 0$; and the Dominant Energy Condition requires $\rho_E \geq |p_i|$. Here, p_i refers to any spatial component of the tensor given in Eq. (40). As is clear from Fig. 3, all these energy conditions are respected in the region occupied by DM.

The comparison between the mass density ρ_M and the energy density ρ_E is shown in Fig. 4, highlighting the limitation of using ρ_M as a substitute for ρ_E when determining the metric under the influence of DM. In the vicinity of the DM spike, particles influenced by the BH exhibit both high density and non-negligible kinetic energy. Consequently, a substantial

portion of the energy sourcing the gravitational field is carried in this kinetic form. Fig. 4 shows that the peak value of ρ_E is approximately 1.50 times that of ρ_M , indicating that about 50% of the energy density in the spike manifests as kinetic energy of the DM particles. Beyond the spike, the difference between the two densities diminishes rapidly, as the maximum velocity of bound particles in the far region becomes increasingly small (with the specific relativistic energy approaching unity). This behavior is consistent with the analysis of the radial potential Eq. (19) in Sect. II.

We present the deviation of numerical metric functions $g(r)$ and $m(r)$, from their Schwarzschild counterparts (g_{Sch} , m_{Sch}) around the spike ($r_H \leq r \leq 10^2$) in the left panels of Fig. 5. The upper left panel shows that the modified metric function $g(r)$ exhibits a slight but yet distinct deviation (the blue line) from the Schwarzschild case, on the order of 10^{-7} – 10^{-6} . Notably, the magnitude of this deviation exceeds that of the energy-momentum tensor itself. The deviation of mass function $m(r)$, shown by the red line in the bottom left panel, reveals a more pronounced difference from the Schwarzschild solution, with deviations of order 10^{-5} .

Although these deviations are small due to the use of Milky Way parameters, our resulting metric exhibits a stronger DM effect than that (g_{EC} , m_{EC}) obtained by an alternative approach adopted in [47]. The ratios of the deviation, defined as $\frac{g - g_{\text{Sch}}}{g_{\text{EC}} - g_{\text{Sch}}}$ and $\frac{m - m_{\text{Sch}}}{m_{\text{EC}} - m_{\text{Sch}}}$, are depicted by the blue dashed line and red dotted line in right panels of Fig. 5, respectively. Particu-

larly, near the peak of the energy density ρ_E (indicated by the thin gray line), the deviations in our results both exceed those of the alternative method by a factor of 2.5. Two key differences distinguish our method. First, we use the full energy density ρ_E Eq. (36) rather than the rest-mass density ρ_M Eq. (35). Second, we consistently include the non-vanishing radial pressure p_r Eq. (37), as derived from the adiabatic growth model. This enhancement is primarily attributed to the inclusion of particle kinetic energy. Consequently, our model captures not only the enhanced density of DM particles near the BH but also the gravitational influence of their significant orbital motion.

V. DISCUSSION

In this work, we have studied the energy-momentum tensor of a DM spike formed through the adiabatic growth of a BH within a galactic halo, and investigated its backreaction on the spacetime geometry. Within the Einstein cluster framework, by adopting the Hernquist density profile as a representative example of DM halo and employing parameters consistent with those of the Milky Way, we derived a statistical formulation of the full tensor, explicitly including the kinetic contribution to the energy density and the anisotropic pressure arising from noncircular orbits. Our results show that the kinetic term enhances the total energy density near the spike by approximately 50% compared with the rest-mass component, while a small but nonvanishing radial pressure introduces mild anisotropy in the stress tensor. Using this tensor as a fixed source in the Einstein equations, we numerically obtained a static, spherically symmetric metric modified by the DM distribution. The resulting spacetime exhibits a slight but measurable deviation from the Schwarzschild solution—roughly a factor of two larger than that found in previous treatments considering only the mass density. Furthermore, the derived energy-momentum tensor satisfies all standard energy condi-

tions, confirming the internal consistency of the model and the physical viability of our relativistic description of the DM spike.

Compared with previous treatments that modeled the DM spike as a static or purely circular-orbit fluid [47], our approach provides a more realistic and dynamical description of the system. The inclusion of kinetic energy and radial pressure naturally leads to stronger deviations of the metric than in earlier simplified analyses, while remaining physically justified since realistic DM spikes around supermassive BHs are expected to be extremely dilute and to generate only weak gravitational fields. This confirms that the semi-consistent iterative procedure employed here captures the dominant physical effects of the DM backreaction while maintaining computational tractability.

Although our model is semi-consistent—the metric and distribution function are obtained iteratively rather than solved simultaneously—it provides a physically grounded step toward a fully self-consistent relativistic description of BHs embedded in DM halos. Achieving complete self-consistency

would require recalculating the DM distribution in a numerically determined metric throughout the adiabatic growth process, which poses a significant computational challenge. While the present analysis adopts a Hernquist profile for the initial halo, the method can be readily generalized to other realistic DM halo models such as NFW [10], Einasto [70], or cored isothermal profiles [71]¹. Extending this framework to include rotating (Kerr) BHs, non-adiabatic evolution, more general DM distributions, and dynamical analyses related to gravitational wave emission will be the focus of future work.

ACKNOWLEDGMENTS

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¹ The latter, however, should be regarded as effective parametrizations rather than equilibrium solutions of collisionless systems, since core-like structures generally require additional physics such as self-interactions or baryonic feedback.

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