ON RELLICH-TYPE ASYMPTOTICS FOR EIGENFUNCTIONS ON RANK ONE SYMMETRIC SPACES OF NONCOMPACT TYPE

PRITAM GANGULY

Abstract. We study eigenfunctions of the Laplace–Beltrami operator Δ_X in exterior domains Ω of rank-one Riemannian symmetric spaces of noncompact type X, a class that includes all hyperbolic spaces. Extending the classical L^2–Rellich theorem for the Euclidean Laplacian, we investigate the asymptotic behavior and L^p–integrability of solutions to the Helmholtz equation

$$\Delta_X f + (\lambda^2 + \rho^2) f = 0 \quad \text{in } \Omega,$$

where $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$ and ρ is the half-sum of positive roots.

We obtain sharp Rellich-type quantitative L^p–growth estimates of f in geodesic annuli, leading to the nonexistence of $L^p(\Omega)$–solutions for the optimal range $1 \leq p \leq 2$ and spectral parameters λ satisfying $|\operatorname{Im}(\lambda)| \leq (2/p - 1)\rho$. As a by-product of our study, we also establish a Rellich-type uniqueness theorem for eigenfunctions in terms of Hardy-type norms. Our results geometrically extend the Euclidean Rellich theorem, revealing how exponential volume growth and the dependence of the L^p–spectrum of Δ_X on p give rise to genuinely non-Euclidean spectral phenomena.

1. Introduction and main results

The study of the asymptotic behavior of eigenfunctions of differential operators, such as the Laplacian and Schrödinger operators, is a central theme in spectral theory and mathematical physics. Such results are particularly valuable for ruling out the presence of eigenvalues embedded in the continuous spectrum of the time-independent Schrödinger operator, a fact of immense importance in quantum mechanics. A classical result in this direction, describing an asymptotic estimate of eigenfunctions at infinity, is the following theorem, originally established by Rellich [34, Satz 1] in 1943.

Theorem 1.1 (Rellich). *Let $\lambda > 0$, and $\Omega := \{x \in \mathbb{R}^n : |x| > R_0\}$ for some $R_0 > 0$. Assume that $0 \neq f \in C^2(\Omega)$ satisfies the Helmholtz equation*

$$\Delta_{\mathbb{R}^n} f + \lambda f = 0 \qquad \text{in } \Omega. \tag{1.1}$$

Then there exists $R_1 > R_0$ sufficiently large, and a constant C (depending on λ, f, and n) such that for all $R > R_1$ one has

$$\int_{R<|x|<2R} |f(x)|^2 \, dx \geq CR. \tag{1.2}$$

As is well known (see [34, Anwendungen 2, p. 59]), a direct consequence of (1.2) is that the Helmholtz equation (1.1) admits no nontrivial L^2–solutions in Ω. We must point out that establishing such a result on exterior domains (complements of bounded regions) is

2020 *Mathematics Subject Classification.* Primary: 43A85, 35J05, Secondary: 47A75, 58J50.

Key words and phrases. Spectrum of the Laplace-Beltrami operator, eigenfunctions, Rellich-type estimates, Riemannian symmetric spaces of noncompact type, spherical functions.

considerably more delicate, whereas in the full space $\mathbb{R}^n$ the nonexistence of L^2–solutions to $\Delta_{\mathbb{R}^n} f + \lambda f = 0$ with $\lambda > 0$ follows immediately from the fact that the Fourier transform provides a unitary equivalence between the Laplacian and multiplication by $|\xi|^2$ on $L^2(\mathbb{R}^n)$. Moreover, Rellich's L^2-asymptotic estimate, combined with unique continuation of Jerison and Kenig [21], implies the absence of positive eigenvalues for the Schrödinger operator $H = -\Delta_{\mathbb{R}^n} + V(x)$ when the potential V is compactly supported. The importance of the compactly supported case becomes clearer when one recalls the work of von Neumann and Wigner, who showed that highly oscillatory potentials can produce positive eigenvalues for the one-dimensional operator $H = -\frac{d^2}{dx^2} + V(x)$ (See [33, p.223]). Later, in 1959, Kato [24] extended this phenomenon to all dimensions $n \geq 2$, constructing potentials V with $(1+|x|)V(x) \in L^\infty(\mathbb{R}^n)$ for which the associated Schrödinger operator admits positive eigenvalues. The subtle nature and usefulness of Rellich's theorem subsequently inspired extensive research. Littman [28] generalized Rellich's result to partial differential equations with constant coefficients, while further developments for Schrödinger operators were due to Kato [24]. Building on this, Agmon [1] and Simon [35] extended Kato's framework to potentials of the form $V = V_1 + V_2$, where V_1 satisfies Kato's condition and V_2 is a real-valued long-range term. In 2003, Ionescu and Jerison [19] proved the absence of positive eigenvalues for H under suitable L^q-decay assumptions on V, removing the earlier compact-support restriction. More recently, Rellich-type theorems have been established for generalized oscillators on $\mathbb{R}^n$, characterizing the growth of eigenfunctions [37], and for discrete Schrödinger operators [20].

Returning to Rellich's paper [34], he also noted another striking consequence of (1.2): if a solution to (1.1) satisfies the size estimate

$$\lim_{|x|\to\infty} |x|^{\frac{n-1}{2}} |f(x)| = 0, \tag{1.3}$$

then necessarily $f = 0$. In other words, nontrivial eigenfunctions with positive eigenvalue cannot decay arbitrarily fast at infinity, unlike harmonic functions ($\lambda = 0$), which may vanish arbitrarily fast even at infinity.

Banerjee and Garofalo [3] recently revisited Rellich-type results, beginning with the observation that the size estimate (1.3) is, in fact, sharp. Indeed, recall that the standard Euclidean spherical functions, given by

$$\phi_\lambda(x) := \int_{S^{n-1}} e^{-i\sqrt{\lambda}\, x\cdot\omega}\, \mathrm{d}\sigma(\omega) = c_n \frac{J_{n/2-1}(\lambda|x|)}{(\lambda|x|)^{n/2-1}}, \qquad x \in \mathbb{R}^n, \tag{1.4}$$

are well known to be radial solutions of the equation $\Delta_{\mathbb{R}^n} f + \lambda f = 0$ for $\lambda > 0$. Using the following well-known asymptotic behavior of the Bessel function J_ν for $\nu > -\frac{1}{2}$ (see, e.g., [13, B.8]),

$$J_\nu(r) = r^{-1/2}\left(\frac{2}{\pi}\right)^{1/2} \cos\left(r - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + R_\nu(r), \qquad |R_\nu(r)| \leq c_\nu r^{-3/2}, \quad (r > 1), \tag{1.5}$$

one observes that ϕ_λ fails to satisfy (1.3). Furthermore, the same asymptotic behavior (1.5) implies that $\phi_\lambda \in L^p(\Omega)$ if and only if $p > \frac{2n}{n-1}$.

Motivated by this observation, Banerjee and Garofalo investigated whether Rellich's theorem admits an L^p extension for $p \leq \frac{2n}{n-1}$; equivalently, whether nontrivial solutions of (1.1) can exist in $L^p(\Omega)$. Their main result, proved in [3], is as follows.

Theorem 1.2 (Banerjee–Garofalo [3])**.** *Let λ and Ω be as in Theorem 1.1. Assume that f is a solution of* (1.1)*. If $f \in L^p(\Omega^*)$ with $\Omega^* := \{x \in \mathbb{R}^n : |x| > R_0 + 4\}$ for some $0 < p \le \frac{2n}{n-1}$, then $f \equiv 0$ in Ω.*

As shown in [3], this L^p version follows from Rellich's L^2 asymptotic estimate (1.2). For $p \in \left(2, \frac{2n}{n-1}\right]$, Hölder's inequality yields an L^p analogue of (1.2), which immediately implies that no nontrivial $L^p(\Omega^*)$ solutions can exist. For $0 < p < 2$, the proof instead relies on a Moser-type submean value estimate.

The L^p extension of Rellich's theorem due to Banerjee and Garofalo represents an important development in the study of Rellich-type problems beyond the classical L^2 framework. As a continuation of this line of work, Banerjee and Garofalo [4] established Rellich-type inequalities for the Baouendi–Grushin operators, using Carleman estimates and weighted integral inequalities adapted to their subelliptic structure. In a subsequent paper [5], they extended Theorem 1.2 to uniformly elliptic divergence-form operators with asymptotically flat coefficients, covering the range $0 < p < \frac{2n}{n-1}$ while leaving the endpoint case open. These contributions fit naturally into a broader program aimed at understanding Rellich-type phenomena for a wider class of differential operators.

In the present work, we investigate analogous questions in the setting of rank-one Riemannian symmetric spaces of noncompact type, a class that includes the real and complex hyperbolic spaces as fundamental examples, where exponential volume growth and curvature play a significant role in determining the behaviour of eigenfunctions. We commence by introducing the necessary notational framework to capture the intricacies of this setting and proceed to state the main results of the paper. Any notation or concept not explicitly explained at this stage will be clarified in Section 2.

Let $X = G/K$ denote a rank-one Riemannian symmetric space of noncompact type. Here, G is a connected, noncompact, semisimple Lie group of real rank one with finite center, and K is a maximal compact subgroup of G. We denote by d the geodesic distance function on X, and by Δ_X the Laplace–Beltrami operator associated with the G-invariant Riemannian metric on X. It is well known that, for any $x \in X$, the volume of a geodesic ball $B(x, r)$ of radius r satisfies

$$|B(x,r)| \asymp e^{2\rho r}, \qquad (r > 1)$$

where $\rho > 0$ is a constant determined by the structure of X (see (2.1)). This exponential volume growth plays a decisive role in the analysis on X.

For $\lambda \in \mathbb{C}$, the elementary spherical functions given by

$$\varphi_\lambda(x) = \int_K e^{-(i\lambda+\rho)H(x^{-1}k)} dk \qquad (x \in X) \tag{1.6}$$

are important eigenfunctions of Δ_X which satisfies $\Delta_X \varphi_\lambda + (\lambda^2 + \rho^2)\varphi_\lambda = 0$. For $\lambda \in \mathbb{R}$, the objects $e^{-(i\lambda+\rho)H(x^{-1}k)}$ are analogous to the functions $x \mapsto e^{-i\lambda x \cdot \omega}$ on $\mathbb{R}^n$, and the above representation (1.6) can be compared with (1.4), the situation in the Euclidean context. This, in view of the self-adjointness, as consequence, implies that the L^2-spectrum of $-\Delta_X$ is given by $S_2(-\Delta_X) = [\rho^2, \infty)$. However, as is well-known, for $1 \le p < \infty$, the L^p-spectrum is given by the following parabolic neighbourhood of the half-line $[\rho^2, \infty)$: ([29], [38, Prop. 2.2])

$$S_p(-\Delta_X) = \{z^2 + \rho^2 : |\operatorname{Im}(z)| \le |\tfrac{2}{p} - 1|\rho\}. \tag{1.7}$$

In other words, unlike the Euclidean Laplacian, the behavior of the L^p-spectrum of Δ_X depends essentially on the value of p. This is one of the most striking differences between spaces of polynomial volume growth and those of exponential volume growth ([16, 17]). As a matter of fact, in contrast to the Euclidean case $\mathbb{R}^n$, where the L^p-Rellich-type asymptotic can be derived from the L^2 case as shown in [3], it is natural in the setting of symmetric spaces to expect an independent L^p-Rellich-type asymptotic, particularly when dealing with eigenvalues having a nonzero imaginary part. As one of the main results of this article, we establish the following asymptotic behavior, in the sense of Rellich, for the eigenfunctions of the Laplace–Beltrami operator in the complement of a cavity.

Theorem 1.3. *Let $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, and set*

$$\Omega := \{x \in X : d(o,x) > R_0\}, \qquad R_0 > 0.$$

Suppose $0 \neq f \in C^2(\Omega)$ satisfies the Helmholtz equation

$$\Delta_X f + (\lambda^2 + \rho^2) f = 0 \qquad \text{in } \Omega. \tag{1.8}$$

Then there exists $R_1 > R_0$ (sufficiently large) such that, for every $R > R_1$, the following estimates hold:

(1) If $\mathrm{Im}(\lambda) = 0$, then

$$\int_{R<d(o,x)<2R} |f(x)|^2 \, dx \;\geq\; C\, R. \tag{1.9}$$

(2) If $1 \leq p < 2$ and $\mathrm{Im}(\lambda) \neq 0$, then

$$\int_{R<d(o,x)<2R} |f(x)|^p \, dx \;\geq\; C \begin{cases} e^{\,p(\gamma_p\rho - |\,\mathrm{Im}(\lambda)|)\, R}, & \text{if } |\,\mathrm{Im}(\lambda)| \neq \gamma_p \rho, \\ R, & \text{if } |\,\mathrm{Im}(\lambda)| = \gamma_p \rho, \end{cases} \tag{1.10}$$

where $\gamma_p = \dfrac{2}{p} - 1$.

Here $C > 0$ denotes a constant depending on X, λ, p (when relevant), and f, but independent of R.

Remark 1.4. The above theorem, in the spirit of Rellich's result (Theorem 1.1), clarifies the quantitative behavior of eigenfunctions across different spectral regimes. In contrast to the Euclidean setting, the present framework naturally includes the case of non-real spectral parameters, which arise naturally in the analysis on symmetric spaces of noncompact type. The relevance of such spectral regions becomes apparent when one examines the structure of the $L^p(X)$–spectrum of Δ_X (viz. (1.7)).

When $\mathrm{Im}(\lambda) = 0$, the theorem shows that the L^2–mass of eigenfunctions in geodesic annuli grows at least linearly in R, in agreement with the classical Euclidean behavior. For $\mathrm{Im}(\lambda) \neq 0$ and $|\,\mathrm{Im}(\lambda)| < \gamma_p\rho$, the L^p–mass exhibits at least exponential growth in R, reflecting the dominant influence of the exponential volume growth of X in this spectral region. At the critical spectral value $|\,\mathrm{Im}(\lambda)| = \gamma_p\rho$, this growth transitions from exponential to linear. In the supercritical regime $|\,\mathrm{Im}(\lambda)| > \gamma_p\rho$, the estimate reveals an exponential decay of the L^p–mass with respect to R, indicating that the decay rate of the eigenfunction outweighs the exponential volume growth of the underlying space.

This trichotomy—exponential growth, linear growth, and exponential decay—is intrinsic to spaces of exponential volume growth and has no direct analogue in the Euclidean setting.

Furthermore, when $|\operatorname{Im}(\lambda)| \le \gamma_p \rho$, the growing lower bounds for the integrals in the preceding theorem demonstrate the *non-integrability* of the corresponding eigenfunctions at infinity. As a consequence, we obtain the following result, which constitutes our second main theorem.

Theorem 1.5. *Let $1 \le p \le 2$, and let Ω be as in Theorem 1.3. Let $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$ satisfy $|\operatorname{Im}(\lambda)| \le \gamma_p \rho$. Assume that f is a solution of (1.8) in Ω such that*

$$\int_\Omega |f(x)|^p \, dx < \infty. \tag{1.11}$$

Then $f = 0$ in Ω.

This result may be regarded as a Liouville-type theorem for eigenfunctions in domains containing a cavity. In view of the structure of the $L^p(X)$ spectrum (1.7), it shows the absence of L^p point spectrum for the Laplace–Beltrami operator in such domains Ω.

Remark 1.6. A few remarks are in order.

(1) **Sharpness of the theorem.** The theorem is sharp in the following sense. For $p > 2$ and any $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, there exist eigenfunctions in Ω with eigenvalue $-(\lambda^2 + \rho^2)$ satisfying (1.11). Moreover, for fixed $1 \le p \le 2$, whenever $|\operatorname{Im}(\lambda)| > \gamma_p \rho$, one can find eigenfunctions in Ω satisfying (1.10) with the same eigenvalue, thereby confirming the *sharpness of the spectral region* described in the theorem. See Remark 3.5 for details.

In establishing this, we employ the elementary spherical function φ_λ and the function Φ_λ appearing in the Harish–Chandra expansion of φ_λ (see (2.8)). We recall that Φ_λ is an eigenfunction of Δ_X, defined away from the identity, with the same eigenvalue as φ_λ (See Section 2.4 for details). In fact, in the analysis of eigenfunctions on Ω, Φ_λ serves as the principal model, while in the full space X, the role is played by φ_λ.

(2) **On the restriction** $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$. Analytically speaking, our approach in proving Theorem 1.3 is based on reducing the Helmholtz equation (1.8) to a hypergeometric differential equation with a regular singularity, via spherical harmonic expansion. The exclusion in the spectral parameter λ is then necessary to ensure the existence and linear independence of the two fundamental solutions near the regular singular point at 1; see Remark 3.2 for further details.

Philosophically, this restriction reflects a deeper harmonic–analytic phenomenon inherent in the Harish–Chandra expansion of the spherical function φ_λ (see (2.8)). The Harish–Chandra c–function is meromorphic with poles along $i\mathbb{Z}$, and the existence and linear independence of the elementary spherical components $\Phi_{\pm\lambda}$ require that these singular parameters be avoided. Moreover, for certain subsets of $i\mathbb{Z}$, the Poisson transform—which provides the analytic characterization of joint eigenfunctions—ceases to be injective. Therefore, the restriction $\lambda \notin i\mathbb{Z}$ is not merely a technical artifact of our method, but rather a condition that aligns naturally with the general spectral theory of symmetric spaces: it guarantees both the analytic regularity of the Harish–Chandra series and the simplicity (nondegeneracy) of the Poisson transform at the given spectral parameter. (See Section 2.4)

(3) **Critical exponent.** In the setting of rank-one symmetric spaces of noncompact type, as noted above, the critical exponent in the theorem is $p = 2$, independent of the dimension of X. This contrasts sharply with the Euclidean case $\mathbb{R}^n$, where the

corresponding result (Theorem 1.2) has the critical endpoint $p = \frac{2n}{n-1}$, which exceeds 2 and depends explicitly on n. This fundamental difference reflects the influence of exponential volume growth and the asymptotic behavior of spherical functions on X.

(4) **Non-positivity of the Laplace–Beltrami operator in exterior domains** As already mentioned, for $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, the functions $\Phi_{\pm\lambda}$ are eigenfunctions of Δ_X in Ω with eigenvalue $-(\lambda^2 + \rho^2)$. From their asymptotic estimates (2.12), it follows that Φ_λ belongs to $L^2(\Omega)$ when $\mathrm{Im}(\lambda) > 0$, whereas $\Phi_{-\lambda}$ does so when $\mathrm{Im}(\lambda) < 0$. This unveils an intriguing analytical feature: although the Laplace–Beltrami operator $-\Delta_X$ is positive on the full space X, its restriction to the exterior region Ω may admit complex eigenvalues (point spectrum).

We now provide a comparison with two other related works addressing Rellich-type theorems in geometric settings. In a recent interesting article, Ballmann, Mukherjee, and Polymerakis [2] investigated spectral properties of Certain Hadamard manifolds. Among other results, they established a Rellich-type theorem in the general framework of a negatively curved Hadamard manifold M that is asymptotically harmonic about a point $\xi \in M_\infty$. This class of manifolds notably includes rank-one symmetric spaces of noncompact type as a special case. More precisely, they proved that the point spectrum of the associated Laplace–Beltrami operator in the complement of a horoball in M with center ξ vanishes. In other words, this result provides the L^2-version of Theorem 1.5 with the domain Ω taken to be the complement of a horoball. It is worth noting that, in general, horoballs are defined as sublevel sets of the associated Busemann function. In the context of a rank-one symmetric space G/K, horospheres are the orbits of conjugates of N in the Iwasawa decomposition $G = KAN$, and horoballs coincide with the corresponding sublevel sets bounded by these horospheres (See for instance [15, Ch.II]). Consequently, horoballs are typically noncompact. In contrast, in the present work we consider the complements of geodesic balls centered at the identity coset $o = eK$, which are compact. Thus, the two geometric configurations—complements of horoballs versus complements of geodesic balls—are fundamentally different. Furthermore, the analytical methods employed in [2], and the present paper are of a distinct nature: while the approach in [2] relies on tools from geometric analysis and asymptotic harmonicity, our proof follows the classical idea of Rellich, based on the spherical harmonic expansion adapted to the noncompact symmetric space setting.

Another related contribution is due to Chen and Liu [10], who investigated scattering theory on real hyperbolic spaces. In their analysis, they proved a somewhat related Rellich-type L^2 uniqueness theorem for solutions to the Helmholtz equation in the exterior of geodesic balls, within the setting of real hyperbolic spaces $\mathbb{H}^n$, where the underlying group structure is given by $SO_0(1, n)/SO(n)$. In comparison, our result extends this L^2-uniqueness property to the more general class of rank-one symmetric spaces of noncompact type, while retaining the same geometric interpretation in terms of complements of geodesic balls.

Next, we turn to the latter part of this article, where we record some interesting phenomena exhibited by eigenfunctions in domains containing a cavity. Setting aside the feature of the L^p spectrum discussed above, Theorem 1.5 may be viewed as a standalone result describing the admissible size estimates of eigenfunctions in Ω in terms of their L^p norms. There are, nevertheless, several well-known results in the literature that characterize eigenfunctions of Δ_X in X satisfying various size estimates as Poisson transforms of suitable boundary data—for instance, functions belonging to appropriate Lebesgue classes or finite measures—on the boundary K/M of X. In what follows, we focus on two such results involving Hardy-type norm estimates, which we now state.

Theorem 1.7 ([6, 8, 18, 27]). *Let $\lambda \in \mathbb{C}$, and $u \in C^\infty(X)$ be such that $\Delta_X u + (\lambda^2 + \rho^2)u = 0$ in X.*

(1) Let $\mathrm{Im}(\lambda) < 0$ or $\lambda = 0$, and $1 < p \leq \infty$. Then there exists $F \in L^p(K/M)$ such that $u = \mathcal{P}_\lambda F$ if and only if

$$\sup_{t>0} \frac{1}{\varphi_{i\,\mathrm{Im}(\lambda)}(a_t)} \left(\int_K |u(ka_t)|^p \, dk \right)^{\frac{1}{p}} < \infty. \tag{1.12}$$

If $p = 1$, then $u = \mathcal{P}_\lambda \mu$ for some signed measure μ on K/M.

(2) Let $\mathrm{Im}(\lambda) = 0$ with $\lambda \neq 0$, and $1 \leq p \leq \infty$. Then there exists $F \in L^p(K/M)$ such that $u = \mathcal{P}_\lambda F$ if and only if

$$\sup_{t>0} e^{\rho t} \left(\int_K |u(ka_t)|^p \, dk \right)^{\frac{1}{p}} < \infty.$$

Here $\mathcal{P}_\lambda F$ stands for the Poisson transform on F, see (2.15).

The part (1) of the above theorem was established by Ben Said et al. [6], while Bousseria-Sami [8] and Ionescu [18] proved part (2) for the range $p \geq 2$. It was further conjectured in [8] that the result should also hold for $1 < p < 2$, which was subsequently confirmed affirmatively by Kumar-Ray-Sarkar[27].

Motivated by the role of such Hardy-type norms in the characterization of eigenfunctions, we establish—as a byproduct of our study of Rellich-type theorems in the present setting—the following result concerning admissible size estimates for eigenfunctions in the complement of a cavity.

Theorem 1.8. *Let $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, $\epsilon > 0$, $1 \leq p \leq \infty$ and let Ω be as in Theorem 1.3. Assume that $f \in C^2(\Omega)$ is a solution of* (1.8) *in Ω. Then there exists an explicit positive function ψ_λ (See* (3.26)*) such that if*

$$\sup_{t>R_0} \frac{t^\epsilon}{\psi_\lambda(a_t)} \left(\int_K |f(ka_t)|^p \, dk \right)^{\frac{1}{p}} < \infty, \tag{1.13}$$

(with the usual modification for $p = \infty$) then $f = 0$ in Ω. Moreover, the condition $\epsilon > 0$ is sharp *in the sense that if $\epsilon = 0$, there exists a nontrivial eigenfunction f in Ω satisfying the estimate* (1.13)

Note that while Theorem 1.8 provides an L^p–characterization of eigenfunctions on the entire symmetric space X via the Poisson transform—where the growth of u along geodesic spheres is governed by Hardy-type norms—our result, Theorem 1.9, concerns eigenfunctions on the *complement of a compact cavity* in X and establishes a *Rellich-type uniqueness* theorem. We also highlight an additional structural feature, stated in the following remark.

Remark 1.9. The function ψ_λ in our result satisfies $\psi_\lambda(a_t) \asymp |\Phi_{-i\,\mathrm{Im}(\lambda)}(a_t)|$ for large t whenever $\mathrm{Im}(\lambda) < 0$ (see (3.27)), where $\Phi_{-i\,\mathrm{Im}(\lambda)}$ denotes the term appearing in the Harish–Chandra expansion of $\varphi_{i\,\mathrm{Im}(\lambda)}$ (see (2.8)) that was used in Theorem 1.7 to characterize eigenfunctions. This function is itself an eigenfunction in Ω (though not necessarily on X) corresponding to the same eigenvalue as $\varphi_{i\,\mathrm{Im}(\lambda)}$; see Section 2.4 for details. Moreover, when

$\operatorname{Im}(\lambda) = 0$, one has $\psi_\lambda(a_t) = e^{-\rho t}$ (see (3.26)). These observations highlight a natural correspondence between our result and the Poisson-type characterization discussed earlier (Theorem 1.7), albeit arising in a distinct geometric setting.

We conclude the introduction with a brief outline of the paper. Section 2 collects the necessary preliminaries on rank-one Riemannian symmetric spaces of noncompact type. In Section 3, we present the proofs of the main results. The paper ends with some concluding remarks, highlighting related questions and possible directions for future research.

Notations. We employ the notation C, c, C_1, c_1, C_2, c_2, etc., for constants whose values may differ at each occurrence. For two nonnegative functions f and g, we write $f \lesssim g$ (respectively, $f \gtrsim g$) to mean that there exists a constant $C > 0$ such that $f \leq Cg$ (respectively, $f \geq Cg$). The notation $f \asymp g$ signifies that both $f \lesssim g$ and $f \gtrsim g$ hold. We also use standard asymptotic notations: $f \sim g$ means that $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$, while $f = o(g)$ means that $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$.

2. Riemannian symmetric spaces of noncompact type

In this section, we introduce the notations and relevant concepts related to the theory of Riemannian symmetric spaces of noncompact type. Most of these are standard and widely available in sources such as [14, 15].

Let X be a Riemannian symmetric space of noncompact type. Then X can be realized as the quotient space $X = G/K$ where G is a connected, noncompact real semi-simple Lie group with finite center and K a maximal compact subgroup of G. The group G naturally acts on X by left translations $l_g : xK \mapsto gxK,\ g \in G$. We denote the base point eK simply by o.

2.1. **Basic structure theory and Riemannian metric.** We denote the Lie algebras of G and K by the corresponding Gothic fraktur notations $\mathfrak{g}$, and $\mathfrak{k}$, respectively. The corresponding Cartan decomposition reads as $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{p}$ where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. The dimension of $\mathfrak{a}$ which is independent of its choice is called the rank of the symmetric space X. In what follows, we assume that $\dim \mathfrak{a} = 1$. Now denoting the real dual of $\mathfrak{a}$ by $\mathfrak{a}^*$, we let $\Sigma \subset \mathfrak{a}^*$ be the restricted system of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. For $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha$ denote the associated root space and we set $\dim \mathfrak{g}_\alpha = m_\alpha$. Fix a positive Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$ which leads to the corresponding set of positive roots Σ^+. Let $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Then $N = \exp \mathfrak{n}$ is a simply connected nilpotent Lie group and $A = \exp \mathfrak{a}$ is an abelian group. Let M be the centralizer of A in K. Now the usual product map $(k, a, n) \mapsto kan$ from $K \times A \times N$ to G is an analytic diffeomorphism which is not a group homomorphism. This leads to the Iwasawa decomposition: $G = KAN$. In view of this decomposition an element $g \in G$ can be uniquely written as $g = k \exp(H(g))n$ where $k \in K, n \in N$ and $H(g) \in \mathfrak{a}$. The group G also admits a polar decomposition $G = K\overline{A^+}K$ a.k.a. the Cartan decomposition where $A^+ := \exp \mathfrak{a}_+$. Let M be the centralizer of A in K. Then we note that the groups M and A normalize N and K/M is the Furstenberg boundary of X.

Now since in our case $\dim \mathfrak{a} = 1$, it is well-known that Σ is either $\{\pm\gamma\}$ or $\{\pm\gamma, \pm 2\gamma\}$ where γ is a positive root, and the associated Weyl group $W = \{\pm I\}$. It is customary to denote the half sum of the positive roots (counted with multiplicities) by ρ, i.e., $\rho = \frac{1}{2}(m_\gamma + 2m_{2\gamma})\gamma$. Let $H_0 \in \mathfrak{a}$ denote the unique element so that $\gamma(H_0) = 1$ leading to the identification of

$\mathfrak{a}$ with $\mathbb{R}$ via the map $t \mapsto tH_0$ $(t \in \mathbb{R})$. Consequently, the maps $t \mapsto t\gamma$ $(t \in \mathbb{R})$, and $z \mapsto z\gamma$ $(z \in \mathbb{C})$ identifies $\mathfrak{a}^*$ with $\mathbb{R}$ and its complexification $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$, respectively. Under this identification, we follow the customary abuse of notation and denote

$$\rho = \rho(H_0) = \tfrac{1}{2}(m_\gamma + 2m_{2\gamma}). \tag{2.1}$$

Moreover, the group A can be realized as $A = \{a_t := \exp(tH_0) : t \in \mathbb{R}\}$ and subsequently $A^+ = \{a_t : t > 0\}$. The Lebesgue measure on $\mathbb{R}$ induces a Haar measure on A which we simply denote by $da_t = dt$.

In view of the Iwasawa decomposition, the map

$$A \times N \longrightarrow X, \qquad (a_t, n) \mapsto a_t n \cdot o,$$

is a surjective diffeomorphism. Consequently, the dimension of X is given by $n = \dim(X) = m_\gamma + m_{2\gamma} + 1$. Moreover, as is well known from the Cartan decomposition, the double cosets Ka_tK and Ka_sK coincide if and only if $t = \pm s$. Hence, every point $x \in X$ can be written in the form $x = ka_t \cdot o$, for some $k \in K$ and a unique $t \geq 0$.

The Killing form B on $\mathfrak{g}$ induces a K-invariant inner product on $\mathfrak{p}$, and thus defines a G-invariant Riemannian metric on X together with an associated Riemannian measure, denoted by dx. Viewing a function f on X as a right K-invariant function on G, the Haar measure dg on G can be normalized so that

$$\int_X f(x)\, dx = \int_G f(g)\, dg.$$

For any integrable function f on X, the integration formula corresponding to the polar decomposition takes the form

$$\int_X f(x)\, dx = \int_K \int_0^\infty f(ka_t)\, J(t)\, dt\, dk, \tag{2.2}$$

where dk denotes the normalized Haar measure on K, and the Jacobian of the polar decomposition is given by

$$J(t) = c_n (2\sinh t)^{m_\gamma + m_{2\gamma}} (\cosh t)^{m_{2\gamma}}. \tag{2.3}$$

The metric induced by the Killing form endows X with the structure of a Riemannian manifold. The corresponding Riemannian distance is denoted by $d(\cdot,\cdot)$, and Δ_X denotes the associated Laplace–Beltrami operator. We now describe the explicit form of Δ_X in polar co-ordinates, which will play a central role in the subsequent analysis. For this, we closely follow Helgason[14, p.309, Ch.II].

2.2. **Polar co-ordinates and Laplace-Beltrami operator.** For $\alpha \in \Sigma^+$, let

$$\mathfrak{k}_\alpha := \{T \in \mathfrak{k} : \ (\mathrm{ad}(H))^2 T = \alpha(H)^2 T \text{ for } H \in \mathfrak{a}\},$$

and $\{T_1^\alpha, \ldots, T_{m_\alpha}^\alpha\}$ a basis of $\mathfrak{k}_\alpha$, orthonormal with respect to the negative of the Killing of $\mathfrak{g}$. We set

$$\omega_\alpha := \sum_{i=1}^{m_\alpha} T_i^\alpha \cdot T_i^\alpha \tag{2.4}$$

where the T_i^α are viewed as left-invariant differential operators on G. Then it is easy to see that ω_α is bi-invariant under M. As proved in [14, Theorem 5.24], the Laplace-Beltrami operator admits the following representation in polar co-ordinates:

$$\Delta_X f(ka\cdot o) = \left(\Delta_A + \sum_{\alpha\in\Sigma^+} m_\alpha (\coth\alpha) A_\alpha\right)_a f(ka\cdot o) \\ + \sum_{\alpha\in\Sigma^+} \sinh^{-2}(\alpha(\log a))((\mathrm{Ad}(a^{-1})\omega_\alpha)f)(ka) \tag{2.5}$$

where Δ_A denotes the Laplacian on $A\cdot o$, and A_α is given by

$$\langle A_\alpha, H\rangle = \alpha(H) \qquad (H\in\mathfrak{a},\ \alpha\in\Sigma^+).$$

Next, we describe a useful closed form expression for Δ_X in the co-ordinate system $\exp(tH_0)\to t$. First note that, since for $f\in C^\infty(A)$,

$$H_0 f(\exp(tH_0) = \frac{d}{dt} f(\exp(tH_0),$$

H_0 is identified with $\frac{d}{dt}$ in this co-ordinate system. Now observe that, as $\mathfrak{a}$ is one-dimensional, there exists C_γ such that $A_\gamma = C_\gamma H_0$. Now observe that

$$B(H_0,H_0) = 2(m_\gamma\gamma(H_0)^2 + m_{2\gamma}(2\gamma(H_0))^2) = 2(m_\gamma + 4m_{2\gamma})$$

whence $A_\gamma = (2(m_\gamma+4m_{2\gamma}))^{-1}\ H_0 = (2(m_\gamma+4m_{2\gamma}))^{-1}\ \frac{d}{dt}$. Similarly, $A_{2\gamma} = (m_\gamma + 4m_{2\gamma}^{-1}\ \frac{d}{dt}$. Subsequently, the Laplacian Δ_A on A takes the form $(2(m_\gamma+4m_{2\gamma}))^{-1}\ \frac{d^2}{dt^2}$. As a result, the Laplace-Beltrami (2.5) takes the following form

$$\Delta_X = \frac{1}{2}(m_\gamma + 4m_{2\gamma})^{-1}\left(\frac{d^2}{dt^2} + (m_\gamma\coth t + 2m_\gamma\coth(2t))\frac{d}{dt}\right) \\ + \sinh^{-2}(t)(\mathrm{Ad}(a_t^{-1})\omega_\gamma) + \sinh^{-2}(2t)(\mathrm{Ad}(a_t^{-1})\omega_{2\gamma}). \tag{2.6}$$

2.3. **Spherical harmonics.** We now describe the system of spherical harmonics adapted to the present setting. To this end, we make use of certain irreducible representations of K admitting M–fixed vectors. Let $\widehat{K_0}$ denote the set of all irreducible unitary representations of K having nontrivial M–fixed vectors. For $\delta\in\widehat{K_0}$, let V_δ be the finite-dimensional Hilbert space on which δ is realized. As shown in [26, Theorem 6], the subspace of V_δ consisting of M–fixed vectors is one-dimensional. Hence, V_δ contains a unique normalized M–fixed vector, denoted by v_1. Let $\{v_1, v_2, \dots, v_{d_\delta}\}$ be an orthonormal basis of V_δ. For each $\delta\in\widehat{K_0}$ and $1\le j\le d_\delta$, define

$$Y_{\delta,j}(kM) = (v_j,\, \delta(k)v_1)_{L^2(K/M)}, \qquad kM\in K/M. \tag{2.7}$$

It follows immediately that $Y_{\delta,1}(eK)=1$ and that $Y_{\delta,1}$ is M–invariant.

Proposition 2.1. *The collection* $\{Y_{\delta,j} : 1\le j\le d_\delta,\ \delta\in\widehat{K_0}\}$ *forms an orthonormal basis of* $L^2(K/M)$.

For a proof, we refer the reader to [15, Theorem 3.5, Section 3, Chapter 5]; see also the discussion at the beginning of [14, Section 2, Chapter 3].

An explicit realization of $\widehat{K_0}$ can be obtained by identifying K/M with the unit sphere in $\mathfrak{p}$. Let $\mathcal{H}^m$ denote the space of homogeneous harmonic polynomials of degree m restricted to the unit sphere. Then the classical spherical harmonic decomposition takes the form

$$L^2(K/M) = \bigoplus_{m=0}^{\infty} \mathcal{H}^m.$$

Each V_δ is contained in some $\mathcal{H}^m$, and the functions $Y_{\delta,j}$ may thus be identified with spherical harmonics.

Theorem 2.2 ([15, Theorem 11.2]). *For each $\delta \in \widehat{K_0}$, there exist unique integers $q \geq p \geq 0$ such that*

$$\delta(\omega_\gamma)\big|_{V_\delta^M} = \frac{p(p+m_{2\gamma}-1) - q(q+m_\gamma+m_{2\gamma}-1)}{2(m_\gamma+4m_{2\gamma})},$$

$$\delta(\omega_{2\gamma})\big|_{V_\delta^M} = -\frac{2p(p+m_{2\gamma}-1)}{m_\gamma+4m_{2\gamma}},$$

with the convention that $p = 0$ whenever $m_{2\gamma} = 0$.

2.4. **Eigenfunctions of the Laplace-Beltrami operator Δ_X.** In this subsection, we briefly recall certain eigenfunctions whose estimates will play a central role in our analysis. We begin with the *elementary spherical functions*. For $\lambda \in \mathbb{C}$, these are defined by

$$\varphi_\lambda(x) = \int_K e^{-(i\lambda+\rho)H(x^{-1}k)}\, dk, \qquad x \in X.$$

It is well known that φ_λ is a K-biinvariant function on G satisfying

$$\varphi_\lambda(o) = 1,\ \varphi_\lambda = \varphi_{-\lambda} \quad \text{for all } \lambda \in \mathbb{C}, \qquad \text{and} \qquad \varphi_\lambda(x) = \varphi_\lambda(x^{-1}) \quad \text{for all } x \in X.$$

Moreover, φ_λ is an eigenfunction of the Laplace–Beltrami operator Δ_X with eigenvalue $-(\lambda^2+\rho^2)$, that is, $\Delta_X\varphi_\lambda + (\lambda^2+\rho^2)\varphi_\lambda = 0$. It follows immediately from the integral representation that the pointwise estimate $|\varphi_\lambda| \leq \varphi_{i\,\mathrm{Im}(\lambda)}$ holds for all $\lambda \in \mathbb{C}$.

To obtain more precise information regarding the behavior of φ_λ away from the identity, we recall the classical *Harish-Chandra expansion* (see Stanton–Tomas [36, Theorem 3.1]; see also [25, p. 7–8]):

$$\varphi_\lambda(a_t) = c(\lambda)\Phi_\lambda(a_t) + c(-\lambda)\Phi_{-\lambda}(a_t), \quad \lambda \in \mathbb{C} \setminus i\mathbb{Z}, \tag{2.8}$$

where

$$\Phi_\lambda(a_t) = e^{(i\lambda-\rho)t} \sum_{k=0}^{\infty} \Gamma_k(\lambda) e^{-2kt}. \tag{2.9}$$

The coefficients $\Gamma_k(\lambda)$ in the above series are defined recursively by $\Gamma_0(\lambda) \equiv 1$, and for $k \geq 1$

$$(k+1)(k+1-i\lambda)\Gamma_{k+1} = \sum_{j=0}^{k} \tfrac{m_\gamma}{2}(\rho+2j-i\lambda)\Gamma_j + \sum_{\substack{j=k+1-2l \\ j,l\geq 0}} m_{2\gamma}(\rho+2j-i\lambda)\Gamma_j.$$

Here $c(\lambda)$ denotes the *Harish–Chandra c–function*, given by [36, (3.2)],

$$c(\lambda) = \Gamma\Big(\frac{m_\gamma}{2}\Big)\,\Gamma\Big(\frac{m_{2\gamma}}{2}\Big)\,\frac{\Gamma(i\lambda)\,\Gamma\Big(\frac{m_\gamma+i\lambda}{2}\Big)}{\Gamma\big(\frac{m_\gamma}{2}+i\lambda\big)\ \Gamma\big(\frac{\rho+i\lambda}{2}\big)}. \tag{2.10}$$

We summarize below several key features of this expansion:

(1) The restriction $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$. The assumption $\lambda \notin i\mathbb{Z}$ ensures that all the coefficients $\Gamma_k(\lambda)$ and $\Gamma_k(-\lambda)$ appearing in the definitions of Φ_λ and $\Phi_{-\lambda}$ are well-defined and that the two solutions remain linearly independent.

From the expression (2.10), it is clear that $c(\lambda)$ is meromorphic with possible poles arising from the *Gamma functions in the numerator* of (2.10). Since $\Gamma(z)$ has poles precisely at $z = 0, -1, -2, \ldots$, these occur exactly when $\lambda \in i\mathbb{Z}_{\geq 0}$. Consequently, both $c(-\lambda)$, and $c(\lambda)$ have no poles in $\mathbb{C} \setminus i\mathbb{Z}$.

(2) The function Φ_λ. The series (2.9) defining Φ_λ converges for $t > c > 0$, away from the identity element $a_0 = e$. Moreover, Φ_λ is an eigenfunction of Δ_X (away from the identity) with eigenvalue $-(\lambda^2 + \rho^2)$. For further details, see Helgason [14, Chapter IV, §5].

(3) Estimates on the coefficients. Gangolli [12] established that there exist positive constants c and d such that

$$|\Gamma_k(\lambda)| \leq c\, k^d \qquad \text{for all } k \geq 1,$$

and furthermore, that these bounds are sharp.

Observe that, from (2.9) and the previously mentioned estimates for the coefficients Γ_k, it follows immediately that

$$\Phi_\lambda(a_t) \sim e^{(i\lambda - \rho)t} \qquad (t \to \infty). \tag{2.11}$$

Consequently, for sufficiently large t, one obtains the asymptotic estimate

$$|\Phi_\lambda(a_t)| \asymp e^{(-\operatorname{Im}(\lambda) - \rho)t}. \tag{2.12}$$

It is well known that the Harish-Chandra c-function $c(\lambda)$ has neither zeros nor poles in the half-plane $\operatorname{Im}(\lambda) < 0$. Within this region, by combining the Harish-Chandra expansion (2.8) with (2.11), one deduces that

$$\lim_{t \to \infty} e^{-(i\lambda - \rho)t}\, \varphi_\lambda(a_t) = c(\lambda).$$

(See [14, Theorem 6.14] for further details.) From this observation, it follows that for $\operatorname{Im}(\lambda) \neq 0$ there exists $t_\lambda > 0$ such that

$$|\varphi_\lambda(a_t)| \asymp e^{(|\operatorname{Im}(\lambda)| - \rho)t} \qquad (t > t_\lambda). \tag{2.13}$$

Also, for $\lambda = 0$, one has the classical estimates

$$e^{-\rho t} \lesssim \varphi_0(a_t) \lesssim (1 + t)e^{-\rho t}, \qquad t > 0.$$

In view of estimate (2.13), for $p > 2$, we have

$$\varphi_\lambda \in L^p(X) \qquad \text{whenever} \qquad |\operatorname{Im}(\lambda)| < \left(1 - \frac{2}{p}\right)\rho. \tag{2.14}$$

Consequently, for every $p > 2$ each spectral parameter λ satisfying the above inequality gives rise to a nontrivial L^p–eigenfunction of $-\Delta_X$. Equivalently, every point of the interior of the $L^p(X)$–spectrum of $-\Delta_X$ (see (1.7)) is contained in the point spectrum of $-\Delta_X$ on $L^p(X)$.

We now turn to a broader class of eigenfunctions. Recall that the complex powers of the Poisson kernel,

$$x \longmapsto e^{-(i\lambda + \rho)H(x^{-1}k)}, \qquad \lambda \in \mathbb{C},\ k \in K,$$

are eigenfunctions of the Laplace–Beltrami operator Δ_X corresponding to the eigenvalue $-(\lambda^2+\rho^2)$. For $\lambda \in \mathbb{C}$ and $F \in L^1(K/M)$, the *Poisson transform* of F is defined by

$$\mathcal{P}_\lambda F(x) = \int_K F(k)\, e^{-(i\lambda+\rho)H(x^{-1}k)}\, dk, \qquad x \in X. \tag{2.15}$$

It is immediate that $\mathcal{P}_\lambda F$ satisfies the eigenvalue equation

$$\Delta_X(\mathcal{P}_\lambda F) + (\lambda^2+\rho^2)\,\mathcal{P}_\lambda F = 0.$$

Moreover, when $F \equiv 1$, the definition yields $\mathcal{P}_\lambda F = \varphi_\lambda$. In fact, by the general theory of harmonic analysis on symmetric spaces, every eigenfunction of the Laplace–Beltrami operator can be realized as a Poisson transform of an appropriate function (or, more generally, of an analytic functional) on the boundary K/M. In particular, for the Dirac measure δ_{k_0} at $k_0 \in K$, one has

$$\mathcal{P}_\lambda \delta_{k_0}(x) = e^{-(i\lambda+\rho)H(x^{-1}k_0)}.$$

We recall that a parameter $\lambda \in \mathbb{C}$ is called *simple* if the map $F \mapsto \mathcal{P}_\lambda F$ (the Poisson transform) is injective (see [15, p. 248]). It is well known (see [15, Theorem 6.5]) that λ is simple if and only if $e(\lambda) \neq 0$, where $1/e(\lambda)$ is the denominator of the Harish–Chandra c–function (2.10). For nonsimple parameters, the Poisson transform acquires a nontrivial kernel and the Harish–Chandra expansion degenerates.

Finally, we recall that eigenfunctions of Δ_X satisfying Hardy-type growth or integrability conditions admit a precise characterization in terms of Poisson transforms. Such representations, formulated via Hardy-type norm estimates, are of particular significance for our analysis and have already been stated in the Introduction (see Theorem 1.7).

3. Proofs of main results

In this section, we present the proofs of our main results, namely Theorems 1.3, 1.5, and 1.8. Before proceeding, we record a few notational remarks for the reader's convenience.

Throughout this section, we denote by Ω the complement of a geodesic ball centered at o, defined by

$$\Omega := \{x \in X : d(o,x) > R_0\}, \qquad \text{for some } R_0 > 0. \tag{3.1}$$

Recall that functions on X are regarded as right K–invariant functions on G. Consequently, the expressions $f(ka_t \cdot o)$ and $f(ka_t)$ represent the same object; hence, we shall use them interchangeably whenever no confusion arises. Moreover, as is customary, we shall not distinguish between integration over K and over K/M for notational simplicity. Throughout the sequel, various constants will appear at different stages. Unless explicitly stated otherwise, these constants may depend on structural parameters of the space X; however, since such dependence plays no essential role in our analysis, we shall omit it from the notation for brevity.

We begin with the following proposition, which provides a fundamental estimate for solutions of the Helmholtz-type equation in Ω.

Proposition 3.1. *Let $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, and let Ω be as in* (3.1)*. Assume that $f \in C^2(\Omega)$ is a non-trivial solution of*

$$\Delta_X f + (\lambda^2+\rho^2) f = 0 \qquad \text{in } \Omega. \tag{3.2}$$

Then, there exists $R_1 > R_0$ such that for any $1 \le p \le \infty$ (with the usual modification for $p=\infty$), the following estimates hold:

(i) ***Case* $\mathbf{Im}(\boldsymbol{\lambda}) \neq \mathbf{0}$.** *There exists a constant $C > 0$, depending on λ and p, such that*

$$\int_K |f(ka_t)|^p\, dk \ \geq \ C\, e^{p(\pm \operatorname{Im}(\lambda)-\rho)t}, \qquad (t > R_1). \tag{3.3}$$

(ii) ***Case* $\mathbf{Im}(\boldsymbol{\lambda}) = \mathbf{0}$**. *There exist constants $A, B, C > 0$, and $\theta \in \mathbb{R}$ depending on λ and other constants related to the structure of X, such that*

$$\left(\int_K |f(ka_t)|^p\, dk\right)^{\frac{1}{p}} \geq Ce^{-\rho t}\big(A^2 + B^2 + 2AB\cos(2\lambda t + \theta)\big)^{1/2}, \quad (t > R_0). \tag{3.4}$$

Proof. Let f be as in the statement of the theorem. Using polar co-ordinates $X_{reg} = KA^+ \cdot o$, there exists $t_0 > R_0$ such that $f \neq 0$ on the sphere of radius t_0 viz. $S_{t_0} = \{ka_{t_0} \cdot o : k \in K\}$. Therefore, in view of the Proposition 2.1, it follows that there exists $\delta \in \widehat{K}$, and $1 \leq j \leq d_\delta$ such that the function

$$u(t) := \int_K f(ka_t \cdot o) Y_{\delta,j}(k)\ dk \qquad (t > R_0) \tag{3.5}$$

is not identically zero as $u(t_0) \neq 0$. Now in view of the hypothesis (3.2), we see that

$$\begin{aligned}
0 &= \int_K \Big(\Delta_X f(ka_t \cdot o) + (\lambda^2 + \rho^2) f(ka_t \cdot o)\Big) Y_{\delta,j}(k)\ dk \\
&= \int_K \Delta_X f(ka_t \cdot o) Y_{\delta,j}(k)\ dk + (\lambda^2 + \rho^2) \int_{K/M} f(ka_t \cdot o) Y_{\delta,j}(k)\ dk \\
&= \int_K \Delta_X f(ka_t \cdot o) Y_{\delta,j}(k)\ dk + (\lambda^2 + \rho^2)\ u(t).
\end{aligned} \tag{3.6}$$

But the explicit form of the Laplace-Beltrami operator (2.6) then shows that

$$\begin{aligned}
&\int_K \Delta_X f(ka_t \cdot o) Y_{\delta,j}(k)\ dk \\
&= \frac{1}{2}(m_\gamma + 4m_{2\gamma})^{-1} \int_K \left(\frac{d^2}{dt^2} + (m_\gamma \coth t + 2m_\gamma \coth(2t))\frac{d}{dt}\right) f(ka_t \cdot o) Y_{\delta,j}(k) dk \\
&\quad + \int_K \big(\sinh^{-2}(t)((\operatorname{Ad}(a_t^{-1})\omega_\gamma) f)(ka_t) + \sinh^{-2}(2t)((\operatorname{Ad}(a_t^{-1})\omega_{2\gamma}) f)(ka_t)\big)\, Y_{\delta,j}(k)\ dk
\end{aligned} \tag{3.7}$$

In order to simplify the last expression, for any $\alpha \in \{\gamma, 2\gamma\}$, recalling the definition of ω_α (See (2.4)), and using the linearity of Ad, we notice that

$$\begin{aligned}
&\int_K ((\operatorname{Ad}(a_t^{-1})\omega_\alpha) f)(ka_t) Y_{\delta,j}(k)\ dk \\
&= \sum_{i=1}^{m_\alpha} \int_K ((\operatorname{Ad}(a_t^{-1})(T_i^\alpha)^2) f)(ka_t) Y_{\delta,j}(k)\ dk \\
&= \sum_{i=1}^{m_\alpha} \int_K \left(\frac{d^2}{ds^2}\bigg|_{s=0} f(ka_t \cdot \exp(s\operatorname{Ad}(a_t^{-1})T_i^\alpha) \cdot o) \right) Y_{\delta,j}(k)\ dk
\end{aligned}$$

which upon using

$$\exp(s\operatorname{Ad}(a_t^{-1})T_i^\alpha) = a_t^{-1} \exp(sT_i^\alpha) a_t$$

transforms to

$$\begin{aligned}\sum_{i=1}^{m_\alpha}\int_K\left(\frac{d^2}{ds^2}\bigg|_{s=0} f(k\cdot\exp(sT_i^\alpha)a_t\cdot o)\right)Y_{\delta,j}(k)\ dk\\ =\sum_{i=1}^{m_\alpha}\frac{d^2}{ds^2}\bigg|_{s=0}\int_K\left(f(k\cdot\exp(sT_i^\alpha)a_t\cdot o)\right)Y_{\delta,j}(k)\ dk\end{aligned}$$

The last expression, after the change of variable $k\mapsto k\cdot\exp(-sT_i^\alpha)$, becomes

$$\sum_{i=1}^{m_\alpha}\int_K f(ka_t\cdot o)\ \frac{d^2}{ds^2}\bigg|_{s=0} Y_{\delta,j}(k\cdot\exp(-sT_i^\alpha))\ dk \tag{3.8}$$

But recalling the definition (2.7), we note that

$$\begin{aligned}\sum_{i=1}^{m_\alpha}\frac{d^2}{ds^2}\bigg|_{s=0} Y_{\delta,j}(k\cdot\exp(-sT_i^\alpha)) &= \sum_{i=1}^{m_\alpha}\frac{d^2}{ds^2}\bigg|_{s=0}(v_j,\ \delta(k)\delta(\exp(-sT_i^\alpha))v_1)_{L^2}\\ &=\left(v_j,\ \delta(k)\ \sum_{i=1}^{m_\alpha}\frac{d^2}{ds^2}\bigg|_{s=0}\delta(\exp(-sT_i^\alpha))v_1\right)_{L^2}\\ &=(v_j,\delta(k)\ \delta(\omega_\alpha)v_1)_{L^2},\end{aligned}$$

which then from the above equations yields

$$\int_K((\mathrm{Ad}(a_t^{-1})\omega_\alpha)f)(ka_t)Y_{\delta,j}(k)\ dk=\int_K f(ka_t\cdot o)(v_j,\delta(k)\ \delta(\omega_\alpha)v_1)_{L^2}\ dk \qquad (\alpha\in\{\gamma,2\gamma\}).$$

Therefore, by using Theorem 2.2, we obtain

$$\int_K((\mathrm{Ad}(a_t^{-1})\omega_\alpha)f)(ka_t)Y_{\delta,j}(k)\ dk=u(t)\times\begin{cases}\frac{p(p+m_{2\gamma}-1)-q(q+m_\gamma+m_{2\gamma}-1)}{2(m_\gamma+4m_{2\gamma})} & \text{if } \alpha=\gamma\\ -\frac{2p(p+m_{2\gamma}-1)}{m_\gamma+4m_{2\gamma}} & \text{if } \alpha=2\gamma.\end{cases}$$

Plugging the above into (3.7), we then have

$$\begin{aligned}\int_K\Delta_X f(ka_t\cdot o)Y_{\delta,j}(k)\ dk &= \frac{1}{2}(m_\gamma+4m_{2\gamma})^{-1}\left(\frac{d^2}{dt^2}+(m_\gamma\coth t+2m_\gamma\coth(2t))\frac{d}{dt}\right)u(t)\\ +&\left(\frac{p(p+m_{2\gamma}-1)-q(q+m_\gamma+m_{2\gamma}-1)}{2(m_\gamma+4m_{2\gamma})}\sinh^{-2}(t)-\frac{2p(p+m_{2\gamma}-1)}{m_\gamma+4m_{2\gamma}}\sinh^{-2}(2t)\right)u(t)\end{aligned}$$

which, using (3.6), shows that $u(t)$ solves

$$\begin{aligned}\Bigg(\frac{d^2}{dt^2}+(m_\gamma\coth t+2m_\gamma\coth(2t))\frac{d}{dt}+\big(p(p+m_{2\gamma}-1)-q(q+m_\gamma+m_{2\gamma}-1)\big)\sinh^{-2}(t)\\ -4p(p+m_{2\gamma}-1)\sinh^{-2}(2t)+2(m_\gamma+4m_{2\gamma})(\lambda^2+\rho^2)\Bigg)u(t)=0.\end{aligned}\tag{3.9}$$

Now writing $l=i\lambda-\rho$, and recalling $2\rho=m_\gamma+2m_{2\gamma}$, by an easy calculation (see [39, p. 543] for a similar computation), we rewrite (3.9) as

$$\begin{aligned}\frac{1}{(\cosh t)^{m_{2\gamma}}(\sinh t)^{m_\gamma+m_{2\gamma}}}\ \frac{d}{dt}\left((\cosh t)^{m_{2\gamma}}(\sinh t)^{m_\gamma+m_{2\gamma}}\frac{du}{dt}\right)\\ +\left(\frac{p(p+m_{2\gamma}-1)}{\cosh^2 t}-\frac{q(q+m_\gamma+m_{2\gamma}-1)}{\sinh^2 t}-l(l+m_\gamma+2m_{2\gamma})\right)u(t)=0\end{aligned}\tag{3.10}$$

The above equation can be transformed into a standard hypergeometric differential equation through suitable substitutions. Specifically, set

$$v(t) = \tanh^{-q}(t)\, \cosh^{-l}(t)\, u(t), \quad \text{and then} \quad z = \tanh^2(t).$$

After a straightforward (though somewhat lengthy) computation, equation (3.10) takes the form

$$\frac{d^2v}{dz^2} + \big[c - (a+b+1)z\big]\frac{dv}{dz} - ab\, v(z) = 0, \qquad (z > C > 0), \tag{3.11}$$

where

$$\begin{cases} a = \frac{1}{2}(q - l - p), \\ b = \frac{1}{2}\big(q - l - p - m_{2\gamma} + 1\big), \\ c = q + \frac{1}{2}\big(m_\gamma + m_{2\gamma} + 1\big). \end{cases} \tag{3.12}$$

Since z is bounded away from the origin, and $c - a - b = i\lambda \notin \mathbb{Z}$ in view of the hypothesis, two linearly independent solutions of this hypergeometric equation are given by (See [30, Ch. 5, Sec. 10.4])

$$v_1(z) = {}_2F_1(a, b; a+b-c+1; 1-z), \quad v_2(z) = (1-z)^{c-a-b}{}_2F_1(c-a, c-b; c-a-b+1; 1-z),$$

where ${}_2F_1$ denotes the Gauss hypergeometric function. For more about hypergeometric functions, we refer the reader to [30, Ch. 5, Sec. 9], see also [31, Ch. 15].

Transforming back to the variable t, and recalling that $l = i\lambda - \rho$, we conclude that for $t > R_0$ there exist constants c_1, c_2 such that

$$u(t) = c_1 u_1(t) + c_2 u_2(t),$$

where u_1 and u_2 are explicitly given by

$$u_1(t) := (\tanh t)^q (\cosh t)^{i\lambda - \rho}\ {}_2F_1\left(\tfrac{q-l+p}{2}, \tfrac{q-l-p-m_{2\gamma}+1}{2}; 1 - i\lambda; 1 - (\tanh t)^2\right), \tag{3.13}$$

and

$$\begin{aligned} u_2(t) &:= (\tanh t)^q (\cosh t)^{-i\lambda - \rho} \\ &\times {}_2F_1\left(\tfrac{p+l+q-m_\gamma+m_{2\gamma}+1}{2}, \tfrac{m_\gamma+2m_{2\gamma}+p+q+l}{2}; 1 + i\lambda; , 1 - (\tanh t)^2\right). \end{aligned} \tag{3.14}$$

Here we have used the relation $c - a - b = i\lambda$, which follows from (3.12).

Using the facts that $\tanh t \to 1$, $\cosh t \sim \frac{1}{2}e^t$ as $t \to \infty$, and the property of the hypergeometric function ${}_2F_1(a, b; c; 0) = 1$, we obtain

$$u_1(t) = e^{(i\lambda + \rho)t}\big(1 + o(1)\big), \qquad u_2(t) = e^{(-i\lambda+\rho)t}\big(1 + o(1)\big), \quad (t \to \infty).$$

Consequently,

$$u(t) = e^{-\rho t}\big(C_1 e^{i\lambda t} + C_2 e^{-i\lambda t}\big)\ (1 + o(1)) \qquad (t \to \infty). \tag{3.15}$$

For convenience, we introduce the notation

$$g_\lambda(t) = C_1 e^{i\lambda t} + C_2 e^{-i\lambda t} \qquad (\lambda \in \mathbb{C}, t > R_0).$$

From the above estimate (3.15) it follows that for $t > R_0$, there exist $C > 0$ such that

$$|u(t)| \geq C e^{-\rho t} |g_\lambda(t)|. \tag{3.16}$$

Therefore, recalling the definition of $u(t)$ from (3.5), for any $1 \leq p \leq \infty$, applying Hölder inequality, we see that

$$Ce^{-\rho t}|g_\lambda(t)| \leq \left| \int_K f(ka_t \cdot o) Y_{\delta,j}(k)\ dk \right| \leq \left(\int_K |f(ka_t)|^p\ dk \right)^{\frac{1}{p}} \left(\int_K |Y_{\delta,j}(k)|^{p'}\ dk \right)^{\frac{1}{p'}}$$

and subsequently, using the fact that $\|Y_{\delta,j}\|_{L^{p'}(K)}$ is always non-zero, we note that there exists a constant $C_3 > 0$ such that

$$e^{-\rho t}|g_\lambda(t)| \leq C_3 \int_K |f(ka_t)|^p\ dk \qquad (t > R_0). \tag{3.17}$$

In what follows, we analyze the behavior of the function $|g_\lambda(t)|$ in different regimes depending on the value of λ.
Case I: $\mathrm{Im}(\lambda) \neq 0$.
We first consider the case where $\mathrm{Im}(\lambda) \neq 0$. First note that

$$g_\lambda(t) = C_1 e^{i\,\mathrm{Re}(\lambda)t} e^{-\mathrm{Im}(\lambda)t} + C_2 e^{-i\,\mathrm{Re}(\lambda)t} e^{\mathrm{Im}(\lambda)t}.$$

Now if either C_1 or C_2 is zero, the it is easy to see that

$$|g_\lambda(t)| = |C_1| e^{-\mathrm{Im}(\lambda)t} \text{ or } |C_2| e^{\mathrm{Im}(\lambda)t}, \tag{3.18}$$

respectively. Therefore, we now assume that both C_1, and C_2 are non-zero.

We next first consider the case of $\mathrm{Im}(\lambda) > 0$. Then, $e^{\mathrm{Im}(\lambda)t}$ being the dominating factor in $g_\lambda(t)$ for large t, we observe that

$$|g_\lambda(t)| \sim |C_2| e^{\mathrm{Im}(\lambda)t} \qquad (t \to \infty).$$

Similarly, when $\mathrm{Im}(\lambda) < 0$, the dominating factor becomes $e^{\mathrm{Im}(\lambda)t}$, and subsequently,

$$|g_\lambda(t)| \sim |C_1| e^{-\mathrm{Im}(\lambda)t} \qquad (t \to \infty).$$

Therefore, using above observations along with (3.18), we conclude that for $\mathrm{Im}(\lambda) \neq 0$, there exists $C^\pm > 0$ (independent of t), such that

$$|g_\lambda(t)| \sim C^\pm e^{\pm\,\mathrm{Im}(\lambda)t} \qquad (t \to \infty). \tag{3.19}$$

Consequently, using the above asymptotics, there exists $R_1 > R_0$ (sufficiently large), and a constant $C > 0$ such that for all $t > R_1$ we have

$$|g_\lambda(t)| \geq C e^{\pm\,\mathrm{Im}(\lambda)t}. \tag{3.20}$$

Therefore, in view of (3.17), for any $1 \leq p \leq \infty$, with standard modification for $p = \infty$, we have

$$e^{p(\pm\,\mathrm{Im}(\lambda) - \rho)t} \leq C_3 \int_K |f(ka_t)|^p\ dk \qquad (t > R_1). \tag{3.21}$$

Case II: $\mathrm{Im}(\lambda) = 0$.

A straightforward calculation involving basic complex analysis gives

$$|g_\lambda(t)|^2 = \left| C_1 e^{i\lambda t} + C_2 e^{-i\lambda t} \right|^2 = |C_1|^2 + |C_2|^2 + 2|C_1||C_2| \cos(2\lambda t + \theta),$$

where $\theta = \arg(C_1\overline{C_2})$. This, in view of (3.17), implies

$$Ce^{-2\rho t}\Big(|C_1|^2+|C_2|^2+2|C_1||C_2|\cos(2\lambda t+\theta)\Big) \leq \Big(\int_K |f(ka_t)|^p\Big)^{\frac{2}{p}}$$

completing the proof of the proposition. □

Remark 3.2. The technical restriction $\lambda \notin i\mathbb{Z}$ on the spectral parameter λ arises directly from the general theory of the hypergeometric differential equation. As seen in the above proof, this exclusion is necessary to ensure the linear independence of the two fundamental solutions v_1 and v_2 of the radial differential equation (3.11) near one of its regular singular points, namely $z = 1$, which is crucial in our analysis.

In general, the local behavior of the solutions of the hypergeometric equation (3.11) near $z = 1$ depends critically on the difference $c - a - b$. When $c - a - b \notin \mathbb{Z}$, the two local solutions around $z = 1$ are linearly independent and can be expressed as analytic functions multiplied by $(1-z)^{c-a-b}$. However, when $c - a - b = 0$ or an integer, the exponents in the corresponding indicial equation coincide, leading to a *resonance*: the second local solution acquires a logarithmic factor. This logarithmic branching behavior destroys the analytic independence of the two solutions near $z = 1$ and significantly complicates their asymptotic analysis. Consequently, such resonant cases are also excluded, as they prevent the construction of regular L^p–estimates required in our argument. For a detailed discussion of the hypergeometric equation near its regular singular points, we refer the reader to [31, Chap. 15], and [30].

We now give

Proof of Theorem 1.3: We divide the proof into two main cases, according to whether the spectral parameter λ is real or has a nonzero imaginary part. We first consider the case $\operatorname{Im}(\lambda) \neq 0$. Let $1 \leq p < 2$. Since $e^{\pm \operatorname{Im}(\lambda)t} \geq e^{-|\operatorname{Im}(\lambda)|t}$, using the polar decomposition of the measure, and the estimate (3.3) of Proposition 3.1, there exists $R_1 > R_0$ such that for any $R > R_1$, we get

$$\int_{R<d(o,\ x)<2R} |f(x)|^p\ dx = \int_R^{2R} J(t)\int_K |f(ka_t)|^p\ dk\ dt \geq C\int_R^{2R} e^{p(-|\operatorname{Im}(\lambda)|-\rho)t}e^{2\rho t}\ dt.$$

Recalling $\gamma_p = \frac{2}{p} - 1$, it is easy to observe that

$$p(-|\operatorname{Im}(\lambda)|-\rho)+2\rho = p(\gamma_p\rho - |\operatorname{Im}(\lambda)|).$$

But then when $|\operatorname{Im}(\lambda)| \neq \gamma_p\rho$, the integral on the right can be estimated as

$$\begin{aligned}
I &= \int_R^{2R} e^{p(-|\operatorname{Im}(\lambda)|-\rho)t}e^{2\rho t}\ dt \\
&= \int_R^{2R} e^{(-|\operatorname{Im}(\lambda)|+\gamma_p\rho)pt}\ dt \\
&= \frac{1}{(\gamma_p\rho - |\operatorname{Im}(\lambda)|)p}\left(e^{2p(\gamma_p\rho-|\operatorname{Im}(\lambda)|)R} - e^{p(\gamma_p\rho-|\operatorname{Im}(\lambda)|)R}\right) \\
&= \frac{1}{(\gamma_p\rho - |\operatorname{Im}(\lambda)|)p}e^{p(\gamma_p\rho-|\operatorname{Im}(\lambda)|)R}\left(e^{p(\gamma_p\rho-|\operatorname{Im}(\lambda)|)R} - 1\right) \\
&= e^{p(\gamma_p\rho-|\operatorname{Im}(\lambda)|)R}L(\lambda, R, p)
\end{aligned}$$

where notice that

$$L(\lambda, R, p) := \begin{cases} \frac{\left(e^{p(\gamma_p\rho - |\operatorname{Im}(\lambda)|)R} - 1\right)}{(\gamma_p\rho - |\operatorname{Im}(\lambda)|)p} & \text{when } |\operatorname{Im}(\lambda)| < \gamma_p\rho \\ \frac{\left(1 - e^{p(\gamma_p\rho - |\operatorname{Im}(\lambda)|)R}\right)}{(|\operatorname{Im}(\lambda)| - \gamma_p\rho)p} & \text{when } |\operatorname{Im}(\lambda)| > \gamma_p\rho \end{cases}$$

Since the factor $e^{p(\gamma_p\rho - |\operatorname{Im}(\lambda)|)R}$ grows rapidly as $R \to \infty$ when $|\operatorname{Im}(\lambda)| < \gamma_p\rho$, while it decays rapidly when $|\operatorname{Im}(\lambda)| > \gamma_p\rho$, we can choose $R_1 > R_0$ sufficiently large so that for all $R > R_1$,

$$L(\lambda, R, p) \geq C(p, \lambda).$$

Consequently,

$$I \geq C(p, \lambda)\, e^{p(\gamma_p\rho - |\operatorname{Im}(\lambda)|)R}.$$

Hence, for any $R > R_1$, we obtain

$$\int_{R<d(o,x)<2R} |f(x)|^p \, dx \ \geq \ C\, e^{p(\gamma_p\rho - |\operatorname{Im}(\lambda)|)R},$$

where the constant C may depend on λ, p, and ρ, but not on R. This completes the proof in the present case.

Next, we consider the critical regime $|\operatorname{Im}(\lambda)| = \gamma_p\rho$. Here the exponential factor vanishes, leaving

$$I = \int_R^{2R} e^{(-|\operatorname{Im}(\lambda)| + \gamma_p\rho)pt}\, dt = \int_R^{2R} 1\, dt = R.$$

Hence,

$$\int_{R<d(o,x)<2R} |f(x)|^p \, dx \ \geq \ CR.$$

Next, we assume that $\operatorname{Im}(\lambda) = 0$. By part (2) of Proposition (3.1)(with $p = 2$), for any $R > R_0$ we have

$$\begin{aligned} C \int_R^{2R} \left(A^2 + B^2 + 2AB\cos(2\lambda t + \theta)\right)\, dt &\leq \int_R^{2R} e^{2\rho t} \int_K |f(ka_t)|^2 \, dk\, dt \\ &\leq \int_{R<d(o,\ x)<2R} |f(x)|^2 \, dx. \end{aligned}$$

The integral on the left can be evaluated explicitly:

$$\int_R^{2R} \left(A^2 + B^2 + 2AB\cos(2\lambda t + \theta)\right)\, dt = (A^2 + B^2)R - \frac{AB}{\lambda}\left(\sin(2\lambda R + \theta) - \sin(4\lambda R + \theta)\right).$$

Choosing $R_1 > R_0$ sufficiently large, there exists $C > 0$ such that for all $R > R_1$ we have

$$(A^2 + B^2)R - \frac{AB}{\lambda}\left(\sin(2\lambda R + \theta) - \sin(4\lambda R + \theta)\right) > CR.$$

Thus, for all $R > R_1$, we arrive at

$$\int_{R<|x|<2R} |f(x)|^2 dx \geq CR.$$

This completes the proof of part (1), and hence the theorem. □

The following remark elucidates the analytical role of the spectral region associated with the preceding theorem. In particular, we investigate the behavior of the eigenfunction f when the parameters p and λ lie outside the ranges considered in Theorem 1.3.

Remark 3.3. Throughout this discussion, let $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, and let f be as in Theorem 1.3. If $\mathrm{Im}(\lambda) \neq 0$, then applying Proposition 3.1 with $p = 2$ gives

$$\int_{R<d(o,x)<2R} |f(x)|^2 \, dx \gtrsim \int_R^{2R} e^{\pm 2\,\mathrm{Im}(\lambda)t} \, dt \;\gtrsim\; C_\lambda \, e^{\pm 2\,\mathrm{Im}(\lambda)R}.$$

Hence the L^2–mass of an eigenfunction in Ω with eigenvalue $-(\lambda^2+\rho^2)$ and $\mathrm{Im}(\lambda) \neq 0$ either grows or decays exponentially on the annulus $\{R < d(o,x) < 2R\}$, depending on the sign of $\mathrm{Im}(\lambda)$.

To illustrate this behaviour, consider the spherical functions Φ_λ for $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$ (see the Harish–Chandra expansion (2.8)). From the asymptotic estimate (2.11), for sufficiently large R,

$$\int_{R<d(o,x)<2R} |\Phi_\lambda(x)|^2 \, dx \;\gtrsim\; C \, e^{-2\,\mathrm{Im}(\lambda)R}.$$

Thus if $\mathrm{Im}(\lambda) < 0$ this lower bound grows exponentially, while if $\mathrm{Im}(\lambda) > 0$ it decays exponentially.
If $p > 2$ and $\mathrm{Im}(\lambda) \neq 0$, then $\gamma_p < 0$, and writing $\gamma_p = -\gamma_{p'}$ (where p' is the conjugate exponent), a careful inspection of the proof of Theorem 1.3 yields, for sufficiently large R,

$$\int_{R<d(o,x)<2R} |f(x)|^p \, dx \;\gtrsim\; C \, e^{-p(\gamma_{p'}\rho \mp \mathrm{Im}(\lambda))R}.$$

Thus, depending on whether

$$\mp \mathrm{Im}(\lambda) > \gamma_{p'}\rho \qquad \text{or} \qquad \mp \mathrm{Im}(\lambda) < \gamma_{p'}\rho,$$

the L^p–mass may exhibit exponential growth or exponential decay. As before, taking $f = \Phi_{\pm\lambda}$ provides explicit examples demonstrating this phenomenon.

We now employ estimates in Theorem 1.3 to establish Theorem 1.5. To this end, we shall make use of a Pizzetti-type mean value property for analytic functions defined on open subsets of X (see [32, (4.1)]; see also [7, (7.5)]).

Lemma 3.4. *Let $U \subset X$ be open and connected, and f be an analytic function on U. Then for sufficiently small $r > 0$, f satisfies*

$$\frac{1}{|S_r(x)|} \int_{S_r(x)} f(y) \; d\sigma(y) = \Gamma(\tfrac{n}{2}) \sum_{m=0}^{\infty} \left(\frac{\sinh \kappa r}{2\kappa} \right)^{2m} L_m f(x), \tag{3.22}$$

where $d\sigma$ is the measure on the sphere $S_t(x)$, and the operator L_m is defined by

$$L_m = \frac{\Delta_X(\Delta_X - (2n+4\beta+4)\kappa^2) \cdots (\Delta_X - (2n+4\beta+4m-4)\kappa^2}{m!\Gamma(m+\frac{n}{2})}.$$

Here β, and κ are constants related to the structure and curvature of X, respectively.

Proof of Theorem 1.5. Let f satisfy (3.2) in Ω, and suppose that $f \in L^p(\Omega)$. We argue by contradiction and assume that $f \neq 0$.

First, consider the case $p = 2$. In this case, $\gamma_p = 0$, and consequently, by the part (1) of Theorem 1.3, f satisfies (1.9), which contradicts the assumption that $f \in L^2(\Omega)$.

Next, assume that $1 \leq p < 2$. First, let $\mathrm{Im}(\lambda) \neq 0$. Under this hypothesis on λ, we have $\gamma_p\rho - |\,\mathrm{Im}(\lambda)| > 0$. In view of part (2) of Theorem 1.3, this implies that f satisfies (1.10), which again contradicts the assumption that $f \in L^p(\Omega)$.

Finally, we consider the remaining case $\mathrm{Im}(\lambda)=0$. To obtain a contradiction, we employ the L^2 estimate (1.9) together with the mean-value type result in Lemma 3.4. To begin with, note that since $-\Delta_X$ is elliptic, the function f is real-analytic on any open, connected subset $U\subset\Omega$. In particular, $f\not\equiv 0$ on U, and hence

$$\|f\|_{L^\infty(\Omega_1)}>0, \qquad \Omega_1:=\{x\in X: d(o,x)>R_0+1\}\subset\Omega.$$

We next show that $\|f\|_{L^\infty(\Omega_1)}<\infty$.

Let $x\in\Omega_1$. Since f is an eigenfunction of Δ_X with eigenvalue $-(\lambda^2+\rho^2)$ in Ω, there exists $r_1>0$ sufficiently small such that, for all $0<t<r_1$, Lemma 3.4 yields the mean value identity

$$\frac{1}{|S_t(x)|}\int_{S_t(x)} f(y)\,d\sigma(y)=f(x)\,\Gamma\big(\tfrac{n}{2}\big)\sum_{m=0}^{\infty}\left(\frac{\sinh(\kappa t)}{2\kappa}\right)^{2m}\frac{\Psi_m(\lambda,n,\beta,\rho)}{m!\,\Gamma(m+\frac{n}{2})}, \tag{3.23}$$

where

$$\Psi_m(\lambda,n,\beta,\rho)=(-1)^m(\lambda^2+\rho^2)\big[(\lambda^2+\rho^2)+(2n+4\beta+4)\kappa^2\big]\cdots\big[(\lambda^2+\rho^2)+(2n+4\beta+4m-4)\kappa^2\big].$$

Define

$$\Psi(t):=\Gamma\big(\tfrac{n}{2}\big)\sum_{m=0}^{\infty}\left(\frac{\sinh(\kappa t)}{2\kappa}\right)^{2m}\frac{\Psi_m(\lambda,n,\beta,\rho)}{m!\,\Gamma(m+\frac{n}{2})}.$$

Then Ψ is continuous and satisfies $\Psi(0)=1$. Since both Ψ and the Jacobian J of the polar decomposition J are continuous and $J(t)>0$ for $t>0$ (see (2.3)), one can choose $0<r<r_1$ small enough so that

$$D(r):=\int_0^r \Psi(t)\,J(t)\,dt\neq 0.$$

(The dependence of Ψ and $D(r)$ on the parameters appearing in Ψ_m is immaterial and will be suppressed in the notation.)

Now, recalling the fact that

$$\frac{1}{|S_t(x)|}\int_{S_t(x)} f(y)\,d\sigma(y)=\int_K f(xka_t)\;dk,$$

from (3.23), we see that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)\,dy=\frac{1}{|B_r(x)|}\int_{B_r(o)} f(xy)\;dy=\int_0^r\int_K f(xka_t)J(t)\;dk\;dt=f(x)\,D(r),$$

where in the second equality we have used the polar decomposition of the measure (See (2.2)). Hence, by Hölder's inequality,

$$|f(x)|\le\frac{1}{|D(r)|\,|B_r(o)|}\int_{B_r(x)}|f(y)|\,dy\le\frac{|B_r(o)|^{\frac{1}{p'}-1}}{|D(r)|}\left(\int_\Omega|f(y)|^p\,dy\right)^{1/p},$$

and therefore $\|f\|_{L^\infty(\Omega_1)}<\infty$.

Combining this bound with the L^2-estimate (1.9) yields, for all $R>\max(R_1,R_0+2)$,

$$CR\le\int_{R<d(o,x)<2R}|f(x)|^2\,dx\le\|f\|_{L^\infty(\Omega_1)}^{2-p}\int_{R<d(o,x)<2R}|f(x)|^p\,dx, \tag{3.24}$$

contradicting the assumption that $f\in L^p(\Omega)$. This establishes the claim when $\mathrm{Im}(\lambda)=0$, and hence completes the proof of the theorem. ∎

Remark 3.5. To examine the sharpness of Theorem 1.5, we construct nontrivial eigenfunctions on Ω with eigenvalue $-(\lambda^2+\rho^2)$ satisfying the L^p-integrability condition (1.11) for pairs (p,λ) with $\lambda \notin i\mathbb{Z}$ not covered by Theorem 1.10.

(i) The case $p>2$, interior of the $L^p(X)$ spectrum. It is well known (see (2.14)) that the elementary spherical functions $\varphi_\lambda \in L^p(X)$ whenever

$$p>2 \quad \text{and} \quad |\operatorname{Im}(\lambda)| < (1-\tfrac{2}{p})\rho.$$

Hence, for such parameters we may simply take $f=\varphi_\lambda$ as an example.

(ii) The complementary region. For the remaining ranges of (p,λ), namely

$$1 \le p < 2,\ |\operatorname{Im}(\lambda)| > \gamma_p\rho, \quad \text{or} \quad p>2,\ |\operatorname{Im}(\lambda)| \ge (1-\tfrac{2}{p})\rho,$$

we use the function Φ_λ appearing in the Harish–Chandra expansion of φ_λ (see (2.8)). From the asymptotic estimate (2.11), for large t and any $1\le p<\infty$,

$$\int_K |\Phi_\lambda(ka_t)|^p\,dk \asymp e^{p(-\operatorname{Im}(\lambda)-\rho)t}. \tag{3.25}$$

Consequently, recalling $\gamma_p = \frac{2}{p}-1$,

$$\int_\Omega |\Phi_\lambda(x)|^p\,dx \asymp \int_{R_0}^\infty e^{p(-\operatorname{Im}(\lambda)-\rho)t}e^{2\rho t}\,dt = \int_{R_0}^\infty e^{-p(\operatorname{Im}(\lambda)-\gamma_p\rho)t}\,dt = \int_{R_0}^\infty e^{-p\nu(\lambda,p)t}\,dt,$$

where $\nu(\lambda,p) := \operatorname{Im}(\lambda)-\gamma_p\rho$.

Hence $\Phi_\lambda \in L^p(\Omega)$ whenever $\nu(\lambda,p)>0$. For $p>2$ (so that $\gamma_p<0$), this condition is automatically satisfied when $\operatorname{Im}(\lambda)>0$, and we take $f=\Phi_\lambda$. If $\operatorname{Im}(\lambda)<0$, we instead choose $f=\Phi_{-\lambda}$. Similarly, for $1\le p<2$, L^p–integrability over Ω requires $|\operatorname{Im}(\lambda)|>\gamma_p\rho$, and we again select $f=\Phi_{\pm\lambda}$ according to the sign of $\operatorname{Im}(\lambda)$.

These constructions show that in all complementary spectral regions, one can find nontrivial L^p–eigenfunctions of $-\Delta_X$ on Ω, thereby confirming the sharpness of Theorem 1.5.

Next, we turn our attention to the proof of Theorem 1.8. This result follows immediately from Proposition 3.1. We begin by introducing the function ψ_λ appearing in the statement of Theorem 1.8. Its definition is naturally motivated by the lower bounds for the L^p–K–means of eigenfunctions established in Proposition 3.1. For $\lambda\in\mathbb{C}$, define the function ψ_λ on $\overline{A^+} = \{a_t : t\ge 0\}$ by

$$\psi_\lambda(a_t) = e^{(-|\operatorname{Im}(\lambda)|-\rho)t}, \qquad t\ge 0, \tag{3.26}$$

and extend it to G as a K–bi-invariant function. The function ψ_λ serves as a convenient comparison profile, reflecting the principal exponential decay rate of eigenfunctions associated with the spectral parameter λ. Furthermore, in view of the estimate (2.9) for Φ_λ, we have the following asymptotic equivalence for large t:

$$\psi_\lambda(a_t) \asymp \begin{cases} |\Phi_{-i\operatorname{Im}(\lambda)}(a_t)|, & \text{if } \operatorname{Im}(\lambda)<0, \\ |\Phi_{i\operatorname{Im}(\lambda)}(a_t)|, & \text{if } \operatorname{Im}(\lambda)>0, \end{cases} \qquad \text{as } t\to\infty. \tag{3.27}$$

Proof of Theorem 1.8. Let $\lambda\in\mathbb{C}\setminus i\mathbb{Z}$, $\epsilon>0$, $1\le p\le\infty$, and let $f\in C^2(\Omega)$ be a solution of (3.2) in Ω satisfying the growth condition (1.13), i.e.,

$$\sup_{t>R_0} \frac{t^\epsilon}{\psi_\lambda(a_t)} \left(\int_K |f(ka_t)|^p\,dk\right)^{\frac{1}{p}} < \infty. \tag{3.28}$$

We argue by contradiction and assume that $f \not\equiv 0$. In what follows, we consider the case $p < \infty$; the case $p = \infty$ can be handled by a straightforward modification of the same argument.

Case 1. $\mathbf{Im}(\boldsymbol{\lambda}) \neq \mathbf{0}$. By part (1) of Proposition 3.1, we have for all $t > R_0$,

$$\int_K |f(ka_t)|^p \, dk \geq C\, e^{p(\pm \operatorname{Im}(\lambda) - \rho)t} \geq C\, e^{p(-|\operatorname{Im}(\lambda)| - \rho)t}.$$

This yields

$$Ct^\epsilon \leq \frac{t^\epsilon}{e^{(-|\operatorname{Im}(\lambda)|-\rho)t}} \left(\int_K |f(ka_t)|^p \, dk \right)^{\frac{1}{p}} = \frac{t^\epsilon}{\psi_\lambda(a_t)} \left(\int_K |f(ka_t)|^p \, dk \right)^{\frac{1}{p}}.$$

Consequently,

$$\sup_{t > R_0} \frac{t^\epsilon}{\psi_\lambda(a_t)} \left(\int_K |f(ka_t)|^p \, dk \right)^{\frac{1}{p}} = \infty,$$

which contradicts (3.28). Hence, we must have $f \equiv 0$ in Ω.

Case 2. $\mathbf{Im}(\boldsymbol{\lambda}) = \mathbf{0}$.

By part (2) of Proposition 3.1, we have

$$\left(\int_K |f(ka_t)|^p \, dk \right)^{\frac{1}{p}} \geq Ce^{-\rho t} U(\lambda, t),$$

where

$$U(\lambda, t) = \big(A^2 + B^2 + 2AB \cos(2\lambda t + \theta) \big)^{\frac{1}{2}}.$$

Since $\psi_\lambda(a_t) = e^{-\rho t}$ whenever $\operatorname{Im}(\lambda) = 0$ (see (3.26)), it follows that

$$\frac{t^\epsilon}{\psi_\lambda(a_t)} \left(\int_K |f(ka_t)|^p \, dk \right)^{\frac{1}{p}} \geq Ct^\epsilon U(\lambda, t).$$

However, it is easy to verify that

$$\sup_{t > R_0} \big(t^\epsilon U(\lambda, t) \big) = \infty,$$

and thus,

$$\sup_{t > R_0} \frac{t^\epsilon}{\psi_\lambda(a_t)} \left(\int_K |f(ka_t)|^p \, dk \right)^{\frac{1}{p}} = \infty,$$

which contradicts (3.28). Therefore, we conclude that $f \equiv 0$ in Ω. Finally, we show that the condition $\epsilon > 0$ in the hypothesis of the theorem is sharp. To this end, we exhibit examples of eigenfunctions on Ω satisfying (3.28) with $\epsilon = 0$.

First, let $\operatorname{Im}(\lambda) > 0$. In this case, $\psi_\lambda(a_t) = e^{(-\operatorname{Im}(\lambda) - \rho)t}$, and we choose $f = \Phi_\lambda$. From the estimate (3.25), it follows that

$$\sup_{t > R_0} \frac{1}{\psi_\lambda(a_t)} \left(\int_K |\Phi_\lambda(ka_t)|^p \, dk \right)^{1/p} < \infty.$$

Similarly, when $\operatorname{Im}(\lambda) < 0$, taking $f = \Phi_{-\lambda}$ yields the same bound.

Finally, when $\operatorname{Im}(\lambda) = 0$, we have $\psi_\lambda(a_t) = e^{-\rho t}$. By Theorem 1.7, one may take $f = \mathcal{P}_\lambda F$ for any $F \in L^p(K/M)$, which also satisfies (3.28) with $\epsilon = 0$.

Hence, from all the cases above, it follows that the assumption $\epsilon > 0$ cannot be relaxed, and the condition in the theorem is therefore sharp. $\square$

4. Concluding remarks

To close, we record a few observations and indicate some open questions that, in our opinion, deserve further attention.

We begin with the following observation. As we have already seen, in the exterior domain Ω, the Laplace–Beltrami operator admits $L^p(\Omega)$ eigenfunctions with eigenvalue $-(\lambda^2+\rho^2)$ for $1 \le p \le 2$ whenever $|\operatorname{Im}(\lambda)| > \gamma_p \rho$, and for all $p > 2$ with any $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, owing to the presence of the spherical solutions $\Phi_{\pm\lambda}$. Thus, as demonstrated throughout this article, a Rellich-type L^p uniqueness result is not possible for $p > 2$. However, in view of the L^2–estimate (1.9), using Hölder's inequality, one can deduce a Rellich-type uniqueness statement expressed in terms of appropriate weighted L^p–norms.

Let Ω be the exterior domain defined in (3.1), and suppose that $f \in C^2(\Omega)$ is a nontrivial solution of the Helmholtz equation (3.2) in Ω for $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$. Then we have the following statement.

- Let $1 \le p < 2$, and let $\lambda = \alpha \pm i\gamma_p\rho$ with $\alpha \in \mathbb{R}$. If
$$\int_\Omega |f(x)|^{p'} e^{2\gamma_p p' \rho\, d(o,x)}\, dx < \infty,$$
then necessarily $f = 0$ in Ω.

We conclude with a few natural questions that arise from the analysis carried out in this article.

(1) As explained earlier, our method—based on the detailed analysis of the corresponding hypergeometric differential equation—does not extend to the spectral parameters $\lambda \in i\mathbb{Z}$. It would be interesting to investigate what happens at these values, in particular, to understand the precise behavior of the eigenfunctions corresponding to such parameters and whether analogous Rellich-type estimates still hold or fail.

(2) A crucial step in our proof is the use of the spherical harmonic expansion, where the K–part (or rotation, so to speak) plays an essential role. In the absence of full rotational symmetry, this approach does not readily extend to Damek–Ricci spaces—the nonsymmetric generalizations of rank-one Riemannian symmetric spaces of noncompact type. It would be interesting to explore whether analogues of Theorems 1.3 and 1.5 can be established for Damek–Ricci spaces, possibly by developing new harmonic analytic tools that bypass the lack of a classical spherical expansion.

(3) In the Baouendi–Grushin setting, Rellich-type inequalities were recently established by Banerjee and Garofalo [4], building upon ideas of Garofalo–Shen [11] and combining Carleman-type estimates, weighted integral inequalities, and the intrinsic homogeneous structure of the operator. A natural direction for further study is to investigate whether analogues of Theorems 1.1 and 1.2 hold in the Heisenberg group, or more generally, in H-type groups.

Acknowledgments

The author would like to thank Agnid Banerjee and Bernhard Krötz for several helpful discussions, and Swagato K. Ray for many illuminating conversations and for suggesting interesting directions. The author is supported by the DST–INSPIRE Faculty Fellowship (DST/INSPIRE/04/2024/001099).

References

[1] S. Agmon, Lower bounds for solutions of Schrödinger equations, *J. Analyse Math.* 23 (1970), 1-25.

[2] W. Ballman, M. Mukherjee, P. Polymerakis, On the spectrum of certain Hadamard manifolds *SIGMA Symmetry Integrability Geom. Methods Appl.* 19 (2023), Paper No. 050, 19 pp.

[3] A. Banerjee, N. Garofalo, An observation on eigenfunctions of the Laplacian, *La Matematica* 3 (2024), no. 4, 1451–1455.

[4] A. Banerjee, N. Garofalo, A Rellich type estimate for a subelliptic Helmholtz equation with mixed homogeneities, arXiv:2311.11559, *J. d'Analyse Mathematique*, to appear.

[5] A. Banerjee, N. Garofalo, Absence of L^p spectrum for asymptotically flat diffusions in region with cavities, arXiv:2507.10728 (2025).

[6] S. Ben Saïd, T. Oshima, N. Shimeno, Fatou's theorems and Hardy-type spaces for eigenfunctions of the invariant differential operators on symmetric spaces. *Int. Math. Res. Not.* 2003, no. 16, 915–931.

[7] C.A. Berenstein, L. Zalcman, Pompeiu's problem on symmetric spaces, *Comment. Math. Helv.*, 55(1980), 593-621.

[8] A. Boussejra, H. Sami, *Characterization of the L^p-range of the Poisson transform in hyperbolic spaces $B(\mathbb{F}^n)$*. *J. Lie Theory* 12 (2002), no. 1, 1–14.

[9] W. Bray, Generalized spectral projections on symmetric spaces of noncompact type: Paley–Wiener theorems, *J. Funct. Anal.* **135** (1996), 206-232. MR1367630

[10] L. Chen, H. Liu, A Scattering theory on hyperbolic spaces, arXiv:2310.16416 (2023).

[11] N. Garofalo and Z. Shen, Absence of positive eigenvalues for a class of subelliptic operators. *Math. Ann.* 304 (1996), no. 4, 701-715.

[12] R. Gangolli, On Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups, *Ann. of Math.*, 93 (1971), 150-165.

[13] L. Grafakos, *Classical Fourier analysis*, Grad. Texts in Math., 249 Springer, New York, 2008, xvi+489 pp. ISBN: 978-0-387-09431-1

[14] S. Helgason, *Groups and Geometric Analysis*, Academic Press, 1984. MR0754767

[15] S. Helgason, *Geometric analysis on Symmetric spaces*, Math. Surveys Monographs 39, Amer. Math. Soc., 1994.

[16] A. Hulanicki, On L_p-spectra of the laplacian on a Lie group with polynomial growth. *Proc. Amer. Math. Soc.* 44 (1974), 482–484.

[17] A. Hulanicki, Subalgebra of $L_1(G)$ associated with Laplacian on a Lie group. *Colloq. Math.* 31 (1974), 259-287.

[18] A. D. Ionescu, On the Poisson transform on symmetric spaces of real rank one. *J. Funct. Anal.* 174 (2000), no. 2, 513–523.

[19] A.D. Ionescu, D. Jerison, On the absence of positive eigenvalues of Schrödinger operators with rough potentials, *Geom. Funct. Anal.* 13 (2003), no. 5, 1029–1081.

[20] H. Isozaki, H. Morioka, A Rellich type theorem for discrete Schr¨odinger operators, *Inverse Probl. Imaging* 8 (2014), no. 2, 475–489.

[21] D. Jerison, C.E. Kenig, Unique continuation and absence of positive eigenvalues for Schrödinger operators, *Ann. of Math. (2)* 121 (1985), 463–494.

[22] K. D. Johnson, Composition series and intertwining operators for the spherical principal series II, *Trans. Amer. Math. Soc.*, **215** (1976), 269-283. MR0385012

[23] K. D. Johnson and N. Wallach, Composition series and intertwining operators for the spherical principal series I, *ibid.*, **229** (1977), 137-173. MR0447483

[24] T. Kato Growth properties of solutions of the reduced wave equation with a variable coefficient. *Comm. Pure Appl. Math.* 12 (1959), 403-425.

[25] T. H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups, in: Special Functions: Group Theoretical Aspects and Applications*, R. Askey, T. H. Koornwinder and W. Schempp (eds.), Reidel, Dordrecht, 1984, 1-85. MR0774055

[26] B. Kostant, On the existence and irreducibility of certain series of representations, *Bull. Amer. Math. Soc.*, **75** (1969), 627-642. MR0245725

[27] P. Kumar, S.K. Ray, R. Sarkar, Characterization of almost L^p-eigenfunctions of the Laplace-Beltrami operator, *Trans. Amer. Math. Soc.* 366 (2014), no. 6, 3191–3225.

[28] W. Littman, Decay at infinity of solutions to partial differential equations with constant coefficients, *Trans. Amer. Math. Soc.* 123 (1966), 449-459.

[29] N. Lohoué, T. Rychener, Die Resolvente von Δ auf symmetrischen Räumen vom nichtkompakten Typ, *Comment. Math. Helv.* 57 (1982), no. 3, 445–468, DOI 10.1007/BF02565869 (German).

[30] F. W. J. Olver, Asymptotics and Special Functions. Wellesley, MA: A. K. Peters. Reprint, with corrections, of original Academic Press edition, 1974.

[31] F. W. J. Olver, and L. C. Maximon, *NIST Handbook of Mathematical functions*, (edited by F. W. F. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark) Chapter 15, National Institute of Standards and Technology, Washington, DC, and Cambridge University Press, Cambridge, 2010. Available online in http://dlmf.nist.gov/15

[32] E.T. Quinto, Mean value extension theorems and microlocal analysis, *Proc. Amer. Math. Soc.*

[33] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, Academic Press, New York-London (1978).

[34] F. Rellich, *Über das asymptotische Verhalten der Lösungen von* $\Delta u + \lambda u = 0$ *in unendlichen Gebieten.* (German) Jber. Deutsch. Math.-Verein. 53 (1943), 57-65.

[35] B. Simon, On positive eigenvalues of one body Schrödinger operators, *Comm. Pure Appl. Math.* 22 (1969), 531–538.

[36] R.J. Stanton, P.A Tomas, Expansions for spherical functions on noncompact symmetric spaces, *Acta Math*, 140(3–4), 251–276 (1978)

[37] T. Tagawa A Rellich type theorem for the generalized oscillator, *Rep. Math. Phys.* 95 (2025), no. 3, 281–302.

[38] M.E. Taylor, L^p-estimates on functions of the Laplace operator, *Duke Math. J.* 58 (1989), no. 3, 773–793.

[39] N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, Translation of Math. Monogr. Vol 22. Amer. Math. Soc. Providence, R.I. 1968.

(Pritam Ganguly) Stat-Math Unit, Indian Statistical Institute, Kolkata, BT Road, Baranagar, Kolkata 700108

Email address: pritam1995.pg@gmail.com