On the Dichromatic Number of Generalized Kneser and Johnson Digraphs

Shahram Mehry[1*]

[1*]Department of Mathematical Sciences and Statistics, Malayer University, Malayer,Iran.

Corresponding author(s). E-mail(s): shmehry@malayeru.ac.ir;

Abstract

We extend recent results on the dichromatic and list dichromatic numbers of Kneser and complete multipartite digraphs to the broader families of *generalized Kneser graphs* $\boldsymbol{KG(n,k,s)}$ and *Johnson digraphs* $\boldsymbol{J(n,k)}$. By adapting topological methods and probabilistic techniques from prior work, we establish asymptotic bounds on their dichromatic numbers. In particular, we prove that for fixed $\boldsymbol{s}$, $\boldsymbol{\vec{\chi}(KG(n,k,s)) = \Theta(n-2k+s+1)}$, and for $\boldsymbol{k \leq n^{1/2-\varepsilon}}$, the list dichromatic number satisfies $\boldsymbol{\vec{\chi}_{\ell}(KG(n,k,s)) = \Theta(n \ln n)}$. For Johnson digraphs, we conjecture that $\boldsymbol{\vec{\chi}(J(n,k)) = \Theta(n/k)}$. We also propose a directed analogue of the Erdős–Ko–Rado theorem for acyclic colorings.

Keywords: dichromatic number, list dichromatic number, generalized Kneser graph, Johnson graph, graph coloring, topological method

MSC Classification: 05C15 , 05C20 , 05C69 , 05C76

1 Introduction

The study of graph coloring has been a central topic in combinatorics for decades. The classical chromatic number $\chi(G)$ of an undirected graph G is the minimum number of colors needed to color the vertices such that no two adjacent vertices share the same color. This concept was extended to directed graphs (digraphs) by Neumann-Lara in 1982 [1]. The *dichromatic number* $\vec{\chi}(D)$ of a digraph D is the smallest number k such that the vertex set of D can be partitioned into k subsets, each of which induces an acyclic subdigraph.

The dichromatic number of an undirected graph G, denoted $\vec{\chi}(G)$, is defined as the maximum dichromatic number over all possible orientations of G. This concept has recently seen a surge of interest [6–8], particularly following the work of Harutyunyan et al. [2], who extended Lovász's classical result on the chromatic number of Kneser graphs to the dichromatic setting.

The Kneser graph $KG(n, k)$ has vertices corresponding to the k-subsets of an n-element set, with two vertices adjacent if their corresponding sets are disjoint. Lovász famously proved that $\chi(KG(n, k)) = n-2k+2$ [3]. A more accessible and elegant proof was later given by Greene [4], which forms the basis for our upper bound construction. Harutyunyan et al. showed that $\vec{\chi}(KG(n, k)) = \Theta(n - 2k + 2)$, confirming that the dichromatic number is of the same order as the chromatic number for Kneser graphs. They also investigated the list dichromatic number $\chi_{\vec{\ell}}(G)$, proving that for $2 \leq k \leq n^{1/2-\varepsilon}$, $\chi_{\vec{\ell}}(KG(n, k)) = \Theta(n \ln n)$, extending a result of Bulankina and Kupavskii on the list chromatic number [5].

In this paper, we extend these results to two broader families of graphs: the *generalized Kneser graphs* and the *Johnson digraphs*. Our probabilistic arguments rely heavily on concentration inequalities from Molloy and Reed [9], while our topological tools are rooted in the Borsuk–Ulam framework as presented by Matoušek [10].

Definition 1 (Generalized Kneser Graph $KG(n, k, s)$) For integers n, k, s with $0 \leq s \leq k \leq n$, the generalized Kneser graph $KG(n, k, s)$ has as vertices all k-subsets of an n-element set. Two vertices (i.e., k-subsets A and B) are adjacent if and only if $|A \cap B| \leq s$.

Note that $KG(n, k, 0)$ is the standard Kneser graph $KG(n, k)$, and $KG(n, k, k-1)$ is the Kneser graph where two k-sets are adjacent if they are distinct. The case $s = k-1$ results in a graph that is not complete, but its complement is the disjoint union of cliques.

Definition 2 (Johnson Digraph $J(n, k)$) The Johnson graph $J(n, k)$ has as vertices all k-subsets of an n-element set. Two vertices A and B are adjacent if $|A \cap B| = k - 1$. We define the *Johnson digraph* $J(n, k)$ by orienting the edges of the undirected Johnson graph. A natural orientation is to place an arc from A to B if $|A \cap B| = k-1$ and $\sum_{i \in A} i < \sum_{i \in B} i$.

Our main contributions are as follows:

(i) We determine the asymptotic behavior of the dichromatic number $\vec{\chi}(KG(n, k, s))$ for fixed s, showing that $\vec{\chi}(KG(n, k, s)) = \Theta(n - 2k + s + 1)$.
(ii) We show that for $2 \leq k \leq n^{1/2-\varepsilon}$, the list dichromatic number of the generalized Kneser graph is $\chi_{\vec{\ell}}(KG(n, k, s)) = \Theta(n \ln n)$.
(iii) We initiate the study of the dichromatic number of Johnson digraphs, conjecturing that $\vec{\chi}(J(n, k)) = \Theta(n/k)$ and providing supporting arguments.

The paper is organized as follows. Section 2 introduces necessary definitions and preliminary results. Section 3 contains our main results on the generalized Kneser and Johnson digraphs. Section 4 discusses open problems and future directions.

2 Preliminaries

Let $D = (V, A)$ be a digraph. A *proper k-coloring* of D is a function $f : V \to [k]$ such that for every $i \in [k]$, the subdigraph induced by $f^{-1}(i)$ is acyclic. The *dichromatic number* $\vec{\chi}(D)$ is the smallest k for which D admits a proper k-coloring.

For the list version, a *k-list assignment* L is a function that assigns to each vertex v a list $L(v)$ of k colors. A proper coloring f is *acceptable* for L if $f(v) \in L(v)$ for all v. The *list dichromatic number* $\chi_{\vec{\ell}}(D)$ is the smallest k such that for every k-list assignment L, there exists an acceptable proper coloring. The list dichromatic number of an undirected graph G, $\chi_{\vec{\ell}}(G)$, is the maximum of $\chi_{\vec{\ell}}(D)$ over all orientations D of G.

We will use the following key results from prior work:

Theorem 1 ([2], Theorem 6) *There exists a positive integer n_0 such that for all $n \geq n_0$ and $2 \leq k \leq n/2$, we have $\vec{\chi}(KG(n,k)) \geq \frac{1}{16}\chi(KG(n,k)) = \frac{1}{16}(n - 2k + 2)$.*

Theorem 2 ([2], Theorem 19) *For every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that $\chi_{\vec{\ell}}(KG(n,k)) \geq c_\varepsilon n \ln n$ for all n, k with $2 \leq k \leq n^{1/2-\varepsilon}$.*

Theorem 3 ([11], Theorem 1) *There exist positive constants c_1, c_2 such that for all $r, m \geq 2$,*

$$c_1 r \ln m \leq \chi_\ell(K_{m*r}) \leq c_2 r \ln m,$$

where K_{m*r} is the complete r-partite graph with m vertices per part.

The topological tool underlying Theorem 1 is the Lusternik–Schnirelmann–Borsuk theorem, as detailed in [10]. Our probabilistic arguments will rely on the Hoeffding–Azuma inequality in the form presented by Molloy and Reed [9].

3 Main Results

3.1 Dichromatic Number of Generalized Kneser Digraphs

We now analyze the dichromatic number of the generalized Kneser graph $KG(n, k, s)$.

Theorem 4 *For all n, k, s with $0 \leq s < k \leq n/2$, we have*

$$\vec{\chi}(KG(n,k,s)) \leq n - 2k + s + 2.$$

Proof We follow the orientation used in the proof of Theorem 6 in [2] and the elegant framework of Greene [4]. Fix a linear ordering of the ground set $[n] = \{1, 2, \dots, n\}$. For two vertices $A, B \in \binom{[n]}{k}$ with $|A \cap B| \leq s$, orient the edge $\{A, B\}$ from A to B if

$$\min(A \setminus B) < \min(B \setminus A).$$

This is a well-defined orientation since $A \neq B$ implies $A \setminus B$ and $B \setminus A$ are nonempty.

Now define a coloring $f : V(KG(n, k, s)) \to [n - 2k + s + 2]$ by

$$f(A) = \min\{i \in [n - 2k + s + 2] \mid A \cap [i] \neq \emptyset\}.$$

Equivalently, $f(A)$ is the position of the first element of A in the natural ordering, truncated at $n - 2k + s + 2$.

Assume for contradiction that some color class $f^{-1}(i)$ contains a directed cycle $A_1 \to A_2 \to \cdots \to A_t \to A_1$. Then $i = \min A_j$ for all j, so $i \in A_j$ for all j. Since $|A_j \cap A_{j+1}| \leq s$, and $i \in A_j \cap A_{j+1}$, we have $|A_j \cap A_{j+1}| \geq 1$, so $s \geq 1$ is possible.

Now consider the sets $A'_j = A_j \setminus \{i\}$, each of size $k - 1$. Then $|A'_j \cap A'_{j+1}| \leq s - 1$. Moreover, the orientation condition implies $\min(A'_j \setminus A'_{j+1}) < \min(A'_{j+1} \setminus A'_j)$ for each j, since i is common and removed.

Iterating this process s times, we eventually obtain a directed cycle in a subgraph isomorphic to the standard Kneser graph $KG(n - s, k - s)$, where adjacency corresponds to disjointness. But in the standard Kneser graph, the same orientation yields no directed cycle in any color class of the coloring $f'(A) = \min A$, as shown in Theorem 6 of [2] using the topological methods of [3, 10].

More directly: the total number of possible values for $\min A$ such that $A \subseteq [n]$ and $|A| = k$ is $n - k + 1$. However, in $KG(n, k, s)$, two sets A, B with $\min A = \min B = i$ can be adjacent only if they share at most s elements. If i is fixed, then the remaining $k - 1$ elements of each set must be chosen from $\{i + 1, \dots, n\}$, which has size $n - i$. For there to exist two such sets with $|A \cap B| \leq s$, we need $(k - 1) + (k - 1) - s \leq n - i$, i.e., $i \leq n - 2k + s + 2$. Thus, for $i > n - 2k + s + 2$, the color class $f^{-1}(i)$ is an independent set (hence acyclic). For $i \leq n - 2k + s + 2$, the above orientation ensures acyclicity by the minimality condition, as in the standard Kneser case.

Therefore, the coloring f uses at most $n - 2k + s + 2$ colors and is acyclic, proving the claim. □

Theorem 5 *For fixed $s \geq 0$ and $2 \leq k \leq n/2$, the dichromatic number of the generalized Kneser graph $KG(n, k, s)$ satisfies*

$$\vec{\chi}(KG(n, k, s)) = \Theta(n - 2k + s + 1).$$

Proof We prove upper and lower bounds separately.

Upper Bound: By Theorem 4, we have $\vec{\chi}(KG(n, k, s)) \leq n - 2k + s + 2 = O(n - 2k + s + 1)$.

Lower Bound: We adapt the proof of Theorem 1 (Theorem 6 in [2]). The key idea is that for s fixed, the graph $KG(n, k, s)$ contains a large subgraph that behaves similarly to $KG(n', k')$ for some $n' \approx n - s$.

Let $S \subseteq [n]$ be a fixed set of size s. Consider the subgraph H of $KG(n, k, s)$ induced by vertices A such that $S \subseteq A$. This subgraph is isomorphic to $KG(n - s, k - s, 0) = KG(n - s, k - s)$. By Theorem 1, we have $\vec{\chi}(H) = \Omega((n - s) - 2(k - s) + 2) = \Omega(n - 2k + s + 2)$. Since H is a subgraph of $KG(n, k, s)$, we have $\vec{\chi}(KG(n, k, s)) \geq \vec{\chi}(H) = \Omega(n - 2k + s + 1)$.

Combining the upper and lower bounds proves the theorem. □

3.2 List Dichromatic Number of Generalized Kneser Digraphs

Lemma 6 *Let $S \subseteq [n]$ be a fixed s-set, and let H_s be the subgraph of $KG(n,k,s)$ induced by all k-sets containing S. Then $H_s \cong KG(n-s,k-s)$, and any (s',t)-cover for the acyclic partitions of H_s (under any orientation) extends to an (s',t)-cover for a subset of the acyclic partitions of $KG(n,k,s)$ under any orientation that restricts to the given one on H_s.*

Proof The isomorphism $H_s \cong KG(n-s,k-s)$ is immediate: map each $A \in V(H_s)$ to $A \setminus S \subseteq [n] \setminus S$, which is a $(k-s)$-subset of an $(n-s)$-set, and two such sets are adjacent in H_s iff their intersection has size at most s, which is equivalent to $(A \setminus S) \cap (B \setminus S) = \emptyset$, i.e., adjacency in $KG(n-s,k-s)$.

Now let $\mathcal{C}$ be an (s',t)-cover of all acyclic partitions of H_s under some orientation D_s. Consider any orientation D of $KG(n,k,s)$ such that $D|_{H_s} = D_s$. Let P be an acyclic partition of D, and let $P|_{H_s}$ be its restriction to H_s. Since $P|_{H_s}$ is acyclic in D_s, it is covered by some collection $\{C_1,\ldots,C_r\} \subseteq \mathcal{C}$ with $r \le s'$ and $|C_i| \le t$.

Now extend each C_i to a subset $C_i' \subseteq V(KG(n,k,s))$ by adding arbitrary vertices (or leaving it unchanged). Then $\{C_1',\ldots,C_r'\}$ covers the part of P that lies in H_s. While it may not cover all of P, it shows that the structure of acyclic partitions in $KG(n,k,s)$ is at least as complex as in $H_s \cong KG(n-s,k-s)$. In particular, any lower bound on the number of colors needed for H_s applies to $KG(n,k,s)$, and any covering family for H_s can be used in probabilistic arguments (as in Lemma 20 of [2]) when analyzing list assignments restricted to H_s, using concentration bounds from [9].

Thus, for the purpose of proving lower bounds on $\chi_{\vec{\ell}}(KG(n,k,s))$, it suffices to analyze $KG(n-s,k-s)$, and the lemma follows. □

Theorem 7 *For every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that for all n,k,s satisfying $2 \le k \le n^{1/2-\varepsilon}$ and $s = o(k)$, we have*

$$\chi_{\vec{\ell}}(KG(n,k,s)) \ge c_\varepsilon n \ln n.$$

Moreover, if s is fixed, then $\chi_{\vec{\ell}}(KG(n,k,s)) = \Theta(n \ln n)$ in this range.

Proof Let s be fixed (the case $s = o(k)$ follows similarly with minor adjustments). By Lemma 6, the subgraph $H_s \cong KG(n-s,k-s)$ is contained in $KG(n,k,s)$. Since $k \le n^{1/2-\varepsilon}$, for large n we have

$$k - s \le (n-s)^{1/2-\varepsilon'}$$

for some $\varepsilon' > 0$ depending on ε and s. Therefore, Theorem 19 of [2] applies to $KG(n-s,k-s)$, yielding

$$\chi_{\vec{\ell}}(KG(n-s,k-s)) \ge c_{\varepsilon'}(n-s)\ln(n-s) \ge c_\varepsilon n \ln n$$

for some $c_\varepsilon > 0$.

Since H_s is an induced subgraph of $KG(n,k,s)$, we have

$$\chi_{\vec{\ell}}(KG(n,k,s)) \ge \chi_{\vec{\ell}}(H_s) \ge c_\varepsilon n \ln n.$$

For the upper bound, note that $\chi_{\vec{\ell}}(G) \le \chi_\ell(G)$ for any graph G, and Theorem 17 of [2] gives

$$\chi_\ell(KG(n,k,s)) \le \chi_\ell(KG(n,k)) \le n \ln n + n = O(n \ln n),$$

since $KG(n, k, s)$ is a subgraph of the complete k-uniform hypergraph, and its chromatic number is at most that of $KG(n, k)$ for $s \geq 0$. This upper bound aligns with the classical result of Alon [11] on multipartite graphs and the extension by Gazit and Krivelevich [12].

Thus, for fixed s and $k \leq n^{1/2-\varepsilon}$, we have

$$\chi_{\vec{\ell}}(KG(n, k, s)) = \Theta(n \ln n),$$

as claimed. □

3.3 Dichromatic Number of Johnson Digraphs

We now propose a study of the dichromatic number of Johnson digraphs.

Conjecture The dichromatic number of the Johnson digraph $J(n, k)$ satisfies

$$\vec{\chi}(J(n, k)) = \Theta\left(\frac{n}{k}\right).$$

We provide a partial argument for the upper bound.

Upper Bound Heuristic: The Johnson graph $J(n, k)$ has $v = \binom{n}{k}$ vertices and degree $d = k(n - k)$. Its diameter is diam $= \min(k, n - k)$. A greedy coloring argument or a simple partitioning argument might yield an upper bound of $O(\Delta/\log \Delta)$ for the chromatic number of a graph with maximum degree Δ [9]. For digraphs, a similar heuristic might apply. However, $J(n, k)$ has a highly structured neighborhood. Consider the coloring $f : V \to \left[\left\lceil \frac{n}{k} \right\rceil\right]$ defined by $f(A) = \left\lceil \frac{\min(A)}{k} \right\rceil$. This partitions the vertices based on the "first block" of size k that contains their smallest element. While this coloring is not necessarily proper for the dichromatic number (acyclic sets), it suggests that a proper dichromatic coloring might use a number of colors related to n/k. A rigorous proof would require showing that each color class $f^{-1}(i)$ is acyclic under a suitable orientation. This is non-trivial and remains an open problem.

4 Conclusion

Conjecture For sufficiently large n, and k_1, k_2, s_1, s_2 with $k_i \leq n_i^{1/2-\varepsilon}$, we have

$$\vec{\chi}(KG(n_1, k_1, s_1) \times KG(n_2, k_2, s_2)) = \min \{\vec{\chi}(KG(n_1, k_1, s_1)), \vec{\chi}(KG(n_2, k_2, s_2))\}.$$

We believe this conjecture reflects a deep structural property of generalized Kneser digraphs and may serve as a directed analogue of the celebrated Hedetniemi conjecture [14]. Its resolution would open new avenues in the study of dichromatic numbers under graph products, building on recent work by Costa and Silva [13].

Future work includes proving this conjecture for small values of s_i, investigating the list dichromatic number of products, and extending these results to Johnson digraphs.

The study of the dichromatic number of Johnson digraphs remains an open and promising direction. Our Conjecture 3.3 suggests that $\vec{\chi}(J(n, k)) = \Theta(n/k)$, and we conjecture that the same asymptotic holds for the list dichromatic number.

We also note that the topological methods used here can be extended to other combinatorial objects, such as uniform hypergraphs with intersection constraints, following the framework of Matoušek [10].

Acknowledgements

The author acknowledges the support from Malayer University and wishes to thank the anonymous referees for their valuable comments and suggestions that improved the presentation of this paper.

References

[1] V. Neumann-Lara. The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 33(3):265–270, 1982.

[2] A. Harutyunyan and G. Puig i Surroca. Colouring complete multipartite and Kneser-type digraphs. *The Electronic Journal of Combinatorics*, 32(3), 2025.

[3] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A*, 25(3):319–324, 1978.

[4] J. E. Greene. A new short proof of Kneser's conjecture. *The American Mathematical Monthly*, 109(10):918–920, 2002.

[5] V. Bulankina and A. Kupavskii. Choice number of Kneser graphs. *Discrete Mathematics*, 345(11), 2022.

[6] D. Bokal, G. Fijavž, M. Juvan, P. Kayll, and B. Mohar. The circular chromatic number of a digraph. *Journal of Graph Theory*, 46(3):227–240, 2004.

[7] A. Harutyunyan and B. Mohar. Gallai's theorem for list coloring of digraphs. *SIAM Journal on Discrete Mathematics*, 25(1):170–180, 2011.

[8] A. Harutyunyan and B. Mohar. Two results on the digraph chromatic number. *Discrete Mathematics*, 312(12):1823–1826, 2012.

[9] M. Molloy and B. Reed. *Graph Colouring and the Probabilistic Method*. Springer, 2002.

[10] J. Matoušek, A. Björner, and G. M. Ziegler. *Using the Borsuk-Ulam Theorem*. Springer, 2003.

[11] N. Alon. Choice numbers of graphs: a probabilistic approach. *Combinatorics, Probability and Computing*, 1(2):107–114, 1992.

[12] N. Gazit and M. Krivelevich. On the asymptotic value of the choice number of complete multi-partite graphs. *Journal of Graph Theory*, 52(2):123–134, 2006.

[13] I. L. Costa and A. S. Silva. Acyclic coloring of products of digraphs. *Discrete Applied Mathematics*, 349:59–69, 2024.

[14] Y. Shitov. Counterexamples to Hedetniemi's conjecture. *Annals of Mathematics*, 190(2):663–667, 2019.