GENERATING ALL AHLFORS CURRENTS BY A SINGLE ENTIRE CURVE

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ABSTRACT. Let X be a compact complex manifold possessing the *Runge approximation property on discs*, meaning that every holomorphic map from a closed disc into X is approximable by a global holomorphic map from $\mathbb{C}$. We construct an entire curve $F : \mathbb{C} \to X$ such that the associated family of concentric holomorphic discs $\{F|_{\overline{\mathbb{D}}_r}\}_{r>0}$ generates all Ahlfors currents on X, thereby settling a conjecture of Sibony.

1. **Introduction**

The investigation of holomorphic entire curves in complex manifolds forms a classical theme in complex geometry (cf. [NW14, Ru21]), originating in Nevanlinna's value distribution theory [Nev25]. This analytic perspective was later enriched by Ahlfors' geometric insights through his covering surface theory [Ahl35].

The advancement of these theories is largely driven by their connection to complex hyperbolicity problems. In a pivotal contribution, McQuillan [McQ98] verified the famous Green–Griffiths conjecture for complex projective surfaces with $c_1^2 > c_2$ by means of *Nevanlinna currents*, which capture the asymptotic behavior of entire curves. Later, Duval [Duv08] employed a simplified variant of Nevanlinna currents, called *Ahlfors currents*, to establish a deep, quantitative refinement of Brody's lemma, thereby obtaining a characterization of complex hyperbolicity in terms of an isoperimetric inequality for holomorphic discs. For further important applications of Ahlfors and Nevanlinna currents, we refer the reader to [DS18, DH18, DV20, HV21, Duj22], among others.

We begin by recalling the basic definitions and notations. Let X be a compact complex manifold endowed with a Hermitian (1,1)-form ω. Throughout this paper, $\mathbb{D}_r$ and $\mathbb{D}(c, r)$ denote the discs in $\mathbb{C}$ of radius r, centered at the origin and at c, respectively. By a holomorphic map $f\colon \overline{\mathbb{D}}_R \to X$, we mean that f is holomorphic on some open neighborhood U of the closed disc $\overline{\mathbb{D}}_R$.

We associate to f a positive bidimension $(1,1)$-current A_f, called the *Ahlfors-type current*, which acts on any smooth $(1,1)$-form η on X by normalized integration:

$$A_f(\eta) = \frac{1}{\mathrm{Area}_\omega(f(\mathbb{D}_R))} \int_{\mathbb{D}_R} f^*\eta. \tag{1.1}$$

The normalization gives A_f the mass $A_f(\omega) = 1$.

From a sequence of nonconstant holomorphic discs $\{f_i\colon \overline{\mathbb{D}}_{r_i} \to X\}_{i\geqslant 1}$ which satisfies the length-area condition

$$\lim_{i\to\infty} \frac{\mathrm{Length}_\omega(f_i(\partial\mathbb{D}_{r_i}))}{\mathrm{Area}_\omega(f_i(\mathbb{D}_{r_i}))} = 0, \tag{1.2}$$

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we obtain an associated sequence of Ahlfors-type currents $\{A_{f_i}\}_{i\geqslant 1}$ with bounded mass. By the Banach–Alaoglu theorem, some subsequence converges weakly to a positive $(1,1)$-current $\mathcal{T}$, which is closed due to the condition (1.2). Such a limit $\mathcal{T}$ is called an *Ahlfors current* on X.

By Ahlfors' lemma (cf. [Duv21, p. 7]), for any entire curve $f\colon \mathbb{C} \to X$, there exists a sequence of increasing radii $\{r_i\}_{i\geqslant 1}$ tending to infinity such that the corresponding concentric holomorphic discs $\{f|_{\overline{\mathbb{D}}_{r_i}}\}_{i\geqslant 1}$ satisfy the length-area condition (1.2). Consequently, one can obtain an associated Ahlfors current $\mathcal{T}$ which encodes the asymptotic geometric behavior of f. Obviously, Ahlfors currents are natural analogues of the Lelong currents of integration over analytic subvarieties [Lel57].

In the context of complex dynamics, Ahlfors currents associated to certain holomorphic curves have been shown to exhibit uniqueness properties [DS18, DV20]. These examples motivate the following fundamental question:

Question 1.1. Are all Ahlfors (resp. Nevanlinna) currents associated to the same entire curve cohomologically equivalent?

Huynh and Xie [HX21] first answered the above question in the negative through explicit counterexamples. This fundamental obstruction rules out a naive analytic intersection theory between entire curves and divisors via pairing of the corresponding Ahlfors (resp. Nevanlinna) currents with Chern forms.

Following his review of an early manuscript [HX21], the late Nessim Sibony (1947–2021) proposed the following striking conjecture: [1]

Conjecture 1.2. *For certain compact complex manifolds X, such as $\mathbb{P}^n$ or complex tori, every Ahlfors current on X can be obtained from a single entire curve $f\colon \mathbb{C} \to X$.*

In the same private correspondence, Sibony pointed out a connection between this conjecture and Problem 9.1 in his joint open problem list with Dinh [DS20], concerning universal entire functions in the sense of Birkhoff [Bir29]. Recently, Dinh-Sibony's Problem 9.1 has spurred a series of recent developments [CHX23, GX24, GX25].

Inspired by Sibony's hint on Birkhoff's method [Bir29], Xie [Xie24] introduced the following concept.

Definition 1.3 ([Xie24, Definition 1.7])**.** A connected complex manifold X equipped with a complete distance function d_X (whose existence is guaranteed by [NO61]) is said to have the *weak Oka-1 property* if the following holds: for any two disjoint closed discs D_1, D_2 contained in a larger closed disc $D \subset \mathbb{C}$, any holomorphic map $F\colon U \to X$ defined on a neighborhood U of $D_1 \cup D_2$, and any error bound $\mathsf{e} > 0$, there exists a holomorphic map $\hat{F}\colon \hat{U} \to X$ on a neighborhood $\hat{U}$ of D such that

$$\mathrm{d}_X\big(\hat{F}(z), F(z)\big) \leqslant \mathsf{e}, \quad \forall z \in D_1 \cup D_2.$$

This class of manifolds includes all Oka manifolds [Fors17] (notably $\mathbb{P}^n$ and complex tori) and embraces the broader class of Oka-1 manifolds (Definition 5.2) introduced by Alarcón and Forstnerič [AF25], which encompasses examples such as Kummer surfaces and elliptic $K3$ surfaces. Further examples can be found in [BW25] (rationally simply connected projective manifolds) as well as in [FL25].

For such manifolds, Xie obtained the following partial answer to Sibony's Conjecture 1.2:

[1] Sibony (personal communication, 2021) pointed out: "I suspect that for $\mathbb{P}^n$, or tori, all the Ahlfors currents can be obtained just using one map; see the discussion on the Birkhoff approach [Bir29] in the enclosed paper [DS20]." The citations are added by the authors.

Theorem 1.4 ([Xie24, Theorem A])**.** *Let X be a compact complex manifold satisfying the weak Oka-1 property. Then there exists an entire curve $f\colon \mathbb{C} \to X$ whose associated holomorphic discs $\{f|_{\overline{\mathbb{D}}(a,r)}\}_{a\in\mathbb{C},\, r>0}$ generate all Nevanlinna and Ahlfors currents on X.*

The proof employs Birkhoff's method to "patch" together infinitely many suitably chosen holomorphic discs $\{f_i\colon \overline{\mathbb{D}}_{\rho_i} \to X\}_{i\geqslant 1}$ (see Lemma 4.1). However, this strategy fails when one is restricted to using concentric discs from a single entire curve $f\colon \mathbb{C} \to X$. The core challenge is that, for any two radii $r_1 < r_2$, the behavior of f on $\overline{\mathbb{D}}_{r_1}$ inherently influences its behavior on $\overline{\mathbb{D}}_{r_2}$. This "pollution" of information makes it unclear how to "amalgamate" the collection $\{f_i\}_{i\geqslant 1}$ into a family of concentric discs from the same entire curve.

Nevertheless, in a recent work [WX25], Wu and Xie were able to verify Sibony's Conjecture 1.2 for complex tori at a cohomological level, by means of Hodge theory and a variant of the "ping-pong" strategy in [Xie24].

Theorem 1.5 ([WX25, Theorem A])**.** *For any complex torus $\mathbb{T}$, there exists an entire curve $f\colon \mathbb{C} \to \mathbb{T}$ such that the concentric holomorphic discs $\{f|_{\overline{\mathbb{D}}_R}\}_{R>0}$ generate all Nevanlinna and Ahlfors currents on $\mathbb{T}$, up to cohomological equivalence.*

Looking back, Sibony's Conjecture 1.2 anticipates the remarkable flexibility of entire curves in Oka geometry (cf. [Fors17]). In this work, we establish this conjecture in full generality for weak Oka-1 manifolds.

We begin by providing two alternative characterizations of weak Oka-1 manifolds.

Definition 1.6. A connected complex manifold X with a complete distance function d_X has the *local Runge approximation property on discs* if for any holomorphic map $f_1\colon U \to X$ defined on a neighborhood U of $\overline{\mathbb{D}}$, and any $\epsilon > 0$, there exists a holomorphic map $f_2\colon V \to X$ defined on a neighborhood V of $\overline{\mathbb{D}}_2$ satisfying $\sup_{z\in\overline{\mathbb{D}}} \mathrm{d}_X(f_1(z), f_2(z)) \leqslant \epsilon$.

Definition 1.7. A connected complex manifold X is said to have the *Runge approximation property on discs*, if any holomorphic map from a neighborhood of a closed disc $D \subset \mathbb{C}$ to X can be approximated uniformly on D by entire curves from $\mathbb{C}$ to X.

The above two properties are actually equivalent to the weak Oka-1 property (Definition 1.3), as established in Proposition 3.1. This class of complex manifolds can be regarded as the antithesis of complex hyperbolic manifolds.

Theorem 1.8 (Main Theorem)**.** *For every compact complex manifold X satisfying the Runge approximation property on discs, there exists an entire curve $F\colon \mathbb{C} \to X$ such that the concentric holomorphic discs $\{F|_{\overline{\mathbb{D}}_r}\}_{r>0}$ generate all Ahlfors currents on X.*

This result reveals a remarkable flexibility of entire curves in Oka geometry. By showing that the concentric discs from a single entire curve are sufficient, it strengthens Theorem 1.4 and thereby fully resolves Sibony's conjecture 1.2 for weak Oka-1 manifolds.

The proof of Theorem 1.8 is based on a delicate induction process, synthesizing ideas from three key sources:

(i) Birkhoff's method of constructing universal entire functions from countable families of holomorphic functions on discs [Bir29];

(ii) the gluing technique in several complex variables, pioneered by Fornaess and Stout [FSt77a, FSt77b];

(iii) the "ping-pong" strategy introduced in [Xie24].

Although the technical machinery involved is sophisticated, the essence of our approach ultimately reduces to the simple identities (4.6) and (4.11).

Our novel inductive construction of such entire curves can be also applied to deduce all previous results, established in [HX21, Xie24, WX25], on Siu's decompositions of Ahlfors currents.

This paper is structured as follows. Section 2 recalls the classical holomorphic approximation theory and explains its application in our setting. Section 3 establishes the equivalence of three characterizations of weak Oka-1 manifolds. The proof of the main theorem is presented in Section 4. Finally, Section 5 discusses three open problems concerning (weak) Oka-1 manifolds.

2. A Holomorphic Gluing Technique

2.1. **Holomorphic Approximation Theory.** We begin by recalling several key results from holomorphic approximation theory, which will play essential roles in our constructions. For a comprehensive treatment, we refer to [FFW20] and [AF21].

Our starting point is the classical theory of holomorphic approximation on the complex plane $\mathbb{C}$. One of the earliest and most influential results in this area is Runge's approximation theorem [Run85]:

Theorem 2.1. *Let K be a compact subset of $\mathbb{C}$ such that its complement $\mathbb{C} \setminus K$ is path connected. Then any holomorphic function f defined on a neighborhood of K can be approximated uniformly on K by a sequence of polynomials.*

The theorem of Mergelyan [Mer52] is a profound generalization of Runge's theorem, providing precise conditions on a compact set $K \subset \mathbb{C}$ to ensure that every function in $\mathcal{A}(K)$ (i.e., continuous on K and holomorphic in its interior) can be uniformly approximated by holomorphic functions on neighborhoods of K (cf. [FFW20, Section 7]). This result has been extensively generalized, notably to the context of $\mathcal{C}^r$-approximation on *admissible sets*.

Definition 2.2 (cf. [FFW20, Definition 3])**.** A compact set N in a Riemann surface M is called *admissible* if it can be written as $N = K \cup E$, where:

- K is a finite union of pairwise disjoint compact domains with piecewise $\mathcal{C}^1$ boundaries.
- E is a finite union of pairwise disjoint smooth Jordan arcs and closed Jordan curves.
- The set E intersects K at most at the endpoints of these arcs, and all such intersections are transverse to the boundary ∂K.

Theorem 2.3 (Mergelyan theorem for manifold-valued maps on admissible sets; cf. [AF21, p. 74])**.** *Let M be a Riemann surface, X be a complex manifold, and N be an admissible set in M. For any $r \in \mathbb{Z}_{\geqslant 0}$, every map $f \in \mathcal{A}^r(N, X)$ can be approximated in the $\mathcal{C}^r(N, X)$ topology by holomorphic maps $U \to X$ defined on some open neighbourhoods U of N.*

Following [FFW20], for $r \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{C}^r(N, X)$ the space of r-times continuously differentiable maps from a neighborhood of N into X, and by $\mathcal{A}^r(N, X)$ the subspace of maps that are holomorphic on the interior $\mathring{N}$.

Remark 2.4. To apply the above theorem to an r-times continuously differentiable map $f\colon N \to X$ that is holomorphic on $\mathring{N}$, we need to extend f to a neighborhood of N. It is a consequence of Whitney's extension theorem [Whit34, FFW20]; see Proposition 2.6.

2.2. **Gluing Two Holomorphic Discs.** The following Propositions 2.5 and 2.6 are standard facts in differential topology; their proofs are therefore omitted.

Proposition 2.5. *Let (M, g) be a complete Riemannian manifold. For any two distinct points $p, q \in M$, any tangent vectors $v_p \in T_pM$, $v_q \in T_qM$, and any $\epsilon > 0$, there exists a smooth curve $\gamma : [0, 1] \to M$ connecting p to q with $\gamma'(0) = v_p$, $\gamma'(1) = v_q$, and length $L(\gamma) < \mathrm{d}_M(p, q) + \epsilon$.* □

This proposition will be used in **Trick 3** below.

Proposition 2.6. *Let M be a Riemann surface and $N \subset M$ an admissible set (see Definition 2.2). Let Y be a smooth manifold and $f : N \to Y$ a continuously differentiable map. Then there exist an open neighborhood V of N in M and a continuously differentiable map $F : V \to Y$ such that $F|_N = f$.* □

In our subsequent application, we take $M = \mathbb{R}^2$ and consider an admissible set $N \subset M$ consisting of two disjoint closed discs (with $\mathcal{C}^\infty$ boundaries) connected transversally by a short line segment (see Figure 1).

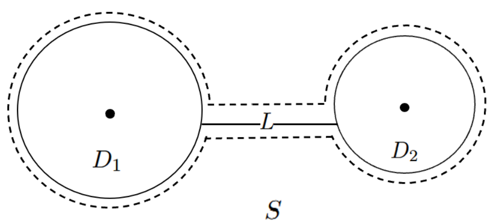

FIGURE 1. Dumbbell-shaped neighborhood of D_1 and D_2.

Lemma 2.7. *Let X be a complex manifold, and let $D_1, D_2 \subset \mathbb{C}$ be two disjoint closed disks. For $i = 1, 2$, let $f_i : U_i \to X$ be a holomorphic map defined on an open neighborhood U_i of D_i. Then, for any $\epsilon > 0$, there exist a dumbbell-shaped domain $S \subset \mathbb{C}$ containing $D_1 \cup D_2$ (see Figure 1) and a holomorphic map $f : S \to X$ such that*

$$\|f - f_i\|_{\mathcal{C}^1} < \epsilon \quad \text{on } D_i \quad \text{for } i = 1, 2.$$

The specific choice of the $\mathcal{C}^1$-norm is immaterial here due to the compactness of $D_1 \cup D_2$. Gluing constructions of this type originate in the work of Fornaess and Stout [FSt77a, FSt77b], and the result is now standard in holomorphic approximation theory.

Proof. Let L be a short line segment connecting D_1 and D_2, and define the compact set $N := D_1 \cup L \cup D_2$ (see Figure 1), which is admissible (see Definition 2.2).

By Proposition 2.5, we can find a continuously differentiable map $g : N \to X$ satisfying $g|_{D_i} = f_i$ for $i = 1, 2$, which consequently belongs to $\mathcal{A}^1(N, X)$ by Proposition 2.6.

By Theorem 2.3, there exists a holomorphic map f defined on a neighborhood Ω of N that approximates g with $\|f - g\|_{\mathcal{C}^1} < \epsilon$ on N. By suitably shrinking Ω, we may take it to be a dumbbell-shaped domain S of the required form. □

3. **Three Equivalent Characterizations of Weak Oka-1 Manifolds**

Proposition 3.1. *The following properties of a complex manifold are equivalent:*

(i) *the weak Oka-1 property (Definition 1.3),*
(ii) *the local Runge approximation property on discs (Definition 1.6),*
(iii) *the Runge approximation property on discs (Definition 1.7).*

Proof. **(i) $\Longrightarrow$ (ii):** Let $f_1 : \overline{\mathbb{D}} \to X$ be a holomorphic map and let $f_2 : \overline{\mathbb{D}}(\frac{3}{2}, \frac{1}{3}) \to X$ be a constant map defined on the closed disc $\overline{\mathbb{D}}(\frac{3}{2}, \frac{1}{3}) \subset \overline{\mathbb{D}}_2$. For any $\epsilon > 0$, by Definition 1.3, there exists a holomorphic map $f : \widehat{U} \to X$ on a neighborhood $\widehat{U}$ of $\overline{\mathbb{D}}_2$ that approximates f_1 with $\mathrm{d}_X(f, f_1) < \epsilon$ on $\overline{\mathbb{D}}$.

(ii) $\Longrightarrow$ (iii): Fix $\epsilon > 0$ and set $\epsilon_k = \epsilon/2^{k+1}$ for each integer $k \geqslant 0$. Beginning with a holomorphic map $f_0 \colon U \to X$ defined on a neighborhood U of $\overline{\mathbb{D}}$, by Definiton 1.6, there exists a holomorphic map $f_1 \colon U_1 \to X$ defined on a neighborhood U_1 of $\overline{\mathbb{D}}_2$ such that $\sup_{z \in \overline{\mathbb{D}}} \mathrm{d}_X(f_1(z), f_0(z)) \leqslant \epsilon_1$.

Proceeding inductively, for each integer $k \geqslant 1$, suppose we have a holomorphic map $f_k \colon U_k \to X$ defined on a neighborhood U_k of $\overline{\mathbb{D}}_{2^k}$. Applying Definition 1.6 again, we obtain a holomorphic map $f_{k+1} \colon U_{k+1} \to X$ defined on a neighborhood U_{k+1} of $\overline{\mathbb{D}}_{2^{k+1}}$ satisfying $\sup_{z \in \overline{\mathbb{D}}_{2^k}} \mathrm{d}_X(f_{k+1}(z), f_k(z)) \leqslant \epsilon_{k+1}$. This iterative process yields a sequence of maps $\{f_k\}_{k \geqslant 0}$ whose limit exists from the error estimate. Denoted by such limit $f \colon \mathbb{C} \to X$. By our construction, there holds

$$\sup_{z \in \overline{\mathbb{D}}} \mathrm{d}_X(f(z), f_0(z)) \leqslant \sum_{k \geqslant 0} \sup_{z \in \overline{\mathbb{D}}_{2^k}} \mathrm{d}_X(f_{k+1}(z), f_k(z)) \leqslant \sum_{k \geqslant 0} \epsilon_{k+1} \leqslant \epsilon.$$

(iii) $\Longrightarrow$ (i): Let D_1, D_2 be disjoint closed discs in a larger closed disc $D \subset \mathbb{C}$. Let $F : U \to X$ be a holomorphic map from a neighborhood U of $D_1 \cup D_2$. Given an arbitrary $\epsilon > 0$, Lemma 2.7 yields a holomorphic map $G \colon S \to X$ on a dumbbell neighborhood S of $D_1 \cup D_2$ such that,

$$\mathrm{d}_X\left(G(z), F(z)\right) \leqslant \epsilon/3, \quad \text{for any } z \in D_1 \cup D_2. \tag{3.1}$$

Since S is simply connected, there exists a conformal map $\phi : S \to \mathbb{D}$. We choose a sufficiently small $\delta > 0$ such that $\phi(D_1 \cup D_2) \subset \overline{\mathbb{D}}_{1-2\delta}$ and, by applying Definition 1.7 to $G \circ \phi^{-1} \colon \overline{\mathbb{D}}_{1-\delta} \to X$, we obtain an entire holomorphic map $H : \mathbb{C} \to X$ satisfying

$$\mathrm{d}_X\left(H(w), G \circ \phi^{-1}(w)\right) \leqslant \epsilon/3, \quad \text{for any } w \in \overline{\mathbb{D}}_{1-\delta}. \tag{3.2}$$

Now choose a compact set $S' \subset S$ containing $D_1 \cup D_2$ with path-connected complement. By Runge's approximation theorem (Theorem 2.1) and compactness argument, there exists a polynomial $\psi : \mathbb{C} \to \mathbb{C}$ approximating ϕ very closely on S' such that $\psi\left(D_1 \cup D_2\right) \subset \mathbb{D}_{1-\delta}$ and such that

$$\mathrm{d}_X\left(G \circ \phi^{-1}(\psi(z)), G \circ \phi^{-1}(\phi(z))\right) \leqslant \epsilon/3, \quad \text{for any } z \in D_1 \cup D_2. \tag{3.3}$$

Define $\widehat{F}(z) \coloneqq H \circ \psi(z) \colon \mathbb{C} \to X$. Summarizing estimates (3.1)–(3.3), we have, for any $z \in D_1 \cup D_2$,

$$\begin{aligned} \mathrm{d}_X\left(\widehat{F}(z), F(z)\right) \leqslant &\, \mathrm{d}_X\left(H \circ \psi(z), G \circ \phi^{-1} \circ \psi(z)\right) + \mathrm{d}_X\left(G \circ \phi^{-1}(\psi(z)), G \circ \phi^{-1}(\phi(z))\right) \\ &+ \mathrm{d}_X\left(G(z), F(z)\right) \leqslant \epsilon. \end{aligned}$$

Thus we conclude the proof. $\square$

4. **Proof of the Main Theorem**

4.1. **Technical preparations.** Our proof of the Main Theorem is based on a sophisticated improvement of the "ping-pong" strategy introduced in [Xie24]. First of all, we need the following key result.

Lemma 4.1 ([Xie24, Observation 3.1])**.** *There exists a countable family of nonconstant holomorphic discs* $\big\{f_i\colon \overline{\mathbb{D}}_{\rho_i} \to X\big\}_{i\geqslant 1}$*, holomorphically defined on larger neighborhoods, which generates all Ahlfors currents on* X*.*

Another key ingredient in our construction is the following lemma.

Lemma 4.2. *For each* $k \geqslant 1$*, let* $\{S_k = \mathbb{D} \cup H_k \cup D\}_{k\geqslant 1}$ *be a shrinking sequence of dumbbell domains, where* D *is an open disk away from* $\mathbb{D}$*, and where* $\{H_k\}_{k\geqslant 1}$ *is a shrinking sequence of connecting strips whose width decrease to* 0 *as* $k \to \infty$ *(see Figure 2). Let* $\{\phi_k\colon \mathbb{D} \to S_k\}_{k\geqslant 1}$ *be a sequence of normalized conformal mappings with* $\phi_k(0) = 0$ *and* $\phi_k'(0) > 0$ *for each* k*. Then* $\{\phi_k\}_{k\geqslant 1}$ *converges to the identity map uniformly on compact set in* $\mathbb{D}$ *as* $k \to \infty$*.*

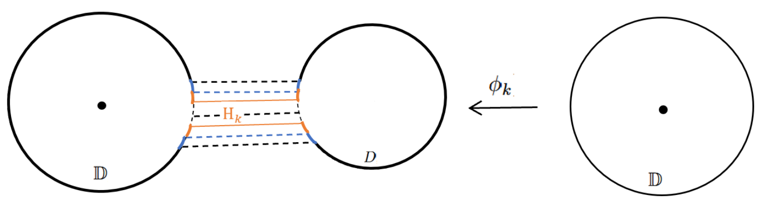

FIGURE 2. Dumbbell domain convergence.

This lemma follows directly from the Carathéodory kernel convergence theorem (cf. [Pom92, Theorem 1.8]). It plays a key role in our "ping-pong" strategy (see **Trick 4**), a term motivated by the configuration in Figure 2, where the two discs resemble a ping-pong ball on opposite sides of a table.

4.2. **A Novel Ping-Pong Induction Process for Entire Curve Construction.** Let $\{g_j\colon \overline{\mathbb{D}}_{r_j} \to X\}_{j\geqslant 1}$ be a sequence of nonconstant holomorphic discs that contains every f_i in Lemma 4.1 infinitely often.

Our objective is to "amalgamate", in a certain sense to be specified later, the family of holomorphic discs $\{g_j\}_{j\geqslant 1}$ into a sequence of concentric discs arising from one entire curve.

Base Case ($n = 1$**).** Fix a countable dense subset $\{\eta_j\}_{j=1}^{\infty}$ of $\mathcal{A}^{1,1}(X)$, the space of smooth $(1,1)$-forms on X endowed with the standard Fréchet topology. We initialize the construction by setting:

$$R_1 := r_1, \quad \epsilon_1 := 1, \quad F_1 := g_1\colon \overline{\mathbb{D}}_{R_1} \to X.$$

Inductive Step: From n **to** $n+1$**.** We now proceed with the inductive construction of F_{n+1} from F_n. The input is the holomorphic disc $F_n\colon U_n \to X$, defined on a neighborhood U_n of $\overline{\mathbb{D}}_{R_n}$, which was obtained in the previous step. We fix an auxiliary relatively compact disc $U_n' \Subset U_n$ containing $\overline{\mathbb{D}}_{R_n}$.

Trick 1: Set an Negligible Error Bound. We choose a sufficiently small error bound $\epsilon_{n+1} > 0$ satisfying

$$\epsilon_{n+1} < \epsilon_n/2, \tag{4.1}$$

such that for any holomorphic map $f\colon U_n' \to X$ approximating F_n closely on U_n':

$$\sup_{z\in U_n'} \mathrm{d}_X\left(f(z), F_n(z)\right) < \epsilon_{n+1},$$

the following two estimates hold:

$$\left|\frac{\mathrm{Length}_\omega\, f(\partial\mathbb{D}_{R_n})}{\mathrm{Area}_\omega\, f(\mathbb{D}_{R_n})} - \frac{\mathrm{Length}_\omega\, F_n(\partial\mathbb{D}_{R_n})}{\mathrm{Area}_\omega\, F_n(\mathbb{D}_{R_n})}\right| < 2^{-n}, \tag{4.2}$$

and for all $j = 1, \ldots, n$,

$$\left|\frac{\int_{\mathbb{D}_{R_n}} f^*\eta_j}{\mathrm{Area}_\omega(f(\mathbb{D}_{R_n}))} - \frac{\int_{\mathbb{D}_{R_n}} F_n^*\eta_j}{\mathrm{Area}_\omega(F_n(\mathbb{D}_{R_n}))}\right| < 2^{-n}. \tag{4.3}$$

The existence of such ϵ_{n+1} can be established by a reductio ad absurdum argument. Indeed, $\mathcal{C}^0$-approximation of F_n by f on the larger domain $U_n' \supset \overline{\mathbb{D}}_{R_n}$ implies $\mathcal{C}^1$-approximation on the compact subset $\overline{\mathbb{D}}_{R_n}$ via Cauchy's estimates. On the other hand, when f is sufficiently close to F_n in the $\mathcal{C}^1$-norm on $\overline{\mathbb{D}}_{R_n}$, the left-hand sides of inequalities (4.2) and (4.3) will automatically be close to zero.

Such an ϵ-trick also played a crucial role in [Xie24]. It enables us to "extend" holomorphic maps from smaller to larger domains (up to negligible errors) by directly applying Runge's approximation theorem, while preserving the desired estimates.

Trick 2: Double the Disc Radius. Set $R_{n+1} := 2R_n$. Applying the local Runge approximation property of X on discs (Definition 1.6), we can find some open neighborhood V_{n+1} of $\overline{\mathbb{D}}_{R_{n+1}}$ and a holomorphic map $G_{n+1}\colon V_{n+1} \to X$ satisfying

$$\sup_{z\in U_n'} \mathrm{d}_X\big(F_n(z), G_{n+1}(z)\big) < \epsilon_{n+1}/8. \tag{4.4}$$

This "extension" requirement is, as a matter of fact, the motivation behind Definition 1.6.

Trick 3: Bridge the Two Discs by an Almost Geodesic to a Dumbbell Domain. To simplify notations, we normalize g_{n+1} to a unit holomorphic disc by setting

$$\widehat{g}_{n+1}(z) := g_{n+1}(r_{n+1}\cdot z)\colon \overline{\mathbb{D}} \to X.$$

Note that this normalization, introduced purely for expository convenience, preserves the associated Ahlfors-type current, i.e., $A_{\widehat{g}_{n+1}} = A_{g_{n+1}}$ by (1.1).

Choose a point $c_{n+1} \in \mathbb{R}_+$ on the positive real axis sufficiently far from the origin such that

$$\overline{\mathbb{D}}_{2R_n} \cap \overline{\mathbb{D}}(c_{n+1}, 1) = \emptyset.$$

Let $L_{n+1} \subset \mathbb{R}$ be the segment on the real axis connecting $\overline{\mathbb{D}}_{R_{n+1}} = \overline{\mathbb{D}}_{2R_n}$ and $\overline{\mathbb{D}}(c_{n+1}, 1)$, with endpoints $q_{n+1} = R_{n+1}$ and $p_{n+1} = c_{n+1} - 1$. Define the compact set

$$K_{n+1} = \overline{\mathbb{D}}_{R_{n+1}} \cup L_{n+1} \cup \overline{\mathbb{D}}(c_{n+1}, 1),$$

which is admissible by Definition 2.2.

For every integer $k \geqslant 1$, define the map

$$\widetilde{g}_{n+1,k}(z) := \widehat{g}_{n+1}\big((z - c_{n+1})^k\big)\colon \overline{\mathbb{D}}(c_{n+1}, 1) \to X.$$

Observe that the associated Ahlfors-type currents and the length-area ratios remain unchanged:

$$(4.5)\qquad A_{\widetilde{g}_{n+1,k}} = A_{g_{n+1}}, \quad \frac{\mathrm{Length}_\omega\, \widetilde{g}_{n+1,k}(\partial \mathbb{D}(c_{n+1},1))}{\mathrm{Area}_\omega\, \widetilde{g}_{n+1,k}(\mathbb{D}(c_{n+1},1))} = \frac{\mathrm{Length}_\omega\, g_{n+1}(\partial \mathbb{D}_{r_{n+1}})}{\mathrm{Area}_\omega\, g_{n+1}(\mathbb{D}_{r_{n+1}})}, \quad \forall\, k \geqslant 1.$$

This is because the identity

$$(4.6)\qquad \frac{k\cdot M}{k\cdot N} = \frac{M}{N}, \quad \forall\, M, N \in \mathbb{C},\ N \neq 0,\ k \neq 0,$$

which directly reflects in the computations.

By Propositions 2.5 and 2.6 and Lemma 2.7, we can conclude that, for every $k \geqslant 1$ there exists a map $\chi_{n+1,k} \in \mathcal{A}^1(K_{n+1}, X)$ linking G_{n+1} and $\widetilde{g}_{n+1,k}$ in the sense:

$$(4.7)\qquad \chi_{n+1,k}\big|_{\overline{\mathbb{D}}_{R_{n+1}}} = G_{n+1}, \quad \text{and} \quad \chi_{n+1,k}\big|_{\overline{\mathbb{D}}(c_{n+1},1)} = \widetilde{g}_{n+1,k},$$

and such that the lengths of the connecting paths $\chi_{n+1,k}(L_{n+1})$ are uniformly bounded:

$$(4.8)\qquad \mathrm{Length}_\omega\left(\chi_{n+1,k}(L_{n+1})\right) \leqslant \mathrm{diam}_\omega(X) + 1 = O(1), \qquad \forall\, k \geqslant 1,$$

where $\mathrm{diam}_\omega(X)$ is the diameter of X with respect to ω. Geometrically, K_{n+1} looks like a dumbbell-shaped domain with an infinitely thin neck L_{n+1}. Its boundary is therefore understood as

$$\partial K_{n+1} = \partial \mathbb{D}_{R_{n+1}} + \partial \mathbb{D}(c_{n+1}, 1) + 2L_{n+1},$$

where the segment L_{n+1} is counted twice for good reasons that will become apparent shortly (see (4.15)).

We choose a sufficiently large integer $k \gg 1$ such that the following two estimates hold:

$$(4.9)\qquad \left| \frac{\mathrm{Length}_\omega\, \chi_{n+1,k}(\partial K_{n+1})}{\mathrm{Area}_\omega\, \chi_{n+1,k}(K_{n+1})} - \frac{\mathrm{Length}_\omega\, g_{n+1}(\partial \mathbb{D}_{r_{n+1}})}{\mathrm{Area}_\omega\, g_{n+1}(\mathbb{D}_{r_{n+1}})} \right| < 2^{-(n+1)},$$

$$(4.10)\qquad \left| \frac{\int_{K_{n+1}} \chi_{n+1,k}^* \eta_j}{\mathrm{Area}_\omega(\chi_{n+1,k}(K_{n+1}))} - \frac{\int_{\mathbb{D}_{r_{n+1}}} g_{n+1}^* \eta_j}{\mathrm{Area}_\omega(g_{n+1}(\mathbb{D}_{r_{n+1}}))} \right| < 2^{-(n+1)}, \quad j = 1, 2, \ldots, n+1.$$

Such a choice of k is possible because, as k tends to infinity, the left-hand sides of both the above inequalities tend to zero. This follows from (4.5), (4.8), and the asymptotic identity

$$(4.11)\qquad \lim_{k\to\infty} \frac{k\cdot M + O(1)}{k\cdot N + O(1)} = \frac{M}{N}, \quad \forall\, M, N \in \mathbb{C},\ N \neq 0,$$

where each occurrence of $O(1)$ denotes a complex number with modulus bounded from above by a positive number independent of k, though these bounded numbers $O(1)$ may differ in different instances.

Trick 4: Round the Dumbbell via Riemann Mapping. Since $\chi_{n+1,k} \in \mathcal{A}^1(K_{n+1}, X)$, by a gluing technique of Fornaess and Stout [FSt77a, FSt77b], or by applying Theorem 2.3, we can find a holomorphic map $h_{n+1,k}$, defined on a neighborhood $S_{n+1,k}$ of K_{n+1}, that approximates $\chi_{n+1,k}$ very closely in the $\mathcal{C}^1$-norm on K_{n+1}, such that in particular

$$(4.12)\qquad \sup_{z\in U_n'} \mathrm{d}_X\left(h_{n+1,k}(z), \chi_{n+1,k}(z)\right) = \sup_{z\in U_n'} \mathrm{d}_X\left(h_{n+1,k}(z), G_{n+1}(z)\right) < \epsilon_{n+1}/8,$$

and such that the estimates (4.9) and (4.10) remain valid when $\chi_{n+1,k}$ is replaced by $h_{n+1,k}$:

$$(4.13)\qquad \left| \frac{\mathrm{Length}_\omega\, h_{n+1,k}(\partial K_{n+1})}{\mathrm{Area}_\omega\, h_{n+1,k}(K_{n+1})} - \frac{\mathrm{Length}_\omega\, g_{n+1}(\partial \mathbb{D}_{r_{n+1}})}{\mathrm{Area}_\omega\, g_{n+1}(\mathbb{D}_{r_{n+1}})} \right| < 2^{-(n+1)},$$

$$\left| \frac{\int_{K_{n+1}} h_{n+1,k}^* \eta_j}{\mathrm{Area}_\omega(h_{n+1,k}(K_{n+1}))} - \frac{\int_{\mathbb{D}_{r_{n+1}}} g_{n+1}^* \eta_j}{\mathrm{Area}_\omega(g_{n+1}(\mathbb{D}_{r_{n+1}}))} \right| < 2^{-(n+1)}, \quad j = 1, 2, \dots, n+1. \tag{4.14}$$

We then select a sufficiently small dumbbell-shaped domain $H_{n+1,k} \Subset S_{n+1,k}$ that "almost contains" K_{n+1}, composed of the interior $\mathring{K}_{n+1} = \mathbb{D}_{R_{n+1}} \cup \mathbb{D}(c_{n+1}, 1)$ together with a very thin neck region that runs parallel to and fully contains L_{n+1}, such that the following inequalities are preserved:

$$\left| \frac{\mathrm{Length}_\omega\, h_{n+1,k}(\partial H_{n+1,k})}{\mathrm{Area}_\omega\, h_{n+1,k}(H_{n+1,k})} - \frac{\mathrm{Length}_\omega\, g_{n+1}(\partial \mathbb{D}_{r_{n+1}})}{\mathrm{Area}_\omega\, g_{n+1}(\mathbb{D}_{r_{n+1}})} \right| < 2^{-(n+1)}, \tag{4.15}$$

$$\left| \frac{\int_{H_{n+1,k}} h_{n+1,k}^* \eta_j}{\mathrm{Area}_\omega(h_{n+1,k}(H_{n+1,k}))} - \frac{\int_{\mathbb{D}_{r_{n+1}}} g_{n+1}^* \eta_j}{\mathrm{Area}_\omega(g_{n+1}(\mathbb{D}_{r_{n+1}}))} \right| < 2^{-(n+1)}, \quad j = 1, 2, \dots, n+1. \tag{4.16}$$

Such a domain $H_{n+1,k}$ exists because, as the width of the neck tends to zero, the left-hand sides of both the above inequalities converge to the corresponding left-hand sides in (4.13) and (4.14), respectively.

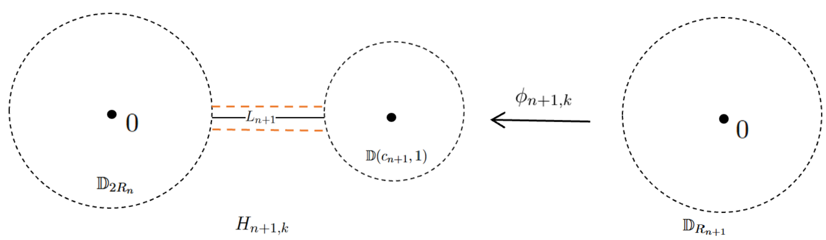

FIGURE 3. Conformal map from dumbbell domain to disc.

Furthermore, by shrinking the neck region if necessary, Lemma 4.2 ensures that the holomorphic map

$$\mu_{n+1} \coloneqq h_{n+1,k} \circ \phi_{n+1,k} \colon \mathbb{D}_{R_{n+1}} \to X$$

approximates $h_{n+1,k}$ closely on U'_n with:

$$\sup_{z \in U'_n} \mathrm{d}_X\left(\mu_{n+1}(z), h_{n+1,k}(z)\right) < \epsilon_{n+1}/8, \tag{4.17}$$

where $\phi_{n+1,k} \colon \mathbb{D}_{R_{n+1}} \to H_{n+1,k}$ denotes the unique conformal mapping normalized by the conditions $\phi_{n+1,k}(0) = 0$ and $\phi'_{n+1,k}(0) > 0$; see Figure 3.

Therefore the inequalities (4.15) and (4.16) read as

$$\left| \frac{\mathrm{Length}_\omega\, \mu_{n+1}(\partial \mathbb{D}_{R_{n+1}})}{\mathrm{Area}_\omega\, \mu_{n+1}(\mathbb{D}_{R_{n+1}})} - \frac{\mathrm{Length}_\omega\, g_{n+1}(\partial \mathbb{D}_{r_{n+1}})}{\mathrm{Area}_\omega\, g_{n+1}(\mathbb{D}_{r_{n+1}})} \right| < 2^{-(n+1)}, \tag{4.18}$$

$$\left| \frac{\int_{\mathbb{D}_{R_{n+1}}} \mu_{n+1}^* \eta_j}{\mathrm{Area}_\omega(\mu_{n+1}(\mathbb{D}_{R_{n+1}}))} - \frac{\int_{\mathbb{D}_{r_{n+1}}} g_{n+1}^* \eta_j}{\mathrm{Area}_\omega(g_{n+1}(\mathbb{D}_{r_{n+1}}))} \right| < 2^{-(n+1)}, \quad j = 1, 2, \dots, n+1. \tag{4.19}$$

This trick, in fact, forms the cornerstone of our novel approach. It provides an appropriate adaptation of Birkhoff's classical technique [Bir29] for the specific purpose of "amalgamating" Ahlfors-type currents in our setting.

Trick 5: Slightly Enlarge the Disc. Take a very small $0 < \delta_{n+1} \ll 1$ such that the holomorphic map

$$F_{n+1}(z) := \mu_{n+1}((1-\delta_{n+1}) \cdot z),$$

defined on a neighborhood U_{n+1} of $\overline{\mathbb{D}}_{R_{n+1}}$, closely approximates μ_{n+1} on $U'_n \Subset \mathbb{D}_{R_{n+1}}$:

$$\sup_{z \in U'_n} \mathrm{d}_X\left(F_{n+1}(z), \mu_{n+1}(z)\right) < \epsilon_{n+1}/8, \tag{4.20}$$

and such that, the estimates (4.18) and (4.19) maintain after replacing μ_{n+1} by F_{n+1}:

$$\left| \frac{\mathrm{Length}_\omega\, F_{n+1}(\partial \mathbb{D}_{R_{n+1}})}{\mathrm{Area}_\omega\, F_{n+1}(\mathbb{D}_{R_{n+1}})} - \frac{\mathrm{Length}_\omega\, g_{n+1}(\partial \mathbb{D}_{r_{n+1}})}{\mathrm{Area}_\omega\, g_{n+1}(\mathbb{D}_{r_{n+1}})} \right| < 2^{-(n+1)}, \tag{4.21}$$

$$\left| \frac{\int_{\mathbb{D}_{R_{n+1}}} F_{n+1}^* \eta_j}{\mathrm{Area}_\omega(F_{n+1}(\mathbb{D}_{R_{n+1}}))} - \frac{\int_{\mathbb{D}_{r_{n+1}}} g_{n+1}^* \eta_j}{\mathrm{Area}_\omega(g_{n+1}(\mathbb{D}_{r_{n+1}}))} \right| < 2^{-(n+1)}, \quad j = 1, 2, \ldots, n+1. \tag{4.22}$$

The argument is based on continuity: as δ_{n+1} tends to zero, the left-hand sides of the two inequalities above converge to their counterparts in (4.18) and (4.19), respectively. Notably, the purpose of this scaling trick is precisely to ensure that F_{n+1} is defined on a slightly larger domain than $\overline{\mathbb{D}}_{R_{n+1}}$, while preserving the required estimates.

This completes the "amalgamation" of g_{n+1} into F_{n+1}.

Combining (4.4), (4.12), (4.17) and (4.20), we conclude that

$$\sup_{z \in U'_n} \mathrm{d}_X\left(F_{n+1}(z), F_n(z)\right) < \epsilon_{n+1}/2. \tag{4.23}$$

This completes the inductive construction at *Step* $n+1$.

4.3. **Final Construction of the Entire Curve.** By the inductive estimates (4.1) and (4.23), our obtained sequence $\{F_n \colon \overline{\mathbb{D}}_{R_n} \to X\}_{n \geqslant 1}$ of holomorphic discs converges uniformly on compact sets to an entire curve $F \colon \mathbb{C} \to X$. Indeed, for any $n \geqslant 1$ and $z \in U'_n$, we have:

$$\mathrm{d}_X(F(z), F_n(z)) \leqslant \sum_{m \geqslant n} \mathrm{d}_X(F_{m+1}(z), F_m(z)) < \sum_{m \geqslant n} \frac{\epsilon_{m+1}}{2} < \sum_{m \geqslant n} \frac{\epsilon_{n+1}}{2^{m-n+1}} = \epsilon_{n+1}.$$

This uniform estimate ensures that F inherits the geometric estimates from the approximating sequence. Specifically, **Trick 1** ensures the following proximity conditions:

$$\left| \frac{\mathrm{Length}_\omega\, F(\partial \mathbb{D}_{R_n})}{\mathrm{Area}_\omega\, F(\mathbb{D}_{R_n})} - \frac{\mathrm{Length}_\omega\, F_n(\partial \mathbb{D}_{R_n})}{\mathrm{Area}_\omega\, F_n(\mathbb{D}_{R_n})} \right| < 2^{-n}, \tag{4.24}$$

and for all $j = 1, \ldots, n$:

$$\left| \frac{\int_{\mathbb{D}_{R_n}} F^* \eta_j}{\mathrm{Area}_\omega(F(\mathbb{D}_{R_n}))} - \frac{\int_{\mathbb{D}_{R_n}} F_n^* \eta_j}{\mathrm{Area}_\omega(F_n(\mathbb{D}_{R_n}))} \right| < 2^{-n}. \tag{4.25}$$

Now, let $\mathcal{T}$ be an Ahlfors current on X. By the construction of the sequence $\{g_j\}_{j \geqslant 1}$ (where each f_i from Lemma 4.1 appears infinitely often), there exists a subsequence $\{g_{n_j}\}_{j \geqslant 1}$ satisfying the length-area condition (1.2) that generates $\mathcal{T}$.

Claim. $\mathcal{T}$ is also generated by the concentric holomorphic discs $\{F|_{\overline{\mathbb{D}}_{R_{n_j}}}\}_{j \geqslant 1}$.

Proof. This sequence clearly satisfies the length-area condition (1.2) by the estimates (4.21) and (4.24). For any η_k in our fixed dense subset $\{\eta_k\}_{k=1}^{\infty}$ of $\mathcal{A}^{1,1}(X)$, the inequalities (4.22) and (4.25) guarantee

$$\lim_{j\to\infty} \frac{\int_{\mathbb{D}_{R_{n_j}}} F^*\eta_k}{\mathrm{Area}_\omega(F(\mathbb{D}_{R_{n_j}}))} = \lim_{j\to\infty} \frac{\int_{\mathbb{D}_{R_{n_j}}} F_{n_j}^*\eta_k}{\mathrm{Area}_\omega(F_{n_j}(\mathbb{D}_{R_{n_j}}))} = \lim_{j\to\infty} \frac{\int_{\mathbb{D}_{r_{n_j}}} g_{n_j}^*\eta_k}{\mathrm{Area}_\omega(g_{n_j}(\mathbb{D}_{r_{n_j}}))} = \mathcal{T}(\eta_k).$$

Therefore the sequence $\{F|_{\overline{\mathbb{D}}_{R_{n_j}}}\}_{j\geqslant 1}$ indeed generates $\mathcal{T}$ by the denseness of $\{\eta_k\}_{k\geqslant 1}$. $\square$

5. Open Problems

To conclude, we would like to emphasize the following challenges in the study of (weak) Oka-1 manifolds.

Question 5.1. Let X be a compact complex manifold with the Runge approximation property on discs. Does there exist an entire curve $f\colon \mathbb{C}\to X$ such that the concentric holomorphic discs $\{f|_{\overline{\mathbb{D}}_r}\}_{r>0}$ generate all Nevanlinna currents on X?

The main difficulty stems from the fact that the definition of Nevanlinna currents (cf., e.g. [WX25]) involves logarithmic integration $\frac{\mathrm{d}t}{t}$ rather than the linear measure $\mathrm{d}t$, which makes a direct adaptation of our method impossible. If the answer is affirmative, this would establish the natural analogue of Sibony's Conjecture 1.2 for Nevanlinna currents.

Definition 5.2 (Oka-1 manifold [AF25])**.** A connected complex manifold X with a complete distance function d_X is *Oka-1* if for any open Riemann surface R, compact Runge set $K\subset R$, discrete sequence $\{a_i\}_{i\geqslant 1}$ in R, continuous map $f\colon R\to X$ holomorphic near $K\cup\{a_i\}_{i\geqslant 1}$, $\epsilon>0$, and $k_i\in\mathbb{N}$, there exists a holomorphic map $F\colon R\to X$ homotopic to f such that:

(1) $\sup_{p\in K}\mathrm{d}_X(F(p),f(p))<\epsilon$,
(2) F agrees with f to order k_i at each a_i.

Question 5.3 ([Xie24, Question 3.4])**.** Does the weak Oka-1 property imply the Oka-1 property?

An affirmative answer, meaning that the Runge approximation property on discs (see Proposition 3.1) implies Oka-1, would provide a one-dimensional analogue of Forstnerič's celebrated theorem [Fors06] that the Runge approximation property on convex sets implies the Oka property.

Conjecture 5.4 ([AF25, Conjecture 9.1])**.** Every rationally connected projective manifold is Oka-1.

Supporting evidence can be found in [CW23]. This conjecture is particularly intriguing from a complex-analytic perspective. The rational connectedness of Fano manifolds, established by Campana [Cam92] and by Kollár–Miyaoka–Mori [KMM92], relies on Mori's celebrated "bend-and-break" method [Mor79]—a technique that uses characteristic p arguments. Establishing the (weak) Oka-1 property for such manifolds would represent significant progress toward the long-standing challenge of constructing rational curves on Fano manifolds via purely analytic method.

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